

# MA2011 Mechatronics System Interfacing Notes

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## Contents

<b>1 Definitions</b>	<b>7</b>
1.1 Measurement system . . . . .	7
1.2 Amplitude linearity . . . . .	8
1.3 Period and frequency . . . . .	10
1.4 Time and frequency domain . . . . .	10
1.5 Magnitude ratio ( $M$ ) . . . . .	11
1.6 Dynamic error ( $\delta$ ) . . . . .	11
1.7 Radicand . . . . .	11
1.8 Cut-off frequency ( $f_c$ ) . . . . .	11
1.9 Frequency response of a filter ( $M$ ) . . . . .	12
1.10 Common mode rejection ratio (CMRR) . . . . .	12
1.11 Analogue-to-digital (A/D) conversion . . . . .	13
1.12 Dithering . . . . .	13
1.13 Transducers . . . . .	13
1.14 Sensors . . . . .	13
1.15 Sensor classification . . . . .	13
1.16 Instrumentation systems . . . . .	14
1.17 Input . . . . .	14
1.18 Output . . . . .	14
1.19 Sensitivity ( $S$ ) . . . . .	14
1.20 Resolution . . . . .	14
1.21 Accuracy . . . . .	14
1.22 Repeatability . . . . .	15
1.23 Types of instrument errors . . . . .	15
1.24 Lorentz's law . . . . .	17
1.25 Faraday's law . . . . .	18

<b>2 Fourier series representation of signals</b>	<b>19</b>
2.1 Fundamental frequency . . . . .	19
2.2 Fourier series representation of a periodical waveform . . . . .	20
2.3 Even functions . . . . .	27
2.4 Odd functions . . . . .	28
2.5 Calculation of the Fourier coefficients . . . . .	29
2.6 Square wave decomposition . . . . .	31
2.7 Complex form of the Fourier series . . . . .	33
2.8 Regular form vs complex form . . . . .	35
2.9 Cosine-only form vs complex form . . . . .	35
2.10 Complex Fourier series decomposition . . . . .	36
2.11 Signal reconstruction . . . . .	38
2.12 Signal approximation . . . . .	38
2.13 Amplitude and phase . . . . .	40
2.14 Fourier spectrum . . . . .	40
2.15 Circuits and Fourier series . . . . .	42
2.16 Conditions for the Fourier series . . . . .	46
2.17 Insights . . . . .	46
<b>3 Bandwidth and frequency response</b>	<b>47</b>
3.1 Decibel scale . . . . .	47
3.2 Frequency response curve (Bode plot) . . . . .	47
3.3 Bandwidth . . . . .	48
<b>4 Periodic functions</b>	<b>50</b>
4.1 Definition . . . . .	50
4.2 Even-function symmetry . . . . .	50
4.3 Odd-function symmetry . . . . .	51
4.4 Properties of symmetric functions . . . . .	51
4.5 Conversion from non-periodic to periodic . . . . .	52
4.6 Examples . . . . .	54
<b>5 Dynamic systems</b>	<b>57</b>
5.1 Example 1 . . . . .	57
5.2 Example 2 . . . . .	58
5.3 Example 3 (second-order system) . . . . .	59
5.4 Measurement system: Ordinary differential equations . . . . .	60

<b>6 Linear systems</b>	<b>61</b>
6.1 Homogeneous equation of a linear system . . . . .	61
6.2 Characteristic equation of a homogeneous equation . . . . .	61
6.3 Roots of the characteristic equation . . . . .	62
6.4 Solving the homogeneous equation . . . . .	63
6.5 Input functions in a linear system . . . . .	64
6.6 Special cases of linear systems . . . . .	64
6.7 Zero-order system . . . . .	65
6.8 First-order system . . . . .	66
6.9 Second-order system . . . . .	70
<b>7 System modelling and analogies</b>	<b>83</b>
7.1 System models . . . . .	83
7.2 Second-order modelling analogies . . . . .	84
7.3 Similarities and differences . . . . .	85
<b>8 Sampling</b>	<b>85</b>
8.1 Sampling rate . . . . .	85
8.2 Analogue vs digital signals . . . . .	86
8.3 Shannon & Nyquist theorem . . . . .	86
8.4 Why don't we sample as fast as possible? . . . . .	88
8.5 Logic behind the minimum sampling rate . . . . .	89
8.6 Theorem . . . . .	89
8.7 Aliasing . . . . .	92
8.8 Applications . . . . .	95
<b>9 Quantisation and encoding</b>	<b>97</b>
9.1 Digitising . . . . .	97
9.2 Pulse code modulation (PCM) . . . . .	97
9.3 Quantisation . . . . .	100
<b>10 Amplifiers</b>	<b>108</b>
10.1 Relationship between output and input . . . . .	108
10.2 Filtering and amplifier linearity . . . . .	108
10.3 Characteristics of amplifiers . . . . .	109
10.4 Operational amplifiers . . . . .	110
10.5 Inverting amplifier . . . . .	116
10.6 Non-inverting amplifier . . . . .	118
10.7 Summer amplifier . . . . .	120
10.8 Difference amplifier . . . . .	121

10.9 Integrator . . . . .	122
10.10 Differentiator . . . . .	123
10.11 Sample and hold circuit . . . . .	124
10.12 Comparator . . . . .	125
10.13 Instrumentation amplifier . . . . .	126
<b>11 Analogue-to-digital (A/D) conversion</b>	<b>128</b>
11.1 Data acquisition (DAQ) devices . . . . .	128
11.2 A/D conversion . . . . .	130
11.3 A/D conversion process . . . . .	131
11.4 A/D converters . . . . .	137
11.5 Flash A/D converters . . . . .	144
11.6 Dual slope converters . . . . .	147
11.7 Sigma-delta A/D converters . . . . .	148
11.8 Comparison of different types of A/D converters . . . . .	149
11.9 Digital-to-analogue (D/A) conversion . . . . .	149
11.10 D/A conversions . . . . .	151
<b>12 1st order systems</b>	<b>153</b>
12.1 Example . . . . .	154
12.2 General (forced) equation . . . . .	155
12.3 Natural (unforced) equation . . . . .	155
12.4 Time constant . . . . .	156
12.5 Response to DC forcing inputs . . . . .	158
12.6 2nd order systems . . . . .	159
<b>13 Measuring temperature</b>	<b>162</b>
13.1 Temperature scales . . . . .	162
13.2 Bimetallic thermometers . . . . .	163
13.3 Resistance temperature detectors (RTD) . . . . .	164
13.4 Thermistors . . . . .	169
<b>14 Measuring displacement (resistive sensors)</b>	<b>171</b>
14.1 Resistive sensors . . . . .	171
<b>15 Measuring forces through displacement</b>	<b>173</b>
15.1 Strain gauges . . . . .	173
15.2 Piezo-resistive force sensor . . . . .	174
<b>16 Measuring displacement (inductive sensors)</b>	<b>175</b>
16.1 Linear variable differential transformer (LVDT) . . . . .	175

<b>17 Measuring displacement (capacitive sensors)</b>	<b>179</b>
17.1 Principle . . . . .	179
17.2 Guard electrode . . . . .	180
17.3 Interfacing with capacitive sensors using AC . . . . .	181
<b>18 Proximity sensors</b>	<b>182</b>
18.1 Hall effect . . . . .	182
18.2 Hall effect sensors . . . . .	183
18.3 Light detectors . . . . .	184
<b>19 Digital encoders</b>	<b>187</b>
19.1 Absolute encoders . . . . .	188
19.2 Incremental encoders . . . . .	189
<b>20 DC Motors</b>	<b>190</b>
20.1 Structure and fields . . . . .	190
20.2 Commutation . . . . .	191
20.3 Generated torque . . . . .	193
20.4 3-pole DC motor . . . . .	194
20.5 Equations . . . . .	195
20.6 DC motor equivalent electrical circuit . . . . .	196
20.7 Steady state speed torque curve . . . . .	197
20.8 Common types of load . . . . .	199
20.9 Maximum output power . . . . .	201
20.10 Maximum efficiency . . . . .	202
20.11 DC motor load matching . . . . .	203
20.12 Driving DC motors . . . . .	204
20.13 Pulse width modulation operating principle . . . . .	205
20.14 Inductive kickback . . . . .	206
20.15 Pulse width modulation circuit . . . . .	207
20.16 Brushless DC motors . . . . .	208
<b>21 Strain gauges</b>	<b>211</b>
21.1 Stress or strain measurements . . . . .	211
21.2 Strain . . . . .	212
21.3 Transverse sensitivity . . . . .	215
21.4 Materials . . . . .	216
21.5 Wheatstone bridge . . . . .	218
21.6 Temperature compensation . . . . .	221
21.7 Half bridge . . . . .	222

21.8 Apparent strain . . . . .	224
21.9 Measuring forces . . . . .	228
21.10 Bridge balancing . . . . .	229

# 1 Definitions

## 1.1 Measurement system

A measurement system consists of 3 components, a transducer, a signal processor and a recorder.

### 1.1.1 Transducer

A transducer is a device that usually converts a physical quality into a time-varying voltage, called an analogue signal.

### 1.1.2 Signal processor

A signal processor is a device that can modify the analogue signal.

### 1.1.3 Recorder

A recorder is a device that displays or records the signal.

### 1.1.4 Input

The input in a measurement system is the physical quantity to be measured.

### 1.1.5 Output

The output in a measurement system is usually the output of the transducer transforming the input into a form compatible with the processor to be processed.

### 1.1.6 Difference

The difference in a measurement system is usually the difference in the input from the output.

### 1.1.7 Characterisation

A good measurement system is characterised by:

- Phase linearity
- Amplitude linearity
- Adequate bandwidth

## 1.2 Amplitude linearity

Amplitude linearity refers to the output always being changed by the same factor multiplied by the change in the input, i.e.

$$V_{out}(t) - V_{out}(0) = \alpha(V_{in}(t) - V_{in}(0))$$

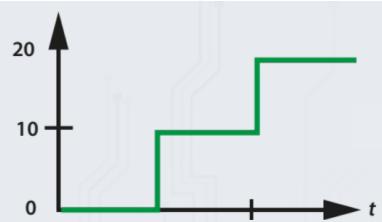
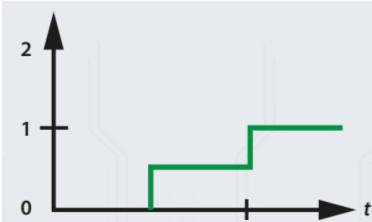
Where:

- $V_{out}(t)$  is the voltage output at time  $t$
- $V_{out}(0)$  is the initial voltage output
- $V_{in}(t)$  is the voltage input at time  $t$
- $V_{in}(0)$  is the initial voltage input
- $\alpha$  is the constant of proportionality, or the scaling factor

### 1.2.1 Remarks

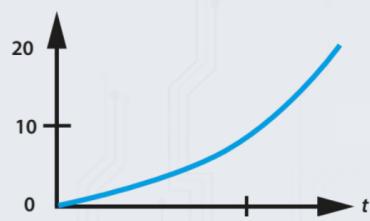
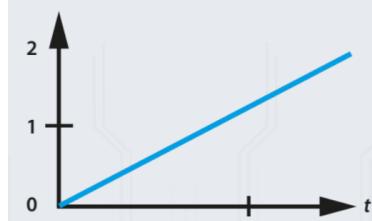
- It is difficult to interpret the output if there is no amplitude linearity.
- A measurement system usually satisfies amplitude linearity over a limited range of input amplitudes, like a spring.
- Linear response of a measurement system usually holds for a limited range of the input rate.
- An ideal measurement system exhibits amplitude linearity for any input amplitude and input rate.

### 1.2.2 Examples



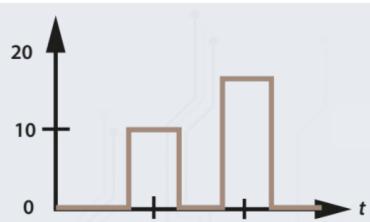
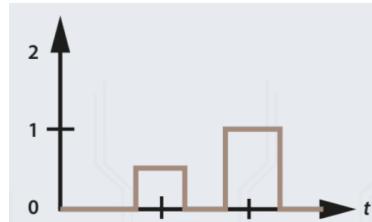
**Linear**

$$V_{out}(t) - V_{out}(0) = 20(V_{in}(t) - V_{in}(0))$$



**Nonlinear**

$\alpha$  is not a constant

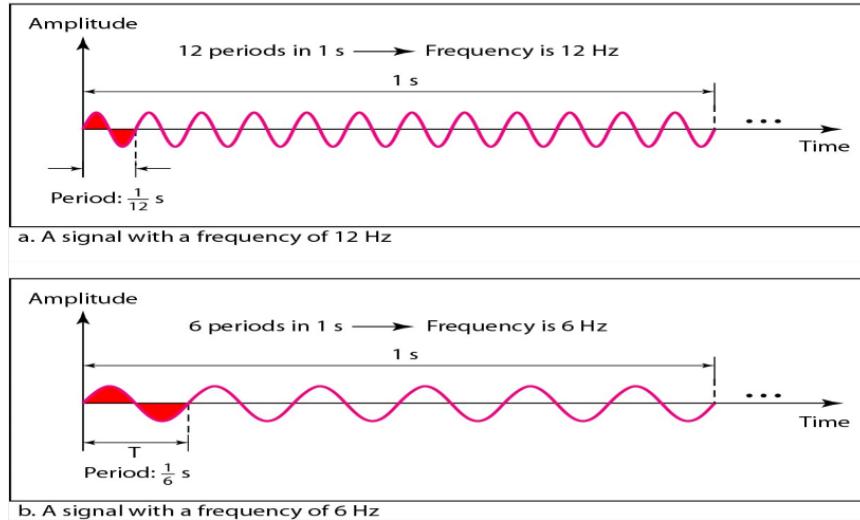


**Nonlinear**

$\alpha=20$  for 1<sup>st</sup> pulse and 15 for the 2<sup>nd</sup> pulse

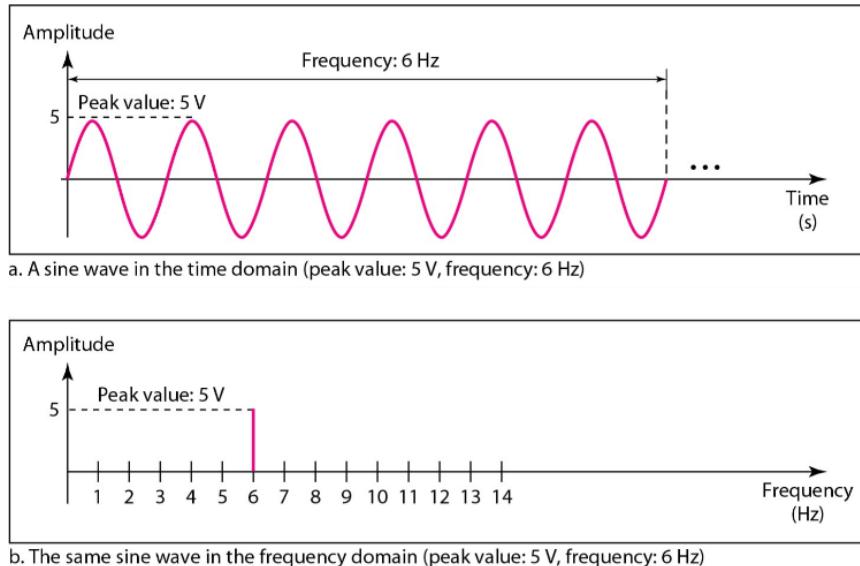
### 1.3 Period and frequency

$T$  is the period in **seconds**, which is the inverse of frequency ( $f$  in Hz).



### 1.4 Time and frequency domain

Below are the time-domain and frequency-domain plots of a sine wave.



## 1.5 Magnitude ratio ( $M$ )

Magnitude ratio, which can be considered as the attenuation, is always less than 1, and is given by:

$$M(\omega) = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

Where:

- $M$  is the magnitude ratio
- $\omega$  is the angular frequency
- $\tau$  is the time constant

## 1.6 Dynamic error ( $\delta$ )

A dynamic error for a first order system is always less than 1, and is given by:

$$\delta(\omega) = 1 - M(\omega)$$

Where:

- $\delta$  is the dynamic error
- $M$  is magnitude ratio

## 1.7 Radicand

Radicand is the quantity inside the square root sign. For example, the radicand of  $\sqrt{3}$  is 3, and the radicand of  $\sqrt{x^2 + 2bx + b^2}$  is  $x^2 + 2bx + b^2$ .

## 1.8 Cut-off frequency ( $f_c$ )

$$f_c = \frac{1}{2\pi RC}$$

Where:

- $f_c$  is the cut-off frequency
- $R$  is the resistance of the resistor ( $\Omega$ )
- $C$  is the capacitance of the capacitor (F)

### 1.8.1 Time constant estimate ( $\tau$ )

$$\tau \approx \frac{1}{f_c}$$

Where:

- $\tau$  is the time constant
- $f_c$  is the cut-off frequency

### 1.9 Frequency response of a filter ( $M$ )

$$M(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{f_c}\right)^2}}$$

Where:

- $M$  is the frequency response
- $f$  is the target frequency
- $f_c$  is the cut-off frequency of the filter

### 1.10 Common mode rejection ratio (CMRR)

- Common mode rejection ratio (CMRR) is the ratio of the difference mode gain to the common mode gain.
- The difference mode gain is the amplification factor for the difference between the input signals.
- The common mode gain is the amplification factor for the average of the input signals.
- For an ideal difference amplifier, the common mode gain is 0, implying an infinite common mode rejection ratio.
- It is desirable to minimise the common mode gain to suppress signals such as noise that are common to both inputs.

## **1.11 Analogue-to-digital (A/D) conversion**

- An electronic integrated circuit which transforms a signal from analogue (continuous) to digital (discrete) form.
- Analogue signals are directly measurable quantities.
- Digital signals only have two states. For the digital computer, we refer to the binary states: 0 and 1.

## **1.12 Dithering**

Dithering is a form of noise that is intentionally applied to randomise quantisation error.

## **1.13 Transducers**

Transducers convert one form of energy into another, and it is not necessary to perform a measurement.

## **1.14 Sensors**

Sensors produce an output signal, which is typically electrical, for the purpose of sensing a physical phenomenon.

## **1.15 Sensor classification**

- Analogue vs digital
  - Light on and off switch vs light dimmer.
- Passive vs active
  - Passive sensors do not require external an external power supply, and they draw energy from the input signal itself.
- Null versus deflection type
  - Null type sensors counteract any deflection due to the measured quantity using an opposing calibrated force.
- Subject of measurement
  - Mechanical, optical, thermal, etc.

## 1.16 Instrumentation systems

- Sensing module, which can be mechanical, thermal, optical, pyrolytic, piezoelectric, etc.
- Conversion module to convert from analogue to digital.
- Pre-processing, which is a module that manipulates the variables.
- Data transmission, which can be wired or wireless, transferred over the internet, etc.
- Presentation or storage to the user.

## 1.17 Input

Input is the stimulus. Some examples include temperature, pressure, and strain.

## 1.18 Output

The output is usually an electrical signal, which is defined using voltage, current, frequency, phase, etc.

## 1.19 Sensitivity ( $S$ )

The sensitivity is defined as:

$$S = \frac{\text{Output variation}}{\text{Input variation}}$$

It is also the slope of the graph of the output ( $f(x)$ ) against the input ( $x$ ).

$$S = \frac{df}{dx}$$

## 1.20 Resolution

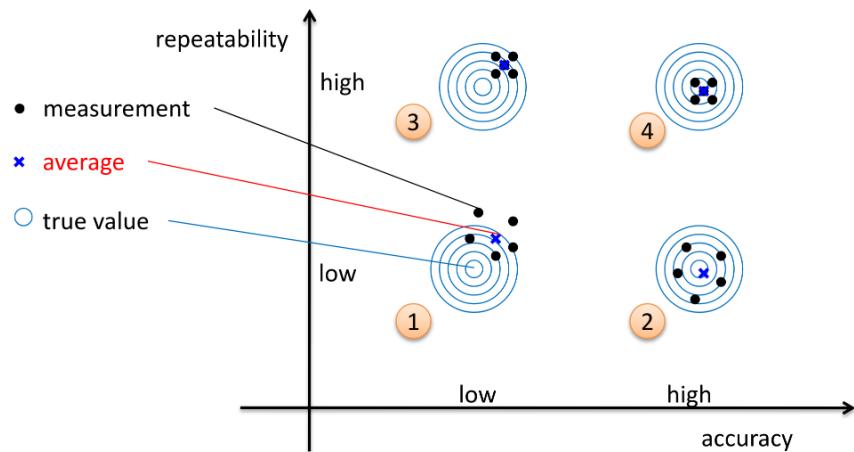
The resolution is the minimum change of the input that can be reliably detected. It is limited by noise, bit-conversion, and many other things.

## 1.21 Accuracy

The accuracy is the difference of the measurement from the true value.

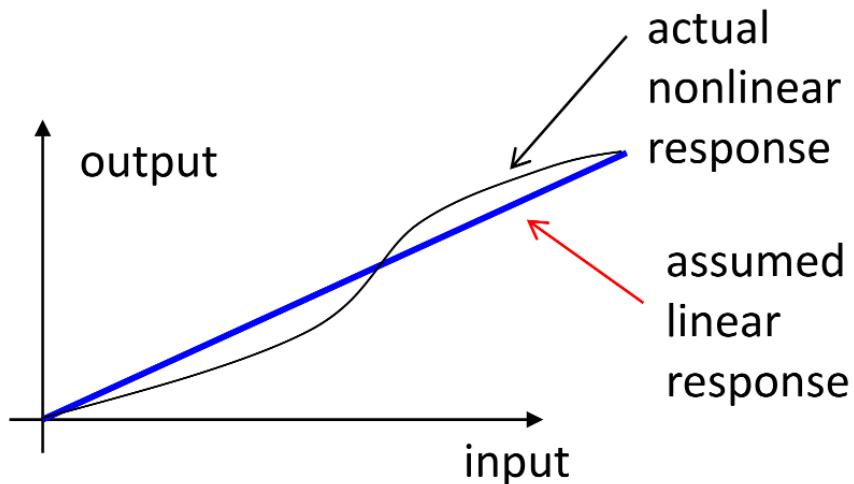
## 1.22 Repeatability

Repeatability is how well a system or device can reproduce an outcome in unchanged conditions.

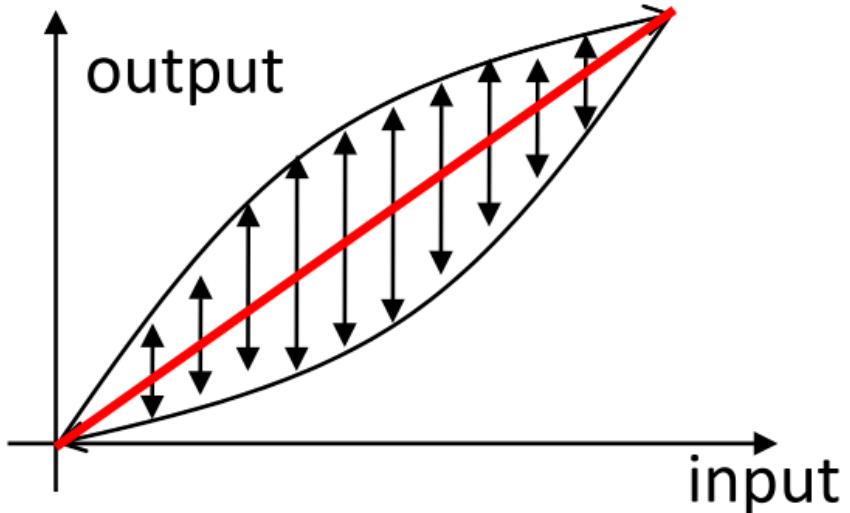


## 1.23 Types of instrument errors

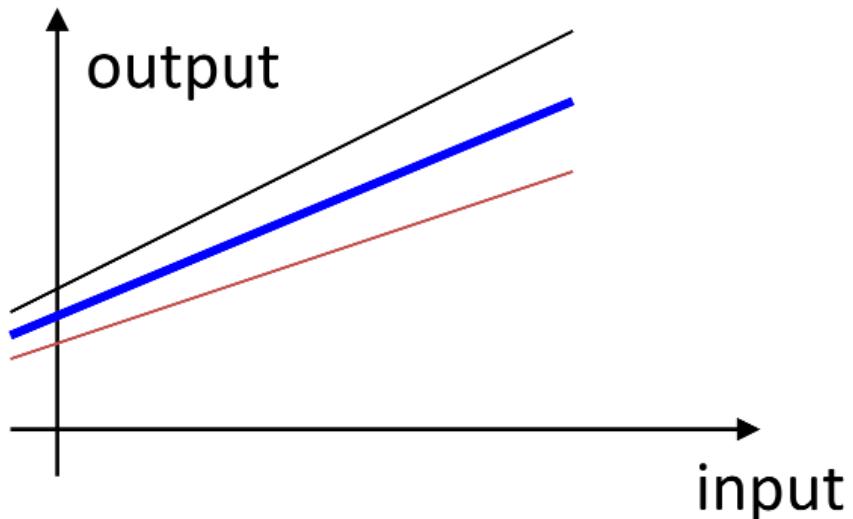
### 1.23.1 Nonlinearity



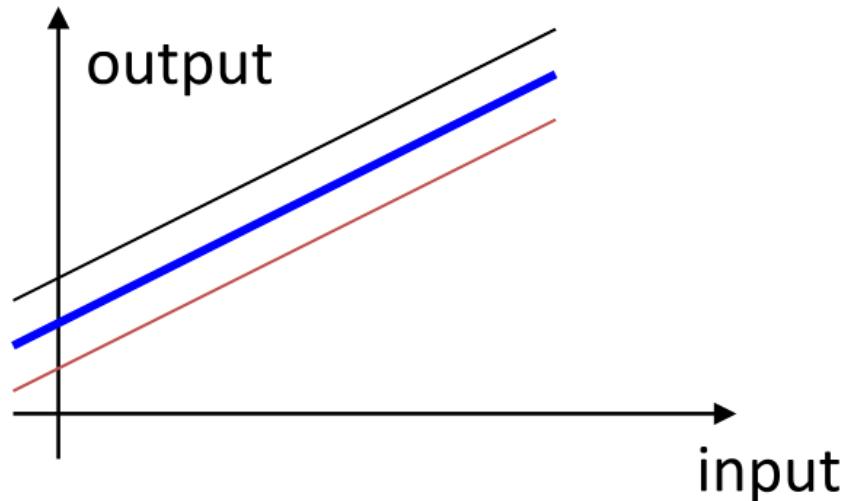
### 1.23.2 Hysteresis



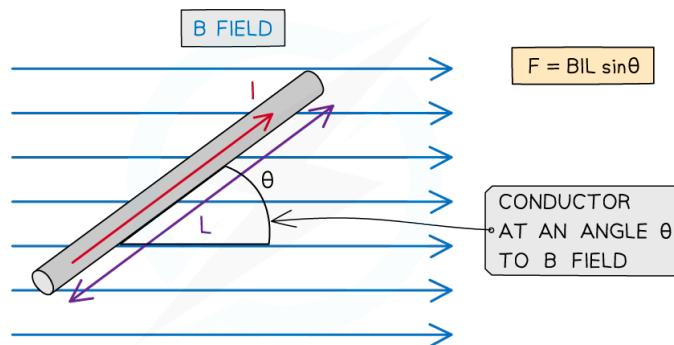
### 1.23.3 Sensitivity error



#### 1.23.4 Zero-shift error



#### 1.24 Lorentz's law



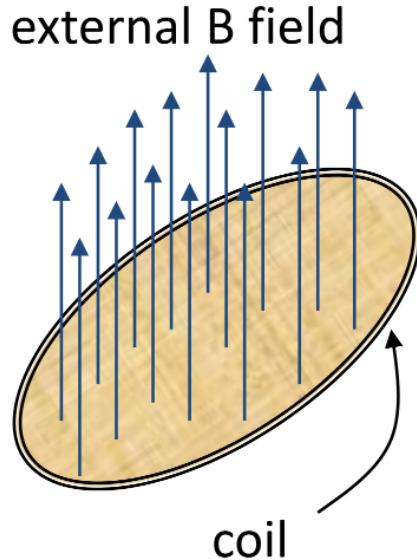
$$\vec{F} = (\vec{i} \times \vec{B})L$$

$$F = ||\vec{F}|| BiL \sin \theta$$

Where:

- $\vec{F}$  is the magnetic force
- $\vec{i}$  is the current
- $\vec{B}$  is the magnetic field
- $L$  is the length of the wire

### 1.25 Faraday's law



$$emf = E = -\frac{d\Phi}{dt}$$

$$\Phi \triangleq \int_{\Sigma} \vec{B} d\vec{\Sigma}$$

Where:

- $emf$  is the electromotive force
- $\Phi$  is the magnetic flux
- $\Sigma$  is the surface whose boundary coincides with the coil
  - It is not **uniquely defined** but  $\text{div } B = 0$ , which means the integral only depends on the boundary.

## 2 Fourier series representation of signals

- Any periodic signal can be represented by a combination of an infinite number of sinusoid terms.
- Easy and standardised way to deal with any periodic signals (simple or complicated) using sine and cosine terms.
- In addition to time domain analysis, frequency domain analysis helps to gain insights of the signals which are fundamental to signal processing, and other mechatronics applications.
- To study bandwidth and phase linearity, which are applied to frequency components of an input signal, it is necessary to review the Fourier series representation of a signal.
- Any periodical waveform can be represented as an infinite series of sine and cosine waveforms of different amplitudes and frequencies.
- Summing up this infinite series gives the original periodical waveform.
- Practically, a finite number of the sine and cosine waveforms can adequately represent a periodical waveform.

### 2.1 Fundamental frequency

Let  $\omega_0$  be the fundamental of first (lowest) harmonic frequency defined as:

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0$$

Where:

- $\omega_0$  is the fundamental angular frequency
- $T$  is the period
- $f_0$  is the fundamental frequency in Hz

The other sine and cosine waveforms have frequencies of integer multiples of  $\omega_0$ .

## 2.2 Fourier series representation of a periodical waveform

The Fourier series representation of a periodical waveform  $f(t)$  is:

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

Where:

- $C_0$  is the DC component of the signal, i.e. the non-periodical part of the waveform, given by:

$$C_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{A_0}{2}$$

Where:

- $T$  is the period
- $f(t)$  is the periodical waveform
- $t$  is the time
- $A_0$  is the initial amplitude of the waveform

- $A_n$  is given by:

$$A_n = \frac{2}{T} \int_0^T f(t) \cos(n\omega_0 t) dt$$

Where:

- $T$  is the period
- $\omega_0$  is the fundamental angular frequency
- $t$  is the time
- $n$  is just a number

- $B_n$  is given by:

$$B_n = \frac{2}{T} \int_0^T f(t) \sin(n\omega_0 t) dt$$

Where:

- $T$  is the period
- $\omega_0$  is the fundamental angular frequency
- $t$  is the time
- $n$  is just a number

Note that  $C_0$  is the average value of the waveform over its period.

### 2.2.1 In general

Given:

$$C_n = \sqrt{A_n^2 + B_n^2}$$

$$\phi_n = -\arctan\left(\frac{B_n}{A_n}\right)$$

Then:

$$\begin{aligned} F(t) &= C_0 + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t) + B_n (\sin n\omega_0 t)) \\ &= C_0 + \sum_{n=1}^{\infty} \sqrt{A_n^2 + B_n^2} \left( \frac{A_n}{\sqrt{A_n^2 + B_n^2}} \cos(n\omega_0 t) + \frac{B_n}{\sqrt{A_n^2 + B_n^2}} \sin(n\omega_0 t) \right) \\ &= C_0 + \sum_{n=1}^{\infty} C_n (\cos(\phi_n) \cos(n\omega_0 t) - \sin(\phi_n) \sin(n\omega_0 t)) \\ &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n) \\ \phi_n &= -\arctan\left(\frac{B_n}{A_n}\right) \\ \cos(\phi_n) &= \frac{A_n}{\sqrt{A_n^2 + B_n^2}} \\ \sin(\phi_n) &= -\frac{B_n}{\sqrt{A_n^2 + B_n^2}} \end{aligned}$$

### 2.2.2 Sine form

Given:

$$C_n = \sqrt{A_n^2 + B_n^2}$$

$$\phi_n^* = \arctan\left(\frac{A_n}{B_n}\right)$$

Then:

$$\begin{aligned} F(t) &= C_0 + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t) + B_n (\sin n\omega_0 t)) \\ &= C_0 + \sum_{n=1}^{\infty} \sqrt{A_n^2 + B_n^2} \left( \frac{A_n}{\sqrt{A_n^2 + B_n^2}} \cos(n\omega_0 t) + \frac{B_n}{\sqrt{A_n^2 + B_n^2}} \sin(n\omega_0 t) \right) \\ &= C_0 + \sum_{n=1}^{\infty} C_n (\sin(\phi_n^*) \cos(n\omega_0 t) + \cos(\phi_n^*) \sin(n\omega_0 t)) \\ &= C_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega_0 t + \phi_n^*) \\ \phi_n^* &= \arctan\left(\frac{A_n}{B_n}\right) \\ \sin(\phi_n^*) &= \frac{A_n}{\sqrt{A_n^2 + B_n^2}} \\ \cos(\phi_n^*) &= \frac{B_n}{\sqrt{A_n^2 + B_n^2}} \end{aligned}$$

### 2.2.3 Example: Square waveform with period T

The square waveform is defined as:

$$f(t) = \begin{cases} 1 & 0 \leq t \leq \frac{T}{2} \\ -1 & \frac{T}{2} \leq t \leq T \end{cases}$$

Then:

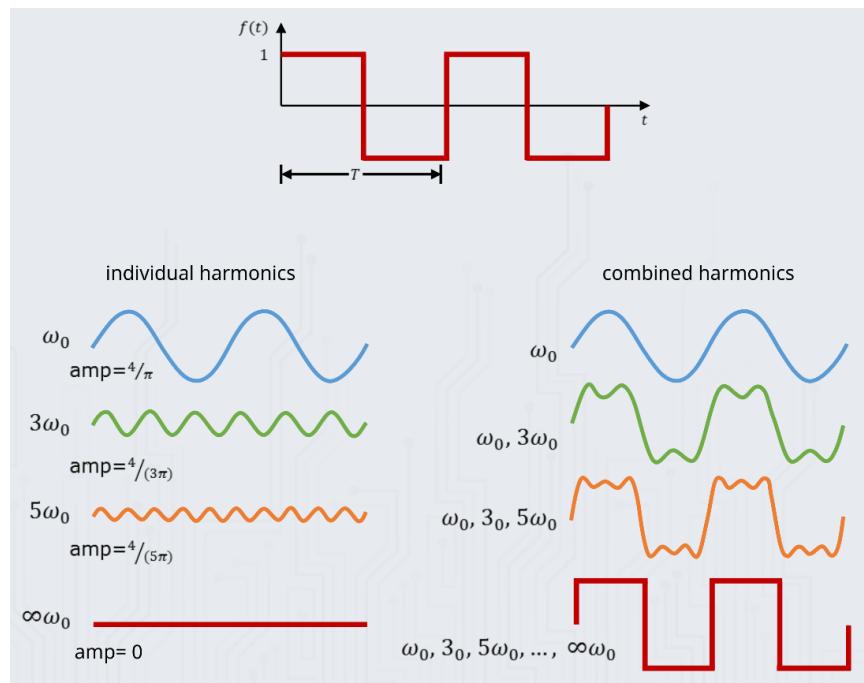
$$A_n = 0$$

$$\begin{aligned} B_n &= \frac{2}{T} \left( \int_0^{\frac{T}{2}} \sin(n\omega_0 t) dt - \int_{\frac{T}{2}}^T \sin(n\omega_0 t) dt \right) \\ &= \frac{2}{T} \left( -\frac{1}{n\omega_0} \cos(n\omega_0 t) \Big|_0^{\frac{T}{2}} + \frac{1}{n\omega_0} \cos(n\omega_0 t) \Big|_{\frac{T}{2}}^T \right) \\ &= \frac{2}{n\pi} (1 - \cos(n\pi)) \\ &= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Therefore:

$$\begin{aligned} F(t) &= \frac{4}{\pi} \sin(\omega_0 t) + \frac{4}{3\pi} \sin(3\omega_0 t) + \frac{4}{5\pi} \sin(5\omega_0 t) + \dots \\ &= \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin((2n-1)\omega_0 t) \end{aligned}$$

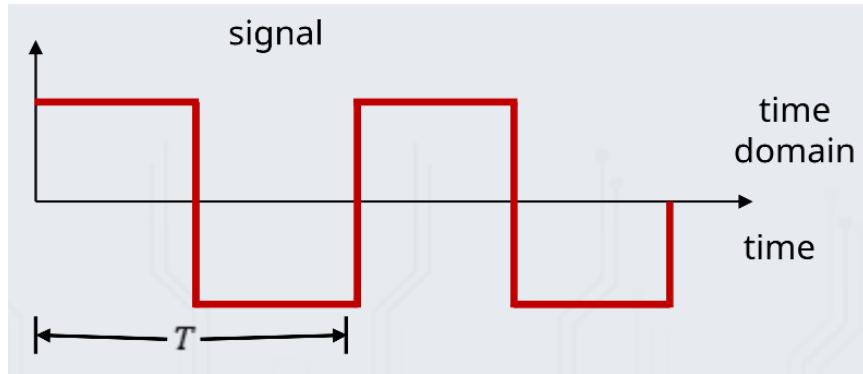
#### 2.2.4 Representation of a square wave



### 2.2.5 Plotting the frequency spectrum of a waveform

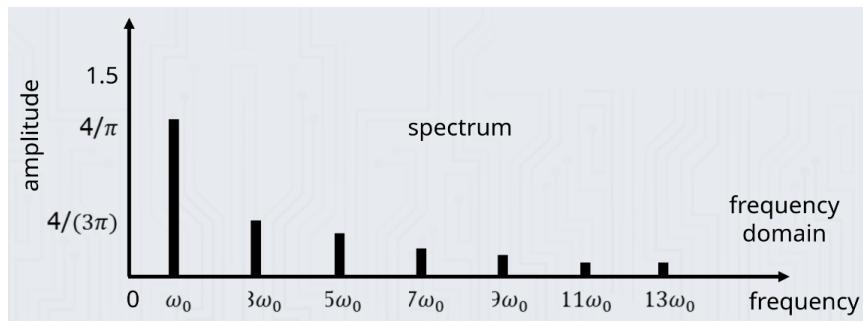
When plotting the frequency spectrum for a signal represented by a Fourier series, use the signal amplitude generated from the equation below:

$$F(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n)$$

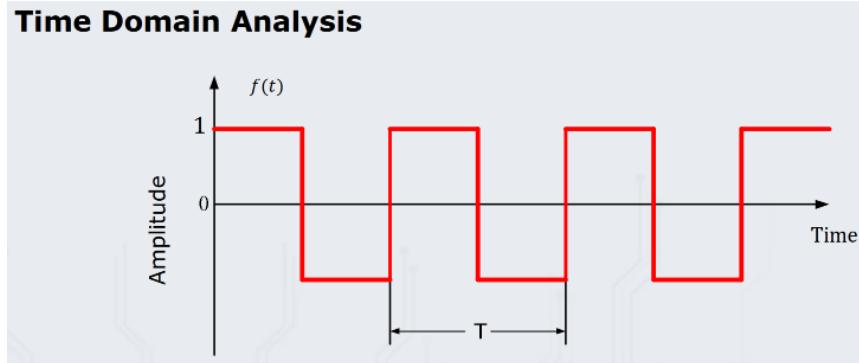


For the square wave above:

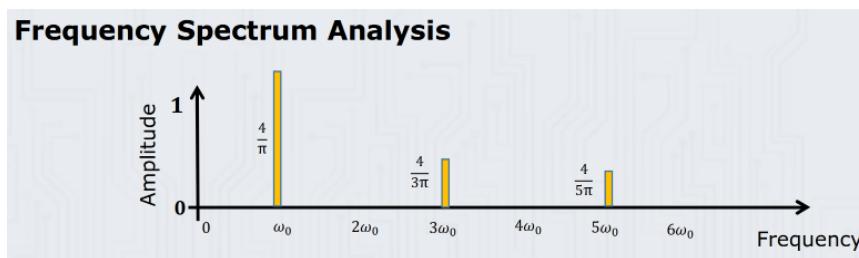
$$F(t) = \frac{4}{\pi} \sin(\omega_0 t) + \frac{4}{3\pi} \sin(3\omega_0 t) + \frac{4}{5\pi} \sin(5\omega_0 t) + \dots$$



### 2.2.6 Time domain analysis



### 2.2.7 Frequency response analysis



### 2.3 Even functions

Even functions can solely be represented by cosine waves, i.e.

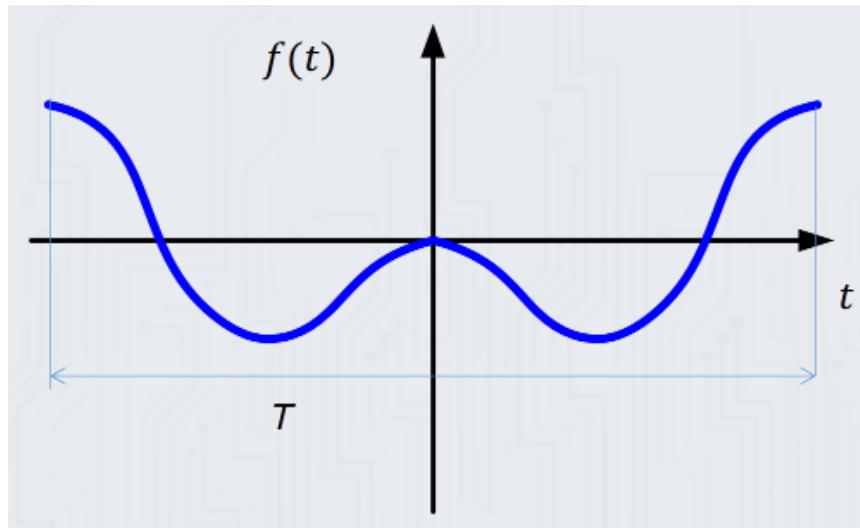
$$B_n = 0$$

Because  $\sin(n\omega_0 t)$  is an odd function,  $f(t) \cdot \sin(n\omega_0 t)$  is an odd function, hence:

$$B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt = 0$$

$$A_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt$$

$$F(t) = C_0 + A_1 \cos(1\omega_0 t) + A_2 \cos(2\omega_0 t) + A_3 \cos(3\omega_0 t) + \dots$$



## 2.4 Odd functions

Even functions can solely be represented by sine waves, i.e.

$$C_0 = 0, \text{ and all } A_n = 0$$

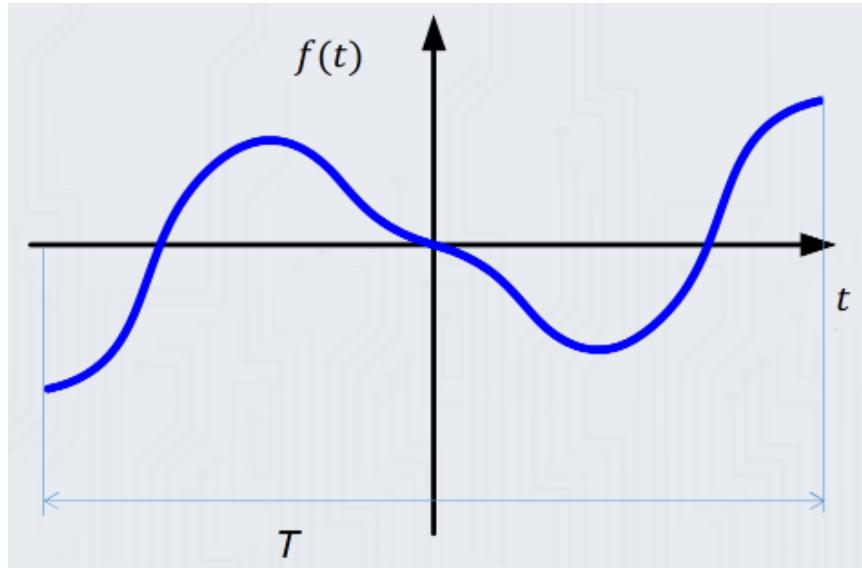
Because  $\cos(n\omega_0 t)$  is an even function,  $f(t) \cdot \cos(n\omega_0 t)$  is an even function, hence:

$$A_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt = 0$$

$$C_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = 0$$

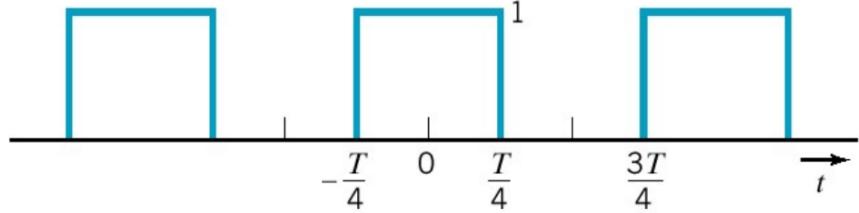
$$B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt = \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt$$

$$F(t) = C_0 + B_1 \sin(1\omega_0 t) + B_2 \sin(2\omega_0 t) + B_3 \sin(3\omega_0 t) + \dots$$



## 2.5 Calculation of the Fourier coefficients

Find the Fourier series for the following periodic waveform:



Function:

$$f(t) = \begin{cases} 1, & t \in [-\frac{T}{4}, \frac{T}{4}] \\ 0, & t \in [\frac{T}{4}, \frac{3T}{4}] \end{cases}$$

Periodic:

$$f(t + T) = f(t)$$

$$\text{Period} = T$$

Symmetry: Even-function

$$f(-t) = f(t)$$

The function is an even function, hence:

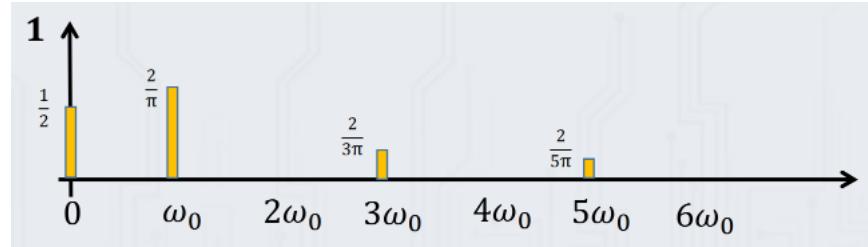
$$B_n = 0$$

$$C_0 = \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{3T}{4}} f(t) dt = \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 dt = \frac{1}{2}$$

$$\begin{aligned} A_n &= \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt \\ &= \frac{4}{T} \int_0^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt \\ &= \frac{4}{T} \int_0^{\frac{T}{4}} 1 \cos(n\omega_0 t) dt \\ &= \frac{4}{n\pi\omega_0 T} \int_0^{\frac{T}{4}} d(\sin(n\omega_0 t)) \\ &= \frac{4}{n\pi\omega_0 t} \left( \sin(n\omega_0 \frac{T}{4}) - 0 \right) \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

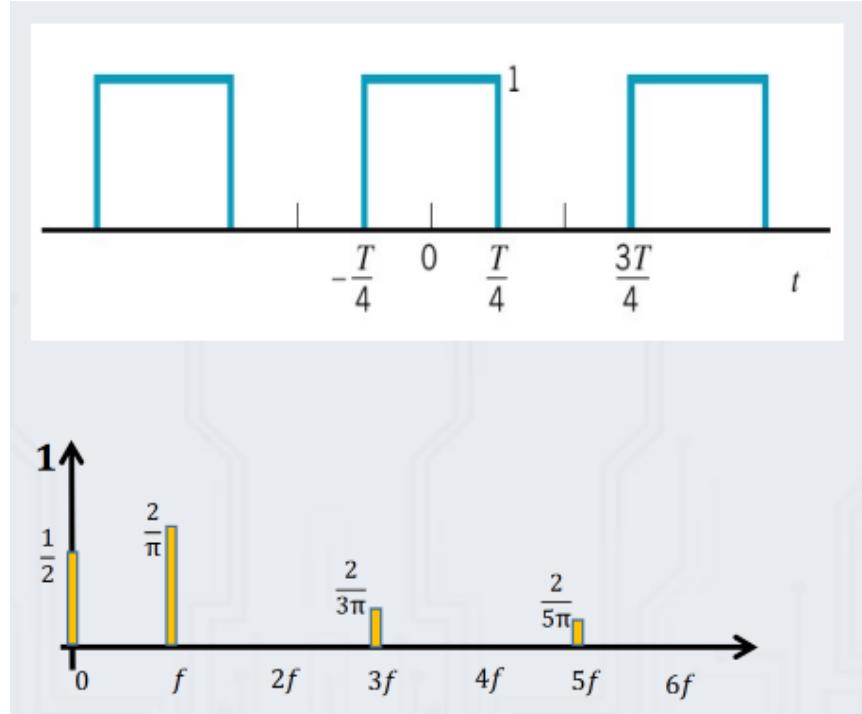
The corresponding Fourier series is:

$$F(t) = \frac{1}{2} + \frac{2}{\pi} \cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{3\pi} \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) + \frac{2}{5\pi} \cos\left(5 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{7\pi} \cos\left(7 \cdot \frac{2\pi}{T} \cdot t\right) + \dots$$



