Brownian Motion on Manifolds

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Introduction

What is Brownian motion?

- Discovered by botanist Robert Brown in 1827 when observing the random motion of particles in fluid resulting from collisions.
- Sparked interest in developing a mathematical construct of this behavior, the Wiener process, which is a continuous-time stochastic process.
- Brownian motion (i.e. in this context, the Wiener process) has connections to many fields of research,
  - such as gradients in neural networks for optimization of deep learning algorithms [1].

Motivation:

- Generalizing Brownian motion on spaces that are not Euclidean can be challenging.

Our work:

Incorporating techniques from differential geometry, we implement a method that generalizes Brownian motion in Euclidean space $\mathbf{R}^n$ to Riemannian manifolds $\mathbf{M} \subset \mathbf{R}^m$, where $\mathbf{R}^m$ corresponds to the Euclidean space in which the manifold is embedded in. The resulting algorithm is scalable, and we demonstrate a concrete example of this method by projecting Brownian motion in $\mathbf{R}^2$ to $\mathbf{S}^2$, a 2-sphere manifold embedded in $\mathbf{R}^3$. All points in $\mathbf{S}^2$ have a unique tangent space (of dimensionality 2) where the steps are sampled from a Gaussian distribution. We consider the problem of efficiently simulating Brownian motion on Riemannian manifolds to examine Birkhoff's Ergodic Theorem. The main objective is to observe how the system evolves over long times to examine how the time-average relates the space-average for the trajectories of the Brownian motion on manifolds. We have started to investigate the applicability of our algorithm and what we can understand from the observed data—including generalizing our methods to more complicated structures and starting to develop an algorithm that computes the probability density estimate of Brownian motion on Riemannian manifolds.

Brownian Motion

The Brownian motion is a continuous-time stochastic process $\{B_t : t \geq 0\}$ such that:

- *Independent Increments:* for $0 \leq t_1 < t_2 < \ldots < t_k$ the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_k} - B_{t_{k-1}}$ are independent.
- *Gaussian Increments:* $B_{t+s} - B_t \sim \mathcal{N}(0, s)$.
  - $B_{t+s} - B_t$ is stationary having a distribution (Gaussian) which is independent of t.
- *Continuous paths:* the sample function $t \to B(\omega)$ is continuous for almost all ω:
  - with probability 1, B_t is continuous with respect to t.

Some important properties of Brownian Motion include:

- $B_t \sim \mathcal{N}(0, t)$, in particular $\mathbb{E}[B_t] = 0$ and $Var[B_t] = t$.
- $\mathbb{E}[B_t B_s] = min(t, s)$ for any $0 \leq s, t$.

Birkhoff's Ergodic Theorem

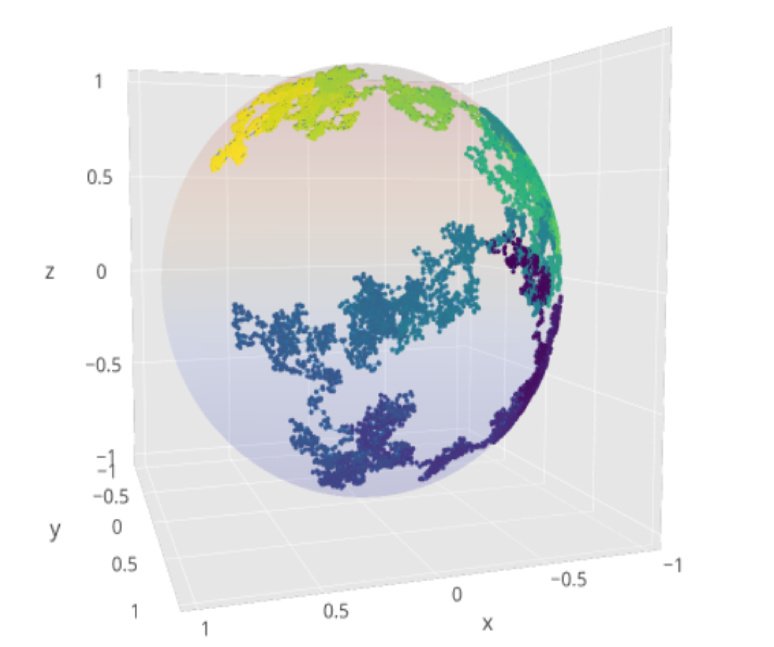
Birkhoff's Ergodic Theorem tells us the following: For any real-valued measurable function $f : \mathcal{M} \to \mathbb{R}$,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s) ds \to \int_{\mathcal{M}} f(x) \mu(dx)$$

for any compact manifold $\mathcal{M}$ and μ is the normalized volume measure [2].

