1) $\begin{bmatrix} 2 & 3 & -1 & a \\ 1 & -1 & 0 & b \\ 0 & 5 & 2 & c \\ 3 & 2 & 1 & a \end{bmatrix} \sim$

$\begin{bmatrix} 0 & 5 & -1 & a-2b \\ 1 & -1 & 0 & b \\ 0 & 5 & 2 & c \\ 0 & 5 & 0 & a-3b \end{bmatrix} \sim$

$\begin{bmatrix} 0 & 0 & -1 & \cancel{a}-2b-(\cancel{a}-3b) \\ 1 & -1 & 0 & b \\ 0 & 0 & 2 & c-(a-3b) \\ 0 & 5 & 0 & a-3b \end{bmatrix} \sim$

$\begin{bmatrix} 0 & 0 & -1 & b \\ 1 & -1 & 0 & b \\ 0 & 0 & 2 & c-a+3b \\ 0 & 5 & 0 & a-3b \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & -1 & 0 & b \\ 0 & 5 & 0 & a-3b \\ 0 & 0 & 1 & -b \\ 0 & 0 & 2 & c-a+3b \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & b \\ 0 & 1 & 0 & \frac{a-3b}{5} \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & c-a+b \end{bmatrix}$$

For the above linear system to be consistent, $c-a+b=0$

(a) $c-a+b = -7-2+3$

$= -6 \neq 0$

$\therefore (2,3,-7,2)$ doesn't belong to span S

(b) $c-a+b = 0-0+0=0$

$\therefore (0,0,0,0)$ belongs to span S.

1c) $c - a + b = 1 - 1 + 1$

$= 1 \neq 0$

$\therefore (1,1,1,1)$ doesn't belong to span S.

2a) Let $\underset{\sim}{w} = (a, b, c)$, $a, b, c \in \mathbb{R}$

$$\begin{bmatrix} 2 & 0 & 0 & a \\ 2 & 0 & 1 & b \\ 2 & 3 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & a \\ 0 & 0 & 1 & a-b \\ 0 & 3 & 1 & c-b \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}a \\ 0 & 3 & 0 & c - \cancel{b} - (a - \cancel{b}) \\ 0 & 0 & 1 & a-b \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}a \\ 0 & 1 & 0 & \frac{1}{3}(c-a) \\ 0 & 0 & 1 & a-b \end{bmatrix}$$

The linear system above is consistent for all $a, b, c \in \mathbb{R}$ and hence span $S = \mathbb{R}^3$

2b) Let $\underset{\sim}{w} = (a, b, c)$, $a, b, c \in \mathbb{R}$

$$\begin{bmatrix} 3 & 2 & 5 & 1 & a \\ 1 & -3 & -2 & 4 & b \\ 4 & 5 & 9 & -1 & c \end{bmatrix} \sim \begin{bmatrix} 0 & 11 & 11 & -11 & a-3b \\ 1 & -3 & -2 & 4 & b \\ 0 & 17 & 17 & -17 & c-4b \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -3 & -2 & 4 & b \\ 0 & 1 & 1 & -1 & \frac{1}{11}(a-3b) \\ 0 & 0 & 0 & 0 & c-4b-\frac{17}{11}(a-3b) \end{bmatrix}$$

The linear system above is only consistent when $c-4b-\frac{17}{11}(a-3b)=0$.

$\therefore$ span $S \neq \mathbb{R}^3$

3a) By inspection, $\{(4,1,2), (-4,10,2)\}$ are linearly independent.

b) By inspection, $\{(-3,0,4), (5,-1,2), (1,1,3)\}$ are linearly independent.

3c) $\begin{bmatrix} -2 & 3 & 6 & 7 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ 1 & 5 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 & -2 & 0 \\ 0 & 13 & 8 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & \frac{7}{2} & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{29}{2} & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{7}{2} & -2 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & -\frac{79}{29} & 0 \\ 0 & 1 & 0 & \frac{3}{29} & 0 \\ 0 & 0 & 1 & \frac{6}{29} & 0 \end{bmatrix}$

$\therefore \{(-2,0,1), (3,2,5), (6,-1,1), (7,0,-2)\}$ is linearly dependent.

3d) Since $\underset{\sim}{0} \in \{(1,5,1), (9,6,1), (0,0,0)\}$,

$\{(1,5,1), (9,6,1), (0,0,0)\}$ is linearly dependent.

4) $\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \sim$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $\underset{\sim}{x} \in V$

$\therefore \underset{\sim}{w} = \underset{\sim}{x}$

$\underset{\sim}{v} = \underset{\sim}{x}$

$\underset{\sim}{u} = \underset{\sim}{x}$

for all $\underset{\sim}{x} \in V$

Since $\underset{\sim}{0}$ is not the only solution to the linear system above, $\{\underset{\sim}{u} - \underset{\sim}{v}, \underset{\sim}{v} - \underset{\sim}{w}, \underset{\sim}{w} - \underset{\sim}{u}\}$ is linearly dependent.

