(a) $\underset{2\times 2}{B}\underset{3\times 2}{A}$ is not defined

b) $\underset{2\times 2}{B}\underset{2\times 3}{C} = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$

c) $D^T - E^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

d) $(D - E)^T = \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} \right)^T$

$= \left(\begin{bmatrix} -5 & 4 & -1 \\ 0 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \right)^T$

$= \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

1e) $DE = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}$

f) $\underset{3\times3}{(D}\underset{3\times2}{A)}^T = \left(\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \right)^T$

$= \left(\begin{bmatrix} 0 & 12 \\ -2 & 1 \\ 11 & 8 \end{bmatrix} \right)^T$

$= \begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$

2) For $AB + BA$ to be well defined,

AB and BA must be defined

$A_{m\times n} B_{r\times s}$: $n = r$

$B_{r\times s} A_{m\times n}$: $s = m$

$\therefore$ A and B must be square matrices of the same size.

3 a) $\operatorname{tr} A = 3 + 2 = 5$

$\operatorname{tr} B = 1 + 0 + 4 = 5$

$\operatorname{tr} E = 6 + 1 - 7 + 0 = 0$

b) $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$

c) $\operatorname{tr}(A+B) = \sum_{i=1}^{n} (a_{ii} + b_{ii})$

$= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$

$= \operatorname{tr}(A) + \operatorname{tr}(B)$ (Proven)

d) $\operatorname{tr}(\alpha A) = \sum_{i=1}^{n} \alpha a_{ii}$

$= \alpha \sum_{i=1}^{n} a_{ii}$

$= \alpha \operatorname{tr}(A)$

4) Proving that the size is the same:

LHS: $(B+C) \in M(r,n)$, $A(B+C) \in M(m,n)$

RHS: $AB \in M(m,n)$, $AC \in M(m,n)$, $AB + AC \in M(m,n)$

$\therefore$ Size of $A(B+C) = AB + AC$

Proving that the entries are the same:

$$[A(B+C)]_{ij} = \sum_{k=1}^{n} A_{ik}(B+C)_{kj}$$

$$= \sum_{k=1}^{n} A_{ik}B_{kj} + A_{ik}C_{kj}$$

$$= (AB + AC)_{ij}$$

5a) $$((A^T)^T)_{ij} = (A^T)_{ji}$$

$$= (A)_{ij}$$

5b) $\left((A \pm B)^T\right)_{ij} = (A \pm B)_{ji}$

$= (A)_{ji} \pm (B)_{ji}$

$= A^T \pm B^T$

c) $\left((\alpha A)^T\right)_{ij} = (\alpha A)_{ji}$

$= \alpha (A)_{ji}$

$= \alpha (A)^T$

5d) Proving that the sizes are the same:

Let $A \in M(m,s)$, $B \in M(s,n)$

$\Rightarrow AB \in M(m,n)$, $(AB)^T \in M(n,m)$

$\Rightarrow B^T \in M(n,s)$, $A^T \in (s,m)$, $B^T A^T \in M(n,m)$

Since $B^T A^T \in M(n,m)$ and $(AB)^T \in M(n,m)$,

the size of $B^T A^T$ and $(AB)^T$ are the same.

Proving that the entries are the same:

$$\left((AB)^T\right)_{ij} = (AB)_{ji}$$

$$= \sum_{k=1}^{n} A_{jk} B_{ki}$$

$$= \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik}$$

$$= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}$$

6a) $A^T \in M(n,m)$

$A^T A \in M(n,n)$

$\therefore$ True, $A^T A$ is indeed a square matrix.

b) $$(A^T A)_{ij} = \sum_{k=1}^{n} (A^T)_{ik} A_{kj}$$

$$= \sum_{k=1}^{n} A_{ki} A_{kj}$$

$$= \sum_{k=1}^{n} (A^T)_{jk} A_{ki}$$

$$= (A^T A)_{ji}$$

$\therefore$ The statement that $A^T A$ is symmetric is true.

