Math Module 1A Lecture Notes

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Contents

1 Sets

A set is basically just a collection of objects, and the objects inside a set are called elements, or points. To say that x is an element of set A is represented by $x \in A$ or $A \ni x$. Similarly, to say that x is not an element of set A is represented by $x \notin A$. A set is usually described using curly braces $\{\}$.

Examples:

$$A = \{1, 2, 3\} \rightarrow 1 \in A, 2 \in A, 3 \in A, 4 \notin A, \pi \notin A$$

$$B = \{\text{Homer}, \text{Marge}, \text{Bart}, \text{Lisa}, \text{Maggie}\}$$

$$\text{Homer} \in B, 1 \notin B, \text{Ned Flanders} \notin B$$

1.1 Describing sets

Common ways to describe sets include:

$$A = \{x : \text{some condition}\}$$

A is the set of objects x for which the condition is true.

$$B = \{x \notin B : \text{some condition}\}$$

B is the set of objects x in Set B for which the condition is true

Examples:

$$\mathbb{Z} = \{x : \text{x is an integer}\} = \{0, 1, -1, 2, -2, ...\}$$

$$A = \{x \in \mathbb{Z} : x = 2k \text{ for some } k \in \mathbb{Z}\} = \{0, 2, -2, 4, -4, ...\}$$

1.2 Subsets and equal sets

Definition: If each element of the set A also belongs to the set B, then A is a subset of B, which is represented by $A \subset B$.

If A is a subset of B and B is also a subset of A, then the sets A and B are considered to be **equal**, which is represented by $A = B$.

1.3 Standard sets

$$\mathbb{N} = \{1, 2, 3, 4, ...\} \text{ is the set of natural numbers}$$

$$\mathbb{Z} = \{..., -2, -1, 0, -1, -2, ...\} \text{ is the set of integers}$$

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \text{ is the set of rational numbers}$$

$$\mathbb{R} \text{ is the set of real numbers}$$

Examples:

$$-\frac{3}{4} \in \mathbb{Q}, \sqrt{2} \notin \mathbb{Q}, \pi \notin \mathbb{Q}, e \notin \mathbb{Q}$$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

1.4 Empty set

A set that has no elements is called an empty set and is denoted by $\{\}$ or $\emptyset$. The empty set does not contain anything, not even the number 0, as a set containing the number 0 actually contains 1 element and is hence not empty.

1.5 Unions

The union $A \cup B$ of the sets A and B is the set:

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or x in both } A \text{ and } B\}$$

Example:

$$A = \{1, 2, 3\}, B = \{-1, 1\}$$

$$A \cup B = \{-1, 1, 2, 3\}$$

1.6 Intersections

The intersection $A \cap B$ of the sets A and B is the set:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Example:

$$A = \{1, 2, 3\}, B = \{-1, 1\}$$

$$A \cap B = \{1\}$$

1.7 Subtracting sets

The set "A minus B", written as $A \setminus B$, is the following set:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

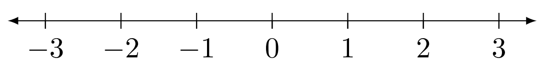
Example:

$$A = \{1, 2, 3\}, B = \{-1, 1\}$$

$$A \setminus B = \{2, 3\}$$

1.8 The real number line

We can represent the set of real numbers geometrically using a number line:



1.9 Irrational numbers

There is no rational number x with the property that $x^2 = 2$. x cannot be properly described by any rational number. However, the real number $\sqrt{2}$ has this property. There are many real numbers that are not rational. Such numbers are called irrational numbers and some important examples are the constants π and e.

$$x \text{ is irrational means that } x \in \mathbb{R} \setminus \mathbb{Q}$$

1.10 Intervals

Intervals are special subsets of real number.

For $a, b \in \mathbb{R}, a < b$, we use the notation:

1. $(a, b] = \{x \in \mathbb{R}, a < x \leq b\}$

2. $[a, b) = \{x \in \mathbb{R}, a \leq x < b\}$

3. $[a, b] = \{x \in \mathbb{R}, a \leq x \leq b\}$

4. $(a, b) = \{x \in \mathbb{R}, a < x < b\}$

5. $[a, \infty) = \{x \in \mathbb{R}, a \leq x\}$
   $(a, \infty) = \{x \in \mathbb{R}, a < x\}$
   $(-\infty, b] = \{x \in \mathbb{R}, x \leq b\}$
   $(-\infty, b) = \{x \in \mathbb{R}, x < b\}$
   $(-\infty, \infty) = \mathbb{R}$

2 Logic

2.1 Statements

In Maths, a statement is either true or false. A statement in maths cannot be both true and false, or be neither true nor false.

