MA2006 Engineering Mathematics Notes

Hankertrix

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Contents

1 Definitions

1.1 Order

The order of the matrix is the number of rows and columns the matrix has. Usually, this is represented as $m \times n$ for a matrix with m rows and n columns.

1.2 Matrix

An $m \times n$ matrix is an array of mn numbers enclosed within a pair of brackets and arranged in m rows and n columns.

1.2.1 Examples

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \pi \\ \sqrt{2} & \frac{1}{2} \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 3 & 9 \end{bmatrix}$$

1.2.2 Notation

Let A be an $m \times n$ matrix. The element in the i-th row and j-th column of A is denoted by a_{ij}.

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

1.3 Square matrix

A square matrix is a matrix where the number of rows is equal to the number of columns, i.e. $m = n$. A square matrix is usually denoted as a $n \times n$ matrix.

1.4 Equal matrices

For two $m \times n$ matrices A and B, they are said to be equal if $a_{ij} = b_{ij}$.

1.5 Adding matrices

If $A + B = C$, then $C = (c_{ij})$ is $m \times n$ and $c_{ij} = a_{ij} + b_{ij}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{bmatrix}$$

1.6 Multiplication of a number to a matrix

If $A = (a_{ij})$ and k is a number, then $cA = (ca_{ij})$ and cA has the same order as A.

$$-2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -6 & -8 \end{bmatrix}$$

We write $(-1)A$ as $-A$. So $B - A = B + (-A)$.

$$\begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2-1 & 3-2 \\ 4-3 & 4-2 \end{bmatrix}$$

1.7 Product of matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times q$ matrices respectively.
If $n = p$, we can form the product matrix AB.
If AB is denoted by $C = (c_{ij})$ then C is $m \times q$ and the element c_{kp} is calculated using the k-th row of A and the p-th column of B as shown below:

$$\begin{aligned} c_{kp} &= \begin{bmatrix} a_{k1} & a_{k2} & \cdots & a_{kN} \end{bmatrix} \begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{Np} \end{bmatrix} \\ &= a_{k1}b_{1p} + a_{k2}b_{2p} + \cdots + a_{kN}b_{Np} \\ &= \sum_{n=1}^{N} a_{kn}b_{nj} \end{aligned}$$

Essentially, we multiply the first row of the first matrix with the first column of the second matrix, then the first row of the first matrix with the second column of the second matrix, and so on. When we have multiplied the first row of the first matrix with all the columns of the second matrix, we repeat the process with the second row of the first matrix and multiply it with all the rows of the second matrix and continue on to the third row of the first matrix and so on.
The resulting product matrix of an $m \times n$ and an $n \times p$ matrix will be an $m \times p$ matrix. Note that the number of **columns** on the **first matrix must be the same** as the number of **rows** of the **second matrix**.

1.7.1 Example

$$P = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2} \qquad Q = \begin{bmatrix} 5 & 1 & 2 & 2 \\ 3 & 3 & 1 & 2 \end{bmatrix}_{2\times 4}$$

We can form PQ but not QP.

$$PQ = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2} \begin{bmatrix} 5 & 1 & 2 & 2 \\ 3 & 3 & 1 & 2 \end{bmatrix}_{2\times 4} = \begin{bmatrix} 11 & 7 & 4 & 6 \\ 27 & 15 & 10 & 14 \\ 43 & 23 & 16 & 22 \end{bmatrix}_{3\times 4}$$

1.8 Identity matrix (I)

An identity matrix is a $n \times n$ matrix (c_{ij}) such that $c_{11} = c_{22} = c_{33} = \ldots = c_{nn} = 1$ and $c_{ij} = 0$. Basically, an identity matrix is a square matrix where the diagonal is all 1, and all the other elements are 0.

1.8.1 Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.8.2 Special property of the identity matrix

The identity matrix multiplied by any matrix will return the matrix back. For any matrix A, $IA = AI = A$.

1.9 Transpose matrix (A^T)

Given A is an $m \times n$ matrix, the transpose of A is the $n \times m$ matrix obtained as follows:
The i-th column of the transpose of A is the i-th row of A.
The transpose of A is denoted by A^T.

1.9.1 Examples

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} \quad B^T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

1.9.2 Transpose of a product matrix

The transpose of a product matrix AB is given by $B^T A^T$, i.e.

$$(AB)^T = B^T A^T$$

1.10 Symmetric matrices

A symmetric matrix is a **square matrix** where $A^T = A$.

1.10.1 Example

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & 6 \\ 5 & 6 & 9 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 0 & 6 \\ 5 & 6 & 9 \end{bmatrix}$$

1.11 Upper triangular matrix

An upper triangular matrix is a **square** matrix where $a_{ij} = 0$ for $i > j$. Basically, an upper triangular matrix has all elements **below** the diagonal as 0.
The transpose of a **lower** triangular matrix is an upper triangular matrix.

$$\begin{bmatrix} 1 & 4 & 5 & 6 \\ 0 & 1 & 6 & 8 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

1.12 Lower triangular matrix

A lower triangular matrix is a **square** matrix where $a_{ij} = 0$ for $i < j$. Basically, an upper triangular matrix has all elements **above** the diagonal as 0.
The transpose of an **upper** triangular matrix is a lower triangular matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 5 & 6 & 2 & 0 \\ 6 & 8 & 0 & 3 \end{bmatrix}$$

1.13 Diagonal matrix

A diagonal matrix is a **square** matrix where $a_{ij} = 0$ for $i \neq j$. Basically, a diagonal matrix has all elements that are not in the diagonal of the matrix as 0.
The transpose of a diagonal matrix is itself, and hence all diagonal matrices are symmetric, i.e. $D^T = D$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

1.14 Matrix form of linear equations

Given a system of N linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2N}x_N = b_2$$

$$\vdots$$

$$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{NN}x_N = b_N$$

a_{ij} is the constant coefficient of the unknown x_j in the i-th equation. b_i is the constant term in the i-th equation.
The system can be written in the matrix form $A\boldsymbol{x} = B$, where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}$$

1.14.1 Example

$$
\begin{aligned}
2x + 3y &= 10 \\
-x + y &= 0
\end{aligned}
$$

$$
\begin{bmatrix} 2x + 3y \\ -x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} 2 & 3 \\ -x & y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}
$$

1.15 Inconsistent system of linear equations

An inconsistent system of linear equations is a system that has no solution.

1.15.1 Example

$$
\begin{aligned}
x + y &= 10 \\
x + y &= 5
\end{aligned}
$$

1.16 Consistent system of linear equations

A consistent system of linear equations is a system that has only one solution, i.e. a **unique** solution, or infinitely many solutions.

1.17 Homogeneous system of linear equations

A homogeneous system of linear equations is a system of equations that **all** equate to 0.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1N}x_N = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2N}x_N = 0$$
$$\vdots$$
$$a_{N1}x_1 + a_{N2}x_2 + a_{N3}x_3 + \ldots + a_{NN}x_N = 0$$

In matrix form, it can be written as $A\boldsymbol{x} = \mathbf{0}$, where $\mathbf{0}$ is:

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

A homogeneous system of linear equations is **always** consistent, as the trivial solution, which is $x_1 = x_2 = \ldots = x_N = 0$, always exists.
Such a system has either only the trivial solution, or has infinitely many solutions (one of which is the trivial solution).

1.18 Vector ($\boldsymbol{x}$)

An N-th dimensional vector is a well-ordered set of N real numbers written in the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

For example, $\left[\begin{smallmatrix} 2 \\ 8 \\ 3 \\ 4 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 \\ 8 \\ 4 \\ 3 \end{smallmatrix}\right]$ are two different 4-th dimensional vectors.

1.18.1 Set of vectors

The set of **all** N-th dimensional vectors forms a vector space denoted by $\mathbb{R}^N$. For example, $\mathbb{R}^3$ is the set of all 3-dimensional vectors and $\left[\begin{smallmatrix} -1 \\ 0 \\ 2 \end{smallmatrix}\right]$ is a member of $\mathbb{R}^3$, i.e.

$$\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^3$$

1.19 Linear combinations of vectors

Let $\boldsymbol{u}$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k$ be vectors in $\mathbb{R}^n$.
$\boldsymbol{u}$ is a **linear combination** of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{v}_k$ if we can find real numbers $a_1, a_2, \ldots, a_{k-1}, a_k$ such that:

$$\boldsymbol{u} = a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \ldots + a_{k-1}\boldsymbol{v}_{k-1} + a_k\boldsymbol{v}_k$$

1.19.1 Example

Express (5, -3, -4) as a linear combination of (1, 1, 0), (3, 0, 1) and (0, 1, 3).
Form the equations:

$$x + 3y + 0z = 5$$

$$x + 0y + z = -3$$

$$0x + y + 3z = -4$$

Solving using Gauss elimination:

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 1 & 0 & 1 & -3 \\ 0 & 1 & 3 & -4 \end{bmatrix} \cong \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -3 & 1 & -8 \\ 0 & 1 & 3 & -4 \end{bmatrix} \cong \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -3 & 1 & -8 \\ 0 & 0 & 10 & -20 \end{bmatrix} \cong \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & -3 & 1 & -8 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad z = -2, \quad y = 2, \quad x = -1$$

1.20 Linearly independent vectors

Let $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{p-1}, \boldsymbol{w}_p$ be vectors in $\mathbb{R}^n$.
We say that $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{p-1}, \boldsymbol{w}_p$ are **linearly independent** if we cannot find any one of these vectors to be a linear combination of the other vectors. To do this, form the homogeneous system:

$$c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2 + \ldots + c_{p-1}\boldsymbol{w}_{p-1} + c_p\boldsymbol{w}_p = 0$$

If $c_1 = c_2 = \ldots = c_{p-1} = c_p = 0$ is the only solution of the system then the vectors are linearly independent.

