Math Module 1B Lecture Notes

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Contents

1 Limits

1.1 The idea of limit

Let $f(x) = \frac{\sin x}{x}$. What happens to $f(x)$ of values of x near 0?

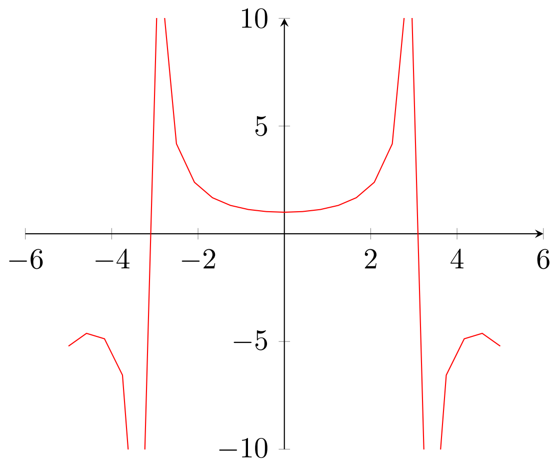
We can observe that:

x	f(x)	x	f(x)
0.1	0.998	-0.1	-0.998
0.01	0.99998	-0.01	0.99998
0.001	0.9999998	-0.001	0.9999998
0.0001	0.999999998	-0.0001	0.999999998

It **looks like** the function values of $f(x)$ get closer and closer to 1 as x approaches 0.

1.2 Plotting approach

We can plot the graph:



1.3 Limits, in rigorous terms

From a maths perspective, "**looks like**" isn't good enough.

But before we can do anything better, we have to **exactly define** what we're talking about in the first place.

Roughly speaking, if $f(x)$ can be made arbitrarily close to some number L by taking the value of the variable x sufficiently close to a, but not equal to a, then we'd like to say that L is f's limit at the point a.

In simpler terms, if $f(x)$ approaches a value L but isn't equal to L, that means that L is f's limit.

1.4 Distance between points

1.4.1 Definition

The distance between two points $x, y \in \mathbb{R}$ is $|x - y|$.

1.4.2 Examples

The distance between 5 and 7 is $7 - 5 = 2$.

The distance between 7 and 5 is also $7 - 5 = 2$.

1.4.3 Thinking in distances

For expressions such as the one below:

$$|x - a| < \delta,\ (\delta > 0)$$

It is usually helpful to think of these expressions in terms of distances, i.e. "the point x is within distance δ from a". Note that for $\delta > 0$:

$$|x - a| < \delta \Leftrightarrow x \in (a - \delta, a + \delta) \tag{1}$$

1.5 Limit points

1.5.1 Definition

Let A be a subset of $\mathbb{R}$. We say that a point $a \in \mathbb{R}$ is a **limit point** of A, if for every $\delta > 0$, there exists a point $x \in A$ such that $0 < |x - a| < \delta$.

Basically, this means that a is a limit point of A if there is a point x such that $0 < |x - a| < \delta$, or that the absolute value of $x - a$ is between 0 and δ.

1.5.2 Example 1

Consider $A = (0, 1] \cup 2$.

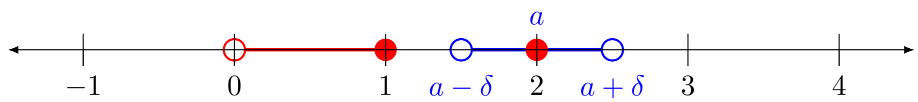
Is 3 a limit point of A?

From the definition of the limit point of A, $a \in \mathbb{R}$ is a limit point of a if for every $\delta > 0$, there exists a $x \in A$ such that $0 < |x - a| < \delta$.

Let $\delta = \frac{1}{2}$. At $a = 3$, there does not exist $x \in A$ such that $0 < |x-a| < \delta$. Hence, $a = 3$ is **not** a limit point of A.

1.5.3 Example 2

Consider $A = (0, 1] \cup 2$.



