Another Look at Portfolio Optimization under Tracking-Error Constraints

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Another Look at Portfolio Optimization under Tracking-Error Constraints

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Abstract

Today, the use of a benchmark portfolio is common practice in the financial management industry. This setup allows the investor to evaluate the added value in line with the risks undertaken. But the relevant concept of risk is relative risk as defined by tracking-error volatility.

The problem of minimizing the volatility of tracking error was originally solved by Roll (1992). He noticed that the optimal portfolios obtained have several undesirable properties and then suggested introducing an additional constraint on the beta of the portfolio.

More recently, Jorion (2003) elegantly tackled this problem again, pointing out that constant-TEV portfolios are described by an ellipse. He showed that because of the flat shape of this ellipse, adding a constraint on total portfolio volatility can substantially improve the performance of the managed portfolio.

This paper looks at the problem from another angle. Instead of considering constant TEV frontiers as Jorion does, we allow tracking error to vary but we fix the risk aversion. It is shown that the resulting optimal portfolios have several desirable properties.

JEL classification : G11, G12, G13.
Keywords : benchmark, tracking-error, portfolio optimization, risk aversion.

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1 Introduction

Today, the use of a benchmark portfolio is common practice in the financial management industry. A benchmark portfolio allows relative performance evaluation of the portfolio manager. The benchmark is usually a combination of large diversified indexes of assets. It generally depends on the investor's preference and is, for the manager, exogenously given. This setup allows the investor to evaluate the added value in line with the risks undertaken. But here, the relevant concept of risk is the relative risk as defined by the tracking-error volatility (TEV), the volatility of the deviation of the active portfolio from the benchmark. Relative risk replaces absolute risk which was used in the traditional portfolio selection problem solved by Markowitz (1952).

The problem of minimizing the volatility of the tracking-error was originally solved analytically by Roll (1992). He observed that the optimal portfolios obtained have several undesirable properties. The main problem with this approach is that it neglects absolute portfolio risk. In order to overcome these drawbacks, Roll (1992) suggests introducing an additional constraint on the beta of the portfolio.

More recently, Jorion (2003) elegantly tackled this problem again pointing out that constant-TEV portfolios are described by an ellipse on the traditional mean–variance plane. Jorion showed that because of the flat shape of this ellipse, adding a constraint on total portfolio volatility can substantially improve the performance of the managed portfolio. Herein, we will call these ellipses iso-tracking curves or frontiers.

This paper looks at the problem from another angle and extends previous results of Bertrand, Prigent and Sobotka (hereafter, BPS) (2001). Instead of considering constant TEV frontiers as Jorion does, we allow tracking error to vary but we fix risk aversion (or, more precisely, aversion to variance). The resulting optimal portfolios lie on what we call an iso-aversion frontier. We derive the explicit equation of these iso-aversion frontiers and study analytically their properties. We will show here that these iso-aversion portfolios also have several desirable properties.

The rest of the paper is organized as follows. Section 2 recalls the main contribution of Roll (1992) and Jorion (2003). Section 3 presents the work of BPS (2001) and derives some new results that will be useful in the rest of the work. Section 4 compares the features of iso-tracking and iso-aversion portfolios.

2 The Tracking-Error Frontier

Roll (1992) has solved the optimization program faced by a portfolio manager with a constraint on the tracking-error volatility of her portfolio relative to her benchmark.

We adopt the same set of notations as Roll (1992) and Jorion (2003)[1].

[1] Bold notations are used for matrix.

$\mathbf{q}$: a $(n \times 1)$ vector of a portfolio weights for the n assets. The portfolio $\mathbf{q_B}$ (*resp.* $\mathbf{q_P}$) is the benchmark (*resp.* the active portfolio).

$\mathbf{x} = \mathbf{q_P} - \mathbf{q_B}$: a $(n \times 1)$ vector of deviations from the benchmark.

$\mathbf{R}$: a $(n \times 1)$ vector of expected returns on all assets in the universe.

$\mathbf{V}$: the $(n \times n)$ covariance matrix of individual asset returns (which is supposed to be invertible)

$\sigma_{ii} = \sigma_i^2$: the variance of the return on asset i.

σ_{ij}: the covariance between returns of assets i and j.

$R_B = \mathbf{q_B'R}$: the expected return on the benchmark.

$\sigma_B^2 = \mathbf{q_B'Vq_B}$: the variance of benchmark return.

$G = \mathbf{x'R}$: excess expected return over benchmark's return.

$T = \sqrt{(\mathbf{q_P} - \mathbf{q_B})' \mathbf{V} (\mathbf{q_P} - \mathbf{q_B})} = \sqrt{\mathbf{x'Vx}}$: the volatility of tracking error (*i.e.* the relative risk).

In this setup, the formal problem of an active portfolio manager is to minimize, for every excess expected return G, the relative risk of her portfolio[2]. Thus, she faces the following problem:

$$\text{Problem 1, } P(1)\text{:} \begin{cases} \underset{\mathbf{x}}{Min}\ \mathbf{x'Vx} \\ st: \quad \mathbf{x'R} = G \\ \quad\quad \mathbf{x'e} = 0 \end{cases} \tag{1}$$

This problem was solved by Roll (1992) and we recall the solution:

$$\mathbf{x} = D(\mathbf{q}_1 - \mathbf{q}_0) \tag{2}$$

$$\mathbf{q_P} = \mathbf{q_B} + D(\mathbf{q}_1 - \mathbf{q}_0) \tag{3}$$

where : $D = \dfrac{G}{R_1 - R_0}$

Parameter D can be interpreted as a relative performance objective.

The portfolios given by 3 exhibit three-fund separation.

The variance of the tracking error and the total variance of the optimal portfolio solutions of $P(1)$ are given respectively by:

$$T^2 = D^2 \left(\sigma_1^2 - \sigma_0^2\right) \tag{4}$$

and

$$\sigma^2 = \sigma_B^2 + T^2 + 2D\sigma_0^2 (R_B/R_0 - 1) \tag{5}$$

[2] Alternatively, it is possible to solve the equivalent problem of maximizing expected excess return subject to a constraint on relative risk, as done by Jorion (2003).