1.21 Inverse matrices (A^{-1})

A square $n \times n$ matrix B is said to be an inverse of a square $n \times n$ matrix A if:

$$AB = BA = I$$

B can also be denoted as A^{-1}.

1.22 Invertible matrices

If a matrix A has an inverse, then we say A is invertible. The inverse of A is denoted by A^{-1}. An invertible matrix has only one unique inverse.

1.22.1 Finding the inverse matrix

Write the matrix A and the identity matrix I side by side and reduce the matrix A to the identity matrix I by using Gauss-Jordan elimination, making sure that all row operations are also applied to the matrix beside A. When A has been reduced to the identity matrix I, the resulting matrix to the side of A is the inverse matrix of A, or A^{-1}.

1.22.2 Inverse of a product matrix

The inverse of a product matrix AB is given by $B^{-1}A^{-1}$, i.e.

$$(AB)^{-1} = B^{-1}A^{-1}$$

1.22.3 Inverse of a 2×2 matrix

Let:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{R}$$

Then A is invertible if and only if $ad - bc \neq 0$, in which case we have:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1.23 Singular matrices

Singular matrices are matrices that are not invertible.

1.24 Minor

The minor M_{ij} of the entry a_{ij} is the determinant of the matrix that remains after the i-th row and j-th column are removed from A.
Some examples:

$$M_{11} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

$$M_{12} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

$$M_{13} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$M_{21} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$$

$$M_{22} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$$

$$M_{23} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$$

$$M_{31} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$M_{32} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$M_{33} \text{ of } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

1.25 Cofactor

The cofactor C_{ij} of the entry a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The factor $(-1)^{i+j}$ in the cofactor $(-1)^{i+j} M_{ij}$ depends on the position of the entry in the matrix:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

1.26 Cofactor expansion

The cofactor expansion along a row or column of A is obtained by multiplying each entry of the row or column with its cofactor, and adding those products together, i.e.
Along the i-th row:

$$\sum_{j=1}^{n} a_{ij} C_i j$$

Along the j-th column:

$$\sum_{i=1}^{n} a_{ij} C_i j$$

1.26.1 Example

Let

$$A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix}$$

Cofactor expansion along first row

$A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix}$ Cofactor exp.:

$$aC_{11} + bC_{12} + cC_{13} = a(-1)^{1+1}M_{11} + b(-1)^{1+2}M_{12} + c(-1)^{1+3}M_{13}$$

$$= aM_{11} - bM_{12} + cM_{13} = a\begin{vmatrix} f & g \\ i & j \end{vmatrix} - b\begin{vmatrix} d & g \\ h & j \end{vmatrix} + c\begin{vmatrix} d & f \\ h & i \end{vmatrix} =$$

$$\begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix} \quad \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix}$$

$$= a(fj - ig) - b(dj - hg) + c(di - hf) =$$

$$= afj + bhg + cdi - aig - bdj - chf = \det A$$

Cofactor expansion along second column

$$A = \begin{bmatrix} a & b & c \\ d & f & g \\ h & i & j \end{bmatrix}$$

Cofactor exp.:

$$bC_{12} + fC_{22} + iC_{32} = b(-1)^3M_{12} + f(-1)^4M_{22} + i(-1)^5M_{32}$$

$$= -b\begin{vmatrix} d & g \\ h & j \end{vmatrix} + f\begin{vmatrix} a & c \\ h & j \end{vmatrix} - i\begin{vmatrix} a & c \\ d & g \end{vmatrix} =$$

$$= -b(dj - hg) + f(aj - hc) - i(ag - dc) =$$

$$= bhg + faj + idc - bdj - fhc - iag = \det A$$

$$afj + bgh + cdi - hfc - iga - jdb$$

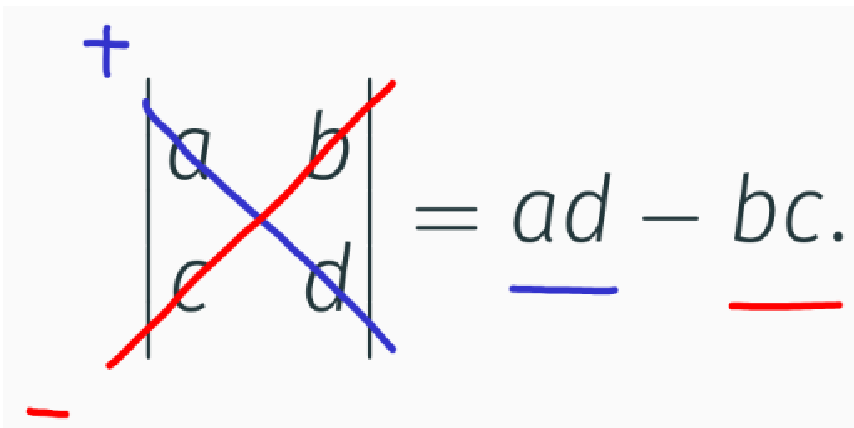
1.26.2 Theorem

Let A be an $n \times n$ matrix. The cofactor expansion along any of its rows or any of its columns will yield the same number.

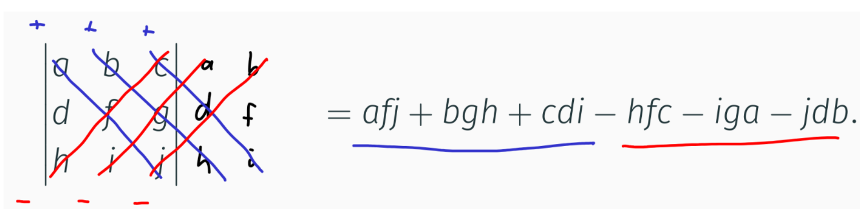
1.27 Determinant of a matrix (det A)

The **determinant** $\det A$ or $|A|$ of a **square** matrix A is a real number.

1.27.1 Determinant of a 2×2 matrix



1.27.2 Determinant of a 3×3 matrix



1.27.3 Definition

Let A be an $n \times n$ matrix.

- If $n = 1$, i.e. $A = [a]$, we define $\det A = a$
- If $n \geq 2$, we define $\det A$ as the number obtained from the cofactor expansion along any row or column of A.

1.27.4 Triangular matrices and diagonal matrices

The determinant of a triangular matrix or a diagonal matrix is the product of all the diagonal elements in the matrix.

1.27.5 Relationship to invertibility

Let A be an $n \times n$ matrix. A is invertible **if and only if** $\det A \neq 0$. Hence, A is singular **if and only if** $\det A = 0$.

1.27.6 Relationship to homogeneous systems

Let $A\boldsymbol{x} = \mathbf{0}$ be a homogeneous system.

- $A\boldsymbol{x} = \mathbf{0}$ has a unique solution given by $\boldsymbol{x} = \mathbf{0}$ **if and only if** $\det A \neq 0$.
- $A\boldsymbol{x} = \mathbf{0}$ has infinitely many solutions **if and only if** $\det A = 0$.

1.27.7 Rules for the manipulation of determinants

$$\det(AB) = \det A \cdot \det B$$
$$\det A = \det A^T$$
$$\det A^{-1} = \frac{1}{\det A}$$
$$\det(kA) = k \det A$$

1.28 Eigenvalues of a matrix (λ)

Given an $n \times n$ matrix A, the eigenvalue is the λ term when finding an $n \times 1$ vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where λ is a real or complex number.

1.29 Eigenvector of a matrix ($\boldsymbol{x}$)

Given an $n \times n$ matrix A, the eigenvector is the $n \times 1$ vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$, where λ is a real or complex number.

1.30 Characteristic equation of a matrix

Given an $n \times n$ matrix A, the equation below is the characteristic equation of matrix A:

$$\det(A - \lambda I) = 0$$

This equation is used to find the eigenvalues (λ) of A. The characteristic equation is a polynomial equation of order n and the matrix A can have up to n distinct eigenvalues.

1.31 Diagonalisable matrix

An $n \times n$ matrix A is said to be **diagonalisable** if there exists an invertible matrix P such that D is a diagonal matrix:

$$D = P^{-1}AP$$

$$A = PDP^{-1}$$

Where:

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

$$P = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots \boldsymbol{x}_{n-1} & \boldsymbol{x}_n \end{bmatrix}$$

λ are the eigenvalues of the matrix and $\boldsymbol{x}$ are the corresponding eigenvectors of the matrix.
A matrix is diagonalisable **if and only if** A has n linearly independent eigenvectors.

