1a) $f(x) = \frac{1}{2-x}$

$$= \frac{1}{2\left(1-\frac{x}{2}\right)}$$

$$= \frac{1}{2} \frac{1}{1-\frac{x}{2}}$$

$$f(x) = \frac{1}{2} \frac{1}{1-\frac{x}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}, \quad x \in (-2, 2)$$

1b) $f(x) = \frac{1}{(2-x)^2}$

$$= \frac{d}{dx} \frac{1}{2-x}$$

$$\frac{d}{dx} \frac{1}{2-x}$$

$$= -1(2-x)^{-2}(-1)$$

$$= \frac{1}{(2-x)^2}$$

$$\therefore f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}} + 0$$

$$= \sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^{n+1}}, \quad x \in (-2, 2)$$

1c) $f(x) = \frac{1}{1+2x}$

$$= \frac{1}{1-(-2x)}$$

$$= \sum_{n=0}^{\infty} (-2x)^n$$

$$= \sum_{n=0}^{\infty} (-2)^n x^n \quad , \; x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

1d) $f(x) = \ln(2-x)$

$$= \ln\left(2\left(1-\frac{x}{2}\right)\right)$$

$$= \ln 2 + \ln\left(1-\frac{x}{2}\right)$$

$$= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(-\frac{x}{2}\right)^n}{n}$$

$$= \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-x)^n}{2^n n}$$

$$= \ln 2 - \sum_{n=1}^{\infty} \frac{x^n}{n 2^n} \quad , \; x \in [-2,)$$

1e) $f(x) = \dfrac{1}{1-(-(x-1))}$

$$= \sum_{n=0}^{\infty} (-(x-1))^n$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n , \quad x \in (0,2)$$

2) From the lecture,

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} , \quad x \in (-1,1]$$

when $x=1$,

$$\arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\therefore \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$$

3a) $\sum_{n=1}^{\infty} nx^n = x\sum_{n=1}^{\infty} nx^{n-1}$

$$= x\frac{d}{dx}\sum_{n=1}^{\infty} x^n$$

$$= x\frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{x}{(1-x)^2}, \quad x \in (-1, 1)$$

3b) $\sum_{n=1}^{\infty} (n(n+1)+2^n) x^{n-1}$

$$= \sum_{n=1}^{\infty} n(n+1) x^{n-1} + \sum_{n=1}^{\infty} 2^n x^{n-1}$$

$$= \frac{d^2}{dx^2} \sum_{n=1}^{\infty} x^{n+1} + 2\sum_{n=1}^{\infty} 2^{n-1} x^{n-1}$$

$$= \frac{d^2}{dx^2} \sum_{n=2}^{\infty} x^n + 2\sum_{n=1}^{\infty} (2x)^{n-1}$$

$$= \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} x^n - 1 - x \right) + 2\sum_{n=0}^{\infty} (2x)^n$$

$$= \frac{d^2}{dx^2} \left(\frac{1}{1-x} - 1 - x \right) + \frac{2}{1-2x}$$

$$= \frac{d}{dx} \left(\frac{1}{(1-x)^2} - 1 \right) + \frac{2}{1-2x}$$

$$= \frac{2}{(1-x)^3} + \frac{2}{1-2x}, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

4) $f(x) = e^{x^2}$

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \qquad f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\therefore \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}$$

For the coefficients of x^{99},

$$0 = \frac{f^{(99)}(0)}{k!}$$

$$f^{(99)}(0) = 0$$

For the coefficients of x^{100},

$$\frac{1}{50!} = \frac{f^{(100)}(0)}{100!}$$

$$f^{(100)}(0) = \frac{100!}{50!}$$

5a) $\lim_{x \to 0} \frac{\sin x - x}{x^2} = \lim_{x \to 0} \frac{\cancel{x} + O(x^3) - \cancel{x}}{x^2}$

$= \lim_{x \to 0} \frac{O(x^3)}{x^2}$

$= 0$

b) $\lim_{x \to 0} \frac{(e^x - 1 - x)^2}{x^2 - \ln(1+x^2)} = \lim_{x \to 0} \frac{(\cancel{1} + \cancel{x} + \frac{x^2}{2} + O(x^3) - \cancel{1} - \cancel{x})^2}{\cancel{x^2} - (\cancel{x^2} - \frac{x^4}{2} + O(x^6))}$

$= \lim_{x \to 0} \frac{\left(\frac{x^2}{2} + O(x^3)\right)^2}{\frac{x^4}{2} + O(x^6)}$