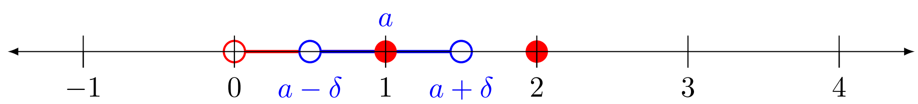
Is 2 a limit point of A?

From the definition of the limit point of A, $a \in \mathbb{R}$ is a limit point of a if for every $\delta > 0$, there exists a $x \in A$ such that $0 < |x - a| < \delta$.

Let $\delta = \frac{1}{2}$. At $a = 2$, there does not exist $x \in A$ such that $0 < |x-a| < \delta$. Hence, $a = 2$ is **not** a limit point of A.

1.5.4 Example 3

Consider $A = (0, 1] \cup 2$.



Is 1 a limit point of A?

From the definition of the limit point of A, $a \in \mathbb{R}$ is a limit point of a if for every $\delta > 0$, there exists a $x \in A$ such that $0 < |x - a| < \delta$.

Let $\delta > 0$. At $a = 1$, there exists a $x \in A$ such that $0 < |x - a| < \delta$. Hence, $a = 1$ **is** a limit point of A.

$\frac{1}{2}$ and 0 is also a limit point of A.

1.6 Definition of a limit

For a function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$ with a as a limit point of A, f approaches a **limit** L if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\lim_{x \to a} f(x) = L$$

$$\Updownarrow$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta, \quad x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

For a visual example, visit this link.

This playlist should help you understand the definition of a limit if the definition above still doesn't make any sense to you.

1.6.1 Example 1

Prove that $\lim_{x \to 2} 17x = 34$.

We want to show that for every $\varepsilon > 0$, there exists $\delta > 0$, such that:

$$0 < |x - 2| < \delta, \quad x \in \mathbb{R} \quad \Rightarrow \quad |17x - 34| < \varepsilon$$

Hence:

$$|17x - 34| = 17|x - 2|$$

$$17|x - 2| < \varepsilon$$

$$|x - 2| < \frac{\varepsilon}{17}$$

Thus, we will need $\delta = \frac{\varepsilon}{17}$.

Let $\varepsilon > 0$. If we choose $\delta = \frac{\varepsilon}{17}$, we have:

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |17x - 34| = 17|x - 2| < 17\delta = \varepsilon$$

So, by definition,

$$\lim_{x \to 2} 17x = 34 \textbf{ (Proven)}$$

1.6.2 Example 2

Prove that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

We want to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - 0| < \delta, \quad x \in \mathbb{R} \quad \Rightarrow \quad \left| x\sin\frac{1}{x} - 0 \right| < \varepsilon$$

Hence:

$$|x\sin\frac{1}{x}| = |x| \cdot |\sin\frac{1}{x}|$$

$$\begin{aligned} |x| \cdot |\sin\frac{1}{x}| &\leq |x| \\ &< \delta = \varepsilon \end{aligned}$$

Thus, we will need $\delta = \varepsilon$.

Let $\varepsilon > 0$. If we choose $\delta = \varepsilon$, we have:

$$\begin{aligned} 0 < |x - 0| < \delta \Rightarrow & \left| x\sin\frac{1}{x} - 0 \right| \\ & \Rightarrow |x| \cdot \left| \sin\frac{1}{2} \right| \leq |x| < \delta = \varepsilon \end{aligned}$$

So, by definition,

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0 \ \textbf{(Proven)}$$

1.6.3 Example 3

Prove that $\lim_{x\to 2} x^2 = 4$.

We want to show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x-2| < \delta, \quad x \in \mathbb{R} \quad \Rightarrow \quad |x^2 - 4| < \varepsilon$$

Hence:

$$|x^2 - 4| = |x+2| \cdot |x-2| < 5|x-2|$$
$$5|x-2| < \varepsilon$$
$$|x-2| < \frac{\varepsilon}{5}$$

If $|x-2| < 1$, then $x \in (1,3)$, so $|x+2| < 5$.