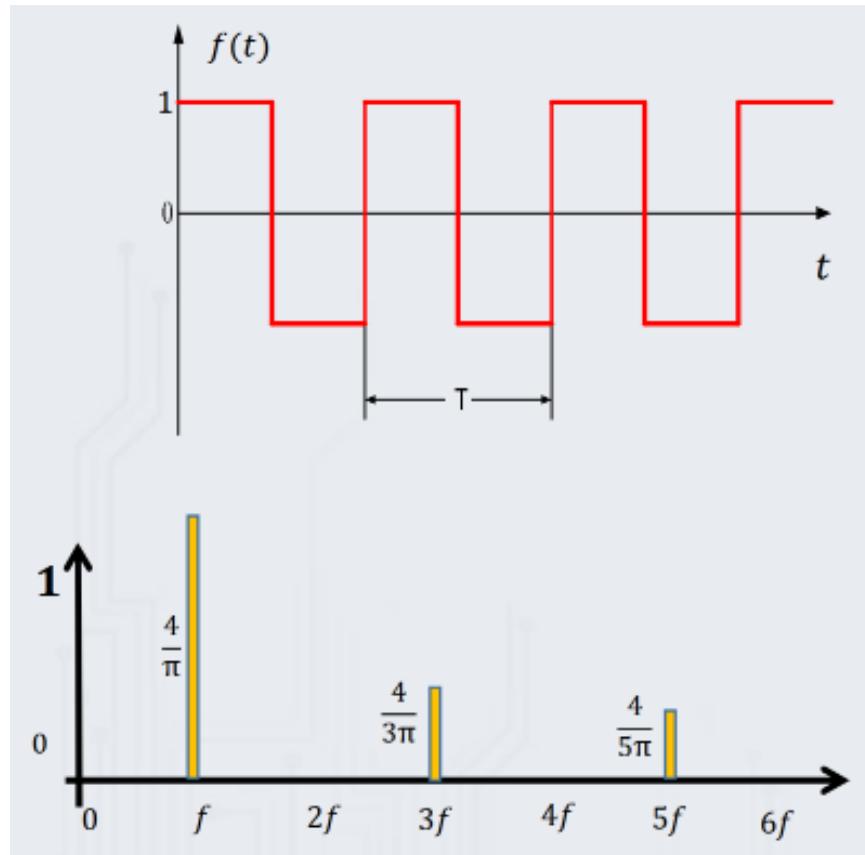
## 2.6 Square wave decomposition

### 2.6.1 Half square wave decomposition



$$F(t) = \frac{1}{2} + \frac{2}{\pi} \cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{3\pi} \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) + \frac{2}{5\pi} \cos\left(5 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{7\pi} \cos\left(7 \cdot \frac{2\pi}{T} \cdot t\right) + \dots$$

### 2.6.2 Full square wave decomposition



$$F(t) = \frac{4}{\pi} \sin(\omega_0 t) + \frac{4}{3\pi} \sin(3\omega_0 t) + \frac{4}{5\pi} \sin(5\omega_0 t) + \dots$$

## 2.7 Complex form of the Fourier series

The standard Fourier series representation is given as:

$$F(t) = C_0 + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t))$$

Using Euler's formulae:

$$\begin{aligned}\cos(n\omega_0 t) &= \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \\ \sin(n\omega_0 t) &= \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}\end{aligned}$$

So:

$$\begin{aligned}e^{jn\omega_0 t} &= \cos(n\omega_0 t) + j \sin(n\omega_0 t) \\ e^{-jn\omega_0 t} &= \cos(n\omega_0 t) - j \sin(n\omega_0 t)\end{aligned}$$

The  $n$ -th harmonic component can be expressed as:

$$\begin{aligned}& A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t) \\ &= A_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + B_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \\ &= A_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t} 2}{+} - j B_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2} \\ &= \frac{A_n - j B_n}{2} e^{jn\omega_0 t} + \frac{A_n + j}{2} e^{-jn\omega_0 t}\end{aligned}$$

Denoting:

$$\begin{aligned}D_n &= \frac{A_n - j B_n}{2}, \quad D_{-n} = \frac{A_n + j B_n}{2} \\ D_0 &= \frac{A_0}{2}\end{aligned}$$

$$A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t) = D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}$$

Therefore, the Fourier series can be expressed as:

$$F(t) = D_0 + \sum_{n=1}^{\infty} (D_n e^{jn\omega_0 t} + D_{-n} e^{-jn\omega_0 t}) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

### 2.7.1 Coefficients

The coefficients  $D_n$  can be evaluated in the following manner:

$$\begin{aligned}
D_n &= \frac{(A_n - jB_n)}{2} \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt - \frac{j}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos(n\omega_0 t) - j \sin(n\omega_0 t)) dt \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega_0 t} dt
\end{aligned}$$

The coefficients  $D_{-n}$  can be evaluated in the following manner:

$$\begin{aligned}
D_{-n} &= \frac{(A_n + jB_n)}{2} \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega_0 t) dt + \frac{j}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)(\cos(n\omega_0 t) + j \sin(n\omega_0 t)) dt \\
&= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{jn\omega_0 t} dt
\end{aligned}$$

Note that  $D_{-n}$  is the complex conjugate of  $D_n$ :

$$D_n = \frac{(A_n - jB_n)}{2}$$

$$D_{-n} = \frac{(A_n + jB_n)}{2}$$

So the Fourier series decomposition has the  $D_n$  in complex form:

$$D_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-jn\omega_0 t} dt \quad n = 0, \pm 1, \pm 2, \dots$$

We have the complex form of the Fourier series:

$$F(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

## 2.8 Regular form vs complex form

### 2.8.1 Regular form

$$\begin{aligned} A_n &= \frac{2}{T} \int_0^T f(t) \cos(n\omega_0 t) dt \quad n = 1, 2, 3, \dots \\ B_n &= \frac{2}{T} \int_0^T f(t) \sin(n\omega_0 t) dt \quad n = 1, 2, 3, \dots \\ F(t) &= C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t) \end{aligned}$$

### 2.8.2 Complex form

$$\begin{aligned} D_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t} dt \\ n &= 0, \pm 1, \pm 2, \dots \\ F(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \end{aligned}$$

## 2.9 Cosine-only form vs complex form

### 2.9.1 Cosine-only form

$$\begin{aligned} C_n &= \sqrt{A_n^2 + B_n^2} \\ \phi_n &= -\arctan\left(\frac{B_n}{A_n}\right) \\ F(t) &= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n) \\ A_{-n} &= \frac{2}{T} \int_0^T f(t) \cos(-n\omega_0 t) dt = A_n \\ B_{-n} &= \frac{2}{T} \int_0^T f(t) \sin(-n\omega_0 t) dt = -B_n \\ C_n &= \sqrt{A_{-n}^2 + B_{-n}^2} = \sqrt{A_n^2 + B_n^2} = C_n \\ \phi_{-n} &= \arctan\left(\frac{B_{-n}}{A_{-n}}\right) = \arctan\frac{B_n}{A_n} = -\phi_n \\ F(t) &= C_0 + \frac{1}{2} \sum_{n=-\infty, n \neq 0}^{\infty} C_n \cos(n\omega_0 t + \phi_n) \end{aligned}$$

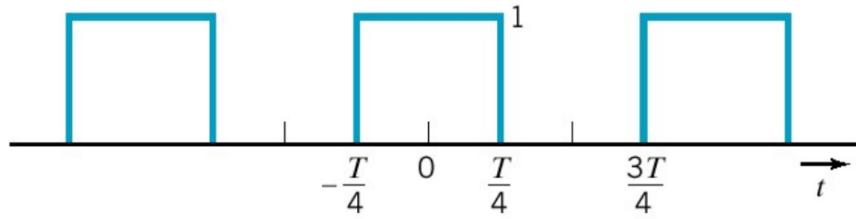
### 2.9.2 Complex form

$$D_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t} dt$$

$$n = 0, \pm 1, \pm 2, \dots$$

$$F(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

### 2.10 Complex Fourier series decomposition



Using  $n\omega_0 = m$ :

$$\begin{aligned} D_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-jn\omega_0 t} dt \\ &= \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 e^{-mt} dt \\ &= \frac{1}{-mT} e^{-mt} \Big|_{-\frac{T}{4}}^{\frac{T}{4}} \\ &= -\frac{1}{mT} \left( e^{\frac{-mT}{4}} - e^{\frac{mT}{4}} \right) \end{aligned}$$

Using Euler's formulae:

$$\cos(n\omega_0 t) = \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}$$

$$\sin(n\omega_0 t) = \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

We have:

$$\begin{aligned}
D_n &= \frac{1}{-in2\pi} \left( e^{\frac{in\pi}{2}} - e^{\frac{-in\pi}{2}} \right) \\
&= \frac{e^{\frac{in\pi}{2}} - e^{\frac{-in\pi}{2}}}{2j} \frac{1}{\pi} \\
&= \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi} \\
D_{-n} &= \frac{\sin\left(\frac{-n\pi}{2}\right)}{-n\pi} \\
&= D_n
\end{aligned}$$

To find  $D_0$ :

$$D_0 = \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 dt = \frac{1}{2}$$

The square signal can be decomposed in complex form:

$$\begin{aligned}
F(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\
&= D_0 + \sum_{n=1}^{\infty} D_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) \\
&= D_0 + \sum_{n=1}^{\infty} 2D_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \\
&= \frac{1}{2} + \sum_{n=1}^{\infty} 2 \frac{\sin\left(\frac{n\pi}{2}\right)}{n\pi} \cos(n\omega_0 t)
\end{aligned}$$

Setting  $\omega_0 = \frac{2\pi}{T}$ , the result is the same as the decomposition using the regular Fourier series:

$$F(t) = \frac{1}{2} + \frac{2}{\pi} \cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{3\pi} \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) + \frac{2}{5\pi} \cos\left(5 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{7\pi} \cos\left(7 \cdot \frac{2\pi}{T} \cdot t\right) + \dots$$

## 2.11 Signal reconstruction

Given:

- DC  $C_0$
- The all harmonic amplitude:  $A_n$  and  $B_n, n = 1, 2, \dots, N$
- The fundamental frequency  $\omega_0$

We can reconstruct the signal by using either one of the following:

$$F(t) = C_0 + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t))$$

$$F(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

## 2.12 Signal approximation

Given:

- DC  $C_0$
- The all harmonic amplitude:  $A_n$  and  $B_n, n = 1, 2, \dots, N$
- The fundamental frequency  $\omega_0$

We can approximate the signal by  $S_N(t)$ :

$$F(t) = \sum_{n=-N}^N D_n e^{jn\omega_0 t}$$

### 2.12.1 Approximation error

A practical calculation of the Fourier series requires that we truncate the series to a finite number of terms.

$$f(t) \approx \sum_{n=-N}^N D_n e^{jn\omega_0 t} = S_N(t)$$

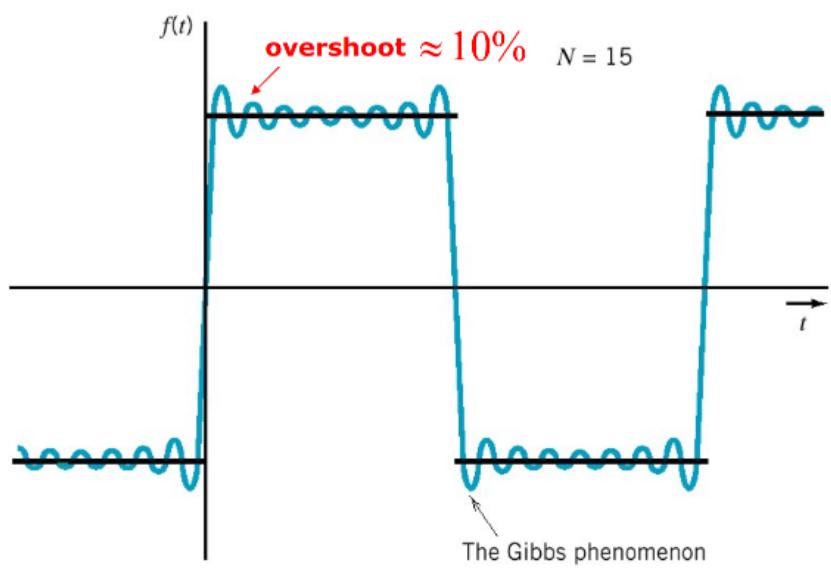
The error for  $N$  terms is:

$$\varepsilon(t) = f(t) - S_N(t)$$

We use the mean-square error (MSE) defined as:

$$\text{MSE} = \frac{1}{T} \int_0^T \varepsilon^2(t) dt$$

MSE is minimum when  $D_n$  is equal to the Fourier series' coefficients.



## 2.13 Amplitude and phase

Based on the cosine form of the Fourier series:

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

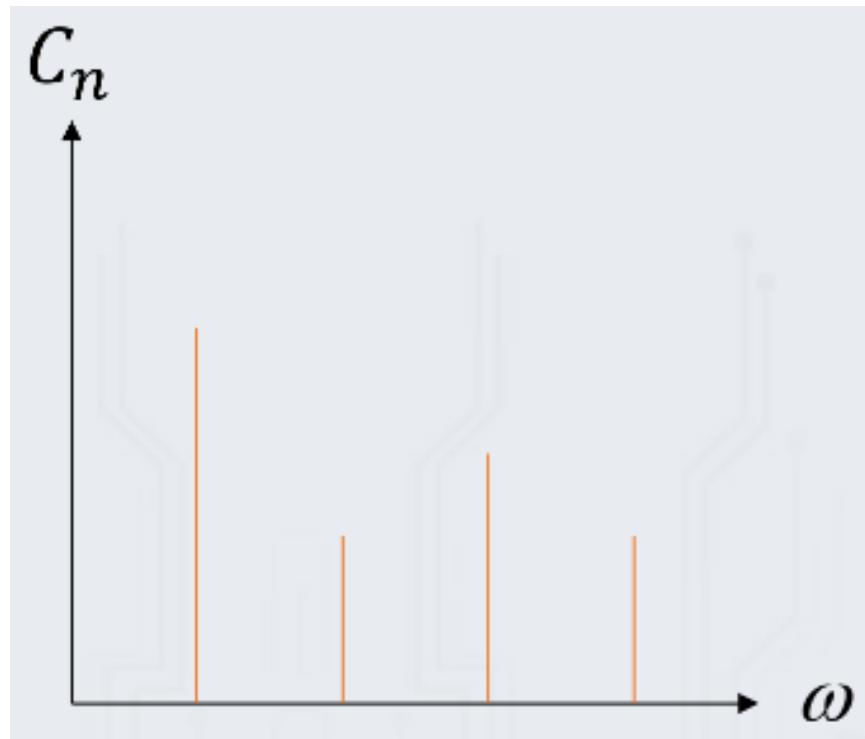
A periodic waveform can be represented by an infinite series of cosine of **single amplitude and phase**.

$$\text{Single amplitude: } C_n = \sqrt{A_n^2 + B_n^2}$$

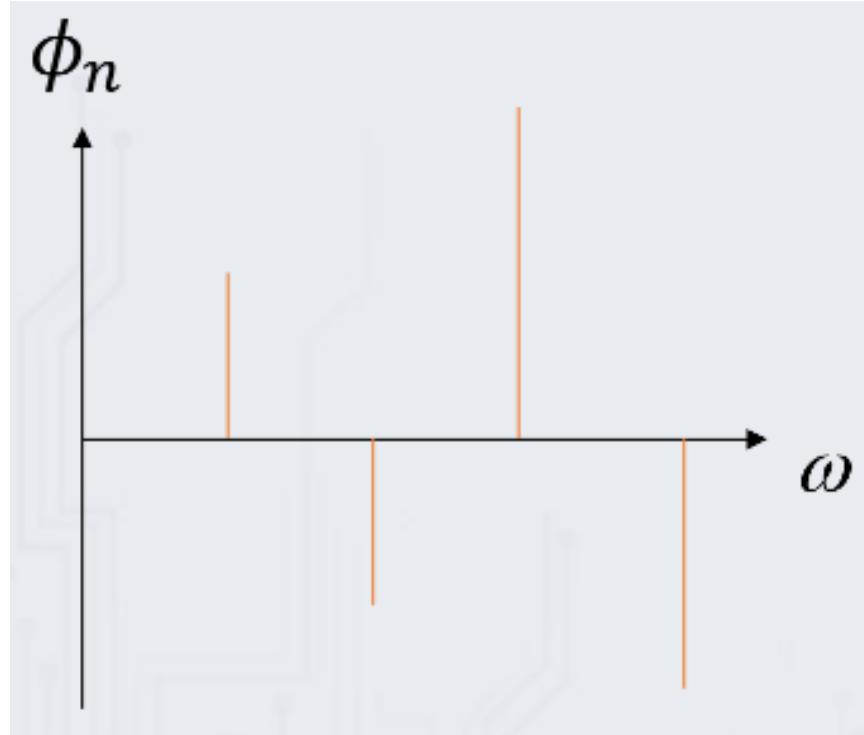
$$\text{Phase (angle): } \phi_n = -\arctan\left(\frac{B_n}{A_n}\right)$$

## 2.14 Fourier spectrum

### 2.14.1 Amplitude spectrum



#### 2.14.2 Phase spectrum



#### 2.14.3 Amplitude, frequency and phase

- **DC:**  $C_0$  is the average value of  $f(t)$
- The  $n^{th}$  **harmonic amplitude:**  $C_n$
- The **fundamental frequency:**  $\omega_0$
- The  $n^{th}$  **fundamental frequency:**  $n\omega_0$
- The  $n^{th}$  **phase angle:**  $\phi_n = -\arctan\left(\frac{B_n}{A_n}\right)$
- The **fundamental term:** For  $n = 1$ , the corresponding sinusoid is  $C_1 \cos(\omega_0 t + \phi_1)$
- The  $n^{th}$  **harmonic term:** The  $n^{th}$  corresponding sinusoid is  $C_n \cos(n\omega_0 t + \phi_n)$

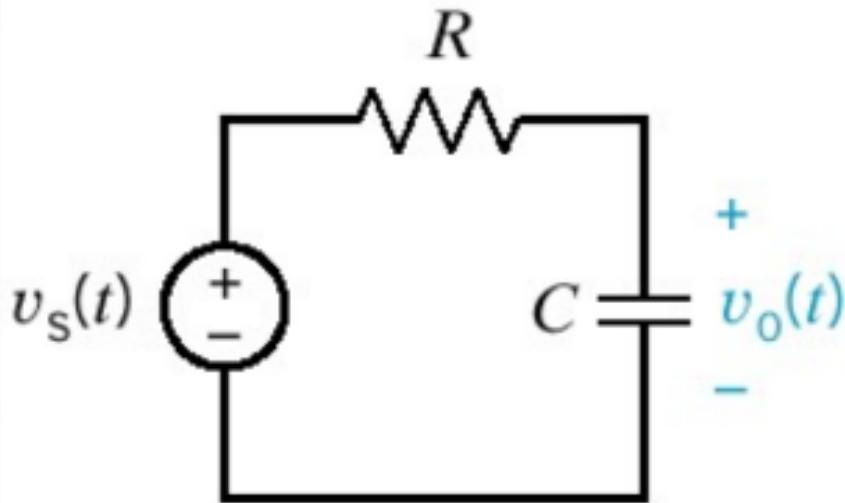
## 2.15 Circuits and Fourier series

It is often desired to determine the response of a circuit excited by a periodic signal  $v_s(t)$ .

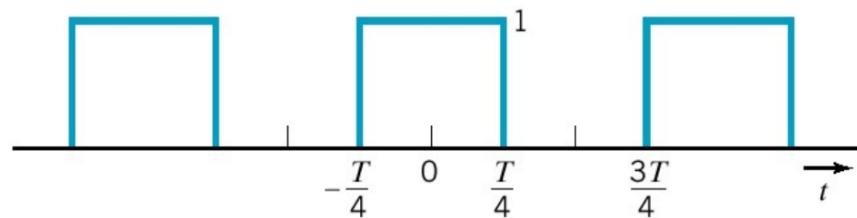
Assume:

$$R = 1 \Omega, \quad C = 2 \text{ F}, \quad T = \pi \text{ sec}$$

And an RC circuit excited by a periodic voltage  $v_s(t)$ , as shown below:

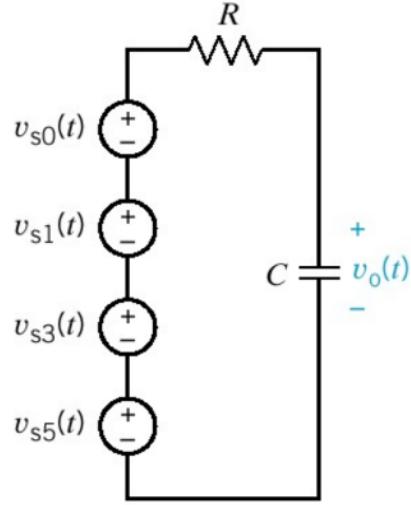


The square signal exciting the RC circuit:



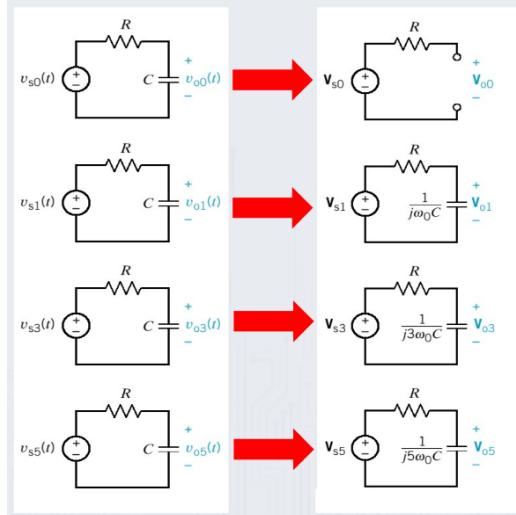
### 2.15.1 Equivalent circuit

In the equivalent circuit below, each voltage source is a term of the Fourier series of the input voltage  $v_s(t)$ .



### 2.15.2 Steady state response of the circuit

Since each input is a sinusoid, we want to find the steady state responses to the sinusoid.



### 2.15.3 Making use of the Fourier series

The Fourier series representation of the square waveform:

$$F(t) = \frac{1}{2} + \frac{2}{\pi} \cos\left(1 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{3\pi} \cos\left(3 \cdot \frac{2\pi}{T} \cdot t\right) + \frac{2}{5\pi} \cos\left(5 \cdot \frac{2\pi}{T} \cdot t\right) - \frac{2}{7\pi} \cos\left(7 \cdot \frac{2\pi}{T} \cdot t\right) + \dots$$

Since  $T = \pi$ , the first 4 terms of  $v_s(t)$  are:

$$v_s(t) \approx \underbrace{\frac{1}{2}}_{v_{s0}(t)} + \underbrace{\frac{2}{\pi} \cos 2t}_{v_{s1}(t)} - \underbrace{\frac{2}{3\pi} \cos 6t}_{v_{s3}(t)} + \underbrace{\frac{2}{5\pi} \cos 10t}_{v_{s5}(t)}$$

The steady state response  $v_0(t)$  can be found using superposition:

$$v_o(t) = v_{o0}(t) + v_{o1}(t) + v_{o3}(t) + v_{o5}(t)$$

### 2.15.4 Getting the impedance of the capacitor

The impedance of the capacitor is:

$$Z_c = \frac{1}{jn\omega_0 C}, \text{ for } n = 0, 1, 3, 5, \dots$$

Because:

$$T = \pi, \quad \omega_0 = \frac{2\pi}{T} = 2 \text{ sec}$$

Since:

$$R = 1 \Omega, \quad C = 2 \text{ F}, \quad T = \pi \text{ sec}$$

We can find:

$$\begin{aligned} V_{on} &= \frac{\frac{1}{jn\omega_0 C}}{R + \frac{1}{jn\omega_0 C}} V_{sn} \\ &= \frac{V_{sn}}{1 + jn\omega_0 CR} \\ &= \frac{V_{sn}}{1 + j4n} \\ &= \frac{(1 - j4n)V_{sn}}{(1 - j4n)(1 + j4n)} \\ &= \frac{(1 - j4n)V_{sn}}{(1 + 16n^2)} \\ &= \frac{1}{\sqrt{1 + 16n^2}} \left( \frac{1}{\sqrt{1 + 16n^2}} - j \frac{4n}{\sqrt{1 + 16n^2}} \right) V_{sn} \end{aligned}$$

Let  $\theta_n = -\arctan(4n)$ :

$$\begin{aligned} V_{on} &= \frac{1}{\sqrt{1+16n^2}}(\cos \theta_n + j \sin \theta_n) V_{sn} \\ &= \frac{1}{\sqrt{1+16n^2}} e^{i\theta_n} V_{sn}, \quad n = 0, 1, 3, 5, \dots \end{aligned}$$

Since  $V_{sn} = |V_{sn}| e^{jn\omega_0 t} = \frac{2}{n\pi} e^{i2nt}$ :

$$\begin{aligned} V_{on} &= \frac{1}{\sqrt{1+16n^2}} e^{i\theta_n} V_{sn} \\ &= \frac{2}{n\pi\sqrt{1+16n^2}} e^{2nt+\theta_n} \end{aligned}$$

When  $n = 0$ :

$$V_{o0} = \frac{1}{2}$$

When  $n = 1$ :

$$\theta_1 = -\arctan(4 \times 1) = -75.96^\circ$$

$$V_{o1} = \frac{2}{1\pi\sqrt{1+16 \times 1^2}} e^{i2 \times 1t(-75.96)} = 0.1544 e^{i(2t-75.96)}$$

When  $n = 3$ :

$$\theta_3 = -\arctan(4 \times 3) = -85.24^\circ$$

$$V_{o3} = \frac{2}{3\pi\sqrt{1+16 \times 3^2}} e^{i2 \times 3t(-85.24)} = 0.0176 e^{i(6t-85.24)}$$

When  $n = 5$ :

$$\theta_5 = -\arctan(4 \times 5) = -87.14^\circ$$

$$V_{o5} = \frac{2}{5\pi\sqrt{1+16 \times 5^2}} e^{i2 \times 5t(-87.14)} = 0.0063 e^{i(10t-87.14)}$$

Therefore:

$$v_o(t) = 0.5 + 0.1544 \cos(2t-75.96^\circ) + 0.0176 \cos(6t-85.24^\circ) + 0.0063 \cos(10t-87.14^\circ)$$

## 2.16 Conditions for the Fourier series

To be described by the Fourier series, the waveform  $f(t)$  must satisfy the following mathematical properties:

- $f(t)$  is a **single-value** function, except at possibly a finite number of points.
- For any  $t_0$ , the integral  $\int_{t_0}^{t_0+T} |f(t)| dt < \infty$ .
- $f(t)$  has a finite number of **discontinuities** within the period  $T$ .
- $f(t)$  has a finite number of **maxima** and **minima** within the period  $T$ .

In practice,  $f(t)$  is usually an amplitude function, so the above 4 conditions are always satisfied.

## 2.17 Insights

### 2.17.1 Frequency response methods

Giving a different kind of insight into a system with insights of unexpected results.

### 2.17.2 Frequency spectrum

Focusing on how signals of different frequencies are represented in a signal thus with insights in terms of the spectrum of the signal.

### 2.17.3 Computer processing

Often, it is easier and more cost-effective to characterise the frequency content of a noise signal than to give a time description of the noise.

### 2.17.4 Applications

Different treatment of different parts of the electromagnetic spectrum means that you can separate the different radio, television and cell phone signals.

### 3 Bandwidth and frequency response

- It is important to estimate the spectrum of a signal when choosing a **measurement** system.
- Ideal **measurement** systems replicate all frequency components of an input signal.
- Practical **measurement** systems have limitations in reproducing all frequencies.

#### 3.1 Decibel scale

The common scale used to **measure** fidelity of a measurement system's reproduction at different frequencies is the decibel scale:

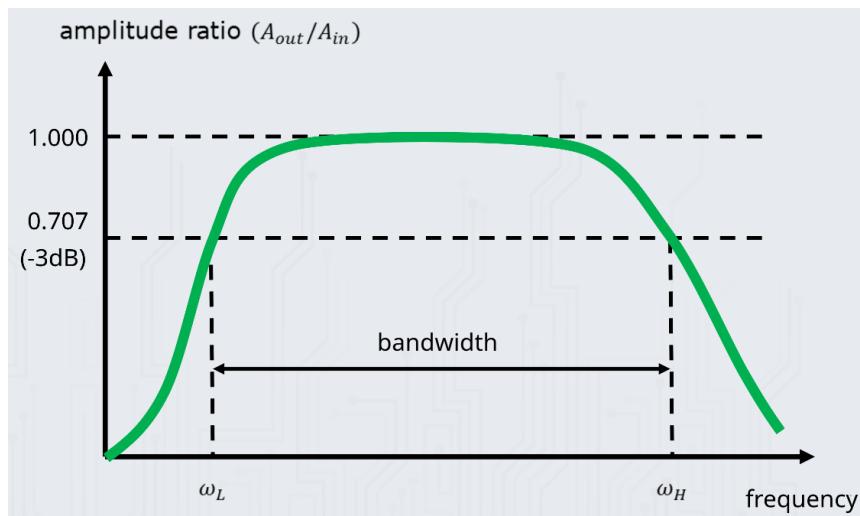
$$dB = 20 \log_{10} \left( \frac{A_{out}}{A_{in}} \right)$$

Where:

- $A_{in}$  is the input amplitude of a harmonic
- $A_{out}$  is the output amplitude of a harmonic

#### 3.2 Frequency response curve (Bode plot)

A frequency response curve or a Bode plot plots  $\frac{A_{out}}{A_{in}}$  versus input frequency.



### 3.3 Bandwidth

Bandwidth is the range of the frequencies where the input is not attenuated, i.e. the amplitude is not reduced, or the volume is not reduced, by more than  $-3\text{ dB}$ , i.e.

$$\text{Bandwidth} = \omega_L \text{ to } \omega_h$$

Where:

- $\omega_L$  is the low cut-off or corner frequency
- $\omega_h$  is the high cut-off or corner frequency

### 3.3.1 Why $-3$ dB?

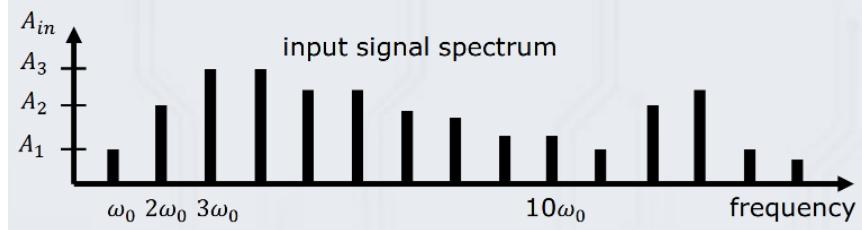
The value comes from half of the output power over the input power, i.e.

$$\begin{aligned}\frac{1}{2} &= \frac{P_{out}}{P_{in}} = \left( \frac{A_{out}}{A_{in}} \right)^2 \\ \Rightarrow \frac{A_{out}}{A_{in}} &= \sqrt{\frac{1}{2}} \\ \Rightarrow \text{dB} &= 20 \log_{10} \sqrt{\frac{1}{2}} \\ &\approx -3 \text{ dB}\end{aligned}$$

### 3.3.2 Example

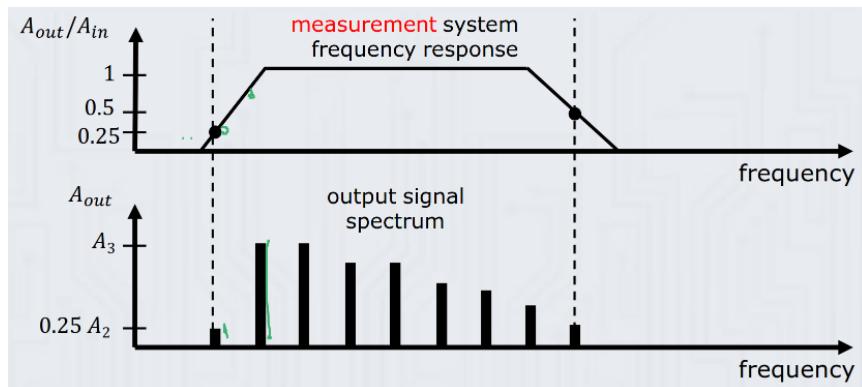
Calculating output amplitude  $A'_i$  given a **measurement** frequency response curve, with the input signal spectrum as:

$$V_{in}(t) = A_1 \sin(\omega_0 t) + A_2 \sin(2\omega_0 t) + A_3 \sin(3\omega_0 t) + \dots$$



The output amplitude  $A'_i$  is calculated as:

$$A'_i = \left( \frac{A_{out}}{A_{in}} \right) A_i$$



## 4 Periodic functions

### 4.1 Definition

A periodic function is any function of time that satisfies the following:

$$f(t + T) = f(t)$$

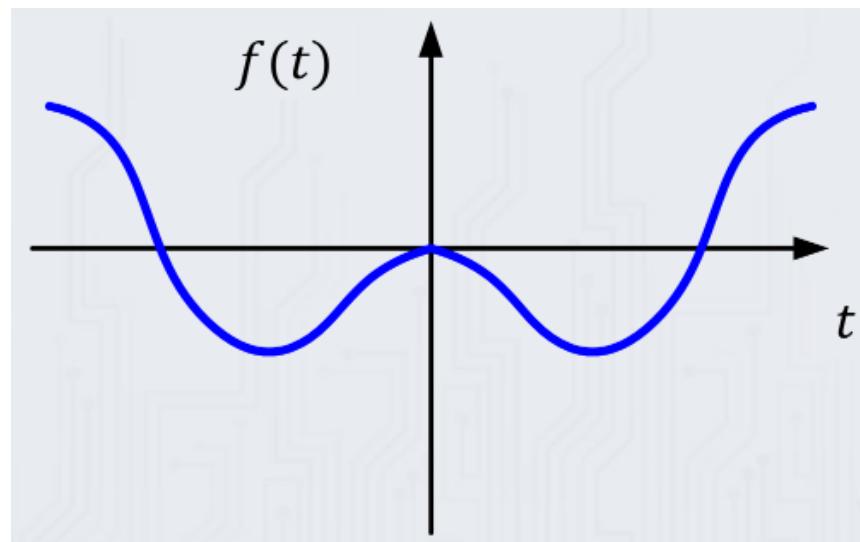
Where:

- $T$  is a constant called the **period** of the function

### 4.2 Even-function symmetry

Any function of time  $f(t)$  that satisfies the below condition is called an **even function**.

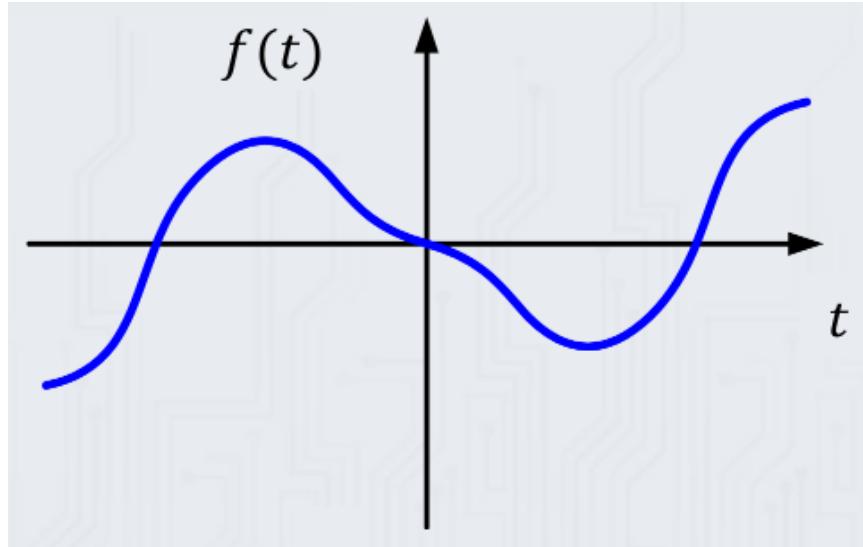
$$f(-t) = f(t)$$



### 4.3 Odd-function symmetry

Any function of time  $f(t)$  that satisfies the below condition is called an **odd function**.

$$f(-t) = -f(t)$$



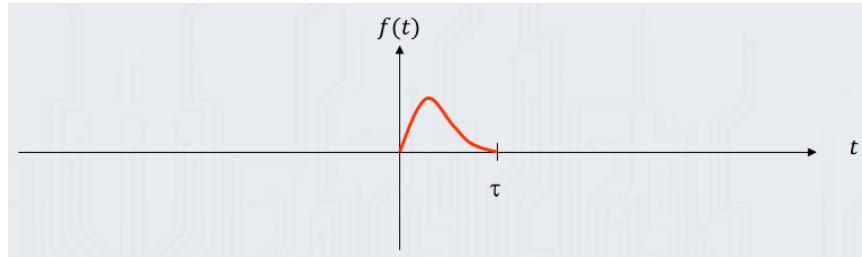
### 4.4 Properties of symmetric functions

Let  $f(t)$  be a periodic function with period  $T$ .

$f(t)$	$\left  \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt \right $
Even	$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = 2 \int_0^{\frac{T}{2}} f(t) dt$
Odd	$\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = 0$

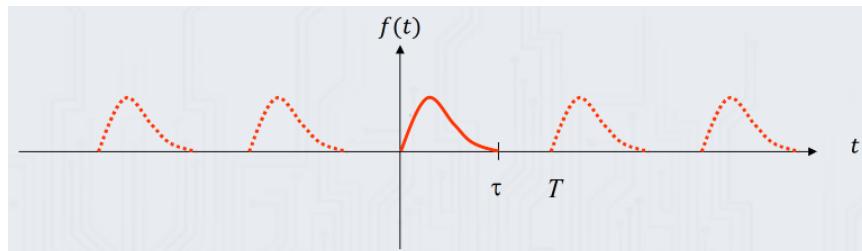
## 4.5 Conversion from non-periodic to periodic

### 4.5.1 Original pattern



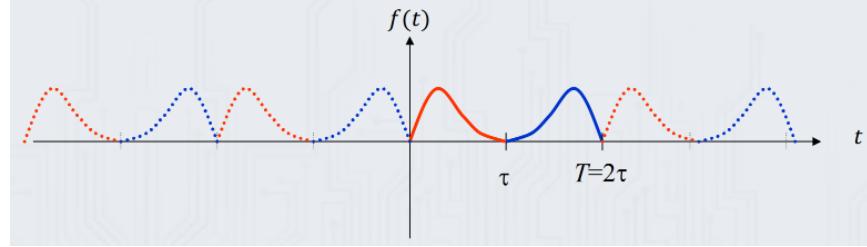
A non-periodic function  $f(t)$  defined over  $(0, t)$  can be expanded into a Fourier series which is defined only in the interval  $(0, t)$ . Note that the original pattern may not necessarily pass the origin.

### 4.5.2 Without considering symmetry



One simple technique that can be applied is to offset the original pattern along the time axis by a distance of  $nT$  ( $\tau < T$ ),  $n = \pm 1, \pm 2, \pm 3, \dots$

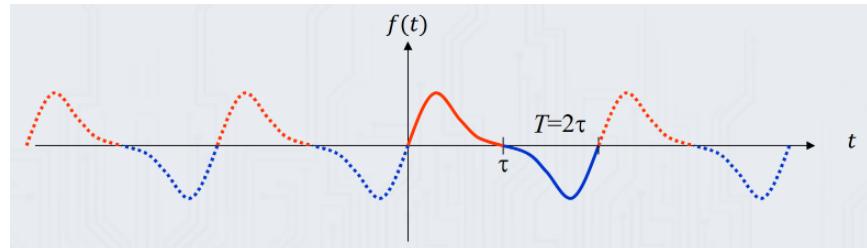
#### 4.5.3 Expansion into even-function symmetry



A second pattern can be created by mirroring the original pattern against an axis  $t = \tau$ .

An even-function symmetric periodic waveform can be generated by offsetting the two patterns merged along the time axis by a distance  $nT(T = 2\tau), n = \pm 1, \pm 2, \pm 3, \dots$

#### 4.5.4 Expansion into odd-function symmetry

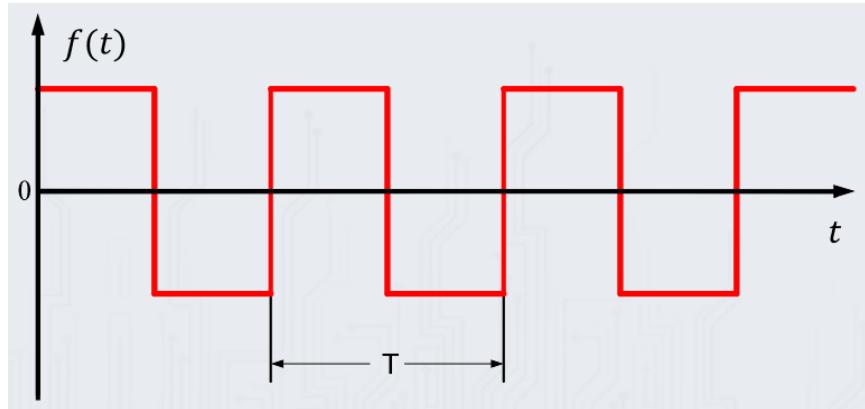


A third pattern can be created by mirroring the original pattern against the time axis and then the axis  $t = \tau$ .

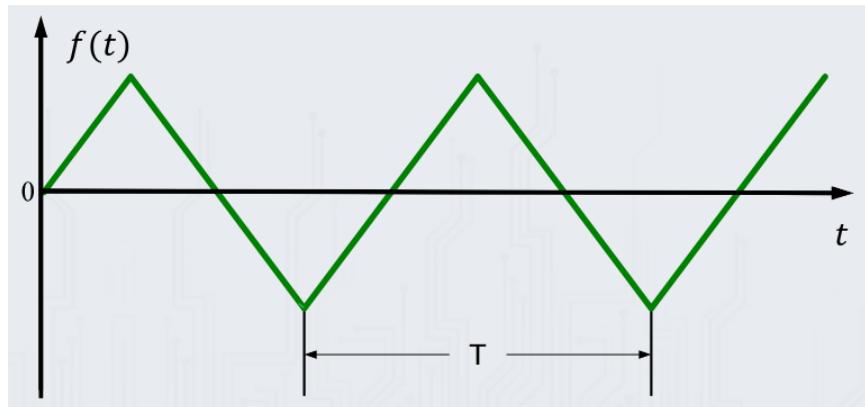
An odd-function periodic waveform can be generated by offsetting the two patterns merged along the time axis by a distance  $nT(T = 2\tau), n = \pm 1, \pm 2, \pm 3, \dots$

## 4.6 Examples

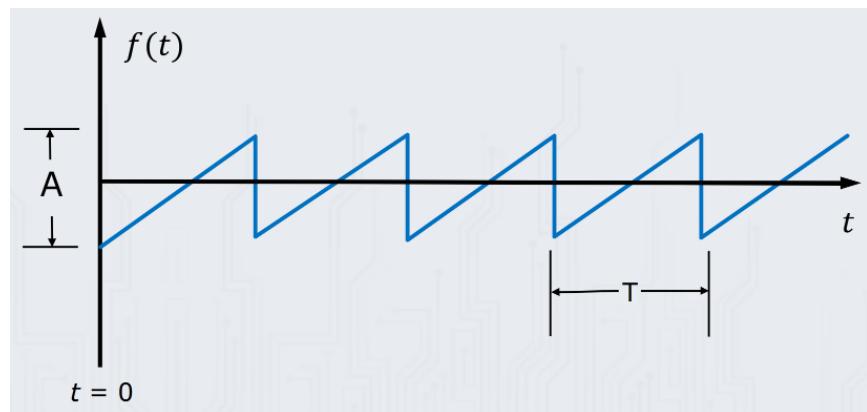
### 4.6.1 Square signal



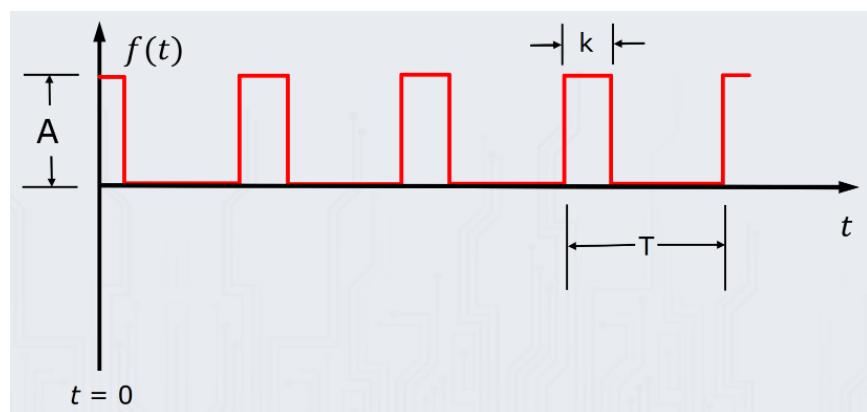
### 4.6.2 Triangular signal



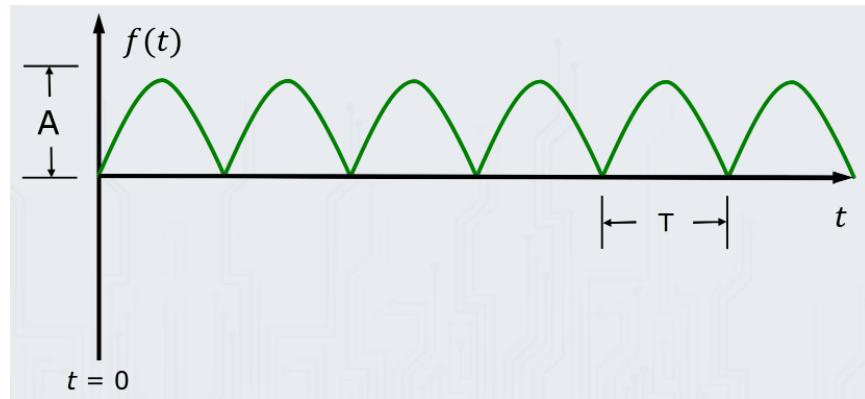
#### 4.6.3 Sawtooth signal



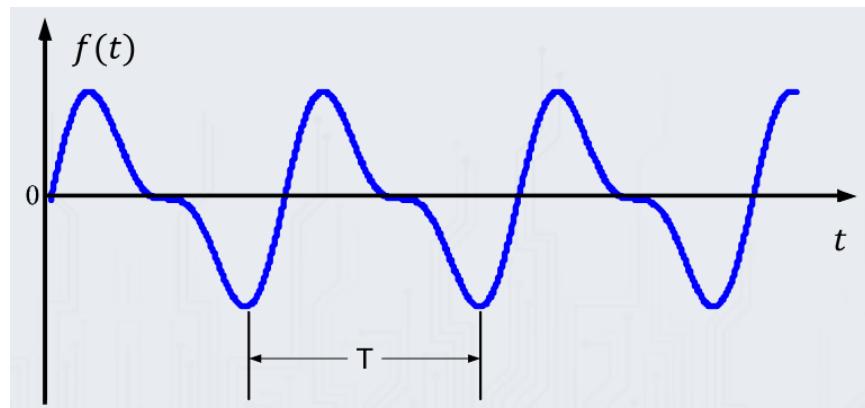
#### 4.6.4 Pulse signal



#### 4.6.5 Rectified signal



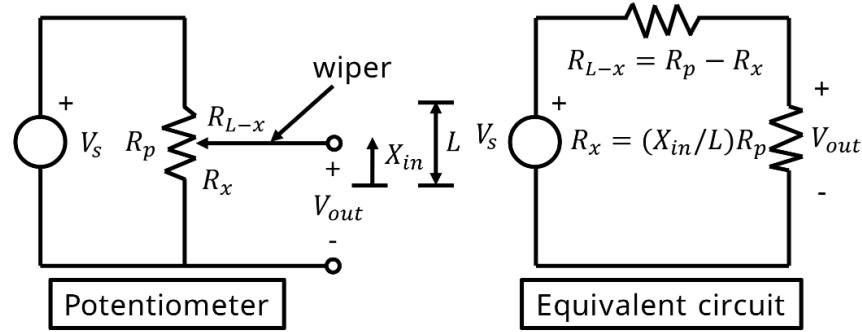
#### 4.6.6 General periodic signal



## 5 Dynamic systems

### 5.1 Example 1

A linear potentiometer used as a position sensor.



