Recitation 2

ENEE324: Engineering Probability

Spring, 2018

The following problems are from the textbook.

Problem 1.3.15.
A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head. Is she right? Does it make a difference if the coin is fair or unfair? How can we generalize Alice's reasoning?

Solution. Sample space $\Omega = \{HH, HT, TH, TT\}$. Define $A = \{HH\}$, $B = \{HH, HT\}$, $C = \{HH, HT, TH\}$. We need to prove Alice's claim,

$$P(A|B) \geq P(A|C)$$

It's easy to show that $P(A|B) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$ and $P(A|C) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$.Therefore, the claim is correct under condition of the fair coin. Now, assume the coin is biased, and $P(H) = p_H$ and $P(T) = p_T$. Then, we have

$$P(A|B) = \frac{p_H^2}{p_H(p_H + p_T)}$$

$$P(A|C) = \frac{p_H^2}{p_H p_H + 2p_H p_T}$$

Therefore, it's obvious that the claim is still satisfied under condition of the biased coin. Given $A \subset B \subset C$ and $A \cap B = A \cap C$, we get the conclusion that the claim is correct as $B \subset C$ and $A \cap B = A \cap C$. $\square$

Problem 1.3.17.
A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives?

Solution. Let A be the event that the batch will be accepted and define A_k is the event that the k^{th} item is not defective. Therefore, use multiplication rule,

$$\begin{aligned} P(A) &= P(A_1 \cap A_2 \cap A_3 \cap A_4) \\ &= P(A_1)\, P(A_2|A_1)\, P(A_3|A_1 \cap A_2)\, P(A_4|A_1 \cap A_2 \cap A_3) \\ &= \frac{95}{100}\frac{94}{99}\frac{93}{98}\frac{92}{97} = 0.812 \end{aligned}$$

$\square$

Problem 1.4.24.
The release of two out of three prisoners has been announced, but their identity is kept secret. One of the prisoners considers asking a friendly guard to tell him who is the prisoner other than himself that will be released, but hesitates based on the following rational: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners(including himself) whose fate is unknown and exactly one of the two will be released. What is wrong with this line of reasoning?

Solution. This is a wrong rationale because it is not based on a correctly specified probabilistic model. In particular, the event where both of the other prisoners are to be released is not properly accounted in the calculation of the posterior probability of release. Therefore, let's name the prisoners A, B, and C, respectively and, without loss of generosity, assume prisoner A is the one who will do the asking.

$$\begin{aligned} P\left(\text{A to be released}|\text{The guard says B}\right) &= \frac{P\left(\text{A and B to be released}\right)}{P\left(\text{The guard says B}\right)} \\ &= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \end{aligned}$$

Similarly,

$$P\left(\text{A to be released}|\text{The guard says C}\right) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Therefore, regardless of the identity revealed by the guard, the probability that A is released is equal to 2/3. □

Problem 1.5.33.
Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased(though the bias is not know exactly). How can they use the biased coin to make a decision so that either option (opera or the movies) is equally likely to be chosen?

Solution. Toss the coin twice and choose opera if got Head-Tail, movie if got Tail-Head. If got two Heads or two Tails, repeat the experiment until getting Head-Tail or Tail-Head to make a fair decision. To prove the event of two Heads and two Tails won't affect the fairness of the experiment, define A_k as the event that the decision was made at k^{th} experiment. Therefore,

$$\begin{aligned} P\left(\text{opera}\right) &= \Sigma_{k=1}^{\infty} P\left(\text{opera}|A_k\right) P\left(A_k\right) \\ &= \frac{1}{2}\Sigma_{k=1}^{\infty} P\left(A_k\right) \\ &= \frac{1}{2}\Sigma_{k=1}^{\infty}\left(1-p\right)^{k-1} p \\ &= \frac{1}{2} \end{aligned}$$

, where p is the probability of event $\{HT, TH\}$. □

Problem 1.5.37.

A cellular phone system services a population of n_1 "voice users" (those who occasionally need a voice connection) and n_2 "data users" (those who occasionally need a data connection). We estimate that at a given time, each user will need to be independent of other users. the data rate for a voice user is r_1 bits/sec and for a data user is r_2 bits/sec. The cellular system has a total capacity of c bits/sec. What is the probability that more users want to use the system that the system can accommodate?

Solution. To crash the system, assume there are simultaneously k_1 voice users and k_2 data users using the system. The probability of the users using the system is

$$P\left(k_1 \text{ voice users and } k_2 \text{ data users}\right) = P\left(k_1 \text{ voice users}\right) P\left(k_2 \text{ data users}\right)$$

$$P\left(k_1 \text{ voice users}\right) = \binom{n_1}{k_1} p_1^{k_1}\left(1-p_1\right)^{n_1-k_1}$$

$$P\left(k_2 \text{ voice users}\right) = \binom{n_2}{k_2} p_2^{k_2}\left(1-p_2\right)^{n_2-k_2}$$

, where p_1 and p_2 are the probability of a in-use voice user and a in-user data user, respectively. The probability that more users want to use the system than the system can accommodate is the sum of all products $P\left(k_1 \text{ voice users}\right) P\left(k_2 \text{ data users}\right)$ as k_1 and k_2 range over all possible values whose total bit rate requirement $k_1 r_1 + k_2 r_2 \geq c$. Thus, the desired probability is

$$\Sigma_{\{(k_1,k_2)|k_1 r_1+k_2 r_2 \geq c, k_1 \leq n_1, k_2 \leq n_1\}} P\left(k_1 \text{ voice users}\right) P\left(k_2 \text{ data users}\right)$$

$\square$