The two following special portfolios of the Markowitz frontier are needed:

Table 1: Two special efficient frontier portfolios

Portfolio	Expected Return	Variance	Proportions
0	$R_0 = \frac{b}{c}$	$\sigma_0^2 = \frac{1}{c}$	$\mathbf{q}_0 = \mathbf{V}^{-1}\frac{\mathbf{e}}{c}$
1	$R_1 = \frac{a}{b}$	$\sigma_1^2 = \frac{a}{b^2}$	$\mathbf{q}_1 = \mathbf{V}^{-1}\frac{\mathbf{R}}{b}$

We use the definitions of the efficient frontier constants[3] introduced by Roll (1977): $a = \mathbf{R}'\mathbf{V}^{-1}\mathbf{R}$, $b = \mathbf{e}'\mathbf{V}^{-1}\mathbf{R}$, $c = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$ and $d = a - \frac{b^2}{c} > 0$ ($\mathbf{e}$ is the $(n \times 1)$ unit vector). Portfolio 0 is the minimum variance portfolio (MVP) and portfolio 1 is located in the return-variance space where a line passing between the origin and the minimum variance portfolio intersects the efficient frontier.

The optimal portfolios given by relation (3) are the sum of the benchmark and a term of deviation which depends only on the two special portfolios and on the excess expected return. This second term does not depend on the benchmark. Thus, two managers operating in the same universe (*i.e.* the same n risky assets) having identical expectations but different benchmarks will carry out the same transactions starting from their initial positions (*i.e.* their benchmark).

The variance of the tracking error given by (4) does not depend on the benchmark. On the other hand, it is positively connected with the excess expected return, G. It is natural that there is a trade-off between excess expected return and relative risk.

The efficient frontier and the tracking-error frontier[4] are plotted in figure 1.

Roll (1992) observed that the optimal portfolios have several undesirable properties. First, they are not optimal in a mean-variance sense as shown in figure (1). Secondly, they are riskier than the benchmark. Finally, they have a beta greater than one, meaning that portfolio managers will generally beat their benchmark in up markets. The main problem with this approach is that it neglects absolute portfolio risk. In order to overcome these drawbacks, Roll (1992) suggests a considerable improvement by introducing an additional constraint on the beta of the portfolio (see Roll (1992)).

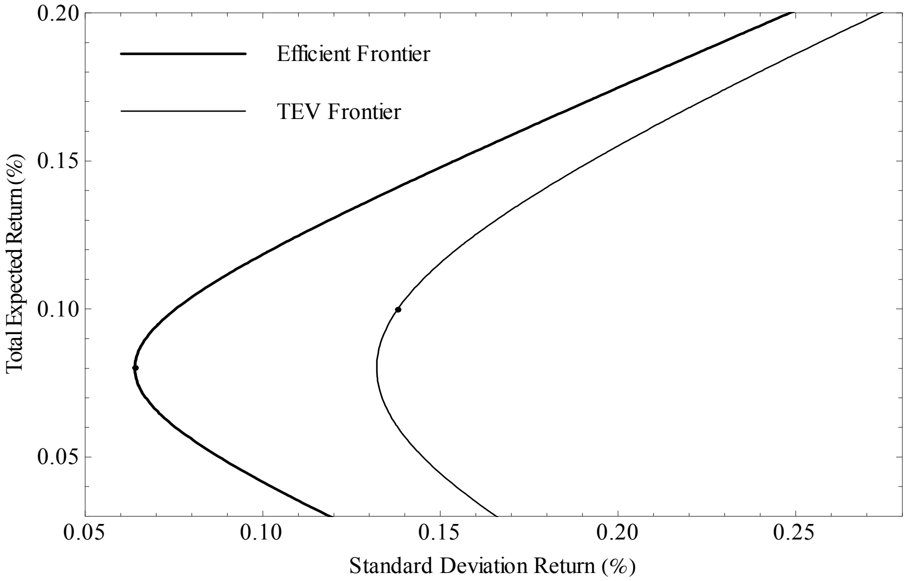
3 The Iso-Tracking Frontier (Jorion 2003)

More recently, Jorion (2003) elegantly tackled this problem again and derives what he calls a constant-TEV Frontier.

[3] Note that the efficient frontier is uniquely determined by the parameters a, b and c or by the parameter d and the MVP (*i.e.* R_0 and σ_0^2).

[4] This frontier is drawn in the usual case where the expected return of the benchmark is greater than the expected return of the minimum variance portfolio, portfolio 0. For the other case, see Roll (1992).

Figure 1: Efficient Frontier and Tracking-Error



He considers a portfolio manager facing the following optimization problem:

$$\text{Problem 2, } P(2)\text{:} \begin{cases} \underset{\mathbf{q_P}}{Max}\ \mathbf{q_P'R} \\ st: \quad (\mathbf{q_P} - \mathbf{q_B})' \mathbf{V} (\mathbf{q_P} - \mathbf{q_B}) = T^2 \\ \qquad \mathbf{q_P'Vq_P} = \sigma_P^2 \\ \qquad \mathbf{q_P'e} = 1 \end{cases} \tag{6}$$

For a fixed amount of tracking-error variance, T^2, the total portfolio variance, σ_P^2, can be varied to obtain the constant-TEV Frontier.

Jorion shows that constant-TEV portfolios are described by an ellipse[5] on the traditional mean–variance plane[6] and studies their properties in a series of theorems.

Figure 2 plots several iso-tracking frontiers for different values of the tracking error. Note that for relatively high tracking error ($T = 24\%$ for example as shown in figure 2), the ellipse does not contain the benchmark portfolio.

Due to the flat shape of this ellipse, Jorion advocated adding a constraint on total portfolio volatility, which can substantially improve the performance of the managed portfolio.