The system behaviour is:

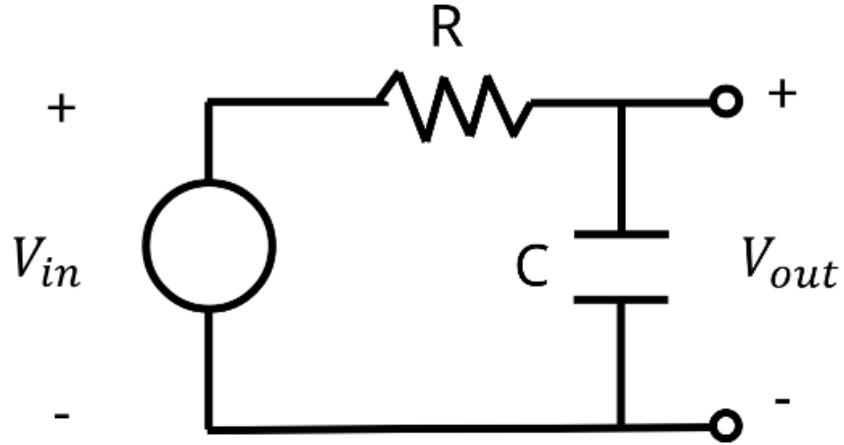
$$V_{out} = \frac{R_x}{R_p} V_s = \frac{V_s}{L} X_{in}$$

Where:

- $X_{in}$  is the wiper displacement with the potentiometer
- $R_p$  is the maximum resistance of the potentiometer
- $R_x$  is the resistance between the potentiometer leads
- $L$  is the maximum amount of wiper travel

## 5.2 Example 2

A resistor-capacitor circuit.



In this system, applying Kirchhoff's Laws and the voltage-current relations for a resistor and capacitor produces a first order linear differential equation relating the output voltage to the input voltage.

The system behaviour is:

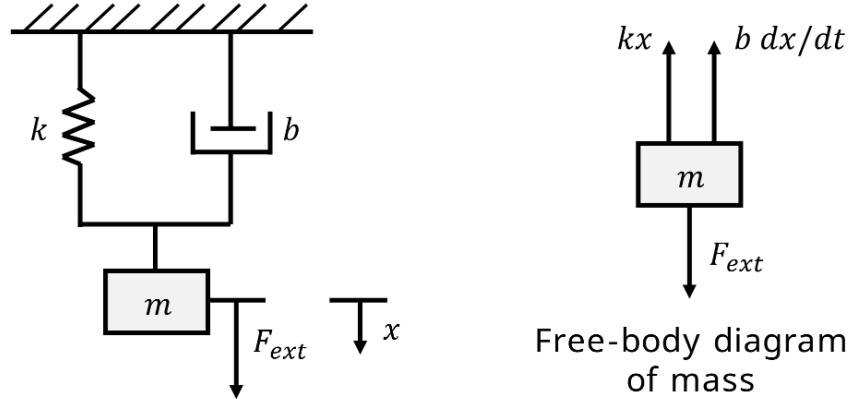
$$RC \frac{dV_{out}}{dt} + V_{out} = V_{in}$$

Where:

- $R$  is the resistance of the resistor
- $C$  is the capacitance of the capacitor
- $V_{out}$  is the output voltage
- $V_{in}$  is the input voltage

### 5.3 Example 3 (second-order system)

A spring damping system.



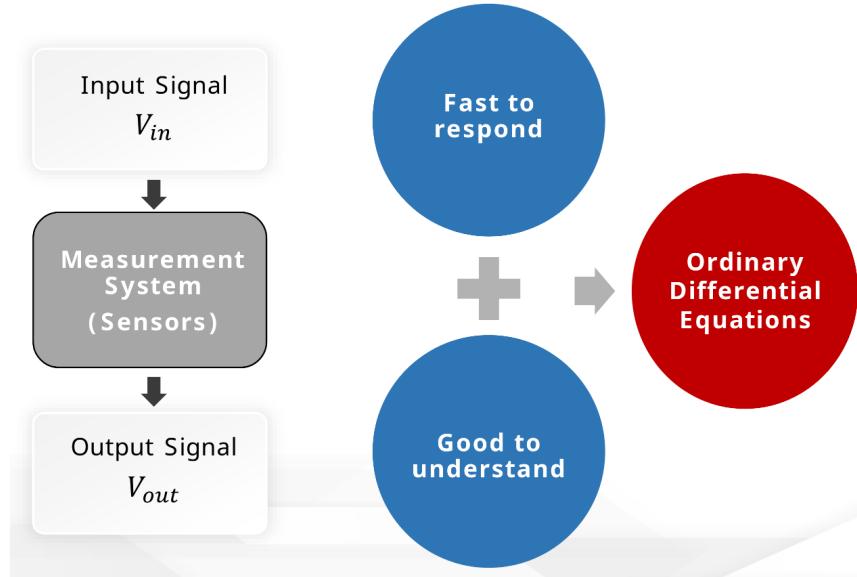
The system behaviour is:

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_{ext}(t)$$

Where:

- $m$  is the mass of the block
- $b$  is the damping coefficient
- $k$  is the spring constant
- $x$  is the displacement of the mass from the equilibrium (rest) position of the mass
- $F_{ext}(t)$  is the external force along the  $x$ -direction

## 5.4 Measurement system: Ordinary differential equations



### 5.4.1 Why ordinary differential equations?

- Ordinary differential equations have time as the only variable.
- Ordinary differential equations can be used to explain the behaviour of a dynamic system.
- At steady state, there is no change, which means there is no need to use ordinary differential equations in steady state.

## 6 Linear systems

- Linear systems are of the form:

$$\begin{aligned} A_n \frac{d^N X_{out}}{dt^N} + A_{N-1} \frac{d^{N-1} X_{out}}{dt^{N-1}} + \cdots + A_1 \frac{d X_{out}}{dt} + A_0 X_{out} \\ = B_M \frac{d^M X_{in}}{dt^M} + B_{M-1} \frac{d^{M-1} X_{in}}{dt^{M-1}} + \cdots + B_1 \frac{d X_{in}}{dt} + B_0 X_{in} \end{aligned}$$

- Alternatively:

$$\sum_{n=0}^N A_n \frac{d^n X_{out}}{dt^n} = \sum_{m=0}^M B_m \frac{d^m X_{in}}{dt^m}$$

- The word "linear" comes from the coefficients:

$$A_n (n = 0, \dots, N) \text{ and } B_m (m = 0, \dots, M)$$

Where:

- $X_{in}$  and  $X_{out}$  are input and output variables
- $A_n$  and  $B_m$  are coefficients
- $N$  is the order of the system

### 6.1 Homogeneous equation of a linear system

$$\sum_{n=0}^N A_n \frac{d^n X_{out}}{dt^n} = 0$$

Where:

- $X_{out}$  is the output variables
- $A_n$  is a coefficient
- $N$  is the order of the system

### 6.2 Characteristic equation of a homogeneous equation

$$\sum_{n=0}^N A_n s^n = 0$$

Where:

- $A_n$  and  $s$  are coefficients
- $N$  is the order of the system

### 6.2.1 Primary ( $N = 1$ )

$$A_1 s + A_0 = 0$$

$$s = \frac{A_0}{A_1}, \text{ if } A_0 \neq 0$$

### 6.2.2 Quadratic ( $N = 2$ )

$$A_2 s^2 + A_1 s + A_0 = 0$$

$$s = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_0 A_2}}{2A_2}, \text{ if } A_2 \neq 0$$

## 6.3 Roots of the characteristic equation

$$\sum_{n=0}^N A_n s^n = 0, A_N \neq 0$$

### 6.3.1 When $N = 1$

Single real root:

$$s_1 = r$$

Where:

- $s_1$  is the coefficient of the characteristic equation
- $r$  is the root

### 6.3.2 When $N = 2$

- Double real roots:

$$s_1 = s_2 = r$$

- Two different real roots:

$$s_1 \neq s_2$$

- Two conjugate roots:

$$s_1 = a + bi, \quad s_2 = a - bi$$

### 6.3.3 When $N = k$

Multiple k-fold real roots:

$$s_1 = \dots = s_k = r$$

## 6.4 Solving the homogeneous equation

### 6.4.1 When $N = 1$

- Single real root:  $s_1 = r$
- General solution for the homogeneous equation:

$$C_0 e^{rt}$$

### 6.4.2 When $N = 2$

1. Two conjugate roots:

$$s_1 = a + bi, \quad s_2 = a - bi$$

General solution for the homogeneous equation:

$$(C_1 \sin(bt) + C_2 \cos(bt))e^{at}$$

2. Two different real roots:

$$s_1 \neq s_2$$

General solution for the homogeneous equation:

$$C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

3. Double real roots:

$$s_1 = s_2 = r$$

General solution for the homogeneous equation:

$$(C_1 + C_2 t)e^{rt}$$

### 6.4.3 When $N = k$

- Multiple k-fold real roots:

$$s_1 = s_2 = \dots = s_k = r$$

- General solution for the homogeneous equation:

$$(C_0 + C_1 t + C_2 t^2 + \dots + C_{k-1} t^{k-1})e^{rt}$$

## 6.5 Input functions in a linear system

- Step input
- Sinusoidal input
- Pulse input
- Square input

## 6.6 Special cases of linear systems

### 6.6.1 Zero-order system

- $M = 0$
- $N = 0$

$$A_0 X_{out} = B_0 X_{in}$$

Where:

- $A_0$  and  $B_0$  are coefficients
- $X_{out}$  and  $X_{in}$  are output and input variables

### 6.6.2 First-order system

- $M = 0$
- $N = 1$

$$A_1 \frac{dX_{out}}{dt} + A_0 X_{out} = B_0 X_{in}$$

Where:

- $A_1$ ,  $A_0$  and  $B_0$  are coefficients
- $X_{out}$  and  $X_{in}$  are output and input variables

### 6.6.3 Second-order system

- $M = 0$
- $N = 2$

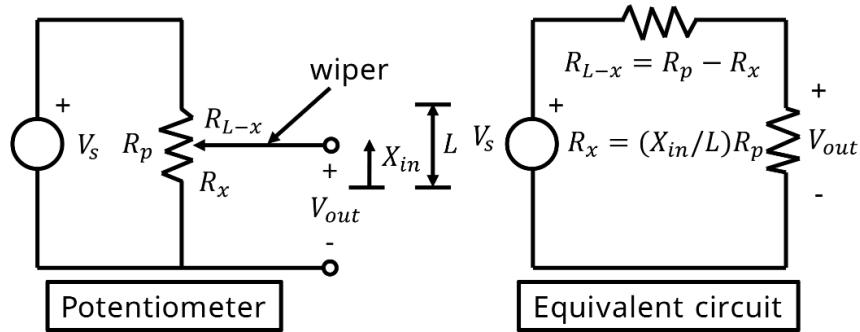
$$A_2 \frac{d^2 X_{out}}{dt^2} + A_1 \frac{dX_{out}}{dt} + A_0 X_{out} = B_0 X_{in}$$

Where:

- $A_1, A_2, A_0$  and  $B_0$  are coefficients
- $X_{out}$  and  $X_{in}$  are output and input variables

### 6.7 Zero-order system

#### 6.7.1 Example



Where:

- $X_{in}$  is the wiper displacement with the potentiometer
- $R_p$  is the maximum resistance of the potentiometer
- $R_x$  is the resistance between the potentiometer leads
- $L$  is the maximum amount of wiper travel

System behaviour:

$$V_{out} = \frac{R_x}{R_p} V_s = \frac{V_s}{L} X_{in}$$

Zero-order system:

$$A_0 X_{out} = B_0 X_{in}$$

### 6.7.2 General zero-order system

$$A_0 X_{out} = B_0 X_{in}$$

$$X_{out} = \frac{B_0}{A_0} X_{in}$$

$$X_{out} = K X_{in}$$

Where:

- $X_{out}$  and  $X_{in}$  are output and input variables
- $K$  is a constant called **gain** or **sensitivity**

### 6.7.3 Remarks

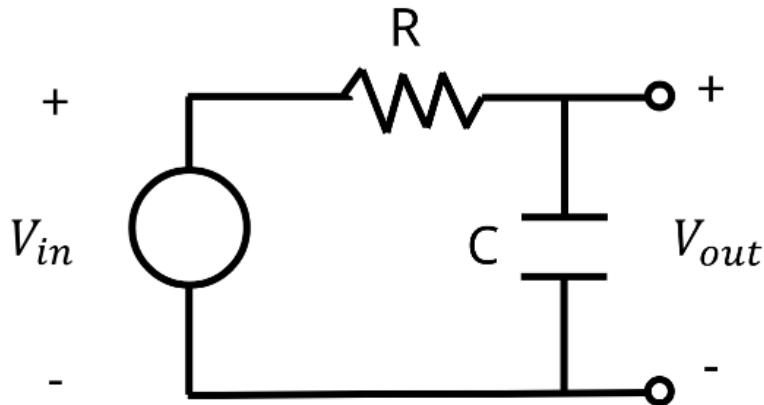
A zero-order system follows the input exactly without any time delay or distortion.

Input Signal  $X_{in} \rightarrow$  Degenerated differential equations  $\rightarrow$  Output signal  $X_{out}$

The input signals can be of any periodic waveform.

## 6.8 First-order system

### 6.8.1 Example



In this system, applying Kirchhoff's Laws and the voltage-current relations for a resistor and capacitor produces a first order linear differential equation relating the output voltage to the input voltage.

System behaviour:

$$RC = \frac{dV_{out}}{dt} + V_{out} = V_{in}$$

### 6.8.2 General first-order system

When  $N = 1$  and  $M = 0$ :

$$\begin{aligned} A_1 \frac{dX_{out}}{dt} + A_0 X_{out} &= B_0 X_{in} \\ \rightarrow \tau \frac{dX_{out}}{dt} + X_{out} &= K X_{in} \\ &= K X_{in} \end{aligned}$$

Where:

- $K = \frac{B_0}{A_0}$  is the static sensitivity
- $\tau = \frac{A_1}{A_0}$  is the time constant

Hence, the first-order system equation can be written as:

$$\tau \frac{dX_{out}}{dt} + X_{out} = K X_{in}$$

Note that in this standard form, the coefficient of the  $X_{out}$  term must be 1, hence:

$$A_0 \neq 0$$

### 6.8.3 Step response of first-order systems

The step input changes instantaneously from 0 to a constant value  $A_{in}$  and is stated mathematically as:

$$X_{in} = \begin{cases} 0 & t < 0 \\ A_{in} & t \geq 0 \end{cases}$$

The output of the system in response to this input is called the step response of the system. For a first-order system, we can find the step response by solving the first-order ordinary differential equation below:

$$\tau \frac{dX_{out}}{dt} + X_{out} = K X_{in}$$

Initial condition:

$$X_{out}(0) = 0$$

Characteristic equation:

$$\tau s + 1 = 0$$

Roots of the characteristic equation:

$$s = -\frac{1}{\tau}$$

#### 6.8.4 Solving the homogeneous equation

- Linear system:

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

- Homogeneous equation:

$$\tau s + 1 = 0$$

- Root:

$$r = -\frac{1}{\tau}$$

- General solution for the homogeneous equation:

$$X_{out_h} = C_0 e^{-\frac{t}{\tau}}$$

Where:

–  $C_0$  is a constant determined later by applying initial conditions

- A particular or steady state solution resulting from the step input  $X_{in} = A_{in}$ :

$$X_{out_p} = KA_{in}$$

- General solutions for the linear system:

$$X_{out} = X_{out_h} + X_{out_p} = C_0 e^{-\frac{t}{\tau}} + KA_{in}$$

#### 6.8.5 Determining the step response of the first-order system

Determining the constant by initial conditions:

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

$$X_{out} = X_{out_h} + X_{out_p} = C e^{-\frac{t}{\tau}} + KA_{in}$$

Applying the initial condition  $X_{out}|_{t=0} = X_{out}(0)$  to this equation gives:

$$X_{out}(0) = C + KA_{in}$$

Thus:

$$C = X_{out}(0) - KA_{in}$$

So, the resulting step response is:

$$X_{out} = X_{out}(0)e^{-\frac{t}{\tau}} + KA_{in}(1 - e^{-\frac{t}{\tau}})$$

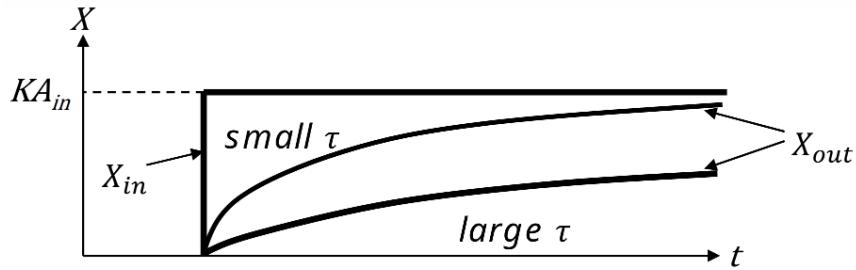
If  $X_{out}(0) = 0$ :

$$X_{out} = KA_{in}(1 - e^{-\frac{t}{\tau}})$$

### 6.8.6 Graph of the step response of the first-order system

$$\begin{aligned}\tau \frac{dX_{out}}{dt} + X_{out} &= KX_{in} \\ \rightarrow X_{out} &= KA_{in}(1 - e^{-\frac{t}{\tau}})\end{aligned}$$

$$\begin{aligned}X_{out} &= X_{out_h} + X_{out_p} = Ce^{-\frac{t}{\tau}} + KA_{in} \\ \rightarrow X_{out}(0) &= 0\end{aligned}$$



Step Response of First-order System

- The graph above represents an exponential rise in the output toward an asymptotic value of  $KA_{in}$ .
- The rate of rise depends only on the time constant  $\tau$ .
- The response is faster for a smaller time constant.
- After one time constant, the output reaches 63.2% of its final value:

$$X_{out}(t = \tau) = KA_{in}(1 - e^{-\frac{\tau}{\tau}}) = 0.632KA_{in}$$

- After four time constants, the step response is:

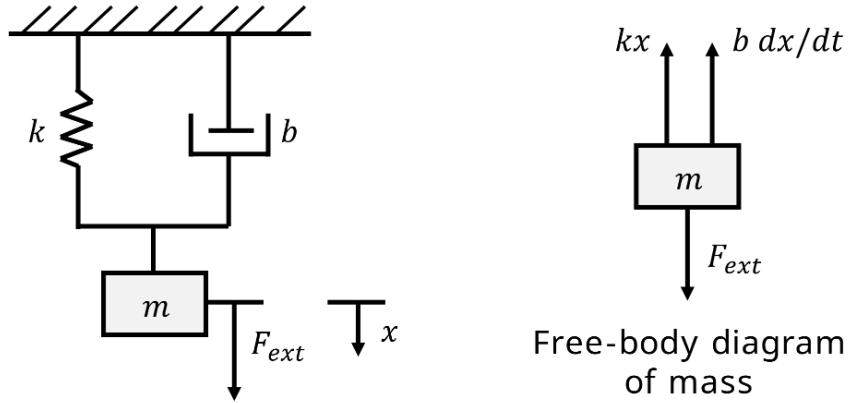
$$X_{out}(t = 4\tau) = KA_{in}(1 - e^{-\frac{4\tau}{\tau}}) = 0.982KA_{in}$$

- Since this value is more than 98% of the steady state value  $KA_{in}$ , we usually assume that a first-order system has reached its steady state value within four time constants.
- When designing a first-order measurement system, look at quantities that affect  $\tau$  and try to reduce them if possible.
- The larger  $\tau$  is, the longer the measurement system takes to respond to an input.

## 6.9 Second-order system

$$A_2 \frac{d^2 X_{out}}{dt^2} + A_1 \frac{dX_{out}}{dt} + A_0 X_{out} = B_0 X_{in}$$

### 6.9.1 Example



Where:

- $m$  is the mass of the block
- $b$  is the damping coefficient
- $k$  is the spring constant
- $x$  is the displacement of the mass from the equilibrium (rest) position of the mass
- $F_{ext}(t)$  is the external force along the  $x$ -direction

System behaviour:

$$m \frac{d^2 x}{dt^2} + b \frac{dx}{dt} + kx = F_{ext}(t)$$

### 6.9.2 Equations

- $M = 0, N = 2$
- Homogeneous equation:

$$A_2 \frac{d^2 X_{out}}{dt^2} + A_1 \frac{dX_{out}}{dt} + A_0 X_{out} = 0$$

- Characteristic equation:

$$A_2 s^2 + A_1 s + A_0, \quad A_2 \neq 0$$

- Roots of the characteristic equation:

$$A_2 s^2 + A_1 s + A_0 = 0, \quad A_2 \neq 0$$
$$s = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_0 A_2}}{2A_2}, \text{ if } A_2 \neq 0$$

### 6.9.3 Solving the homogeneous equation

- Homogeneous equation:

$$\sum_{n=0}^N A_n \frac{d^n X_{out}}{dt^n} = 0$$

- Characteristic equation:

$$\sum_{n=0}^N A_n s^n = 0, \quad A_N \neq 0$$

- Two conjugate roots:

$$s_1 = a + bi, \quad s_2 = a - bi$$

General solution for the homogeneous equation:

$$(C_1 \sin(bt) + C_2 \cos(bt))e^{at}$$

- Two different real roots:

$$s_1 \neq s_2$$

General solution for the homogeneous equation:

$$C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

- Double real roots:

$$s_1 = s_2 = r$$

General solution for the homogeneous equation:

$$(C_1 + C_2 t) e^{rt}$$

#### 6.9.4 Unforced response of a second-order system

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

Characteristic equation of the second-order system:

$$ms^2 + bs + k = 0$$

Roots of the characteristic equation:

$$s_1 = -\frac{b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

$$s_2 = -\frac{b}{2m} - \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

#### 6.9.5 Unforced response without damping, with $b = 0$

$$m \frac{d^2x}{dt^2} + kx = 0, \quad ms^2 + k = 0$$

Roots of the second order system:

$$s_1 = i\sqrt{k}m, \quad s_2 = -i\sqrt{k}m$$

Homogeneous solution:

$$x_h(t) = A \cos\left(\sqrt{\frac{k}{m}}t\right) + B \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Coefficients  $A$  and  $B$  could be determined by the initial conditions:

$$x(t=0), \quad \left. \frac{dx(t)}{dt} \right|_{t=0}$$

Natural frequency of undamped oscillatory motion with radian frequency:

$$\omega_n = \sqrt{\frac{k}{m}}$$

Under this frequency, the undamped system would naturally oscillate if the spring were stretched and the mass is released and allowed to move without any external force ( $F_{ext} = 0$ )

### 6.9.6 Unforced response without damping, with $b \neq 0$

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0, \quad ms^2 + bs + k = 0$$

If the Radicand = 0, the roots of the second-order system are:

$$\text{Radicand} = \sqrt{b^2 - 4mk} = 0 \rightarrow b^2 = 4mk$$

$$s_1 = s_2 = -\frac{b}{2m} = -\sqrt{\frac{4mk}{4m^2}} = -\sqrt{\frac{k}{m}}$$

Homogeneous solution:

$$x_h(t) = (A + Bt)e^{-\omega_n t} A = \pi r \omega_n^2 = \sqrt{\frac{k}{m}}$$

Coefficients  $A$  and  $B$  could be determined by the initial conditions:

$$x(t=0), \quad \left. \frac{dx(t)}{dt} \right|_{x=0}$$

Solution:

$$x_h(t) = (A + Bt)e^{-\omega_n t}$$

This represents an exponential decaying motion.

For critical damping, if the radicand = 0, the critical damping constant is:

$$b_c = 2\sqrt{mk} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

For non-critical damping, if the radicand  $\neq 0$ , the damping ratio is:

$$\zeta = \frac{b}{b_c} = \frac{b}{2\sqrt{mk}}$$

Note:

1. Damping ratio is a measure of the proximity to critical damping.
2. A critically damped system has a damping ratio of 1.

### 6.9.7 Properties

- Homogeneous equation:

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

- Characteristic equation:

$$ms^2 + bs + k = 0$$

- Roots of the second-order system

$$s = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Because:

$$\begin{aligned} b_c &= 2\sqrt{mk} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n \\ \zeta &= \frac{b}{b_c} = \frac{b}{2\sqrt{mk}} \end{aligned}$$

- 2 different real roots of the second-order system:

$$\begin{aligned} s &= \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} \\ &= \frac{-\frac{b}{b_c} \pm \sqrt{\frac{b^2}{b_c^2} - \frac{4mk}{b_c^2}}}{2 \cdot \frac{m}{b_c}} \\ &= \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\frac{1}{\omega_n}} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \end{aligned}$$

Where:

- $\zeta$  is the damping ratio
- $\omega_n$  is the natural frequency

### 6.9.8 Under damped system ( $\zeta < 1$ , with 2 complex conjugate roots)

- Roots

$$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1 - \zeta^2}$$

$$s_2 = -\zeta\omega_n - j\omega_n\sqrt{1 - \zeta^2}$$

- Homogeneous solution:

$$x_h(t) = e^{-\zeta\omega_n t} \left[ A \cos(\omega_n \sqrt{1 - \zeta^2} t) + B \sin(\omega_n \sqrt{1 - \zeta^2} t) \right]$$

- This motion represents damped oscillation consisting of sinusoidal motion with exponentially decaying amplitude.

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

- The frequency of oscillation is called the damped natural frequency.

### 6.9.9 Overdamped system ( $\zeta > 1$ , with 2 real roots)

- Roots:

$$s_1 = -\zeta\omega_n + \omega_n + \sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n + \sqrt{\zeta^2 - 1}$$

- Homogeneous solution:

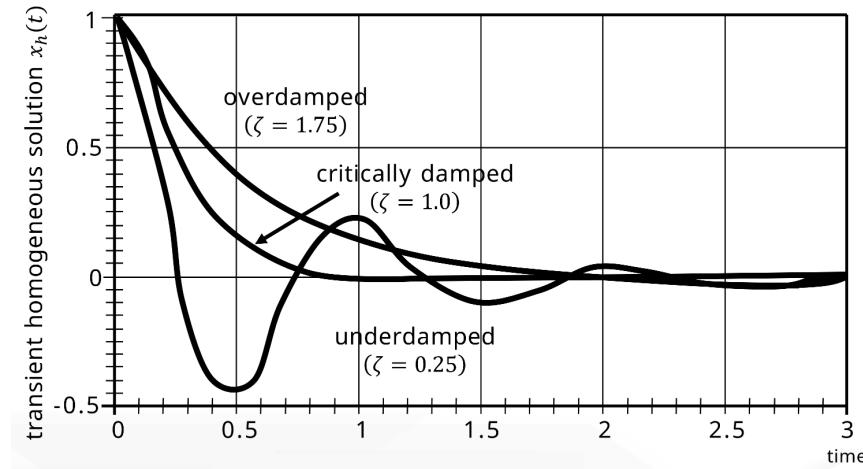
$$x_h(t) = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

- This motion represents an exponential decaying output.

### 6.9.10 Graphs of unforced responses

With initial conditions:

$$x(0) = 1, \quad \left. \frac{dx(t)}{dt} \right|_{t=0} = 0$$



### 6.9.11 Summary

Let's define:

$$\text{Natural frequency, } \omega_n = \sqrt{\frac{k}{m}}$$

$$\text{Damping ratio, } \zeta = \frac{b}{2\sqrt{km}}$$

We have:

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = 0$$

The characteristic equation is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Whose roots are:

$$s_1 = -\zeta\omega_n + \omega_n + \sqrt{\zeta^2 - 1}$$

$$s_2 = -\zeta\omega_n - \omega_n + \sqrt{\zeta^2 - 1}$$

As  $F_{ext} = 0$  (unforced).

When  $\zeta = 0$  (undamped):

$$x_h(t) = a \cos(\omega_n t) + B \sin(\omega_n t)$$

When  $\zeta < 1$  (under damped):

$$x_h(t) = e^{-\zeta\omega_n t} \left[ A \cos \left( \omega_n \sqrt{1 - \zeta^2} t \right) + B \sin \left( \omega_n \sqrt{1 - \zeta^2} t \right) \right]$$

When  $\zeta > 1$  (overdamped):

$$x_h(t) = A e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + B e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

Where the coefficients  $A$  and  $B$  are determined from the initial conditions.

### 6.9.12 Forced response of a second-order system

A second-order system will have forced response when  $F_{ext}(t) \neq 0$ . For the second-order system:

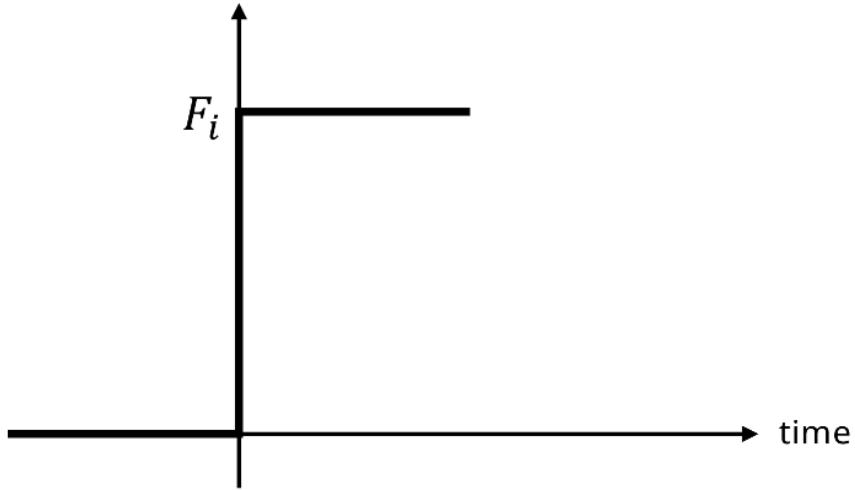
$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_{ext}(t)$$

Its solution can be obtained by combining a general solution ( $x_h(t)$ ) of its homogeneous equation, and a particular solution ( $x_p(t)$ ) of the second-order system.

$$x(t) = x_h(t) + x_p(t)$$

When the external force has step input:

$$F_{ext} = \begin{cases} 0 & t < 0 \\ F_i & t \geq 0 \end{cases}$$



It is easy to see that the second-order system  $m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_{ext}(t)$  has a particular solution:

$$x_p(t) = \frac{F_i}{k}$$

Because:

$$\frac{kF_i}{k} = F_i$$

### 6.9.13 Solving the homogeneous equation of the forced response of a second-order system

The homogeneous equation can be solved using the same technique developed for the unforced response of the second-order system.

As  $F_{ext} = 0$  (unforced response).

When  $\zeta = 0$  (undamped):

$$x_h(t) = A \cos(\omega_n t) + B \sin(\omega_n t)$$

When  $\zeta = 1$  (critically damped):

$$x_h(t) = (A + Bt)e^{-\omega_n t}$$

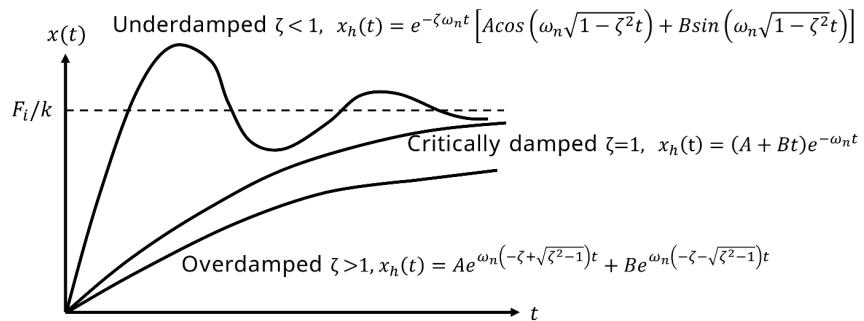
When  $\zeta < 1$  (under damped):

$$x_h(t) = e^{-\zeta \omega_n t} [A \cos(\omega_n \sqrt{1 - \zeta^2} t) + B \sin(\omega_n \sqrt{1 - \zeta^2} t)]$$

When  $\zeta > 1$  (overdamped):

$$x_h(t) = Ae^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + Be^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

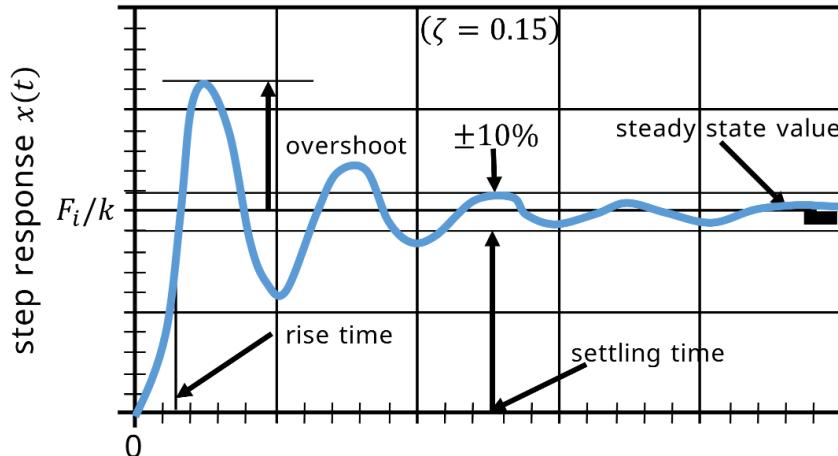
### 6.9.14 Graphs of forced responses



Initial conditions:

$$x(t=0) = 0, \quad \left. \frac{dx(t)}{dt} \right|_{t=0} = 0$$

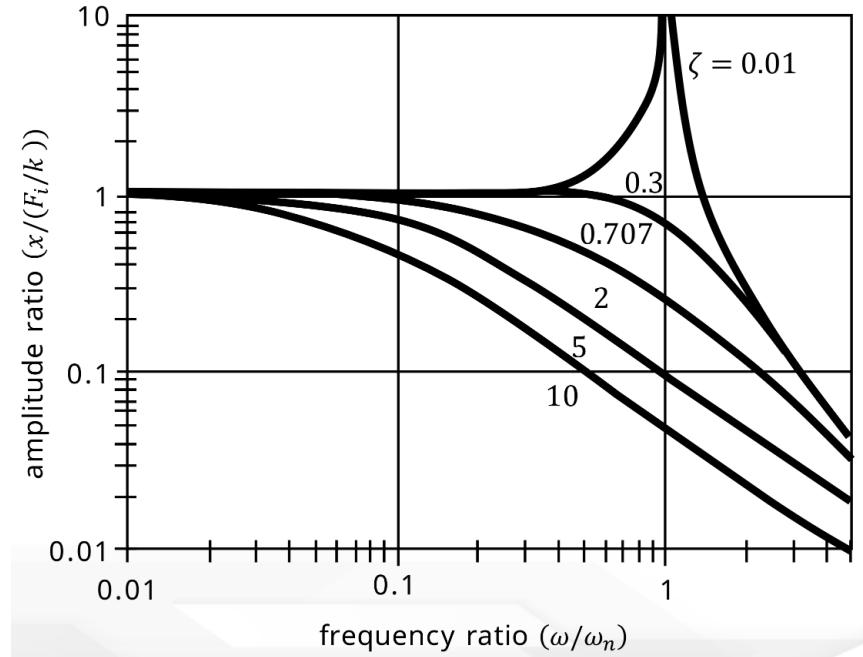
### 6.9.15 Characteristics of a graph of forced response



Where:

- "Steady state value" refers to the value where the system reaches after all transients have dissipated.
- "Rise time" refers to the time required for the system to go from 10% to 90% of the steady state value.
- "Over-shoot" is a measure of the amount the output exceeds the steady state value.
- "Settling time" refers to the time required for the system to settle to within an amplitude band whose width is a specific  $\pm 10\%$  of the steady state value.

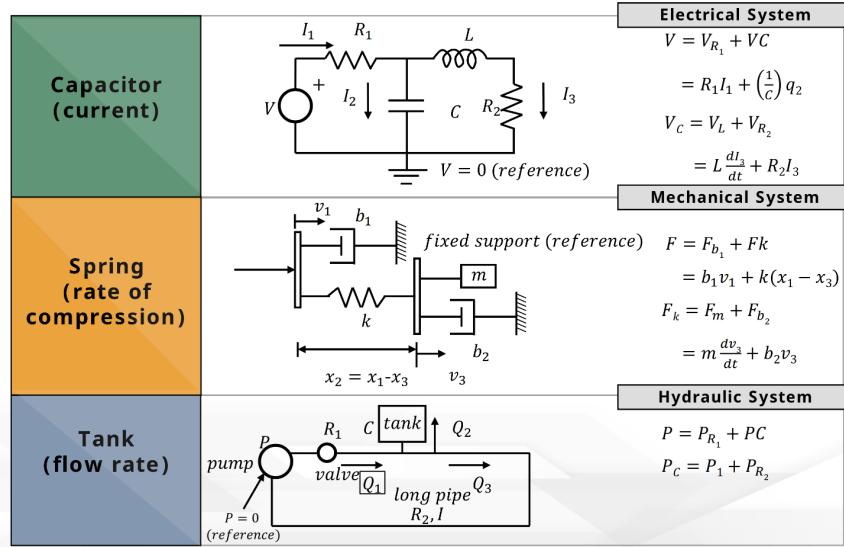
### 6.9.16 Forced response amplitude ratio vs frequency ratio graph



Note that  $\zeta = \frac{1}{\sqrt{2}} \approx 0.707$  provides the best amplitude linearity over the largest bandwidth.

## 7 System modelling and analogies

### 7.1 System models



The systems on the right-hand side of the image are the system that are analogous to the models on the left-hand side.

## 7.2 Second-order modelling analogies

Generic quantity	Mechanical translation	Mechanical rotation	Electrical	Hydraulic
Effort ( $E$ )	Force ( $F$ )	Torque ( $T$ )	Voltage ( $V$ )	Pressure ( $P$ )
Flow ( $F$ )	Speed ( $v$ )	Angular speed ( $\omega$ )	Current ( $i$ )	Volumetric flow rate ( $Q$ )
Displacement ( $q$ )	Displacement ( $x$ )	Angular displacement ( $\theta$ )	Charge ( $q$ )	Volume ( $V$ )
Momentum ( $p$ )	Linear momentum ( $p = mv$ )	Angular momentum ( $h = J\omega$ )	Flux linkage ( $I = N\Phi = Li$ )	$\frac{\text{Momentum}}{\text{Area}}$ ( $\Gamma = IQ$ )
Resistor ( $R$ )	Damper ( $b$ )	Rotatory damper ( $B$ )	Resistor ( $R$ )	Resistor ( $R$ )
Capacitor ( $C$ )	Spring ( $\frac{1}{k}$ )	Torsion spring ( $\frac{1}{k}$ )	Capacitor ( $C$ )	Tank ( $C$ )
Inertia ( $I$ )	Mass ( $m$ )	Moment of inertia ( $J$ )	Inductor ( $L$ )	Inertance ( $I$ )
Inertia energy storage (special case)	$F = \dot{p}$ ( $F = ma$ )	$T = \dot{h}$ ( $T = J\alpha$ )	$V = \dot{\lambda}$ ( $V = L \frac{di}{dt}$ )	$P = \dot{\Gamma}$ $\left( P = I \frac{dQ}{dt} \right)$
Capacitor energy storage	$F = kx$	$T = k\theta$	$V = \frac{1}{C}q$	$P = \frac{1}{C}V$
Dissipative	$F = bv$	$T = B\omega$	$V = Ri$	$P = RQ$

## 7.3 Similarities and differences

### 7.3.1 Similarities

- Mathematical representation
- Mathematical solution
- Mathematical properties

### 7.3.2 Differences

- Constants (coefficients, or parameters)
- Physical meanings of these parameters of the system

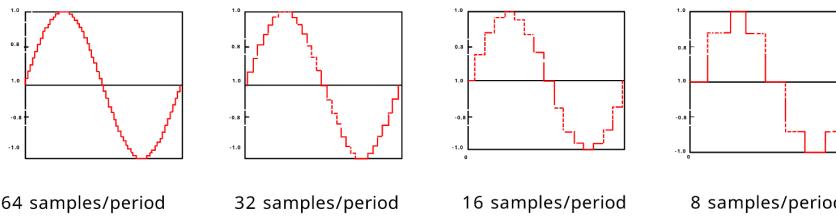
### 7.3.3 Analogies

- For those parameters among the different system types: Resistors, valves, mass, inertia, ...
- System terms: Effort, flow, displacement, momentum, resistance, capacitance, ...

## 8 Sampling

### 8.1 Sampling rate

- Higher sampling rates allow the waveform to be more accurately represented.
- Low sampling rates may lead the waveform to be less accurately represented.



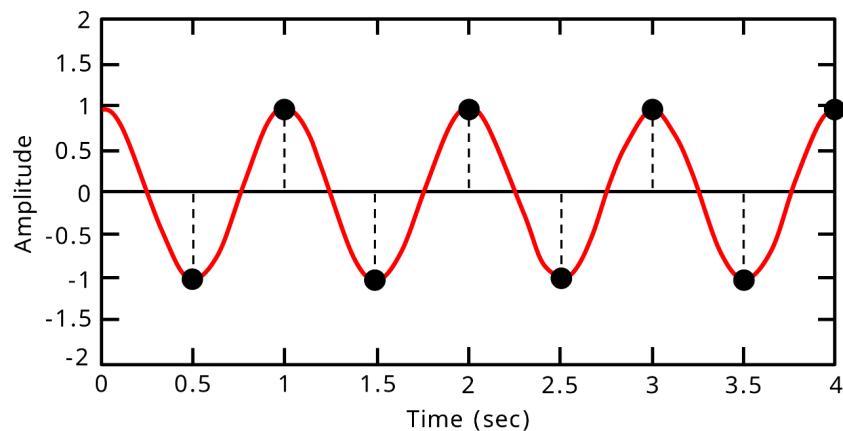
## 8.2 Analogue vs digital signals

Analogue signal	Digital signal
Continuous	Discrete
Generated via analogue devices	Sampled in a fixed interval
Not coded	Coded value
Original signal	Sequential data array

## 8.3 Shannon & Nyquist theorem

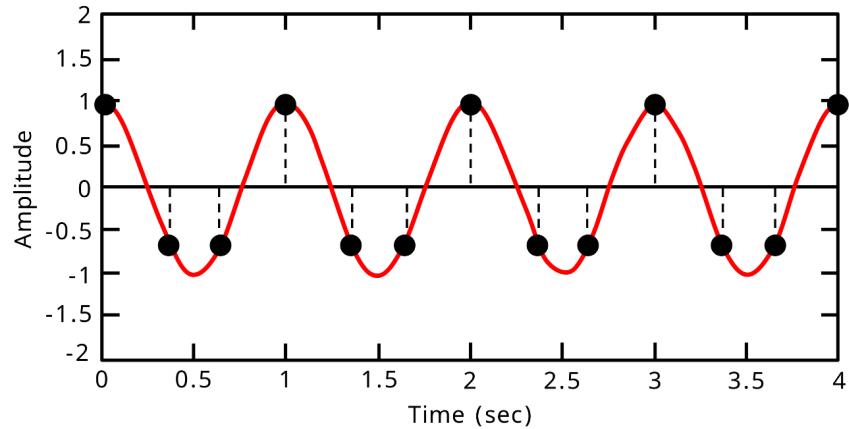
The best explanation for the Shannon & Nyquist sampling theorem is this YouTube video.

### 8.3.1 Sampling 1 Hz sine wave at 2 Hz



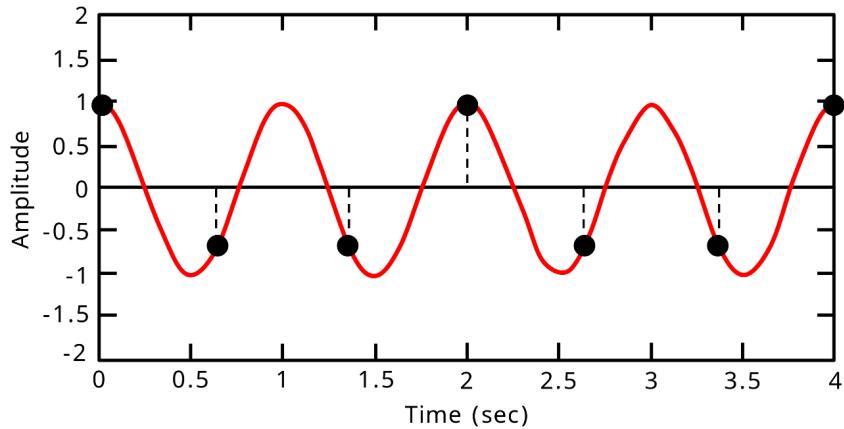
There are sufficient samples to capture each peak and trough of the signal.

### 8.3.2 Sampling 1 Hz sine wave at 3 Hz



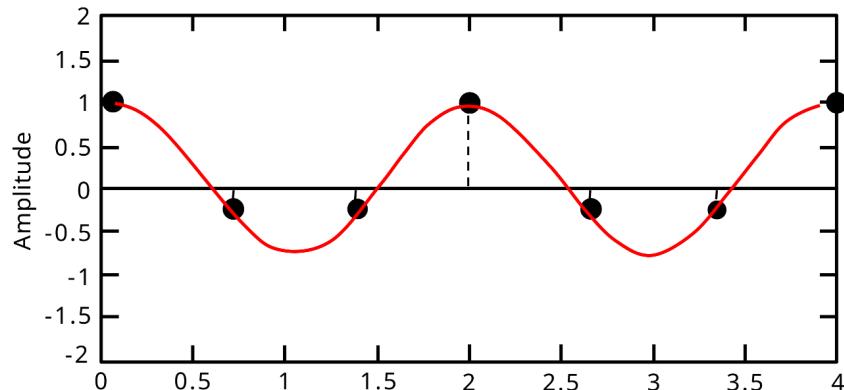
There are more than enough samples to capture the variations in the signal.

### 8.3.3 Sampling 1 Hz sine wave at 1.5 Hz



There isn't enough samples to capture all the peaks and troughs in the signal, which results in information being lost.

The signal may also be misinterpreted as a 0.5 Hz signal, as shown below:



### 8.4 Why don't we sample as fast as possible?

- Sampling as fast as possible results in huge amounts of data.
- It also requires high speed software.
- A lot of storage is needed to store the data.

