1) $\sum_{x=1}^{\infty}(a_n+b_n)$

Suppose $\sum_{n=1}^{\infty}(a_n+b_n)$ exists, which means it converges

$$\sum_{n=1}^{\infty}(a_n+b_n)-\sum_{n=1}^{\infty}a_n$$

$$=\cancel{\sum_{n=1}^{\infty}a_n}+\sum_{n=1}^{\infty}b_n-\cancel{\sum_{n=1}^{\infty}a_n}$$

$$=\sum_{n=1}^{\infty}b_n$$

Since $\sum_{x=1}^{\infty}b_n$ diverges, the sum does not exist. Hence, $\sum_{n=1}^{\infty}(a_n+b_n)$ diverges.

2) The sum of $\sum_{n=1}^{\infty}(a_n+b_n)$ can either be convergent or divergent.

For a divergent series,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n+1} \text{ is divergent}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{1}{n+1}\right) \text{ is divergent.}$$

For a convergent series,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \text{ is divergent}$$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n+1} \text{ is divergent}$$

$$\sum_{n=1}^{\infty} \left((-1)^n + (-1)^{n+1}\right) \text{ is convergent}$$

and its sum is 0.

3) $\sum_{k=1}^{n} (-1)^k$

4) $\lim_{n \to \infty} a_n = l$

For every $\varepsilon > 0$, there exists a number N such that

$$n > N \Rightarrow |a_n - l| < \varepsilon$$

Choosing $\varepsilon = 10$, by definition, there exists a N such that there exists $|a_n - l| < 10$,

$$-l + 10 < a_n < l + 10$$

$\therefore$ The "tail" is bounded and before that, there are only finitely many values, $a_1, a_2, a_3, \ldots, a_N$.

Take $M = \max\{|a_1|, \ldots, |a_n|, |-l+10|, |l+10|\}$

$\therefore |a_n| \leq M$ and hence the sequence $(a_n)_{n=1}^{\infty}$ is bounded.

5) $a_n = r_n b_n \qquad \lim_{n\to\infty} r_n = l$

$l \neq 0$: $\sum a_n$ converges $\Leftrightarrow$ $\sum b_n$ converges.

When $l = 0$:

$\sum a_n$ converges $\not\Rightarrow$ $\sum b_n$ converges

$\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges

$$\lim_{\substack{n\to\infty \\ b_n \neq 0}} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{r_n b_n}{b_n} = \lim_{n\to\infty} r_n = 0$$

$b_n = 0 \Rightarrow a_n = 0$

6a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

$$\left|\frac{\sin n}{n^2}\right| = \frac{|\sin n|}{n^2}$$

Since $|\sin n| \leq 1$

$$0 \leq \left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series $p > 1$)

By the comparison test,

$\displaystyle\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$ converges.

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges

absolutely.

6b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$

$$= \sum_{n=1}^{3} (-1)^n \frac{\ln n}{n} + \sum_{n=4}^{\infty} (-1)^n \frac{\ln n}{n}$$

$$= \frac{\ln 2}{2} - \frac{\ln 3}{3} + \sum_{n=4}^{\infty} (-1)^n \frac{\ln n}{n}$$

Since $\frac{\ln 4}{4} \geq \frac{\ln 5}{5} \geq \frac{\ln 6}{6} \geq \cdots \geq 0$

and $\lim_{n \to \infty} \frac{\ln n}{n} = 0$

By Leibniz' Test,

$\sum_{n=4}^{\infty} (-1)^n \frac{\ln n}{n}$ converges.

Since $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} = \frac{\ln 2}{2} - \frac{\ln 3}{3} + \sum_{n=4}^{\infty} (-1)^n \frac{\ln n}{n}$,

$\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ converges.

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

6b) Since $\frac{\ln n}{n} > \frac{1}{n}$, and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

By the comparison test, (p-series $p \leq 1$)

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| \text{ diverges.}$$

$$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} \text{ converges conditionally.}$$

6c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$

$$\lim_{n \to \infty} \frac{\frac{1}{\sqrt[3]{n^2+1}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{\sqrt[3]{n^2+1}}$$

$$= \lim_{n \to \infty} \frac{n}{n^{\frac{2}{3}} \sqrt[3]{1+\frac{1}{n^2}}}$$

$$= \lim_{n \to \infty} \frac{\sqrt[3]{n}}{\sqrt[3]{1+\frac{1}{n^2}}}$$

$$= \infty > 0$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p \leq 1$),

by the limit comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+1}}$ diverges

6d) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)^{\cancel{2}}}{n^2 \cancel{(n+1)}!}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n^2}$$

$$= 0 < 1$$

By the ratio test,

$\sum_{n=1}^{\infty} \frac{n^2}{n!}$ converges.