Examples:

$$4 < 7 \rightarrow \text{True}$$

$$4 = 7 \rightarrow \text{False}$$

$$\text{There are infinitely many prime numbers.} \rightarrow \text{True}$$

$$\text{All prime numbers are odd.} \rightarrow \text{False}$$

$$\text{The real part of every non-trivial zero of the Riemann zeta function is } 1/2. \rightarrow ?$$

2.2 Examples of non-statements

4.

Hello!

This sentence is false.

2.3 Negating a statement

The negation *not* P of a statement P, is a statement that is false when P is true and is true when P is false.

Truth table:

P	not P
T	F

More examples:

P	not P
x = 7	$x \neq 7$
x < 7	$x \geq 7$
All NTU students are younger than 30.	There exists an NTU student that is at least 30.
There exists a professor in NTU that is sane.	All professors in NTU are insane.

2.4 "For all" statements

"For all" is represented by "$\forall$".

2.5 "There exists" statements

"There exists" is represented by "$\exists$".

2.6 Interactions between "for all" and "there exists" statements

In general, the negation of "$\forall x \in A, P(x)$ is true", is "$\exists x \in A, P(x)$ is false". Similarly, the negation of "$\exists x \in A, P(x)$ is true", is "$\forall x \in A, P(x)$ is false".

3 Open sets

A set $A \subset \mathbb{R}$ is open if for every $x \in A$, there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$. The set A is not open when there exists $x \in A$ such that for every $\delta > 0$, $(x - \delta, x + \delta) \notin A$.

To make this definition easier to understand, let's assume δ to be a very small number. Let's look at the set of $(3, 5)$. If we pick a x value that is very close to the boundary, like 4.9999 ($x \neq 5$ as the set doesn't include 5), there's still a value of δ that is greater than 0 ($\delta > 0$) that can be added to 4.9999 that will not cause $(x+\delta)$ to exceed the bounding value. In this case, δ would be 0.000001. Similarly, for the lower bound, we can pick a x value that is very close to the boundary, such as 3.0001 ($x \neq 3$ as the set doesn't include 3), and there will still be a non-zero δ that is greater than 0 ($\delta > 0$) that can be subtracted from x that will not cause $(x - \delta)$ to be lower than the lower bound. In this case, δ would be 0.00001. Hence, $(3, 5)$ would be an open set.

Now, let's look at the set of $[3, 5]$. If we pick a x value that is at the boundary (remember that the set includes the boundaries), like 5, there's no value of δ that is greater than 0 ($\delta > 0$) that can be added to 5 that will not cause $(x+\delta)$ to exceed the bounding value. Similarly, for the lower bound, if we pick a x value that is at the boundary (the set includes the boundaries), such as 3, and there's no value of δ that is greater than 0 ($\delta > 0$) that can be subtracted from x that will not cause $(x - \delta)$ to be lower than the lower bound. Hence, $(3, 5)$ would not be an open set.

Examples (suppose $a < b$):

1. $(a, b) \rightarrow$ Open

2. $(a, b] \rightarrow$ Not open

3. $[a, b] \rightarrow$ Not open

4. $(a, \infty) \rightarrow$ Open

5. $[a, \infty) \rightarrow$ Not open

3.1 Not open sets

A set is **not open** means that **there exists** $x \in A$ **such that for every** $\delta > 0$**,** $(x - \delta, x + \delta) \not\subset A$.

4 Closed sets

A set $A \subset \mathbb{R}$ is closed if its "complement" $\mathbb{R} \setminus A$ is open.

Examples (suppose $a < b$):

1. $(a, b) \rightarrow$ Not closed

$\mathbb{R} \setminus (a, b) = (-\infty, a] \cup [b, \infty)$ is not open. Hence (a, b) is not closed.

2. $(a, b] \rightarrow$ Not closed

$\mathbb{R} \setminus (a, b] = (-\infty, a] \cup [b, \infty)$ is not open. Hence $(a, b]$ is not closed.