## 8.5 Logic behind the minimum sampling rate

- We need to sample a digital signal at a rate more than 2 times the **maximum frequency** ( $f_{max}$ ) **component** in the signal to retain all frequency components.
- To faithfully represent the analogue signal, the digital samples must be taken at a frequency  $f_s$ , such that:

$$f_s > 2f_{max}$$

Where:

- $f_s$  is the **sampling rate** (not sampling frequency)
- $f_{max}$  is the maximum frequency in the signal, also known as the **Nyquist frequency**

- If we approximate a signal by a truncated Fourier series ( $N$  terms), the maximum frequency component is the **highest harmonic frequency**. Hence, the time interval between the digital samples is:

$$\Delta t = \frac{1}{f_s}$$

## 8.6 Theorem

$$F(t) = \sum_{n=0}^N C_n \cos(n\omega_0 t + \phi_n)$$

$$f(t) = \sum_{n=-N}^N D_n e^{jn\omega_0 t}$$

Where:

- $N$  is the maximum frequency component of the signal
- $f_s$  is the sampling rate
- $f_{max}$  is the Nyquist frequency

### 8.6.1 Shannon-Nyquist Theorem

$$f_s > 2f_{max}$$

Where:

- $f_s$  is the sampling rate
- $f_{max}$  is the Nyquist frequency

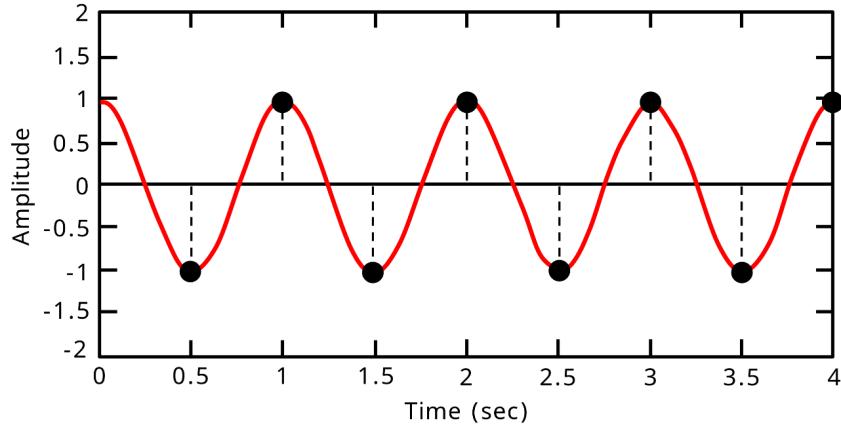
### 8.6.2 Time interval between the digital samples ( $\Delta t$ )

$$\Delta t = \frac{1}{f_s}$$

Where:

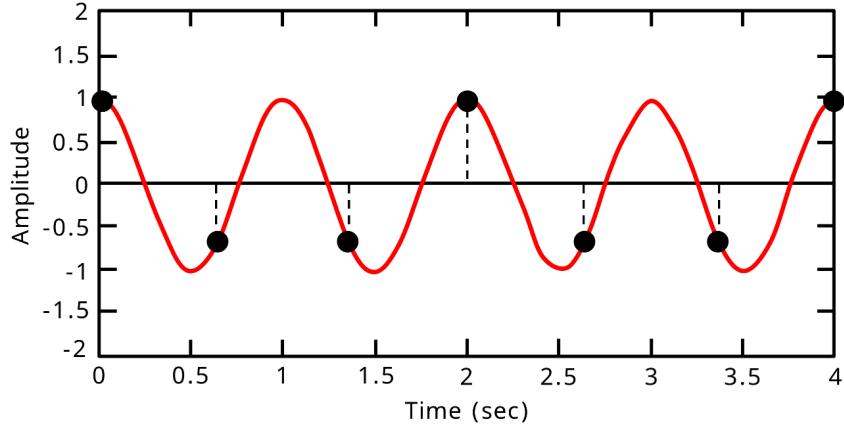
- $\Delta t$  is the time interval between the digital samples
- $f_s$  is the sampling rate

### 8.6.3 Sampling 1 Hz sine wave at 2 Hz



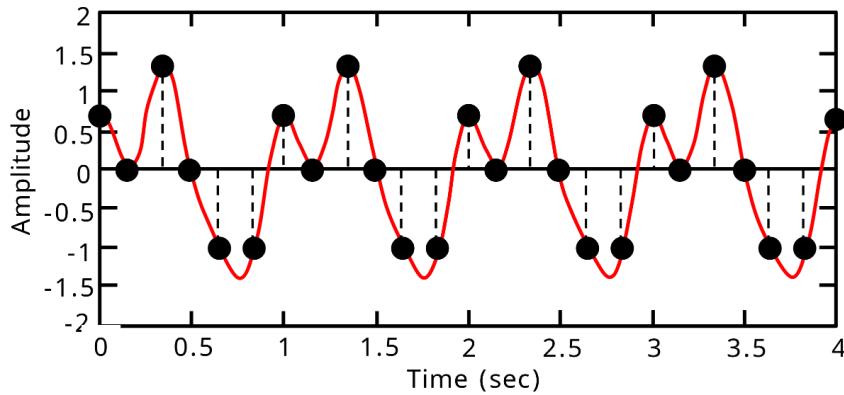
- Maximum frequency component:  $N = 1$
- Nyquist frequency:  $f_{max} = 1 \text{ Hz}$
- Sampling rate:  $f_s = 2 \text{ Hz}$
- Time interval between digital samples:  $\Delta t = \frac{1}{2} \text{ sec}$

#### 8.6.4 Sampling 1 Hz sine wave at 1.5 Hz



- Maximum frequency component:  $N = 1$
- Nyquist frequency:  $f_{max} = 1 \text{ Hz}$
- Sampling rate:  $f_s = 1.5 \text{ Hz}$
- Time interval between digital samples:  $\Delta t = \frac{1}{1.5} = \frac{2}{3} \text{ sec}$

#### 8.6.5 Sampling a sine wave with multiple frequencies at 6 Hz

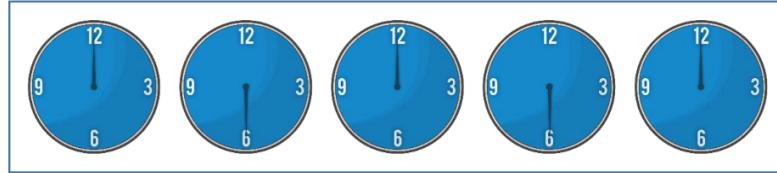


- Maximum frequency component:  $N = 3$
- Nyquist frequency:  $f_{max} = 3 \text{ Hz}$
- Sampling rate:  $f_s = 6 \text{ Hz}$
- Time interval between digital samples:  $\Delta t = \frac{1}{6} \text{ sec}$

## 8.7 Aliasing

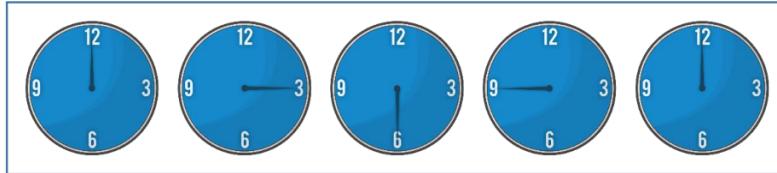
### 8.7.1 Sampling of a clock with only one hand

- Sampling a clock at double the Nyquist frequency:



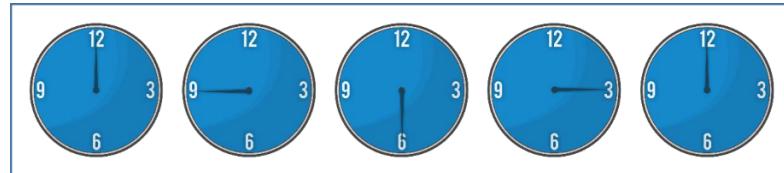
- Sampling rate:  $f_s = \frac{1}{30}$  Hz
- Nyquist frequency:  $f_{max} = \frac{1}{60}$  Hz
- Double Nyquist frequency:  $f_{max} = \frac{1}{30}$  Hz
- Aliasing occurs as the receiver cannot tell if the clock is moving forward or backwards.

- Sampling of a clock above double the Nyquist frequency:



- Sampling rate:  $f_s = \frac{1}{15}$  Hz
- Nyquist frequency:  $f_{max} = \frac{1}{60}$  Hz
- Double Nyquist frequency:  $f_{max} = \frac{1}{30}$  Hz
- No aliasing occurs as the receiver can tell that the clock is moving forward.

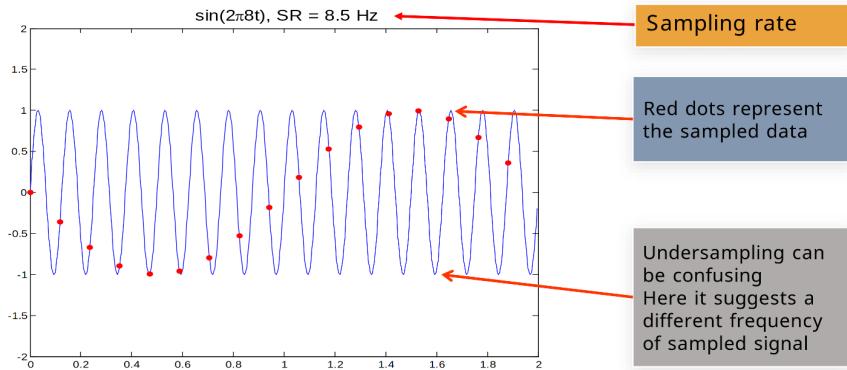
- Sampling of a clock under double the Nyquist frequency:



- Sampling rate:  $f_s = \frac{1}{45}$  Hz
- Nyquist frequency:  $f_{max} = \frac{1}{60}$  Hz
- Double Nyquist frequency:  $f_{max} = \frac{1}{30}$  Hz
- Aliasing occurs as the receiver thinks that the clock is moving backwards instead of forward.

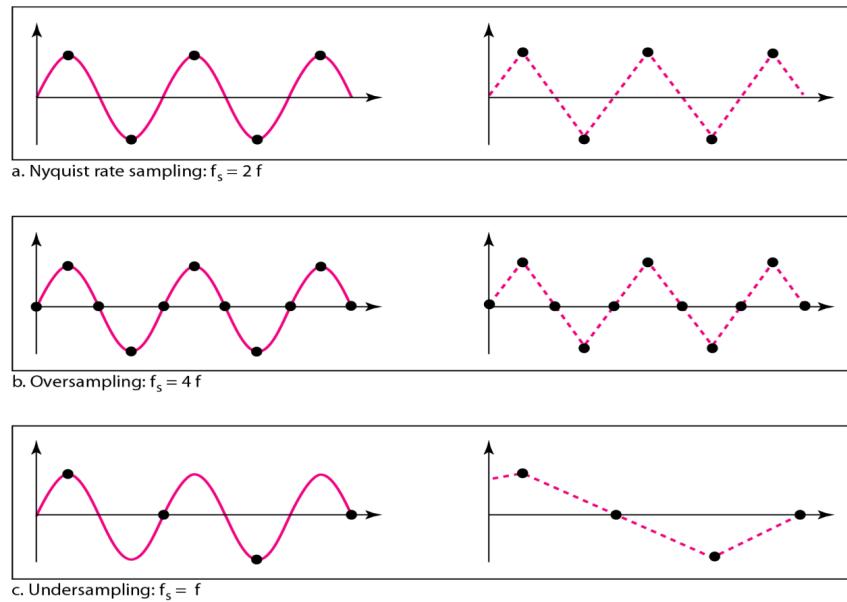
### 8.7.2 Undersampled signal

Below is a signal with a frequency of 8 Hz sampled at a rate of 8.5 Hz



An undersampled signal can confuse you about its frequency when reconstructed as the sampling rate is too low.

### 8.7.3 Reconstruction of a sampled sine wave



#### 8.7.4 Frequency of aliased signal ( $f_a$ )

The frequency of an aliased signal ( $f_a$ ) is given as:

$$f_a = |f_s \cdot i - f_n|$$

Where:

- $f_a$  is the frequency of the aliased signal
- $f_s$  is the sampling rate
- $i$  is the closest integer multiple of the sampling rate to the signal being aliased
- $f_n$  is the frequency of the signal being aliased

For example, if the signal is  $f_n = 21$  Hz and is sampled with  $f_s = 10$  Hz, then the aliased frequency would be  $|i \cdot f_s - f_n| = |2 \cdot 10 - 21| = 1$  Hz

#### 8.7.5 Capturing the shape of the waveform

Even though sampling at twice the **Nyquist frequency** will ensure that you measure the correct frequency of your signal, it will not be sufficient to capture the shape of the waveform.

If the shape of the waveform is desired, you should sample at a rate approximately **10 times the Nyquist frequency**.

### 8.8 Applications

#### 8.8.1 Recording audio

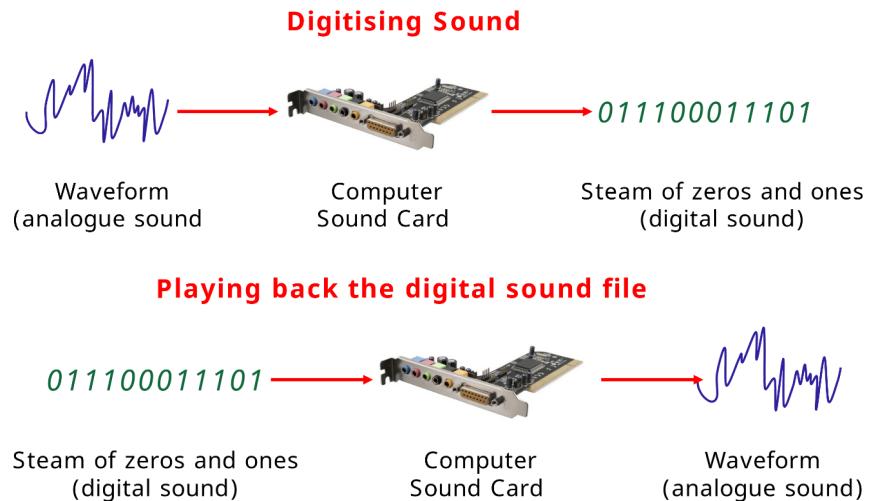
- The range of human hearing is 20 – 20,000 Hz.
- We lose high frequency response with age.
- Women generally have better response than men.
- To reproduce an audio signal of 20 kHz requires a sampling rate of at least 40 kHz.
- **Below** the sampling rate of 40 kHz, aliasing will occur, according the Shannon-Nyquist Theorem.

### 8.8.2 Digital voice telephone transmission

- Voice data for telephone purposes is limited to frequencies less than 4 kHz.
- According to the Shannon-Nyquist Theorem, it would take 8,000 samples  $2 \cdot 4,000$  to capture a 4,000 Hz signal perfectly.
- Generally, one byte is recorded per sample (256 levels). One byte is eight bits of binary data.
- $8 \text{ bits} \cdot 8,000 \text{ samples per second} = 64 \text{ kbps}$  over a circuit.

## 9 Quantisation and encoding

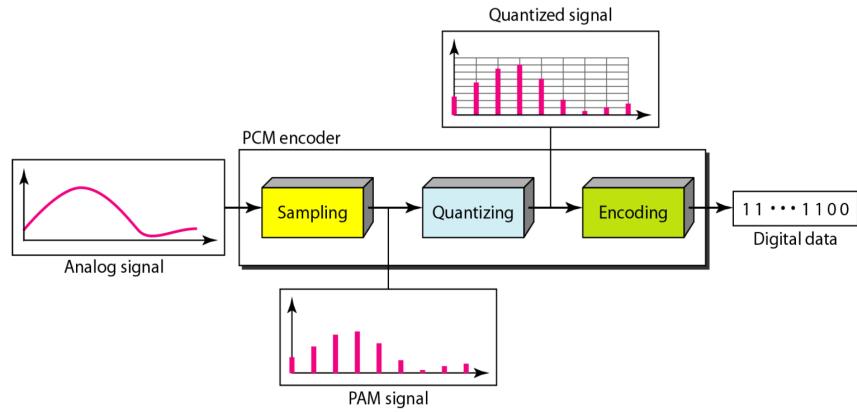
### 9.1 Digitising



### 9.2 Pulse code modulation (PCM)

- Pulse code modulation consists of three steps to digitise an analogue signal:
  1. Sampling
  2. Quantisation
  3. Binary encoding
- Before we sample, we have to filter the signal to limit the maximum frequency of the signal as it affects the sampling rate.
- Filtering should ensure that we do not distort the signal by removing high frequency components that affect the signal shape.

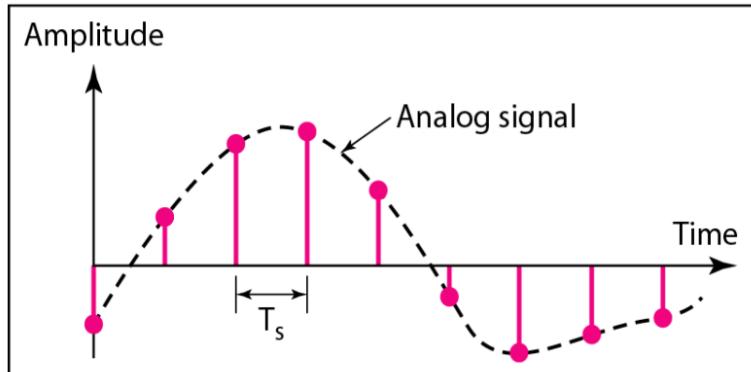
### 9.2.1 Components of a PCM encoder



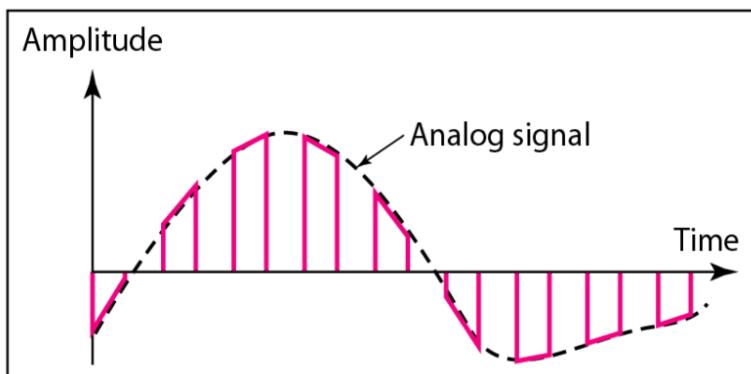
### 9.2.2 Sampling methods and pulse amplitude modulation (PAM)

- The analogue signal is sampled every  $T_s$  secs.
- $T_s$  is known as the sampling interval.
- $f_s = \frac{1}{T_s}$  is called the sampling rate or sampling frequency.
- There are 3 sampling methods:
  1. Ideal, which is an instant pulse at each sampling instant.
  2. Natural, which is a pulse of short width with varying amplitude.
  3. Flat top, which is to sample and hold the value. It is similar to the natural sampling method, but with a constant amplitude value.
- This process is known as pulse amplitude modulation (PAM) and the outcome is a signal with analogue (non-integer) values.

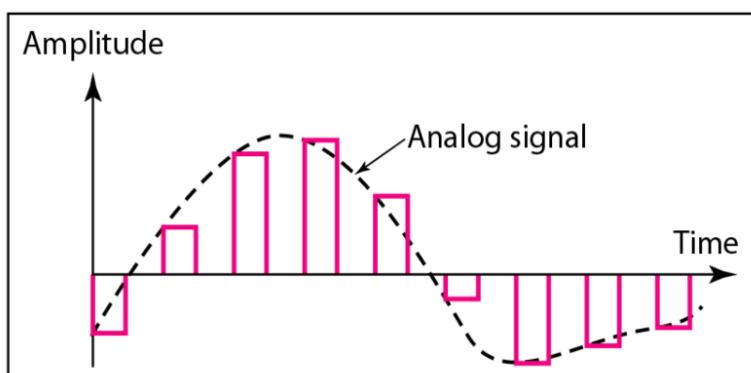
### 9.2.3 Images of the sampling methods



a. Ideal sampling



b. Natural sampling



c. Flat-top sampling

### 9.3 Quantisation

- Sampling results in a series of pulses of varying amplitude values ranging between two limits: a minimum and a maximum value.
- The amplitude values are finite between the two limits.
- We need to map the finite amplitude values onto a finite set of known values.
- This is achieved by dividing the distance between the minimum and maximum into  $L$  zones, each of height  $\Delta$

$$\Delta = \frac{\max - \min}{L}$$

### 9.3.1 Analogue quantisation size (code width) ( $Q$ )

$$Q = \frac{V_{max} - V_{min}}{N}$$

Where:

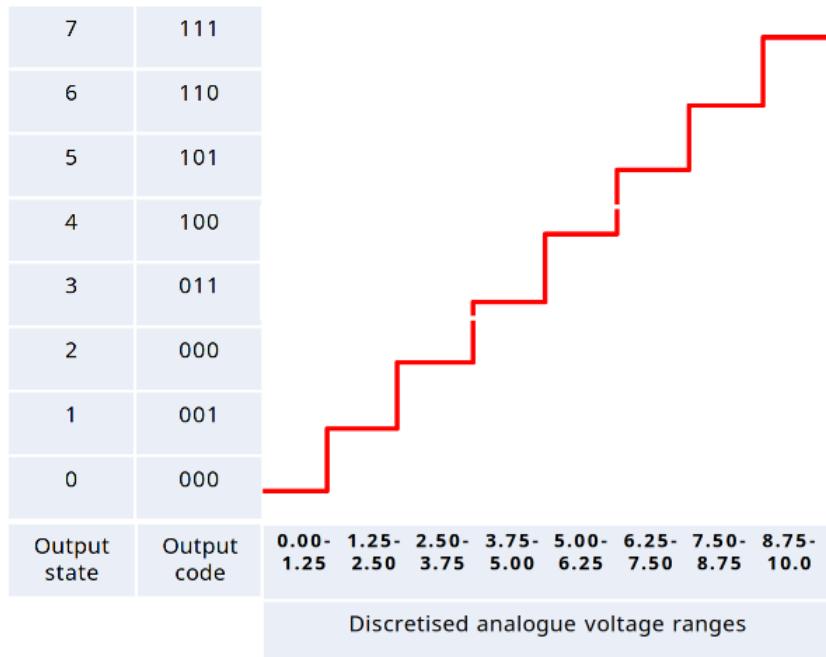
- $Q$  is the analogue quantisation size
- $V_{max}$  is the maximum voltage value
- $V_{min}$  is the minimum voltage value
- $N$  is the number of zones

Example:

- Given  $N = 8, V_{max} = 10\text{ V}, V_{min} = 0\text{ V}$ .
- Analogue quantisation size of code width:  $Q = \frac{V_{max}-V_{min}}{N} = \frac{10-0}{8} = 1.25\text{ V}$
- This means that the amplitude of the digitised signal has an error of at most  $1.25\text{ V}$ .
- Therefore, the A/D converter can only resolve a voltage within  $1.25\text{ V}$  of the exact analogue voltage.

### 9.3.2 Quantisation vs encoding

- Quantisation is the transformation of a continuous analogue input into a set of discrete output states.
- Encoding is the assignment of a digital code word or number to each output state.



- Each output state covers a subrange of the overall voltage range.
- The step-stair signal represents the states of a digital signal generated by sampling a linear ramp of an analogue signal occurring over the voltage range.
- The figure shows how a continuous voltage range is divided into discrete output states, each of which is assigned a unique code.

### 9.3.3 Analogue-to-digital (A/D) converter

- An analogue-to-digital converter is an electronic device that converts an analogue voltage to a digital code.
- The output of the analogue-to-digital converter can be directly interfaced to a digital device, like a microcontroller or a computer.
- The resolution of an analogue-to-digital converter is the number of bits used to digitally approximate the analogue value of the input.
- The number of possible states  $N$  is equal to the number of bit combinations that can be produced from the converter:

$$N = 2^n$$

Where:

- $N$  is the number of possible states
  - $n$  is the number of bits
- Most commercial analogue-to-digital converters are an 8, 10 or 12-bit device, with 256 ( $2^8$ ), 1024 ( $2^{10}$ ), or 4096 ( $2^{12}$ ) states respectively.

### 9.3.4 Mid-points

- The midpoint of each zone is assigned a value from 0 to  $L - 1$ , resulting in  $L$  values.
- Each sampling falling in a zone is then approximated to the value of the midpoint.

### 9.3.5 Quantising zones and mid-points

- Assume a voltage signal with amplitudes  $V_{min} = -20\text{ V}$  and  $V_{max} = 20\text{ V}$ .
- Using  $L = 8$  quantisation levels.
- Zone width:  $\Delta = \frac{20 - (-20)}{8} = 5$
- The 8 zones are:
  - -20 to -15
  - -15 to -10
  - -10 to -5
  - -5 to 0
  - 0 to +5
  - +5 to +10
  - +10 to +15
  - +15 to +20
- The mid-points are:
  - -17.5
  - -12.5
  - -7.5
  - -2.5
  - 2.5
  - 7.5
  - 12.5
  - 17.5

### 9.3.6 Assigning codes to zones

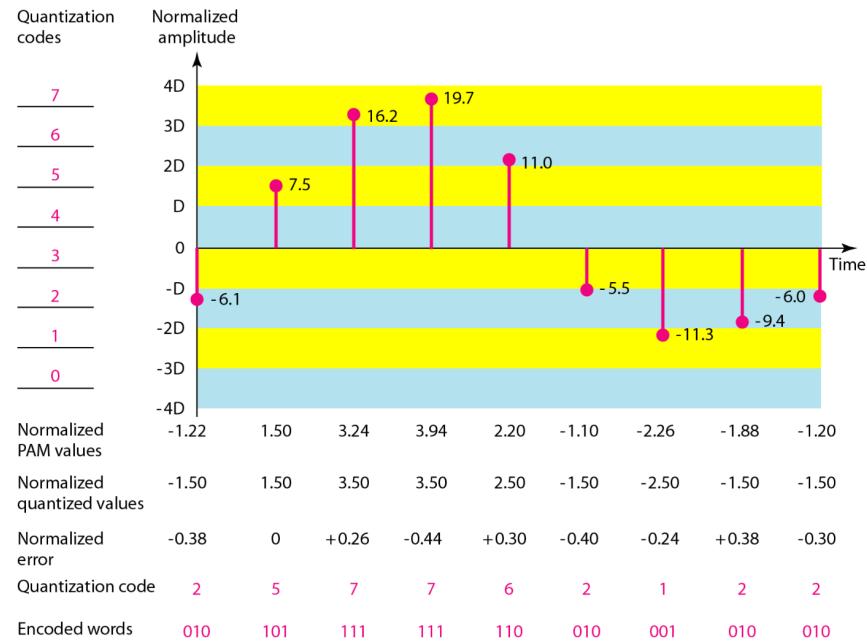
- Each zone is then assigned a binary code.
- The number of bits required to encode the zones, or the number of bits per sample, is obtained as follows:

$$n_b = \log_2 L$$

Where:

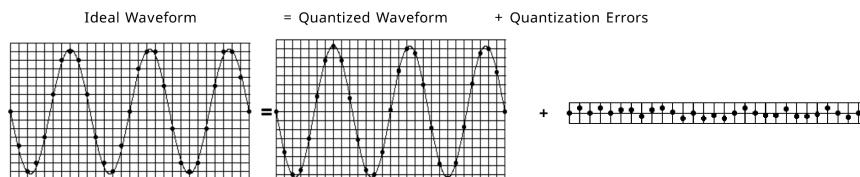
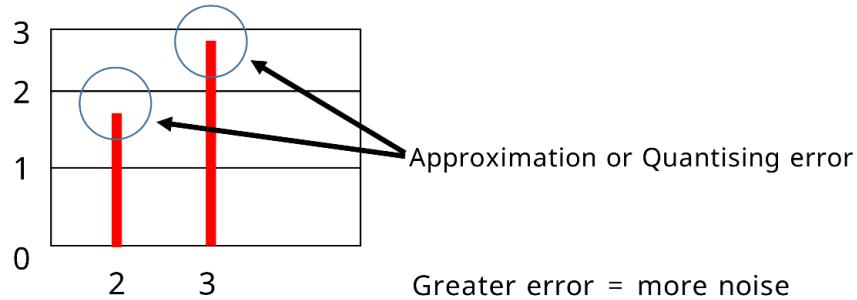
- $n_b$  is the number of bits to encode the zone.
- $L$  is the number of zones
- In the example above,  $n_b = 3$ .
- The 8 zone codes are therefore:
  - 000
  - 001
  - 010
  - 011
  - 100
  - 101
  - 110
  - 111
- Assigning codes to the zones:
  - 000 will refer to zone -20 to -15
  - 001 will refer to zone -15 to -10
  - 010 will refer to zone -10 to -5
  - 011 will refer to zone -5 to 0
  - 100 will refer to zone 0 to +5
  - 101 will refer to zone +5 to +10
  - 110 will refer to zone +10 to +15
  - 111 will refer to zone +15 to +20

### 9.3.7 Quantisation and encoding of a sampled signal



### 9.3.8 Quantisation error

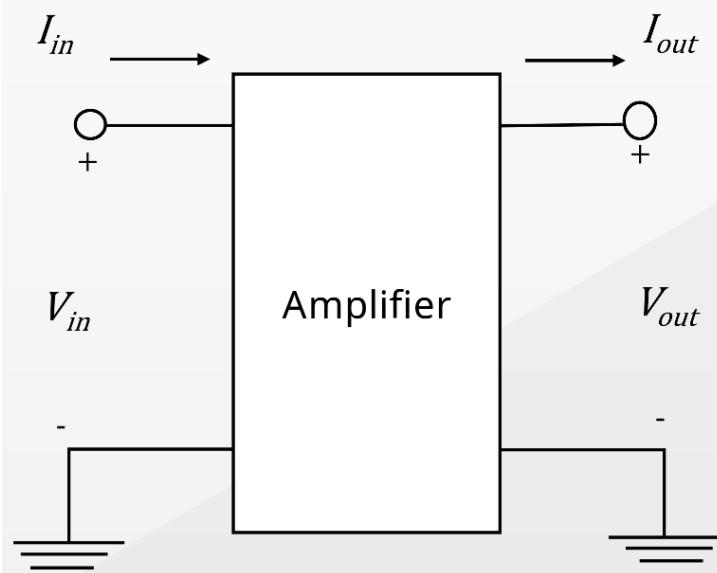
- When a signal is quantised, an error is introduced as the encoded signal is an approximation of the actual amplitude value.
- The difference between the actual and encoded value (mid-point) is known as the quantisation error.
- The greater the number of zones, the smaller the width of the zone ( $\Delta$ ), which results in smaller errors.
- However, increasing the number of zones will also increase the number of bits required to encode the samples, which will increase the bit rate.



## 10 Amplifiers

An amplifier increases the amplitude of a signal without affecting the phase of the different components of the signal. This means the voltage gain should be constant for all frequencies.

### 10.1 Relationship between output and input



$$V_{out} = A_v V_{in}$$

Where:

- $A_v$  is the gain. Ideally,  $A_v$  is constant for all frequencies, but there is a bandwidth associated with cut-off frequencies.

### 10.2 Filtering and amplifier linearity

- Amplifiers are designed for certain frequencies instead of all frequencies.
- Output characteristics are governed by the amplifier's bandwidth.
- There are associated cut-off frequencies (thresholds) for amplifiers.

### 10.3 Characteristics of amplifiers

- Size
- Cost
- Power consumption
- Input impedance
- Output impedance
- Gain
- Bandwidth

#### 10.3.1 Input impedance ( $Z_{in}$ )

Most amplifiers are designed to have:

- Large input impedance
- As little current as possible is drawn from the input

The input impedance  $Z_{in}$  is given by:

$$Z_{in} = \frac{V_{in}}{I_{in}}$$

Where:

- $Z_{in}$  is the input impedance
- $V_{in}$  is the input voltage
- $I_{in}$  is the input current

The input impedance should be large to have little current drawn from the input.

### 10.3.2 Output impedance ( $Z_{out}$ )

- The voltage drop  $\Delta V_{out}$  is a measure of how much the output voltage drops with the output current.
- Most of the amplifiers are designed to have a very small output impedance, so the output voltage will not change much as the output current changes.

Output impedance  $Z_{out}$  is:

$$Z_{out} = \frac{\Delta V_{out}}{I_{in}}$$

Where:

- $\Delta V_{out}$  is the voltage drop measured relative to the output voltage with no current. The output impedance should be small to have little change when the output current changes.

## 10.4 Operational amplifiers

### 10.4.1 Characteristics

1. Low-cost.
2. Versatile integrated circuits.
3. Single chip consisting of internal transistors, resistors, and capacitors.
4. Combined with external discrete components to create a wide variety of signal processing circuits.

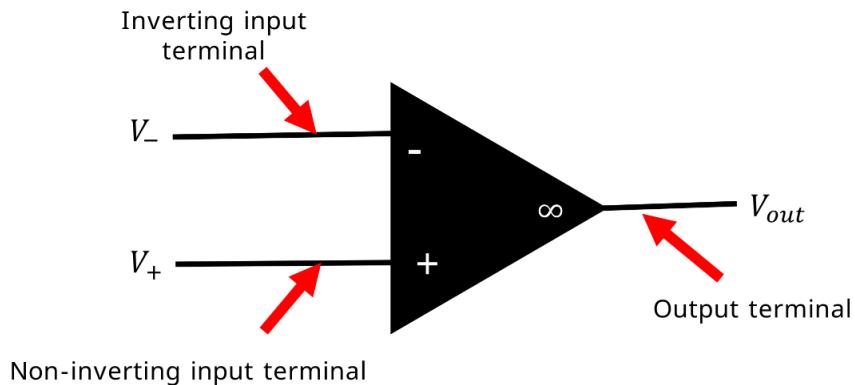
### 10.4.2 Basic block of amplifiers

Amplifiers	Integrators	Summers
A/D converters	D/A converters	Differentiators
Active filters	Sample & hold amplifiers	Comparators

#### 10.4.3 Functions

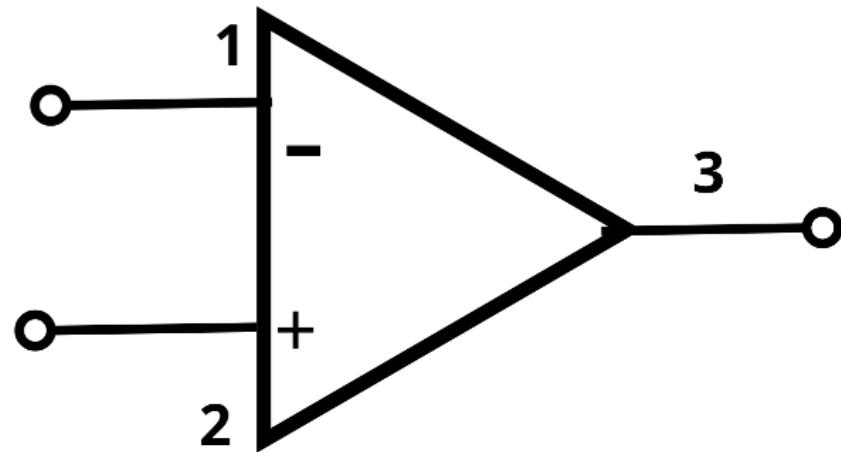
- Inverting amplifiers
- Non-inverting amplifiers
- Summer amplifiers
- Difference amplifiers
- Integrator amplifiers
- Differentiator amplifiers

#### 10.4.4 Schematic and nomenclature



- A differential input
  - The inverting input (-)
  - The non-inverting input (+)
- Single output
- Infinite gain ( $\infty$ )
- The voltages are all referenced to a common ground.

#### 10.4.5 Output voltage

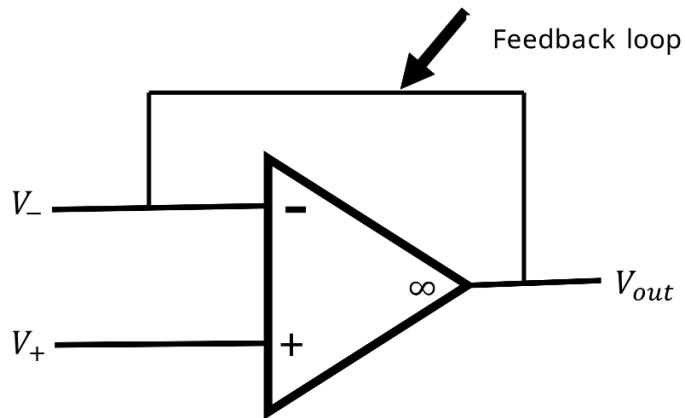


$$V_{out} = A_v V_{in}$$

$$V_3 = A(V_2 - V_1)$$

The output voltage is proportional to the difference between the two inputs of the amplifier.

#### 10.4.6 How to control the gain?

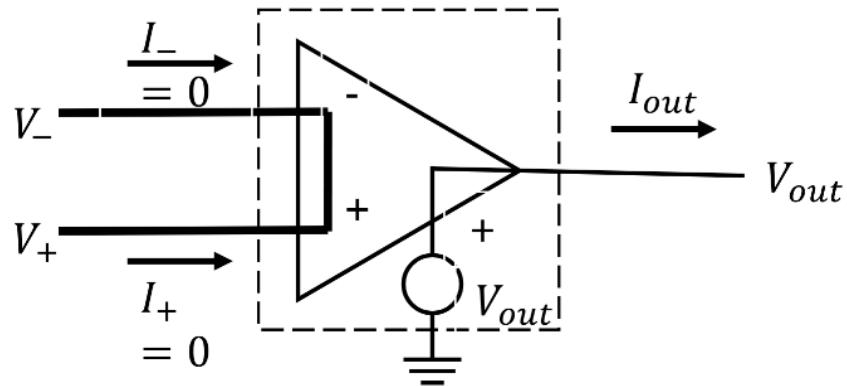


The feedback loop is connected from the output to the inverting input (-).

#### 10.4.7 Closed loop vs open loop configuration

Closed loop configuration	Open loop configuration
When feedback is present	When feedback is absent
Stabilisation of the amplifiers	Considerable instability due to the high gain
Control of the gain	Seldom used

#### 10.4.8 Ideal model for operational amplifiers



Infinite impedance at both inputs.

- No current is drawn from the input circuits.
- Therefore,  $I_+ = I_- = 0$ .

Infinite gain, assuming no current flow between the short of the two inputs.

- The difference between the input voltages must be 0, otherwise the output would be infinite.
- Therefore,  $V_+ = V_-$ .

Zero output impedance.

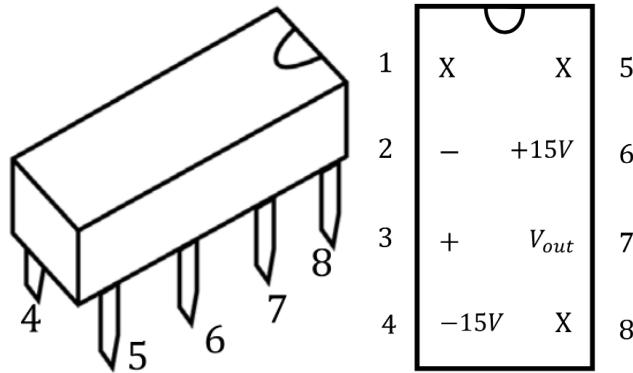
- The output voltage does not depend on the output current.

Note that  $V_{out}$ ,  $V_+$  and  $V_-$  are all referenced to the same ground, and there must be feedback between the output and the inverting input for stable linear behaviour.

#### 10.4.9 Summary of the ideal operational amplifier

- The ideal operational amplifier has infinite impedance at both inputs, so no current is drawn from the input circuit:  $I_+ = I_- = 0$ .
- It has infinite gain, so the difference between input voltages is zero:  $V_+ = V_-$ .
- It has zero output impedance, so the output voltage does not depend on output current.
- The open-loop gain is a very large, and can be considered as infinite.
- The input impedances of the two terminals are very large, and can be considered as infinite.
- The output impedance is very small, and can be considered as zero.

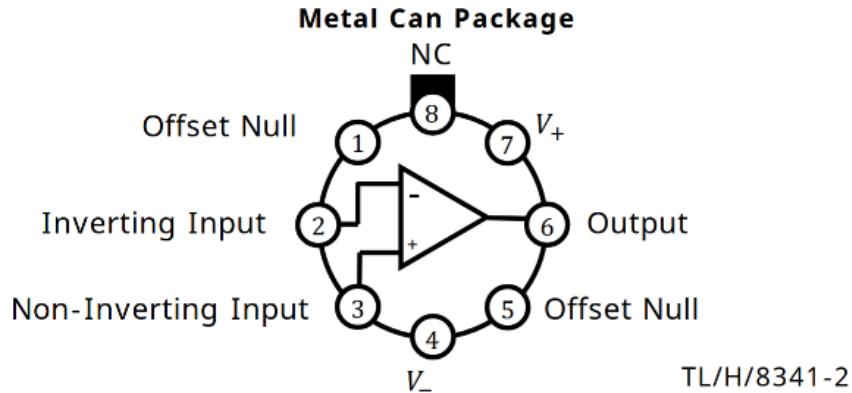
#### 10.4.10 Real operational amplifier



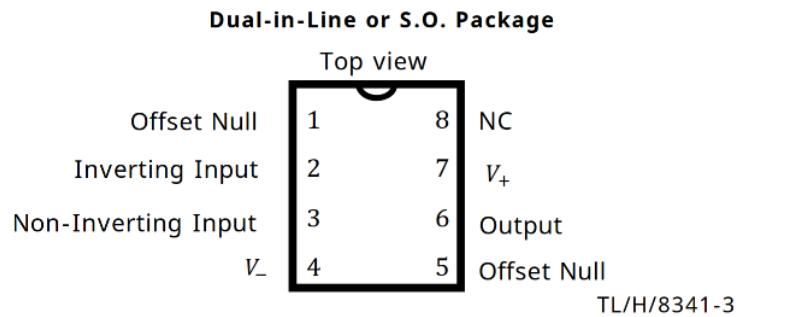
Packaging

- Eight pins and dual inline package (DIP) integrated circuit or a chip.
- 741 is the designation of a general purpose operational amplifier by many manufacturers.

#### 10.4.11 Pin configuration (pin-out)



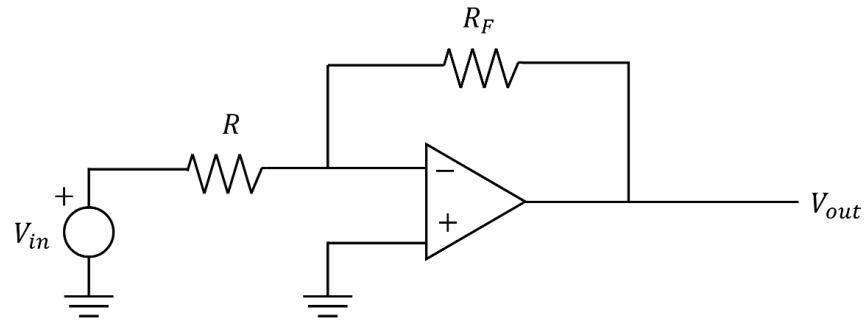
Order Number LM741H, LM741AH, LM741CH or LM741EH  
See NS Package Number H08C



Order Number LM741J, LM741AJ, LM741CJ, LM741CM, LM741CN or LM741EN  
See NS Package Number J08A, M08A or N08E

- One indentation or spot
- The pins are numbered counterclockwise
- Pin 2: Inverting input (-)
- Pin 3: Non-inverting input (+)
- Pin 4: External power supply (-15 V)
- Pin 7: External power supply (15 V)
- Pin 6: The operational amplifier output
- Pins 1, 5 and 8: Not normally used, no connections are required

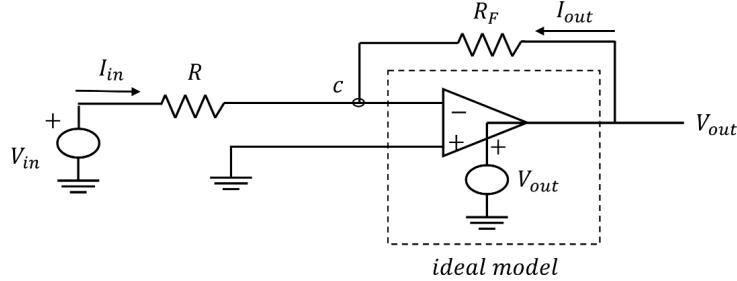
## 10.5 Inverting amplifier



An inverting amplifier inverts and amplifies the input voltage.

- It is constructed by connecting two external resistors to an operational amplifier.
- This circuit inverts and amplifies the input voltage.
- The resistor  $R_F$  forms the feedback loop.
  - The loop always goes from the output to the inverting input of the operational amplifier, so the feedback loop is negative.

### 10.5.1 Equivalent circuit



At node C:

$$I_{in} = -I_{out}, \quad V_c = 0$$

Where:

- $I_{in}$  is the input current
- $I_{out}$  is the output current
- $V_c$  is the voltage at node C

Since no current flows into inputs of the operational amplifier, voltage across the resistor  $R$  is  $V_{in} - V_c = V_{in}$ , from Ohm's law:

$$V_{in} = I_{in}R$$

Where:

- $V_{in}$  is the input voltage
- $I_{in}$  is the input current
- $R$  is the resistance of the resistor  $R$

Voltage across the resistor  $R_F$  is  $V_{out} - V_c = V_{out}$ , from Ohm's law:

$$V_{out} = I_{out}R_F = -I_{in}R_F$$

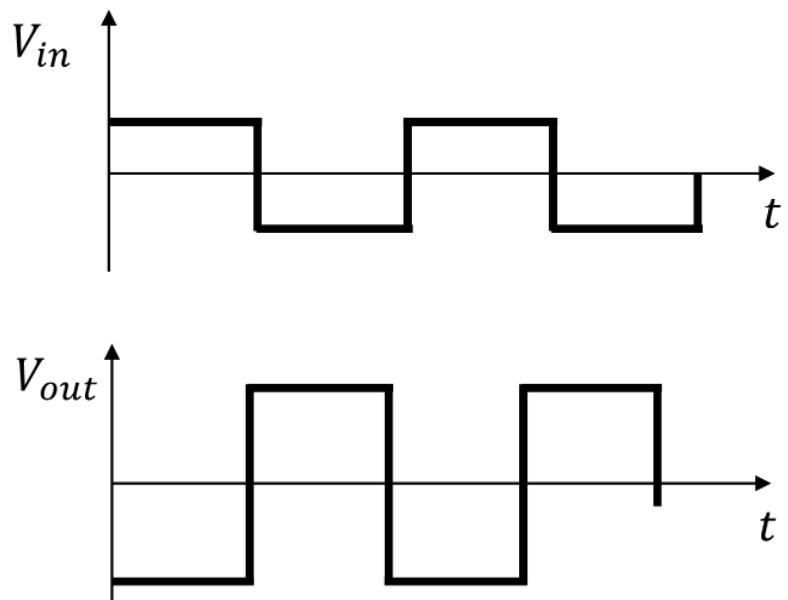
$$\frac{V_{out}}{V_{in}} = -\frac{R_F}{R}$$

Where:

- $V_{out}$  is the output voltage
- $V_{in}$  is the input voltage
- $R_F$  is the resistance of resistor  $R_F$
- $R$  is the resistance of resistor  $R$

### 10.5.2 Characteristics

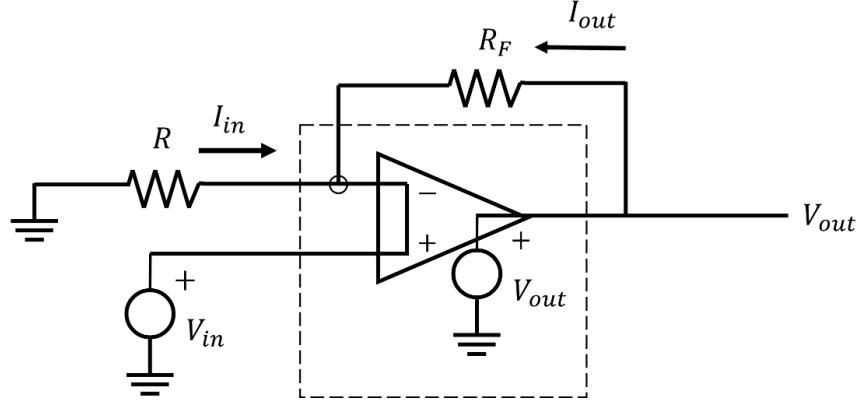
- The voltage gain of the amplifier is determined simply by the external resistors  $R_F$  and  $R$ .
- The voltage gain of the amplifier is always negative.
- An example of an inverting (-) amplifier:



### 10.6 Non-inverting amplifier

A non-inverting amplifier amplifies the input voltage without inverting the signal.

### 10.6.1 Equivalent circuit



At node  $C$ :

$$V_c = V_{in}, \quad V_{in} = -I_{in}R, \quad I_{in} + I_{out} = 0$$

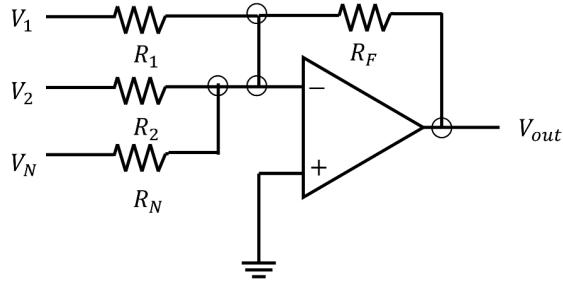
Voltage across  $R_F$ :

$$\begin{aligned} V_{out} &= I_{out}R_F + V_{in} \\ \therefore \frac{V_{out}}{V_{in}} &= \frac{I_{out}R_F + V_{in}}{-I_{in}R} \\ &= \frac{I_{out}R_F - I_{in}R}{-I_{in}R} \\ &= \frac{-I_{in}R_F - I_{in}R}{-I_{in}R} \\ &= 1 + \frac{R_F}{R} \end{aligned}$$

### 10.6.2 Summary

- A non-inverting amplifier amplifies the input without inverting the signal.
- It has a positive gain greater than or equal to 1.