Thus, we will need $\delta = min\{1, \frac{\varepsilon}{5}\} > 0$.
Let $\varepsilon > 0$. If we choose $\delta = min\{1, \frac{\varepsilon}{5}\}$, we have:

$$0 < |x-2| < \delta \quad \Rightarrow \quad |x-2| < \frac{\varepsilon}{5}$$

And

$$\begin{aligned} |x-2| < 1 \quad \Rightarrow \quad & |x^2 - 4| = |x+2| \cdot |x-2| < 5|x-2| \\ & \qquad \because\ |x-2| < 1 \ \text{ so } \ x \in (1,3) \\ & \qquad \text{and } |x+2| < 5 \\ \Rightarrow \quad & |x^2 - 4| < 5|x-2| < \varepsilon \quad \because\ |x-2| < \frac{\varepsilon}{5} \end{aligned}$$

Note that the $\because$ stands for because.

So, by definition:

$$\lim_{x\to 2} x^2 = 4 \ \textbf{(Proven)}$$

1.6.4 Limits are independent of function values

It is very important to realise that, given a function $f(x)$, the value $f(a)$ does not affect the limit $\lim_{x\to a} f(x)$. To actually find this limit, we don't need to consider $f(a)$ and we don't care about whether $f(a)$ is defined.

If there **exists** a number $L \in \mathbb{R}$ such that $\lim_{x\to a} f(x) = L$, we say that **"f(x) has a limit as x approaches a"** or that **"the limit $\lim_{x\to a} f(x)$ exists"**.

On the contrary, if no such $L \in \mathbb{R}$ exists, we say that **"$f(x)$ has no limit as x approaches a"** or that **"the limit $\lim_{x\to a} f(x)$ does not exist"**.

Note that this has **nothing to do** with whether or not $L = 0$. A zero limit is still a limit.

1.7 Limit Laws

1.7.1 Theorem

Consider $f : A_1 \to \mathbb{R}, g : A_2 \to \mathbb{R}$. Suppose a is a limit point of $A_1 \cap A_2$, and $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$, then:

1.
$$\begin{aligned}\lim_{x\to a}(Af(x) + Bg(x)) &= Al + Bm \\ &= A \cdot \lim_{x\to a} f(x) + B \cdot \lim_{x\to a} g(x)\end{aligned}$$

2.
$$\begin{aligned}\lim_{x\to a}(f(x)g(x)) &= lm \\ &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)\end{aligned}$$

3.
$$\begin{aligned}\lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{l}{m}, \text{ provided } m \neq 0 \\ &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\end{aligned}$$

4.
$$\begin{aligned}\lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{l}, \text{ provided } n \in \mathbb{N} \text{ and } l \geq 0 \text{ if } n \text{ is even} \\ &= \sqrt[n]{\lim_{x\to a} f(x)}\end{aligned}$$

5. L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \text{ when } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ and } g'(x) \neq 0$$

1.7.2 Proof of $\lim_{x\to a}(Af(x)+Bg(x)) = Al + Bm$

Suppose that:

$$\lim_{x\to a} f(x) = l,\ \lim_{x\to a} = m.$$

Let $\varepsilon > 0$. By our assumptions, there exists a $\delta_1, \delta_2 > 0$ such that:

$$0 < |x-a| < \delta_1,\ x \in A_1, \quad \Rightarrow \quad |f(x)-l| < \frac{\varepsilon}{2(|A|+1)},$$

$$0 < |x-a| < \delta_2,\ x \in A_2, \quad \Rightarrow \quad |g(x)-m| < \frac{\varepsilon}{2(|B|+1)}$$