1.31.1 For symmetric matrices

If A is a symmetric matrix, i.e. $A = A^T$, then:

$$P^{-1} = P^T$$

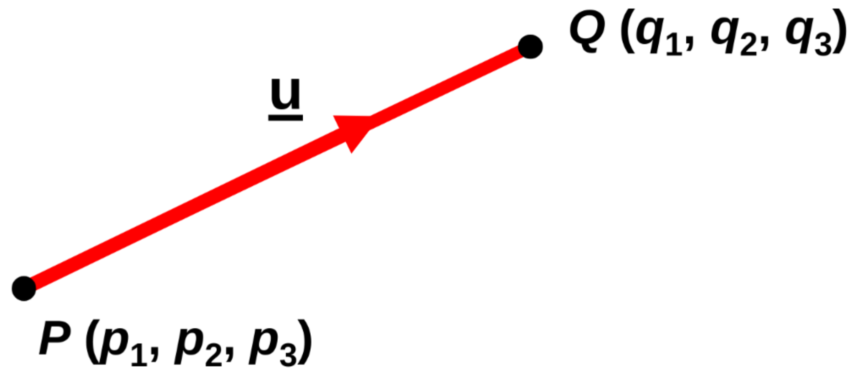
$$A = PDP^T$$

1.32 Vector

A vector is a quantity that has magnitude and direction.

1.32.1 Notation

Let $\boldsymbol{u}$ be a vector given by the directed line $\overrightarrow{PQ}$.



$$\boldsymbol{u} = \overrightarrow{PQ} = (q_1 - p_1)\boldsymbol{i} + (q_2 - p_2)\boldsymbol{j} + (q_3 - p_3)\boldsymbol{k}$$

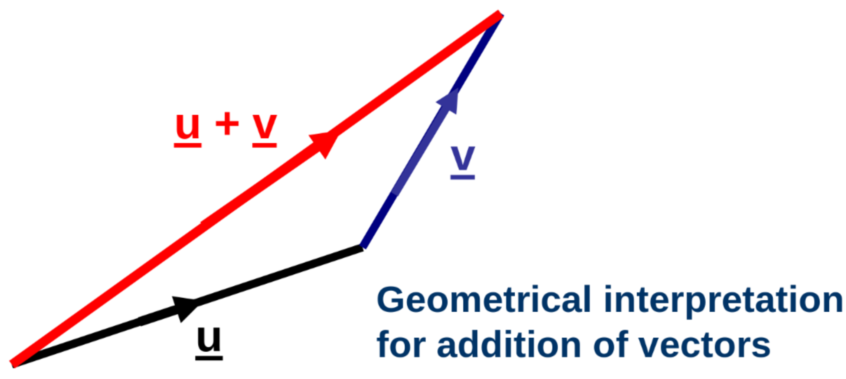
1.33 Vector addition

If

$$\boldsymbol{u} = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} = (a, b, c)$$
$$\boldsymbol{v} = p\boldsymbol{i} + q\boldsymbol{j} + r\boldsymbol{k} = (p, q, r)$$

Then:

$$\boldsymbol{u} + \boldsymbol{v} = (a + p)\boldsymbol{i} + (b + q)\boldsymbol{j} + (c + r)\boldsymbol{k} = (a + p, b + q, c + r)$$



1.34 Magnitude of a vector ($|\boldsymbol{x}|$)

If $\boldsymbol{x} = (x_1, x_2, x_3)$, then:

$$|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

1.35 Norm of a vector ($||\boldsymbol{x}||$)

The norm of a vector gives its magnitude.

$$\text{If } \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

Then:

$$||\boldsymbol{x}|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

1.36 Multiplication of a vector with a scalar

Let $\boldsymbol{u}$ be a vector (a, b, c) and $k \in \mathbb{R}$, then:

$$k\boldsymbol{u} = k(a, b, c) = (ka, kb, kc)$$

$$|k\boldsymbol{u}| = |k||\boldsymbol{u}|$$

If k is positive, then $k\boldsymbol{u}$ points in the **same** direction as $\boldsymbol{u}$.
If k is negative, then $k\boldsymbol{u}$ points in the **opposite** direction as $\boldsymbol{u}$.

1.37 Unit vector

Let $\boldsymbol{u}$ be a vector (a, b, c), then the **unit** vector $\boldsymbol{v}$ is given by:

$$\begin{aligned} \boldsymbol{v} &= \frac{1}{|\boldsymbol{u}|}\boldsymbol{u} \\ &= \frac{1}{\sqrt{a^2 + b^2 + c^2}}(a, b, c) \\ &= \frac{a}{a^2 + b^2 + c^2}\boldsymbol{i} + \frac{b}{a^2 + b^2 + c^2}\boldsymbol{j} + \frac{c}{a^2 + b^2 + c^2}\boldsymbol{k} \end{aligned}$$

We say that $\boldsymbol{v}$ is obtained by normalising $\boldsymbol{u}$.

1.38 Dot product

Let $\boldsymbol{u} = (a, b, c)$ and $\boldsymbol{v} = (p, q, r)$ be two vectors. We define the dot product $\boldsymbol{u} \cdot \boldsymbol{v} = ap + bq + cr$.

$$(a, b, c) \cdot (p, q, r) = ap + bq + cr$$

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{u} &= (a, b, c) \cdot (a, b, c) \\ &= a^2 + b^2 + c^2 \\ &= |\boldsymbol{u}|^2 \end{aligned}$$

$$(\alpha \boldsymbol{u} + \beta \boldsymbol{v}) \cdot (\gamma \boldsymbol{p} + \epsilon \boldsymbol{q}) = \alpha\gamma \boldsymbol{u} \cdot \boldsymbol{p} + \alpha\epsilon \boldsymbol{u} \cdot \boldsymbol{p} + \beta\gamma \boldsymbol{v} \cdot \boldsymbol{p} + \beta\epsilon \boldsymbol{v} \cdot \boldsymbol{p}$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta$$

Where:

- θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$

If $\boldsymbol{u} \cdot \boldsymbol{v}$ is zero then $\boldsymbol{u}$ and $\boldsymbol{v}$ are perpendicular.

1.39 Cross product

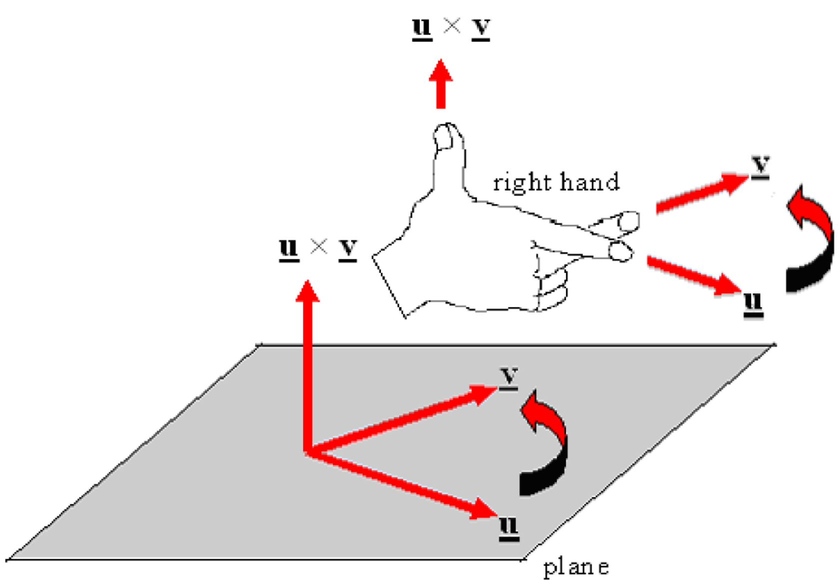
Let $\boldsymbol{u} = (a, b, c)$ and $\boldsymbol{v} = (p, q, r)$ be two vectors. The cross product of a vector is the determinant of the matrix when laying the vectors out as shown below:

$$
\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= (a, b, c) \times (p, q, r) \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a & b & c \\ p & q & r \end{vmatrix} \\
&= (br - cq)\boldsymbol{i} + (cp - ar)\boldsymbol{j} + (aq - bp)\boldsymbol{k} \\
&= (br - cq, cp - ar, aq - bp) \\
&= |\boldsymbol{u}||\boldsymbol{v}| \cos\theta
\end{aligned}
$$

Where:

- θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$

The resulting vector $\boldsymbol{u} \times \boldsymbol{v}$ from the cross product is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$.



1.40 Plane

The equation of a plane is given in the forms below:

$$ax + by + cz = k, k \in \mathbb{R}$$

$$\vec{n} \cdot \vec{r} = 0$$

Where:

- $\vec{n}$ is the normal vector of the plane, given by (a, b, c)
- $\vec{r}$ is the position vector of any point on the plane

1.40.1 Parametrising the plane

$$x = \alpha_1 s + \beta_1 u + \gamma_1$$

$$y = \alpha_2 s + \beta_2 u + \gamma_2$$

$$z = \alpha_3 s + \beta_3 u + \gamma_3$$

Where:

- s and u are free parameters, i.e. $s, u \in \mathbb{R}$.
- x, y and z are expressed in terms of linear functions of s and u.

