ENEE324 Recitation Problems #1 Spring 2018

From the textbook,
Problem 1.1.3*
Prove $A \cup (\cap_{n=1}^{\infty} B_n) = \cap_{n=1}^{\infty} (A \cup B_n)$

Please refer to the solution for the problem 3* on page 53 in the textbook.

Problem 1.2.5
Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

Define A = {Geniuses students} and B = {Students love chocolate}.
Given $P(A) = 0.6, P(B) = 0.7, P(A \cap B) = 0.4$.
Therefore, $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = 0.1$

Problem 1.2.8
You enter a special kind of chess tournament, in which you play one game with each of three opponents, but you get to choose the order in which you play your opponents, knowing the probability of a win against each. You win the tournament if you win two games in a row, and you want to maximize the probability of winning. Show that it is optimal to play the weakest opponent second, and that the order of playing the other two opponents does not matter.

Assume three opponents A, B, and C with probability of wining them $p_A, p_B, and\ p_C$, respectively. Assume B is the weakest opponent, $p_B > p_A\ and\ p_B > p_C$. The probability that you win in condition of B being at the second is

$$P(winning) = p_B(p_A + (1 - p_A)p_C)$$

where the order is ABC or CBA.
Now, consider the order of CAB or BAC, $P(winning) = p_A(p_C + (1 - p_C)p_A)$. And the order of ACB or BAC, $P(winning) = p_C(p_A + (1 - p_A)p_B)$
Given B is the weakest, we get

$$p_B(p_A + (1 - p_A)p_C) \geq p_A(p_C + (1 - p_C)p_A)$$
$$p_B(p_A + (1 - p_A)p_C) \geq p_C(p_A + (1 - p_A)p_B)$$

Therefore, choosing the weakest opponent at 2nd round is the optimal solution.

Problem 1.2.10
Prove $P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A) + P(B) - 2P(A \cap B)$

Since $(A \cap B^c)$ and $(A^c \cap B)$ are disjoint sets,

$$P\big((A \cap B^c) \cup (A^c \cap B)\big) = P(A \cap B^c) + P(A^c \cap B)$$
$$= P(A) - P(A \cap B) + P(B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$$

Problem 1.2.12*
Prove the inclusion-exclusion formula by induction

$$P(\cup_{k=1}^{n} A_k) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i \leq j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i \leq j \leq k \leq n} P(A_i \cap A_j \cap A_k) - \cdots$$
$$+ (-1)^{n-1} P(\cap_{k=1}^{n} A_k)$$

Base step, n = 1 gives you $P(A_1) = P(A_1)$. It's satisfied.
Suppose it's satisfied for n subsets, now we need to show it's still satisfied for n+1 subsets as the following formula,

$$P(\cup_{k=1}^{n+1} A_k) = \sum_{i=1}^{n+1} P(A_i) - \sum_{1\le i\le j\le n+1} P(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n+1} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^n P(\cap_{k=1}^{n+1} A_k)$$

From the beginning,
$P(\cup_{k=1}^{n+1} A_k) = P(\cup_{k=1}^{n} A_k \cup A_{n+1})$, just expand the right
$= P(\cup_{k=1}^{n} A_k) + P(A_{n+1}) - P(\cup_{k=1}^{n} A_k \cap A_{n+1})$, use the formula with n = 2

For the last term in the equation,
$P(\cup_{k=1}^{n} A_k \cap A_{n+1})$
$= P(\cup_{k=1}^{n} (A_k \cap A_{n+1}))$, by the distributive property
$= \sum_{i=1}^{n} P(A_i \cap A_{n+1}) - \sum_{1\le i\le j\le n} P(A_i \cap A_j \cap A_{n+1}) + \sum_{1\le i\le j\le k\le n} P(A_i \cap A_j \cap A_k \cap A_{n+1}) - \cdots + (-1)^{n-1} P(\cap_{k=1}^{n} A_k \cap A_{n+1})$, by the formula

For the first two terms in the question,
$P(\cup_{k=1}^{n} A_k) + P(A_{n+1})$
$= \sum_{i=1}^{n} P(A_i) - \sum_{1\le i\le j\le n} P(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(\cap_{k=1}^{n} A_k) + P(A_{n+1})$, expand by the formula

$$= \sum_{i=1}^{n+1} P(A_i) - \sum_{1\le i\le j\le n} P(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(\cap_{k=1}^{n} A_k)$$

Therefore,
$P(\cup_{k=1}^{n+1} A_k)$

$$= \sum_{i=1}^{n+1} P(A_i) - \sum_{1\le i\le j\le n} P(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(\cap_{k=1}^{n} A_k) - \sum_{i=1}^{n} P(A_i \cap A_{n+1}) + \sum_{1\le i\le j\le n} P\big(A_i \cap A_j \cap A_{n+1}\big) - \sum_{1\le i\le j\le k\le n} P\big(A_i \cap A_j \cap A_k \cap A_{n+1}\big) + \cdots - (-1)^{n-1} P(\cap_{k=1}^{n} A_k \cap A_{n+1})$$

$$= \sum_{i=1}^{n+1} P(A_i) - \sum_{1\le i\le j\le n+1} P(A_i \cap A_j) + \sum_{1\le i\le j\le k\le n+1} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^n P(\cap_{k=1}^{n+1} A_k)$$

Q.E.D.