6) $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & k & 6 & 1 \\ 3 & 6 & k & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & k-4 & 4 & -1 \\ 0 & 0 & k-3 & 0 \end{bmatrix}$

i) When $k \neq 3, k \neq 4$, the system has a unique solution.

ii) When $k = 3$, the system has infinite solutions

iii) When $k = 4$, the system has no solutions

7) $2ax + ay + bz = 2 - a$

when $x = 1, y = 1, z = -1,$

$2a + a - b = 2 - a$

$3a - b = 2 - a$

$4a - b = 2 \quad - (1)$

when $x = 2, y = 1, z = -2$

$4a + a - 2b = 2 - a$

$6a - 2b = 2$

$3a - b = 1$

$b = 3a - 1 \quad - (2)$

Sub (2) into (1)

$4a - (3a - 1) = 2$

$a + 1 = 2$

$a = 1$

7) Sub $a=1$ into (2)

$$b = 3(1) - 1$$

$$b = 2$$

$\therefore a = 1$ and $b = 2$

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 2 & 4 & 2 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 2 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x + z = 0$ $\qquad$ $y = 1$

$x = -z$

$\therefore$ the general solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

8a)

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 5 & 1 & 0 & 0 \\ 4 & 1 & 6 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & -5 & 5 & 1 & 0 & -2 \\ 0 & -15 & 6 & 0 & 1 & -4 \\ 1 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & -15 & 6 & 0 & 1 & -4 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{4}{5} & 0 & \frac{-3}{5} \\ 0 & 1 & -1 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & -9 & -3 & 1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 4 & \frac{4}{5} & 0 & -\frac{3}{5} \\ 0 & 1 & -1 & -\frac{1}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{9} & -\frac{2}{9} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{8}{15} & \frac{4}{9} & \frac{13}{45} \\ 0 & 1 & 0 & \frac{2}{15} & \frac{1}{9} & \frac{8}{45} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{9} & -\frac{2}{9} \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -\frac{8}{15} & \frac{4}{9} & \frac{13}{45} \\ \frac{2}{15} & \frac{1}{9} & \frac{8}{45} \\ \frac{1}{3} & -\frac{1}{9} & -\frac{2}{9} \end{bmatrix}$$

8b) $\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 4 & -1 & 0 & 1 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 4 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -5 & 3 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -3 & 3 & -2 & -2 \end{array}\right] \sim$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & \frac{2}{3} & \frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -\frac{13}{3} & -\frac{7}{3} \\ 0 & 1 & 0 & -1 & \frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & -1 & \frac{2}{3} & \frac{2}{3} \end{array}\right]$

$$\therefore A^{-1} = \begin{bmatrix} 4 & -\frac{13}{3} & -\frac{7}{3} \\ -1 & \frac{5}{3} & \frac{2}{3} \\ -1 & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

a) $(A^2)^{-1}A^2 = I$

$(A^2)^{-1}AA = I$

$(A^2)^{-1}AAA^{-1} = IA^{-1}$

$(A^2)^{-1}A = A^{-1}$

$(A^2)^{-1}AA^{-1} = A^{-1}A^{-1}$

$(A^2)^{-1}I = A^{-1}A^{-1}$

$(A^2)^{-1} = (A^{-1})^2$ (shown)

10) $A^2 + A - 4I = 0$

$$\frac{1}{2}A^2 + \frac{1}{2}A - 2I = 0$$

$$\frac{1}{2}A^2 - \frac{1}{2}A + A - 2I + I = I$$

$$\frac{1}{2}A^2 - \frac{1}{2}A + A - I = I$$

$$\frac{1}{2}A(A - I) + A - I = I$$

$$\left(\frac{1}{2}A + 1\right)(A - I) = I$$

$$\left(\frac{1}{2}A + 1\right) = I(A - I)^{-1}$$

$$(A - I)^{-1} = \frac{1}{2}A + 1$$

(shown)