1.41 Surfaces

Points (x, y, z) on a surface in 3D space may be described by a single equation in x, y and z, which is:

$$F(x, y, z) = k, k \in \mathbb{R}$$

In parametric form, the x, y and z coordinates on a surface are described by functions of two parameters s and u, namely:

$$x = f(s, u)$$

$$y = g(s, u)$$

$$z = h(s, u)$$

The functions above are the solutions of $F(x, y, z) = k$. For a plane, f, g and h are linear functions of s and u.

1.41.1 Examples

Plane (flat) surface:

$$2x + y + z = 1$$

Spherical surface:

$$(x-1)^2 + (y-2)^2(z-3)^2 = 16$$

1.41.2 Parametrising the surface

Let the surface S be given by $(x-1)^2+(y-2)^2+(z-3)^2 = 16$. Let $z-3 = \rho$.

$$\begin{aligned}(x-1)^2 + (y-2)^2 + (z-3)^2 &= 16 \\ (x-1)^2 + (y-2)^2 + \rho^2 &= 16 \\ (x-1)^2 + (y-2)^2 &= 16 - \rho^2 \qquad (1)\end{aligned}$$

The left-hand side of the equation is always positive, hence:

$$16 - \rho^2 \geq 0$$

$$-4 \leq \rho \leq 4$$

Using the trigonometric identity:

$$\cos^2\theta + \sin^2 = 1$$

Let $a = \sqrt{16-\rho^2}$:

$$(a\cos\theta)^2 + (a\sin\theta)^2 = a^2 \qquad (2)$$

Comparing (1) and (2):

$$x - 1 = \sqrt{16-\rho^2}\cos\theta$$

$$y - 2 = \sqrt{16-\rho^2}\sin\theta$$

Hence, a possible parametric representation is:

$$\left.\begin{array}{l} x = 1 + \sqrt{16-\rho^2}\cos\theta \\ y = 2 + \sqrt{16-\rho^2}\sin\theta \\ z = 3 + \rho \end{array}\right\} \text{ for } \begin{array}{c} -4 \leq \rho \leq 4 \\ 0 \leq \theta < 2\pi \end{array}$$

1.42 Curves in 3D

A curve may be formed by the intersection of two surfaces. As a surface is described by an equation in x, y and z, finding all the points on a curve is like solving 2 equations in 3 unknowns x, y and z. One of the unknowns can be set to be a free parameter to solve for the other two unknowns (in terms of the free parameter).
Hence, all points on a curve can be expressed in parametric form as:

$$x = F(s)$$

$$y = G(s)$$

$$z = H(s)$$

For a straight line:

$$x = at + p$$

$$y = bt + q$$

$$z = ct + r$$

Where:

- s and t are free parameters, i.e. $s, t \in \mathbb{R}$

1.43 Derivative of a vector function

Let f be a scalar function $f(x)$ and $\boldsymbol{F}$ be a vector function $\boldsymbol{F}(u)$. The derivative of a scalar function is:

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Where:

- $f(x+h) - f(x)$ is the change in output
- h is the change in input

Likewise, the derivative of the vector function follows as:

$$\boldsymbol{F}'(u) = \frac{d\boldsymbol{F}}{du} = \lim_{h \to 0} \frac{\boldsymbol{F}(u+h) - \boldsymbol{F}(u)}{h}$$

For $\boldsymbol{F}(u) = (p(u), q(u), r(u))$:

$$\frac{d\boldsymbol{F}}{du} = \left(\frac{dp}{du}, \frac{dq}{du}, \frac{dr}{du} \right)$$

1.43.1 Example

$$\begin{aligned}
\boldsymbol{F}(u) &= (\sin 2u, u^3 + 2u^2, 2u) \\
\frac{d\boldsymbol{F}}{du} &= (2\cos 2u, 3u^2 + 4u, 2) \\
\frac{d^2\boldsymbol{F}}{du^2} &= (-4\sin 2u, 6u + 4, 0)
\end{aligned}$$

1.43.2 Product rule

Let g be a scalar function of one variable x and $\boldsymbol{G}$ is a vector function of x.

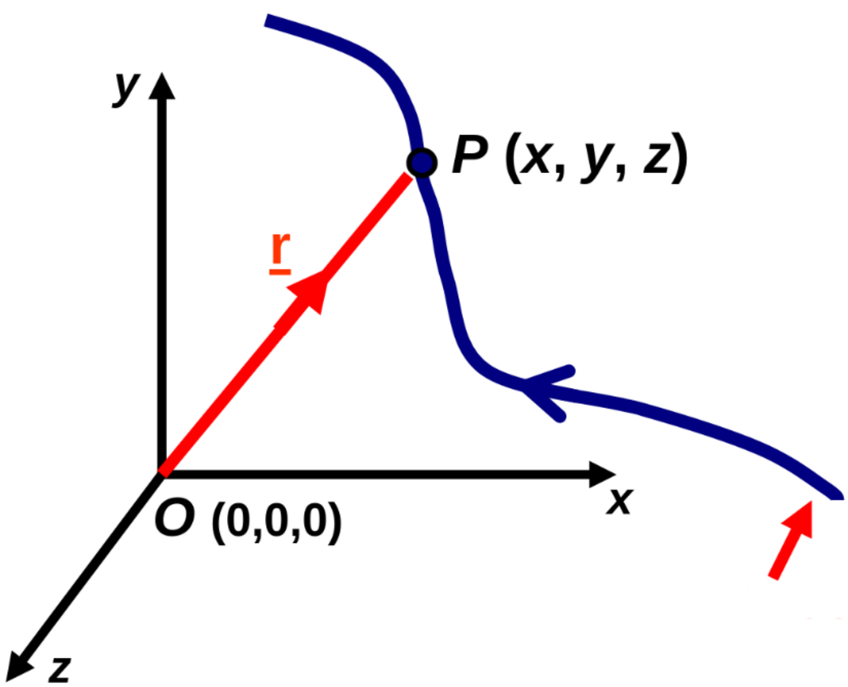
$$\frac{d}{dx} g(x)\boldsymbol{G}(x) = g(x)\frac{d\boldsymbol{G}}{dx} + \frac{dg}{dx}\boldsymbol{G}$$

1.43.3 Chain rule

Let F be a vector function given by $F(u) = f(x(u), y(u), z(u))$:

$$\frac{dF}{du} = \frac{\partial f}{\partial x} \cdot \frac{dx}{du} + \frac{\partial f}{\partial y} \cdot \frac{dy}{du} + \frac{\partial f}{\partial z} \cdot \frac{dz}{du}$$

1.44 Motion of a particle



1.44.1 Position of the particle

The position of a particle is changing with respect to time (t):

$$\begin{aligned} x &= xt \\ y &= yt \\ z &= zt \end{aligned}$$

1.44.2 Position or displacement of the particle

The position or displacement of the particle is with respect to the origin (O):

$$\begin{aligned} \boldsymbol{r}(t) &= (x(t) - 0)\boldsymbol{i} + (y(t) - 0)\boldsymbol{j} - (z(t) - 0)\boldsymbol{k} \\ &= (x(t), y(t), z(t)) \\ &= x(t)\boldsymbol{i} + y(t) + \boldsymbol{j} + z(t)\boldsymbol{k} \end{aligned}$$

1.44.3 Velocity of the particle

Velocity is the rate of change of displacement with respect to time (t):

$$\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

1.44.4 Speed of the particle

Speed is the magnitude of the velocity:

$$\left|\frac{d\boldsymbol{r}}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

1.44.5 Acceleration of the particle

Acceleration is the rate of change of velocity with respect to time.

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2x}{dt^2}\boldsymbol{i} + \frac{d^2y}{dt^2}\boldsymbol{j} + \frac{d^2z}{dt^2}\boldsymbol{k}$$

1.45 Newton's second law

Let $\boldsymbol{F} = (F_x, F_y, F_z)$:

$$\begin{aligned}
\boldsymbol{F} &= m\frac{d^2\boldsymbol{r}}{dt^2} \\
&= m\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt}\right) \\
&= ma
\end{aligned}$$

$$\begin{aligned}
F_x &= m\frac{d^2x}{dt^2} \\
F_y &= m\frac{d^2y}{dt^2} \\
F_z &= \frac{d^2z}{dt}
\end{aligned}$$

Where:

- $\boldsymbol{F}$ is the force vector on the object
- m is the mass of the object
- $\boldsymbol{r}$ is the displacement vector of the object
- t is the time
- a is the acceleration of the object

1.46 Vector differential operator (∇)

The vector differential operator is defined as:

$$\begin{aligned}
\nabla &= \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)
\end{aligned}$$

1.47 Gradient operator (grad)

The gradient operator is essentially the same as the ∇ operator.

$$\text{grad } f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

1.48 Normal vectors

1.48.1 Curves

A curve in 2D space is given in the form $F(x, y) = c, c \in \mathbb{R}$. **A** normal vector to the curve is given by:

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right)$$

1.48.2 Surfaces

A surface in 3D space is given in the form $F(x, y, z) = c, c \in \mathbb{R}$. **A** normal vector to the curve is given by:

$$\nabla F|_{(x,y,z)=(x_0,y_0,z_0)}$$

1.49 Divergence operator (div)

Let $\boldsymbol{F}$ be a vector function:

$$\begin{aligned} \text{div } \boldsymbol{F} &= \nabla \cdot \boldsymbol{F} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \boldsymbol{F} \end{aligned}$$