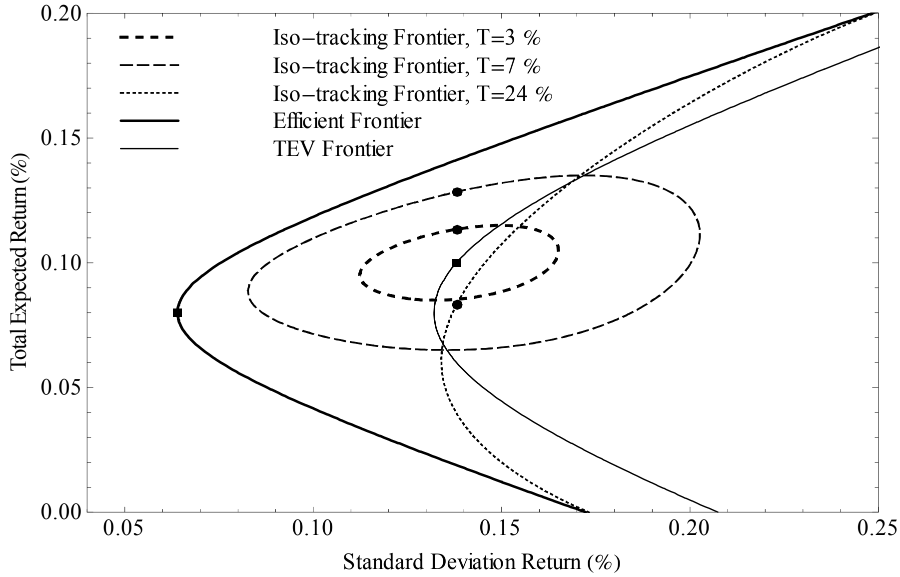
When the level of tracking error has been fixed, the portfolio manager must select a portfolio on the corresponding iso-tracking frontiers. Jorion suggests that the manager should choose the portfolio with the same total risk as the benchmark. This portfolio lies on the top part of the ellipse, which is rather flat. Three such portfolios are plotted in figure 2 for $T = 3\%$, $T = 7\%$ and $T = 24\%$ (they are represented by the three points).

This constraint on total risk is most useful when on the one hand, the tracking-error is

[5] Notice that in the mean-standard deviation plane, the locus of the constant-TEV portfolios is no longer an ellipse.

[6] Herein, we will call these curves iso-tracking frontiers.

Figure 2: Iso-Tracking Frontiers



lower and on the other hand, the benchmark is less efficient.

4 The Iso-aversion or iso-IR Frontier (BPS 2001)

BPS (2001) consider the problem of a portfolio manager who maximizes the mean-variance criterion under a tracking-error constraint. This means reintroducing into the optimization program aversion both to absolute risk and to relative risk (*i.e.* tracking error).

4.1 Derivation of the frontiers

The portfolio manager faces the following optimization problem:

$$\text{Problem 3, } P(3)\text{:} \begin{cases} \underset{\mathbf{q_P}}{Max}\ \mathbf{q_P'R} - \frac{\phi}{2}\mathbf{q_P'Vq_P} \\ st: \quad (\mathbf{q_P} - \mathbf{q_B})' \mathbf{V} (\mathbf{q_P} - \mathbf{q_B}) = T^2 \\ \qquad \mathbf{q_P'e} = 1 \end{cases} \tag{7}$$

The parameter ϕ is defined as the marginal rate of substitution between the expected return of the portfolio and its variance. In what follows, we will call it risk aversion.

The solution is given by:

$$\begin{cases} \mathbf{q_P} = \mathbf{q_B} + \frac{1}{\phi+\lambda}\left(-\phi\mathbf{q_B} + \mathbf{V}^{-1}(\mathbf{R} - \mu\mathbf{e})\right) \\ \text{where} \begin{cases} \lambda + \phi = \frac{\sqrt{P(\phi)}}{T} \\ P(\phi) = \phi^2\Delta_2 - 2\phi\Delta_1 + d \\ \mu = \frac{b-\phi}{c} = R_0 - \phi\sigma_0^2 \end{cases} \end{cases} \tag{8}$$

where:

$$\Delta_1 = R_B - R_0 \tag{9}$$
$$\Delta_2 = \sigma_B^2 - \sigma_0^2 \tag{10}$$

For a given value of the risk aversion parameter, ϕ, an optimal portfolio solution to $P(3)$ is obtained for each value of the tracking error, T.

In order to show that these portfolios exhibit three-fund separation like portfolios on the TEV and on the iso-tracking frontiers, we rewrite expression (8) as:

$$\mathbf{q_P} = \mathbf{q_B} + \frac{T}{\sqrt{P(\phi)}} \left(-\phi \mathbf{q_B} + b\mathbf{q_1} - (b - \phi)\, \mathbf{q_0}\right)$$

Jorion work inspired us to revisit our previous work and take it a stage further. We derive the analytic expression of what we call an iso-aversion frontier: the set of optimal portfolios for a given value of risk aversion, ϕ.

Theorem 1 [7]**:** *Each iso-aversion frontier can be described by a right/left opening hyperbola in the mean–standard deviation plane, centered at* $\left(0, -\frac{\beta}{2\alpha}\right)$ *:*

$$\left(\frac{\sigma_P}{\sqrt{\Delta}}\right)^2 - \left(\frac{R_P + \frac{\beta}{2\alpha}}{\sqrt{\Delta/\alpha}}\right)^2 = 1 \tag{11}$$

Figure 3 represents the graph of some of these hyperbolas.

As risk aversion increases, curves move towards the left of the graph (towards portfolios of lesser absolute risk). When there is strong risk aversion, the manager will tend to choose portfolios whose absolute risk is lower than that of the benchmark. When risk aversion becomes (at the limit) infinite, the optimal portfolio coincides with the minimum variance portfolio of the efficient frontier if we accept a sufficient volatility for the tracking error.

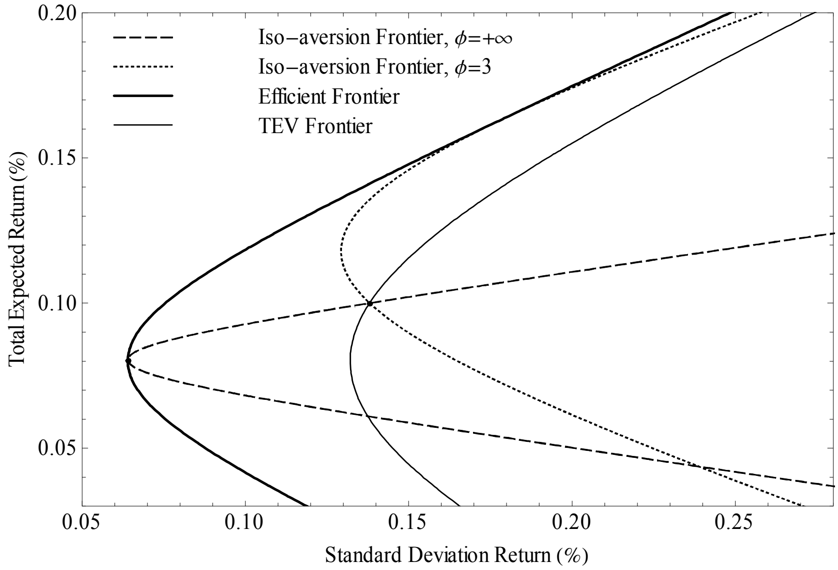
Moreover, for some values of ϕ (*e.g.* $\phi = 3$ on figure 3), portfolios which dominate the benchmark in a mean-variance sense can be obtained.

