Math Module 1B Tutorial

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Contents

1 Question 1

1.1 (a)

$$\begin{aligned}\frac{x^3+8}{x+2} &= \frac{(x+2)(x^2-2x+4)}{x+2}\\ &= x^2-2x+4\end{aligned}$$

Hence:

$$\begin{aligned}\lim_{x\to -2}\frac{x^3+8}{x+2} &= \lim_{x\to -2} x^2-2x+4\\ &= (-2)^2-2(-2)+4\\ &= 12\end{aligned}$$

1.2 (b)

$$\begin{aligned}\frac{\sqrt{x}-\sqrt{4-x}}{x-2} &= \frac{\sqrt{x}-\sqrt{4-x}}{x-2}\left(\frac{\sqrt{x}+\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\right)\\ &= \frac{x-(4-x)}{(x-2)(\sqrt{x}+\sqrt{4-x})}\\ &= \frac{2x-4}{(x-2)(\sqrt{x}+\sqrt{4-x})}\\ &= \frac{2(x-2)}{(x-2)(\sqrt{x}+\sqrt{4-x})}\\ &= \frac{2}{\sqrt{x}+\sqrt{4-x}}\end{aligned}$$

Hence:

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{4-x}}{x-2} &= \lim_{x\to 2}\frac{2}{\sqrt{x}+\sqrt{4-x}}\\ &= \frac{2}{\sqrt{2}+\sqrt{4-2}}\\ &= \frac{2}{2\sqrt{2}}\\ &= \frac{1}{\sqrt{2}}\end{aligned}$$

1.3 (c)

$$
\begin{aligned}
\frac{\sqrt[3]{1+\sin x}-1}{x} &= \frac{\sqrt[3]{1+\sin x}-1}{x}\left(\frac{(\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1}{(\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1}\right)\\
&= \frac{1+\sin x-1}{x((\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1)}\\
&= \frac{\sin x}{x((\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1)}\\
&= \frac{\sin x}{x}\left(\frac{1}{(\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1}\right)
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\lim_{x\to 0}\frac{\sqrt[3]{1+\sin x}-1}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\left(\frac{1}{(\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1}\right)\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{(\sqrt[3]{1+\sin x})^2+\sqrt[3]{1+\sin x}+1}\\
&= 1\cdot\frac{1}{(\sqrt[3]{1+\sin 0})^2+\sqrt[3]{1+\sin 0}+1}\\
&= 1\cdot\frac{1}{(\sqrt[3]{1})^2+\sqrt[3]{1}+1}\\
&= 1\cdot\frac{1}{1^2+1+1}\\
&= 1\cdot\frac{1}{1+1+1}\\
&= 1\cdot\frac{1}{3}\\
&= \frac{1}{3}
\end{aligned}
$$

1.4 (d)

Using L'Hôpital's rule:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0}\frac{d\sin 4x}{dx}\left(\frac{d\sin 6x}{dx}\right)^{-1}\\
&= \lim_{x\to 0}(4\cos 4x)(6\cos 6x)^{-1}\\
&= \lim_{x\to 0}\frac{4\cos 4x}{6\cos 6x}\\
&= \frac{4\cos 4(0)}{6\cos 6(0)}\\
&= \frac{4}{6}\\
&= \frac{2}{3}
\end{aligned}$$

Without using L'Hôpital's rule:

$$\begin{aligned}
\frac{\sin 4x}{\sin 6x} &= \frac{4x\sin 4x}{4x}\cdot\left(\frac{6x\sin 6x}{6x}\right)^{-1}\\
&= 4x\left(\frac{\sin 4x}{4x}\right)\cdot\frac{1}{6x}\left(\frac{\sin 6x}{6x}\right)^{-1}\\
&= \frac{4x}{6x}\left(\frac{\sin 4x}{4x}\right)\left(\frac{\sin 6x}{6x}\right)^{-1}\\
&= \frac{4}{6}\left(\frac{\sin 4x}{4x}\right)\left(\frac{\sin 6x}{6x}\right)^{-1}
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{x\to x}\frac{\sin 4x}{\sin 6x} &= \lim_{x\to 0}\frac{4}{6}\left(\frac{\sin 4x}{4x}\right)\left(\frac{\sin 6x}{6x}\right)^{-1}\\
&= \frac{4}{6}(1)(1)^{-1}\\
&= \frac{4}{6}(1)(1)\\
&= \frac{2}{3}
\end{aligned}$$

