Math Module 4A Cheat Sheet

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Contents

1 Definitions

1.1 Indefinite integral

An indefinite integral represents the collection of **all** antiderivatives of a function $f(x)$.

$$\int f(x)\,dx$$

1.1.1 Example

We know that all the antiderivatives of the function $f(x) = 2x$ have the form $F(x) = x^2 + C$, where C is an arbitrary constant, so we express this fact by writing:

$$\int 2x\,dx = x^2 + C$$

1.2 Trigonometric polynomial

A trigonometric polynomial is a sum of terms like:

$$\cos^m x \sin^n x$$

1.3 Division algorithm

Let $a, b \in \mathbb{Z}, b \neq 0$. Then there exist unique $q, r \in \mathbb{Z}$ such that:

$$a = qb + r, \quad 0 \leq r \leq |b|$$

The integer q above is called the **quotient** and r is called the **remainder**, when a is divided by b.

1.4 Polynomial division

Dividing a **polynomial** $P(x)$ by a **polynomial** $Q(x)$ means finding polynomials $q(x), r(x)$ such that:

$$P(x) = q(x)Q(x) + r(x), \quad \deg\, r(x) < \deg\, Q(x)$$

q is called the **quotient** and r is the **remainder** when p is divided by Q.

2 List of basic indefinite integrals

1. $$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C \text{ for } a \neq -1$$

2. $$\begin{aligned}\int \frac{1}{x}\,dx &= \ln|x| + C\\ &= \ln x + C \text{ for } x > 0\\ &= \ln(-x) + C \text{ for } x < 0\end{aligned}$$

3. $$\int e^x\,dx = e^x + C$$

4. $$\int \sin x\,dx = -\cos x + C$$

5. $$\int \cos x\,dx = \sin x + C$$

6. $$\int \frac{1}{\cos^2 x}\,dx = \tan x + C$$

7. $$\int \frac{1}{\sin^2 x}\,dx = -\cot x + C$$

8. $$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

9. $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

3 Integration by parts

Since:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

We get:

$$f(x)g(x) + C = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$$

I.e.

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

4 Change of variables (substitution)

By the chain rule we have:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

So:

$$\int f'(g(x))g'(x)\, dx = f(g(x)) + C$$

Using the notation:

$$u = g(x),\ du = g'(x)\, dx$$

We get:

$$\int f'(u)\, du = f(u) + C$$

The trick below works well if we can identify $g'(x)$ as a factor of the integrand.

$$\int f'(g(x))g'(x)\, dx = \begin{bmatrix} u = g(x) \\ du = g'(x)\, dx \end{bmatrix} = \int f'(u)\, du$$

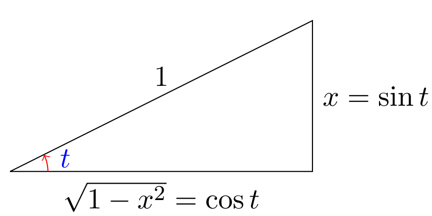
5 Inverse substitution

Sometimes, an inverse substitution $x = h(t)$, $dx = h'(t)\, dt$, works well. In inverse substitutions, we want h to be one-to-one so that $t = h^{-1}(x)$.

$$\int f(x)\, dx = \int f(h(t))h'(t)\, dt$$

5.1 Substitution for $\sqrt{1-x^2}$

Find $\int \sqrt{1-x^2}\, dx$. For integrals where $\sqrt{1-x^2}$ appears and there is no other obvious way forward, the follow idea often works:



The inverse substitution:

$$x = \sin t$$
$$\sqrt{1-x^2} = \cos t$$

In general:

$$\cos t = \pm\sqrt{1-\sin^2 t} = \pm\sqrt{1-x^2}$$

But for our inverse substitution, we have:

$$t = \arcsin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

So:

$$\cos t \geq 0$$
$$\cos t = \sqrt{1-x^2}$$

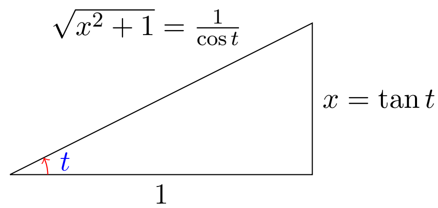
The equations we have:

$$x = \sin t$$
$$dx = \cos t\, dt$$
$$\sqrt{1-x^2} = \cos t$$

Finding $\int \sqrt{1-x^2}\, dx$:

$$\begin{aligned}
\int \sqrt{1-x^2}\, dx &= \int cos^2 t\, dt \\
&= \int \frac{1+\cos 2t}{2}\, dt \\
&= \frac{t}{2} + \frac{\sin 2t}{4} + C \\
&= \frac{t}{2} + \frac{2\sin t \cos t}{4} + C \\
&= \frac{1}{2}\arcsin x + \frac{2 \cdot 2x\sqrt{1-x^2}}{4} + C \\
&= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C
\end{aligned}$$

5.2 Substitution for $\sqrt{x^2+1}$



In general,

$$\begin{aligned}\frac{1}{\cos^2 t} &= \frac{\sin^2 t + \cos^2 t}{\cos^2 t} \\ &= \tan^2 t + 1 \\ &= x^2 + 1\end{aligned}$$