Notice too that all these curves contain the benchmark. This is a desirable property for practical applications and will be supplemented by proposition 8 below.

When $\phi = 0$, the iso-aversion frontier merges with the TEV frontier. So the tracking-error frontier is readily obtained as the special case where an individual maximizing the mean-variance criterion under a constraint on the tracking error is neutral towards absolute risk ($\phi = 0$).

[7]The proofs are gathered in the appendixes.

Figure 3: Iso-aversion Frontiers for different risk aversions



4.2 Properties of the frontiers

On a practical level, the problem of determining the risk aversion level remains. First, the following result is proved.

Theorem 2 *: Every iso-aversion curve has a unique tangency point with the efficient frontier. This tangency occurs at* $T = \frac{P(\phi)}{\phi}$ *for*[8] $\phi \neq 0$. *The variance of the portfolio return is then equal to* $\sigma_P^2 = \frac{d}{\phi^2} + \sigma_0^2$.

We will propose here to choose the risk aversion, ϕ^* (see corollary 3 below), for which the curve solution of $P(3)$ is tangent to the efficient frontier in a portfolio which has the same risk as the benchmark.

Corollary 3 *: The iso-aversion curve that is tangent to the efficient frontier at the benchmark variance is defined by the following coefficient of risk aversion:*

$$\phi^* = \sqrt{\frac{d}{\Delta_2}}$$

We can easily check that for ϕ^**, the variance of the managed portfolio is equal to the variance of the benchmark:* $\sigma_P^2 = \frac{d}{\phi^{*2}} + \sigma_0^2 = \sigma_B^2$. *We name it: iso-*$\phi^*$ *frontier.*

As shown in the appendix, it is the risk aversion level for which the vertical center of the hyperbola is maximum. In other words, it is the hyperbola that has the minimum variance portfolio with the highest expected return.

[8]Recall that for $\phi = 0$, the iso-aversion frontier is simply the tracking-error frontier which has no point of tangency with the efficient frontier.

We can be said to be choosing a level of risk aversion that is implicit to the benchmark through Δ_2 and to the efficient frontier through d. We thus obtain a curve on which it is possible to choose a portfolio according to the desired volatility and/or to the desired tracking-error.

Note that the coefficient of risk aversion, ϕ^*, is negatively related to the curvature of the efficient frontier through d. Indeed, as d increases, the efficient frontier becomes less curved, its slope steepens. Thus, as d increases, the slope of the tangent at the efficient portfolio with the same volatility as the benchmark increases too. It is quite natural that the resulting new iso-ϕ^* frontier obtains for a higher level of risk aversion, ϕ^*.

Moreover, it is negatively related to the horizontal distance between the benchmark and the MVP, Δ_2. This is because when Δ_2 increases (wether σ_B^2 increases or σ_0^2 decreases), the slope of the tangent at the efficient portfolio with the same volatility as the benchmark decreases, meaning that ϕ^* decreases.

In the following table, we report the values of ϕ^* for a reasonable range of variations for the parameters d and Δ_2.

Table 2: Risk Aversion ϕ^*

	Δ_2									
	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
$d = 0.1$	3.16	3.02	2.89	2.77	2.67	2.58	2.50	2.42	2.36	2.29
$d = 0.2$	4.47	4.26	4.08	3.92	3.78	3.65	3.54	3.43	3.33	3.24
$d = 0.3$	5.48	5.22	5	4.80	4.63	4.47	4.33	4.20	4.08	3.97
$d = 0.4$	6.33	6.03	5.77	5.55	5.34	5.16	5.00	4.85	4.71	4.59
$d = 0.5$	7.07	6.74	6.45	6.20	5.98	5.77	5.59	5.42	5.27	5.13
$d = 0.6$	7.75	7.39	7.07	6.79	6.55	6.32	6.12	5.94	5.77	5.62
$d = 0.7$	8.37	7.98	7.64	7.34	7.07	6.83	6.61	6.42	6.24	6.07
$d = 0.8$	8.94	8.53	8.16	7.84	7.56	7.30	7.07	6.86	6.67	6.49

A lot of empirical work has been done on estimating the coefficient of relative risk aversion. For example, Bliss and Panigirtzoglou (2004, p. 432) or Corrado and Miller (2006, p 104)) provide a summarizing table of these findings. Although there seems to be no consensus among researchers, it appears to be accepted that the coefficient[9] must be greater than 3.5. Thus, the values reported in table 2 are in line with what is found in experimental and empirical studies.

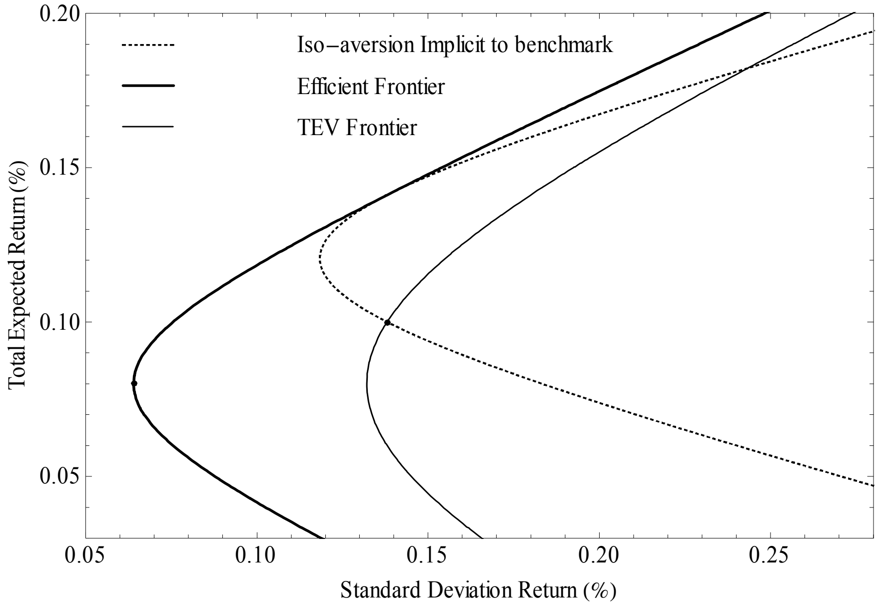
Of course, if the portfolio manager has a precise idea of risk aversion level of the investor, she can select that particular value instead of ϕ^*.