3. $[a, b] \rightarrow$ Closed

$\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is open. Hence $[a, b]$ is closed.

4. $(a, \infty) \rightarrow$ Not closed

$\mathbb{R} \setminus (a, \infty) = (-\infty, a]$ is not open. Hence (a, ∞) is not closed.

5. $[a, \infty) \rightarrow$ Closed

$\mathbb{R} \setminus [a, \infty) = (-\infty, a)$ is open. Hence (a, ∞) is closed.

6. $\mathbb{R} \rightarrow$ Closed

$\mathbb{R} \setminus \mathbb{R} = \emptyset$ is open. Hence $\mathbb{R}$ is closed.

7. $\emptyset \rightarrow$ Closed

$\mathbb{R} \setminus \emptyset = \mathbb{R}$ is open. Hence $\emptyset$ is closed.

From the examples, we can see that the set $(a, b]$ is neither open nor closed and the sets $\mathbb{R}$ and $\emptyset$ are both closed and open at the same time. $\mathbb{R}$ and $\emptyset$ are known as clopen sets, which are sets that are both closed and open. This means that open and closed sets are **not mutually exclusive** and both can happen at the same time.

In general, an open set is usually a set that does not include its bounding values while a closed set is a set that includes its bounding values, but this is not always the case, as seen from the sets $(a, b]$, $\mathbb{R}$ and $\emptyset$.

5 Logical AND

Given two statements P and Q, the statement P **AND** Q is true when both P and Q are true, and false otherwise.

Truth table:

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

6 Logical OR

The statement P **OR** Q is false when **both** P and Q are false, and true otherwise.

Truth table:

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

Examples:

P	Q	*not* Q	(P AND *not* Q)	*not* (P AND *not* Q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

7 Implications

Many statements in maths have the form "if something, then something". For example:

- If x is an even integer, then x^2 is an even integer.
- If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

  We often use implications in our day-to-day life.

- If it rains, then I'll bring an umbrella.

7.1 Notation

The statement "if A, then B" can also be expressed as $A \Rightarrow B$, which means A implies B.

7.1.1 Example 1

$$x \text{ is an even integer. } \Rightarrow x^2 \text{ is an even integer.}$$

7.1.2 Example 2

If $x > 0$ then $x^2 > x^{\frac{1}{2}}$. The statement is false, as when $x = 1$, $x > 0$ but $x^2 = x^{\frac{1}{2}}$. The implication mentioned above is false because it is possible for "$x > 0$" to be true without "$x^2 > x^{\frac{1}{2}}$" to be true. The first statement does not guarantee the second and hence the implication is false. This is usually demonstrated using a counterexample.

The truth value of the above implication does not "depend on x". It is irrelevant that $x^2 > x^{\frac{1}{2}}$ for some $x > 0$. The above implication is simply **false**.

7.1.3 Example 3

If p_k is the kth prime number, i.e.

$$p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$$

And if n is a positive integer, then

$$1 + \prod_{k=1}^{n} p_k = 1 + p_1 \cdot ... \cdot p_n$$

Is also a prime number.

n	$1 + \prod_{k=1}^{n} p_k$
1	$1 + 2 = 3 \rightarrow$ prime
2	$1 + 2 \cdot 3 = 7 \rightarrow$ prime
3	$1 + 2 \cdot 3 \cdot 5 = 31 \rightarrow$ prime
4	$1 + 2 \cdot 3 \cdot 5 \cdot 7 = 211 \rightarrow$ prime
5	$1 + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2311 \rightarrow$ prime
6	$1 + 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 59,509 \rightarrow$ **not prime**

Hence, the implication is **false**.

7.1.4 Example 4

If a and b are positive real numbers, then:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$\dfrac{a+b}{2}$ is the arithmetic mean of a and b.

$\sqrt{ab}$ is the geometric mean of a and b.

The implication is **true**, as for $a, b > 0$, we have

$$\begin{aligned}\frac{a+b}{2} - \sqrt{ab} &= \frac{a}{2} - \sqrt{ab} + \frac{b}{2} \\ &= (\sqrt{\frac{a}{2}} - \sqrt{\frac{b}{2}})^2 \\ &\geq 0\end{aligned}$$

7.1.5 Example 5

If x is a real number and $x^2 < 0$, then x is a pink elephant. This statement is **true**.

