Math Module 2B Notes

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April 29, 2025

Contents

1 Definitions

1.1 Derivative

Given a function $f(x)$, it's **derivative** at a point a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the above limit exists, then we say the function f is **differentiable** at the point a. Also, when we say that the derivative $f'(a)$ **exists**, we mean that the limit above exists (it's a real number), which means f is differentiable at a.

By putting $x = a + h$, we can rewrite the expression as:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

1.1.1 Notation for the set of functions differentiable at $x = a$

The set of functions differentiable at $x = a$ is denoted by $D(a)$, so if we write $f \in D(a)$, it means that the limit $f'(a)$ exists.

1.1.2 Interpretation

In the sciences, derivative means **rate of change**.

Example: If $s(t)$ is the distance travelled (m) at time t (s), then $s'(t)$ is the velocity ($\mathrm{m\,s^{-1}}$) at time t and $s''(t)$ is the acceleration ($\mathrm{m\,s^{-2}}$) at time t.

Example: If $Q(t)$ is the amount of charge (C) that's passed through a cross-section of a wire at time t (s), then $Q'(t)$ is the current A ($\mathrm{C\,s^{-1}}$)

1.2 Powers of x

$$x^r = e^{r \ln x}$$

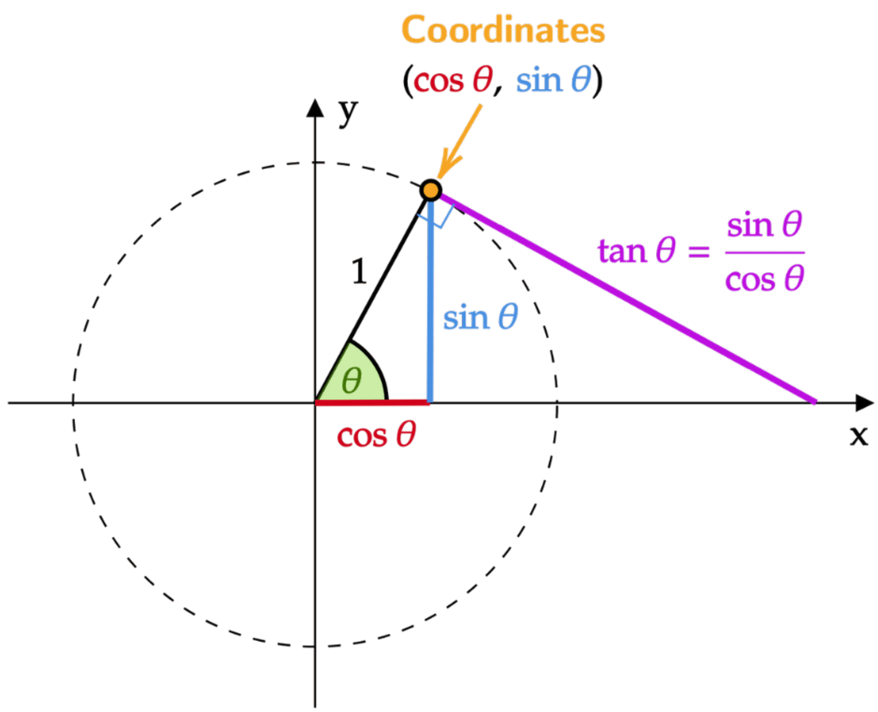
1.3 Unit circle

A unit circle is a circle with **radius 1** and is **centred** at the **origin**.

1.4 $\cos\theta$ and $\sin\theta$ for arbitrary $\theta \in \mathbb{R}$

Let $\theta in \mathbb{R}$ and consider the point (x, y) on the **unit circle** whose angle with the positive x-axis is θ (counterclockwise for $\theta > 0$, clockwise for $\theta < 0$). We define:

$$\cos\theta = x, \quad \sin\theta = y$$



1.5 Global maximum

Consider a function $f : A \to \mathbb{R}$. A point $x_0 \in A$ is called a **point of global maximum** if:

$$f(x) \leq f(x_0), \quad \text{for all } x \in A$$

1.6 Global minimum

Consider a function $f : A \to \mathbb{R}$. A point $x_0 \in A$ is called a **point of global minimum** if:

$$f(x) \geq f(x_0), \quad \text{for all } x \in A$$

1.7 Local maximum

Consider a function $f : A \to \mathbb{R}$. A point $x_0 \in A$ is called a **point of local maximum** if there exists some open interval $(a, b) \ni x_0$ such that:

$$f(x) \leq f(x_0), \quad \text{for all } x \in (a, b) \cap A$$

1.8 Local minimum

Consider a function $f : A \to \mathbb{R}$. A point $x_0 \in A$ is called a **point of local minimum** if there exists some open interval $(a, b) \ni x_0$ such that:

$$f(x) \geq f(x_0), \quad \text{for all } x \in (a, b) \cap A$$

1.9 Critical point

A **critical point** of a function $f : A \to \mathbb{R}$ is $c \in A$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

1.10 Extremum

Extremum just means the maximum or minimum value of a function.