[9]Using data for the United Kingdom, Blake (1996) found that relative risk aversion coefficients are much higher than most previous studies have found. They range from 7.88 to 47.60 across different investor's wealth ranges, with a weighted average value of 35.04.

In a study of the composition of the household portfolio, Flavin and Yamashita (2002) used mainly levels of risk aversion of 2, 4 and 8.

Figure 4 illustrates the result (the risk aversion implicit to the benchmark is, in the example, $\phi^* = 4,023$).

Figure 4: Iso-aversion Frontiers for the risk aversion implicit to the benchmark



In the following result, we study the maximal reduction in variance with respect to the variance of the benchmark that is attainable on the iso-ϕ^* frontier.

Proposition 4 *As the benchmark moves away from the efficient frontier (i.e., either Δ_1 decreases and/or Δ_2 increases), the maximal reduction in variance that is attainable on the iso-IR^* frontier rises. Moreover, as the curvature of the efficient frontier decreases (i.e., as d and thus also the eccentricity of the efficient frontier increases), the maximal reduction in variance increases.*

In the following proposition, we establish that each iso-aversion frontier is uniquely characterized by its associated information ratio.

Proposition 5 *All the portfolios on a iso-aversion frontier have the same value for information ratio, IR:*

$$IR(\phi, d, \Delta_1, \Delta_2) \equiv \frac{R_P - R_B}{T} = \frac{d - \Delta_1 \phi}{\sqrt{P(\phi)}}$$

Note that it is equal for all portfolios because it is independent of the tracking-error, T. Moreover, we can easily check that $IR(\phi, d, \Delta_1, \Delta_2)$ is a decreasing function of ϕ.

We also get:

$$\begin{aligned} IR(0, d, \Delta_1, \Delta_2) &= \sqrt{d} \\ \lim_{\phi \to +\infty} IR(\phi, d, \Delta_1, \Delta_2) &= \frac{-\Delta_1}{\sqrt{\Delta_2}} \end{aligned}$$

This last relationship comes from the property stating that the covariance of the rate of return on the MVP and that on any portfolio (i.e. the benchmark) is always equal to the variance of the rate of return on the MVP (see the appendix of Roll (1977)*).*

Thus, the iso-aversion frontiers can be renamed iso-IR frontiers. At the same time, we can now reinterpret the iso-ϕ^* frontier in terms of the associated information ratio, $IR^* \equiv IR(\phi^*, d, \Delta_1, \Delta_2)$, the information ratio of the efficient portfolio with the same absolute risk as the benchmark. The benchmark can now be seen as an objective both in terms of information ratio and in terms of absolute risk. The iso-ϕ^* frontier can be renamed the iso-IR^* frontier.

The IR being constant along an iso-IR frontier, the portfolio manager can choose a portfolio that exhibits the desired level of absolute risk knowing that her constraint on tracking error permitting, she will be able to pick a portfolio on the upper part of the iso-IR frontier.

We now turn our attention to the beta of the portfolios on the iso-IR frontiers.

Proposition 6 *The beta of the portfolios on the iso-IR frontiers are given by :*

$$\beta(T, \phi, d, \Delta_1, \Delta_2) \equiv \frac{\mathbf{q_P}'\mathbf{V}\mathbf{q_B}}{\mathbf{q_B}'\mathbf{V}\mathbf{q_B}} = 1 + \frac{T}{\sqrt{P(\phi)}} \frac{\Delta_1 - \phi\Delta_2}{\sigma_B^2}$$

we can easily check that :

$$\begin{aligned} \beta(T, \phi, d, \Delta_1, \Delta_2) &> (\textit{resp.} <) 1 \Longleftrightarrow \phi < (\textit{resp.} >) \frac{\Delta_1}{\Delta_2} \\ \beta(T, \phi, d, \Delta_1, \Delta_2) &= 1 \Longleftrightarrow \phi = \frac{\Delta_1}{\Delta_2} \end{aligned}$$

When the beta is equal to one, it becomes independent of T. Thus, all the portfolios along that iso-IR frontier exhibit the same beta of one. Note that this is the only case in which all the portfolios on an iso-IR frontier have the same beta. This frontier is the same as the Roll-beta-frontier for beta equal to one. It is also the only case in which an iso-IR frontier can coincide with a Roll-beta-frontier.

Corollary 7 *: The portfolios on the iso-IR^* frontier, $\phi^* = \sqrt{\frac{d}{\Delta_2}}$, have a beta smaller than one. Moreover, it is decreasing in T.*

5 Comparison of iso-IR and iso-tracking portfolios

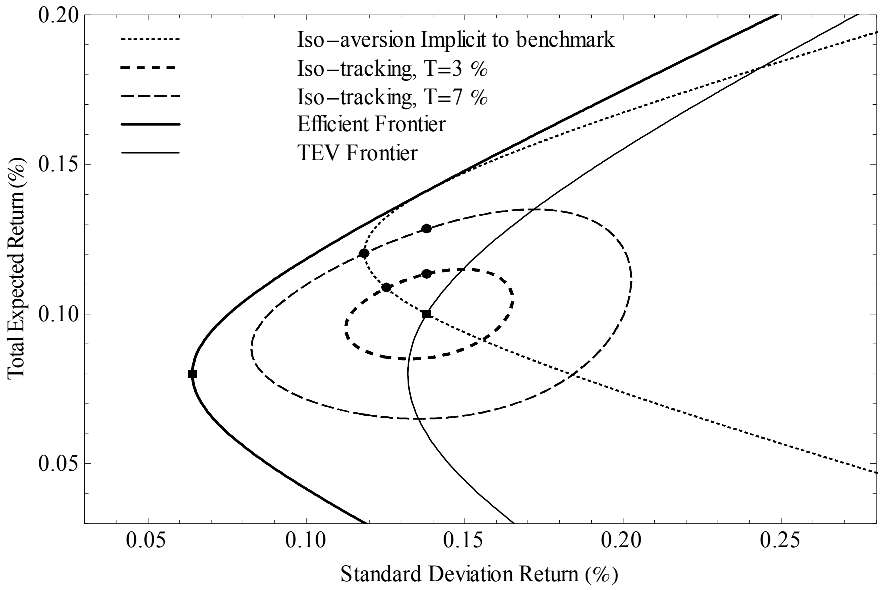
As we have looked at two ways of overcoming the difficulties of the tracking-error frontier, it seems natural to attempt a comparison. Each solution has its own strengths and weaknesses.