If we let:

$$A = \{x \in \mathbb{R} : x^2 < 0\} = \emptyset$$

The last statement can be phrased as "for all $x \in A, x$ is a pink elephant". The negation of this statement would be "there exists $x \in A$, such that x is not a pink elephant". Since the negation of the original statement is **false**, the original statement must be **true**.

7.2 Another way to think about implications

The implication $P \Rightarrow Q$ is another way of saying *not*(P AND *not* Q).

P	Q	*not* (P AND *not* Q)
T	T	T
T	F	F
F	T	T
F	F	T

$$\downarrow$$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

8 Equivalences

An implication has a "direction". For example, the implication $x = 2 \Rightarrow x^2 = 4$ is **true** but if we turn it around, $x^2 = 4 \Rightarrow x = 2$, we get something **false**.

If we instead consider the implication $x = 2$ or $x = -2 \Rightarrow x^2 = 4$, which is still true, we see that the "reverse" implication $x^2 = 4 \Rightarrow x = 2$ or $x = -2$ also holds. Hence, the two statements "$x = 2$ or $x = -2$" and "$x^2 = 4$" are **equivalent**.

The statement $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$ can be written as $P \Leftrightarrow Q$. In this case, we say that P and Q are equivalent, or P if and only if Q.

9 Contrapositive statement

Suppose that you **know** that the implication "if it rains, then your lecturer carries an umbrella" is a **true** statement. One day, you see your lecturer not carrying an umbrella, so you conclude that "if your lecturer does not carry an umbrella, then it does not rain".

The statements $P \Rightarrow Q$ and *not* $Q \Rightarrow$ *not* P are equivalent.

P	Q	$P \Rightarrow Q$	*not* Q	*not* P	*not* Q $\Rightarrow$ *not* P
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

10 Functions

10.1 Definition

Consider two sets A and B. A **function** $f : A \to B$ is a rule that assigns to each element $x \in A$ exactly one element $f(x) \in B$ called the value of the function f at the point x.

Put simply, a function takes a set of inputs, A and returns a set of outputs B. One input can only have one output.

The set A is called the **domain** of f, and B is called the **codomain** of f. We also say that $f : A \to B$ is a function **from** A **to** B.

10.1.1 Example 1

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$$

$$f(1) = 1, f(-2) = 4$$

10.1.2 Example 2

Let $A = \{\text{Homer}, \text{Marge}, \text{Bart}, \text{Lisa}, \text{Maggie}\}$ and define $f : A \to \mathbb{N}$ by $f(x) =$ the age of x in years.

$$f(\text{Homer}) = 38$$
$$f(\text{Marge}) = 34$$
$$f(\text{Bart}) = 10$$
$$f(\text{Lisa}) = 8$$
$$f(\text{Maggie}) = 1$$

$$f(\text{Ned Flanders}) = \text{undefined}$$
$$\text{Ned Flanders} \notin A$$

10.1.3 Important note

For a function f with domain A, $f(x)$ is only defined for $x \in A$.

So for a function:

$$f : [0, \infty) \to \mathbb{R}$$
$$f(x) = x^2$$

$f(-2) \neq 4$, instead $f(-2) = \text{undefined}$ as $f(x)$ is only defined for $x \in A$.

10.2 Sequences

10.2.1 Definition

A function $f : A \to \mathbb{R}$ where A is a subset of $\mathbb{N}$ is called a **sequence**.

1. Example 1

   The function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 1 + \frac{(-1)^n}{n}$ is a sequence. We have:

   $$f(1) = 1 + \frac{-1}{1} = 0, f(2) = 1 + \frac{1}{2} = \frac{3}{2}, f(3) = 1 + \frac{-1}{3} = \frac{2}{3}, \text{ etc.}$$

   For sequences, we often use the notation a_n instead of $f(n)$ and

   $$(a_n), (a_n)_{n=1}^{\infty}, (a_1, a_2, a_3, ...), \text{ etc. instead of } f$$

2. Example 2

   The sequence in the example 1 would more commonly be described as

   $$(a_n)_{n=1}^{\infty}, \text{ where } a_1 = 1+\frac{-1}{1} = 0, a_2 = 1+\frac{1}{2} = \frac{3}{2}, a_3 = 1+\frac{-1}{3} = \frac{2}{3}, \text{ etc.}$$