1.50 Curl operator (curl)

Let $\boldsymbol{F}$ be a vector function:

$$\begin{aligned} \text{curl } \boldsymbol{F} &= \nabla \times \boldsymbol{F} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \boldsymbol{F} \end{aligned}$$

1.51 Laplacian operator (∇^2)

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2} \right)$$

1.52 Leibniz theorem

If we can find a function $F(x)$ such that $\frac{dF}{dx} = f(x)$, then:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

1.53 Line element (ds)

For a line $\boldsymbol{r}$ that can be parametrised with t:

$$ds = \left\|\frac{d\boldsymbol{r}}{dt}\right\|$$

For example:

$$\begin{aligned}
\boldsymbol{r} &= (x, y, z) \\
ds &= \left\|\frac{d}{dt}(x, y, z)\right\| \\
&= \left\|\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)\right\| \\
&= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
\end{aligned}$$

1.54 Length of a curve (arc length)

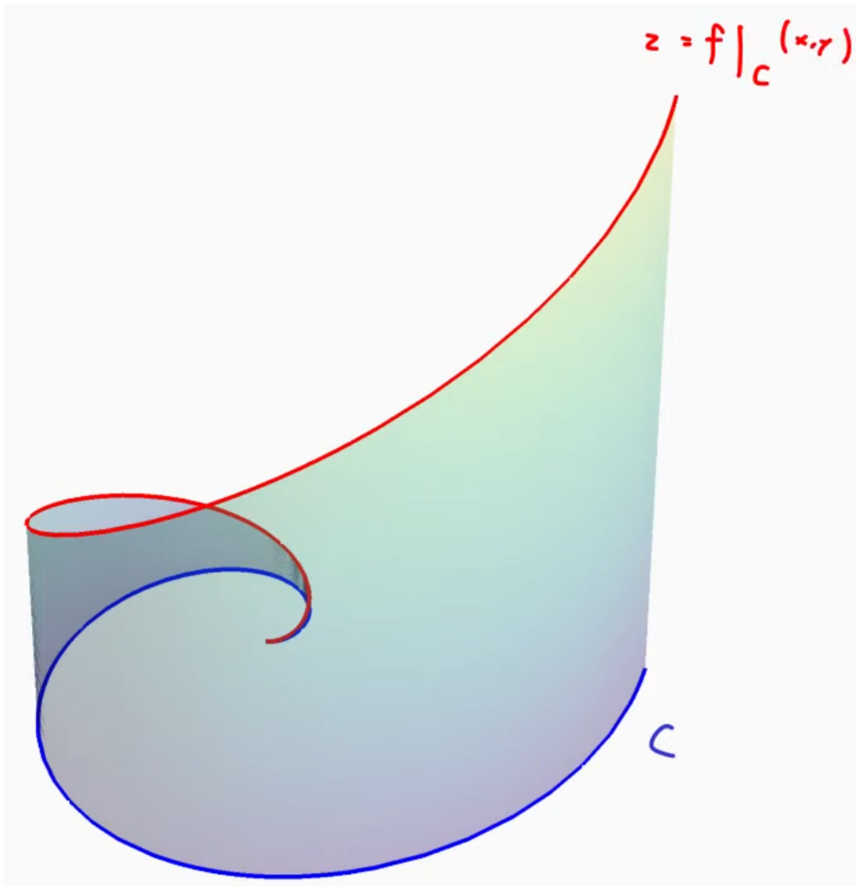
For a smooth curve C:

$$\int_C ds$$

Where:

- $\int_C ds$ in context of a particle's motion is the distance travelled by the particle.

1.55 Line integral of a scalar function (area under a curve)



For a smooth curve C and a scalar function f:

$$\int_C f(\boldsymbol{x})\,ds = \int_C f(x,y,z)\,ds$$

1.56 Line integral of a vector function (work done)

The line integral of a vector function can be thought of as the work done by the vector function. For a smooth curve C parametrised by $\boldsymbol{x} = \boldsymbol{r}(t), t \in [a, b]$ and a vector function $\boldsymbol{F}$:

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\boldsymbol{r} &= \int_C \boldsymbol{F} \cdot \boldsymbol{U} \, ds \\ &= \int_a^b \boldsymbol{F}(\boldsymbol{r}(t)) \cdot \boldsymbol{r'}(t) \, dt\end{aligned}$$

Where:

- $d\boldsymbol{r}$ is the infinitesimal position or displacement vector in the context of a particle's motion.
- $\boldsymbol{U}$ is the unit vector representing the direction of travel in the context of a particle's motion.
- ds is the infinitesimal distance of each section of the curve, or the infinitesimal arc length of the curve.
- $\boldsymbol{r}$ is the position vector of the particle in the context of a particle's motion.
- $\boldsymbol{r'}$ is the derivative of the position vector of the particle with respect to time t in the context of a particle's motion. In other words, $\boldsymbol{r'}$ is the velocity of the particle.

1.57 Infinitesimal surface area element (dS)

If the equation for a surface S can be written as $z = f(x, y)$, then the relationship between the infinitesimal **surface area** element dS and the infinitesimal **area** element dA is:

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA$$

1.58 Surface area of a surface

For a smooth surface S:

$$\iint_S dS$$

Where:

- dS is the infinitesimal surface area element.

1.59 Surface integral of a scalar function

For a smooth surface S and a scalar function $f(\boldsymbol{x})$, where $\boldsymbol{x} = (x, y) = (r\cos\theta, r\sin\theta)$:

$$\begin{aligned}\iint_S f(\boldsymbol{x})\, dS &= \iint_R f(x, y) dy dx \\ &= \iint_R f(r, \theta)\, r dr d\theta\end{aligned}$$

Where:

- dS is the infinitesimal surface area element.

1.60 Surface integral of a vector function (flux)

The surface integral of a vector function can be thought of as the flux through the surface. For a smooth surface S and a vector function $\boldsymbol{F}$:

$$\iint_S \boldsymbol{F} \cdot \hat{n}\, dS$$

Where:

- $\hat{n}$ is the unit normal vector to the surface, i.e. the vector is perpendicular to the surface, and has a magnitude of 1.
- dS is the infinitesimal surface area element.

1.61 Volume integral

For a volume T:

$$\iiint_T dV$$

Where:

- dV is the infinitesimal volume element.

1.62 Green's theorem

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy$$

Where:

- C is a closed curve that is positively oriented. A positively oriented curve is a curve that has the region R bounded by the curve on the **left** side as we walk along the curve, with our head facing the direction of the curve.
- R is the region bounded by the closed curve C.

1.63 Stoke's theorem

Stoke's theorem essentially states that the line integral of a vector function, or the work done by the vector function, is equal to surface integral of the curl of the vector function. It is the multidimensional version of Green's theorem. For a smooth curve C and a vector function $\boldsymbol{F}$:

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint_S (\text{curl } \boldsymbol{F}) \cdot \hat{n}\,dS = \iint_S (\nabla \times \boldsymbol{F}) \cdot \hat{n}\,dS$$

Where:

- $\hat{n}$ is the unit normal vector to the surface, i.e. a vector that is perpendicular to the surface with a magnitude of 1.
- dS is the infinitesimal surface area element.

1.64 Gauss' divergence theorem

Gauss' divergence theorem essentially states that the surface integral of a vector function, or the flux through a surface, is equal to the volume integral of the divergence of the vector function. For a smooth surface S and a vector function $\boldsymbol{F}$:

$$\iint_S F \cdot \hat{n}\,dS = \iiint_T \text{div } \boldsymbol{F}\,dV = \iiint_T \nabla \cdot \boldsymbol{F}\,dV$$

Where:

- $\hat{n}$ is the unit normal vector to the surface, i.e. a vector that is perpendicular to the surface with a magnitude of 1.
- dS is the infinitesimal surface area element.
- dV is the infinitesimal volume element.

1.65 Conservative vector fields

1.65.1 Two-dimensional vector fields

A vector field $\boldsymbol{F}(x, y) = f(x, y)\boldsymbol{i} + g(x, y)\boldsymbol{j}$ is considered **conservative** if:

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

1.65.2 Vector fields with 3 or more dimensions

A vector field $\boldsymbol{F}$ is considered **conservative** if:

$$\text{curl } \boldsymbol{F} = \nabla \times F = \boldsymbol{0}$$

1.66 Potential function

- If the vector field $\boldsymbol{F}(x, y) = f(x, y)\boldsymbol{i} + g(x, y)\boldsymbol{j}$ is conservative, we can find a function $\phi(x, y)$ such that $\frac{\partial \phi}{\partial x} = f(x, y)$ and $\frac{\partial \phi}{\partial y} = g(x, y)$.
- This function $\phi(x, y)$ is called the **potential function** of $\boldsymbol{F}(x, y)$.
- With this potential function, we can easily get the integral of the vector field using the Leibniz theorem.