As Jorion (2003, p. 72)) observes: "The graph in Figure 2 (figure 1 here) shows an unintended effect of TE optimization: Instead of moving toward the true efficient frontier (i.e., up and to the left of the benchmark), the TE frontier moves up and to the right. This outcome increases the total volatility of the portfolio, which is a direct result of focusing myopically on excess returns instead of total returns"

Note that the iso-IR^* frontier exhibits the property desired by Jorion.

The figure 5 plots the iso-IR^* frontier as well as two iso-tracking frontiers for $T = 3\%$ and $T = 7\%$. The figure plots the iso-IR^* portfolios such that $T = 3\%$ and $T = 7\%$. The portfolios of the iso-tracking frontier with the same volatility as the benchmark for $T = 3\%$ and $T = 7\%$ are also plotted.

Figure 5: Comparison of iso-aversion and iso-tracking portfolios



For the iso-IR^* frontier, the relationship between expected returns and risk is, as expected, nonlinear as opposed to Jorion's solution.

The portfolios on the iso-IR^* frontier are less risky than the benchmark, except for a very high level of tracking error. We know that some investors, while convinced of the importance of a benchmarked portfolio, also want a low level of absolute risk.

5.1 Dominated portfolios

For the iso-IR^* frontier, it may seem astonishing that dominated portfolios can be chosen for low level of tracking error. But, as this frontier takes into account both absolute risk and relative risk, when the constraint on relative risk is slackened enough, it naturally becomes possible to obtain dominant portfolios in a mean-variance sense. Note that the same holds for the portfolios of the iso-tracking frontiers with the same volatility as the benchmark.

Moreover, for iso-tracking frontiers, there is a level of tracking error from which the expected return of the optimal portfolio decreases. It can even become lower than the expected return of the benchmark, as can be seen on figure 2 for a tracking error of 24%. For sufficiently high level of tracking error, *e.g.* $T = 25$ % in the example, the portfolio of the iso-tracking frontier with the same risk as the benchmark does not exist.

5.2 Information ratio and beta

The portfolios on the iso-IR^* frontier have the same information ratio as shown in proposition 5. The following result establishes that the same does not hold for iso-tracking frontiers with the same total risk as the benchmark.

Proposition 8 *The information ratio as a function of the tracking-error decreases for portfolios of the iso-tracking frontiers with the same total risk as the benchmark .*

So, in order to increase tracking error, the portfolio manager tends to decrease the information ratio of the portfolio. Thus, the extra tracking error is not rewarded by a sufficient extra return. Basing their decisions on the information ratio alone, portfolio managers have no incentive to move away from the benchmark. Moreover, as the total risk of the portfolio is set equal to the total risk of the benchmark, taking total risk into account does not give more incentive to move away from the benchmark, a rather disturbing feature.

We have already proved that the beta of the portfolios on the iso-IR^* frontier is smaller than one. The same holds for the portfolios of iso-tracking frontiers with the same volatility as the benchmark. This property comes from the definition of the tracking error itself and from the equal variance of portfolios and benchmark :

$$\begin{aligned} T^2 &= \sigma_P^2 + \sigma_B^2 (1 - 2\beta_P) \\ &= 2\sigma_B^2 (1 - \beta_P) \end{aligned}$$

$$\Longleftrightarrow \beta_P = 1 - \frac{T}{2\sigma_B^2} < 1, \ \forall \, T > 0$$

It is important to bear in mind this feature: all the portfolios are defensive with respect to the benchmark. If the portfolio manager anticipates a rise in the market, she must use, in the context of iso-IR frontiers, the result in proposition 6 to set up a portfolio with beta greater than one. Jorion (2003) does not address this issue beyond Roll (1992).

5.3 Proportions of assets in portfolios

We study the variations of the asset proportions of the portfolios of the iso-IR frontiers and of iso-tracking frontiers.

Proposition 9 *The weight of a given asset changes monotically along all iso-IR frontiers. The weight of a given asset does not change monotically along all iso-tracking frontiers.*

This result shows that it is easier for a portfolio manager to pass from one portfolio to another along an iso-IR frontier than to do the same with portfolios of iso-tracking frontiers with the same total risk as the benchmark. Indeed, in the latter case, some large moves in asset weights can occur.

5.4 Implied risk aversion

Here, we compare portfolios on the iso-IR^* frontier and portfolios on the iso-tracking frontiers (with the same volatility as the benchmark) by evaluating their implied risk aversion for the same tracking-error level. The risk aversion for portfolios on the iso-IR^* frontier is obtained directly by corollary 3. For the iso-tracking frontiers, we compute the risk aversion, ϕ, such that the iso-IR frontier and the iso-tracking frontier intersect in a portfolio with the same volatility as the benchmark.

The results appear in the following tables. We use the same data as Jorion (2003): $d = 0.25$, $R_0 = 8\%$, $\sigma_B = 13,8\%$.

The risk aversion parameters for the iso-IR^* frontiers are given in table 3. They are independent of Δ_1 and of T.

Table 3: Risk Aversion for iso-IR* portfolios

σ_0	ϕ^*
6%	4.023
8%	4.447
10%	5.258

The implied risk aversion parameters for iso-tracking portfolios with the same volatility as the benchmark are reported in table 4.

Table 4: Implied Risk Aversion for iso-tracking portfolios by TEV

	TEV									
	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
Δ_1= 0%										
	Risk Aversion: ϕ									
σ_0= 6%	0.162	0.325	0.489	0.656	0.826	1.001	1.181	1.368	1.563	1.768
σ_0= 8%	0.198	0.397	0.599	0.804	1.014	1.231	1.456	1.692	1.942	2.207
σ_0= 10%	0.277	0.556	0.84	1.131	1.433	1.748	2.081	2.438	2.824	3.25
Δ_1= 1%										
	Risk Aversion: ϕ									
σ_0= 6%	0.807	0.968	1.13	1.295	1.463	1.635	1.813	1.997	2.19	2.393
σ_0= 8%	0.986	1.182	1.38	1.582	1.789	2.002	2.224	2.456	2.702	2.963
σ_0= 10%	1.376	1.649	1.927	2.211	2.506	2.814	3.14	3.489	3.867	4.283
Δ_1= 2%										
	Risk Aversion: ϕ									
σ_0= 6%	1.448	1.603	1.758	1.916	2.077	2.243	2.413	2.59	2.775	2.969
σ_0= 8%	1.767	1.953	2.141	2.333	2.529	2.732	2.943	3.164	3.397	3.645
σ_0= 10%	2.463	2.716	2.973	3.238	3.511	3.797	4.099	4.423	4.773	5.16

Most of the time, implied risk aversion coefficients associated with these portfolios are very low. To achieve risk aversion around 3.5, it is necessary to allow the tracking error to be at least 10% for $\Delta_1 = 0\%$, 8% for $\Delta_1 = 1\%$ and 6% for $\Delta_1 = 2\%$ (with $\sigma_0 = 10\%$ for each case).

