

Math Module 4B Cheat Sheet

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1 Definitions

1.1 Differential equation (d.e or DE)

A differential equation is an equation involving one or more derivatives of an unknown function. The order of the highest derivative occurring in a differential equation is called the **order** of the differential equation.

1.1.1 Examples

$$\frac{dy}{dx} + x^2 = y$$

$$\frac{d^2y}{dx^2} = -k^2y$$

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^5 + \cos x = 0$$

$$\sin\left(\frac{dy}{dx}\right) + \arctan y = 1$$

The equations above are all differential equations in the unknown function $y(x)$.

1.2 Solution to a differential equation

Consider an n th order differential equation in the unknown y . A function $y(x)$ which is (at least) n times differentiable on an interval I is called a **solution** to the differential equation on I , if the substitution $y = y(x)$, $y' = y'(x)$, \dots , $y^{(n)} = y^{(n)}(x)$ reduces the differential equation to an identity valid for all $x \in I$.

A solution to a differential equation is sometimes also called a **particular solution**.

The **general** solution to a differential equation is the collection of all (particular) solutions.

1.3 Initial-value problems (i.v.p)

An n th-order differential equation together with n **initial conditions** of the form:

$$\begin{aligned}y(x_0) &= y_0 \\y'(x_0) &= y_0 \\&\vdots \\y^{(n-1)}(x_0) &= y_{n-1}\end{aligned}$$

Where y_0, y_1, \dots, y_{n-1} are constants, is called an **initial-value problem**. A solution to an initial value problem is a function that satisfies both the stated differential equations and all the initial conditions.

1.4 Separable differential equations

A first-order differential equation is called **separable** if it can be written in the form:

$$p(y)y' = q(x)$$

A separable differential equation is just a differential equation that we can separate x and y .

If p and q are continuous, we can solve $p(y)y' = q(x)$ by integrating both sides with respect to x :

$$\begin{aligned}\int p(y)y' dx &= \int q(x) dx \\ \int p(y) dy &= \int q(x) dx\end{aligned}$$

1.5 Linear differential equation

A differential equation that can be written in the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

Where a_0, a_1, \dots, a_n and F are functions of x only, is called a **linear** differential equation of order n .

Basically, the degree of all terms in y is **at most one**.

1.5.1 Examples

$$I \cdot y'' + x^2 y + (\sin x)y = e^x \quad (1)$$

$$xy''' + 4x^2 y' - \frac{2}{1+x^2}y = 0 \quad (2)$$

The equations above ((1) and (2)) are **linear** differential equations.

$$y'' + x \sin(y') - xy = x^2 \quad (3)$$

$$y'' + x^2 y' + y^2 = 0 \quad (4)$$

The equations above ((3) and (4)) are **nonlinear** differential equations.

1.6 Linear initial value problems

Let a_1, a_2, \dots, a_n, F be functions that are continuous on an interval I . Then, for any $x_0 \in I$, the initial-value problem below has a unique solution on I .

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

$$y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$$

1.7 First-order linear differential equations

A differential equation that can be written in the form:

$$a(x)\frac{dy}{dx} + b(x)y = r(x)$$

Where $a(x)$, $b(x)$, and $r(x)$ are functions defined on an interval (a, b) , is called a **first-order linear differential equation**.

If $a(x) \neq 0$ on (a, b) , we have the **standard form**:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Where $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{r(x)}{a(x)}$.

Let $P(x)$ be an antiderivative to $p(x)$ and multiply with $e^{P(x)}$ to get:

$$e^{P(x)}\frac{dy}{dx} + p(x)e^{P(x)}y = q(x)e^{P(x)}$$

$$\frac{d}{dx} \left(e^{P(x)}y \right) = q(x)e^{P(x)}$$

We can solve the problem by integration.

1.8 Homogeneous differential equation

Homogeneous just means that the differential equation is equal to 0.

1.9 Second-order linear differential equation

A **second-order linear** differential equation, has the form:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$$

Here, a_0, a_1, a_2 and F are functions defined on an interval I . If $F(x) = 0$ for all $x \in I$, we say that the equation is **homogeneous**.

If $a_0(x) \neq 0$ on I , dividing gives the **standard form**:

$$y'' + p(x)y' + q(x)y = f(x)$$

1.9.1 Theorem

For a **homogeneous** linear differential equation:

$$a(x)y'' + b(x)y' + c(x)y = 0$$

If $y_1(x)$ and $y_2(x)$ are two solutions on the interval I , then any **linear combination** below is also a solution on I :

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

Where C_1, C_2 are constants.