1.67 Periodic functions

Let $f(x)$ be a well-defined function for $-\infty < x < \infty$. $f(x)$ is said to be periodic with period p, where p is a non-zero constant, if $f(x)$ satisfies the property:

$$f(x + p) = f(x), x \in \mathbb{R}$$

1.67.1 Sum and product of periodic functions

Let $f(x)$ and $g(x)$ be periodic with period p. Then:

- $f(x) + g(x)$ is periodic with period p
- $f(x)g(x)$ is also periodic with period p

1.67.2 Integral of a periodic function

If $f(x)$ is periodic with period p, then:

$$\int_{x=c}^{x=c+p} f(x)\, dx \text{ has the same value no matter what } c \text{ is.}$$

1.67.3 Examples

$$f(x) = \sin(x), -\infty < x < \infty$$
$$g(x) = \cos(x), -\infty < x < \infty$$

Both of the functions above are periodic with periods of 2π, because:

$$\sin(x + 2\pi) = \sin(x), \quad \cos(x + 2\pi) = \cos(x)$$

1.68 Fourier series of a periodic function

The Fourier series of a periodic function $f(x)$ with period $2L, L \in \mathbb{R}^+$, is given by the series:

$$a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

Where:

$$a_0 = \frac{1}{2L} \int_{\alpha}^{\alpha+2L} f(x)\, dx$$
$$a_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos\left(\frac{n\pi x}{L}\right)\, dx$$
$$b_n = \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$
$$n = 1, 2, 3, \ldots$$

1.69 Condition for a periodic function to be equal to its Fourier series

Let a periodic function be $f(x)$. For $f(x)$ to be equal to its Fourier series:

$$a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi s}{L}\right) + b_n \sin\left(\frac{n\pi s}{L}\right) \right\} = \frac{1}{2}\left(\lim_{x \to s^-} f(x) + \lim_{x \to s^+} f(x) \right)$$

If $f(x)$ is continuous at $x = s$, then the Fourier series of $f(x)$ at $x = s$ is equal to $f(x)$, i.e.

$$a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} = f(x) \text{ where } f(x) \text{ is continuous}$$

1.70 Odd functions

A function $f(x)$ is said to be **odd** over the interval $-a < x < a, a \in \mathbb{R}^+$ if $f(-x) = -f(x)$ for $-a < x < a$.

If $f(x)$ is **odd** over $-a < x < a$, then:

$$\int_{-a}^{a} f(x)\,dx = 0$$

1.70.1 Example

$$S(x) = \sin\left(\frac{n\pi x}{L}\right) \text{ is odd over } -L < x < L$$

1.71 Even functions

A function $f(x)$ is said to be **even** over the interval $-a < x < a, a \in \mathbb{R}^+$ if $f(-x) = f(x)$ for $-a < x < a$.

If $f(x)$ is **even** over $-a < x < a$, then:

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{2a} f(x)\,dx$$

1.71.1 Example

$$C(x) = \cos\left(\frac{n\pi x}{L}\right) \text{ is even over } -L < x < L$$

1.72 Sum and product of even and odd functions

1.72.1 $f(x)$ and $g(x)$ are both odd over $-a < x < a$

- $f(x) + g(x)$ is also **odd** over $-a < x < a$
- $f(x)g(x)$ is **even** over $-a < x < a$

1.72.2 $f(x)$ and $g(x)$ are both even over $-a < x < a$

- $f(x) + g(x)$ is also **even** over $-a < x < a$
- $f(x)g(x)$ is also **even** over $-a < x < a$

1.72.3 $f(x)$ is odd while $g(x)$ is even over $-a < x < a$

- $f(x) + g(x)$ is neither odd nor even over $-a < x < a$
- $f(x)g(x)$ is **odd** over $-a < x < a$

1.73 Fourier series of an odd periodic function

The Fourier series of an **odd** periodic function $f(x)$ is called the Fourier sine series, and is given by:

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

Where:

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \quad \because f(x) \text{ and } \sin\left(\frac{n\pi x}{L}\right) \text{ are both odd} \end{aligned}$$

1.73.1 Extending a continuous function

Suppose $g(x)$ is a continuous function over $0 < x < l, l \in \mathbb{R}$. We can extend $g(x)$ to become an **odd** periodic function of period $2l$ by letting $g(x) = -g(-x)$:

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l$$

Where:

$$b_n = \frac{2}{l}\int_{0}^{l} g(x)\sin\left(\frac{n\pi x}{l}\right)\,dx$$

$$n = 1, 2, 3, \ldots$$

1.74 Fourier series of an even periodic function

The Fourier series of an **even** periodic function $f(x)$ is called the Fourier cosine series, and is given by:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where:

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ &= \frac{1}{L}\int_{0}^{L} f(x)\,dx \quad \because f(x) \text{ is even} \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \quad \because f(x) \text{ and } \cos\left(\frac{n\pi x}{L}\right) \text{ are both even} \end{aligned}$$

1.74.1 Extending a continuous function

Suppose $g(x)$ is a continuous function over $0 < x < l, l \in \mathbb{R}$. We can extend $g(x)$ to become an **even** periodic function of period $2l$ by letting $g(x) = g(-x)$:

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right), \quad 0 < x < l$$

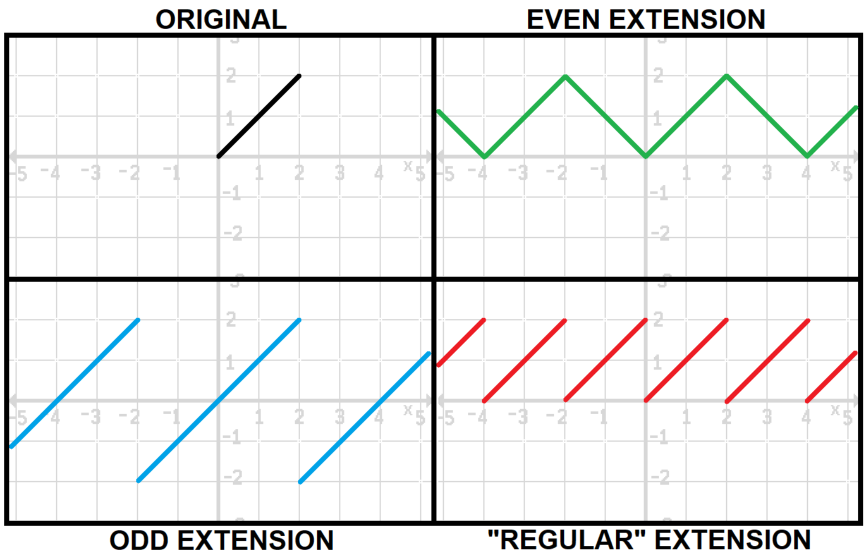
Where:

$$a_0 = \frac{1}{l}\int_{0}^{l} g(x)\,dx$$

$$a_n = \frac{2}{l}\int_{0}^{l} g(x)\cos\left(\frac{n\pi x}{l}\right)\,dx$$

$$n = 1, 2, 3, \ldots$$

1.75 Periodic extensions of continuous functions



1.76 Complex Fourier series

Let $f(x)$ be a periodic function with period $2L, L \in \mathbb{R}^+$. The complex Fourier series of $f(x)$ is given by:

$$c_0 + \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad i = \sqrt{-1}$$

Where:

$$c_0 = \frac{1}{2L} \int_{\alpha}^{\alpha+2L} f(x)\, dx$$

$$c_n = \frac{1}{2L} \int_{\alpha}^{\alpha+2L} f(x) e^{-\frac{in\pi x}{L}}\, dx \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

1.77 Laplace transform of a function ($\mathcal{L}$)

Let $f(t)$ be a well-defined function for $t \geq 0$. The Laplace transform is given by:

$$\mathcal{L}\{f(t)\} = \int_{t=0}^{t \to \infty} f(t) e^{-st}\, dt$$

Where:

- s is the Laplace transform parameter

1.78 Inverse Laplace transform ($\mathcal{L}^{-1}$)

The inverse Laplace transform is given by:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

1.79 Heaviside unit step function ($u(t-a)$)

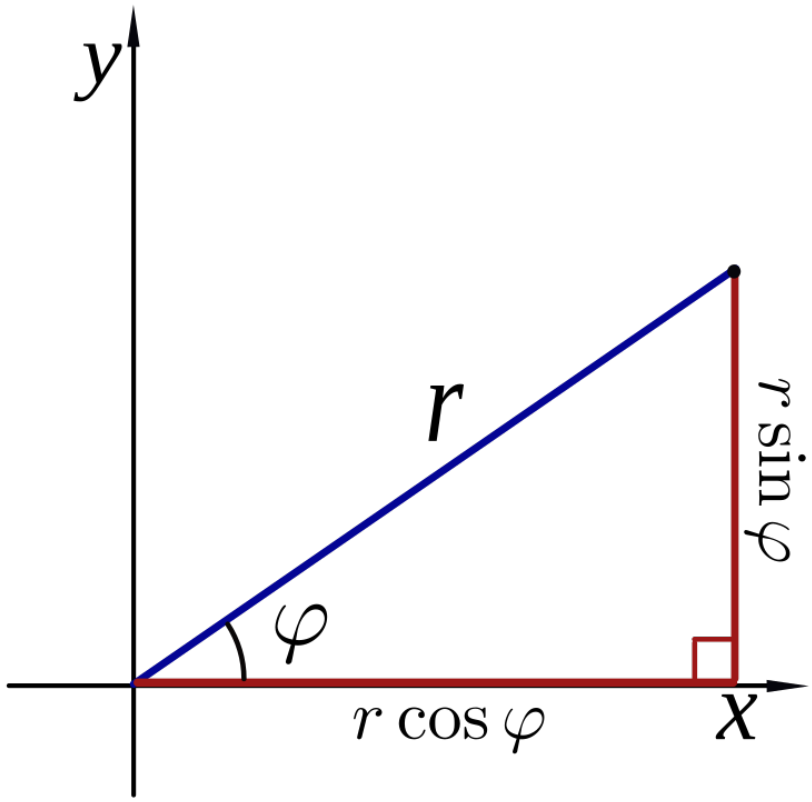
$$u(t-a) = \begin{cases} 0 & \text{for } t \leq a \\ 1 & \text{for } t > a \end{cases}$$