5a) Since $\underset{\sim}{0} \in \operatorname{span} S$, $\operatorname{span} S \neq \emptyset$

Let $\underset{\sim}{u}, \underset{\sim}{v} \in \operatorname{span} S$,

$\underset{\sim}{u} + \underset{\sim}{v}$

$= 1\underset{\sim}{u} + 1\underset{\sim}{v} \in \operatorname{span} S$ by definition of span S

Let $k \in \mathbb{R}$,

$k\underset{\sim}{u} \in \operatorname{span} S$ by definition of span S

$\therefore$ span S is a subspace of V

b) When $\underset{\sim}{0} \in S$, there will be infinitely many solutions to the homogeneous linear system created from the vectors in S, due to the presence of a row with entirely zeros.

$\therefore$ S is linearly dependent if $\underset{\sim}{0} \in S$.

5c) S is linearly dependent

$$\Updownarrow$$

There exist a $k_1, \ldots, k_m \in \mathbb{R}$ that is <u>not</u> all zero and $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_m \in S$ such that $k_1\underset{\sim}{v}_1 + \ldots + k_m\underset{\sim}{v}_m = 0$

Suppose $m \geq 2$,

There exists an $1 \leq i \leq m$ such that $k_i \neq 0$.

$$k_i\underset{\sim}{v}_i = -\sum_{j \neq i} k_j\underset{\sim}{v}_j$$

$$k_i\underset{\sim}{v}_i = -(k_1\underset{\sim}{v}_1 + \ldots + k_m\underset{\sim}{v}_m - k_i\underset{\sim}{v}_i)$$

$$\underset{\sim}{v}_i = \sum_{j \neq i}\left(-\frac{k_j}{k_i}\right)\underset{\sim}{v}_j$$

Suppose $m = 1$,

$k_1 \neq 0$

$k_1\underset{\sim}{v}_1 = 0$

$\underset{\sim}{v}_1 = \underset{\sim}{0} \in S$

Let $\underset{\sim}{w} \in S$ such that $\underset{\sim}{w} \neq \underset{\sim}{0}$,

$$\underset{\sim}{v}_1 = 0\underset{\sim}{w}$$

$\therefore$ At least one vector in S is a linear combination of the others.

5c) There exists $\underset{\sim}{v}, \underset{\sim}{v}_1, \dots, \underset{\sim}{v}_m \in S$ such that

$$v = k_1 \underset{\sim}{v}_1 + \dots + k_m \underset{\sim}{v}_m, \quad k_1, \dots, k_m \in \mathbb{R}$$

$$k_1 \underset{\sim}{v}_1 + \dots + k_m \underset{\sim}{v}_m + (-1)\underset{\sim}{v} = \underset{\sim}{0}$$

Since the coefficient of $\underset{\sim}{v}$ is $-1 \neq 0$, S is linearly dependent.

6a) Let $W = \{$ all vectors of the form $(a,b,c,0)\}$

$\therefore$ $x \in W$ if and only if

$$x = a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0),$$
$$a, b, c \in \mathbb{R}$$

By inspection, $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ is linearly independent.

A basis is $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ and the dimension of the space is 3.

6b) Let $W = \{$ all vectors of the form (a,b,c,d), where $d = a+b$ and $c = a-b \}$

$\therefore$ $x \in W$ if and only if

$$x = a(1,0,1,1) + b(0,1,-1,1), \quad a, b \in \mathbb{R}$$

By inspection, $\{(1,0,1,1), (0,1,-1,1)\}$ is linearly independent.

$\therefore$ A basis is $\{(1,0,1,1), (0,1,-1,1)\}$ and the dimension of the space is 2.

6c) Let $W = \{$all vectors of the form (a,b,c,d) where $a=b=c=d\}$

$\therefore\ x \in W$ if and only if

$$x = a(1,1,1,1),\ a \in \mathbb{R}$$

By inspection, $\{(1,1,1,1)\}$ is linearly independent

$\therefore$ A basis is $\{(1,1,1,1)\}$ and the dimension is of the space is 1.

7a) Let $W = \{$all 3×3 diagonal matrices$\}$

$\therefore\ x \in W$ if and only if

$$x = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},$$

$$a,b,c \in \mathbb{R}$$

By inspection $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ is linearly independent.

$\therefore$ A basis is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right\}$ and the dimension of the space is 3.

7b) Let $W = \{\text{all } 3\times 3 \text{ symmetrical matrices}\}$

$\therefore x \in W$ if and only if

$$x = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + d\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix} + e\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix} + f\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}, \quad a,b,c,d,e,f \in \mathbb{R}$$

By inspection, $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\}$ is linearly independent.

$\therefore$ a basis is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right\}$ and the dimension of the space is 6.

7c) Let $W = \{\text{all upper triangular matrices}\}$

$\therefore x \in W$ if and only if

$$x = a\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + b\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} + c\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix} + d\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix} + e\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix} + f\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},$$

$a,b,c,d,e,f \in \mathbb{R}$

By inspection, $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\right\}$ is linearly independent.