10.3 Different ways of describing functions

A function can be described in any way. In fact, just using words is perfectly fine as long as the meaning is clear and unambiguous. Some particularly common ways are:

Explicit formulae like:

$$f(x) = sin(1 + x^3),$$
$$g(y) = \frac{1+y}{1-y},$$
$$a_n = 2^n.$$

Implicit formulae like:

$$\sin g(t) = t, \qquad -\pi \le g(t) \le \frac{\pi}{2},$$

Recurrent formulae for a sequence, like:

$$a_1 = 2, a_{n+1} = 2a_n,$$
$$\text{so } a_1 = 2, a_2 = 2a_1 = 4, a_3 = 2a_3 = 8, \text{ etc.}$$
$$a_1 = 1, a_2 = 2, a_{n+1} = a_n + a_{n-1}$$
$$\text{so } a_1 = 1, a_2 = 2, a_3 = a_2 + a_1 = 3, a_4 = a_3 + a_2 = 5, \text{ etc.}$$

10.4 The graph of a function

A function $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$ can be represented by its **graph**.

The graph of $f : A \to \mathbb{R}$ is formally defined as a set of pairs (x, y).

$$G_f = \{(x, f(x)) : x \in A\}$$

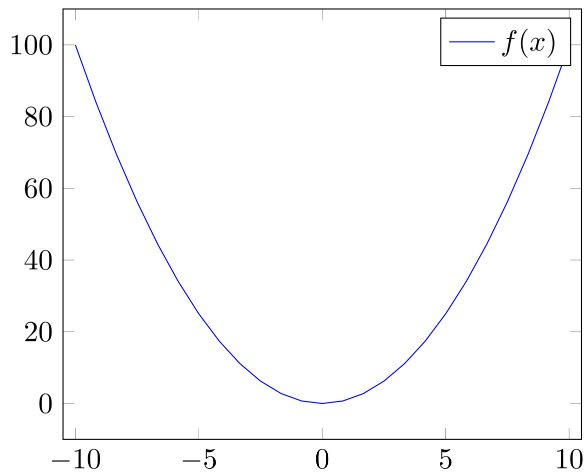
10.5 The domain of a function

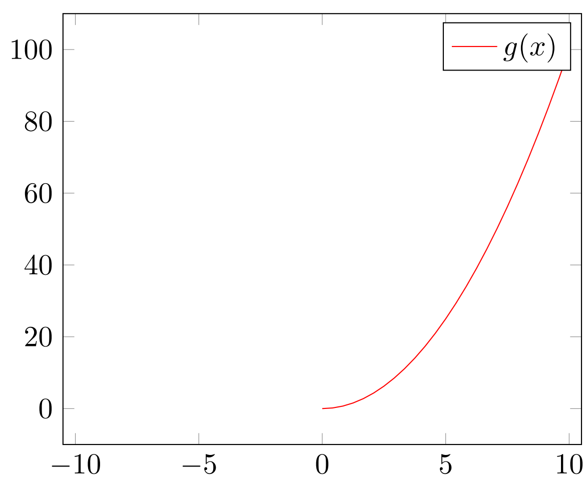
For a function $f : A \to B$, the set A is called the **domain** of f. Note that the domain is part of the definition of a function, so:

$$f : \mathbb{R} \to \mathbb{R} \quad f(x) = x^2, \text{ and } g : [0, \infty] \to \mathbb{R} \quad g(x) = x^2$$

Are **different** functions.

Below is the graph of $f(x)$ versus $g(x)$:





10.5.1 The natural domain of a function

Usually, we will simply state an expression for f without saying what the domain is. For example $f(x) = \sqrt{x-1}$, where the natural domain of f is $\{x : x - 1 \geq 0\} = [1, \infty)$.

If we don't say anything about the domain, we will assume that it is the largest set A, for which $f(x)$ 'makes sense' when $x \in A$. We also call this the **natural domain** of f.

10.6 Image and range of a function

10.6.1 Definition

Consider a function $f : A \to B$.

For $K \subset A$, the set $f(K) = \{f(x) : x \in K\} \subset B$, is called the **image** of the set K. The image of $f(A)$ of the whole domain A is called the range of the function f.

