1) $p(x,y,z) = 3x^2 + x^2 z^2 e^{-2y}$

$\nabla p = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$

$\frac{\partial p}{\partial x} = 6x + 2z^2 x e^{-2y}$

$\frac{\partial p}{\partial y} = -2x^2 z^2 e^{-2y}$

$\frac{\partial p}{\partial z} = 2zxe^{-2y}$

$\nabla p \Big|_{(x,y,z)=(1,0,2)} = (6+2(2)^2, -8, 4)$

$= (14, -8, 4)$

a) In the direction $(1,-2,2)$:

Magnitude $= \sqrt{1^2+(-2)^2+2^2}$

$= 3$

Unit vector $= \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$

Projection of $(14,-8,4)$ on $(1,-2,2)$:

$(14,-8,4)\cdot\left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \frac{38}{3}$

b) $(14,-8,4)\cdot(a,b,c)=0$

$$14a-8b+4c=0$$

The vectors are the vectors $a\underset{\sim}{i}+b\underset{\sim}{j}+c\underset{\sim}{k}$ such that $14a-8b+4c=0$.

c) Let θ be the angle between $(14,-8,4)$ and the direction vector $\vec{v}$.

$$(14,-8,4)\cdot\vec{v}$$
$$=\frac{38}{3}|\vec{v}||\cos\theta|$$

To maximise $(14,-8,4)\cdot\vec{v}$,

$$\cos\theta=\pm1$$
$$\theta=0^\circ,180^\circ$$

$\therefore \vec{v}=\pm(14,-8,4)$

Greatest rate $=|\pm(14,-8,4)|$

$$=\sqrt{14^2+8^2+4^2}$$
$$=2\sqrt{69}$$

2) Let S be the surface given by:

$$z = 1 + 2x^2 + 3y^2$$

$$z - 2x^2 - 3y^2 = 1$$

A normal vector $\vec{n}$ to the surface is given by:

$$\frac{\partial S}{\partial x} = -4x$$

$$\frac{\partial S}{\partial y} = -6y$$

$$\frac{\partial S}{\partial z} = 1$$

$$\therefore (-4x, -6y, 1)$$

At $(1,2,15)$,

$$\vec{n} = (1,2,15) + t(-4,-12,1)$$

$$x = 1 - 4t$$

$$y = 2 - 12t$$

$$z = 15 + t$$

3) Let $\underset{\sim}{G} = (a, b, c)$, $\underset{\sim}{F} = (d, e, f)$

$$\operatorname{div}(\underset{\sim}{F} \times \underset{\sim}{G})$$

$$= \nabla \cdot (\underset{\sim}{F} \times \underset{\sim}{G})$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[(a, b, c) \times (d, e, f)\right]$$

$$\begin{array}{ccccc} i & j & k & i & j \\ a & b & c & a & b \\ d & e & f & d & e \end{array}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left[bf\underset{\sim}{i} + cd\underset{\sim}{j} + ae\underset{\sim}{k} - db\underset{\sim}{k} - ec\underset{\sim}{i} - fa\underset{\sim}{j}\right]$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (bf - ec, cd - fa, ae - db)$$

$$= -\left(b\frac{\partial f}{\partial x} + f\frac{\partial b}{\partial x}\right) + e\frac{\partial c}{\partial x} + c\frac{\partial e}{\partial x} - \left(c\frac{\partial d}{\partial y} + d\frac{\partial c}{\partial y}\right) +$$

$$f\frac{\partial a}{\partial y} + a\frac{\partial f}{\partial y} - \left(a\frac{\partial e}{\partial z} + e\frac{\partial a}{\partial z}\right) + d\frac{\partial b}{\partial z} + b\frac{\partial d}{\partial z}$$

$$= b\left(\frac{\partial d}{\partial z} - \frac{\partial f}{\partial x}\right) + f\left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right) + c\left(\frac{\partial e}{\partial x} - \frac{\partial d}{\partial y}\right) +$$

$$e\left(\frac{\partial c}{\partial x} - \frac{\partial a}{\partial z}\right) + d\left(\frac{\partial b}{\partial z} - \frac{\partial c}{\partial y}\right) + a\left(\frac{\partial f}{\partial y} - \frac{\partial e}{\partial z}\right)$$