2 Derivative as the slope of the tangent

Let $f(x)$ be a function that is differentiable at $x = a$. The line passing through the points $(a, f(a))$ and $(a+h, f(a+h))$ on f's graph, is given by:

$$y = \frac{f(a+h) - f(a)}{h}(x - a) + f(a)$$

As h tends to 0, this line approaches the **tangent line** at $(a, f(a))$.

$$\text{Slope of the line} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$

The limit of the line is $f'(a)$ as $h \to 0$. Hence, $f'(a)$ is the slope of the tangent.

The equation for the tangent line would be:

$$y = f'(a) \cdot (x - a) + f(a)$$

This equation only makes sense if $f'(a)$ is a finite real number, which means $f \in D(a)$.

For an interactive graph illustrating the derivative as the slope of the tangent, go to this link.

3 Differentiability implies continuity

If a function f is differentiable at a point a, then f is also continuous at the point a. That means that $D(a) \subset C(a)$.

Proof:

Suppose f is differentiable at a, which means:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{=} f'(a) \text{ exists}$$

Then:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot h + f(a) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a)
\end{aligned}$$

We just proved:

$$f \text{ differentiable at } x = a \Rightarrow f \text{ continuous at } x = a$$

3.1 Contrapositive form

This theorem is most often used in its contrapositive form:

If f is **not continuous** at $x = a$, then it is also **not differentiable** at $x = a$.

3.2 The reverse does not hold true

f being continuous at $x = a$ does **not** mean that f is differentiable at $x = a$.

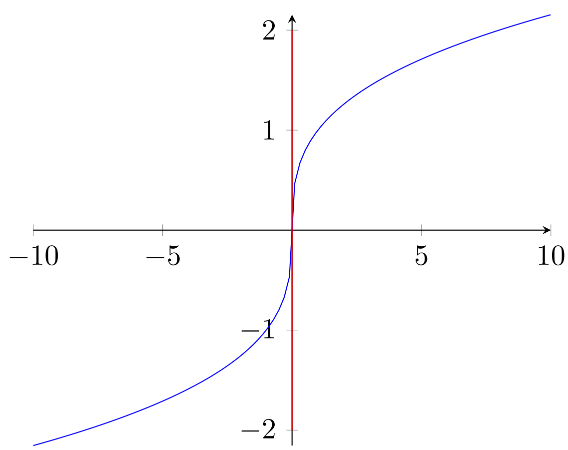
3.2.1 Example 1

Let $f(x) = \sqrt[3]{x}$:

$f(x)$ is continuous at $x = 0$ as it is an elementary function, but is it differentiable at $x = 0$?

$$\begin{aligned}\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h} \\ &= \lim_{h \to 0} h^{-\frac{2}{3}} \\ &= \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} \\ &= +\infty\end{aligned}$$

Hence, $\lim_{h \to 0} f(x)$ does not exist, and thus f is **not** differentiable at $x = 0$.



The graph of $f(x) = \sqrt[3]{x}$ has a vertical tangent at $(0, 0)$, which means it is **not** differentiable at $(0, 0)$.

3.2.2 Example 2

Let $f(x) = \sin|x|$:

$f(x)$ is continuous at $x = 0$ as it is an elementary function, but is it differentiable at $x = 0$?