The linearity principle holds **only** for differential equations that are **both homogeneous** and **linear**. The result is stated above for a second-order linear homogeneous equation, but the analogous result holds for n th order linear homogeneous equations.

1.10 Linearly dependent functions

Two functions defined on the interval I are said to be **linearly dependent** on I , if one function is a **scalar (constant) multiple** of another function.

1.10.1 Example

$$\begin{aligned}y_1(x) &= \cos 2x \\y_2(x) &= 3(1 - 2 \sin^2 x)\end{aligned}$$

$$\begin{aligned}y_2(x) &= 3(1 - 2 \sin^2 x) \\&= 3 \left(1 - 2 \cdot \frac{1 - \cos 2x}{2} \right) \\&= 3(1 - (1 - \cos 2x)) \\&= 3 \cos 2x \\&= 3y_1(x)\end{aligned}$$

Since $y_2(x) = 3y_1(x)$, $y_1(x)$ and $y_2(x)$ are **linearly dependent**.

1.11 Linearly independent functions

Two functions defined on the interval I are said to be **linearly independent** if one function is **not** a scalar (constant) multiple of another function.

1.11.1 Example

$$\begin{aligned}y_1(x) &= e^x \\ y_2(x) &= xe^x\end{aligned}$$

Since neither e^x nor xe^x is a constant multiple of the other, the two functions are **linearly independent**.

1.11.2 Theorem

Let I be an interval and consider the equations:

$$y'' + p(x)y' + q(x)y = f(x) \quad (3)$$

$$y'' + p(x)y' + q(x)y = 0 \quad (4)$$

Let $y_1(x), y_2(x)$ be **linearly independent** solutions of (4) and $y_p(x)$ a solution of (3) on I . Then:

- The general solution of (4) on I is:

$$y(x) = C_1y_1(x) + C_2y_2(x), \quad C_1, C_2 \in \mathbb{R}$$

- The general solution to (3) on I is:

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x), \quad C_1, C_2 \in \mathbb{R}$$

So, the **general solution** to the non-homogeneous differential equation $y'' + p(x)y' + q(x)y = f(x)$ is of the form:

$$y(x) = y_c(x) + y_p(x)$$

Where $y_c(x) = C_1y_1(x) + C_2y_2(x)$ is the general solution to the associated homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0$$

And y_p is a particular solution to:

$$y'' + p(x)y' + q(x)y = f(x)$$

1.12 Characteristic equation

Consider a homogeneous linear differential equation of order 2 with **constant coefficients**.

$$ay'' + by' + cy = 0$$

Where $a, b, c \in \mathbb{R}, a \neq 0$.

Since, in order to satisfy the equation, we want constant multiples of y and its derivatives to cancel, we should look for solutions of this differential equation of the form $y(x) = e^{rx}$ (since derivatives of y are constant multiples of y). Trying this:

With $y(x) = e^{rx}$, we get:

$$y'(x) = re^{rx}, \quad y''(x) = r^2e^{rx}$$

Substituting the above equation into the differential equation $ay'' + by' + cy = 0$, we get:

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0$$

Since $e^{rx} > 0$, the equation is satisfied only if:

$$ar^2 + br + c = 0$$

The quadratic equation above is called the **characteristic equation** for the differential equation above.

2 Solving first-order differential equations (FODEs)

In order to solve first-order differential equations of the form:

$$\frac{dy}{dx} + p(x)y = q(x) \quad (1)$$

We will use a method called the integrating factor. Introducing an integrating factor called μ to the equation above:

$$\mu \frac{dy}{dx} + \mu p(x)y = \mu q(x)$$

Considering the product rule to simplify the above equation:

$$\frac{d}{dx}(uv) = u \frac{du}{dx} + v \frac{du}{dx}$$

Comparing this with the left-hand side of the original first-order differential equation, we have:

$$u = \mu \quad (2)$$

$$\frac{du}{dx} = \frac{dy}{dx} \quad (3)$$

$$v = y \quad (4)$$

$$\frac{du}{dx} = \mu p(x) \quad (5)$$

Equations (2) and (5) yield:

$$\frac{d\mu}{dx} = \frac{du}{dx} = \mu p(x)$$

Solving this with the separation of variables:

$$\frac{d\mu}{dx} = \mu p(x)$$

$$\frac{1}{\mu} \frac{d\mu}{dx} = p(x)$$

$$\int \frac{1}{\mu} d\mu = \int p(x) dx$$

$$\ln |\mu| = \int p(x) dx$$

$$\mu = e^{\int p(x) dx} \quad (6)$$

Substituting (6) into (1):