3) $b\left(\frac{\partial d}{\partial z}-\frac{\partial f}{\partial x}\right)+f\left(\frac{\partial a}{\partial y}-\frac{\partial b}{\partial x}\right)+c\left(\frac{\partial e}{\partial x}-\frac{\partial d}{\partial y}\right)$

$+e\left(\frac{\partial c}{\partial x}-\frac{\partial a}{\partial z}\right)+d\left(\frac{\partial b}{\partial z}-\frac{\partial c}{\partial y}\right)+a\left(\frac{\partial f}{\partial y}-\frac{\partial e}{\partial z}\right)$

$=(a,b,c)\cdot\left(\frac{\partial f}{\partial y}-\frac{\partial e}{\partial z},\ \frac{\partial d}{\partial z}-\frac{\partial f}{\partial x},\ \frac{\partial e}{\partial x}-\frac{\partial d}{\partial y}\right)+$

$(d,e,f)\cdot\left(\frac{\partial b}{\partial z}-\frac{\partial c}{\partial y},\ \frac{\partial c}{\partial x}-\frac{\partial a}{\partial z},\ \frac{\partial a}{\partial y}-\frac{\partial b}{\partial x}\right)$

$\text{curl } F=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(d,e,f)$

$$\begin{array}{ccccc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ d & e & f & d & e \end{array}$$

$=\left(\frac{\partial f}{\partial y}-\frac{\partial e}{\partial z},\ \frac{\partial d}{\partial z}-\frac{\partial f}{\partial x},\ \frac{\partial e}{\partial x}-\frac{\partial d}{\partial y}\right)$

$\text{curl } G=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times(a,b,c)$

$$\begin{array}{ccccc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ a & b & c & a & b \end{array}$$

$=\left(\frac{\partial c}{\partial y}-\frac{\partial b}{\partial z},\ \frac{\partial a}{\partial z}-\frac{\partial c}{\partial x},\ \frac{\partial b}{\partial x}-\frac{\partial a}{\partial y}\right)$

3) $\therefore \operatorname{div}(\underset{\sim}{F} \times \underset{\sim}{G})$

$= (a, b, c) \cdot \operatorname{curl} \underset{\sim}{F} +$

$(d, e, f) \cdot -\operatorname{curl} \underset{\sim}{G}$

$= \underset{\sim}{G} \cdot \operatorname{curl} \underset{\sim}{F} - \underset{\sim}{F} \cdot \operatorname{curl} \underset{\sim}{G}$

4) $q = \nabla \phi$

$$\phi = f(x, y, z) = (a, b, c)$$

$$\nabla \times q = \nabla \times (\nabla \phi)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

$$\begin{array}{ccccc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} & \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{array}$$

$$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial y \partial z}, \frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial x \partial y}\right)$$

$$= (0, 0, 0)$$

5) $\nabla^2 \phi = 0$

$$\nabla \cdot \nabla \phi = 0$$

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

$$\frac{\partial \phi}{\partial x} = 2xp\left(x^2+y^2+z^2\right)^{p-1}$$

$$\frac{\partial \phi}{\partial y} = 2yp\left(x^2+y^2+z^2\right)^{p-1}$$

$$\frac{\partial \phi}{\partial z} = 2zp\left(x^2+y^2+z^2\right)^{p-1}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4x^2 p(p-1)\left(x^2+y^2+z^2\right)^{p-2} + 2p\left(x^2+y^2+z^2\right)^{p-1}$$

$$\frac{\partial^2 \phi}{\partial y^2} = 4y^2 p(p-1)\left(x^2+y^2+z^2\right)^{p-2} + 2p\left(x^2+y^2+z^2\right)^{p-1}$$

$$\frac{\partial^2 \phi}{\partial z^2} = 4z^2 p(p-1)\left(x^2+y^2+z^2\right)^{p-2} + 2p\left(x^2+y^2+z^2\right)^{p-1}$$

5) $\nabla^2 \phi = 0$

$$4p(p-1)(x^2+y^2+z^2)^{p-1} + 6p(x^2+y^2+z^2)^{p-1} = 0$$

$$(4p(p-1)+6p)(x^2+y^2+z^2)^{p-1} = 0$$

$$4p^2 - 4p + 6p = 0$$

$$4p^2 + 2p = 0$$

$$p(2p+1) = 0$$

$$\therefore p = 0 \quad \text{or} \quad p = -\frac{1}{2}.$$