Let $\delta = min\{\delta_1, \delta_2\}$. Then:

$$0 < |x-a| < \delta,\ x \in A_1 \cap A_2$$

$$\Downarrow$$

$$0 < |x-a| < \delta_1\ x \in A_1$$
$$0 < |x-a| < \delta_2\ x \in A_2$$

$$\Downarrow$$

$$\begin{aligned} |Af(x)+Bg(x)-(Al+Bm)| &\leq |A||f(x)-l| + |B||g(x)-m| \\ &< \frac{\varepsilon}{2}\frac{|A|}{|A|+1} + \frac{\varepsilon}{2}\frac{|B|}{2|B|+1} \\ &< \varepsilon \end{aligned}$$

The proofs for the other laws are also quite similar.

1.7.3 Deriving the limits of polynomials

Proving $\lim_{x \to a} = a$:

For $\varepsilon > 0$, let $\delta = \varepsilon$. Then:

$$0 < |x - a| < \delta \Rightarrow |x - a| < \delta = \varepsilon,$$

So, $\lim_{x \to a} x = a$.

Proving $\lim_{x \to a} 1 = 1$:

For $\varepsilon > 0$, let $\delta = 1$. Then:

$$0 < |x - a| < \delta \Rightarrow |1 - 1| < 0 < \varepsilon,$$

So, $\lim_{x \to a} 1 = 1$.

Then, using $\lim_{x \to a}(Af(x) + Bg(x)) = Al + Bm$ that we just proved, we can conclude that:

$$\begin{aligned} \lim_{x \to a}(c_1 x + c_0) &= \lim_{x \to a}(c_1 x + c_0 \cdot 1 \\ &= c_1 a + c_0 \end{aligned}$$

Also, using $\lim_{x \to a} f(x)g(x) = lm$, we can conclude that:

$$\begin{aligned} \lim_{x \to a} x^2 &= \lim_{x \to a} x \cdot x \\ &= a \cdot a \\ &= a^2 \end{aligned}$$

Using $\lim_{x \to a}(Af(x) + Bg(x)) = Al + Bm$ again, we can get:

$$\lim_{x \to a}(c_2 x^2 + c_1 x + c_0) = c_2 a^2 + c_1 a + c_0$$

Repeating this argument, we get that for any **polynomial** $p(x)$:

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_2 x^2 + c_1 x + c_0,$$

We have:

$$\begin{aligned}\lim_{x\to a} p(x) &= c_n a^n + c_{n-1}a^{n-1} + \ldots + c_2 a^2 + c_1 a + c_0)\\ &= p(a)\end{aligned}$$

This property also holds for some other functions as well, as long as a is in the domain of the function. A few examples are:

1. $\lim_{x\to a} \sin x = \sin a$
2. $\lim_{x\to a} \cos x = \cos a$
3. $\lim_{x\to a} e^x = e^a$
4. $\lim_{x\to a} \ln x = \ln a$

A function with this property is said to be **continuous**.

1.7.4 Reformulated limit laws

In each of the laws below, the equation only applies **if the limit on the right-hand side exists and the expression makes sense**. Otherwise, you cannot directly apply the laws below.

1. $\lim_{x\to a}(Af(x) + Bg(x)) = A\lim_{x\to a} f(x) + B\lim_{x\to a} g(x)$
2. $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
3. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
4. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

1.7.5 Example 1

Evaluate $\lim_{x\to 2} \frac{x^2-x-2}{x-2}$:

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-x-2}{x-2} &\neq \frac{2^2-2-2}{2-2} \\ &\neq \frac{0}{0}\end{aligned}$$

We cannot apply the limit laws directly, so we divide x^2-x-2 by $x-2$ using long division, we get:

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-x-2}{x-2} &= \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} \\ &= \lim_{x\to 2}(x+1) \\ &= 2+1 \\ &= 3\end{aligned}$$

1.7.6 Example 2

Evaluate $\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1}$:

$$\begin{aligned}\lim_{x\to a} \frac{\sqrt{2-x}-1}{x-1} &\neq \frac{\sqrt{2-1}-1}{1-1} \\ &\neq 0\end{aligned}$$