## 10.7 Summer amplifier



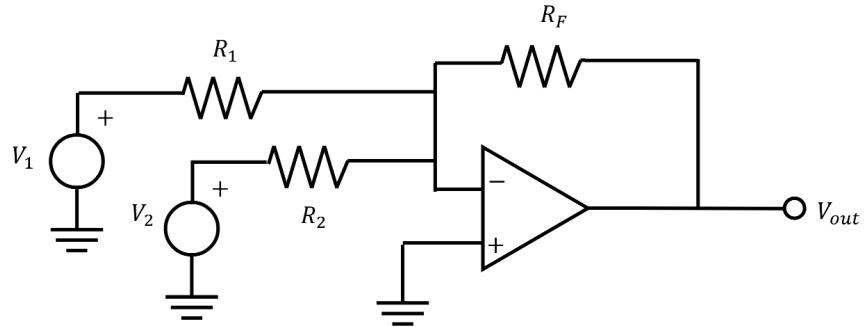
$$V_{outN} = -\frac{R_F}{R_N} V_N$$

$$V_{out1} = -\frac{R_F}{R_1} V_1$$

$$V_{out2} = -\frac{R_F}{R_2} V_2$$

$$V_{out} = - \left( \frac{R_F}{R_1} V_1 + \frac{R_F}{R_2} V_2 + \frac{R_F}{R_N} V_N \right)$$

### 10.7.1 Equivalent circuit

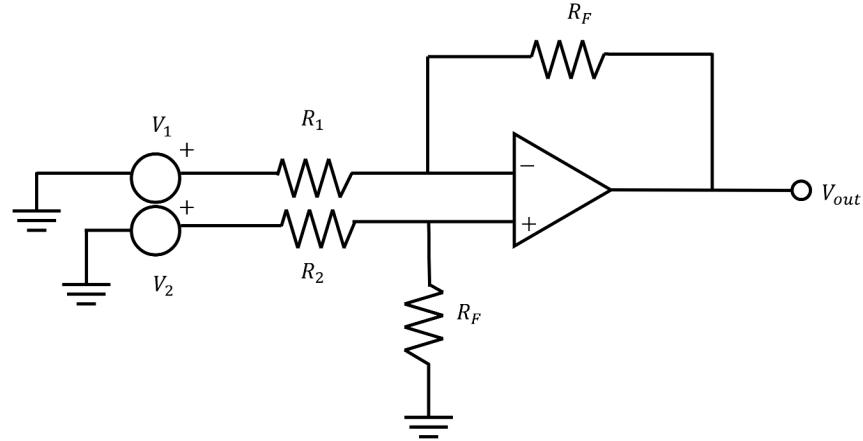


$$\frac{V_1}{R_1} + \frac{V_2}{R_2} = -\frac{V_{out}}{R_F}$$

- The summer amplifier is also known as the adder.
- It adds another analogue signal.
- The output is the negative sum of the inputs.

## 10.8 Difference amplifier

A difference amplifier circuit is used to subtract analogue signals.



$$V_1 - I_1 R_1 = V_2 - I_2 R_2 = I_2 R_F$$

$$V_{out} = -I_1 R_F - I_1 R_1 + V_1$$

Hence:

$$I_2 = \frac{V_2}{R_F + R_2} \rightarrow I_1 = \frac{V_1}{R_1} - \frac{V_2}{R_F + R_2} \frac{R_F}{R_1}$$

So:

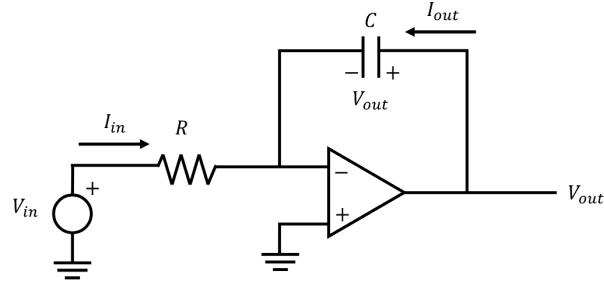
$$V_{out} = V_1 - (R_F + R_1) \left( \frac{V_1}{R_1} - \frac{V_2}{R_F + R_2} \frac{R_F}{R_1} \right)$$

If  $R_1 = R_2 = R$ ,

$$V_{out} = \frac{R_F}{R_1} (V_2 - V_1)$$

## 10.9 Integrator

The integrator circuit is created by replacing the feedback resistor of the inverting operation amplifier circuit with a capacitor.

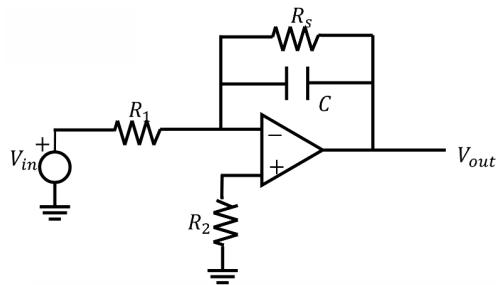


$$\frac{dV_{out}}{dt} = \frac{I_{out}}{C} = -\frac{I_{in}}{C} = -\frac{V_{in}}{RC}$$

So:

$$V_{out}(t) = -\frac{1}{RC} \int_0^t V_{in}(\tau) d\tau$$

### 10.9.1 Practical integrator



$$C \frac{dV_{out}}{dt} + \frac{V_{out}}{R_s} = I_{out} = -I_{in} = -\frac{V_{in}}{R_1}$$

So:

$$\frac{dV_{out}}{dt} + \frac{1}{CR_s} V_{out} = \frac{1}{R_1 C} V_{in}$$

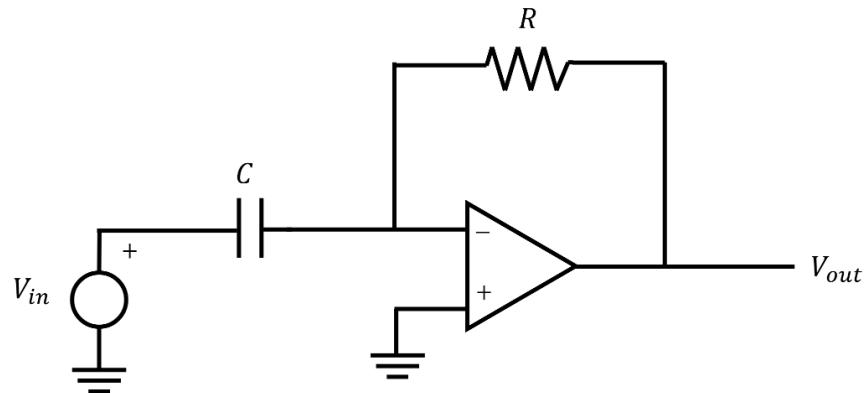
Should choose:

$$R_s > 10R_1, \quad R_2 = -\frac{R_1 R_s}{R_1 + R_s}$$

The reason is  $R_2$  is an approximation of the parallel combination of  $R_1$  and  $R_s$  to minimise the DC offset due to the input current bias.

### 10.10 Differentiator

The input resistor of the inverting operational amplifier circuit is replaced by a capacitor to form a differentiator circuit.



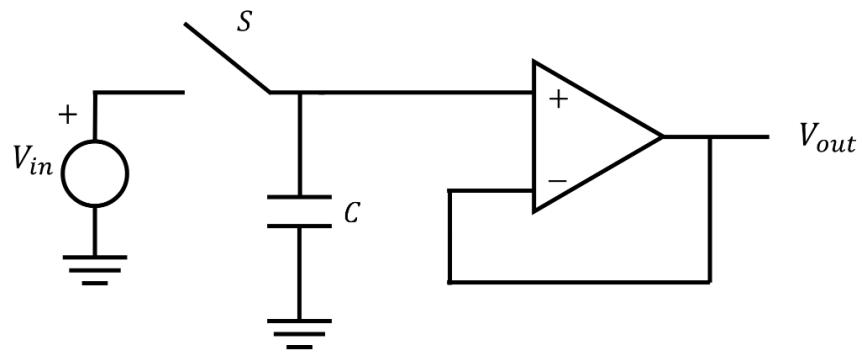
$$\frac{dV_{in}}{dt} = \frac{I_{in}}{C} = -\frac{I_{out}}{C} = -\frac{V_{out}}{RC}$$

So:

$$V_{out} = -RC \frac{dV_{in}}{dt}$$

### 10.11 Sample and hold circuit

1. It is extensively used in analogue-to-digital conversion.
2. Its signal value must be stabilised while it is converted to a digital representation.
3. It consists of voltage-holding capacitor and a voltage follower.
4. It works while the switch is closed.



When switch  $S$  is closed:

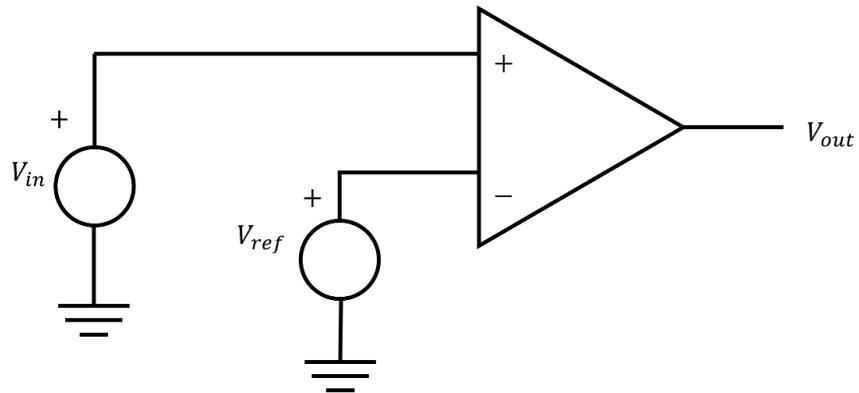
$$V_{out}(t) = V_{in}(t)$$

$$V_{out}(t - t_{sampled}) = V_{in}(t_{sampled})$$

The capacitor  $C$  should be one with low leakage.

## 10.12 Comparator

- A comparator is used to determine whether one signal is greater than another.
- The comparator is an example of an operational amplifier circuit where there is no negative feedback and the circuit exhibits infinite gain.
- The result is that the operational amplifier saturates.
- Saturation means that the output remains at its most positive or most negative output value.



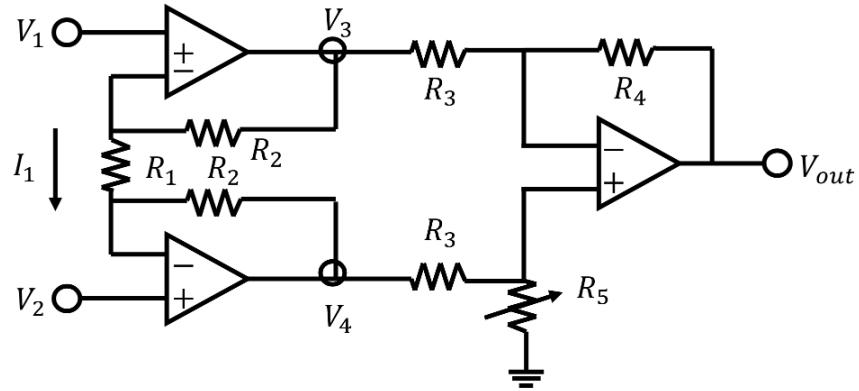
$$V_{out} = \begin{cases} +V_{sat}, & V_{in} > V_{ref} \\ -V_{sat}, & V_{in} < V_{ref} \end{cases}$$

Where:

- $V_{sat}$  is the saturation voltage of the comparator. Most comparators are specially built.

### 10.13 Instrumentation amplifier

- An instrumentation amplifier is used for subtracting analogue signals.
- It does not invert the signal, like a non-inverting amplifier.



The left side:

$$V_3 - V_1 = I_1 R_2$$

$$V_2 - V_4 = I_1 R_2$$

$$V_1 - V_2 = I_1 R_1$$

The right side:

$$V_3 - I_3 R_3 = V_4 - I_4 R_3 = I_4 R_5$$

$$V_{out} = -I_3 R_4 - I_3 R_3 + V_3$$

Where:

- $I_3$  is the current through  $R_3$
- $I_4$  is the current through  $R_4$

So:

$$\begin{aligned}V_3 &= \left( \frac{R_2}{R_1} + 1 \right) V_1 - \frac{R_2}{R_1} V_2 \\V_4 &= \left( \frac{R_2}{R_1} + 1 \right) V_2 - \frac{R_2}{R_1} V_1 \\V_{out} &= \frac{R_5(R_3 + R_4)}{R_3(R_3 + R_5)} V_4 - \frac{R_4}{R_3} V_3\end{aligned}$$

If  $R_4 = R_5$ , then:

$$V_{out} \left[ \frac{R_4}{R_3} \left( 1 + 2 \cdot \frac{R_2}{R_1} \right) \right] (V_2 - V_1)$$

So if  $V_1 = V_2$ , then  $V_{out} = 0$ . In practice, we need a variable resistor  $R_2$  to tune such that  $R_4 = R_5$ .

#### 10.13.1 Why use instrumentation amplifiers?

- A difference amplifier may be satisfactory for low-impedance source, but its input impedance is too low for high-output impedance source.
- If the levels of the input signals are very low and the signals include noise, the difference amplifier is unable to extract a satisfactory difference signal.
- The instrumentation amplifier is a solution for this problem.

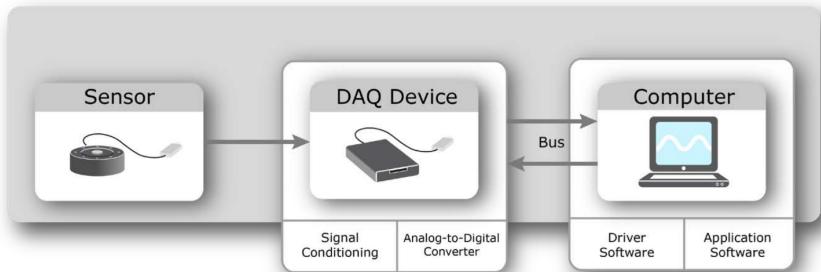
### 10.13.2 Characteristics

- The instrumentation amplifier has very high input impedance.
- Large common mode rejection ratio (CMRR), which is the ratio of the difference mode gain to the common mode gain.
- The difference mode gain is the amplification factor for the difference between the input signals.
- The common mode gain is the amplification factor for the average of the input signals.
- For an ideal difference amplifier, the common mode gain is 0, implying an infinite common mode rejection ratio.
- It is desirable to minimise the common mode gain to suppress signals such as noise that are common to both inputs.
- The instrumentation amplifier also has the capability to amplify low-level signals in a noisy environment, which is often a requirement in applications with differential output and signal conditioning.
- It also has a consistent bandwidth over a large range of gains.

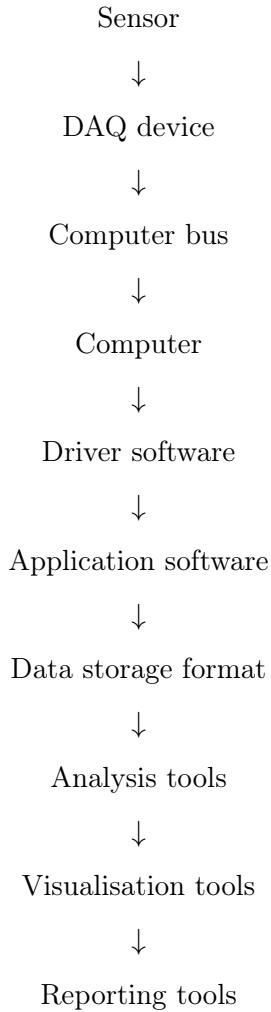
## 11 Analogue-to-digital (A/D) conversion

### 11.1 Data acquisition (DAQ) devices

**Data acquisition (DAQ) products with sensors**



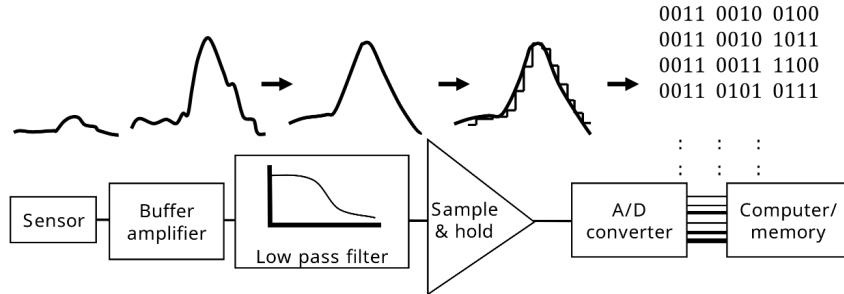
### 11.1.1 Flow chart



### 11.1.2 Examples

Sensor	Phenomenon
Thermocouple, thermistor	Temperature
Photo sensor	Light
Microphone	Sound
Strain gage, piezoelectric transducer	Force and pressure
Potentiometer, optical encoder	Position and displacement
Accelerometer	Acceleration
pH electrode	pH

## 11.2 A/D conversion



### 1. Buffer amplifier

- Isolates the output from the input.
- Provides a signal in a range close to but not exceeding the full input voltage range of the A/D converter.

### 2. Low pass filter

- Necessary to remove any undesirable high-frequency components in the signal that could produce aliasing.
- The cut-off frequency of the low-pass filter is less than half of the sampling rate.

### 3. Sample and hold amplifier

- This amplifier maintains a fixed input value from an instantaneous sample during the short conversion time of the A/D converter.

### 4. A/D converter

- The converter should have a resolution and analogue quantisation size appropriate for the system and the signal.

### 5. Computer and memory

- The computer must properly interface with the A/D converter system to store and process the data.
- It also needs to have sufficient memory and storage to store the data.

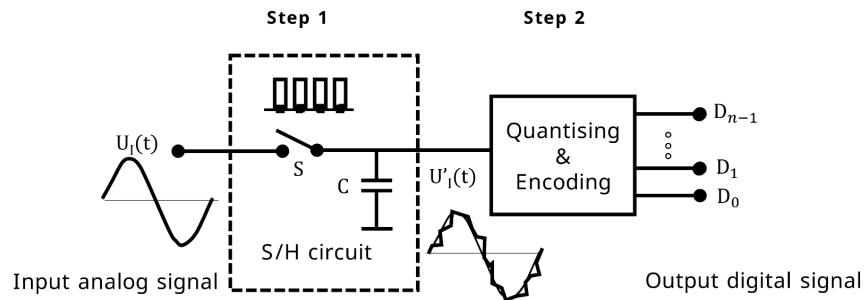
### 11.2.1 Definition

- An electronic integrated circuit which transforms a signal from analogue (continuous) to digital (discrete) form.
- Analogue signals are directly measurable quantities.
- Digital signals only have two states. For the digital computer, we refer to the binary states: 0 and 1.

### 11.2.2 Why do we need analogue-to-digital conversion?

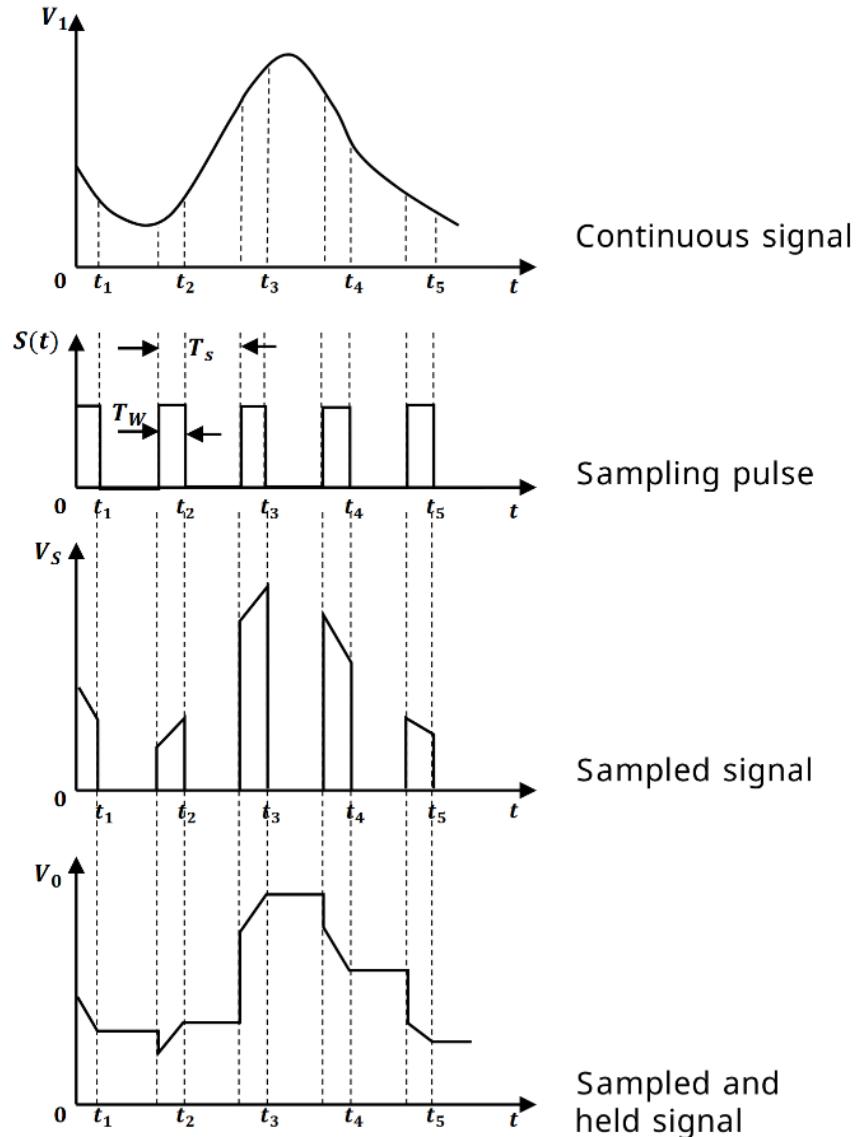
- Microprocessors can only perform complex processing on digitised signals.
- When signals are in digital form, they are less susceptible to the deleterious effects of additive noise.
- A/D conversion provides a link between the analogue world of transducers and the digital world of signal processing and data handling.

## 11.3 A/D conversion process



1. Sampling and holding (S/H)
2. Quantising and encoding (Q/E)

### 11.3.1 Sampling and holding



- Holding the signal benefits the accuracy of the A/D conversion.
- The minimum sampling rate should be at least twice the highest data frequency of the analogue signal.

### 11.3.2 Resolution

- The resolution is the smallest change in the analogue signal that will result in a change to the digital output.

$$\Delta V = \frac{V_{ref}}{2^n}$$

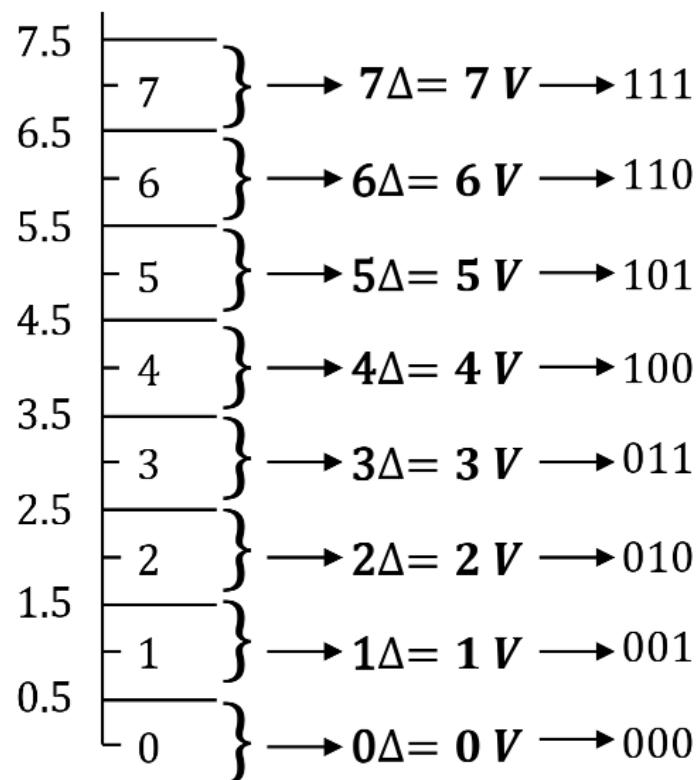
Where:

- $\Delta V$  is the resolution
  - $n$  is the number of bits in the digital output
  - $2^n$  is the number of states
  - $V_{ref}$  is the reference voltage range
- The resolution represents the quantisation error inherent in the conversion of the signal to digital form.

### 11.3.3 Quantising and encoding

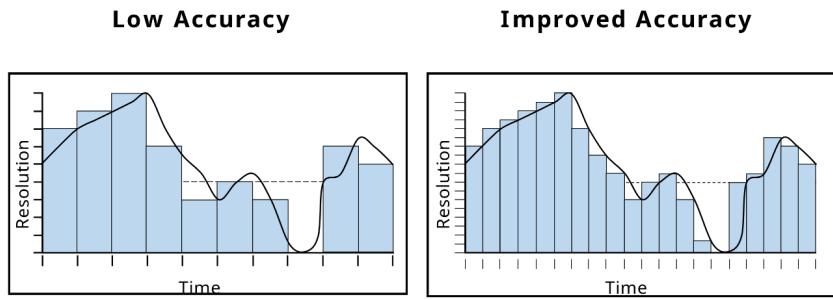
- Quantising refers to partitioning the reference signal range into a number of discrete quanta, then matching the input signal to the correct quantum.
- Encoding refers to assigning a unique digital code to each quantum, then allocating the digital code to the input signal.

Analog signal      Digital Output in binary



#### 11.3.4 Ways to improve the accuracy of the A/D conversion

1. Increase the resolution, which improves the accuracy in measuring the amplitude of the analogue signal.
2. Increasing the sampling rate, which increases the maximum frequency that can be measured.



#### 11.3.5 Advantages of A/D conversion

- A digital signal is superior to an analogue signal, as it is more robust to noise and can easily be recovered, corrected and amplified.
- For this reason, most analogue signals will be changed to their digital forms.

#### 11.3.6 Applications of A/D conversion

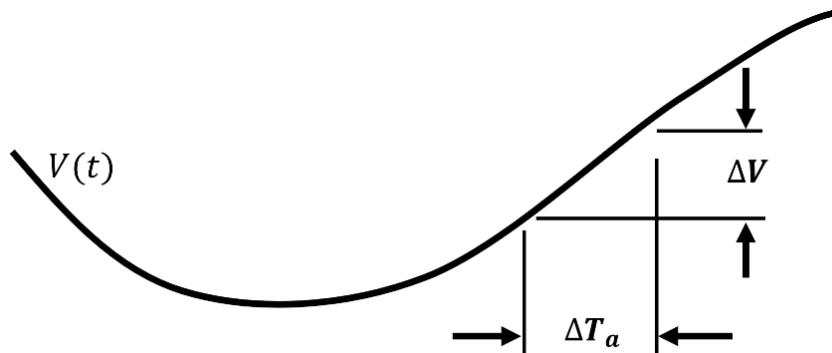
- Analogue-to-digital converters are used virtually everywhere where an analogue signal has to be processed, stored, or transported in digital form.
- Some examples include:
  - Cell phones
  - Thermocouples
  - Digital oscilloscopes
  - Digital voltmeters

### 11.3.7 Time taken for the A/D conversion

- The setting time depends on:
  - The design of the converter.
  - The method used for conversion.
  - The speed of the components used in the electronic design.
- Because the analogue signal changes continuously, the uncertainty when the conversion occurs (in the sample time window), causes the corresponding uncertainty in the digital value.
- This is of particular concern if there is no sample and hold amplifier on the analogue-to-digital input.

### 11.3.8 Aperture time

- The aperture time refers to the duration of the time between each reading of the analogue-to-digital converter.
- It is associated with any error in the digital output due to changes in the input during this time.
- The relationship between the aperture time and the uncertainty in the input amplitude is shown below:



During the aperture time, the input signal changes by  $\Delta V(t)$ , where:

$$\Delta V(t) = \frac{dV(t)}{dt} \Delta T_a$$

## **11.4 A/D converters**

Design principles of A/D converters

1. Successive approximation.
2. Flash or parallel encoding.
3. Single-slope and dual-slope integration.
4. Switched capacitor.
5. Delta sigma.

Other principles include:

- Voltage-to-frequency.
- Staircase ramp or single slope.
- Charge balancing or redistribution.
- Tracking, synchronising or resolving.

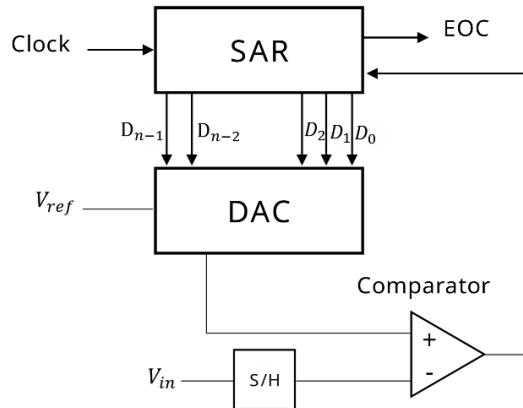
Note that design principles 1 (successive approximation) and 2 (flash or parallel encoding) occurs the most often.

### **11.4.1 Successive approximation**

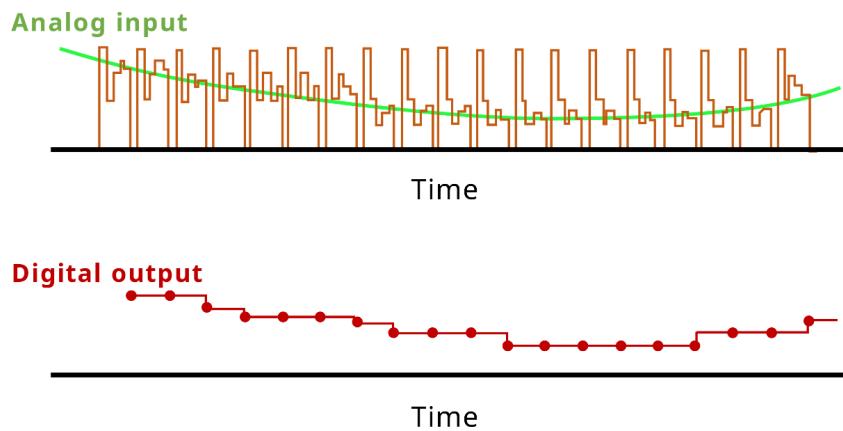
1. A/D converters designed based on successive approximation is very widely used as it is relatively fast and cheap.
2. A successive approximation A/D converter uses a digital-to-analogue (D/A) converter in a feedback loop.
3. When the start signal is sent, the sample and hold (S & H) amplifier latches the analogue input.
4. The control unit begins an iterative process, where the digital value is approximated, converted to an analogue value with the D/A converter, and compared to the analogue input with the comparator.
5. When the D/A output equals the analogue input, the end signal is set by the control unit and the correct digital output is available at the output.

#### 11.4.2 Successive approximation A/D converter circuit

- The circuit uses an n-bit digital-to-analogue converter to compare the results from the digital-to-analogue converter and the original analogue results.
- It uses a successive approximation register (SAR) to supply an approximate digital code to the digital-to-analogue converter of  $V_{in}$ .
- It compares the change in digital output to bring it closer to the input value.
- The circuit uses closed-loop feedback conversion.



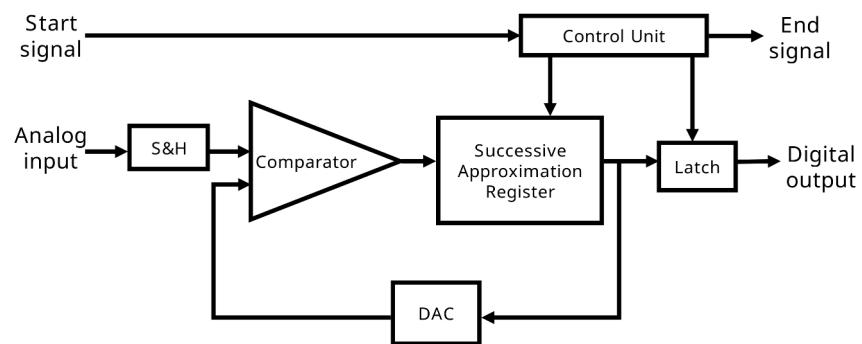
Output:



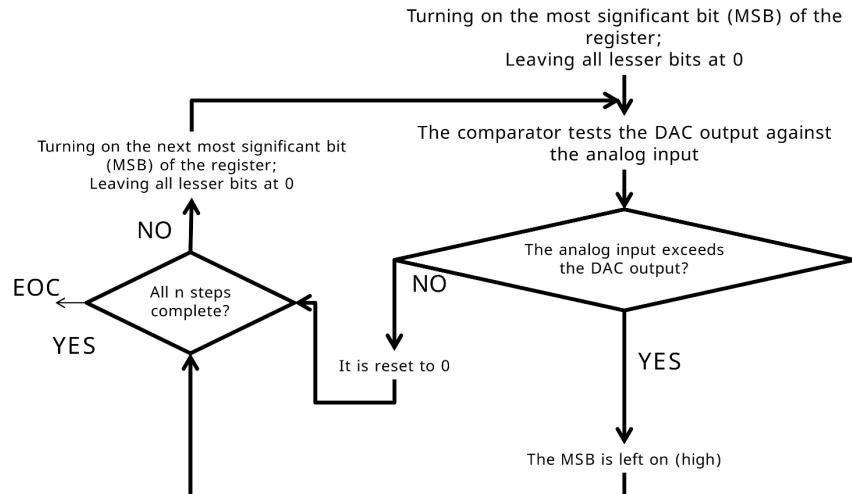
#### 11.4.3 Successive approximation pros and cons

Pros	Cons
High speed and good reliability	For higher resolution successive approximation, analogue-to-digital converters will be slower
Medium accuracy compared to other analogue-to-digital converter types.	Speed limited to about 5 milliseconds per sample.
Good tradeoff between speed and cost.	
Capable of yielding the binary number in a serial format (one bit at a time).	

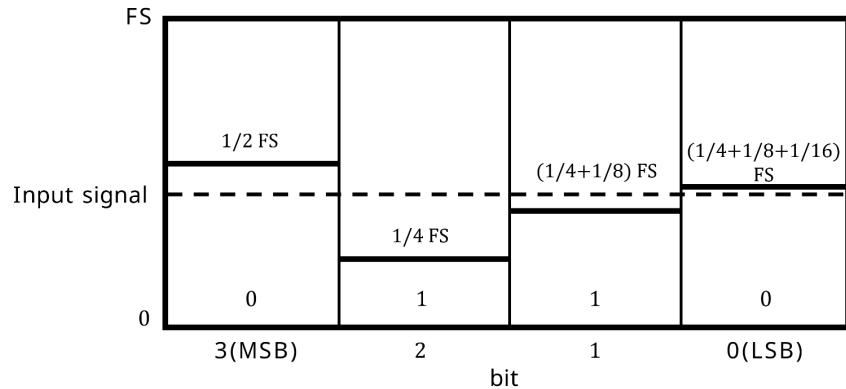
#### 11.4.4 Successive approximation flow chart



#### 11.4.5 A/D converter flow chart



#### 11.4.6 Example of a 4-bits A/D converter



The digital result is 0110. A higher resolution will produce more accurate results.

#### 11.4.7 Conversion time

- An n-bit successive approximation A/D converter has a conversion time of  $n\Delta T$ , where  $\Delta T$  is the cycle time of the digital-to-analogue converter and the control unit.
- The typical conversion time for 8, 10, or 12-bit successive approximation A/D converters ranges from 1 to 100  $\mu$ s

#### 11.4.8 Example of a 10-bit A/D converter

- Number of bits:  $n = 10$
- Voltage input:  $V_{in} = 0.6 \text{ V}$
- Reference voltage:  $V_{ref} = 1 \text{ V}$

Bit	Voltage
9	0.5
8	0.25
7	0.125
6	0.0625
5	0.03125
4	0.015625
3	0.0078125
2	0.00390625
1	0.001952125
0	0.0009765625

Number of possible states:  $N = 2^n = 1024$

$$\begin{aligned}\text{Resolution: } \Delta V &= \frac{V_{max} - V_{min}}{N} \\ &= \frac{1 \text{ V}}{1024} \\ &= 0.0009765625 \times V_{ref}\end{aligned}$$

#### 11.4.9 Process of calculating the most significant bit (bit 9)

1. Divide  $V_{ref}$  by 2,  $V = \frac{V_{ref}}{2} = 0.5$ .
2. Compare  $V$  with  $V_{in}$ .
3. If  $V_{in}$  is greater than  $V$ , turn the most significant bit (MSB) on (set to 1).
4. If  $V_{in}$  is less than  $V$ , turn the most significant bit off (set to 0).
5.  $V_{in} = 0.6$  V and  $V = 0.5$ .
6. Since  $V_{in} > V$ ,  $MSB = 1$

MSB	MSB-1	MSB-2	MSB-3	...					
1									

#### 11.4.10 Process of calculating the most significant bit - 1 (bit 8)

1.  $V = \frac{V_{ref}}{2} + \frac{V_{ref}}{4} = 0.5 + 0.25 = 0.75$  V.
2. Compare  $V_{in}$  to  $V$ .
3. Since  $0.6 < 0.75$ , the current most significant bit is turned off (set to 0).

MSB	MSB-1	MSB-2	MSB-3	...					
1	0								

#### 11.4.11 Process of calculating the most significant bit - 2 (bit 7)

1. Go back to the last voltage that caused it to be turned on (bit 9) and add it to  $\frac{V_{ref}}{8}$ .
2. Hence,  $V = \frac{V_{ref}}{2} + \frac{V_{ref}}{8} = 0.5 + 0.125 = 0.625 \text{ V}$
3. Since  $0.6 < 0.625$ , the current most significant bit is turned off (set to 0).

MSB	MSB-1	MSB-2	MSB-3	...					
1	0	0							

#### 11.4.12 Process of calculating the most significant bit - 3 (bit 6)

1. Go back to the last voltage that caused it to be turned on (bit 9) and add it to  $\frac{V_{ref}}{16}$ .
2. Hence,  $V = \frac{V_{ref}}{2} + \frac{V_{ref}}{16} = 0.5 + 0.0625 = 0.5625 \text{ V}$
3. Since  $0.6 > 0.5625$ , the current most significant bit is turned on (set to 1).

MSB	MSB-1	MSB-2	MSB-3	...					
1	0	0	1						

This process continues for all the remaining bits:

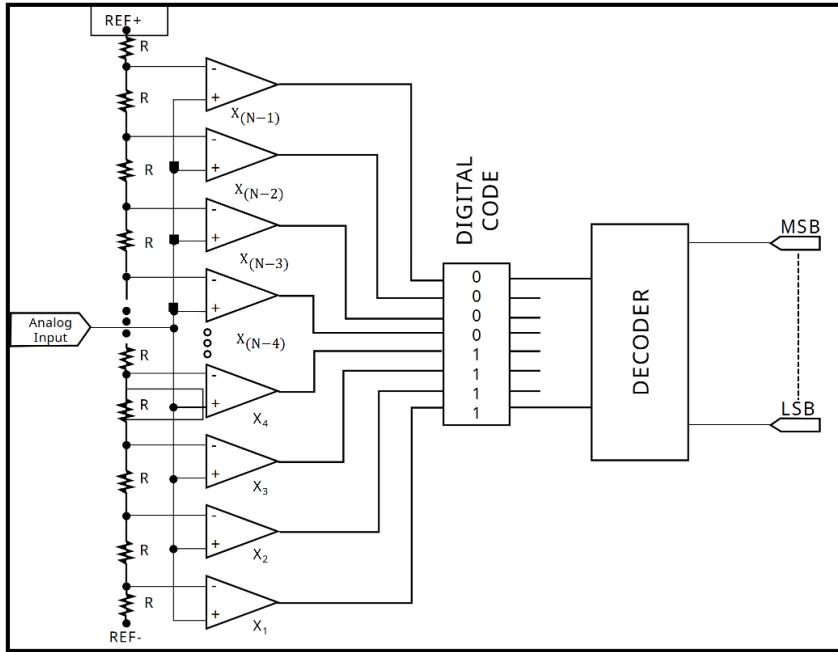
MSB	MSB-1	MSB-2	MSB-3	MSB-4	MSB-5	MSB-6	MSB-7	MSB-8	LSB
1	0	0	1	1	0	0	1	1	0

Results:

$$\frac{1}{2} + \frac{1}{16} + \frac{1}{32} + \frac{1}{256} + \frac{1}{512} = 0.599\,609\,375 \text{ V}$$

## 11.5 Flash A/D converters

- Has  $N - 1$  comparators.
- Has  $N$  resistors.



### 11.5.1 How does it work?

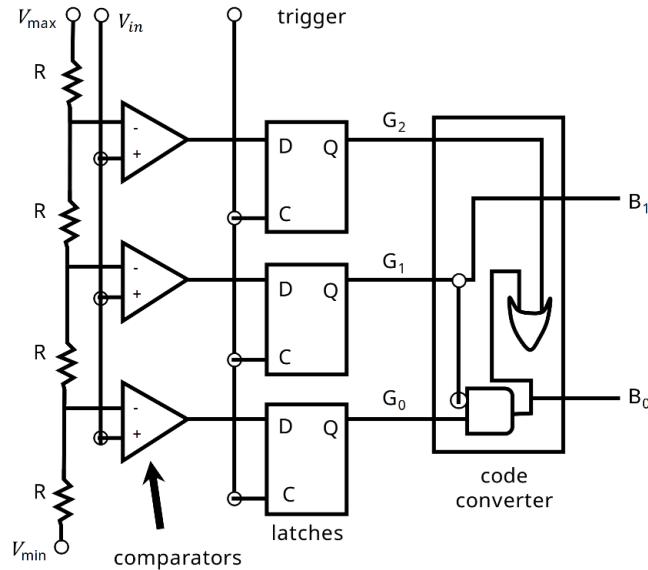
- It uses the  $N$  resistors to form a ladder voltage divider, which divides the reference voltage into  $N$  equal intervals.
- It uses the  $N - 1$  comparators to determine in which of these  $N$  voltage intervals the input voltage  $V_{in}$  lies.
- The combination logic then translates the information provided by the output of the comparators.
- This analogue-to-digital converter does not require a clock, so the conversion time is set by the settling time of the comparators and the propagation time of the combinational logic.

### 11.5.2 Pros and cons of flash A/D converters

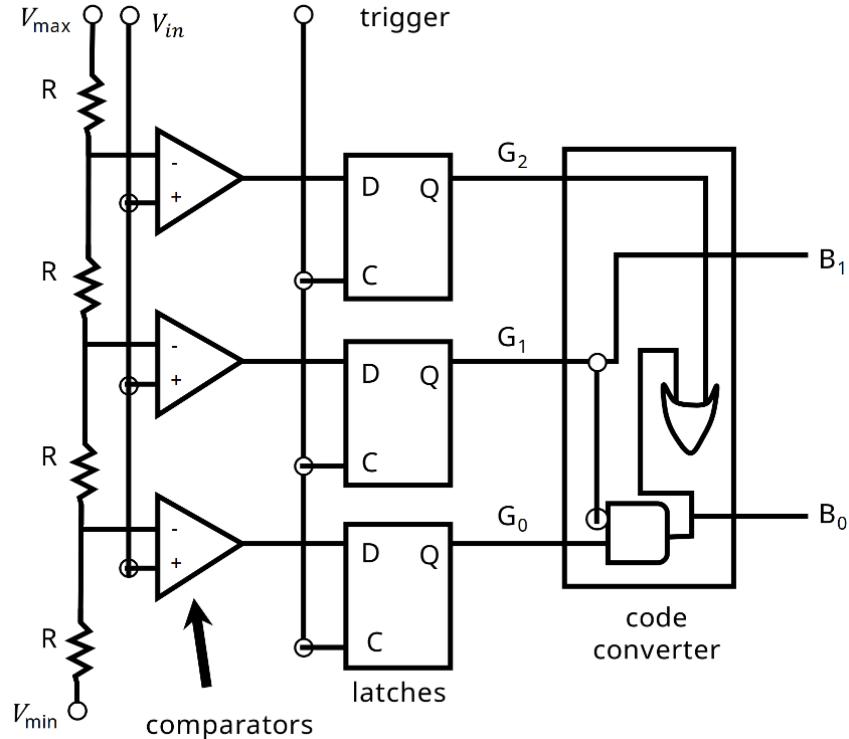
Pros	Cons
Very fast.	Expensive.
Very simple operational theory.	Prone to produce glitches in the output.
Speed is only limited by gate and comparator propagation delay.	Each additional bit of resolution requires twice the comparators.

### 11.5.3 Characteristics

- The fastest type of analogue-to-digital converter.
- It consists of a bank of input comparators acting in parallel to identify the signal level.
- The figure below shows a 2-bit converter with a resolution for output states.
- The output of the latches is in a coded form, which is easily converted to the required binary output with combinational logic.



#### 11.5.4 Output of a 2-bit flash converter



State	Code (G <sub>2</sub> G <sub>1</sub> G <sub>0</sub> )	Binary (B <sub>1</sub> B <sub>0</sub> )	Voltage Range
0	000	00	0 - 1
1	001	01	1 - 2
2	011	10	2 - 3
3	111	11	3 - 4

This assumes:

- An input voltage range of 0 to 4 V.
- The voltage range is set by the V<sub>min</sub> and V<sub>max</sub>.
- The code converter is a simple combinational logic circuit.
- For a 2-bit converter, the relationship between the code bit G<sub>i</sub> and the binary bits B<sub>i</sub> are:

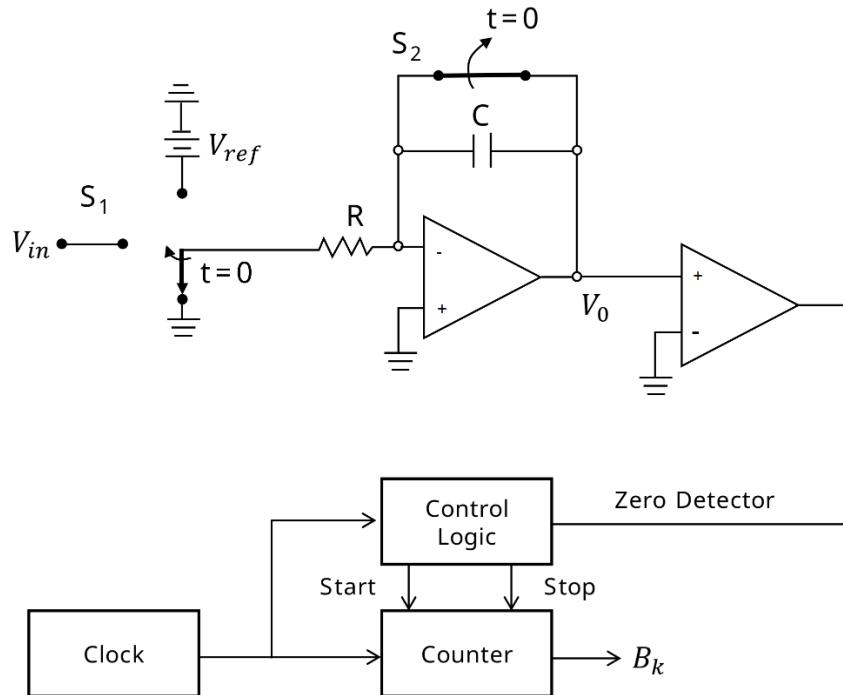
$$V_0 = G_0 \cdot \overline{G_1} + G_2$$

$$B_1 = G_1$$

## 11.6 Dual slope converters

Components:

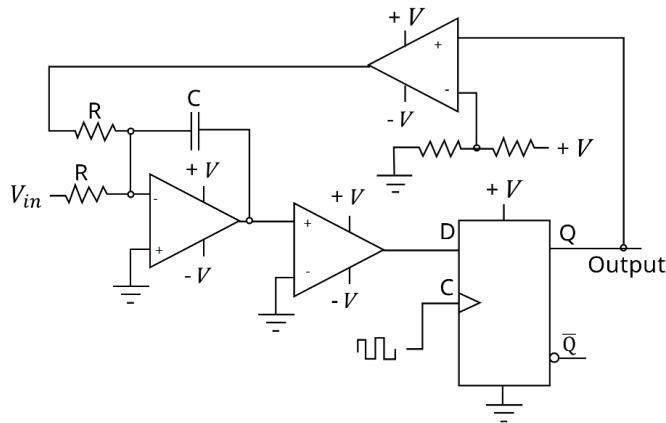
- Integrator
- Electronically controlled switches
- Counter
- Clock
- Control logic
- Comparators
- Resistor
- Capacitor



## 11.7 Sigma-delta A/D converters

Components:

- Resistors
- Capacitor
- Comparators
- Control logic
- Digital-to-analogue converter



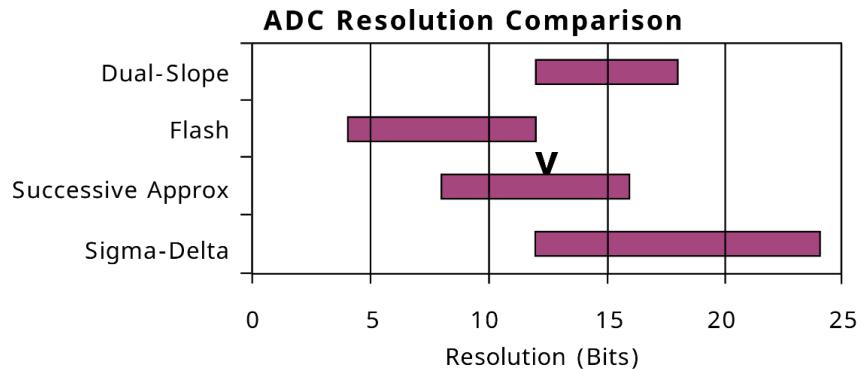
### 11.7.1 How does it work?