1.79.1 Difference of two Heaviside unit step functions

$$u(t-\alpha) - u(t-\beta) = \left\{ \begin{matrix} 1 & \text{for } \alpha < t < \beta \\ 0 & \text{for } t < \alpha \text{ or } t > \beta \end{matrix} \right\} \text{ for } 0 \leq \alpha < \beta$$

2 Other coordinate systems

2.1 Polar coordinates

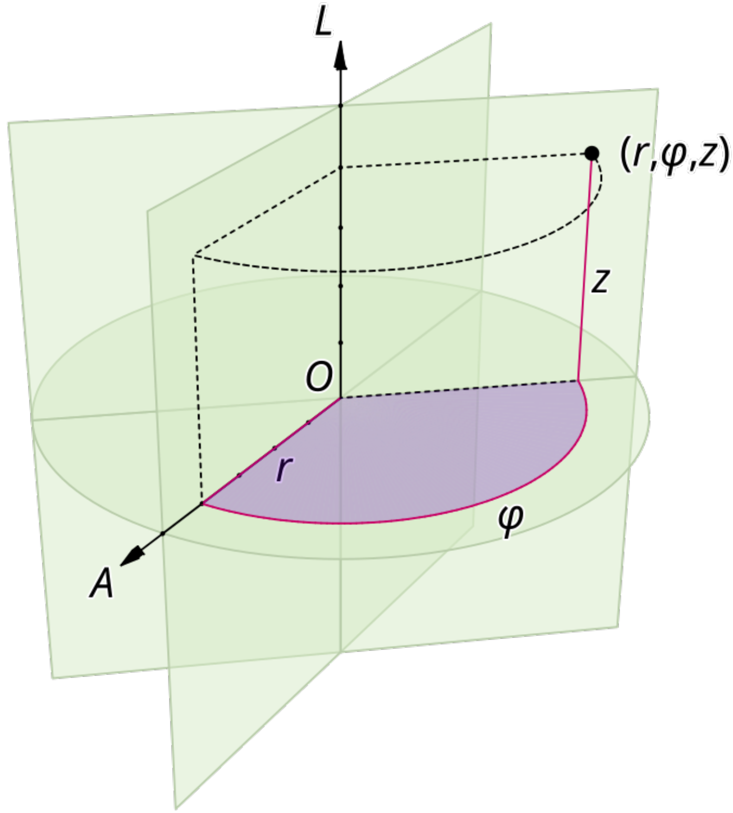


$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

Infinitesimal area element, $dA = dxdy = rdrd\varphi$

2.2 Cylindrical coordinates



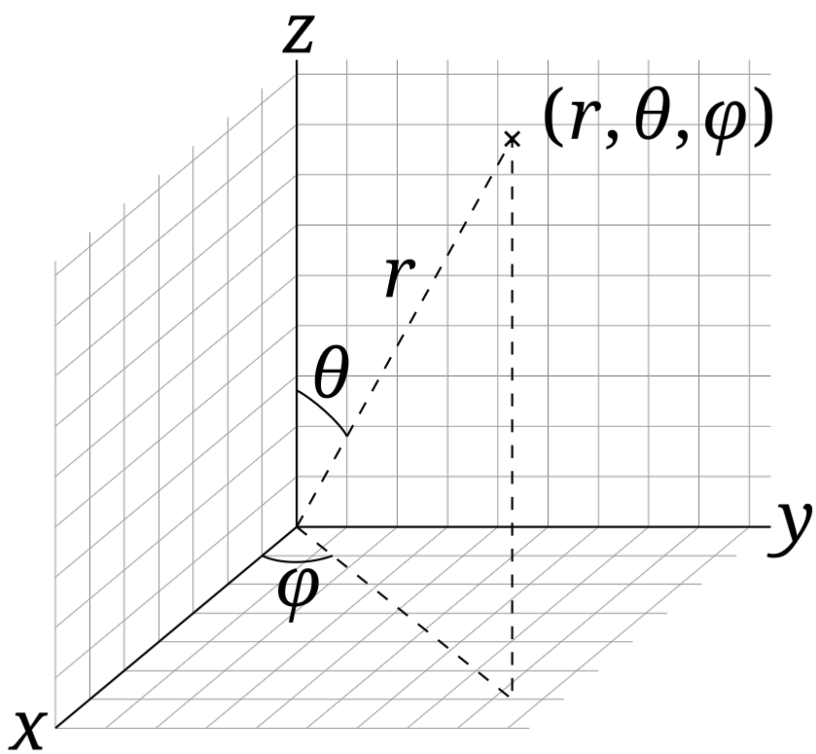
$$x = r\cos\varphi$$

$$y = r\sin\varphi$$

$$z = z$$

Infinitesimal volume element, $dV = dxdydz = rdrd\varphi dz$

2.3 Spherical coordinates



$$x = r\cos\varphi\sin\theta$$

$$y = r\sin\varphi\sin\theta$$

$$z = r\cos\theta$$

Infinitesimal volume element, $dV = dxdydz = r^2 dr d\varphi d\theta$

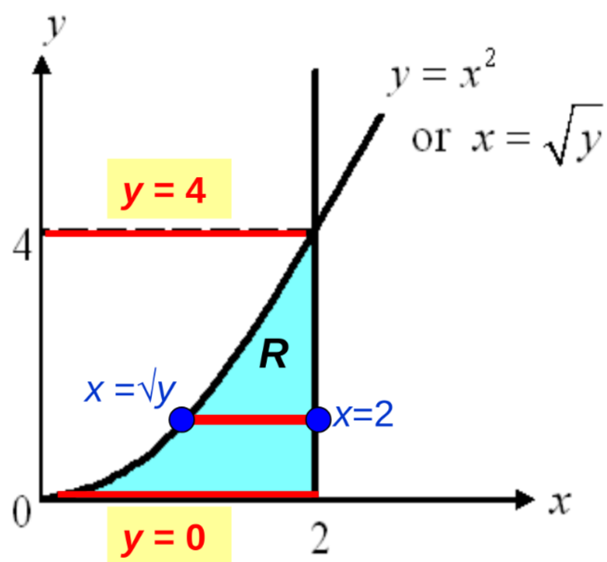
3 Figuring out the integration limits for multiple integrals in Cartesian coordinates

1. Choose a variable to integrate with respect to first. For a function $f(x, y)$, it can be either x or y.
2. Keep the other variable constant. If we choose x, we keep y constant, and if we choose y, we keep x constant.
3. Draw a lot of lines to cover the region R in the axis of the variable that is kept constant. If y is kept constant, we draw a lot of **horizontal lines**. If x is kept constant, we draw a lot of **vertical lines**.

3.1 Example of the 3rd step

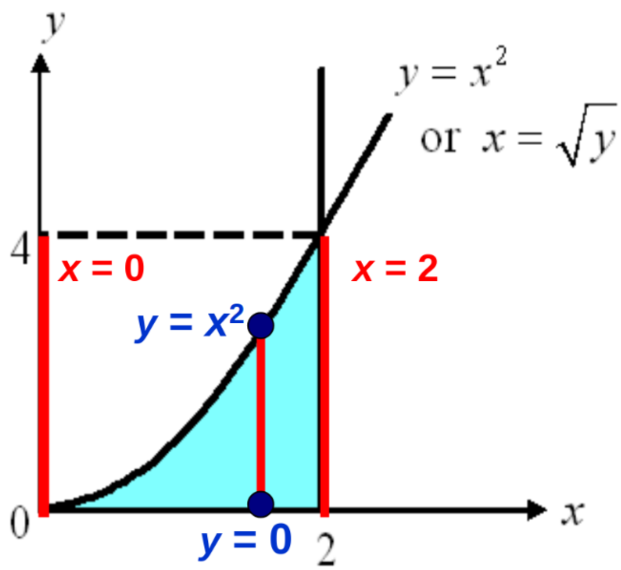
The example below is $\iint_R (3x^2 + y)\, dA$, where R is the region bounded by the curve $y = x^2$, the line $x = 2$ and the positive x-axis.

3.1.1 Keeping y constant



- We can see that the first **horizontal line** is $y = 0$ and the last **horizontal line** is $y = 4$.
- Each **horizontal line** starts on the curve $x = \sqrt{y}$ and ends on the line $x = 2$.