Once again, we cannot apply the limit laws directly, so we multiply $\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1}$ by $\frac{\sqrt{2-x}+1}{\sqrt{2-x}+1}$

$$\begin{aligned}\lim_{x\to 1} \frac{\sqrt{2-x}-1}{x-1} &= \lim_{x\to 1} \frac{(\sqrt{2-x}-1)(\sqrt{2-x}+1)}{(x-1)(\sqrt{2-x}+1)} \\ &= \lim_{x\to 1} \frac{2-x-1}{(x-1)(\sqrt{2-x}+1)} \\ &= \lim_{x\to 1} \frac{1-x}{-(1-x)(\sqrt{2-x}+1)} \\ &= \lim_{x\to 1} \frac{1}{-(\sqrt{2-x}+1)} \\ &= -\frac{1}{2}\end{aligned}$$

1.7.7 Incorrect example

The following is **wrong**:

$$\begin{aligned}
\lim_{x \to a} x \cdot \frac{1}{x} &= \lim_{x \to 0} x \cdot \lim_{x \to 0} \frac{1}{x} \\
&= 0 \cdot \lim_{x \to 0} \frac{1}{x} \\
&= 0
\end{aligned}$$

Instead, we should do this:

$$\begin{aligned}
\lim_{x \to 0} \cdot \frac{1}{x} &= \lim_{x \to 0} 1 \\
&= 1
\end{aligned}$$

2 Squeeze Theorem

Suppose $f(x) \leq g(x) \leq h(x)$, for $x \in I \setminus \{a\}$, where I is some open interval containing the point a. Then:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \quad \Rightarrow \quad \lim_{x \to a} g(x) = L$$

2.1 Proof

Suppose:

$$f(x) \leq g(x) \leq h(x), \text{ for } x \in I \setminus \{a\} \tag{1}$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \tag{2}$$

Let $\varepsilon > 0$. Since I is open and $a \in I$, there exists a $\delta_1 > 0$ such that $(a - \delta_1, a + \delta_1) \subset I$. And by using equation (2), there exists a $\delta_2, \delta_3 > 0$ such that:

$$0 < |x - a| < \delta_2,\ x \in dom\, f \ \Rightarrow\ |f(x) - L| < \varepsilon \ \Rightarrow\ L - \varepsilon < f(x)$$

$$0 < |x - a| < \delta_3,\ x \in dom\, h \ \Rightarrow\ |h(x) - L| < \varepsilon \ \Rightarrow\ h(x) < L + \epsilon$$

Note that *dom* represents the **domain** of the function.

Let $\delta = min\{\delta_1, \delta_2, \delta_3\}$. Using equation (1), we get:

$$0 < |x - a| < \delta$$

$$\Downarrow$$

$$x \in I \setminus \{a\}$$

$$0 < |x - a| < \delta_2$$

$$0 < |x - a| < \delta_3$$

$$\Downarrow$$

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

$$\Downarrow$$

$$|g(x) - L| < \varepsilon$$

2.2 Another useful result

For $f : A \to \mathbb{R}$, we have:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a} |f(x) - L| = 0$$

2.2.1 Proof

$$\lim_{x\to a} f(x) = L$$

$$\Updownarrow$$

$$\begin{array}{l}\text{For every } \varepsilon > 0, \text{ there exists a } \delta > 0 \\ \text{such that } 0 < |x-a| < \delta, \; x \in A\end{array} \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

$$\Updownarrow$$

$$\begin{array}{l}\text{For every } \varepsilon > 0, \text{ there exists a } \delta > 0 \\ \text{such that } 0 < |x-a| < \delta, \; x \in A\end{array} \quad \Rightarrow \quad ||f(x) - L| - 0| < \varepsilon$$

$$\Updownarrow$$

$$\lim_{x\to a} |f(x) - L| = 0$$

2.2.2 Example 1

Evaluate $\lim_{x\to 0} x \sin \frac{1}{x}$ (again):