Method: Projection of Brownian motion from tangent space to the manifold

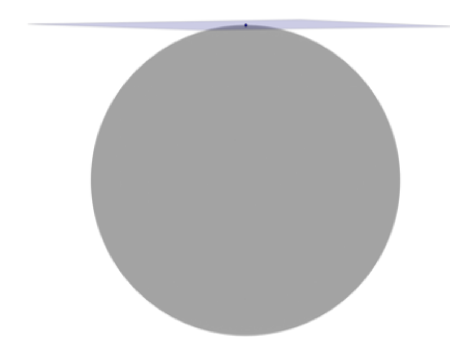
Representation of Brownian motion on a 2-sphere embedded in 3 dimensional Euclidean space



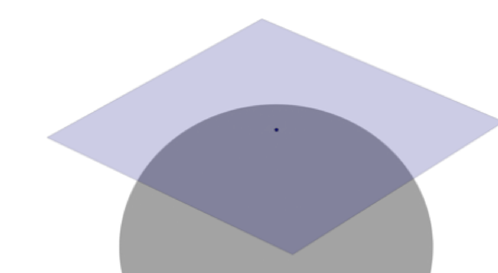
Steps are generated in $\mathbf{R}^2$ and projected on to $\mathbf{S}^2$ embedded in $\mathbf{R}^3$

$B_s \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 0$ and $\sigma^2 = 1/N$

(Top) A step is generated in the tangent plane at the pole, drawing random numbers from a Gaussian Distribution. The updated point in the tangent plane is projected onto the sphere-- preserving the separation between the updated point and the pole.



(Middle) This process is then repeated, but starting at the new point's unique tangent space on the sphere.



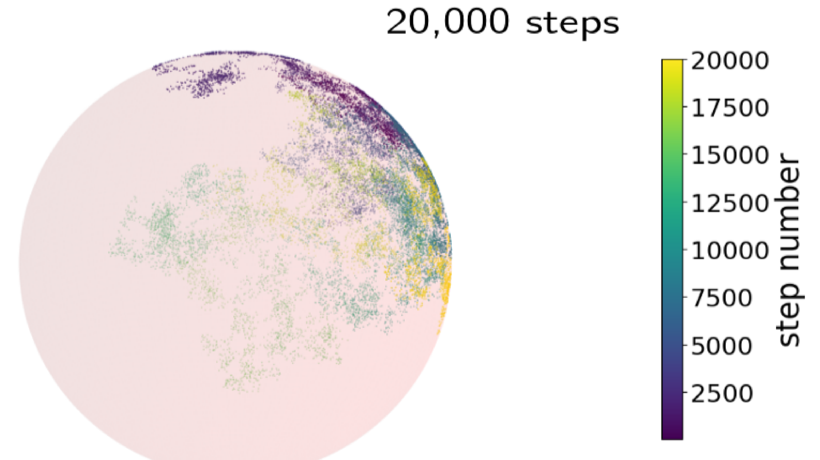
(Bottom) A representation of many steps generated using this method.