- The input is over sampled and goes to the integrator.
- The integration is then compared to the ground.
- It then iterates and produces a serial bit stream.
- The output is a serial bit stream with the number of 1's proportional to the  $V_{in}$ .
- With this arrangement, the sigma-delta modulator automatically adjusts its output to ensure that the average error at the quantiser output is zero.
- The integrator value is the sum of all past values of the error. Hence, whenever there is a non-zero error value, the integrator value just keeps building until the error is once again forced to zero.

### 11.7.2 Pros and cons of sigma-delta A/D converters

Pros	Cons
High resolution	Slow due to over sampling
No need for precision components	Only good for low bandwidth

## 11.8 Comparison of different types of A/D converters



Type	Speed (Relative)	Cost (relative)
Dual-slope	Slow	Medium
Flash	Very fast	High
Successive approximation	Medium fast	Low
Sigma-delta	Slow	Low

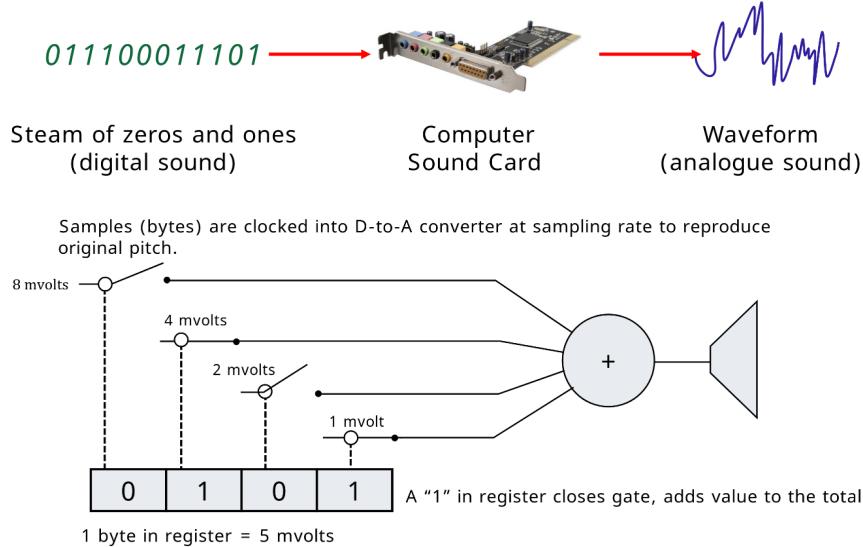
- Adding more resolution is a simple matter of adding more resistors, comparators and latches.
- The combinational logic code converter would also be different.
- Unlike with the successive approximation converter, adding resolution does not increase the time required for a conversion.

## 11.9 Digital-to-analogue (D/A) conversion

- It is to reverse the process of A/D conversion by changing a digital value to an analogue voltage.
- Digital-to-analogue conversion allows a computer to interface with external analogue circuits and devices.

### 11.9.1 Example of a D/A converter

#### Playing back the digital sound file



### 11.9.2 Problems with D/A conversion

- Finite word length.
  - Most systems today do 16-bit digitising.
  - Hence, there are 65536 different levels.
- The loudest sounds need room, so the normal sounds don't make use of the entire range.
  - Problems occur at low levels where sounds are represented by only one or two bits, which results in a lot of distortion.
- Dithering adds low level broadband noise.

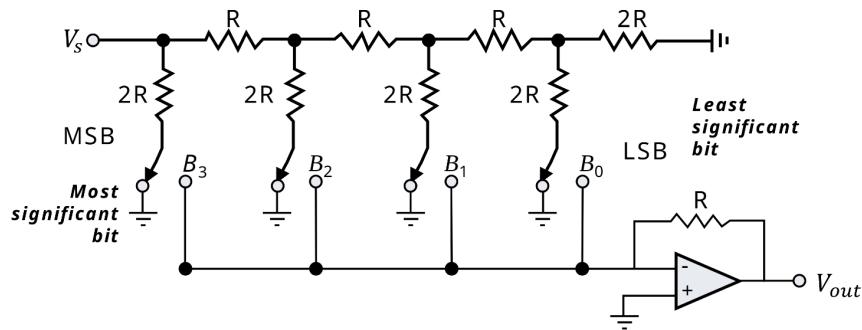
## 11.10 D/A conversions

### 11.10.1 How to do D/A conversions?

The simplest type of D/A converter is a resistor ladder network connected to an inverting summer operational amplitude circuit. Below is a 4-bit  $R - 2R$  resistor ladder network which requires only two precision resistance values  $R$  and  $2R$ .

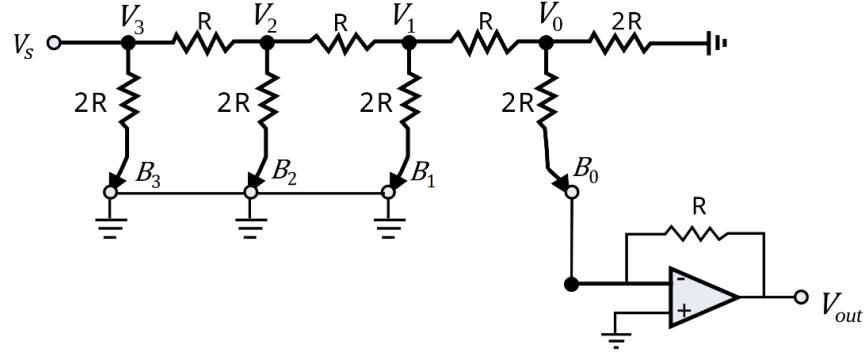
Note:

- The digital input to the digital-to-analogue converter is a 4-bit binary number represented by bits  $B_0, B_1, B_2$ , and  $B_3$ .
- $B_0$  is the least significant bit and  $B_3$  is the most significant bit.
- Each bit in the circuit controls a switch between the ground and the inverting input of the operational amplifier.



### 11.10.2 When $B_0$ of the D/A converter is 0001

$B_0$  is the least significant bit. If the bit number is 0001, then the  $B_0$  switch connects to the operational amplifier, while the others are grounded.



Since the inverting operational amplifier is grounded, we have:

$$V_{out_0} = -\frac{1}{2}V_0$$

$$V_0 = \frac{1}{2}V_1, \quad V_1 = \frac{1}{2}V_2, \quad V_2 = \frac{1}{2}V_3 = \frac{1}{2}V_s$$

So:

$$V_{out_0} = -\frac{1}{16}V_s$$

$$V_{out_1} = -\frac{1}{8}V_s, \quad V_{out_2} = -\frac{1}{4}V_s, \quad V_{out_3} = -\frac{1}{2}V_s$$

Total output:

$$V_{out} = \sum_{i=0}^{n-1} B_i V_{out_i}$$

## 12 1st order systems

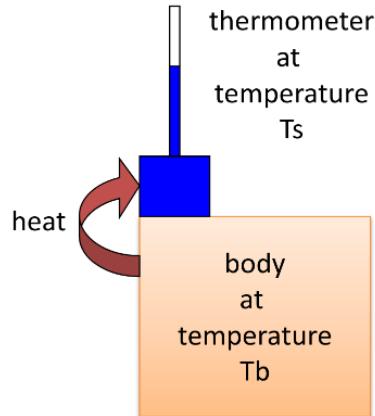
In general, the time response of a first order system is:

$$x(t) = a + be^{-\frac{t}{\tau}}$$

Where:

- $x$  is the response of the system
- $t$  is the time
- $a$  and  $b$  are arbitrary constants to be determined
- $\tau$  is the time constant

## 12.1 Example



Thermal measurements:

- Heat:

$$q = \frac{T_b - T_s}{R}$$

Where:

- $R$  is the thermal resistance

- Change in heat:

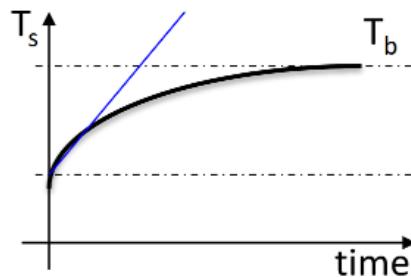
$$\frac{dT_s}{dt} = \frac{q}{C}$$

Where:

- $q$  is the heat
- $C$  is the thermal capacitance

- If  $\frac{dT_b}{dt} = 0$ :

$$T_s(t) = T_{s0} + (T_b - T_{s0})(1 - e^{-\frac{t}{RC}})$$



## 12.2 General (forced) equation

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = f(t)$$
$$x(0) = x_0$$

Where:

- $\tau$  is the time constant
- $f(t)$  is the forced input
- $x_0$  is the initial condition

## 12.3 Natural (unforced) equation

$$\frac{dx_N(t)}{dt} + \frac{x_N(t)}{\tau} = 0$$
$$x(0) = x_0$$
$$x_N(t) = x_0 e^{-\frac{t}{\tau}}$$

Satisfying the initial conditions:

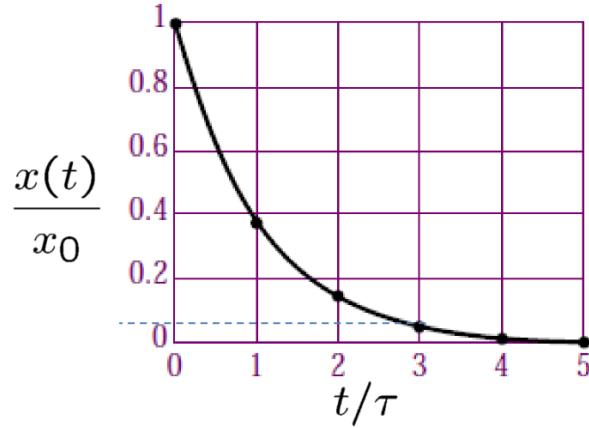
$$x_N(t) = K e^{-\frac{t}{\tau}}$$

$$x_N(0) = K = x_0$$

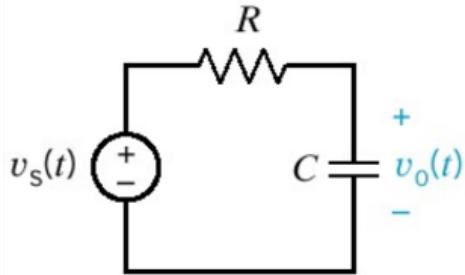
Where:

- $\tau$  is the time constant
- $f(t)$  is the forced input
- $x_0$  is the initial condition

## 12.4 Time constant



### 12.4.1 RC circuits

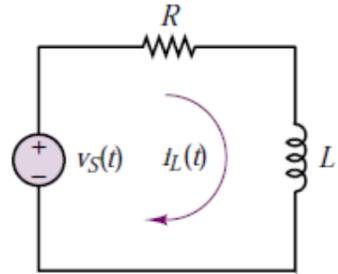


$$RC = \frac{V}{I} \cdot \frac{Q}{V} = \frac{Q}{\frac{Q}{t}} = t$$

Where:

- $R$  is the resistance of the resistor
- $C$  is the capacitance of the capacitor
- $V$  is the voltage
- $I$  is the current
- $Q$  is the total charge
- $t$  is the time

#### 12.4.2 RL circuits



$$\frac{L}{R} = \frac{V}{\frac{I}{t}} \cdot \frac{I}{V} = t$$

Where:

- $R$  is the resistance of the resistor
- $L$  is the inductance of the inductor
- $V$  is the voltage
- $I$  is the current
- $t$  is the time

## 12.5 Response to DC forcing inputs

Given:

$$\frac{dx(t)}{dt} + \frac{x(t)}{\tau} = F_0$$

$$x(0) = x_0$$

Where:

$$f(t) = F_0$$

Looking for a particular (forced) solution  $x_F(t)$ , and considering a **DC** steady-state solution:

$$\cancel{\frac{dx_{SS}}{dt}} + \frac{x_{SS}}{\tau} = F_0$$

$$\frac{x_{SS}}{\tau} = F_0$$

$$x_{SS} = F_0\tau = x_\infty$$

Determining a general solution:

- Including the natural solution:

$$x(t) = x_N(t) + x_{SS}(t) = Ke^{-\frac{t}{\tau}} + x_\infty$$

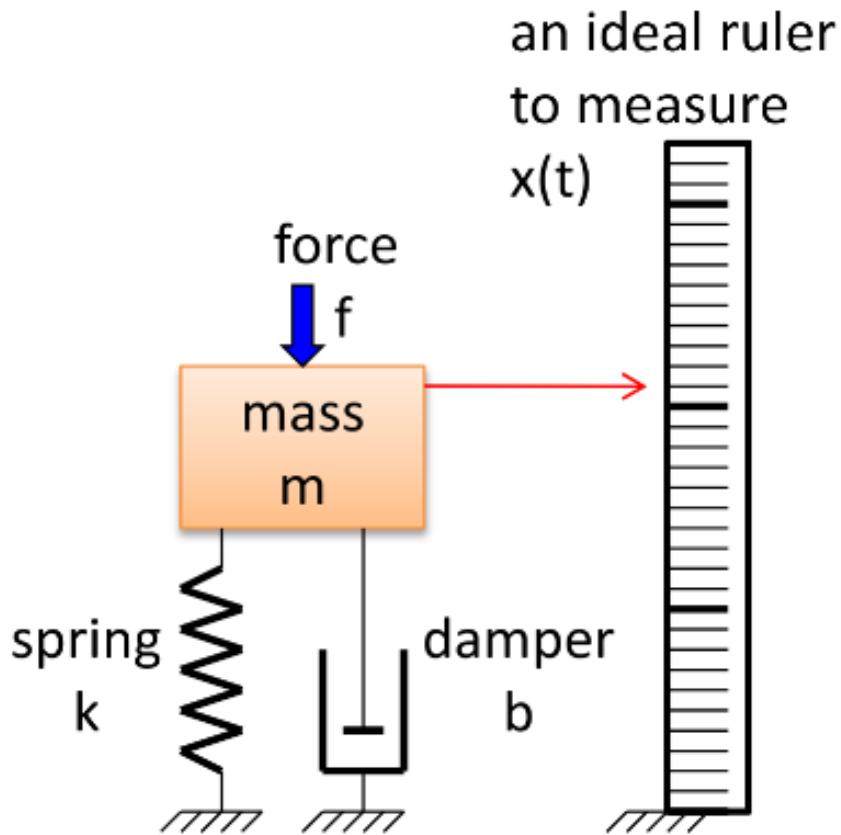
- Satisfying the initial condition:

$$x(0) = K + x_\infty = x_0$$

Hence, the general solution is:

$$x(t) = (x_0 - x_\infty)e^{-\frac{t}{\tau}} + x_\infty$$

## 12.6 2nd order systems



We can indirectly estimate the force from a displacement measurement:

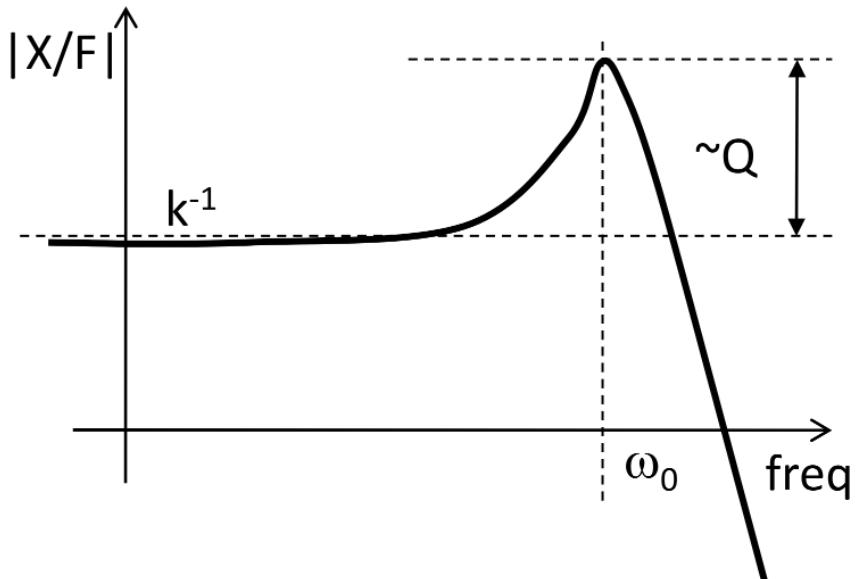
- Dynamic equations:

$$F = m\ddot{x} + b\dot{x} + kx$$

- Frequency response:

$$\frac{X}{F} = \frac{k^{-1}}{-\frac{\omega^2}{\omega_0^2} + \frac{j\omega}{Q\omega_0} + 1}$$

### 12.6.1 Frequency response



$$\frac{X}{F} = \frac{k^{-1}}{-\frac{\omega^2}{\omega_0^2} + \frac{j\omega}{Q\omega_0} + 1}$$

- Resonance:

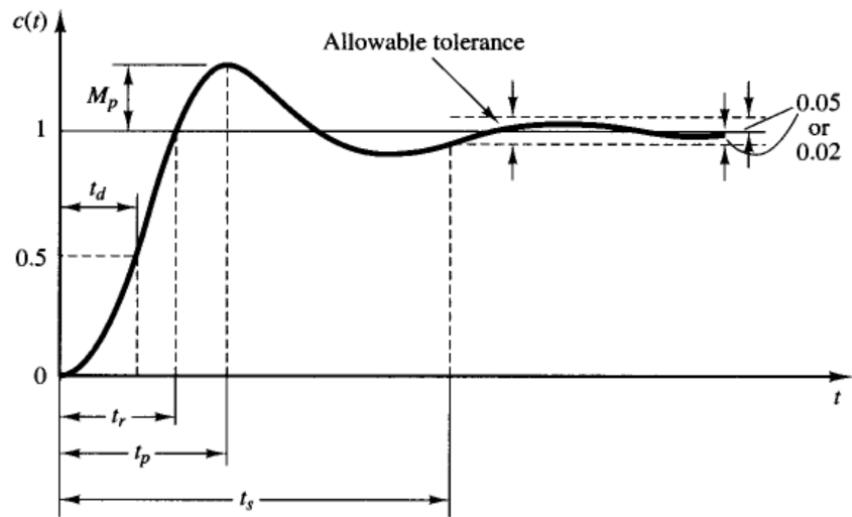
$$\omega_0^2 = \frac{k}{m}$$

- Mechanical  $Q$ :

$$Q^2 = \frac{km}{b}$$

When  $Q = 0.5$ , the system is critically damped.

### 12.6.2 Time response



- Rise time
- Settling time
- Overshoot

## 13 Measuring temperature

### 13.1 Temperature scales

$$C = K - 273.15$$

$$F = 1.8 \cdot C + 32$$

Where:

- $C$  is the temperature in degree Celsius
- $K$  is the temperature in Kelvin
- $F$  is the temperature in degree Fahrenheit

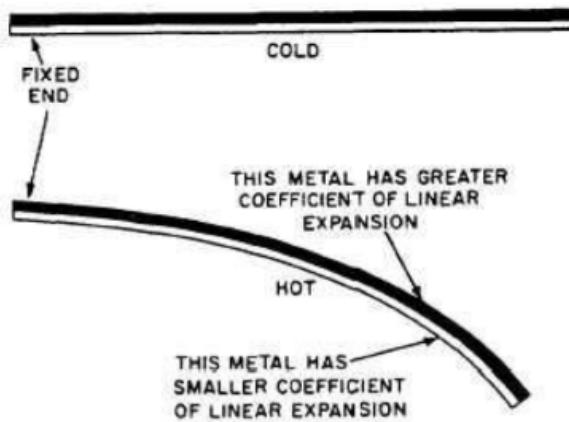
These temperature scales are based on:

- Fixed-points, such as the temperatures at phase transitions, triple points, etc.
- Size of the degree, such as  $\frac{1}{100}$  of the difference between icy and boiling water.
- Interpolation in between fixed points, like does  $50^{\circ}\text{C}$  correspond to the level of mercury which is halfway between the  $0^{\circ}\text{C}$  and  $100^{\circ}\text{C}$  levels?

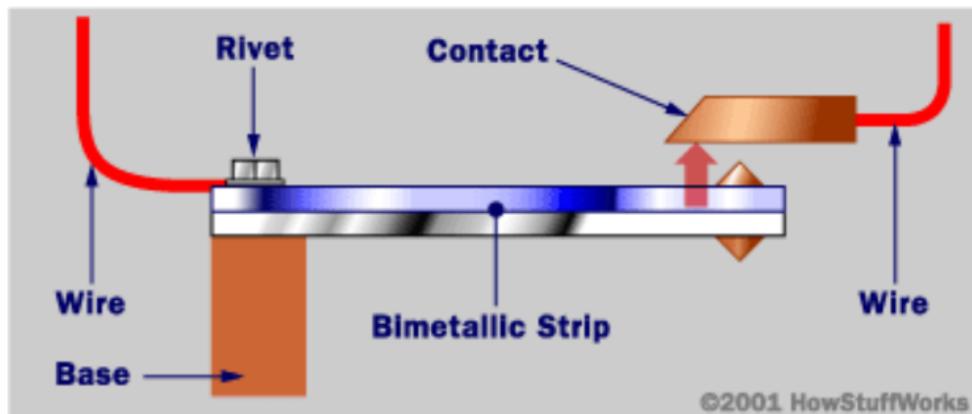
Temperature scales are standardised using the ITS-90 standard.

## 13.2 Bimetallic thermometers

- Makes use of differential thermal expansion of different metals.
  - Metal A and B bonded at temperature T1.
  - Bending occurs at different temperatures.



- Furnace thermostat, which makes uses of a switch to control the temperature.



### 13.3 Resistance temperature detectors (RTD)

- Resistance temperature detectors are based on changes of resistance with temperature.
  - Usually, they are a **metal wire** on insulating support, which eliminates mechanical strain.
  - They are also encased, to minimise the influence from the environment, such as corrosion.



### 13.3.1 Linearity range

- For a given material, a linear relationship can be assumed for a **limited range**.

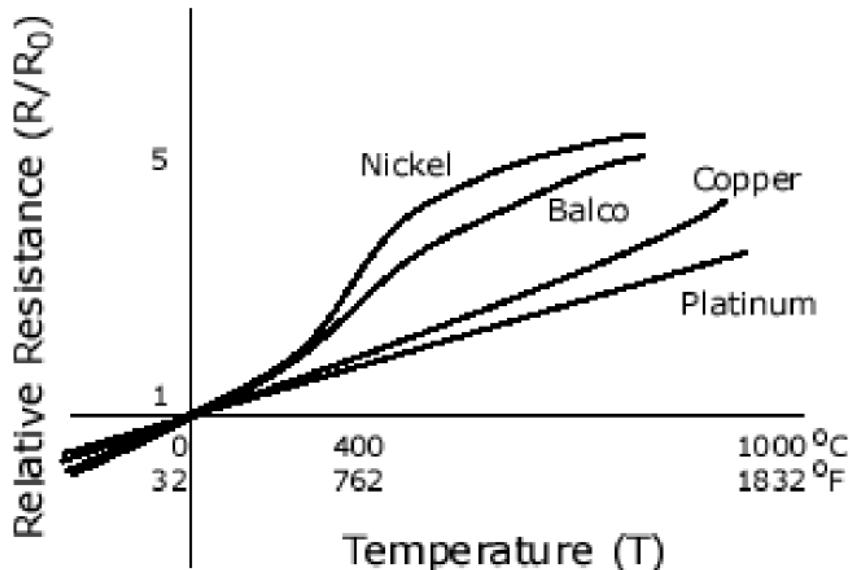
$$\frac{R}{R_0} = 1 + \alpha(T - T_0)$$

Where:

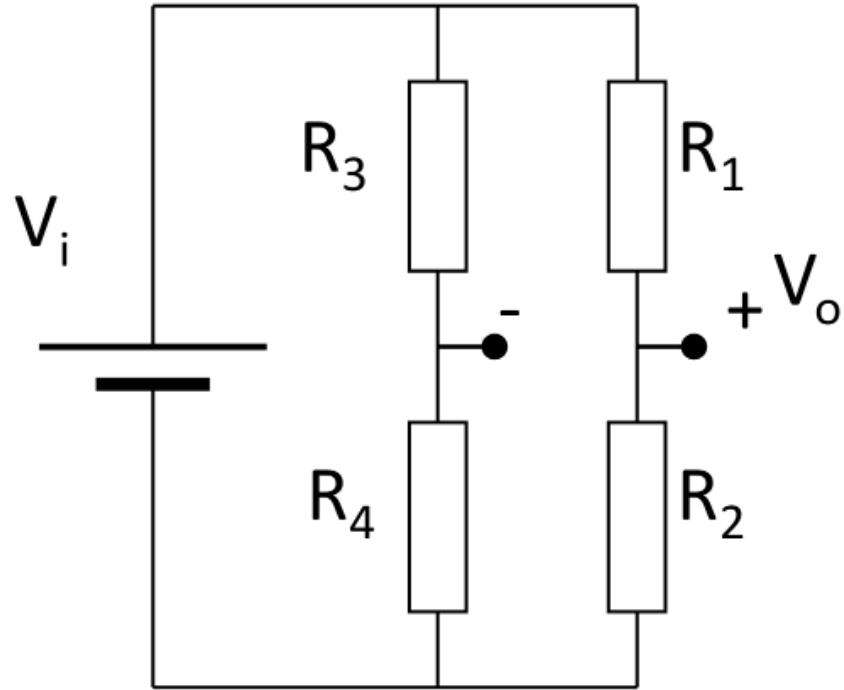
- $R$  is the resistance at temperature  $T$
- $R_0$  is the resistance at temperature  $T_0$
- $\alpha$  is the temperature coefficient

- For platinum:

- $\pm 0.3\%$  over the range  $0 - 200^\circ\text{C}$
- $\pm 1.2\%$  over the range  $200 - 800^\circ\text{C}$



### 13.3.2 Wheatstone bridge



- Bridge equations:

$$\frac{V+}{V_i} = \frac{R_2}{R_1 + R_2}$$

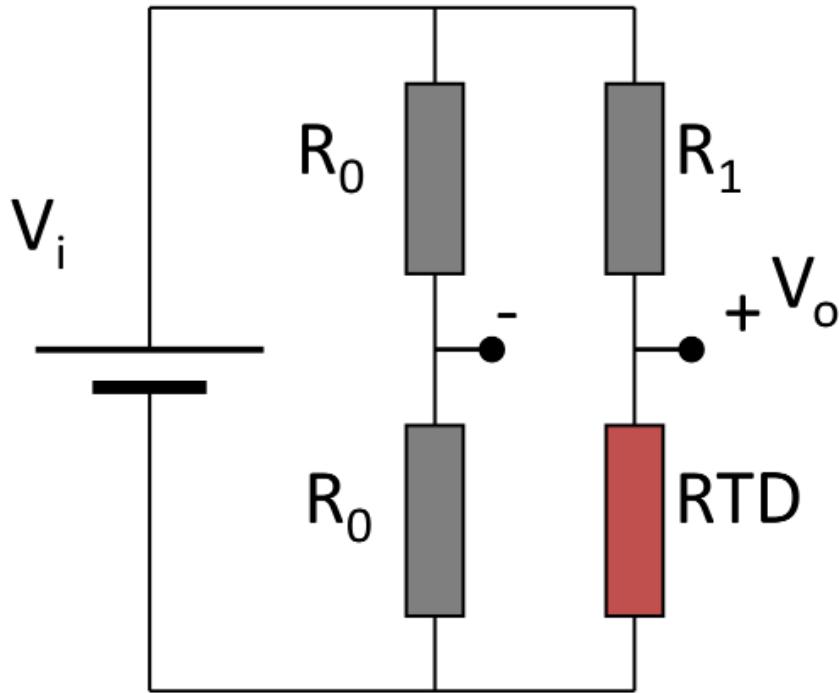
$$\frac{V-}{V_i} = \frac{R_4}{R_3 + R_4}$$

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4}$$

- Bridge balance condition:

$$V_o = 0 \Leftrightarrow R_1 R_4 = R_2 R_3$$

### 13.3.3 Numerical example



- An RTD forms one arm of an equal-arm Wheatstone bridge:

$$R_0 = R_1 = 25 \Omega$$

At  $0^\circ\text{C}$ ,  $RTD = 25 \Omega$  and  $\alpha = 0.003925 \text{ }^\circ\text{C}^{-1}$

- If the  $R_3$  required to balance the bridge is  $37.36 \Omega$ , find the temperature of the RTD.
- Solution using the bridge-balance condition:

$$\cancel{R_0} * RTD = R_1 * \cancel{R_0}$$

$$RTD = R_1 = 37.36 \Omega$$

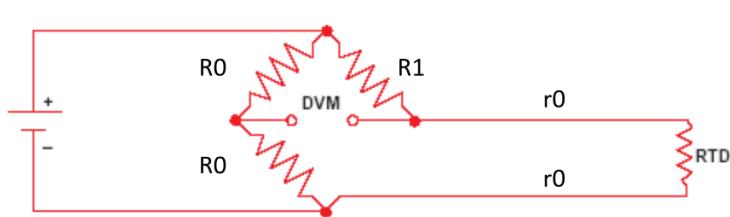
For  $\frac{R_{RTD}}{R_0} = 1 + \alpha(T - T_0)$ :

$$\frac{37.36 \Omega}{25 \Omega} = 1 + 0.003925(T - 0)$$

$$T = 126 \text{ }^\circ\text{C}$$

### 13.3.4 Measurements

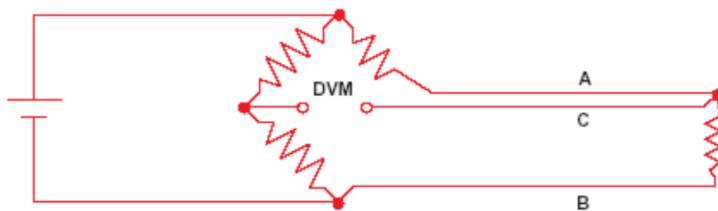
- Wheatstone bridge has **low resistance** (conductors) and is subject to **self-heating**.
- Lead-wire effects:
  - 2-wires:



Long wires are also subject to temperature-resistance changes:

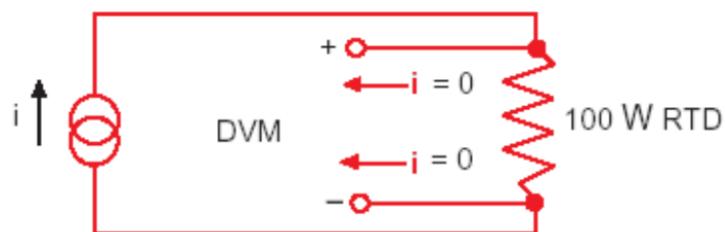
$$RTD + 2 \cdot r_0 = R_1$$

- 3-wires:



$$RTD + r_0 = R_1 + R_0 \rightarrow RTD = R_1$$

- 4-wires:



### 13.4 Thermistors

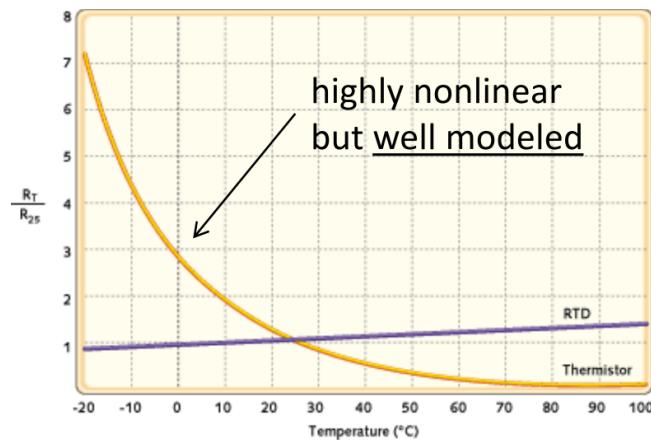
Thermistors are thermally sensitive resistors.

- They are made of ceramic-like **semiconductors**.
  - $R_0$  is much larger than *RTD*.
- The resistance decreases rapidly with temperature.
  - High-sensitivity
  - Ruggedness
  - Fast time-response

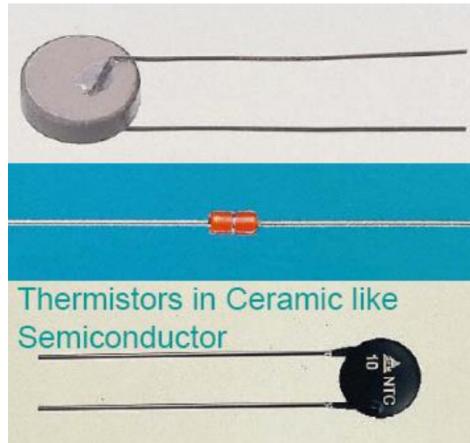
$$R = R_0 e^{\beta \left( \frac{1}{T} - \frac{1}{T_0} \right)}$$

Where:

- $R$  is the current resistance
- $R_0$  is the initial resistance
- $\beta$  is the material constant
- $T$  is the current temperature in Kelvin (K)
- $T_0$  is the initial temperature in Kelvin (K)



### 13.4.1 Examples



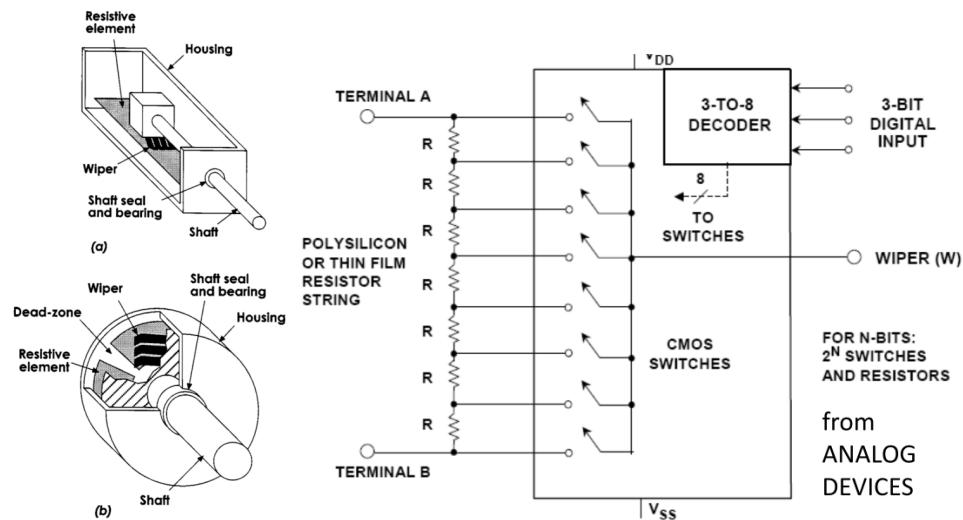
## 14 Measuring displacement (resistive sensors)

### 14.1 Resistive sensors

- Potentiometer, also known as a "pot".
  - **3-terminal** electromechanical device based on a conductive **wiper** sliding against a fixed, resistive element.
  - Many varieties of varying qualities and for different functions. Some examples include rheostats, trimmers, volume control, etc.

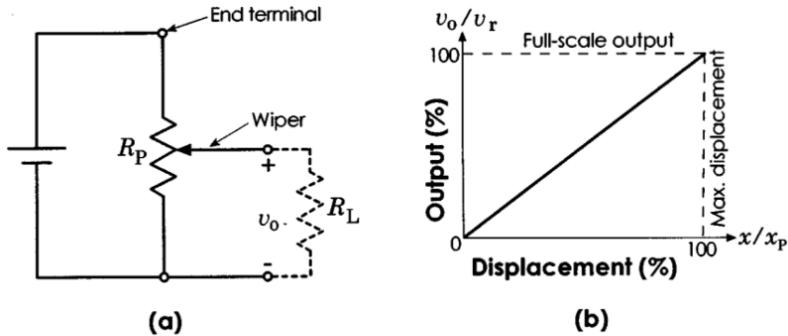


- Precision potentiometers, which are manually or **digitally** tunable.

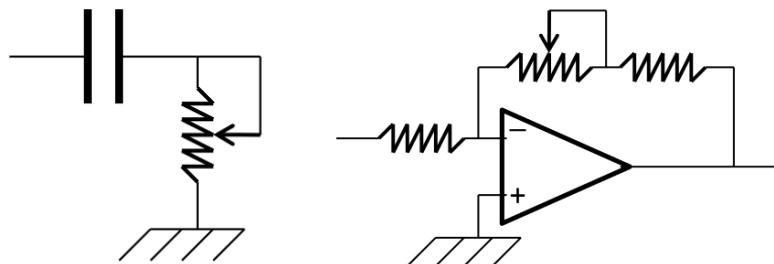


### 14.1.1 Using potentiometers in electrical circuits

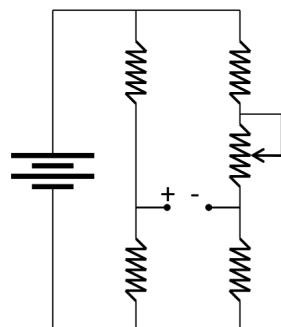
- Voltage divider



- Variable resistance



tunable filters variable gain amplifier



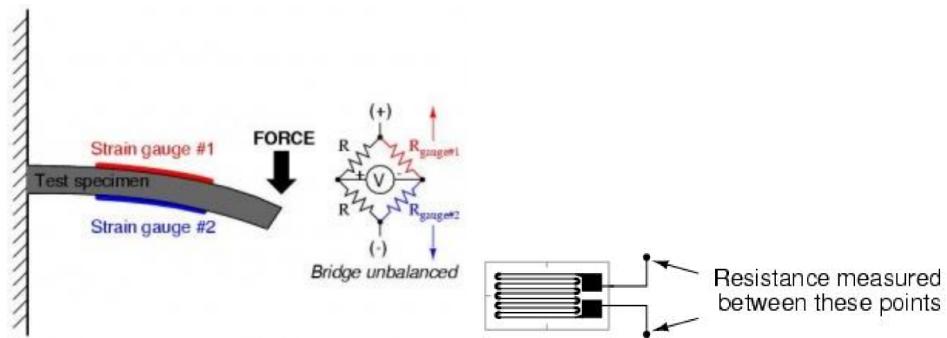
Wheatstone bridge  
with adjustable offset

## 15 Measuring forces through displacement

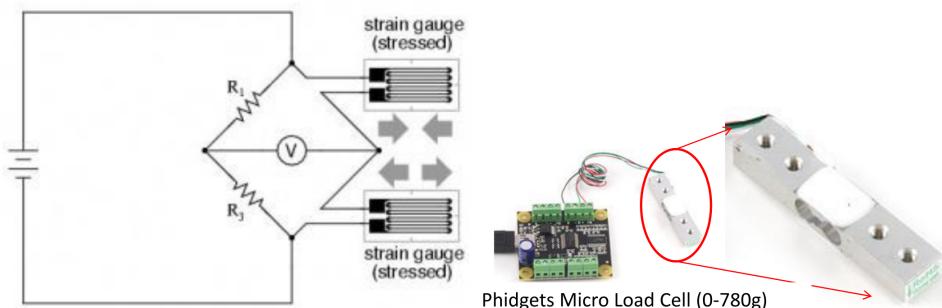
Force sensors.

### 15.1 Strain gauges

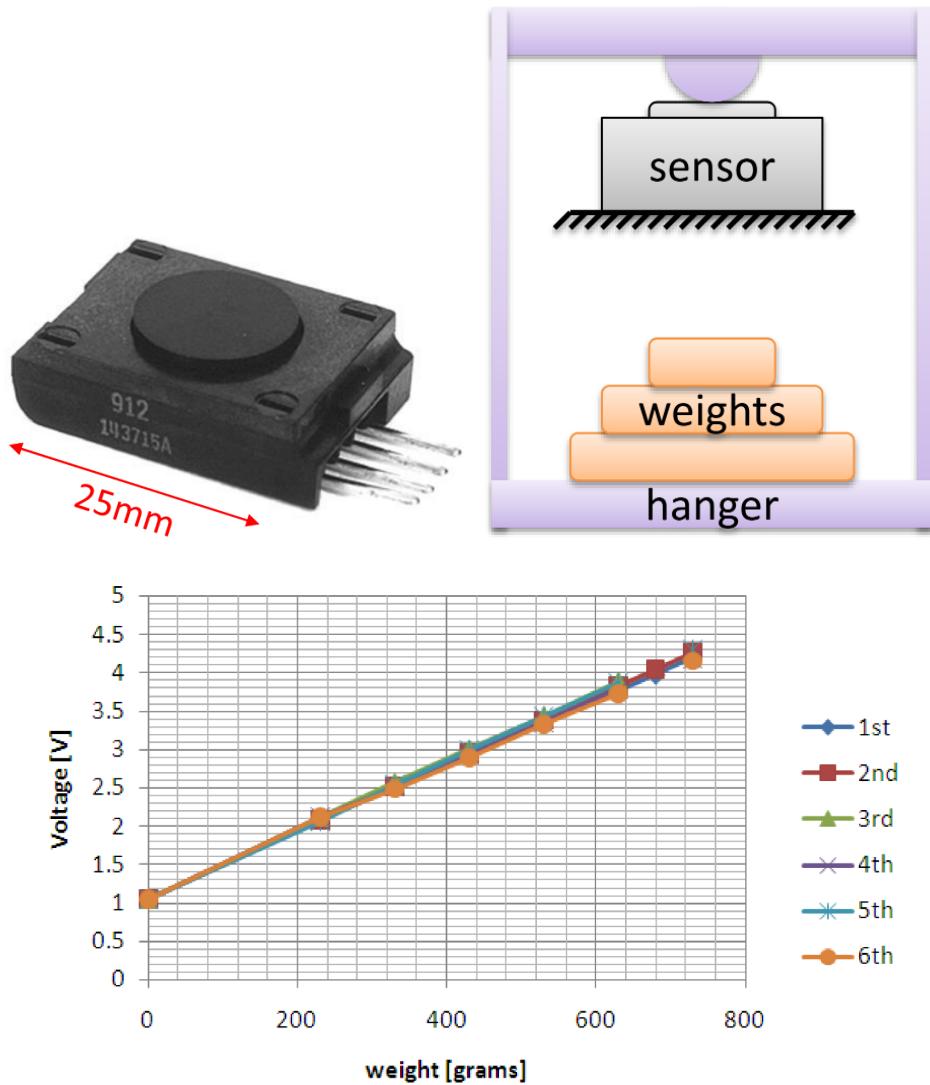
Force measurements are always made indirectly via deformations.



Half-bridge strain gauge circuit



## 15.2 Piezo-resistive force sensor



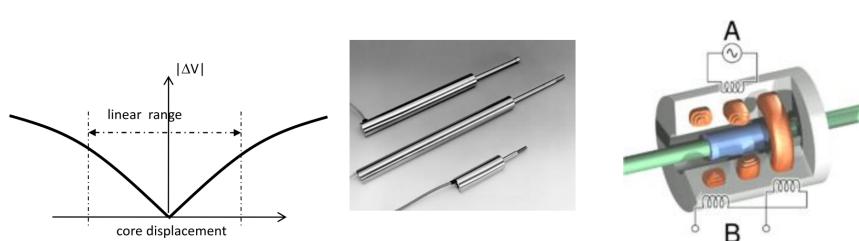
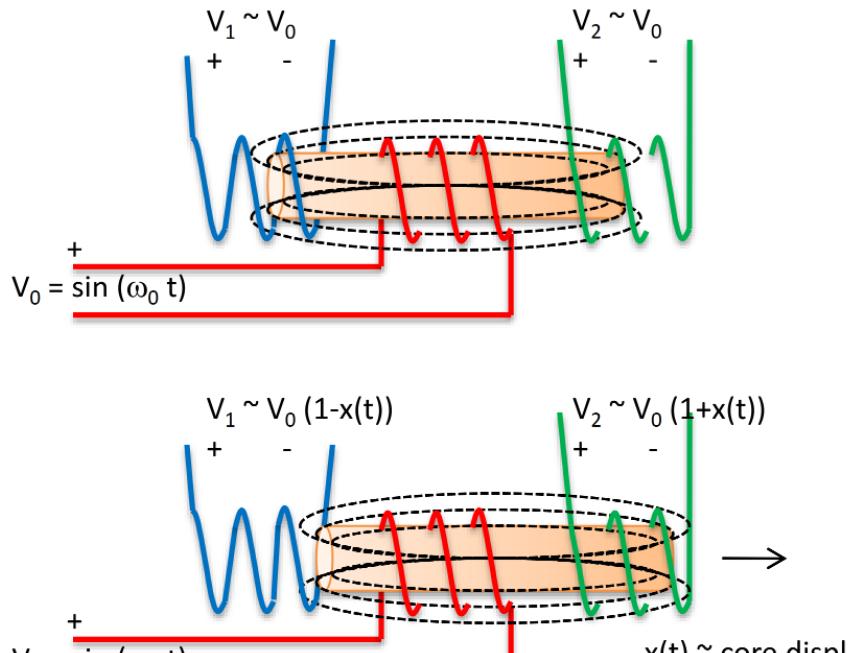
## 16 Measuring displacement (inductive sensors)

### 16.1 Linear variable differential transformer (LVDT)

A linear variable differential transformer is a type of electrical transformer that measures **linear** displacement.