So:

$$\frac{1}{\cos t} = \pm\sqrt{x^2+1}$$

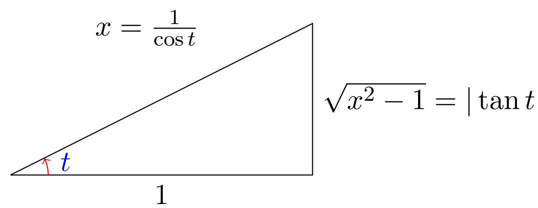
But for our inverse substitution, we take:

$$t = \arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

So $\cos t > 0$, i.e.

$$\frac{1}{\cos t} = \sqrt{x^2+1}$$

5.3 Substitution for $\sqrt{x^2-1}$



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\begin{aligned}
\sqrt{x^2-1} &= \sqrt{\frac{1}{\cos^2 t} - 1} \\
&= \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} \\
&= \sqrt{\frac{\sin^2 t}{\cos^2 t}} \\
&= \sqrt{\tan^2 t} \\
&= |\tan t| = \begin{cases} \tan t & \text{for } t \in \left[0, \frac{\pi}{2}\right) \\ -\tan t & \text{for } t \in \left(\frac{\pi}{2}, \pi\right] \end{cases}
\end{aligned}
$$

6 Integration of trigonometric polynomials

These equations below are useful:

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

6.1 Even powers

If all powers are even, we can use those formulas to reduce its degree.

$$\begin{aligned}
\int \cos^2 x \sin^2 x \, dx &= \int \frac{1 + \cos 2x}{2} \cdot \frac{1 - \cos 2x}{2} \, dx \\
&= \frac{1}{4} \int (1 - \cos^2 2x) \, dx \\
&= \frac{1}{4} \int \sin^2 2x \\
&= \frac{1}{8} \int (1 - \cos 4x) \, dx \\
&= \frac{1}{8} \left(x - \frac{\sin 4x}{4} \right) + C
\end{aligned}$$

6.2 One odd power

If at least one power is odd, we can make a clever substitution.

$$\begin{aligned}
\int \sin^3 x \cos^4 x \, dx &= \int \sin^2 x \cos^4 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^4 \sin x \, dx \\
&\begin{bmatrix} u = \cos x \\ du = -\sin x \, dx \end{bmatrix} \\
&= -\int (1 - u^2) u^4 \, du \\
&= -\int u^4 - u^6 \, du \\
&= \frac{u^7}{7} - \frac{u^5}{5} + C \\
&= \frac{\cos^7 x}{7} - \frac{\cos^5}{5} + C
\end{aligned}$$

7 Factoring polynomials

Each polynomial $Q(x)$ can be factorised:

$$Q(x) = A(x - x_1)(x - x_2) \cdots (x - x_n)$$

Where $x_1, \ldots, x_n$ are the roots. Some x_i might be complex. But if the coefficients of Q are real, complex roots occur only in couples:

$$a - bi, \quad a + bi$$

For such pairs of complex roots, multiplying the corresponding factors gives:

$$(x - a + bi)(x - a - bi) = (x - a)^2 + b^2$$

So, any polynomial is a product of linear and quadratic polynomials, where each quadratic factor has no real root. The power of each factor in the product is called the **multiplicity**.

8 Guessing roots

If a polynomial with integer coefficients has an integer root, we can guess it.

If all the coefficients of a polynomial $Q(x)$ are integers and the root x is integer, then x divides the constant term.

8.1 Example

Factorise $Q(x) = x^5 - 2x^3 - 2x^2 - 3x - 2$.

Any integer root of Q must divide by -2, so possible integer roots are $\pm 1, \pm 2$. Substituting, we see that -1 is a root, so $x + 1$ is a factor of Q.

Doing long division:

$$Q(x) = (x + 1)(x^4 - x^3 - x^2 - x - 2)$$

Again, any integer roots of $x^4 - x^3 - x^2 - x - 2$ must divide by -2 so again, possible integer roots are $\pm 1, \pm 2$. Testing, we find that -1 is a root, so we divide by $(x + 1)$ again.

Doing long division:

$$\begin{aligned} Q(x) &= (x + 1)(x^3 - x^2 - x - 2) \\ &= (x + 1)^2(x^3 - 2x^2 + x - 2) \end{aligned}$$

Again, any integer roots of $x^3 - 2x^2 + x - 2$ must divide by -2 so again, possible integer roots are $\pm 1, \pm 2$. Testing, we find that 2 is a root, so we divide by $(x - 2)$.

Doing long division:

$$\begin{aligned} Q(x) &= (x + 1)(x^3 - x^2 - x - 2) \\ &= (x + 1)^2(x^3 - 2x^2 + x - 2) \\ &= (x + 1)^2(x - 2)(x^2 + 1) \end{aligned}$$

Since $x^2 + 1$ has no real roots, we are done.

