Linear systems

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = \sum_{m=0}^{M} B_m \frac{d^m X_{in}}{dt^m}$$

Characteristic equation:

$$\sum_{n=0}^{N} A_n s^n = 0$$

1) Answer is (d).

2) 1^{st} order system:

$$\tau \frac{dX_{out}}{dt} + X_{out} = K X_{in}$$

Characteristic equation:

$$\tau s + 1 = 0$$

$$s = -\frac{1}{\tau}$$

∴ The homogeneous solution is

$$X_{out_h} = Ce^{-\frac{t}{\tau}}$$

A particular solution to the equation is:

$$X_{out_p} = AX_{in}$$

$$X_{out_p}' = 0$$

$$X_{in} = A_{in}$$

$$\tau \frac{dX_{out_p}}{dt} + X_{out_p} = KA_{in}$$

$$\tau(0) + AA_{in} = KA_{in}$$

$$A = K$$

$$\therefore X_{out_p} = KA_{in}$$

2) A general solution to the equation is:

$$X_{out} = X_{out_h} + X_{out_p}$$
$$= Ce^{-\frac{t}{\tau}} + KA_{in}$$

When $X_{out} = 0$, $X_{in} = 0$

$$0 = Ce^{-\frac{0}{\tau}} + KA_{in}$$
$$0 = C + KA_{in}$$
$$C = -KA_{in}$$
$$X_{out} = -KA_{in}e^{-\frac{t}{\tau}} + KA_{in}$$
$$= KA_{in}(1 - e^{-\frac{t}{\tau}})$$

From the question, $K = 1$

$$\therefore X_{out} = A_{in}(1 - e^{-\frac{t}{\tau}})$$

When $t \to \infty$, $X_{out} = 100$,

$$100 = A_{in} \lim_{t \to \infty}(1 - e^{-\frac{t}{\tau}})$$
$$A_{in} = 100$$
$$\therefore X_{out} = 100(1 - e^{-\frac{t}{\tau}})$$

When $t = 1.2$, $X_{out} = 80$,

$$80 = 100(1 - e^{-\frac{1.2}{\tau}})$$
$$\frac{80}{100} = 1 - e^{-\frac{1.2}{\tau}}$$
$$\frac{1}{5} = e^{-\frac{1.2}{\tau}}$$
$$\ln\left(\frac{1}{5}\right) = -\frac{1.2}{\tau}$$
$$\tau = \frac{-1.2}{\ln\left(\frac{1}{5}\right)}$$
$$= 0.7456019215$$
$$\approx 0.75$$
$$\therefore X_{out} = 100(1 - e^{-\frac{t}{0.75}})$$

When $t = 1.5$,

$$\text{Error} = 100 - 100(1 - e^{-\frac{1.5}{0.75}})$$
$$= 13.3748061$$
$$\approx 13.4 \text{ units}$$

3) Dynamic error ≤ 0.02

$$\delta(w) \leq 0.02$$

$$1 - M(w) \leq 0.02$$

$$0.98 \leq M(w) \leq 1$$

$$0.98 \leq \frac{1}{\sqrt{1+(w\tau)^2}} \leq 1$$

$$\frac{50}{49} \geq \sqrt{1+(w\tau)^2} \geq 1$$

$$1 \leq 1+(w\tau)^2 \leq \frac{2500}{2401}$$

$$0 \leq w^2\tau^2 \leq \frac{99}{2401}$$

$$0 \leq w^2 \leq \frac{99}{2401\tau^2}$$

$$0 \leq w \leq \sqrt{\frac{99}{2401(2)^2}}$$

$$0 \leq w \leq 0.1015293303$$

$$0 \leq w \leq 0.1 \text{ rads}^{-1}$$

$\therefore w_{max} = 0.1 \text{ rads}^{-1}$

4) $30 \leq \text{Amplitude} \leq 40$

$f = 10Hz$

$$\begin{aligned} \omega &= 2\pi f \\ &= 2\pi(10) \\ &= 20\pi \end{aligned}$$

The output signal could be:

$35 + 5\sin(20\pi t \pm \phi)$

$$\delta(\omega) < 0.01$$

$$1 - \frac{1}{\sqrt{1+(\omega\tau)^2}} < 0.01$$

$$-\frac{1}{\sqrt{1+(\omega\tau)^2}} < -0.99$$

$$1 \geq \frac{1}{\sqrt{1+\omega^2\tau^2}} > 0.99$$

$$1 \leq \sqrt{1+\omega^2\tau^2} < \frac{100}{99}$$

$$1 \leq 1+\omega^2\tau^2 < \frac{10000}{9801}$$

$$0 \leq \omega^2\tau^2 < \frac{199}{9801}$$

$$0 \leq \tau < \sqrt{\frac{199}{9801\omega^2}}$$

$$0 \leq \tau < \sqrt{\frac{199}{9801(20\pi)^2}}$$

$$\tau < 0.00226783511$$

$$\tau < 2.27\,ms$$