- It has variable coupling via sliding ferromagnetic core.
  - One primary coil driven by AC, usually in kHz.
  - Two secondary coils
- Differential voltage:

$$V_{out} = \Delta V = V_2 - V_1 \simeq x(t)V_0$$



### 16.1.1 Amplitude modulation

The amplitude of the output voltage is modulated by the physical displacement:

$$\Delta V \approx V_0 x(t) = \sin(\omega_0 t) \sin(\omega_x t)$$

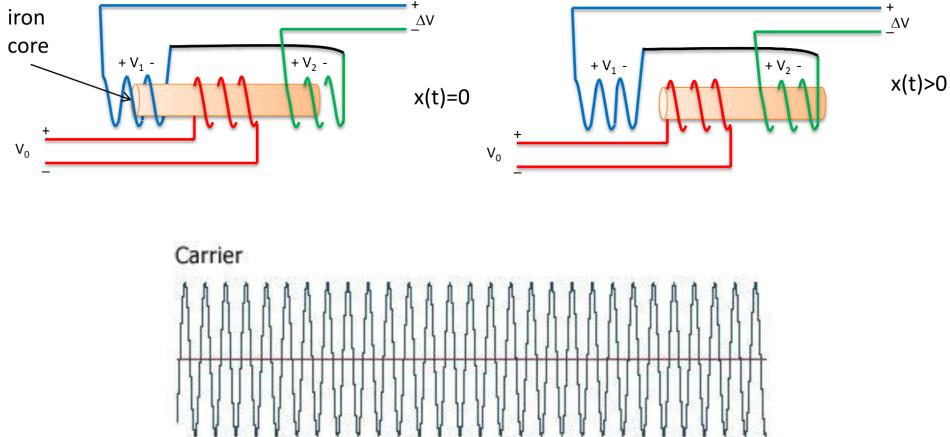


Figure 1:  $V_0 = \sin(\omega_0 t)$

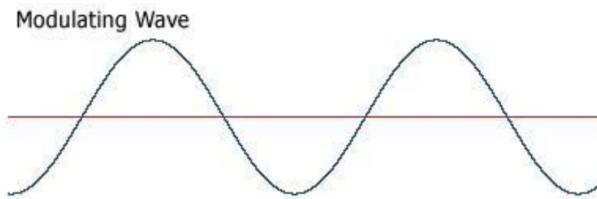


Figure 2:  $x(t) = \sin(\omega_x t)$

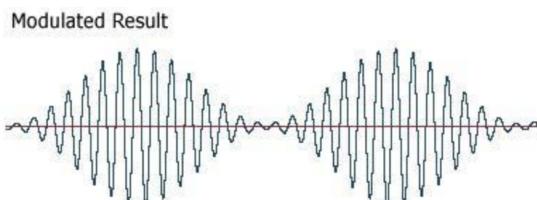
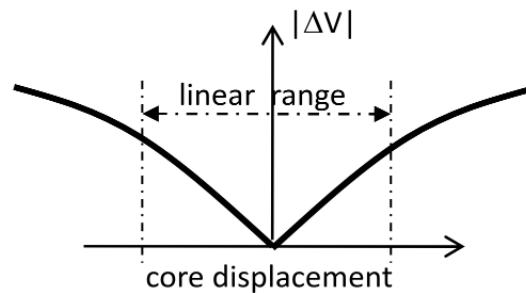
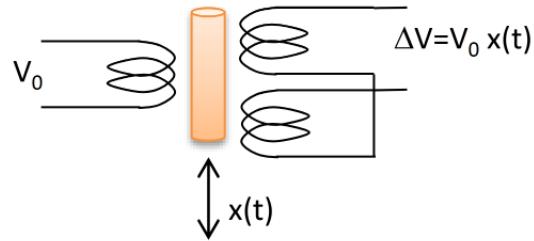


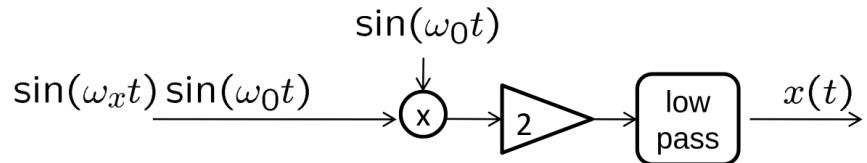
Figure 3:  $\Delta V = \sin(\omega_0 t) \sin(\omega_x t)$

### 16.1.2 Amplitude demodulation



$$V_0 = \sin(\omega_0 t)$$

$$x(t) = \sin(\omega_x t)$$



↓

$$\sin(\omega_x t) \sin^2(\omega_0 t)$$

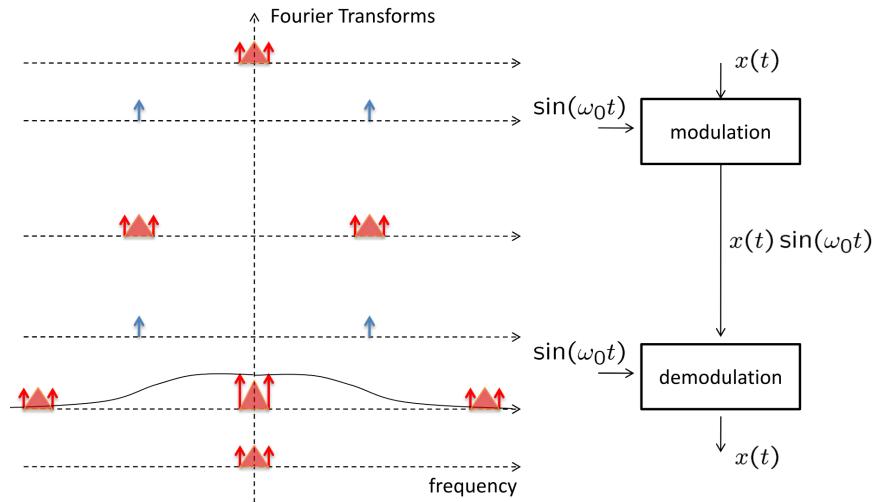
↓

$$\sin(\omega_x t) \frac{1 - \cos(2\omega_0 t)}{2}$$

↓

$$\underbrace{\frac{1}{2} \sin(\omega_x t)}_{\text{Low frequency}} + \underbrace{\frac{\sin(t(2\omega_0 t + \omega_x)) - \sin(t(2\omega_0 - \omega_x))}{4}}_{\text{Unwanted high frequency, to be filtered out}}$$

### 16.1.3 Amplitude modulation and demodulation



## 17 Measuring displacement (capacitive sensors)

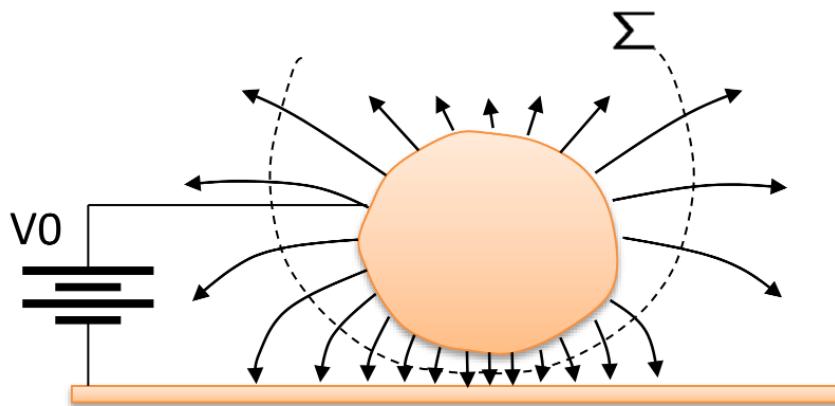
### 17.1 Principle

- Capacitance definition:

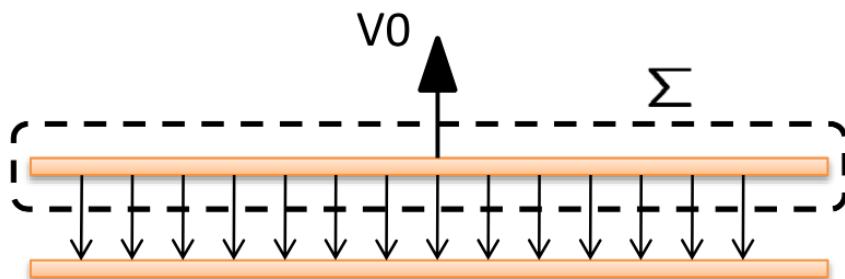
$$C = \frac{Q}{V}$$

- In the ideal case, we have infinite parallel plates.
- Used for proximity sensing.

Gauss' Law:

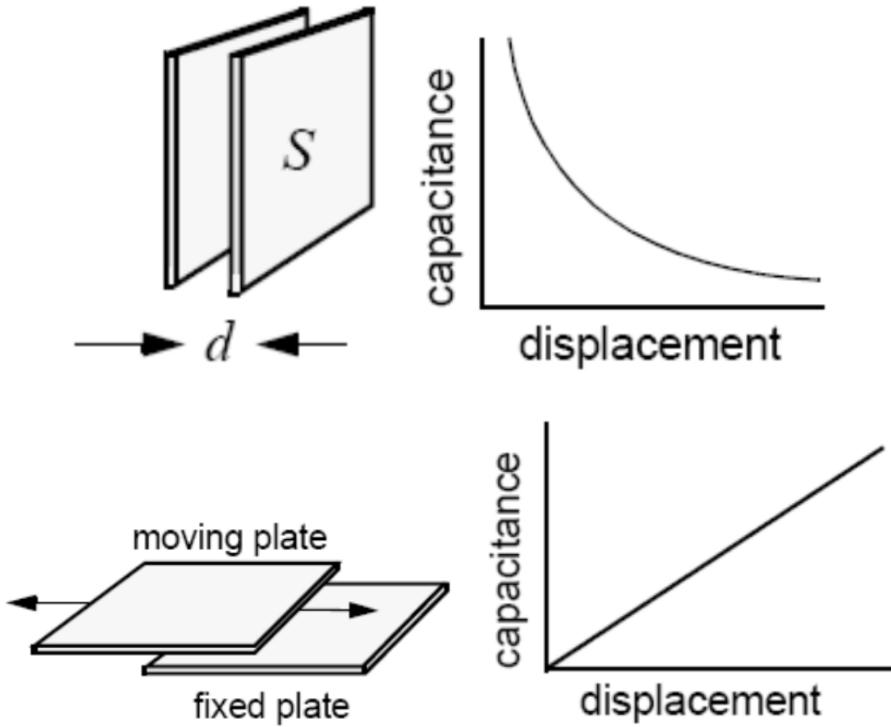


$$Q = \iint_{\Sigma} \epsilon_0 \epsilon_r E \, dS$$



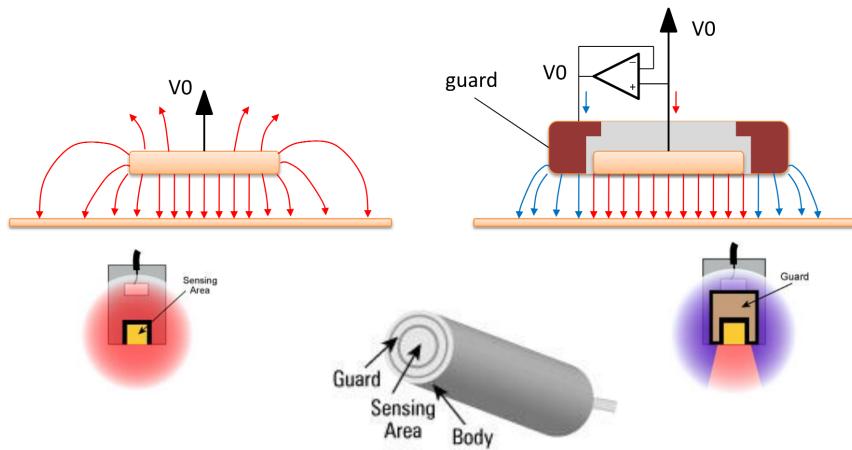
$$C = \frac{Q}{V} = \frac{\epsilon_0 \epsilon_r E S}{Ed} = \frac{\epsilon_0 \epsilon_r S}{d}$$

### 17.1.1 Types of plate movement



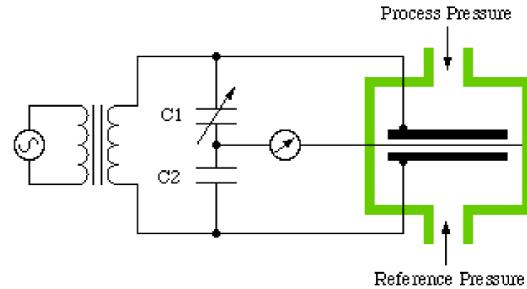
## 17.2 Guard electrode

The guard electrode limits field-fringing effects.

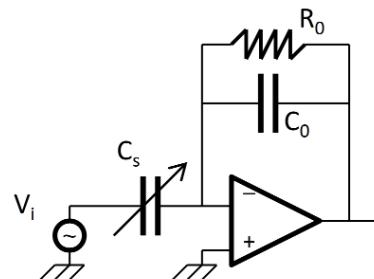


### 17.3 Interfacing with capacitive sensors using AC

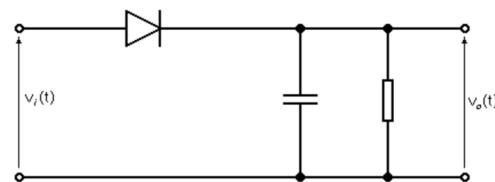
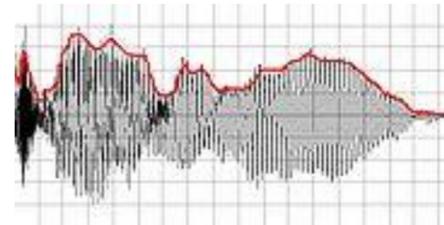
- AC bridge:



- AC driver circuit:



- Envelope demodulator, which is the simplest kind of demodulation for non-negative signals.

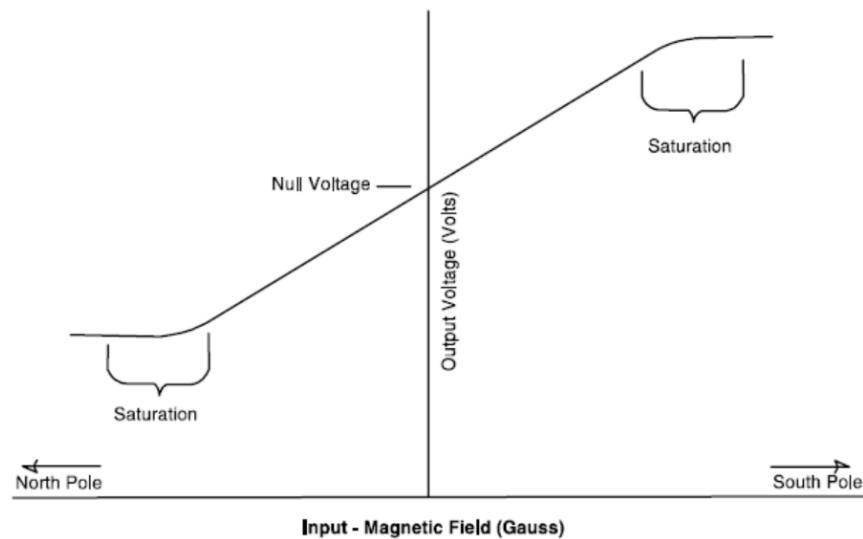
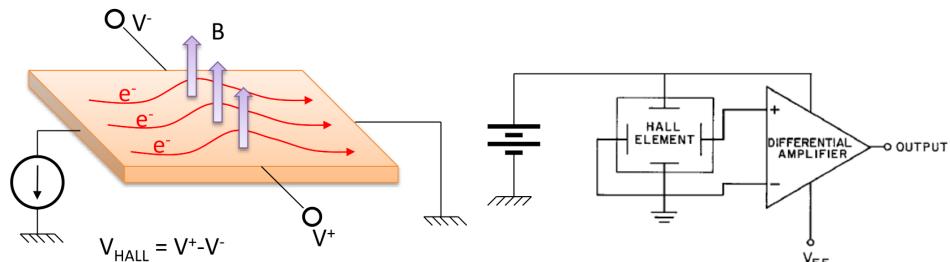


## 18 Proximity sensors

### 18.1 Hall effect

The Lorentz force is defined as:

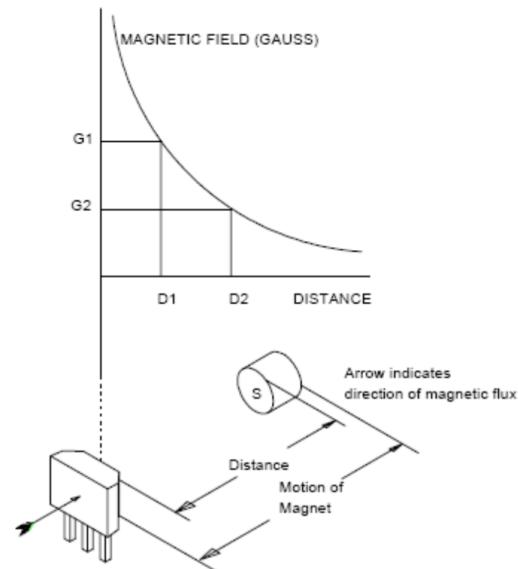
$$\vec{F} = q\vec{v} \times \vec{B}$$



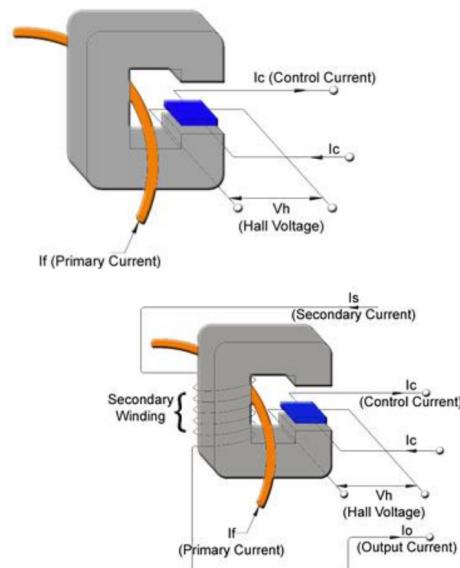
## 18.2 Hall effect sensors

### 18.2.1 Proximity sensor

Hall effect proximity sensors have a contactless switch.

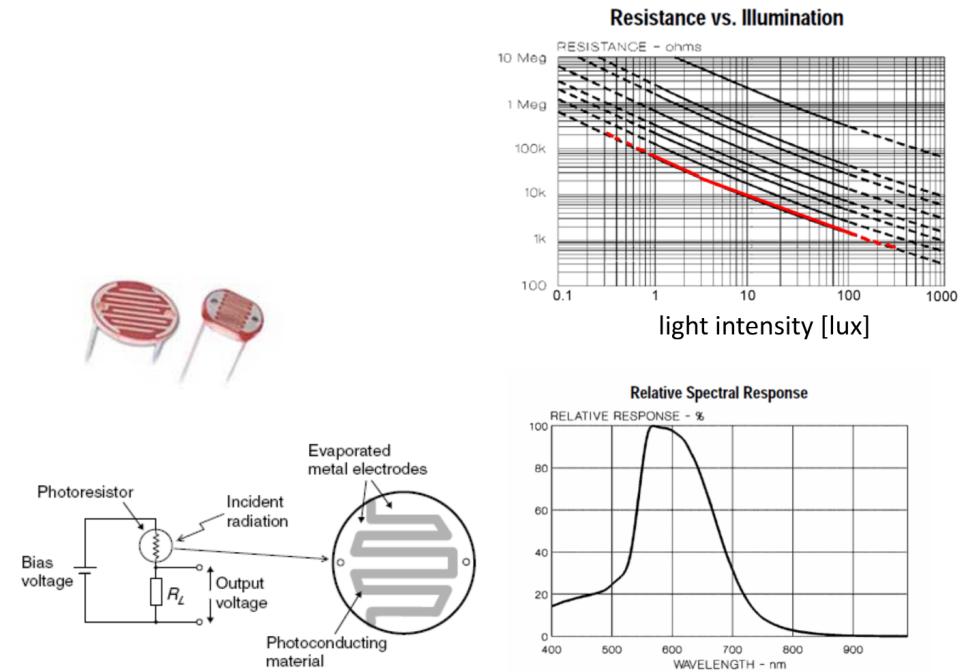


### 18.2.2 Current sensor



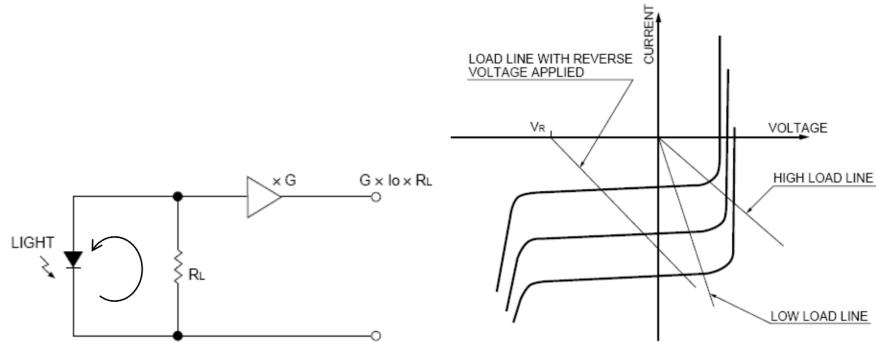
## 18.3 Light detectors

### 18.3.1 Photo-resistors

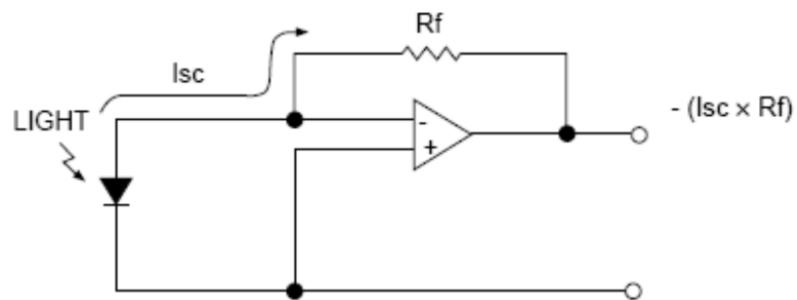


### 18.3.2 Photo-diodes

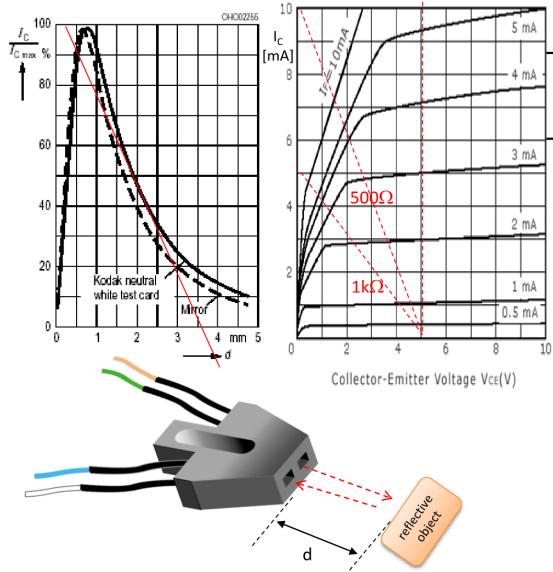
- Load resistance



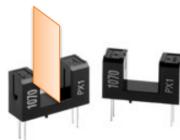
- Operational amplifier circuit



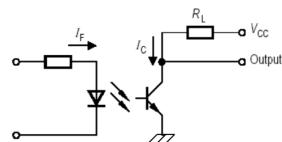
### 18.3.3 Photo-transistors



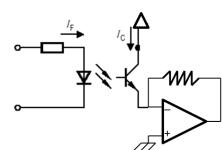
- Transmissive type:
  - Photo-interrupter



- Reflective type:
  - Resistive load

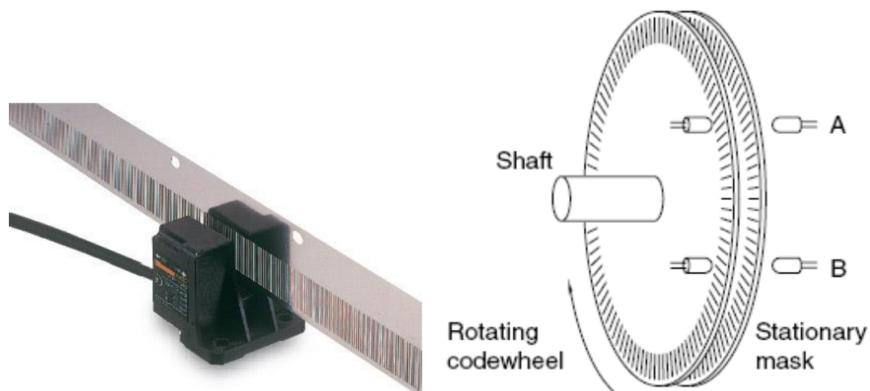


- Current-voltage operational amplifier



## 19 Digital encoders

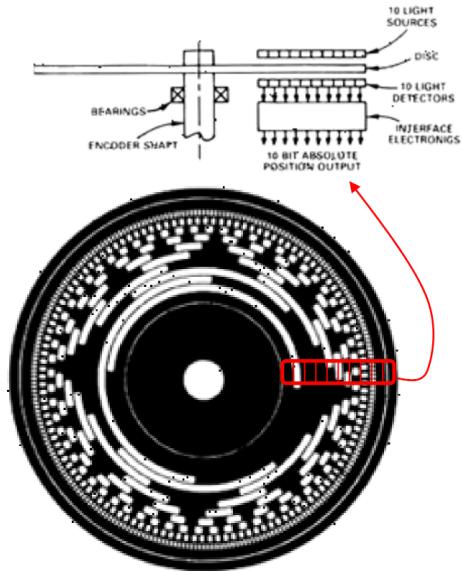
- Digital encoders convert either linear or rotary motion into a sequence of digital pulses.
- They are made using optical transmitter and receiver pairs, with a glass or plastic material photographically patterned.



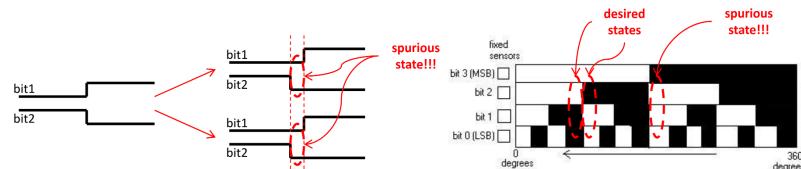
- Alternatively, they can also make use of hall effect sensors that are coupled with magnetic rings and bars.



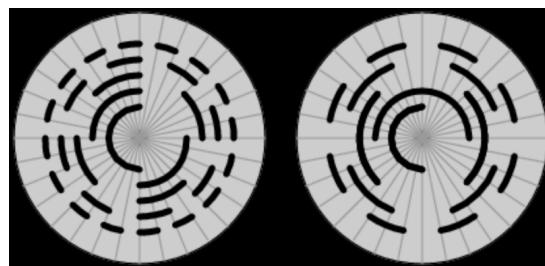
## 19.1 Absolute encoders



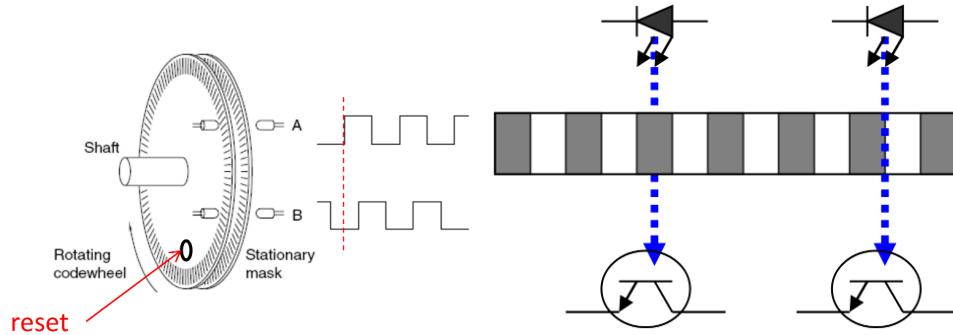
- There are  $n$  TX and RX pairs for coding  $2^n$  sectors.
- Angular n-bits encoders have a resolution of  $\frac{360^\circ}{2^n}$ .
- They are more expensive, as they require  $n$  TX and RX pairs.
- Spurious states may arise from contemporary transitions.



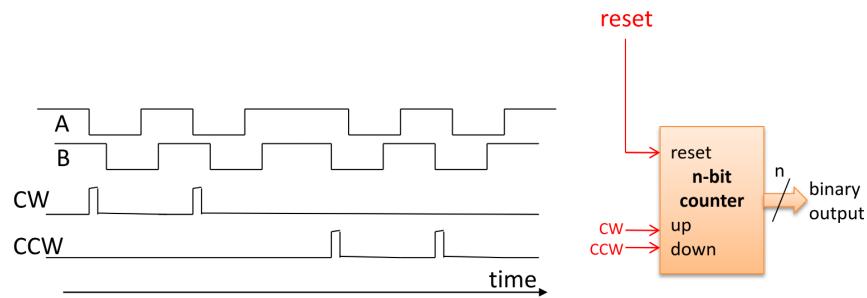
- Gray code can be used instead of natural binary code to ensure that there are no contemporary transitions.



## 19.2 Incremental encoders

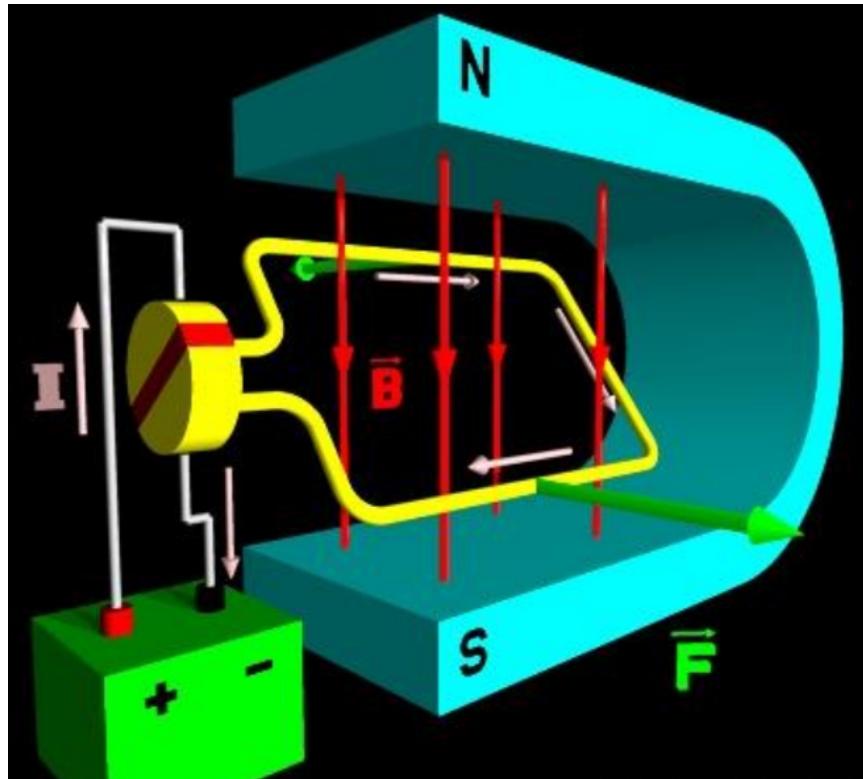


- Incremental encoders have a simpler design.
  - A single pair of TX and RX is insufficient to encode the direction.
  - Hence, 2 TX and RX pairs plus a "reset" position are required.
- The signals are quadrature, which means the signals are  $\frac{1}{4}$  cycle out-of-phase.



## 20 DC Motors

### 20.1 Structure and fields

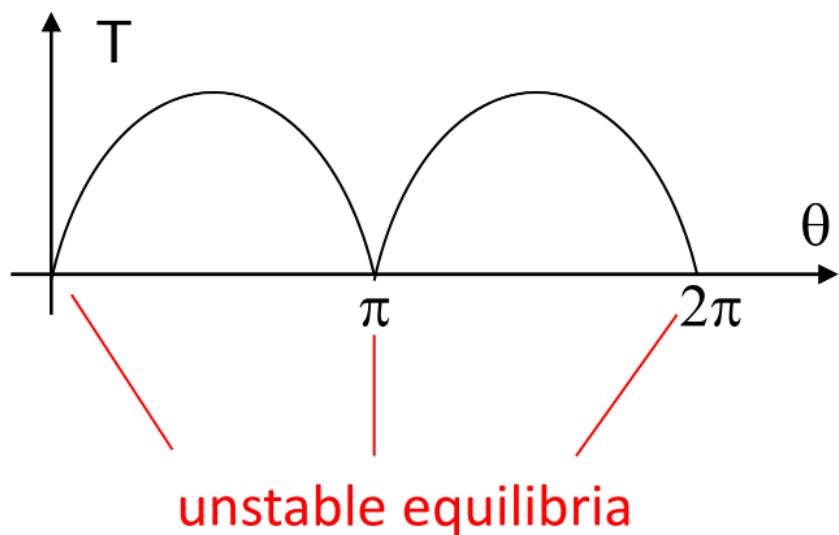
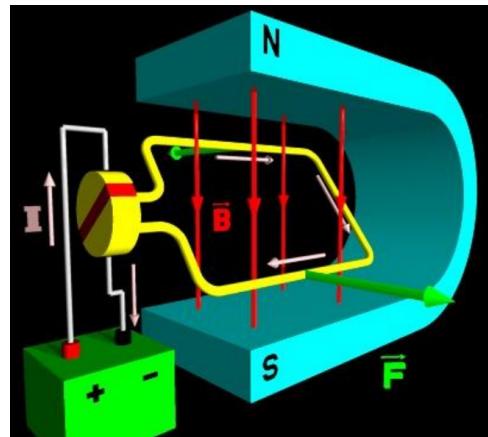


- Stator is the external magnet, which is fixed.
- Rotor is the internal magnet, which rotates.
- The stator field and the rotor fields are always orthogonal to produce the maximum torque.

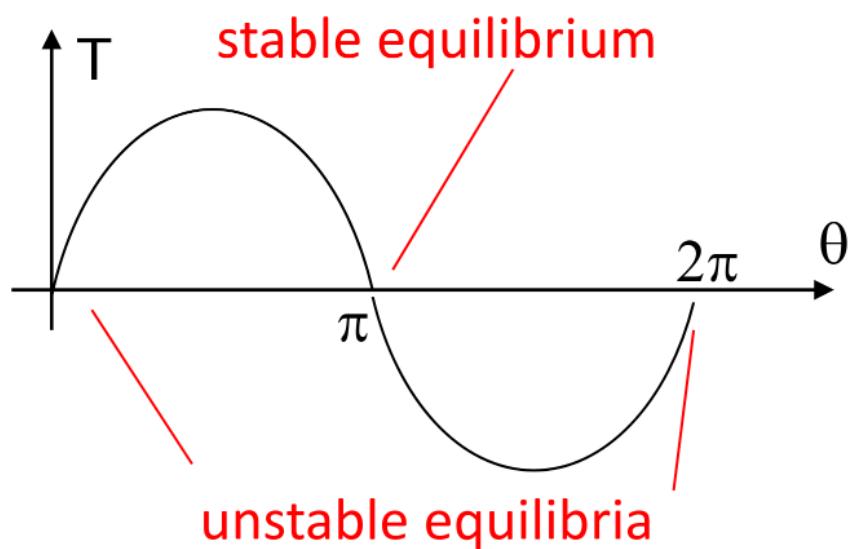
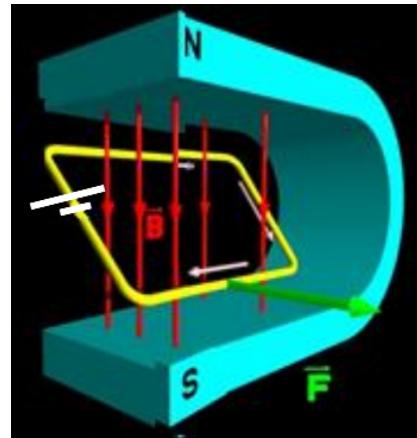
## 20.2 Commutation

When torque  $T = 0$ , the motor is in equilibrium.

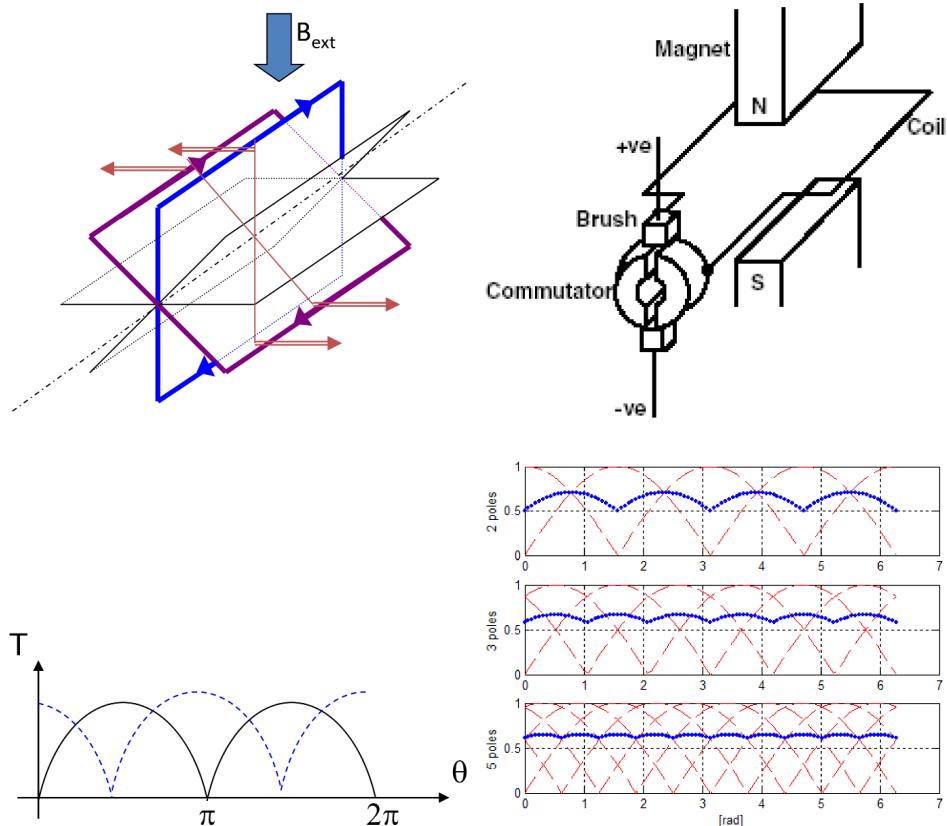
### 20.2.1 With commutation



### 20.2.2 Without commutation

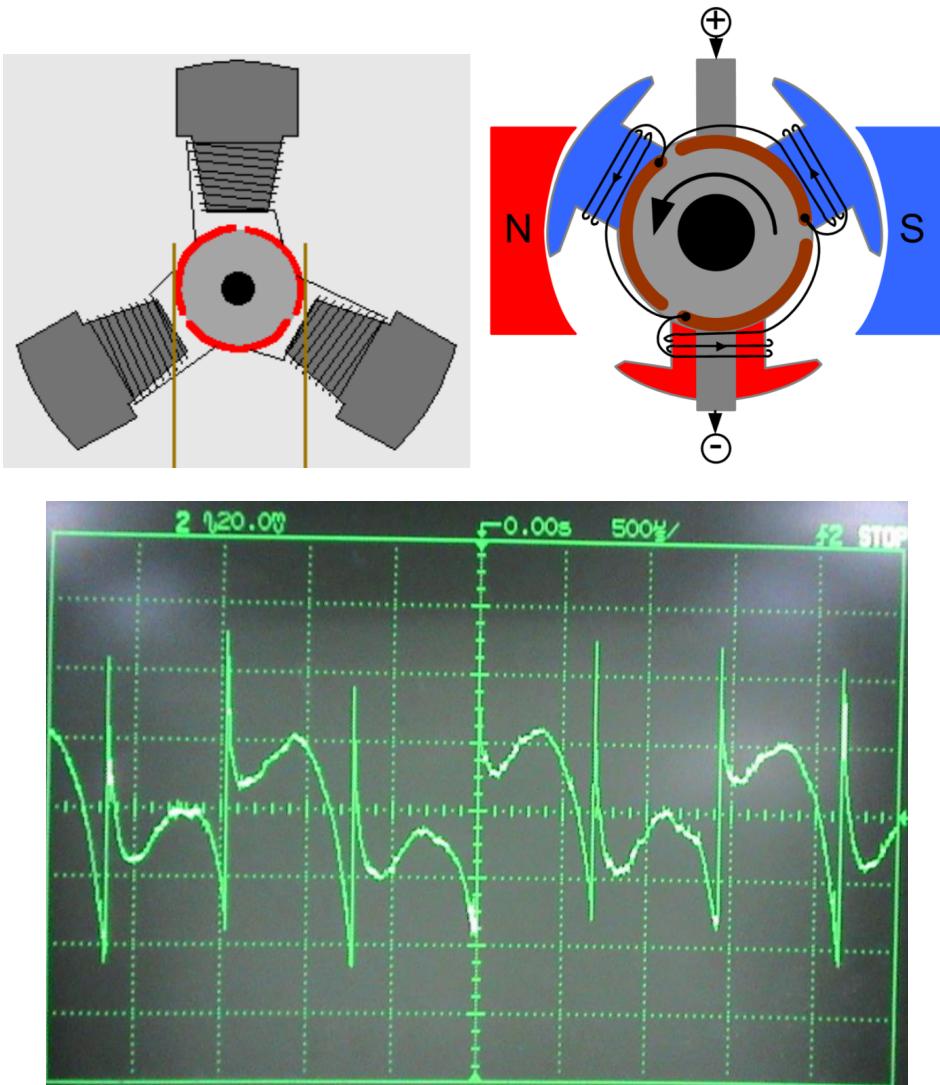


### 20.3 Generated torque

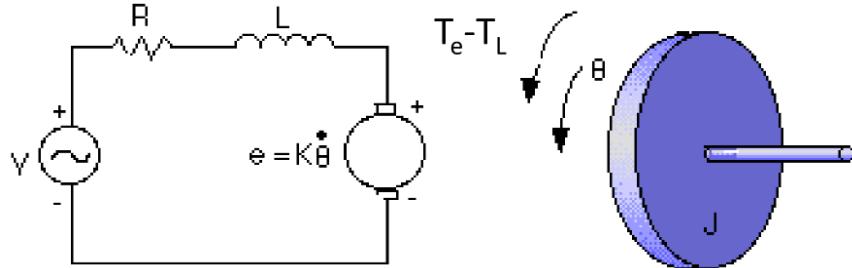


The larger the number of poles, the more constant the torque, which means the torque produced by the motor is more independent of the rotor position.

## 20.4 3-pole DC motor



## 20.5 Equations



- Armature equation

$$V = R_i + L \frac{di}{dt} + e$$

- Mechanical equation

$$J\dot{\omega} + b\omega = T_e - T_L$$

- Electro-mechanical coupling

$$T_e = K_t i$$

$$e = K_e \omega$$

$$T_e \omega = ei \quad \Leftrightarrow \quad K_e = k_t \triangleq K_a$$

Where:

- $V$  is the voltage
- $R$  is the resistance
- $L$  is the inductance of the circuit
- $e$  is the electromotive force
- $J$  is the radius of the motor
- $\omega$  is the angular velocity
- $T_e$  is the load torque
- $T_L$  is the electromagnetic torque
- $K_a$  is the armature constant

## 20.6 DC motor equivalent electrical circuit

- Let:
  - Voltage be an analogue of torque.
  - Current be an analogue of speed.
- Then:
  - Inductance is an analogue of inertia.
  - Resistance is an analogue of damping.
  - Capacitance is an analogue of compliance.
  - Electrical power  $VI$  is an analogue of mechanical power  $T\omega$ .
- Note that a **mechanical parallel** is an analogue of the **electrical series**.

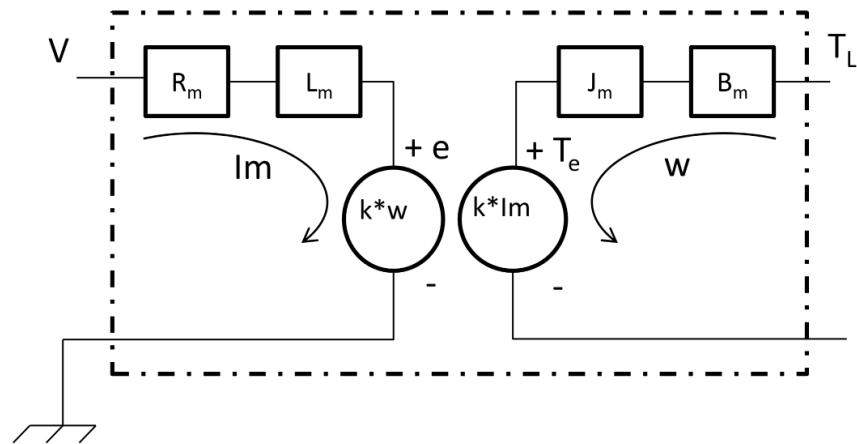


Figure 4: DC motor equivalent electromechanical model.

## 20.7 Steady state speed torque curve

$$V = Ri + L \frac{di}{dt} + K_a \omega$$

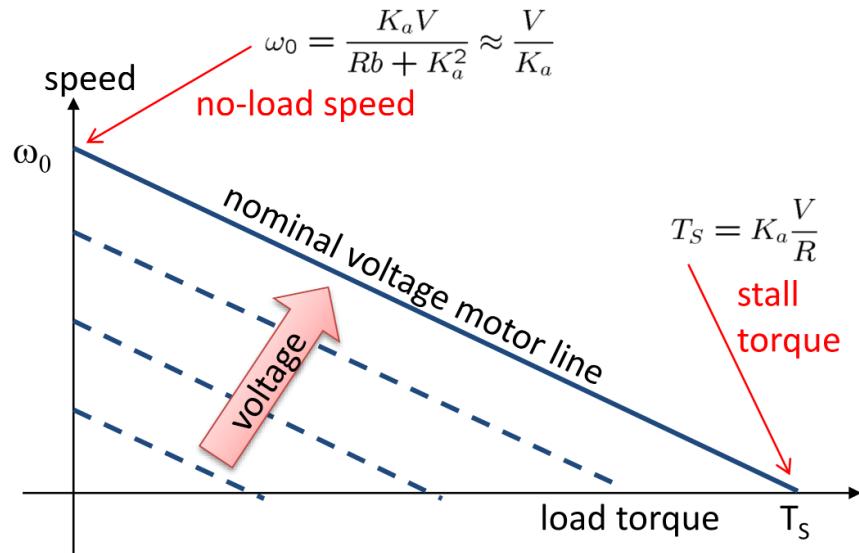
$$J \frac{d\omega}{dt} + b\omega + T_L = K_a i$$

$$V - K_a \omega = R_i$$

$$i = \frac{1}{K_a} b\omega + T_L$$

$$K_a V - K_a^2 \omega = R b \omega + R T_L$$

$$\omega = \frac{K_a V}{R T_L} R b + K_a^2$$

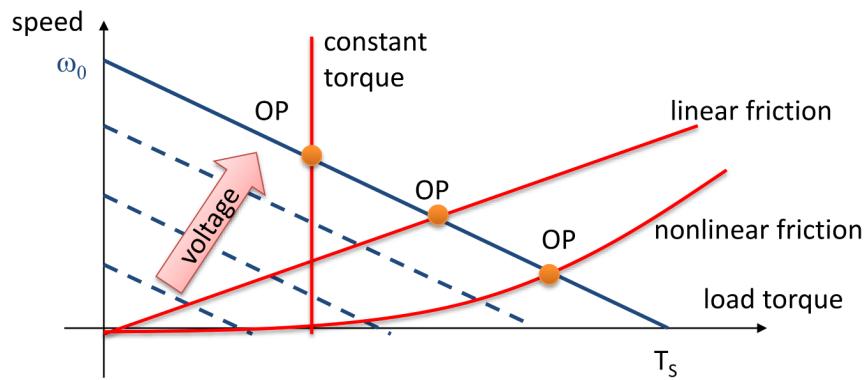


### 20.7.1 Load lines

Loads can be of various kinds, such as:

- Friction:  $T_L = b_L\omega$
- Constant torque:  $T_L = \text{const}$
- Inertial:  $T_L = I = \frac{d\omega}{dt} = 0 @ \text{Steady state}$
- Nonlinear:  $T_L = f(\omega)$

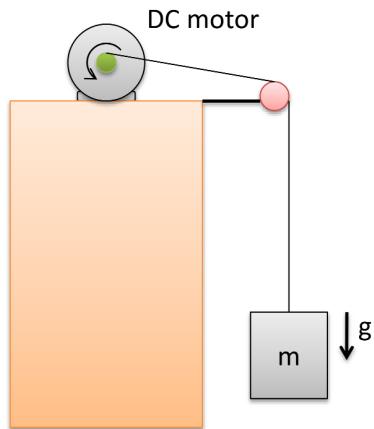
OP is the operating point in the diagram below.