10.6.2 Example 1

$$\text{Let } A = \{\text{Homer}, \text{Marge}, \text{Bart}, \text{Lisa}, \text{Maggie}\}$$
$$\text{and define } f : A \to \mathbb{N} \text{ by } f(x) = \text{ the age of } x$$
$$\text{Also, let } K = \{\text{Bart}, \text{Lisa}, \text{Maggie}\}.$$

Remember that:

$$f(\text{Homer}) = 39$$
$$f(\text{Marge}) = 34$$
$$f(\text{Bart}) = 10$$
$$f(\text{Lisa}) = 8$$
$$f(\text{Maggie}) = 1$$

Then the "range of f" $= f(A) = \{f(x) : x \in A\} = \{38, 34, 10, 8, 1\}$

$$f(K) = \{f(x) : x \in K\} = \{10, 8, 1\}$$

10.6.3 Example 2

$$f(x) = \sqrt{1 - x^2}$$

Domain $= \{x : 1 - x^2 \geq 0\} = \{x : x^2 \leq 1\} = [-1, 1]$

Range $= \{\sqrt{1 - x^2} : x \in [-1, 1] = [0, 1]\}$

11 Properties of functions

11.1 Increasing and decreasing functions

11.1.1 Definition

A function $f : A \to \mathbb{R}$ where $A \subset \mathbb{R}$, is said to be:

Increasing if $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

Decreasing if $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Monotonic if it is either increasing or decreasing.

Furthermore, if $B \subset A$, we say that f is increasing/decreasing/monotonic **on B** if either of the above conditions hold of $x_1, x_2 \in B$.

11.2 Strictly increasing and decreasing functions

11.2.1 Definition

A function $f : A \to \mathbb{R}(A \subset \mathbb{R})$ is said to be:

Strictly increasing if $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Strictly decreasing if $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Monotonic if it is either increasing or decreasing.

Furthermore, if $B \subset A$, we say that f is strictly increasing/decreasing/monotonic **on B** if either of the above conditions hold of $x_1, x_2 \in B$.

11.2.2 Examples

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \text{ is not increasing as:}$$
$$-1 < 0 \text{ but } f(-1) > f(0)$$

$$f : [0, \infty] \to \mathbb{R}, f(x) = x^2 \text{ is strictly increasing as:}$$
$$x_1, x_2 \in (0, \infty) \text{ and } x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

$$f : \mathbb{R} \to \mathbb{R}, f(x) = 1 \text{ is increasing}$$
$$x_1, x_2 \in \mathbb{R} \text{ and } x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$
$$\text{But not strictly increasing as:}$$
$$0, 1 \in \mathbb{R} \text{ and } 0 < 1 \text{ but } f(0) = f(1)$$

11.3 Bounded functions

11.3.1 Definition

A function $f : A \to \mathbb{R}$ is **bounded** if there exists a $M > 0$ such that $|f(x)| \leq M$, for all $x \in A$.

In simpler terms, a function that is bounded is a function that doesn't approach $+\infty$ or $-\infty$.

Note that $|f(x)| \leq M \Leftrightarrow -M \leq f(x) \leq M$.

A function that is not bounded is said to be **unbounded**. Furthermore, if $B \subset A$, we say that f is bounded **on B** if the above inequality holds for all $x \in B$.

11.3.2 Example 1

$$\sin x, \ \ \cos x \text{ are bounded.}$$

The domain of $\sin x$ is $\mathbb{R}$. For all $x \in \mathbb{R}$, we have $|\sin x| \leq 1$, so $\sin x$ is bounded. The same is true for $\cos x$.

11.3.3 Example 2

x^2 is bounded on the interval $[-2, 1]$, as for $x \in [-2, 1]$ we have $|x^2| \leq 4$, so x^2 is bounded on $[-2, 1]$.

11.4 Locally bounded functions

11.4.1 Definition

A function $f : A \to \mathbb{R}$ is **locally bounded at point** $a \in A$ if there exists some $\delta > 0$ such that f is bounded on $A \cap (a - \delta, a + \delta)$.

A function that is **locally bounded** means that f is locally bounded at every $a \in A$.

11.4.2 Example 1

Given the definition of f below:

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{x}$$

Show that f is locally bounded.

Let A be the domain of f, i.e. $A = \mathbb{R} \setminus \{0\}$, and suppose $a \in A$.

