1a) $k \cdot (\underset{\sim}{0} + \underset{\sim}{u}) = k \cdot \underset{\sim}{u}$ (property 1d)

$k \cdot \underset{\sim}{0} + k \cdot \underset{\sim}{u} = k \cdot \underset{\sim}{u}$ (property 2c)

$k \cdot \underset{\sim}{0} = \underset{\sim}{0}$

b) Assume $k \neq 0$,

$\underset{\sim}{u} = (\frac{1}{k} \cdot k) \cdot \underset{\sim}{u}$

$\underset{\sim}{u} = \frac{1}{k} \cdot (k \cdot \underset{\sim}{u})$

Since $k \cdot \underset{\sim}{u} = \underset{\sim}{0}$

$\underset{\sim}{u} = \frac{1}{k} \cdot \underset{\sim}{0}$

$\underset{\sim}{u} = \underset{\sim}{0}$ (property 2)

c) $\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0}$ (property 1e)

$-1 \cdot (\underset{\sim}{u} - \underset{\sim}{u}) = -1 \cdot \underset{\sim}{0}$

$-1 \cdot \underset{\sim}{u} - 1 \cdot (-\underset{\sim}{u}) = \underset{\sim}{0}$ (property 2 and 2b)

$-1 \cdot \underset{\sim}{u} = -\underset{\sim}{u}$

By property 1b,

$-\underset{\sim}{u} + \underset{\sim}{u} = \underset{\sim}{0} \quad \therefore -1 \cdot \underset{\sim}{u} = -\underset{\sim}{u}$

2) Let Hippo be $\underset{\sim}{0}$.

$$\mathcal{V} = \{\underset{\sim}{0}\}$$

$\underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$, $k \cdot \underset{\sim}{0} = \underset{\sim}{0}$ for all $k \in \mathbb{R}$

Property 1a:

$$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0} \in \mathcal{V}$$

Property 1b:

$$\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$$
$$\underset{\sim}{v} + \underset{\sim}{u} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$$

$\therefore \underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$, for $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{V}$

Property 1c:

$$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = (\underset{\sim}{0} + \underset{\sim}{0}) + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$$

$$\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{0} + (\underset{\sim}{0} + \underset{\sim}{0}) = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0}$$

$\therefore (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w})$, for $\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \in \mathcal{V}$

2) Property 1d:

$$\underset{\sim}{u} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{0} = \underset{\sim}{0} = \underset{\sim}{u}, \text{ for all } \underset{\sim}{u} \in V$$

Property 1e:

$$\underset{\sim}{u} + (-\underset{\sim}{u}) = \underset{\sim}{0} + -1 \cdot \underset{\sim}{0}$$

$$= \underset{\sim}{0} + \underset{\sim}{0} \quad (k \cdot \underset{\sim}{0} = \underset{\sim}{0})$$

$$= \underset{\sim}{0} \text{ for } \underset{\sim}{u} \in V$$

Property 2a:

$$k \cdot \underset{\sim}{u} = \underset{\sim}{0} \in V \text{ for } \underset{\sim}{u} \in V$$

Property 2b:

$$k(\underset{\sim}{u} + \underset{\sim}{v}) = k(\underset{\sim}{0} + \underset{\sim}{0}) = k(\underset{\sim}{0}) = \underset{\sim}{0} \text{ for } \underset{\sim}{u}, \underset{\sim}{v} \in V$$

Property 2c:

$$(k+l)\underset{\sim}{u} = (k+l)\underset{\sim}{0} = \underset{\sim}{0}$$

$$k\underset{\sim}{u} + l\underset{\sim}{u} = k\underset{\sim}{0} + l\underset{\sim}{0} = \underset{\sim}{0}$$

$$\therefore (k+l)\underset{\sim}{u} = k\underset{\sim}{u} + l\underset{\sim}{u} = \underset{\sim}{0} \text{ for } \underset{\sim}{u} \in V$$

2) Property 2d

$$k(l\underset{\sim}{u}) = k\underset{\sim}{0} = \underset{\sim}{0}$$

$$(kl)\underset{\sim}{u} = (kl)\underset{\sim}{0} = \underset{\sim}{0}$$

$$\therefore k(l\underset{\sim}{u}) = (kl)\underset{\sim}{u} \text{ for } \underset{\sim}{u} \in \mathcal{V}$$

Property 2e

$$1\underset{\sim}{u} = 1(\underset{\sim}{0}) = \underset{\sim}{0} = \underset{\sim}{u} \text{ for } \underset{\sim}{u} \in \mathcal{V}$$

$\therefore \langle \mathcal{V}, +, \cdot \rangle$ is a vector space.

3ai) $\mathcal{W} \neq \emptyset$ as $(0,0,0) \in \mathcal{W}$

$$\underset{\sim}{u} + \underset{\sim}{v} = (u,0,0) + (v,0,0)$$

$$= (u+v, 0, 0) \in \mathcal{W}$$

for $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{W}$, $u, v \in \mathbb{R}$

$$k\underset{\sim}{u} = k(u,0,0) = (ku,0,0) \in \mathcal{W}$$

for $\underset{\sim}{u} \in \mathcal{W}$, $u \in \mathbb{R}$

$\therefore \mathcal{W}$ is a subspace of $\mathcal{V}$

3aii) $k\underset{\sim}{u} = k(u,1,1) = (ku,k,k) \notin W$ or $\underset{\sim}{0} \notin W$

$\therefore$ $W$ is not a subspace of $V$

bi) $W \neq \emptyset$ as $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

$$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \in W \text{ for } \underset{\sim}{u}, \underset{\sim}{v} \in W$$

$$\because (a+e) + (d+h)$$
$$= (a+d) + (e+h)$$
$$= tr(\underset{\sim}{u}) + tr(\underset{\sim}{v})$$
$$= 0 + 0$$
$$= 0$$

$$k\underset{\sim}{u} = k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \in W \text{ for } u \in W$$

$$\because ka + kd = k(a+d)$$
$$= k\,tr(\underset{\sim}{u})$$
$$= k0$$
$$= 0$$

$\therefore$ $W$ is a subspace of $V$.

ii) $\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in W$

$$\begin{bmatrix} 5 & 5 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} \notin W$$

$$\because \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} = 4 \neq 0$$

$\therefore W$ is not a subspace of $V$.