A Simulation of Brownian Motion on $\mathbf{S}^2$

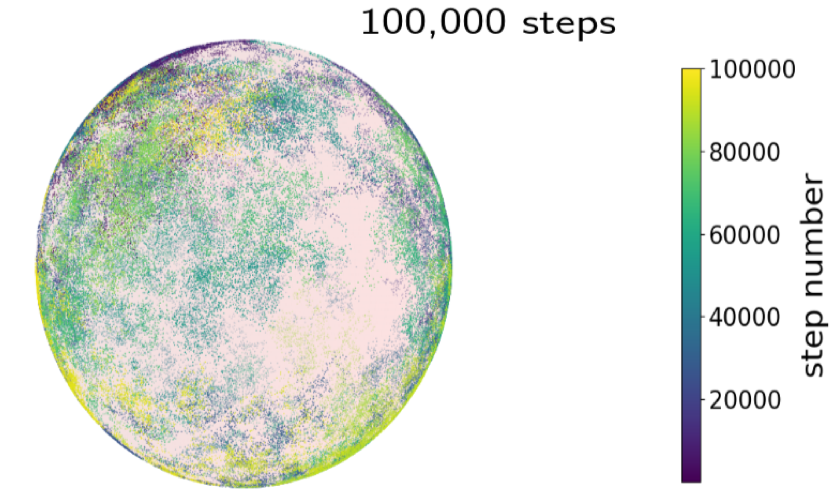
Radius of Sphere: 1

Speed Parameter (step size): 0.00032

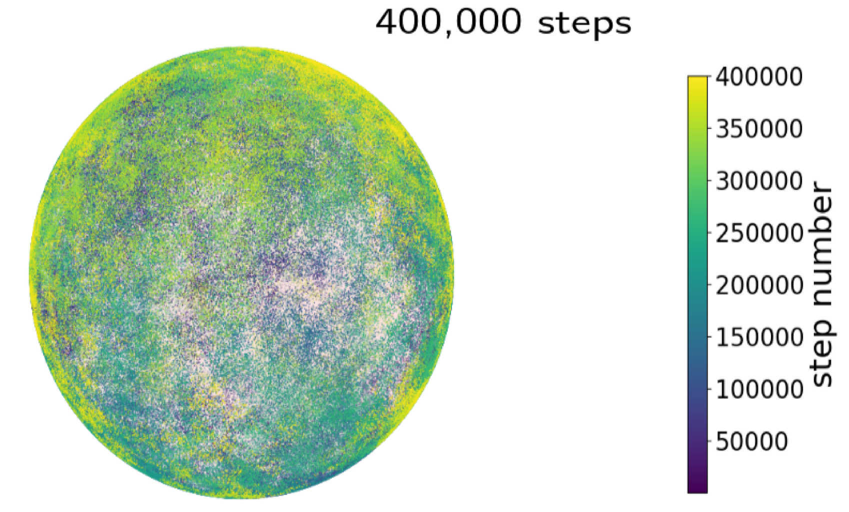
20,000 steps



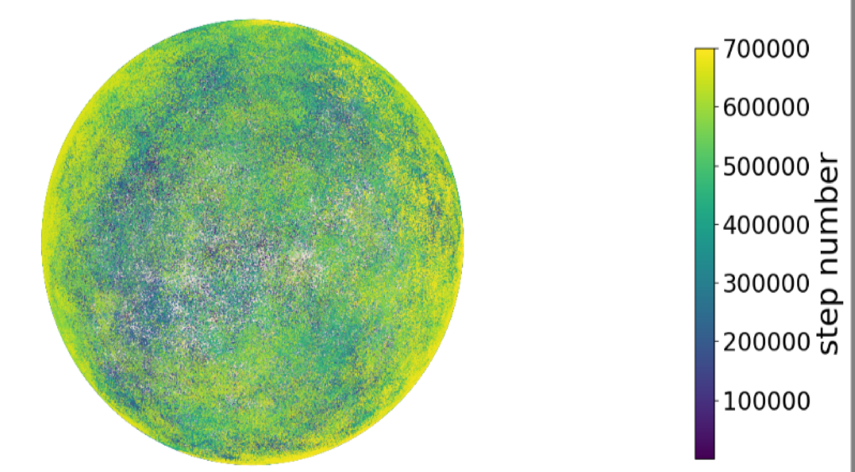
100,000 steps



400,000 steps



700,000 steps



Head and Tail Data in 3-dimensional Euclidean Space

first 11 paths beginning at (0,0,1)