Note that $a \neq 0$, so a is within the domain of f.

Let $\delta = \frac{|a|}{2},\ M = \frac{2}{|a|}$, where $M, \delta > 0$ and

$$|x| > \frac{|a|}{2},\ \text{for } x \in (a - \delta, a + \delta) \cap A.$$

Therefore,

$$\begin{aligned}
|f(x)| = \frac{1}{|x|} &< \frac{1}{\frac{|a|}{2}} \\
&< \frac{2}{|a|} \\
&< M,\ \text{for } x \in (a - \delta, a + \delta) \cap A
\end{aligned}$$

Hence, f is bounded on $(a - \delta, a + \delta) \cap A$. Thus, f is locally bounded.

11.4.3 Example 2

Given the definition of g below:

$$g : \mathbb{R} \to \mathbb{R},\ g(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that g is not locally bounded.

The definition of a locally bound function f is that for each $a \in \mathbb{R}$, there exists $\delta > 0$ such that f is bounded on $a - \delta, a + \delta$.

So for a function f that is **not** locally bound, there exists some $a \in \mathbb{R}$, such that f is not bounded on $a - \delta, a + \delta$.

The domain A of g is $\mathbb{R}$. Since $0 \in A$, and for all $\delta > 0, g$ is unbounded on $(0 - \delta, 0 + \delta) \cap A = (-\delta, \delta)$. That means g is not locally bounded at 0 and thus g is **not locally bounded**.

11.5 Unbounded functions

11.5.1 Definition

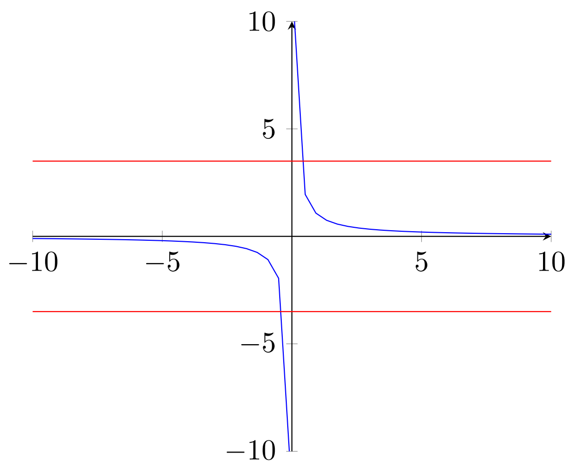
From the definition of a bounded function, which is "a function $f : A \to \mathbb{R}$ is **bounded** if there exists a $M > 0$ such that $|f(x)| \leq M,$ for all $x \in A$".

A function that is not bounded is said to be **unbounded**, which is "f is bounded if for all $M > 0$, there exists $x \in A$ such that $|f(x)| > M$".

11.5.2 Example

Is the function $f(x) = \frac{1}{x}$ bounded or unbounded?

For the graph below, $y = f(x) = \frac{1}{x}$ is in blue, and $y = -M$ and $y = M$ are in red.



Domain of $f = \mathbb{R} \setminus \{0\}$

Let $M > 0$, and take $x = \frac{1}{2M}, x \in \mathbb{R} \setminus \{0\}$

$$|f(x)| = \left| \frac{1}{\frac{1}{2M}} \right| = |2M| = 2M > 0$$

Hence, f is unbounded.

11.6 Odd and even functions

11.6.1 Definition

A function $f : A \to \mathbb{R}$ is said to be:

Odd if $x \in A \Rightarrow -x \in A$ and $f(-x) = -f(x)$.

The graph of $y = f(x)$ is symmetric about $(0, 0).

Even if $x \in A \Rightarrow -x \in A$ and $f(-x) = f(x)$.

The graph of $y = f(x)$ is symmetric about the y-axis.

11.7 Periodic functions

11.7.1 Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **periodic** if there exists some $T > 0$ such that $f(x + T) = f(x),$ for all $x \in \mathbb{R}$.

The number T is called a **period** for f.

11.8 Examples

$\sin x$ and $\cos x$ are both periodic with period 2π. $\sin x$ is odd, $\cos x$ is even. x^2 is even, x^3 is odd. e^x is neither odd nor even. Given any function $f(x)$, the function $g(x) = f(|x|)$ is even.