3ci) Let $u(x) = -1$ for all $x \in \mathbb{R}$

$u(x) \in W$

$$(-1u)(x) = -u(x)$$
$$= -(-1)$$
$$= 1 \notin W$$

$\therefore W$ is not a subspace of $V$.

ii) $\underset{\sim}{0} \notin W$, hence $W$ is not a subspace of $V$.

iii) $W \neq \emptyset$ as $o(x) = 0$ for all $x \in \mathbb{R}$, $o(x) \in W$

$$(f+g)(x) = f(x) + g(x)$$
$$f(2) + g(2) = 0 + 0 = 0$$
$$\therefore (f+g)(x) \in W$$

3c iii) $(kf)(x) = kf(x)$

$$(kf(2)) = kf(2)$$
$$= k0$$
$$= 0$$
$$\therefore (kf)(x) \in W$$

iv) Let $o(x) = 0$ for all $x \in \mathbb{R}$, o is a constant function and hence differentiable.

$o \in W$, $\therefore \underset{\sim}{0} \in W$ $\because$ $ao'(x) + bo(x) = 0 + 0 = 0$

$(f+g)(x) = f(x) + g(x)$ $\quad (f+g)' = f' + g'$

$(f+g)' \in C'$

$$a(f+g)'(x) + b(f+g)(x)$$
$$= af'(x) + ag'(x) + bf(x) + bg(x)$$
$$= af'(x) + bf(x) + ag'(x) + bg(x)$$
$$= 0 + 0$$
$$= 0$$
$$\therefore (f+g)(x) \in W$$

$$(kf)(x) = kf(x)$$

$$a(kf)'(x) + b(kf)(x)$$
$$= akf'(x) + bkf(x)$$
$$= k(af'(x) + bf(x)) \rightarrow$$
$$= k(0)$$
$$= 0$$

$\therefore (kf)(x) \in W$

$\therefore W$ is a subspace of $V$.

3 (v) Let $g(x) = C$ for all $x \in \mathbb{R}$, $g \in W$

$$(2g)(x) = 2g(x) = 2C > C$$

$\therefore W$ is not a subspace of $V$

vi) Let $o(x) = 0$ for all $x \in \mathbb{R}$, $o \in W$

$W \neq \emptyset$ as $o \in W$.

For $f, g \in W$, there exists a $C, D$ such that

$f \in W_C$, $g \in W_D$

$$|f(x)| \leq C, \quad |g(x)| \leq D$$

$$|(f+g)(x)| = |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq C + D$$

$$\therefore f+g \in W_{C+D} \subseteq W$$

Let $f \in W$, $\lambda \in \mathbb{R}$, there exist a $C$ such that $f \in W_C$.

$$|f(x)| \leq C \text{ for all } x \in \mathbb{R}$$

$$|(\lambda f)(x)| = |\lambda f(x)| = |\lambda||f(x)| \leq |\lambda| C$$

$$\therefore \lambda f \in W_{|\lambda|C} \subseteq W$$

$\therefore W$ is a subspace of $V$.

4a) Since $\mathcal{U}$ and $\mathcal{W}$ both contain $\underset{\sim}{0}$,

$\underset{\sim}{0} \in \mathcal{U} \cap \mathcal{W}$

$\therefore \mathcal{U} \cap \mathcal{W} \neq \emptyset$

Let $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{U} \cap \mathcal{W}$,

Since $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{U}$ and $\underset{\sim}{u}, \underset{\sim}{v} \in \mathcal{W}$, and

$\underset{\sim}{u} + \underset{\sim}{v} \in \mathcal{U}$ and $\underset{\sim}{u} + \underset{\sim}{v} \in \mathcal{W}$,

$\underset{\sim}{u} + \underset{\sim}{v} \in \mathcal{U} \cap \mathcal{W}$.

Let $k \in \mathbb{R}$

Since $\underset{\sim}{u} \in \mathcal{U}$ and $\underset{\sim}{u} \in \mathcal{W}$, and

$k\underset{\sim}{u} \in \mathcal{U}$ and $k\underset{\sim}{u} \in \mathcal{W}$,

$k\underset{\sim}{u} \in \mathcal{U} \cap \mathcal{W}$.

$\therefore \mathcal{U} \cap \mathcal{W}$ is a subspace of $\mathcal{V}$.

4b) Let $\mathcal{U} = \{(x, 0)$ for all $x \in \mathbb{R}\}$

$\mathcal{W} = \{(0, y)$ for all $y \in \mathbb{R}\}$

Both $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $V \in \mathbb{R}^2$

$(1,0) \in \mathcal{U}, \mathcal{W} \in (0,1)$

$(1,0) + (0,1) = (1,1) \notin \mathcal{U} \cup \mathcal{W}$

$\therefore \mathcal{U} \cup \mathcal{W}$ is not a subspace of $V \in \mathbb{R}^2$