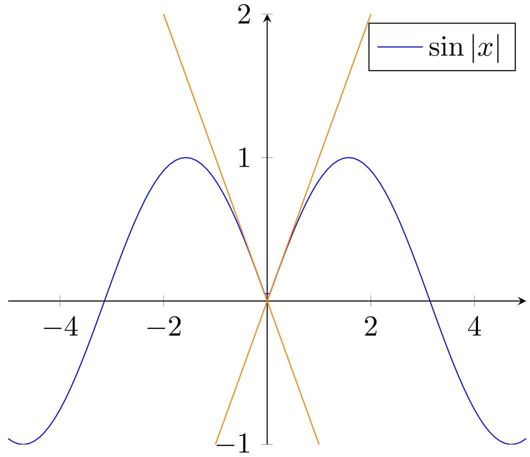
$$\begin{aligned}
\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0} \frac{\sin|h| - \sin|0|}{h} \\
&= \lim_{h \to 0} \frac{\sin|h|}{h}
\end{aligned}$$

Getting the left-hand limit:

$$\begin{aligned}
\lim_{h \to 0-} \frac{\sin|h|}{h} &= \lim_{h \to 0-} \frac{\sin(-h)}{h} \\
&= \frac{-\sin h}{h} \\
&= -1
\end{aligned}$$

Getting the right-hand limit:

$$\begin{aligned}
\lim_{h \to 0+} \frac{\sin|h|}{h} &= \lim_{h \to 0+} \frac{\sin(h)}{h} \\
&= \frac{\sin h}{h} \\
&= 1
\end{aligned}$$



Since $\lim_{h\to 0-}\frac{\sin|h|}{h} \neq \lim_{h\to 0+}\frac{\sin|h|}{h}$, $\lim_{h\to 0}\frac{\sin|h|}{h}$ does not exist and thus, f is not differentiable at 0.

4 Differentiation rules

Given two functions f and g, and given that the right-hand side makes sense, we have:

1. Sum rule, for any real constants c, d: $(cf + dg)'(x) = cf'(x) + dg'(x)$
2. Product rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
3. Ratio rule (Quotient rule): $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
4. Chain rule: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Note the condition that the right-hand side must make sense for the equation to hold, which means the **derivative** for f and g **must exist**. In particular, the theorem tells us that:

- If f and g are both differentiable at x, then so is $cf + dg$.
- If f and g are both differentiable at x, then so is fg.
- If f and g are both differentiable at x and $g(x) \neq 0$, then $\frac{f}{g}$ is differentiable at x.
- If g is differentiable at x and f is differentiable at $g(x)$, then $f(g(x))$ is differentiable at x.

If the right-hand side does not make sense, then the theorem gives us **no information**.

4.1 Example 1

If $f(x) = x \sin x$, what is $f'(x)$?

$$f'(x) = 1 \cdot \sin x + x \cdot \cos x$$

In particular, since both x and $\cos x$ are differentiable, the product rule tells us that $x \cdot \sin x$ is differentiable.

4.2 Example 2

If $f(x) = x \cdot |x|$, what is $f'(0)$?

4.2.1 Common error

Since $|x|$ has no derivative at $x = 0$, $f'(0)$ does not exist.

WRONG

4.2.2 Correct approach

Since $|x|$ has no derivative at $x = 0$, the product rule does not apply. We will have to figure this out by other methods, such as using the definition.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(h)}{h} \\ &= \lim_{h \to 0} \frac{h|h| - 0}{h} \\ &= \lim_{h \to 0} |h| \\ &= 0 \end{aligned}$$

5 Derivatives of some elementary functions

5.1 Notation

$$\frac{d}{dx}f(x) = f'(x)$$

5.2 Derivatives

1. $\frac{d}{dx}C = 0$
2. $\frac{d}{dx}x^{\alpha} = \alpha x^{\alpha-1}$
3. $\frac{d}{dx}e^x = e^x$
4. $\frac{d}{dx}\ln x = \frac{1}{x}$
5. $\frac{d}{dx}\sin x = \cos x$
6. $\frac{d}{dx}\cos x = -\sin x$
7. $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$
8. $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$
9. $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$

6 Derivatives and extreme points

If a is a local maximum or minimum point of a function f whose domain contains some interval $(a-\delta, a+\delta)$ for some $\delta > 0$, and if f is differentiable at a, then $f'(a) = 0$.

6.1 Finding extreme points

Using this theorem, points of local maxima or minima may only occur at points c where:

1. $f'(c) = 0$
2. $f'(c)$ does not exist
3. There is no $\delta > 0$ such that $(c-\delta, c+\delta)$ is contained in the domain of f

Points where case 1 or case 2 happens, are called **critical points**. If the domain is a closed bounded interval $[a, b]$, case 3 above occurs exactly at the endpoints a and b.