6e) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Since $\frac{1}{\ln n} > \frac{1}{n} \geq 0$,

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, (p-series $p \leq 1$)

By the comparison test,

$\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

6f) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$

When $p \leq 0$, $\lim_{n\to\infty} \frac{1}{(\ln n)^p} = \lim_{n\to\infty} (\ln n)^{-p} \neq 0$

By the limit test,

the series diverges.

When $p > 0$,

$$(\ln n)^p << n << e^n$$

$$\lim_{n\to\infty} \frac{(\ln n)^p}{n} = \lim_{x\to\infty} \frac{(\ln x)^p}{x}$$

$$= p \lim_{x\to\infty} \frac{(\ln x)^{p-1}}{x}$$

$$= p(p-1)\ldots(p-i) \lim_{x\to\infty} \frac{(\ln x)^{p-1-i}}{x}$$

$$= 0$$

Eventually, $(\ln n)^p < n$, i.e., $\frac{1}{n} < \frac{1}{(\ln n)^p}$ when

n is large.

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, by the comparison test,

$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ diverges.

6g) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

$$\lim_{n\to\infty} \frac{\frac{1}{(\ln n)^{n+1}}}{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \frac{\cancel{(\ln n)^n}}{(\ln n)^{\cancel{n+1}}}$$

$$= \lim_{n\to\infty} \frac{1}{\ln n}$$

$$= 0 < 1$$

By the ratio test,

$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ converges.

6h) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$\int_2^{+\infty} \frac{1}{x \ln x}\, dx$$

$$= \Big[\ln(\ln x)\Big]_2^{\infty}$$

$$= \infty$$

$\therefore$ By the integral test, $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

6i) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

$u = \ln x \quad \frac{du}{dx} = \frac{1}{x}$

$$\int_2^{+\infty} \frac{1}{x(\ln x)^2}\,dx$$

$$= \int_{\ln 2}^{+\infty} \frac{1}{u^2}\,du$$

$$= \left[-\frac{1}{u}\right]_{\ln 2}^{\infty}$$

$$= \left[-\frac{1}{\ln x}\right]_2^{\infty}$$

Since $\left[-\frac{1}{\ln x}\right]_2^{\infty}$ converges, by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

6j) $\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$

$$0 \le \frac{1}{n^2 \ln n} < \frac{1}{n^2}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges

(p-series $p > 1$)

by the comparison test,

$\sum_{n=2}^{\infty} \frac{1}{n^2 \ln n}$ converges

6k) $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n \cancel{(n+1)!}}{\cancel{n!}(n+1)^{\cancel{n+1}}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

$\therefore$ By the ratio test $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

6l) $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} \frac{n^n 2^{\cancel{n+1}} \cancel{(n+1)!}}{(n+1)^{\cancel{n+1}} \cancel{2^n} \cancel{n!}}$$

$$= 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= 2 \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{2}{e} < 1$$

$\therefore$ By the ratio test, $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ converges.

6 m) $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

$$\lim_{n\to\infty} \frac{\frac{3^{n+1}(n+1)!}{(n+1)^{n+1}}}{\frac{3^n n!}{n^n}} = \lim_{n\to\infty} \frac{3^{\cancel{n}+1}\cancel{(n+1)}!\, n^n}{\cancel{3^n}\,\cancel{n!}\,(n+1)^{n\cancel{+1}}}$$

$$= \lim_{n\to\infty} \frac{3n^n}{(n+1)^n}$$

$$= 3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{3}{e} > 1$$

$\therefore$ By the ratio test, $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ diverges.

6 n) $\sum_{n=1}^{\infty} e^{-\sqrt{n}}$

Since $\int_1^{+\infty} e^{-\sqrt{x}}\,dx$ converges,

by the integral test,

$\sum_{n=1}^{\infty} e^{-\sqrt{n}}$ converges.

7) A function would be:

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

$$f(x) = \sin(\pi x), \; x \in \mathbb{R}$$