$= \lim_{x \to 0} \frac{\frac{x^4}{4} + x^2 O(x^3) + O(x^6)}{\frac{x^4}{2} + O(x^6)}$

$= \lim_{x \to 0} \frac{\frac{x^4}{4} + O(x^5)}{\frac{x^4}{2} + O(x^6)}$

$= \lim_{x \to 0} \frac{\frac{1}{4} + O(x)}{\frac{1}{2} + O(x^2)} = \frac{\frac{1}{4}}{\frac{1}{2}}$

$= \frac{1}{2}$

5c) $\lim_{x\to 0} \dfrac{2\sin 3x - 3\sin 2x}{5x - \arctan 5x}$

$$= \lim_{x\to 0} \frac{2\left(3x - \frac{(3x)^3}{6} + o(x^5)\right) - 3\left(2x - \frac{(2x)^3}{6} + o(x^5)\right)}{\cancel{5x} - \left(\cancel{5x} - \frac{(5x)^3}{3} + o(x^5)\right)}$$

$$= \lim_{x\to 0} \frac{\cancel{6x} - 9x^3 + o(x^5) \cancel{- 6x} + 4x^3 + o(x^5)}{\frac{125x^3}{3} + o(x^5)}$$

$$= \lim_{x\to 0} \frac{-5x^3 + o(x^5)}{\frac{125x^3}{3} + o(x^5)}$$

$$= \frac{-5 + o(x^2)}{\frac{125}{3} + o(x^2)}$$

$$= -\frac{3}{25}$$

5d) $\lim_{x \to 0} \dfrac{\sin(\sin x) - x}{x(\cos(\sin x) - 1)}$

$$= \lim_{x \to 0} \frac{\sin x - \frac{\sin^3 x}{3!} + o(\sin^5 x) - x}{x\left(\cancel{1} - \frac{\sin^2 x}{2} + o(\sin^4 x) \cancel{-1}\right)}$$

$$= \lim_{x \to 0} \frac{\sin x - \frac{\sin^3 x}{3!} + o(x^5) - x}{x\left(-\frac{\sin^2 x}{2} + o(x^4)\right)}$$

$$= \lim_{x \to 0} \frac{\cancel{x} - \frac{x^3}{3!} + o(x^5) - \frac{(x + o(x^3))^3}{3!} + o(x^5) \cancel{-x}}{x\left(-\frac{(x + o(x^3))^2}{2} + o(x^4)\right)}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3!} + o(x^5) - \frac{x^3 + o(x^5)}{3!} + o(x^5)}{x\left(-\frac{x^2 + o(x^4)}{2} + o(x^4)\right)}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3!} - \frac{x^3}{3!} + o(x^5)}{-\frac{x^3}{2} + o(x^5)} = \lim_{x \to 0} \frac{-\frac{2}{6}}{-\frac{1}{2}}$$

$$= \frac{2}{3}$$

6a) $y = \sum_{n=0}^{\infty} a_n x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + xy' + y = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + a_0 + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + n a_n + a_n \right] x^n = 0$$

$$2a_2 + a_0 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} + (n+1) a_n \right] x^n = 0$$

6a) Comparing coefficients of x_n:

$$\cancel{(n+1)}(n+2)a_{n+2} + \cancel{(n+1)}a_n = 0$$

$$(n+2)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{n+2}$$

6b) $1 = y(0)$

$$= \sum_{n=0}^{\infty} a_n 0^n$$

$$= a_0$$

$0 = y'(0)$

$$= \sum_{n=1}^{\infty} n a_n 0^{n-1}$$

$$= a_1$$

$\therefore a_0 = 1, \ a_1 = 0$

From (a),

$$(n+2) a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{1}{n+2} a_n$$

For $n \in \mathbb{Z}^+$, $a_{2n+1} = 0$

$a_2 = -\frac{1}{2}, \ a_4 = \frac{1}{4(2)}, \ a_6 = -\frac{1}{6(4)(2)}$

$$a_{2n} = \frac{(-1)^n}{2n(2n-2)(2n-4)\ldots(2)}$$

6b) $$a_{2n} = \frac{(-1)^n}{2^n (n)(n-1)(n-2)\dots(1)}$$

$$a_{2n} = \frac{(-1)^n}{2^n (n!)} \text{ for } n \in \mathbb{Z}^+$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (n!)} x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-x^2}{2}\right)^n$$

$$= e^{\frac{-x^2}{2}}$$