Guess, from the graph or otherwise: $\lim_{x\to 0} x \sin \frac{1}{x} = 0$

Proof:

Note that:

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0 \quad \Leftrightarrow \quad \lim_{x\to 0} \left| x \sin \frac{1}{x} - 0 \right| = 0$$

$$0 \leq \left| x \sin \frac{1}{x} \right| = |x| \left| \sin \frac{1}{x} \right| \leq |x|$$

By the squeeze theorem:

$$\lim_{x\to 0} \left| x \sin \frac{1}{x} \right| = 0$$

Hence:

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0$$

2.2.3 Example 2

Evaluate $\lim_{x\to 0} e^{\sin(\cot x)} x^4$

Guess, from the graph or otherwise: $\lim_{x\to 0} e^{\sin(\cot x)} x^4 = 0$

Proof:

By the squeeze theorem,

$$|e^{\sin(\cot x)} x^4 - 0| = e^{\sin(\cot x)} \cdot |x^4|$$

Because the range of $\sin(\cot x)$ is $x \in [-1, 1]$, the range of $e^{\sin(\cot x)}$ will be $x \in [e^{-1}, e^1]$, which is less than e:

$$e^{\sin(\cot x)} \cdot |x^4| \leq e \cdot |x^4|$$

$$e \cdot |x^4| \to 0 \text{ as } x \to 0$$

Hence:

$$\lim_{x\to 0} e^{\sin(\cot x)} x^4 = 0$$

2.3 A lemma

For $0 < x < \frac{\pi}{2}$, we have:

$$x \cos^2 x < \sin x < x$$

If f and g are **even** functions such that $f(x) < g(x)$, for $x \in (0, a)$, then we also have:

$$f(x) < g(x), \text{ for } x \in (-a, 0)$$

2.3.1 Proof

Suppose f and g are even and:

$$f(x) < g(x), \text{ for } x \in (0, a)$$

Then for $x \in (-a, 0)$, let $u = x$ so $u \in (0, a)$. We get:

$$f(x) = f(-u) = f(u) < g(u) = g(-u) = g(x)$$

We showed that $x \cos^2 x < \sin x < x$ for $x \in (0, \frac{\pi}{2})$.

Since $x > 0$, we get:

$$\cos^2 x < \frac{\sin x}{x} < 1, \text{ for } x \in (0, \frac{\pi}{2})$$

Since all three expressions above are even functions, the two inequalities extend to $(-\frac{\pi}{2}, 0)$. Hence:

$$\cos^2 x < \frac{\sin x}{x} < 1, \text{ for } x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$$

2.3.2 Example 1

Find $\lim_{x\to 0} \sin x$:

Using the lemma:

$$\frac{\sin x}{x} < 1, \text{ for } x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$$

Note that for $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, we have:

$$\left|\frac{\sin x}{x}\right| = \frac{\sin x}{x} < 1 \text{ for } x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$$

So for $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$:

$$0 \le |\sin x| \le |x|$$

Because by using the squeeze theorem, $|\sin x|$ approaches 0 and $|x|$ approaches 0 when x approaches 0.

Hence:

$$\lim_{x\to 0} \sin x = 0$$

2.3.3 Example 2

Find $\lim_{x\to 0}\cos x$:

For $x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}), \cos x > 0$, so:

$$\begin{aligned}\cos x &= |\cos x| \\ &= \sqrt{\cos^2 x} \\ &= \sqrt{1-\sin^2 x}\end{aligned}$$

Since $\sin^2 x \to 0$ as $x \to 0$:

$$\begin{aligned}\sqrt{1-\sin^2 x} &\to \sqrt{1} \\ &\to 1\end{aligned}$$

Hence:

$$\lim_{x\to 0}\cos x = 1$$

2.3.4 Example 3

Find $\lim_{x\to 0}\frac{\sin x}{x}$:

We showed that:

$$\cos^2 x < \frac{\sin x}{x} < 1 \text{ for } x \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$$

So $\lim_{x\to 0}\frac{\sin x}{x} = 1$

3 Limits at infinity

We also want to consider a function's behaviour as x becomes very large (either positive or negative). For example, it is clear that the values of $f(x) = 4 - \frac{2}{x}$ can be made arbitrarily close to 4 by choosing x to be large enough.