1.5 (e)

$$\begin{aligned}
\frac{\sin^3 2x}{x^3} &= \frac{1}{x^3}\left(\frac{2x\sin 2x}{2x}\right)\left(\frac{2x\sin 2x}{2x}\right)\left(\frac{2x\sin 2x}{2x}\right) \\
&= \frac{8x^3}{x^3}\left(\frac{\sin 2x}{2x}\right)^3 \\
&= 8\left(\frac{\sin 2x}{2x}\right)^3
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin^3 2x}{x^3} &= \frac{1}{8}\left(\frac{\sin 2x}{2x}\right)^3 \\
&= 8(1)^3 \\
&= 8
\end{aligned}$$

1.6 (f)

$$
\begin{aligned}
\frac{\sqrt{5x^2+3}}{-2x+5} &= \frac{\sqrt{x^2\left(5+\frac{3}{x^2}\right)}}{5-2x} \\
&= \frac{\sqrt{x^2}\sqrt{5+\frac{3}{x^2}}}{5-2x} \\
&= \frac{|x|\sqrt{5+\frac{3}{x^2}}}{5-2x} \\
&= \frac{|x|\sqrt{5+\frac{3}{x^2}}}{x\left(\frac{5}{x}-2\right)}
\end{aligned}
$$

Hence:

$$
\lim_{x\to-\infty} \frac{\sqrt{5x^2+3}}{-2x+5} = \frac{|x|\sqrt{5+\frac{3}{x^2}}}{x\left(\frac{5}{x}-2\right)}
$$

Since $x \to -\infty$, $|x| = -x$, $\frac{3}{x^2} \to 0$ and $\frac{5}{x} \to 0$:

$$
\begin{aligned}
\lim_{x\to-\infty} \frac{|x|\sqrt{5+\frac{3}{x^2}}}{x\left(\frac{5}{x}-2\right)} &= \lim_{x\to-\infty} \frac{-x\sqrt{5+\frac{3}{x^2}}}{x\left(\frac{5}{x}-2\right)} \\
&= \lim_{x\to-\infty} \frac{-\sqrt{5+\frac{3}{x^2}}}{\left(\frac{5}{x}-2\right)} \\
&= \frac{-\sqrt{5+0}}{0-2} \\
&= \frac{-\sqrt{5}}{-2} \\
&= \frac{\sqrt{5}}{2}
\end{aligned}
$$

1.7 (g)

$$\begin{aligned}
\sqrt{x^4+6x^2}-x^2 &= (\sqrt{x^4+6x^2}-x^2)\left(\frac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}\right)\\
&= \frac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}\\
&= \frac{6x^2}{\sqrt{x^4\left(1+\frac{6}{x^2}\right)}+x^2}\\
&= \frac{6x^2}{\sqrt{x^4}\sqrt{1+\frac{6}{x^2}}+x^2}\\
&= \frac{6x^2}{|x^2|\sqrt{1+\frac{6}{x^2}}+x^2}\\
&= \frac{6x^2}{x^2\sqrt{1+\frac{6}{x^2}}+x^2} \qquad \because \quad x^2>0\\
&= \frac{6x^2}{x^2(\sqrt{1+\frac{6}{x^2}}+1)}\\
&= \frac{6}{1+\sqrt{1+\frac{6}{x^2}}}
\end{aligned}$$

Hence:

$$\lim_{x\to\infty}\sqrt{x^4+6x^2}-x^2 = \lim_{x\to\infty}\frac{6}{1+\sqrt{1+\frac{6}{x^2}}}$$

Since $\frac{6}{x^2}\to 0$ as $x\to\infty$:

$$\begin{aligned}
\lim_{x\to\infty}\frac{6}{1+\sqrt{1+\frac{6}{x^2}}} &= \frac{6}{1+\sqrt{1+0}}\\
&= \frac{6}{2}\\
&= 3
\end{aligned}$$

2 Question 2

Using the definition of a limit, which is for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta,\ x \in A \ \Rightarrow\ |f(x) - L| < \varepsilon$$

For $\varepsilon > 0$, $x \notin \mathbb{Q}$, $f(x) = 0$, hence:

$$|0 - 0| = 0 < \varepsilon$$

Thus, the limit for $f(x)$ exists when $x \notin \mathbb{Q}$.