For investors that care about absolute risk and that exhibit a level of risk aversion in line with what is found in empirical work, it seems more relevant to select a portfolio on the iso-IR^* frontier.

6 Concluding remarks

To overcome the drawbacks of tracking-error optimization already highlighted by Roll (1992), Jorion (2003) shows that adding a constraint on volatility of constant-TEV portfolios can improve the performance of the managed portfolio.

We show here that another perspective is possible, extending the results of BPS (2001). Instead of considering constant TEV frontiers as Jorion does, we allow tracking error to vary but we fix risk aversion. We obtain a set of iso-aversion (iso-IR) frontiers ranging from the tracking-error frontier ($\phi = 0$) to a frontier that contains the minimum variance portfolio ($\phi = +\infty$). Recall that all these frontiers contain the benchmark.

We show that the optimal portfolios on an iso-aversion frontier have several interesting properties. In particular, they all have the same information ratio, allowing a portfolio to be selected on a frontier according to desired volatility and/or to desired tracking error. Moreover, the weight of a given asset changes monotically along all iso-IR frontiers.

We suggest that the manager choose a risk aversion implicit to the benchmark for a given market parametrization (*i.e.* d and the MVP). The portfolios on this iso-ϕ^* (iso-IR*) frontier are less risky than the benchmark and dominate it in a mean-variance sense. In addition, the iso-ϕ^* frontiers are obtained for levels of risk aversion consistent with empirical evidence. To sum up, these portfolios may be an interesting alternative to those recommended by Jorion (2003).

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APPENDIX

Here, we derive the shape of the Iso-aversion frontiers in the mean–standard deviation plane. We then examine various properties both of iso-aversion frontiers and of iso-tracking frontiers.

Proof of theorem 1:

First, we take T such that $\mathbf{q'_P R} = R_P$, with $\mathbf{q_P}$ from (8). Then, we compute $\mathbf{q'_P V q_P} = \sigma_P^2$ with the preceding expression of the tracking error and obtain the following quadratic equation in the expected return and the variance of the solution portfolio of $P(3)$:

$$\sigma_P^2 - \alpha R_P^2 - \beta R_P - \gamma = 0 \tag{A-1}$$

Then, rearrange terms in the preceding equation to obtain :

$$\left(\frac{\sigma_P}{\sqrt{\Delta}}\right)^2 - \left(\frac{R_P + \frac{\beta}{2\alpha}}{\sqrt{\Delta/\alpha}}\right)^2 = 1 \tag{A-2}$$

with:

$$-\frac{\beta}{2\alpha} = R_0 + \phi \frac{d\Delta_2 - \Delta_1^2}{P(\phi)} \tag{A-3}$$

$$\alpha = \frac{P(\phi)}{(d - \Delta_1 \phi)^2} > 0 \tag{A-4}$$

$$\Delta = \gamma - \frac{\beta^2}{4\alpha} = \sigma_0^2 - \frac{\Delta_1^2 - d\Delta_2}{P(\phi)} \tag{A-5}$$

The relation (A-2) is a hyperbola because $\alpha > 0$ and is thus of the opposite sign to the coefficient of σ_P^2 in (A-1).

The term Δ is strictly positive because the term $(d\Delta_2 - \Delta_1^2)$ is also positive since the benchmark is within the efficient set. Thus, the hyperbola is right/left opening in the mean–standard deviation plane.

We will now analyze the properties of this set of hyperbolas.

- *Center of the hyperbola.* The horizontal center of the hyperbola is zero. The vertical center of the hyperbola is $-\frac{\beta}{2\alpha}$. When risk aversion is zero (*i.e.* the hyperbola is the tracking-error frontier) or infinite, the vertical center is thus the expected return of the minimum variance portfolio, R_0. Recall that it is also the vertical center of the efficient frontier.

- The vertical center of the hyperbola, as a function of risk aversion, ϕ, reaches a maximum at $\phi = \sqrt{\frac{d}{\Delta_2}}$. We will see below in the corollary of theorem 2 that this value of ϕ plays a particular role.

- The expression of the asymptotes to the hyperbola is given by:

$$R_P = \pm \frac{d - \Delta_1 \phi}{\sqrt{P(\phi)}} \sigma_P - \frac{\beta}{2\alpha}$$

- The two asymptotes reduce to one horizontal line for $\phi = \frac{d}{\Delta_1}$: $R_P = R_B$.

- For this value of ϕ, the iso-aversion frontier is the half-line $R_P = R_B$ for $\sigma_P \geq \sqrt{\frac{1}{d}\left(cR_B^2 - 2aR_B + b\right)}$.

- Study of the slope of the asymptote:

$$\lim_{\phi \to +\infty} \frac{d - \Delta_1 \phi}{\sqrt{P(\phi)}} = \pm \frac{\Delta_1}{\Delta_2}$$

$$\lim_{\phi \to 0} \frac{d - \Delta_1 \phi}{\sqrt{P(\phi)}} = \pm \sqrt{d}$$

As risk aversion tends toward zero, the iso-aversion frontier becomes the tracking-error frontier. Its asymptote is the same as that of the Markowitz efficient frontier.

Proof of theorem 2:

We will first show that the iso-aversion frontier and the efficient frontier intersect each other at a unique point. Knowing that it is impossible to build portfolios that lie outside the efficient frontier, the unique intersection point is then the tangency point between the two frontiers.