$\therefore$ a basis is $\left\{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}\right\}$

and the dimension of the space is 6.

8a) Let $W = \{$all polynomials in $p \in P_3$ such that $p(0) = 0\}$

$\therefore x \in W$ if and only if

$x(0) = 0$

By inspection, $\{x, x^2, x^3\}$ is linearly independent.

$\therefore$ a basis is $\{x, x^2, x^3\}$ and the dimension of the space is 3.

b) Let $W = \{$all polynomials in $p \in P_3$ such that $p(0) = p(2) = 0\}$

$\therefore x \in W$ if and only if

$p(0) = 0$ and $p(2) = 0$

By inspection, $\{x^2 - 2x, x^3 - 2x^2\}$ is linearly independent.

$\therefore$ a basis is $\{x^2 - 2x, x^3 - 2x^2\}$ and the dimension of the space is 2.

9a) Since $\underset{\sim}{0} \in N(A)$, $N(A) \neq \emptyset$

Let $\underset{\sim}{u}, \underset{\sim}{v} \in N(A)$,

$A(\underset{\sim}{u} + \underset{\sim}{v})$

$= A\underset{\sim}{u} + A\underset{\sim}{v}$

$= \underset{\sim}{0} + \underset{\sim}{0}$ $\because A\underset{\sim}{u} = \underset{\sim}{0}$, $A\underset{\sim}{v} = \underset{\sim}{0}$

$= \underset{\sim}{0}$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} \in N(A)$

Let $k \in \mathbb{R}$,

$A(k\underset{\sim}{u})$

$= kA\underset{\sim}{u}$

$= k(\underset{\sim}{0})$ $\because A\underset{\sim}{u} = \underset{\sim}{0}$

$= \underset{\sim}{0}$

$\therefore k\underset{\sim}{u} \in N(A)$

$\therefore N(A)$ is a subspace of $\mathbb{R}^n$.

9bi) Let $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$A\underset{\sim}{x} = \underset{\sim}{0}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -16 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 16x_3$$

$$x_2 = 19x_3$$

$$x_3 = x_3$$

$$\therefore \underset{\sim}{x} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

By inspection, $\{(16, 19, 1)\}$ is linearly independent.

$\therefore$ A basis is $\{(16, 19, 1)\}$ and the dimension is 1.

9bii) Let $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

$$A\underset{\sim}{x} = \underset{\sim}{0}$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & -7 & -7 & -4 & 0 \\ 0 & 7 & 7 & 4 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 7 & 7 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + x_3 = \frac{2}{7}x_4$$

$$x_1 = \frac{2}{7}x_4 - x_3$$

$$x_2 + x_3 = -\frac{4}{7}x_4$$

$$x_2 = -\frac{4}{7}x_4 - x_3$$

$$x_3 = x_3$$

$$x_4 = x_4$$

9bii) $x_1 = \frac{2}{7}x_4 - x_3$

$x_2 = -\frac{4}{7}x_4 - x_3$

$x_3 = x_3$

$x_4 = x_4$

$$\underset{\sim}{x} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ -4 \\ 0 \\ 7 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

By inspection, $\{(-1,-1,1,0), (2,-4,0,7)\}$ is linearly independent.

$\therefore$ A basis is $\{(-1,-1,1,0), (2,-4,0,7)\}$ and the dimension is 2.

9biii) Since A^{-1} exists,

$$A\underset{\sim}{x} = \underset{\sim}{0}$$

$$AA^{-1}\underset{\sim}{x} = \underset{\sim}{0}$$

$$\underset{\sim}{x} = \underset{\sim}{0}$$

By definition, $\{\underset{\sim}{0}\}$ is linearly independent.

$\therefore$ A basis is $\varnothing$ by definition and the dimension is 0.

10) $P_n = \{\text{Polynomials of degree} \leq n\}$

$= \text{span}\{1, x, x^2, \ldots, x^n\} \subseteq \mathcal{C}(\mathbb{R})$

$\dim P_n = n+1$

a) $f_j : \mathbb{R} \mapsto \mathbb{R}$, $f_j(x) = x^j$, $j \geq 0$

$\{f_j \mid j = 0, 1, \ldots, n\}$ is linearly independent.

$\dim \{f_j \mid j = 0, 1, \ldots, n\} = n+1$

$\{f_j \mid j = 0, 1, \ldots, n\} \in \mathcal{C}(\mathbb{R})$

$\therefore$ For any $n \geq 0$, we can find $n+1$ independent vectors in $\mathcal{C}(\mathbb{R})$.

10b) Suppose that $\mathcal{C}(\mathbb{R})$ is finite dimensional, that is, $\dim \mathcal{C}(\mathbb{R}) = m$.

Taking $S = \{1, x, x^2, \ldots, x^m\}$

S is linearly independent.

$\dim S = m+1 > m = \dim \mathcal{C}(\mathbb{R})$

$\therefore$ S is linearly dependent, which is a contradiction as S is linearly independent.

$\therefore$ $\mathcal{C}(\mathbb{R})$ must be infinite dimensional.