For $\varepsilon > 0$, $x \in \mathbb{Q}$:

$$0 < |x - 0| < \delta$$
$$|x| < \delta$$
$$f(|x|) < \varepsilon$$
$$x < \varepsilon$$

Hence, let $\delta = \varepsilon$ and we have:

$$0 < |x| < \delta \ \Rightarrow\ |f(x)| < \delta = \varepsilon$$

Thus, $\lim_{x\to 0} f(x)$ exists.

Guessing the limit of $f(x)$ to be 0 when $x \to 0$:

$$\lim_{x\to 0} f(x) = 0$$

Proving the limit of $f(x)$ is 0 when $x \to 0$ using the squeeze theorem:

$$\lim_{x\to 0} f(x) = 0 \quad \Leftrightarrow \quad \lim_{x\to 0} |f(x) - 0| = 0$$
$$\lim_{x\to 0} |f(x)| = 0$$

Hence:

$$\lim_{x\to 0} f(x) = 0$$

3 Question 3

3.1 (a)

To know if Master Yoda can complete all of his tasks on time, we need to take the sum to infinity of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}$:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{2^k} &= \frac{\frac{1}{2}}{1 - \frac{1}{2}} \\
&= \frac{\frac{1}{2}}{\frac{1}{2}} \\
&= 1
\end{aligned}$$

Since 2 is a finite number, Master Yoda will be able to complete all of his tasks in finite time.

3.2 (b)

Taking the sum to n of the series $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots, \frac{1}{n^2+n}$:

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k^2+k} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= \frac{1}{1} - \frac{1}{2} \\
&+ \frac{1}{2} - \frac{1}{3} \\
&+ \frac{1}{3} - \frac{1}{4} \\
&+ \ldots \\
&+ \frac{1}{n-2} - \frac{1}{n-1} \\
&+ \frac{1}{n-1} - \frac{1}{n} \\
&+ \frac{1}{n} - \frac{1}{n+1} \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

When $x \to \infty$, $\frac{1}{n+1} \to 0$ and hence, the sum to infinity of the above series is:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2+k} &= 1 - 0 \\
&= 1
\end{aligned}$$

Since 1 is a finite number, Master Yoda will be able to complete all of his tasks in finite time.

4 Question 4

4.1 (a)

Squaring $|x+y|$:

$$\begin{aligned} |x+y|^2 &= (x+y)^2 \\ &= x^2 + 2xy + y^2 \end{aligned}$$

Squaring $|x| + |y|$

$$\begin{aligned} (|x|+|y|)^2 &= |x|^2 + 2|x||y| + |y|^2 \\ &= x^2 + 2|x||y| + y^2 \quad \because\ |x|^2 = x^2 \text{ and } |y|^2 = y^2 \end{aligned}$$

Since $|x||y| = xy$ when $x, y > 0$ or $x, y < 0$ and $|x||y| > xy$ when $x < 0, y > 0$ and $x > 0, y < 0$, $|x||y| \geq xy$. This means $|x+y|^2 \leq (|x|+|y|)^2$.

Since $|x+y|^2 \leq (|x|+|y|)^2$, $|x+y| \leq |x|+|y|$ (**Proven**).

4.2 (b)

Squaring $||x| - |y||^2$:

$$\begin{aligned} ||x| - |y||^2 &= (|x| - |y|)^2 \\ &= |x|^2 - 2|x||y| + |y|^2 \\ &= x^2 - 2|x||y| + y^2 \quad \because\ |x|^2 = x^2 \text{ and } |y|^2 = y^2 \end{aligned}$$

Squaring $|x-y|^2$:

$$\begin{aligned} |x-y|^2 &= (x-y)^2 \\ &= x^2 - 2xy + y^2 \end{aligned}$$

Since $|x||y| = xy$ when $x, y > 0$ or $x, y < 0$ and $|x||y| > xy$ when $x < 0, y > 0$ and $x > 0, y < 0$, $|x||y| \geq xy$. This means $||x| - |y||^2 \leq |x-y|^2$.

Since $||x| - |y||^2 \leq |x-y|^2$, $||x| - |y|| \leq |x-y|$ (**Proven**).