We can express this by writing:

$$\lim_{x\to+\infty} 4 - \frac{2}{x} = 4$$

3.1 Definition

Suppose f is defined on some interval (a, ∞). We say that $f(x)$ has a limit L as x approaches positive infinity, and write $\lim_{X \to +\infty} f(x) = L$, if for every $\varepsilon > 0$, there exists a number R such that:

$$x > R \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Likewise, for f defined on some interval $(-\infty, b)$, we say that $f(x)$ has a limit L as x approaches negative infinity, and write $\lim_{x \to -\infty} f(x) = L$, if for every $\varepsilon > 0$, there exists a number R such that:

$$x < R \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Limits at infinity follow the same limit laws as normal limits, so we can use limit laws to conclude that for any **positive** integer n, we also have:

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

When evaluating a limit at infinity, a common technique is to factor out the highest possible power.

3.2 Example 1

Show that $\lim_{x \to +\infty} = 0$.

Let $\varepsilon > 0$ and $R = \frac{1}{\varepsilon}$. We get:

$$\begin{aligned} x > R \Rightarrow \left| \frac{1}{x} - 0 \right| &= \frac{1}{|x|} \\ &= \frac{1}{x} \\ &< \frac{1}{R} = \varepsilon \end{aligned}$$

$$x > R = \frac{1}{\varepsilon} > 0$$

3.3 Example 2

Show that $\lim_{x \to -\infty} = 0$.

Let $\varepsilon > 0$ and $R = \frac{1}{\varepsilon}$. We get:

$$\begin{aligned} x < R \Rightarrow \left| \frac{1}{x} - 0 \right| &= \frac{1}{|x|} \\ &= \frac{1}{x} \\ &< \frac{1}{R} = \varepsilon \end{aligned}$$

$$x < R = \frac{1}{\varepsilon} > 0$$

3.4 Example 3

Find:

$$\lim_{x \to +\infty} \frac{x^3 + 4x - 5}{7x^3 + 3}$$

As $x \to +\infty$:

$$\begin{aligned} \frac{x^3 + 4x - 5}{7x^3 + 3} &= \frac{x^3 \left(1 + \frac{4}{x^2} - \frac{5}{x^3}\right)}{x^3 \left(7 + \frac{3}{x^3}\right)} \\ &= \frac{1 + \frac{4}{x^2} - \frac{5}{x^3}}{7 + \frac{3}{x^3}} \\ &= \frac{1 + \frac{4(1)}{x^2} - \frac{5(1)}{x^3}}{7 + \frac{3(1)}{x^3}} \\ &\to \frac{1}{7} \text{ as } x \to +\infty \end{aligned}$$

This is because all terms with $\frac{1}{x^n}$, where $n \in \mathbb{Z}^+$, approach 0 when $x \to +\infty$.

Hence:

$$\lim_{x \to +\infty} \frac{x^3 + 4x - 5}{7x^3 + 3} \to \frac{1}{7} \text{ as } x \to +\infty$$

3.5 Incorrect example

Find:

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\begin{aligned}
\frac{\sqrt{2x^2+1}}{3x-5} &= \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{\sqrt{x^2}\cdot\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{\sqrt{2+\frac{1}{x^2}}}{\left(3-\frac{5}{x}\right)}
\end{aligned}$$

Since $\frac{1}{x}$ and $\frac{5}{x}$ approach 0 when $x\to\infty$:

$$\frac{\sqrt{2+\frac{1}{x^2}}}{\left(3-\frac{5}{x}\right)} \to \frac{\sqrt{2}}{3} \text{ as } x\to-\infty$$