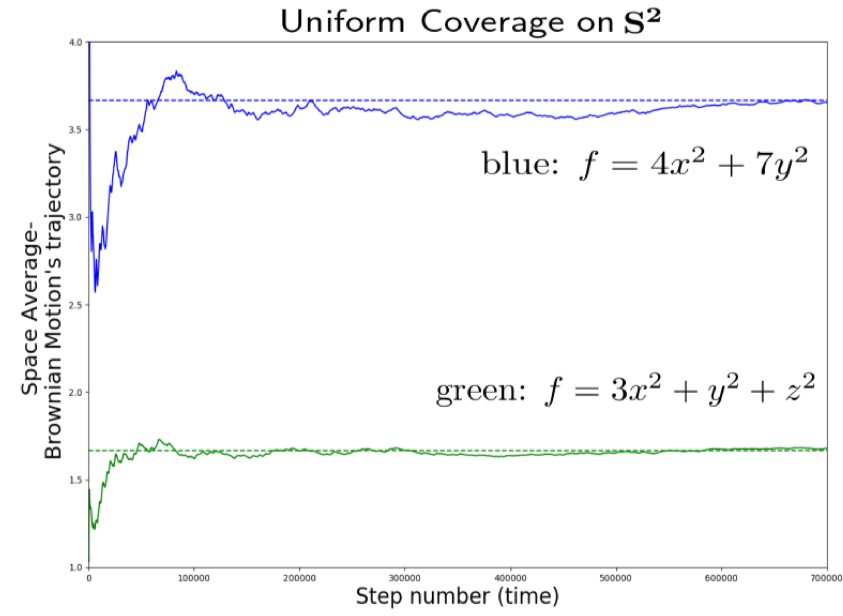
Step Number	X	Y	Z
1	-0.164752	-0.926234	0.33904
2	-0.154385	-0.934937	0.319466
3	-0.151615	-0.937956	0.311853
4	-0.158393	-0.9318	0.32659
5	-0.144568	-0.935898	0.321241
6	-0.153818	-0.939522	0.306006
7	-0.143957	-0.936943	0.318455
8	-0.15115	-0.933532	0.325073
9	-0.138033	-0.939587	0.313247
10	-0.139228	-0.931037	0.337322
11	-0.166231	-0.920309	0.354117

last 11 paths ending at (0,0,1)

Step Number	X	Y	Z
699990	-0.0113684	0.0976607	0.995155
699991	0.0161934	0.0993037	0.994925
699992	-0.00376535	0.0814738	0.996668
699993	-0.0101033	0.0806412	0.996692
699994	-0.00427355	0.0545876	0.9985
699995	-0.0155452	0.055774	0.998322
699996	-0.018099	0.0353122	0.999212
699997	-0.0280907	0.018602	0.999432
699998	-0.0118644	0.0268064	0.99957
699999	-0.0242817	0.0183192	0.999537
700000	2.07421e-18	-5.93504e-19	1

Success, rotation back to the pole.

Brownian motion trajectories: Uniform Coverage on $\mathbf{S}^2$



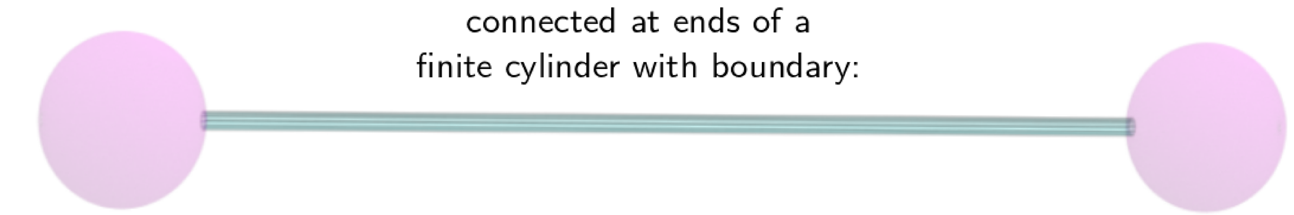
For both test functions after a sufficient amount of time, the time-average of the Brownian motion's trajectory over f converges to the space-average.