So, for a **continuous** function $f : [a, b] \to \mathbb{R}$, we know that:

- Global minimum and maximum points **exist** in $[a, b]$, by the max/min theorem
- These points must be points $c \in [a, b]$ where
  1. c is a critical point of f, so $f'(c) = 0$, or $f'(c)$ does not exist, **or**
  2. Any of the endpoints a or b

By comparing the function values at these points, we can deduce what the global maximum and minimum values and points are.

6.1.1 Example 1

Find, on the interval $[0, 5]$, the smallest and largest values of:

$$f(x) = \frac{x^3}{3} - x^2 - 3x$$

f is continuous on $[0, 5]$ which is closed bounded interval, so we know that global maximum and minimum points exist in $[0, 5]$, by the max/min theorem.

Let's look at the critical points:

$$f'(x) = x^2 - 2x - 3$$

$$\begin{aligned} f'(x) &= 0 \\ x^2 - 2x - 3 &= 0 \\ x = 1 \pm \sqrt{1+3} \\ x = 1 \pm 2 \end{aligned}$$

Since $x = -1 \notin [0, 5]$:

$$x = 3$$

How about the endpoints?

$$x = 0, x = 5$$

Calculate f at these points:

$$f(0) = 0$$

$$\begin{aligned} f(3) &= \frac{3^3}{3} - 3^2 - 3 \cdot 3 \\ &= -9 \ \textbf{(Smallest)} \end{aligned}$$

$$\begin{aligned} f(5) &= \frac{5^5}{5} - 5^2 - 5 \cdot 5 \\ &= \frac{5}{3} \ \textbf{(Biggest)} \end{aligned}$$

Since the global maximum and minimum must be found among $x = 0, x = 3, x = 5$, we can conclude that $f(3) = -9$ is the smallest and $f(5) = \frac{5}{3}$ is the largest value of f on interval $[0, 5]$.

6.1.2 Example 2

Find if possible, the largest and smallest values of $f(x) = \frac{1}{x}$ on the interval $(0, 1)$.

Note that $(0, 1)$ is not a closed interval, so the max/min theorem gives no information. However, for any $a \in (0, 1)$, we have:

$$\frac{a}{2} \in (0,1), \quad f\left(\frac{a}{2}\right) > f(a), \qquad \frac{1+a}{2} \in (0,1), \quad f\left(\frac{1+a}{2}\right) < f(a)$$

So, for every $a \in (0, 1)$, there exist points in $(0, 1)$ where f is bigger and points in $(0, 1)$ where f is smaller.

Hence, there is no biggest and no smallest value of f in $(0, 1)$.

6.1.3 Example 3

Find, on the interval $[-2, 2]$, the largest and smallest values of:

$$f(x) = \frac{3}{2}x^{\frac{2}{3}} - x$$

f is continuous on the closed bounded interval $[-2, 2]$, so there are global maximum and minimum points in $[-2, 2]$.

Critical points:

$$f'(x) = x^{-\frac{1}{3}} - 1 \text{ for } for x \neq 0$$

$$f'(0) \text{ does not exist}$$

$$\begin{aligned} f'(x) &= 0 \\ x^{-\frac{1}{3}} - 1 &= 0 \\ x^{-\frac{1}{3}} &= 1 \\ x &= 1 \in [-2, 2] \end{aligned}$$

The critical points are $x = 0$ and $x = 1$.

The endpoints are $x = -2$ and $x = 2$.

Comparing the values at the points:

$$f(-2) = \frac{3}{2} \cdot 4^{\frac{2}{3}} + 2 = 5.77976315 \textbf{ (Biggest)}$$

$$f(0) = \frac{3}{2} \cdot 0^{\frac{2}{3}} - 0 = 0 \textbf{ (Smallest)}$$

$$f(1) = \frac{3}{2} \cdot 1^{\frac{2}{3}} - 1 = \frac{1}{2}$$

$$f(2) = \frac{3}{2} \cdot 4^{\frac{2}{3}} - 2 = 1.77976315$$

Since the global maximum and minimum must be found among $x = -2, x = 0, x = 1, x = 2$, we can conclude that $f(0) = 0$ is the smallest and $f(-2) = 5.77976315$ is the largest value of f on interval $[-2, 2]$.

7 Zero derivative is not sufficient

By our observations for a **differentiable** $f : (a, b) \to \mathbb{R}, c \in (a, b)$:

$$f \text{ has a point of local extremum at } x = c \quad \Rightarrow f'(c) = 0$$

However, the reverse implication does **not** hold.

Example: For $f(x) = x^3$, f is differentiable and $f'(0) = 0$ but $x = 0$ is not a point of local extremum.