3.1.2 Keeping x constant



- We can see that the first **vertical line** is $x = 0$ and the last **vertical line** is $x = 2$.
- Each **vertical line** starts on the line $y = 0$ and ends on the curve $y = x^2$.

4 Figuring out the integration limits for multiple integrals in polar coordinates

1. Choose a variable to integrate with respect to first. For a function $f(r, \theta)$, it can be either r or θ.
2. Keep the other variable constant. If we choose r, we keep θ constant, and if we choose θ, we keep r constant.
3. Draw a lot of lines to cover the region R in the axis of the variable that is kept constant. If θ is kept constant, we draw a lot of **radial lines**, which are lines that come outwards from the centre of the circle. If r is kept constant, we draw a lot of **concentric circles** whose radii slowly increases.

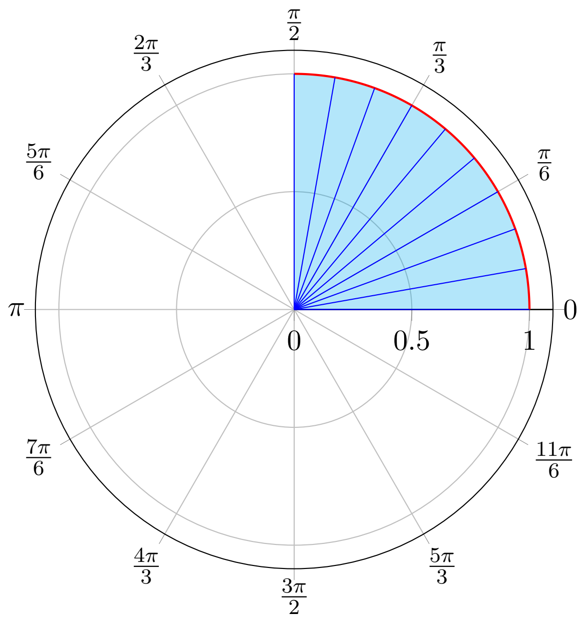
4.1 Example of the 3rd step

The example below is $\iint_R x^2 + y^2 \, dA$, where R is the region bounded by the equation $x^2 + y^2 \leq 1$, the line $x = 0$ and line $y = 0$.

In polar form, the equation would be:

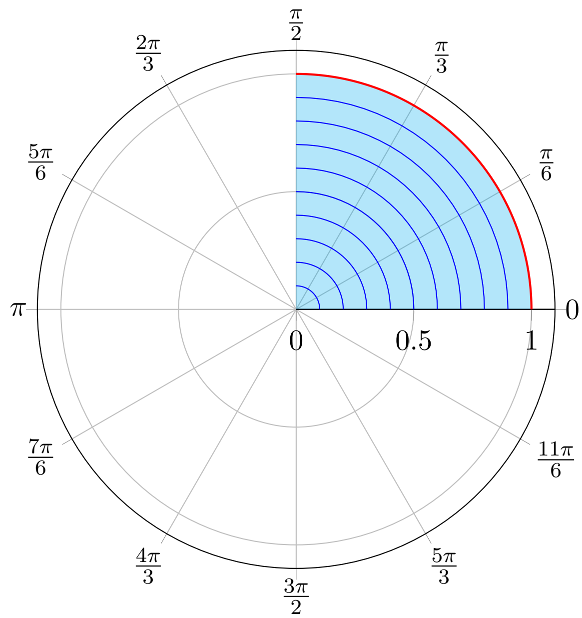
$$\iint_R (r\cos^2\theta + r\sin^2\theta)\, r dr d\theta$$

4.1.1 Keeping θ constant



- We can see that the first **radial line** is $\theta = 0$ and the last **horizontal line** is $\theta = \frac{\pi}{2}$.
- Each **radial line** starts at the point $r = 0$ and ends on the circular arc $r = 1$.

4.1.2 Keeping r constant



- We can see that the first **concentric circle** is $r = 0$ and the last **concentric circle** is $r = 1$.
- Each **concentric circle** starts at the line $\theta = 0$ and ends on the line $\theta = \frac{\pi}{2}$.

5 Mathematical formulas

5.1 Trigonometric identities

5.1.1 Basic and Pythagorean identities

$$\csc x = \frac{1}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\cot x = \frac{1}{\tan x}$$
$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$
$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec x$$
$$1 + \cot^2 x = \csc^2 x$$

5.1.2 Angle sum and different identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\sin(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

5.1.3 Double angle identities

$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

5.1.4 Half angle identities

$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}$$

$$\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

$$\sin^2 x = \frac{1}{2}\left[1-\cos 2x\right]$$

$$\cos^2 x = \frac{1}{2}\left[1+\cos 2x\right]$$

$$\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

5.1.5 Sum identities

$$\sin P + \sin Q = 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$$

$$\sin P - \sin Q = 2\cos\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$$

$$\cos P + \cos Q = 2\cos\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$$

$$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$$

5.2 Standard derivatives

$$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arccos x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$$

$$\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$$

5.3 Standard integrals

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|\sqrt{x^2-a^2}+x\right|$$

$$\int \tan x\,dx = \ln|\sec x|$$

$$\int \cot x\,dx = \ln|\sin x|$$

$$\int \csc x\,dx = -\ln|\csc x+\cot x|$$

$$\int \sec x\,dx = -\ln|\sec x+\tan x|$$

$$\int x\cos(px)\,dx = \frac{1}{p^2}(\cos(px)+px\sin(px))$$

$$\int x\sin(px)\,dx = \frac{1}{p^2}(\sin(px)-px\cos(px))$$

6 Laplace transforms

6.1 Elementary functions

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \text{ for } s > a$$

$f(t)$	$\mathcal{L}\{f(t)\}$	Conditions/Remarks
1	$\frac{1}{s}$	$s > 0$
t^{n-1}	$\frac{(n-1)!}{s^n}$	$s > 0, n = 1, 2, 3, \cdots$
$t^{n-1}e^{at}$	$\frac{(n-1)!}{(s-a)^n}$	$s > a, a$ is a constant $n = 1, 2, 3, \cdots$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s > 0,\ \omega$ is a constant
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s > 0,\ \omega$ is a constant
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|,\ a$ is a constant
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|,\ a$ is a constant
$u(t-a)$	$\frac{e^{-as}}{s}$	$s > 0,\ a$ is a nonnegative constant $u(t-a)$ is the unit step function

6.2 Properties

- Suppose $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exists for $s > c_f$ and $s > c_g$ respectively, then:

  $$\mathcal{L}\{\alpha f(t)+\beta g(t)\} = \alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\} \text{ for } s > \max(c_f, c_g), \quad \alpha, \beta \in \mathbb{R}$$

- Suppose $\mathcal{L}\{f(t)\} = F(s)$ for $s > c_f$, then:

  $$\mathcal{L}\{e^{at} f(t)\} = F(s-a) \text{ for } s > a + c_f, \quad a \in \mathbb{R}$$

- Suppose $\mathcal{L}\{f(t)\} = F(s)$ for $s > c_f$, then:

  $$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n F}{ds^n} \text{ for } s > c_f, \quad n = 1, 2, 3, \ldots$$

- Suppose $\mathcal{L}\{f(t)\}$ exists for $s > c_f$, then for $s > c_f$:

  $$\mathcal{L}\left\{\frac{df}{dt}\right\} = s\mathcal{L}\{f(t)\} - f(0)$$

  $$\mathcal{L}\left\{\frac{d^2 f}{dt^2}\right\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

  $$\mathcal{L}\left\{\frac{d^3 f}{dt^3}\right\} = s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$$

- Suppose $\mathcal{L}\{f(t)\}$ exists for $s > c_f$, then:

  $$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \text{ for } s > c_f, \quad a \geq 0$$

  Where:

  - u is the Heaviside unit step function

- Suppose $\mathcal{L}\{f(t)\}$ exists for $s > c_f$, then:

  $$\mathcal{L}\left\{\int_{\tau=0}^{\tau=t} f(\tau)\, d\tau\right\} = \frac{1}{s}F(s) \text{ for } x > \max(0, c_f)$$

6.3 Table of properties

Function	LT of function	Remarks
$f(t)$	$F(s)$	$\mathcal{L}\{f(t)\} = F(s)$ for $s > c_f$
$g(t)$	$G(s)$	$\mathcal{L}\{g(t)\} = G(s)$ for $s > c_g$
(1) $\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	α, β constants $s > \max(c_f, c_g)$.
(2) $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t) = d^n f/dx^n$ $(n \geq 1)$ $s > c_f$.
(3) $e^{at} f(t)$	$F(s-a)$	a constant $s > a + c_f$
(4) $\displaystyle\int_{\tau=0}^{\tau=t} f(\tau)d\tau$	$\dfrac{1}{s}F(s)$	$s > \max(0, c_f)$
(5) $u(t-a)f(t-a)$	$e^{-as}F(s)$	Unit-step function $u(t-a)$, $s > c_f$
(6) $t^n f(t)$	$(-1)^n F^{(n)}(s)$	$F^{(n)}(s) = d^n F/ds^n$ $(n \geq 1)$ $s > c_f$