(dotted lines) The integral of each function over the surface area of a unit ball were calulated and normailzed after dividing by the surface area. (blue dotten line: 3.666 | green dotten line: 1.666)

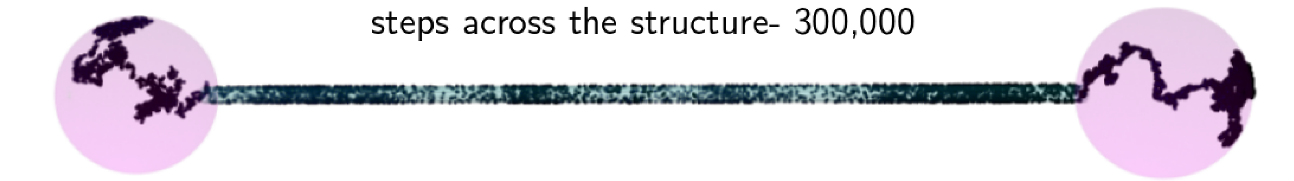
Constructing Brownian Motion as a Diffusion Process

Brownian motion is a Markov process with continuous sample paths (i.e a diffusion process).

Compact manifold featuring 2-spheres connected at ends of a finite cylinder with boundary:



Example simulation: steps across the structure- 300,000



Questions- how does the rate of convergence of the motion change when:

1. the length of the cylinder is modified?
2. the size of the openings connecting the 2-spheres and the cylinder are modified?

Density Estimation on $\mathbf{S}^n$ [3] [4]

A result of differential geometry: density updates are computed by:

$$d\vec{x} = |\det J_\phi| d\vec{u}$$

which gives the relationship between differentials (infinitesimal volumes).

Using the Jacobian of the transform $\phi : \mathbf{R}^n \to \mathbf{R}^m$ with $m > n$ the relationship between volumes is given by the following formula:

$$d\vec{x} = \sqrt{\det \mathbf{G}} d\vec{u} \quad where \quad \mathbf{G} = J_\phi^T J_\phi$$

is the metric induced by embedding ϕ in the tangent space of $\mathbf{S}^n$. The density updates for $\vec{x} \in \mathbf{S}^n$ with $\vec{u} \in \mathbf{R}^n$ can be computed in conjunction with our method for the projection $\phi(u) : \mathbf{R}^n \to \mathbf{S}^n \subset \mathbf{R}^{n+1}$.

The determinant of the metric $\mathbf{G}(\mathbf{x})$ that is associated with this transform is given by:

$$\mathbf{G} = J_\phi(\mathbf{x})^T J_\phi(\mathbf{x}) = \left(\frac{2}{\mathbf{x}^T\mathbf{x} + 1}\right)^{2n}$$

Conclusion and Discussion

We show that after a long time (large sample size and small steps), in regards to Brownian motion on $\mathbf{S}^2$, the system evolves to where it has no memory of its initial state. That is, given a sequence of random trajectories on the manifold, the current state of the Brownian path captures relevant historical information, but, once known, is independent of the past (i.e. a Markovian system, where the current state characterizes the process). Also, all parts of the manifold are visited without any systematic period. Thus,the time-average of the Brownian motion trajectory equals the space-average almost everywhere (i.e. ergodic), and the probability of finding all initial points on the manifold are expected to be the same on the unit interval (i.e. uniformly distributed).

The future goal is to increase the representative power of Brownian motion on manifolds by expanding the software package that I have developed (`brownian-manifold`) and its related tutorials. This includes continuing to improve methods for motion in different diffusion scenarios. Additionally, it is important to to continually develop and think about algorithms that can be used in conjunction with our method, such as approximating density, to understand what can know about the data (i.e. inference).

Software and Interactive Visuals

Visit our Plot.ly page to interact with manifolds- `https://plot.ly/~besser2/`

The `brownian-manifold` Package

`brownian-manifold` is the package developed for this project. All source code and a tutorial on how to use `brownian-manifold` is on GitHub– this includes how to make informative plots- `https://github.com/hankbesser/brownian-manifold`

References

[1] D. Balduzzi, M. Frean, L. Leary, J. Lewis, K. Wan-Duo Ma, and B. McWilliams, "The Shattered Gradients Problem: If resnets are the answer, then what is the question?," *ArXiv e-prints*, Feb. 2017.

[2] L. R. Bellet, *Ergodic Properties of Markov Processes*, pp. 1–39. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006.

[3] D. Jimenez Rezende and S. Mohamed, "Variational Inference with Normalizing Flows," *ArXiv e-prints*, May 2015.

[4] M. C. Gemici, D. Rezende, and S. Mohamed, "Normalizing Flows on Riemannian Manifolds," *ArXiv e-prints*, Nov. 2016.