A portfolio that belongs to an iso-aversion frontier has a variance given by:

$$\begin{aligned}\sigma_P^2 &= \mathbf{q_P'Vq_P} \\ &= \sigma_B^2 - 2\frac{\phi}{\sqrt{P(\phi)/T}}\sigma_B^2 + \left(\frac{\phi}{\sqrt{P(\phi)/T}}\right)^2 \sigma_B^2 + \frac{2}{\sqrt{P(\phi)/T}}(R_B - \mu) \\ &+ \left(\frac{1}{\sqrt{P(\phi)/T}}\right)^2 \left(-2\phi(R_B-\mu) + \left(a - 2b\mu + c\mu^2\right)\right)\end{aligned}$$

Moreover, this portfolio also belongs to the efficient frontier if its variance is equal to:

$$\sigma_P'^2 = \frac{1}{d}\left(R_P^2 - \frac{2bR_P}{c} + \frac{a}{c}\right)$$

knowing that:

$$\begin{aligned}R_P &= \left(\mathbf{q_B} + \frac{1}{\sqrt{P(\phi)/T}}\left(-\phi\mathbf{q_B} + \mathbf{V}^{-1}(\mathbf{R} - \mu\mathbf{e})\right)\right)' \mathbf{R} \\ &= R_B + \frac{1}{\sqrt{P(\phi)/T}}(-\phi R_B + a - b\mu) \end{aligned} \tag{A-6}$$

We seek the values of T, if they exist, which are solutions of:

$$\sigma_P^2 - \sigma_P'^2 = 0 \tag{A-7}$$

The equation $(A-7)$ can be written as:

$$\alpha_3 T^2 + \alpha_2 T + \alpha_1 = 0$$

where

$$\begin{aligned}\alpha_1 &= \sigma^{*2} - \sigma_B^2 \\ \alpha_2 &= -2\alpha_1 z(\phi) \\ \alpha_3 &= \alpha_1 z(\phi)^2\end{aligned}$$

We use $z(\phi) = \frac{\phi}{\sqrt{P(\phi)}}$. The variance term σ^{*2} is the variance of the portfolio of the efficient frontier that has the same expected return as the benchmark.

The equation $(A-7)$ becomes:

$$\alpha_1 z(\phi)^2 T^2 - 2\alpha_1 z(\phi) T + \alpha_1 = 0$$

Its discriminant is:

$$delta = 4.z(\phi)^2 - 4.z(\phi)^2 = 0$$

Hence, the equation $(A-7)$ in T has a double root which is given by:

$$T = \frac{P(\phi)}{\phi} \text{ for } \phi \neq 0$$

The two curves have a unique point of intersection, which is also a tangency point.

Proof of corollary 3:

The proof of the corollary is straightforward. We seek the values of ϕ, if they exist, which are solutions of:

$$\begin{aligned} & \sigma_P^2 - \sigma_B^2 = 0 \\ \Longleftrightarrow \quad & \tfrac{d}{\phi^2} + \sigma_0^2 - \sigma_B^2 = 0 \end{aligned}$$

The solutions are:

$$\phi^* = \sqrt{\frac{d}{(\sigma_B^2 - \sigma_0^2)}} = \sqrt{\frac{d}{\Delta_2}},\ \phi^{**} = -\sqrt{\frac{d}{(\sigma_B^2 - \sigma_0^2)}} = -\sqrt{\frac{d}{\Delta_2}}$$

The root ϕ^{**} must be ruled out because it is negative.

Proof of proposition 4: The difference between the variance of the benchmark and the variance of the minimum variance portfolio of the iso-IR^* frontier is a function of d, Δ_1, Δ_2 and is denoted $f(d, \Delta_1, \Delta_2)$:

$$f(d, \Delta_1, \Delta_2) \equiv \sigma_B^2 - \sigma_0^2(\phi^*) = \frac{\left(d - \Delta_1 \sqrt{\frac{d}{\Delta_2}}\right) \Delta_2}{2d}$$

Its partial derivatives as well as their signs are given by:

$$\frac{\partial f}{\partial \Delta_1} = -\frac{1}{2}\sqrt{\frac{\Delta_2}{d}} < 0;\ \frac{\partial f}{\partial \Delta_2} = \frac{1}{2} - \frac{\Delta_1}{4\sqrt{\Delta_2 d}} > 0;\ \frac{\partial f}{\partial d} = \frac{\Delta_1 \sqrt{\frac{d}{\Delta_2}} \Delta_2}{4d^2} > 0.$$

Proof of propositions 5, 6, 8 and corollary 7:

By straightforward computations.

Proof of proposition 8: The information ratio of a portfolio on an iso-tracking frontier is given by :

$$IR(T) = \frac{\sqrt{T^2 \left(T^2 - 4\Delta_2\right)\left({\Delta_1}^2 - d\Delta_2\right)} - T^2\Delta_1}{2T\Delta_2}$$

Its derivative with respect to T is always negative, for all values of Δ_1, Δ_2 and d:

$$\frac{\partial IR(T)}{\partial T} = \frac{T^2\left(\Delta_1^2 - d\Delta_2\right) - \Delta_1\sqrt{T^2\left(T^2 - 4\Delta_2\right)\left(\Delta_1^2 - d\Delta_2\right)}}{2\Delta_2\sqrt{T^2\left(T^2 - 4\Delta_2\right)\left(\Delta_1^2 - d\Delta_2\right)}} < 0, \ \forall T.$$

Proof of proposition 9:

- Iso-IR frontiers:

From equations (8) and $(A-6)$, we obtain :

$$\begin{aligned}
\mathbf{q_P} &= \mathbf{q_B} + \frac{R_P - R_B}{(-\phi R_B + a - b\mu)}\left(-\phi \cdot \mathbf{q_B} + \mathbf{V}^{-1}\left(\mathbf{R} - \mu \cdot \mathbf{e}\right)\right) \\
&= \mathbf{q_B} + \frac{R_P - R_B}{d - \Delta_1\phi}\left(-\phi \cdot \mathbf{q_B} + \mathbf{V}^{-1}\left(\mathbf{R} - \mu \cdot \mathbf{e}\right)\right)
\end{aligned}$$

Then, the gradient vector of q_p with respect to R_P is a constant vector given by the following expression :

$$\frac{\partial \mathbf{q_P}}{\partial R_P} = \frac{1}{d - \Delta_1\phi}\left(-\phi \cdot \mathbf{q_B} + \mathbf{V}^{-1}\left(\mathbf{R} - \mu \cdot \mathbf{e}\right)\right)$$

- Iso-tracking frontiers:

  It is sufficient to find a counterexample.