3.5.1 Explanation of why the example is incorrect

For $a\geq 0$:

$$x=\sqrt{a} \quad\Leftrightarrow\quad x^2=a,\ x\geq 0$$

Hence:

$$x=\sqrt{a^2} \quad\Leftrightarrow\quad x^2=a^2,\ x\geq 0 \quad\Leftrightarrow\quad x=|a|$$

$$\sqrt{a^2}=|a|$$

For example:

$$\sqrt{(-2)^2}=\sqrt{4}=2=|-2|$$

The example automatically assumed that $\sqrt{x^2}$ is x without considering what x will be when approaching the limit, as x in $\sqrt{x^2}$ can be either x or $-x$.

3.5.2 Important note

Do not write $\sqrt{a^2} = \pm a$, as it is not correct. The square root is a **function** of its argument and has **only one** (non-negative) value.

For example, the solutions to the equation $x^2 = a$ for $a > 0$ are $x = \pm\sqrt{a}$. Do not write the solution as $x = \sqrt{a}$ and claim that $\sqrt{a}$ has both a positive and negative value.

3.6 Corrected example

Find:

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5}$$

$$\begin{aligned}
\frac{\sqrt{2x^2+1}}{3x-5} &= \frac{\sqrt{x^2\left(2+\frac{1}{x^2}\right)}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{\sqrt{x^2}\cdot\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)}
\end{aligned}$$

For $x < 0$, $|x| = -x$:

$$\begin{aligned}
\frac{|x|\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} &= \frac{-x\sqrt{2+\frac{1}{x^2}}}{x\left(3-\frac{5}{x}\right)} \\
&= \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}
\end{aligned}$$

Since $\frac{1}{x^2}$ and $\frac{1}{x}$ approach 0 as $x \to -\infty$:

$$\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} \to -\frac{\sqrt{2}}{3} \text{ as } x \to -\infty$$

Hence:

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2+1}}{3x-5} \to -\frac{\sqrt{2}}{3} \text{ as } x \to -\infty$$

4 Limit of a sequence

4.1 Definition

We say that a sequence (a_n) has the limit L and write $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$, there exists a number N such that:

$$n > N \quad \Rightarrow \quad |a_n - L| < \varepsilon$$

The limits of sequences are evaluated with similar methods to other forms of limits.

4.2 Example

Find:

$$\lim_{n\to\infty}(\sqrt{n^2+2n}-n)$$

$$\begin{aligned}
\sqrt{n^2+2n}-n &= \frac{(\sqrt{n^2+2n}-n)(\sqrt{n^2+2n}+n)}{\sqrt{n^2+2n}+n} \\
&= \frac{n^2+2n-n^2}{\sqrt{n^2+2n}+n} \\
&= \frac{2n}{\sqrt{n^2+2n}+n} \\
&= \frac{2n}{\sqrt{n^2\left(1+\frac{2}{n}\right)}+n} \\
&= \frac{2n}{|n|\cdot\sqrt{1+\frac{2}{n}}}
\end{aligned}$$

Since $|n| = n$ for $n > 0$:

$$\begin{aligned}
\frac{2n}{|n|\cdot\sqrt{1+\frac{2}{n}}} &= \frac{2n}{n\sqrt{1+\frac{2}{n}}+n} \\
&= \frac{2n}{n(\sqrt{1+\frac{2}{n}}+1)} \\
&= \frac{2}{\sqrt{1+\frac{2}{n}}+1}
\end{aligned}$$

Since $\frac{2}{n}\to 0$ when $n\to\infty$:

$$\begin{aligned}
\frac{2}{\sqrt{1+\frac{2}{n}}+1} &\to \frac{2}{\sqrt{1}+1} \\
&\to \frac{2}{2} \\
&\to 1 \text{ when } n\to\infty
\end{aligned}$$