$$e^{\int p(x) dx} \frac{dy}{dx} + e^{\int p(x) dx} y = e^{\int p(x) dx} q(x)$$

Using equations (2), (3), (4), (5) and (6) with the product rule:

$$\frac{d}{dx} \left(e^{\int p(x) dx} y \right) = e^{\int p(x) dx} q(x)$$

Integrating both sides with respect to x , we get:

$$e^{\int p(x) dx} y = \int q(x) e^{\int p(x) dx} dx$$

In summary, reduce the given first-order differential equation into the form $\frac{dy}{dx} + p(x)y = q(x)$, then find the integrating factor with $\mu = e^{\int p(x) dx}$ and multiply every term by it. Apply the product rule to obtain $\frac{d}{dx} \left(e^{\int p(x) dx} y \right) = e^{\int p(x) dx} q(x)$. Then integrate both sides with respect to x and solve for y .

3 Solving second-order differential equations (SODEs)

3.1 Solving linear homogeneous second-order differential equations

To solve linear (degree of all terms is at most 1) homogeneous (the equation is equal to 0) second-order differential equations of the form:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0$$

We are finding the **general solution** of y . First, we need to identify the **characteristic or auxiliary equation** of the second-order differential equation. It is given by:

$$am^2 + bm + c = 0$$

Using the quadratic formula,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We now have three cases for the different types of roots.

3.1.1 Case 1: Roots are real and distinct ($b^2 - 4ac > 0$)

The general solution is:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Where C_1 and C_2 are constants to be found.

3.1.2 Case 2: Roots are real and equal ($b^2 - 4ac = 0$)

The general solution is:

$$y = (C_1 + C_2 x) e^{mx}$$

3.1.3 Case 3: Roots are complex ($b^2 - 4ac < 0$)

The general solution is:

$$y = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x)), \quad m = \alpha + \beta i$$

3.2 Solving linear non-homogeneous second-order differential equations

To solve linear non-homogeneous second-order differential equations of the form:

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$$

We must first find the **complimentary function**, which is the function $y = q(x)$. When this function is substituted into the second-order differential equation, the right-hand side is 0 (similar to the **general solution** of a linear homogeneous second-order differential equation). After which, we must find the **particular solution**, which is the function $y = p(x)$. When this function is substituted into the second-order differential equation, it gives us $f(x)$. Finally, the general solution to the linear non-homogeneous second-order differential equation is given by:

$$y = \text{Complimentary function} + \text{Particular solution}$$

To find the particular solution, we must consider 3 cases.

3.2.1 Case 1: $f(x)$ is a polynomial of degree n , $f(x) = a_0 + a_1 x + \dots + a_n x^n$

The particular solution is a polynomial with degree equal to the degree of $f(x)$.

$$p(x) = b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n$$

3.2.2 Case 2: $f(x) = (c_0 + c_1x + c_2x^2 + \dots + c_nx^n)e^{kx}$, $c_n \in \mathbb{R}$

1. The complimentary function does not have e^{kx}

The particular solution is:

$$p(x) = (c_0 + c_1x + c_2x^2 + \dots + c_nx^n)e^{kx}$$

2. The complimentary function has e^{kx} but not xe^{kx}

The particular solution is:

$$p(x) = x(c_0 + c_1x + c_2x^2 + \dots + c_nx^n)e^{kx}$$

3. The complimentary function has e^{kx} and xe^{kx}

The particular solution is:

$$p(x) = x^2(c_0 + c_1x + c_2x^2 + \dots + c_nx^n)e^{kx}$$

4. The complimentary function is $q(x) = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x))$

The particular solution is:

$$p(x) = pe^{kx}$$

3.2.3 Case 3: $f(x) = k \cos(ax), k \sin(ax)$ **or** $k \cos(ax) + r \sin(ax)$

1. The complimentary function does not have $A \cos(ax) + B \sin(ax)$

The particular solution is:

$$p(x) = p \cos(ax) + q \sin(ax)$$

2. The complimentary function has $A \cos(ax) + B \sin(ax)$

The particular solution is:

$$p(x) = x(p \cos(ax) + q \sin(ax))$$

3.2.4 After the particular solution is found

Once we find the particular solution, we must find its first and second derivatives, $p'(x)$ and $p''(x)$. After which, we substitute them into the original second-order differential equation to find the constants p and q . And now, the full general solution to the linear non-homogeneous second-order differential equation is:

$$y = q(x) + p(x)$$