9 Integrating a rational function

Given a rational function:

$$f(x) = \frac{P(x)}{Q(x)} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_0}{x^m + b_{m-1} + \cdots + b_0}$$

A partial fraction is an expression from the following list:

1. Dx^k
2. $\dfrac{C}{(x-c)^k}$
3. $\dfrac{Ax+B}{(x^2+px+q)^k}$, where the function $x^2 + px + q$ has no real root

9.1 Step 1

If deg $P \geq$ deg Q, divide $P(x)$ by $Q(x)$:

$$\begin{aligned} f(x) &= \frac{P(x)}{Q(x)} \\ &= \frac{q(x)Q(x) + r(x)}{Q(x)} \\ &= q(x) + \frac{r(x)}{Q(x)} \end{aligned}$$

$q(x)$ is a polynomial, so it can be integrated.

$$\text{deg } r < \text{deg } Q$$

9.2 Step 2

Factorise $Q(x)$ into linear and irreducible quadratic factors:

$$Q(x) = A(x-c_1)^{l_1} \cdots (x-c_\alpha)^{l_\alpha}[(x-a_1)^2 + b_1^2]^{q_1} \cdots [(x-a_\beta)^2 + b_\beta^2]^{q_\beta}$$

9.3 Step 3

Each factor $(x-c)^l$ in $Q(x)$, gives us partial fractions:

$$\frac{C_1}{x-c}, \frac{C_2}{(x-c)^2}, \ldots, \frac{C_l}{(x-c)^l}$$

And each factor $[(x-a)^2 + b^2]^q$ gives us partial fractions:

$$\frac{A_1x + B_1}{(x-a)^2 + b^2}, \frac{A_2x + B_2}{[(x-a)^2 + b^2]^2}, \ldots, \frac{A_qx + B_q}{[(x-a)^2 + b^2]^q}$$

9.4 Example

Find:

$$\int \frac{x^6 + 2x^4 + x^2 + x + 1}{x^5 + 2x^3 + x}\,dx$$

First step:

$$\frac{x(x^5 + 2x^3 + x) + x + 1}{x^5 + 2x^3 + x} = x + \frac{x+1}{x^5 + 2x^3 + x}$$

Second step:

$$\begin{aligned} x^5 + 2x^3 + x &= x(x^4 + 2x^2 + 1) \\ &= x(x^2+1)^2 \end{aligned}$$

Third step:

$$\frac{x+1}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2}$$

Calculating, we have:

$$\begin{aligned} &\frac{a}{x} + \frac{bx+c}{x^2+1} + \frac{dx+e}{(x^2+1)^2} \\ &= \frac{a(x^4 + 2x^2 + 1) + x(bx+c)(x^2+1) + x(dx+e)}{x(x^2+1)^2} \\ &= \frac{(a+b)x^4 + cx^3 + (2a+b+d)x^2 + (c+e)x + a}{x(x^2+1)^2} \end{aligned}$$

Comparing coefficients with the original expression:

$$\frac{x+1}{x(x^2+1)^2}$$

$$\begin{array}{cccccccccc} \text{a} & + & \text{b} & & & & & & & = 0 \\ & & & & \text{c} & & & & & = 0 \\ \text{2a} & + & \text{b} & & & + & \text{d} & & & = 0 \\ & & & & \text{c} & & & + & \text{e} & = 1 \\ & & & & & & & & \text{a} & = 1 \end{array}$$

I.e.

$$a = 1,\ b = -1,\ c = 0,\ d = -1,\ e = 1$$

So the integrand is:

$$x + \frac{1}{x} - \frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}$$

$$\int x\,dx = \frac{x^2}{2} + C_1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C_2$$

$$\begin{aligned}
\int \frac{x}{x^2+1}\,dx &= \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \\
&= \frac{1}{2}\int \frac{1}{u}\,du \\
&= \frac{1}{2}\ln|u| + C_3 \\
&= \frac{1}{2}\ln(x^2+1) + C_3
\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{(x^2+1)^2}\,dx &= \begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \\
&= \frac{1}{2}\int \frac{1}{u^2}\,du \\
&= -\frac{1}{2u} + C_4 \\
&= -\frac{1}{2(x^2+1)} + C_4
\end{aligned}$$

$$\begin{aligned}
\int 1 \cdot \frac{1}{x^2+1}\,dx &= \frac{x}{x^2+1} - \int x \cdot \frac{-2x}{(x^2+1)^2}\,dx \\
&= \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{(x^2+1)^2}\,dx \\
&= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\,dx - 2\int \frac{1}{(x^2+1)^2}\,dx
\end{aligned}$$

So:

$$\begin{aligned}\int \frac{1}{(x^2+1)^2} &= \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\int \frac{1}{x^2+1}\,dx \\ &= \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan x + C_5\end{aligned}$$

Wrapping it all up:

$$\begin{aligned}&\int \frac{x^6+2x^4+x^2+x+1}{x^5+2x^3+x}\,dx \\ &= \frac{x^2}{2} + \ln|x| - \frac{1}{2}\ln(1+x^2) + \frac{1}{2(1+x^2)} + \frac{x}{2(1+x^2)} + \frac{1}{2}\arctan x + C\end{aligned}$$