7a) $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$

when $i > j$, $(A)_{ij} = 0$, $(B)_{ij} = 0$,

$$(A+B)_{ij} = 0 + 0$$
$$= 0$$

$\therefore$ The statement that $A+B$ is upper triangular is true.

b) $$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$
$$= \sum_{k=1}^{i-1} A_{ik} B_{kj} + \sum_{k=i}^{n} A_{ik} B_{kj}$$

when $i > j$,

$k \leq i-1 \Rightarrow a_{ik} = 0$

$k \geq i \Rightarrow b_{kj} = 0$

$(AB)_{ij} = 0 + 0 = 0$

$\therefore$ The statement AB is an upper triangular matrix is true.

8) (a) is a linear equation.

(b) is not a linear equation.

(c) is not a linear equation.

9a) $7x - 5y = 3$

$7x = 3 + 5y$

$x = \dfrac{3+5y}{7}$

$y = t,\ x = \dfrac{3+5t}{7},\ t \in \mathbb{R}$

b) $-8u + 2v - 5w + 6x = 1$

$8u - 2v + 5w - 6x = -1$

$u = \frac{1}{8}(2v - 5w + 6x - 1)$

$v = \alpha,\ w = \beta,\ x = \gamma,\ u = \frac{1}{8}(2\alpha - 5\beta + 6\gamma - 1),$

$\alpha, \beta, \gamma \in \mathbb{R}$

10) $w = 6s - 3t - 2,$

$v = s,$

$x = 7 - 4t,$

$y = 8 - 5t,$

$z = t,$

$s, t \in \mathbb{R}$

11) $\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$$x + y + 2z = 8$$
$$y - 5z = -9$$
$$z = 2$$

$\therefore z = 2, y = 1, x = 3$

12) $\begin{bmatrix} 2 & 1 & -2 & -2 & -2 \\ 1 & -1 & 2 & -1 & -1 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$x = t - 1,$

$y = 2s,$

$z = s,$

$u = t,$

$s, t \in \mathbb{R}$

13 i) $ax_1^2 + bx_1 + c = y_1$

$ax_2^2 + bx_2 + c = y_2$

$ax_3^2 + bx_3 + c = y_3$

$$\begin{bmatrix} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{bmatrix}$$ (shown)

ii) $$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 11 \\ 4 & -2 & 1 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -13 \\ 0 & -6 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & -6 & -3 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 6 & 42 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$\therefore a = 3, b = -4, c = 7$

$\therefore y = 3x^2 - 4x + 7$

(14) $$\begin{bmatrix} 1 & 1 & 2 & a \\ 1 & 0 & 1 & b \\ 2 & 1 & 3 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & -1 & -1 & b-a \\ 0 & -1 & -1 & c-2a \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & -1 & -1 & c-2a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-2a+a-b \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 & a \\ 0 & 1 & 1 & a-b \\ 0 & 0 & 0 & c-a-b \end{bmatrix}$$

$\therefore\ c-a-b=0 \Longrightarrow c=a+b$

15a) $$\begin{bmatrix} 2 & 1 & 3 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\therefore\ x_1=0,\ x_2=0,\ x_3=0$

15b)

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & -8 & 0 \\ 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 31 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$w = t,$

$x = -t,$

$y = t,$

$z = 0,$

$t \in \mathbb{R}$

16) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & (a+2) \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-2) & (a-14) \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & -7 & (a^2-2) & (a-14) \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & (a^2-16) & (a-4) \end{bmatrix}$

The system has no solutions when $a=-4$.

The system has infinitely many solutions when $a=4$.

The system has one solution when $a \neq \pm 4$.

17) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & -3 & 3 & 1 & -2 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \sim$

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 6 & -2 & -2 \end{array}\right] \sim$

$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \sim$

$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$

a) $x_1 = \frac{4}{3}$, $x_2 = -\frac{2}{3}$, $x_3 = -\frac{1}{3}$

b) $x_1 = \frac{1}{3}$, $x_2 = \frac{1}{3}$, $x_3 = -\frac{1}{3}$