Recitation 3

ENEE324: Engineering Probability

Spring, 2018

The following problems come from the textbook.

Problem 1.6.50.
Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independent of everyone else, and ignore the additional complication presented by leap years (i.e. assume that nobody is born on February 29). What is the probability that each person has a distinct birthday?

Solution. There are n persons. Each one of them has 365 choices for the birthday. Therefore, all of possible ways without restrictions is 365^n. With constraint that no two persons has the same birthday, assuming $n \leq 365$, the first person has 365 choices for birthday, the second person has 364 choices, the third has 363, so on so forth. Therefore, the possibility that each person has a distinct birthday is

$$P\left(\text{the distinct birthday}\right) = \frac{365 \times 364 \times 363... \times (365 - n + 1)}{365^n}$$

□

Problem 1.6.55.
Eight rooks are placed in distinct squares of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another, i.e., that there is no row or column with more than one rook?

Solution. Assume all rooks are different. The first rook has $8 \times 8 = 64$ choices for placements, the second has 63, so on so forth. Therefore, all of possible ways is the permutation of 64 choosing 8, which is $\frac{64!}{8!}$. To avoid from being captured by other rooks, the first rook still has 64 choices. However, the second rook has $7 \times 7 = 49$ choices because the illegal column and row claimed by the first rook have been removed. Similarly, the third rook has 36 choices, and so on so forth. Therefore, the probability is

$$P\left(\text{all rooks are safe from each other}\right) = \frac{64 \times 49 \times 36... \times 4 \times 1}{\frac{64!}{8!}}$$

Now, assume all rooks are no different, all of possible ways become combination of 64 choosing 8, which is $\binom{64!}{8!}$. Therefore, the possibility is

$$P\left(\text{all rooks are safe from each other}\right) = \frac{64 \times 49 \times 36... \times 4 \times 1}{\binom{64!}{8!}}$$

□

Problem 2.2.2.
You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Possion PMF. (For simplicity, exclude birthdays on February 29.)

Solution. It's obvious that binomial random variable can well model the problem. Let X as a random variable be the number of person who has the same birthday as you.

$$p_X\left(k=1\right)=P\left(X=1\right)=\binom{500}{1}\frac{1}{365}(\frac{364}{365})^{499}=0.3584$$

Since $n=500$ and $p=\frac{1}{365}$, Possion parameter $\lambda=\frac{500}{365}$. By Possion probability,

$$p_X\left(k=1\right)=P\left(X=1\right)=\frac{\lambda^1}{1!}e^{-\lambda}=0.3481$$

, which closely agrees with the binomial probability. □

Problem 2.2.5.
A packet communication system consists of a buffer that stores packets from some source, and a communication line that receives packets from the buffer and transmits them to a receiver. The system operates in time-slot pairs. In the first slot, the system stores a number of packets that are generated by the source according to be stored is a given integer b, and packets arriving to a full buffer are discarded. In the second slot, the system transmits either all the stored packets or c packets (whichever is less). Here, c is a given integer with 0 < c < b.

1. Assuming that at the beginning of the first slot the buffer is empty, find the PMF of the number of packets stored at the end of the first slot and at the end of the second slot.

2. What is the probability that some packets get discarded during the first slot?

Solution. Let X be the number of stored packets at the end of the first time slot. For the case that the buffer is not full, the probability that $X=k<b$ is same as the probability that k packets generated by the source.

$$p_X\left(k\right)=\frac{\lambda^k}{k!}e^{-\lambda},\text{for } k=0,1,2,...b-1$$

When the buffer is full, the probability is equal to the probability that more than b packets generated by the source. Therefore,

$$p_X\left(b\right)=\Sigma_{k=b}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}$$

Now, let Y be the number of stored packets at the end of the second time slot. Based on the problem description, $Y=X-min\left\{X,c\right\}$.Thus, for the case that less than c packets stored in the buffer,

$$p_Y\left(0\right)=\Sigma_{k=0}^{c}\frac{\lambda^k}{k!}e^{-\lambda}$$

For the case that the number of stored packets is between c and b,

$$p_Y(k) = p_X(k+c) = \frac{\lambda^k}{k!}e^{-\lambda}, \text{for } k = 1, 2, ...b - c - 1$$

For the case that more than b packets stored in the buffer,

$$p_Y(b-c) = p_X(b)$$

The probability that some packets get discarded in the first slot is same as the probability that more than b packets generated by the source. Thus, it's equal to

$$p_X(b) = \Sigma_{k=b}^{\infty}\frac{\lambda^k}{k!}e^{-\lambda}$$

$\square$