## 20.8 Common types of load

1. Constant torque:

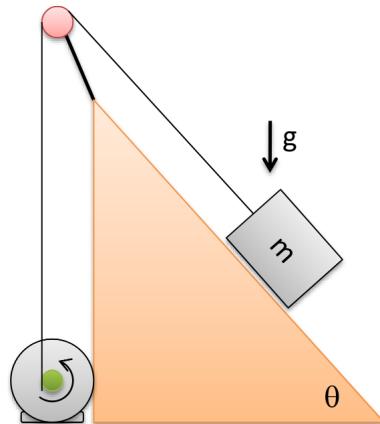
$$T_L^{(1)} = mgr$$



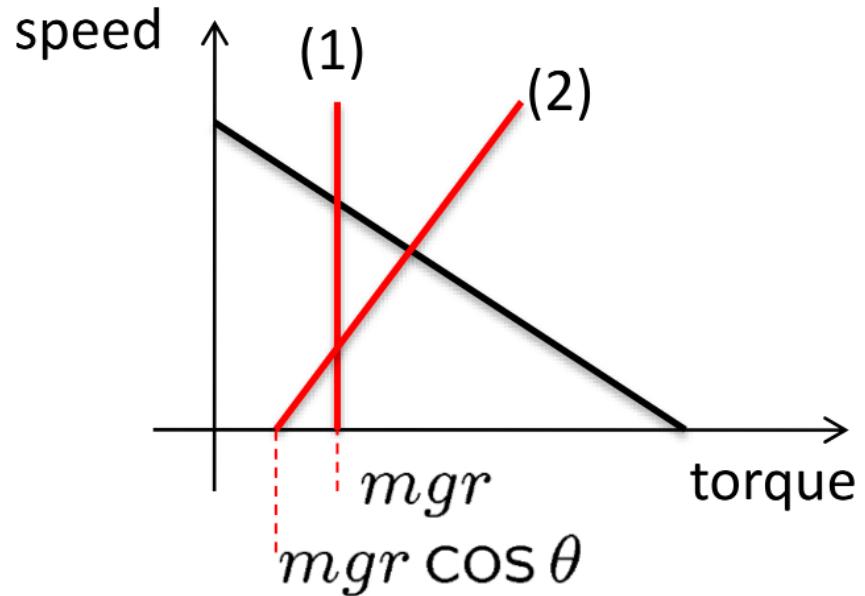
2. Constant torque with friction:

$$T_L^{(2)} = mgr \cos \theta + b_L v$$

$$T_L^{(2)} = mgr \cos \theta + b_L \omega r$$



Graph of the common types of loads:



## 20.9 Maximum output power

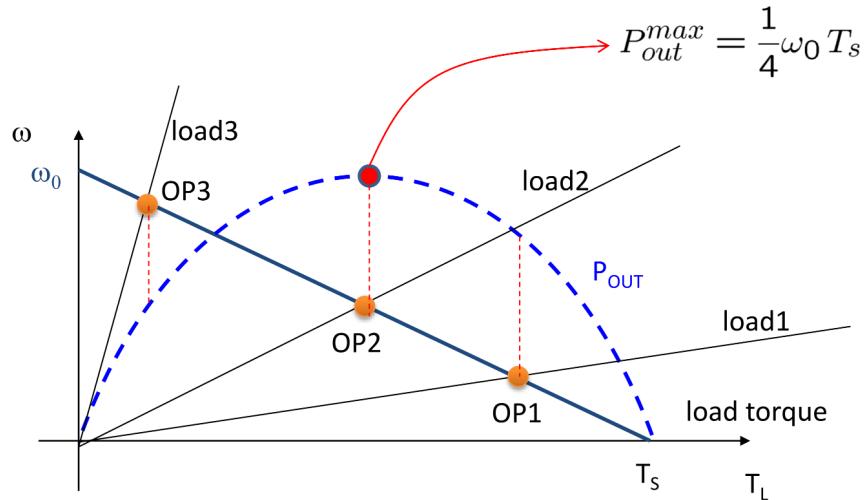
At nominal voltage  $V$ :

$$P_{out} = \omega T_L$$

The power output is the **area** in the speed-torque graph. Using a friction load:

$$T_L = b_L \omega_L$$

OP is the operating point in the diagram below.



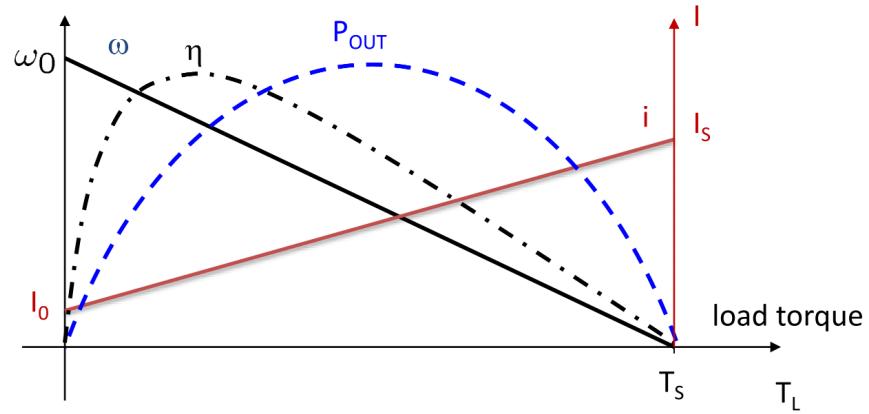
Hence, the maximum output power is:

$$P_{out}^{max} = \frac{1}{4} \omega_0 T_s$$

## 20.10 Maximum efficiency

At nominal voltage  $V$ :

$$\eta = \frac{\text{power in}}{\text{power out}} = \frac{\omega T_L}{VI}$$



## 20.11 DC motor load matching

- Figure out the mechanical load of the motor. For a friction load:

$$T_L = b_L \omega_L$$

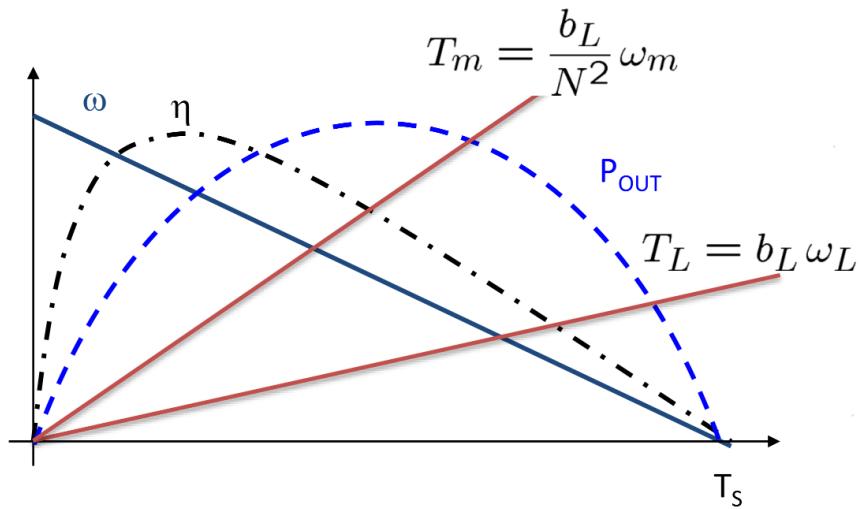
- Figure out the gear ratio  $N$ :

$$T_L = N T_m$$

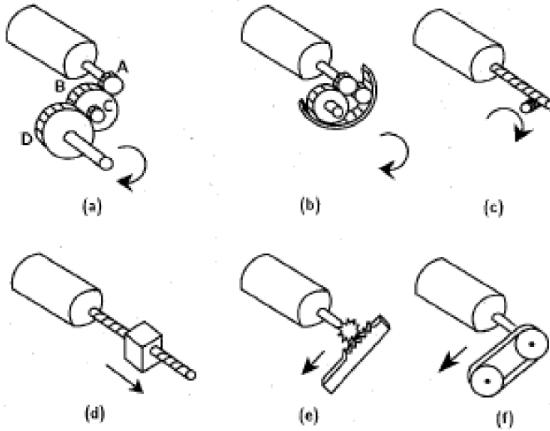
$$\omega_m = N \omega_L$$

- Use the torque and angular speed to turn it into voltage ( $V$ ) and current ( $I$ ).

You can either design for maximum power or maximum efficiency.



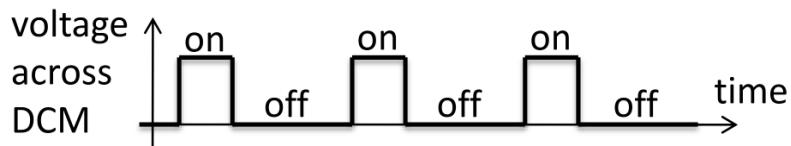
### 20.11.1 Types of gears



- (a) Spur gears mesh pairs of gears with different numbers of teeth to achieve speed reduction.
- (b) Planetary gears have several gears meshed in an outer ring for a large speed reduction.
- (c) Worm gears produce rotary motion at right angles to the shaft.
- (d) A lead screw and nut can create linear motion, as can (e) a rack-and-pinion system, and (f) belt-and-pulley-drives.

### 20.12 Driving DC motors

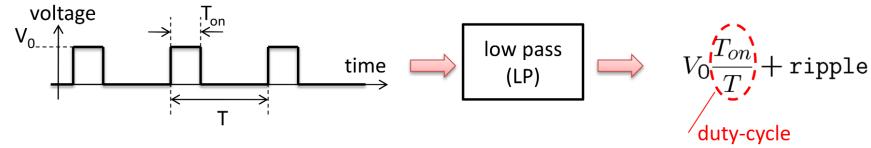
- Using power amplifiers to drive DC motors is possible but is typically avoided.
- This is due to **large power dissipation** and **over-heating of the amplifier**.
- It is preferable to continuously switch the motor on and off using pulse width modulation (PWM).



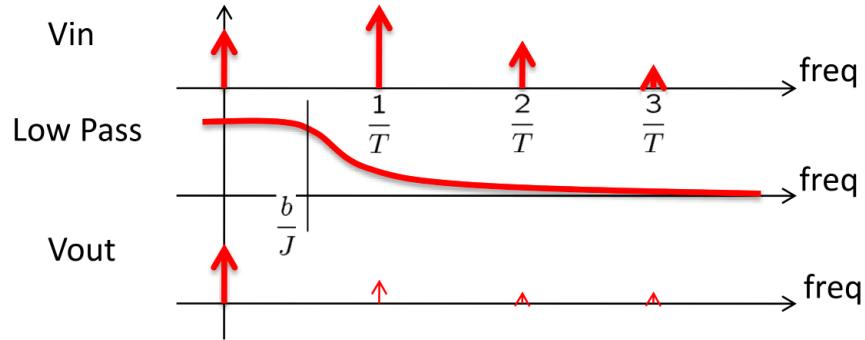
### 20.13 Pulse width modulation operating principle

The DC motor is in fact a 2nd order low pass filter.

$$\omega = \frac{K_a V - (R + L_s)T_L}{(L_s + R)(J_s + b) + K_a^2}$$



Frequency analysis:



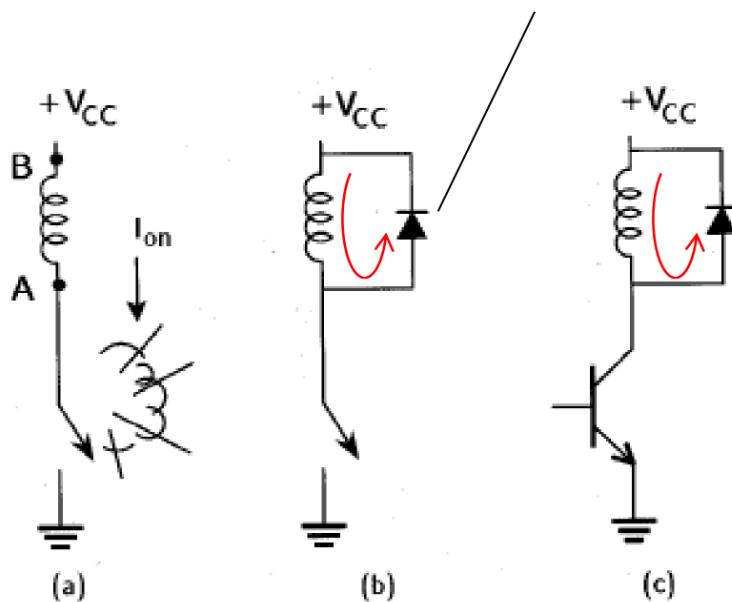
## 20.14 Inductive kickback

- There is voltage across the inductor:

$$V = L \frac{di}{dt}$$

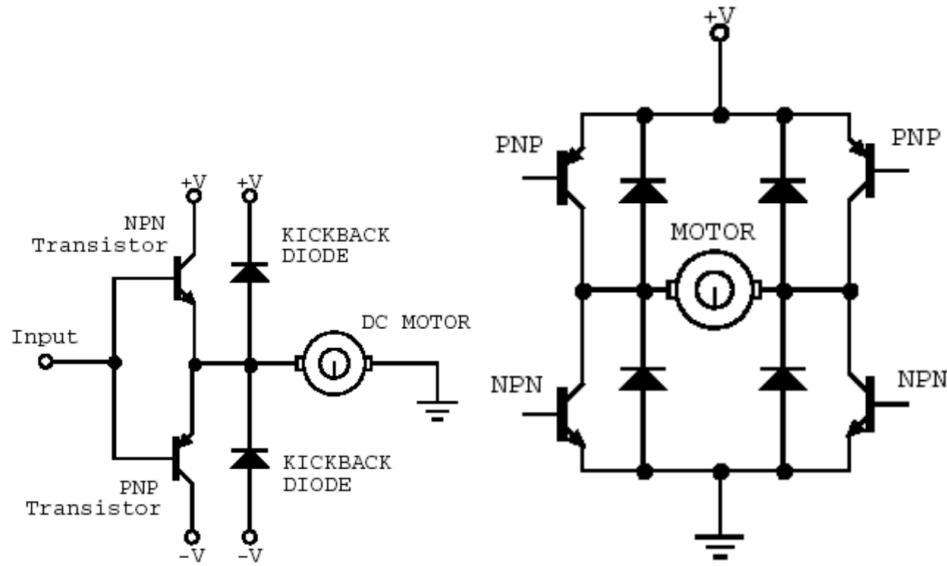
- If the current starts decreasing, the voltage  $v = v_B - v_A$  quickly decreases.
- Hence, voltage  $v_A$  quickly increases.

flyback / kickback  
diode

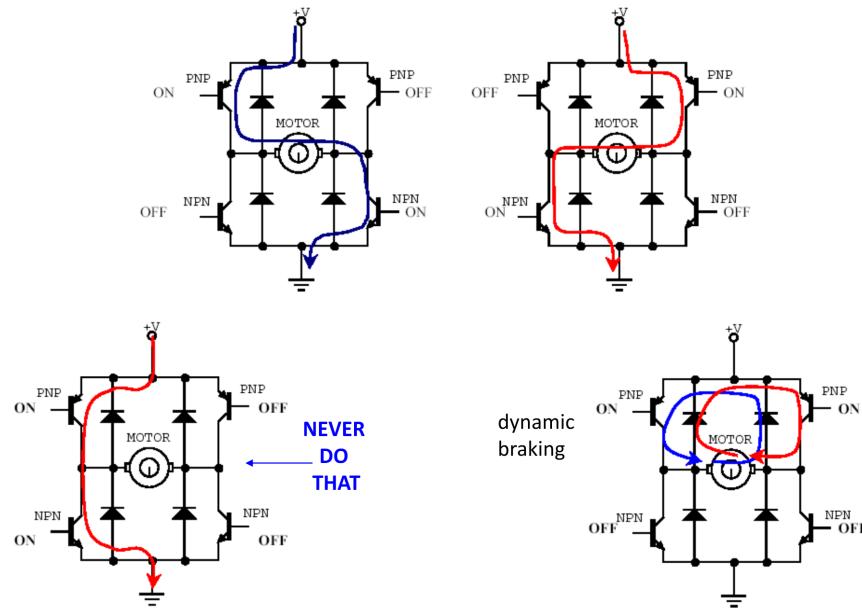


- (a) The steady-state current through an inductor  $I_{on}$ , cannot immediately go to 0 at A when the switch is opened. The changing current induces a voltage across the inductor, making the potential at A greater than at B, causing the switch or relay to arc over.
- (b) Flyback diodes protect switches from blowing up.
- (c) Transistor switches must be protected in the same manner.

### 20.15 Pulse width modulation circuit

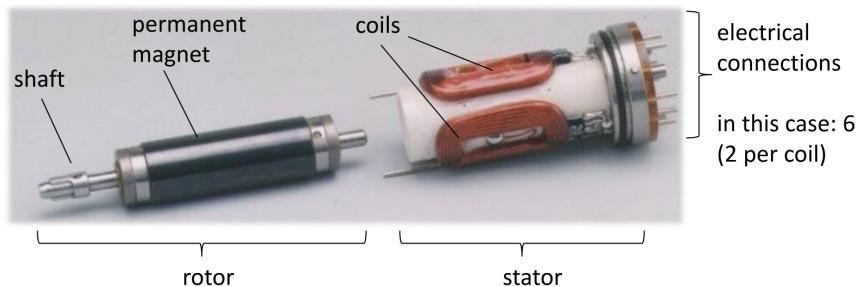


For the diagrams above, the diagram on the left is a circuit for bidirectional current flow through a DC motor, and the diagram on the right is a typical H-bridge circuit, including kickback diodes to protect against inductive kickback.



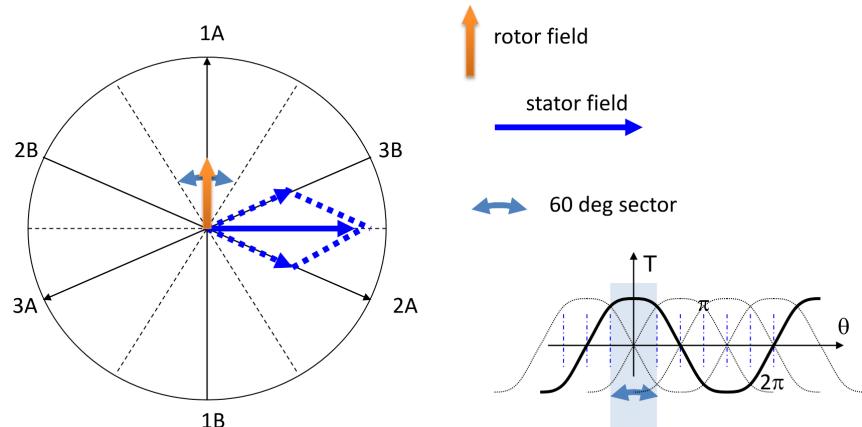
## 20.16 Brushless DC motors

- A brushless DC motor is similar to a brushed DC motor, but has its insides out.
- The stator field is rotating instead of remaining static.
- The rotor field is given by a permanent magnet and stays in place.



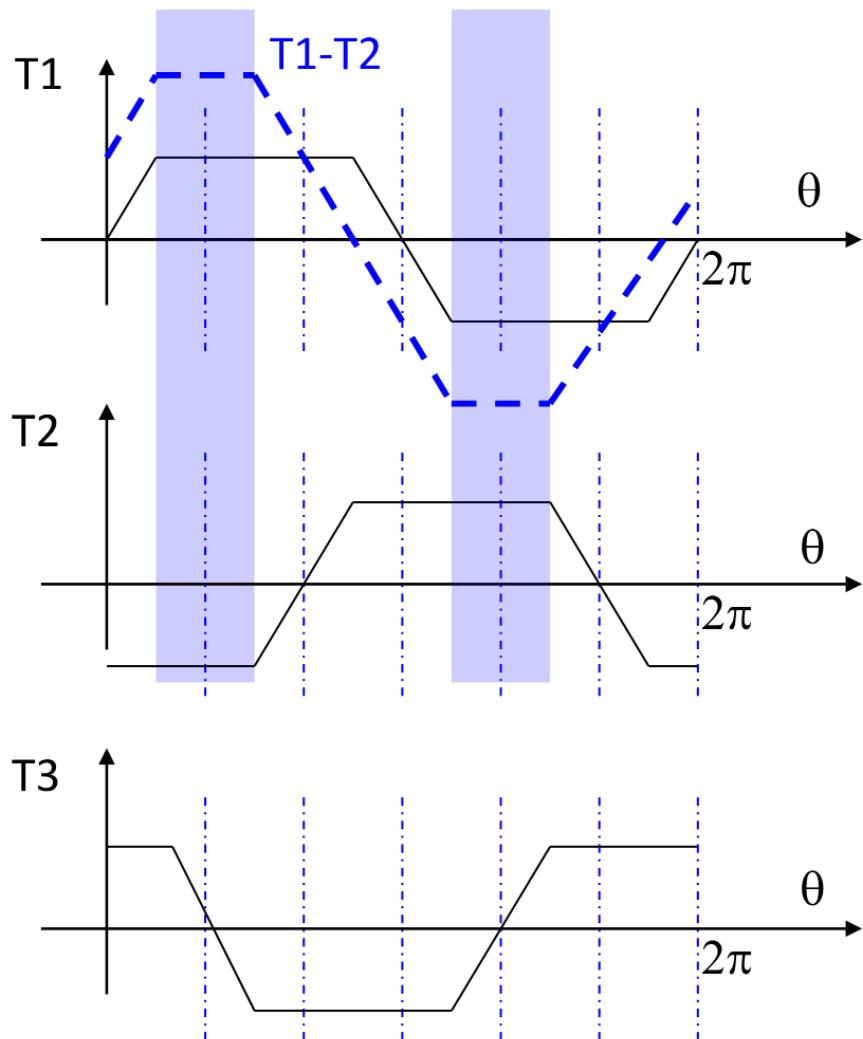
### 20.16.1 Driving strategy

To maximise the output torque, try to keep the rotor and the stator field orthogonal as much as possible.



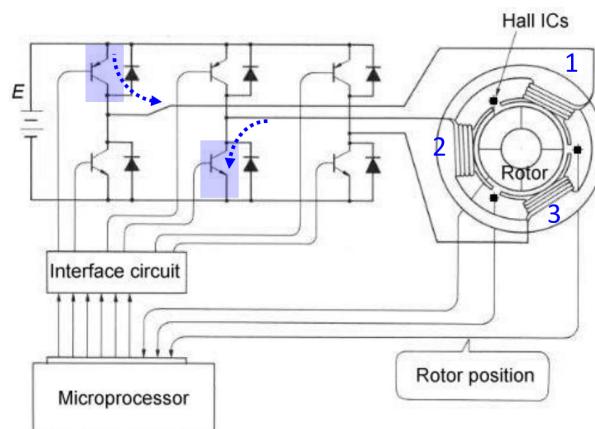
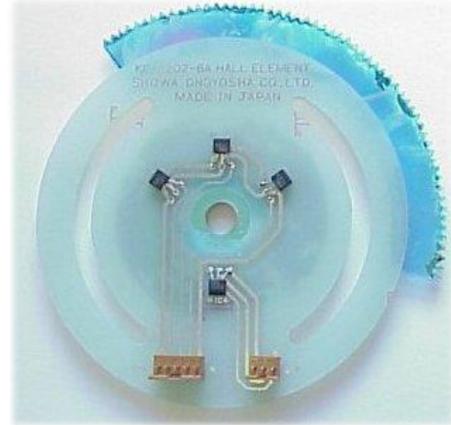
### 20.16.2 Trapezoidal brushless DC motors

- $T_j$  is the torque due to the  $j$ -th coils.
- These motors always drive two coils at a time with opposite currents  $+I$  and  $-I$ .
- The total torque  $T_i - T_j$  always has a constant zone.
  - Use torque  $T_1 - T_2$  when  $\theta = \frac{\pi}{3} \pm \frac{\pi}{6}$ .
  - You will need to know the **angular position of the rotor**.



### 20.16.3 Getting a desired torque

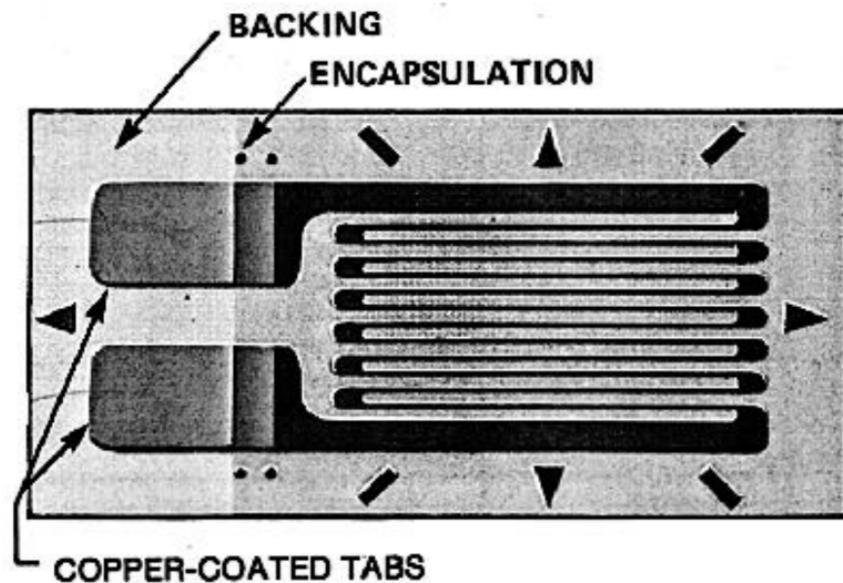
- Detect the rotor position via encoders, typically Hall effect sensors.
- Select the appropriate switches to determine the desired  $T_i - T_j$  torque.



## 21 Strain gauges

### 21.1 Stress or strain measurements

- Strain measurements are important to determine safe loading conditions of mechanical structures.
- The stress or force measurements are typically derived indirectly from strain and displacement measurements.
- Electrical resistance strain gauges have the following characteristics:
  - Thin metal foil, typically constantan.
  - The thin metal foil is patterned onto plastic backing material.
  - This backing is then bonded onto mechanical structures, and the stress is inferred from solid mechanics principles.

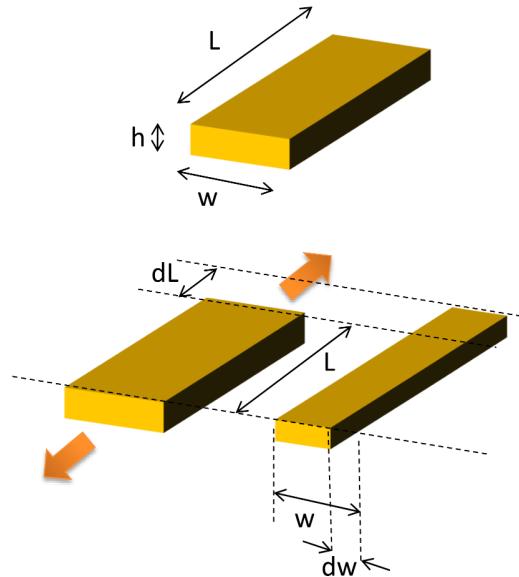


## 21.2 Strain

- Strain quantifies the amount of deformation of a body, and it is given by:

$$S = \frac{dL}{L}$$

- It is non-dimensional, as it is defined as a relative change ( $\frac{dL}{L}$ ).
- Typical material undergo "micro strains" from  $10^{-6}$  (ppm) up to a few %.
- Strain can either be positive (tensile strain) or negative (compressive strain).



### 21.2.1 Poisson's ratio ( $\nu$ )

Poisson's ratio is typically from 0.3 (steel) to 0.5 (rubber). It is defined as follows:

$$\nu = \frac{\text{lateral strain}}{\text{axial strain}}$$

### 21.2.2 Resistance of a rectangular conductor

$$r = \rho \frac{L}{A}, \quad A = wh$$

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{dL}{L} - \frac{dA}{A}$$

$$\frac{dR}{R} = \frac{d\rho}{\rho} + \frac{dL}{L} - \left( \frac{dw}{w} + \frac{dh}{h} \right)$$

- Axial strain:

$$S = \frac{dL}{L}$$

- Lateral strain:

$$\frac{dw}{w} = \frac{dh}{h} = -\nu \frac{dL}{L} = -\nu S$$

$$\begin{aligned} \frac{dR}{R} &= \frac{d\rho}{\rho} + \frac{dL}{L} - \left( \frac{dw}{w} + \frac{dh}{h} \right) \\ &= \frac{d\rho}{\rho} + (1 + 2\nu)S \\ &= \left( \underbrace{\frac{d\rho}{\rho} \frac{1}{S}}_{\text{piezoresistivity}} + 1 + 2\nu \right) S \\ &= \mathcal{G}S \end{aligned}$$

### 21.2.3 Gauge factor ( $\mathcal{G}$ ):

$$\mathcal{G} = \frac{dR}{R} \frac{1}{S} = \frac{1}{R} \frac{\partial R}{\partial S} = \frac{d\rho}{\rho} \frac{1}{S} + 1 + 2\nu$$

$$dR = dR^S = \frac{\partial R}{\partial S} S$$

Note that we are only considering changes of resistance due to strain  $dR = dR^S$ .

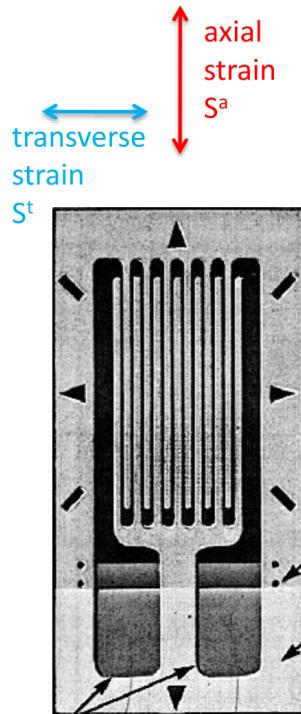
Gauge factor of various materials:

Material	Gauge factor
Nickel	-12.6
Manganese	+0.07
Nichrome	+2.0
Constantan	+2.1
Soft Iron	+4.2
Carbon	+20
Platinum	+4.8

### 21.3 Transverse sensitivity

$$dR = R_0 \mathcal{G} S$$

- The larger  $R_0$  is, the larger  $dR$  is.
- Long and thin wires allow for larger  $R_0$ , given that the wires are aligned with axial strain ( $S^a$ ).
- Practically, long wires are assembled in the form of a serpentine.
- The end-loops of these wires are aligned with the transverse axis, and are made thicker to reduce sensitivity to transverse strain ( $S^t$ ).



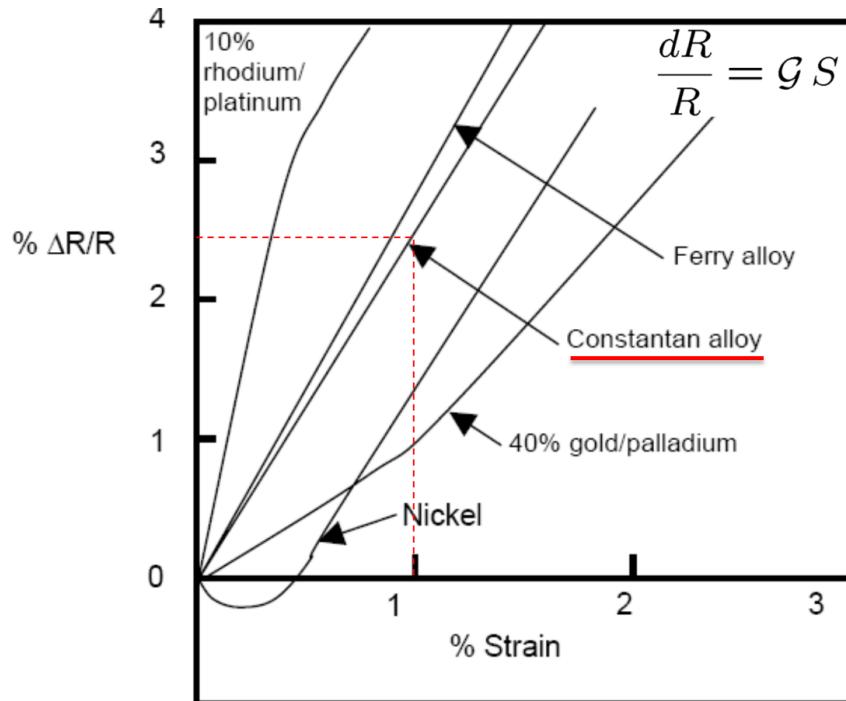
## 21.4 Materials

- The best materials to use constantan and ferry alloys.
- Typical strain ranges are:

$$S = 1 - 10^4 \mu\text{S}$$

Which is from 1 ppm to 1%.

- $\mathcal{G}$  is roughly 2.
  - $\frac{dR}{R}$  is in the same order of magnitude as  $S$ .
  - However, there is a challenge in detecting small resistance changes.



#### 21.4.1 Numerical example

$$dR = R = \mathcal{G}S$$

- $\mathcal{G}$  is roughly 2
- $R_0$  is roughly  $100 - 1,000 \Omega$
- Strain is in the order of  $10 - 10^4 \mu\text{S}$  (micro strain)
- Strain is non-dimensional, and  $1 \mu\text{S} = 10^{-6}$  (e.g.  $1 \mu\text{m m}^{-1}$ )

$$dR = (100 \Omega) \times 2 \times (100 \mu\text{S} = 0.02 \Omega)$$

- Transverse sensitivity is in the order of 1%.
- How do we sense such small changes?

## 21.5 Wheatstone bridge

- Bridge equations:

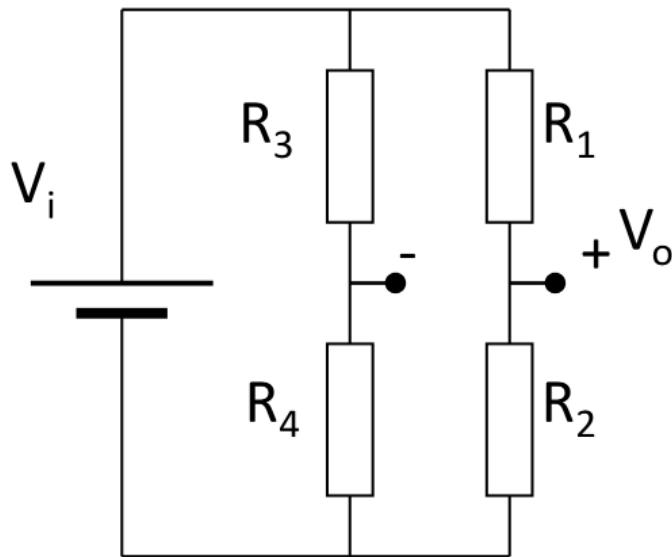
$$\frac{V^+}{V_i} = \frac{R_2}{R_1 + R_2}$$

$$\frac{V^-}{V_i} = \frac{R_4}{R_3 + R_4}$$

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} - \frac{R_4}{R_3 + R_4}$$

- Bridge balance condition:

$$V_o = 0 \quad \Leftrightarrow \quad R_1 R_4 = R_2 R_3$$

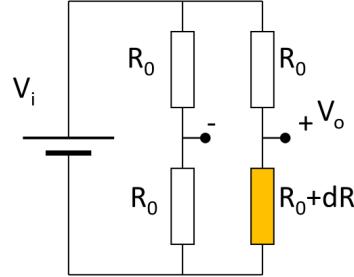


### 21.5.1 1st order approximation

- When  $R_1 = R_2$  and  $R_3 = R_4$ , it implies balance.
- The first order approximation is acceptable up to a few percent S. Note that  $1\% S = 10^4 \mu S$ .

$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_2}{R_2} - \frac{dR_1}{R_1} + \frac{dR_3}{R_3} - \frac{dR_4}{R_4} \right)$$

### 21.5.2 Quarter bridge



Consider:

$$R_1 = R_3 = R_4 = R_0$$

$$R_G = R_0 + dR$$

The bridge output is:

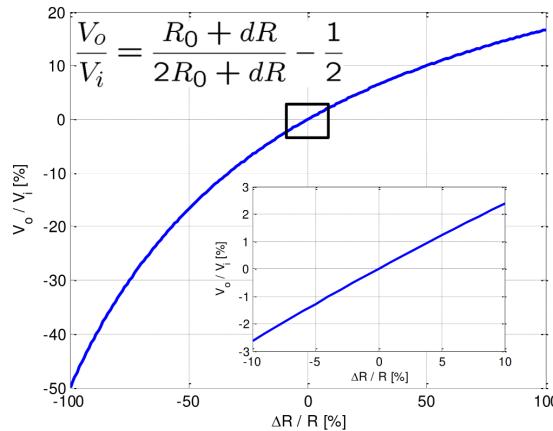
$$\frac{dV_o}{V_i} = \frac{R_0 + dR}{2R_0 + dR} - \frac{1}{2}$$

The Taylor expansion:

$$\frac{dV_o}{V_i} = \frac{1}{4} \frac{dR}{R_0} - \frac{1}{4} \frac{dR^2}{R_0^2} + \dots$$

Hence:

$$\frac{dV_o}{V_i} \simeq \frac{1}{4} \frac{dR}{4R_0} = \frac{1}{4} \mathcal{G}S$$



### 21.5.3 Quarter bridge connection types

- 2-wire connection

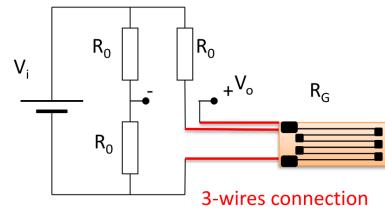
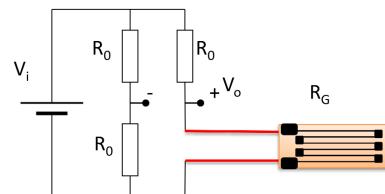
- $R_w$  is the long wires' resistance, which has the following characteristics:
  - \* It is as high as a few ohms.
  - \* The resistance is temperature dependent.
  - \* There is unbalancing effects.

- 3-wire connection

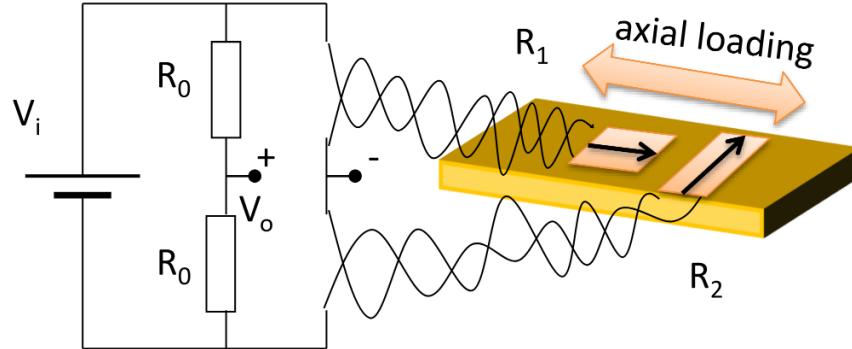
- No current in the 3rd wire.
- The bridge is balanced.
- There is an attenuated gauge factor.

$$\frac{V_o}{V_i} = \frac{1}{4} \frac{dR}{R_0 + R_w} = \frac{1}{4} \frac{R_0}{R_0 + R_w} \frac{dR}{R_0} = \frac{1}{4} \mathcal{G}^* S$$

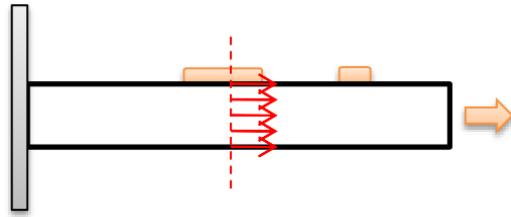
$$\mathcal{G}^* = \mathcal{G} \frac{R_0}{R_0 + R_w} \leq \mathcal{G}$$



## 21.6 Temperature compensation



### 21.6.1 Dummy gauges



- Dummy gauges are mounted in close thermal contact but **not bonded**.
- Strain gauge:  $R_1 = R_0 + dR_1$
- Dummy gauge:  $R_2 = R_0 + dR_2$

$$dR_1 = \frac{\partial R_1}{\partial S} S + \frac{\partial R_1}{\partial T} dT$$

$$dR_2 = \frac{\partial R_2}{\partial T} dT$$

Because the strain gauge and dummy gauge are technologically similar gauges in thermal contact:

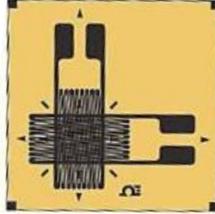
$$\frac{\partial R_1}{\partial T} dT = \frac{\partial R_2}{\partial T} dT$$

Hence:

$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_1}{R_0} - \frac{dR_2}{R_0} \right) = \frac{1}{4R_0} \left( \frac{\partial R_1}{\partial S} S + \frac{\partial R_1}{\partial T} dT - \frac{\partial R_2}{\partial T} dT \right)$$

$$\frac{dV_o}{V_i} = \frac{1}{4R_0} \left( \frac{\partial R_1}{\partial S} S \right) = \frac{1}{4} \mathcal{G} S$$

### 21.6.2 Poisson gauges



- Poisson gauges are mounted in close thermal contact and bonded.
- Strain gauge:  $R_1 = R_0 + dR_1$
- Poisson gauge:  $R_2 = R_0 + dR_2$

$$dR_1 = \frac{\partial R_1}{\partial S} S + \frac{\partial R_1}{\partial T} dT$$

$$dR_2 = -\nu \frac{\partial R_2}{\partial S} dS + \frac{\partial R_2}{\partial T} dT$$

Because the strain gauge are Poisson gauge are technologically similar gauges bonded together in thermal contact:

$$\frac{\partial R_1}{\partial T} dT = \frac{\partial R_2}{\partial T} dT$$

$$\frac{\partial R_1}{\partial S} = \frac{\partial R_2}{\partial S}$$

Hence:

$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_1}{R_0} - \frac{dR_2}{R_0} \right) = \frac{1}{4R_0} \left( \frac{\partial R_1}{\partial S} S + \frac{\partial R_1}{\partial T} dT + \nu \frac{\partial R_2}{\partial S} S - \frac{\partial R_2}{\partial T} dT \right)$$

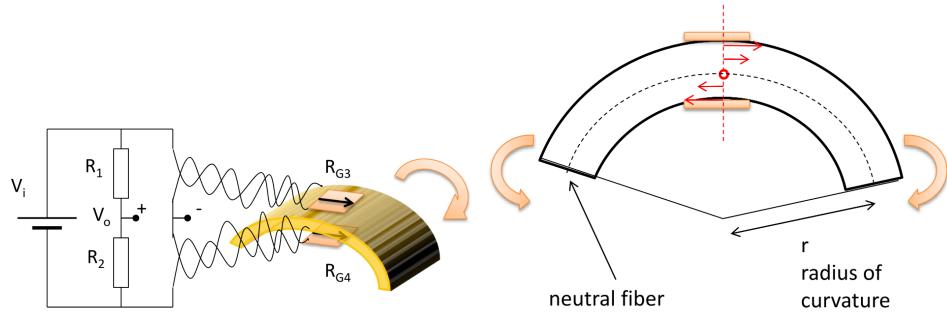
$$\frac{dV_o}{V_i} = \frac{1}{4R_0} \left( \frac{\partial R_1}{\partial S} (1 + \nu) S \right) = \frac{1}{4} G (1 + \nu) S$$

### 21.7 Half bridge

Half bridges have two active strain gauges which enhances the sensitivity of the bridge.

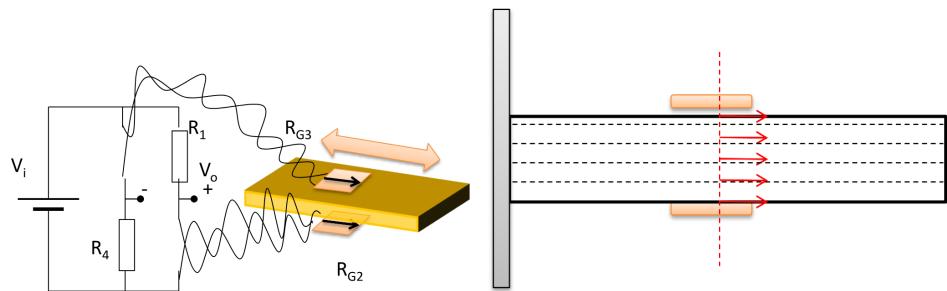
$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_2}{R_2} - \frac{dR_1}{R_1} + \frac{dR_3}{R_3} - \frac{dR_4}{R_4} \right)$$

### 21.7.1 Bending



$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_3}{R_3} - \frac{dR_4}{R_4} \right) = \frac{1}{2} G S^b$$

### 21.7.2 Axial loading



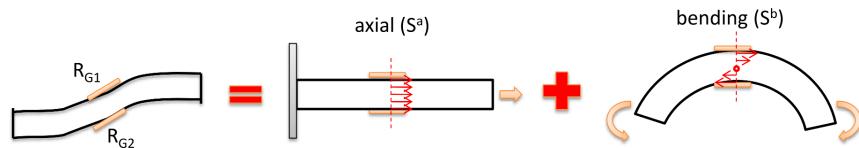
$$\frac{dV_o}{V_i} = \frac{1}{4} \left( \frac{dR_2}{R_2} + \frac{dR_3}{R_3} \right) = \frac{1}{2} G S^a$$

## 21.8 Apparent strain

- Apparent strain is manifested as any change in gauge resistance which is not due to the strain being measured.
- For example, combinations of different mechanical loading, like:

$$S_1 = S^a + S^b$$

$$S_2 = S^a - S^b$$



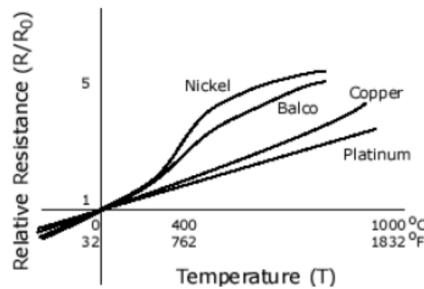
### 21.8.1 Thermal effects

- Resistance changes might be due to a combination of strain and temperature.

$$dR = \frac{\partial R}{\partial S} S + \frac{\partial R}{\partial T} dT = \frac{\partial R}{\partial S} \left( S + \underbrace{\left( \frac{\partial R}{\partial S} \right)^{-1} \frac{\partial R}{\partial T} dT}_{\text{apparent strain } S^T} \right)$$

- Apparent strain due to temperature.  $dR^T$  is the resistance change solely due to temperature in the equation below:

$$S^T = \left( \frac{\partial R}{\partial S} \right)^{-1} \frac{\partial R}{\partial T} dT = \frac{1}{G} \frac{1}{R} \frac{\partial R}{\partial T} dT = \frac{1}{G} \frac{dR^T}{R}$$

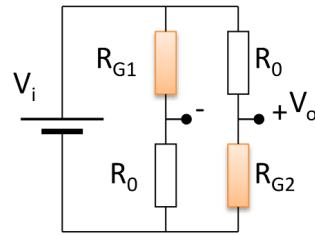
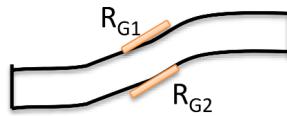


### 21.8.2 Half bridge example

Ultimately, sensitivity to loading condition and temperature is determined by the electrical configuration.

1. Configuration 1:

$$S_{G1} = S^a + S^b + S^T$$



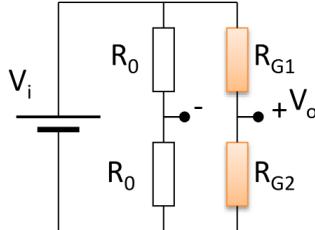
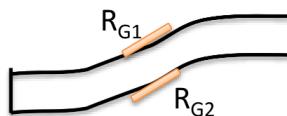
$$S_{G2} = S^a - S^b + S^T$$

$$\frac{dV_o}{V_i} = \frac{1}{4}\mathcal{G}(S_{G1} + S_{G2}) = \frac{1}{2}\mathcal{G}(S^a + S^T)$$

- The configuration is sensitive to axial strain.
- It can compensate for bending but not temperature.

2. Configuration 2:

$$S_{G1} = S^a + S^b + S^T$$



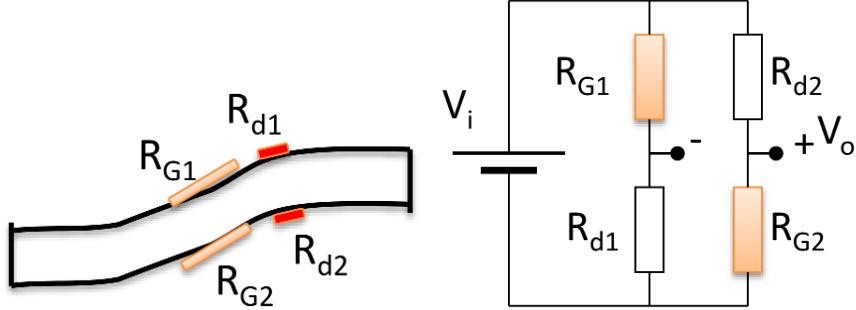
$$S_{G2} = S^a - S^b + S^T$$

$$\frac{dV_o}{V_i} = \frac{1}{4}\mathcal{G}(-S_{G1} + S_{G2}) = \frac{1}{2}\mathcal{G}S^b$$

- The configuration is sensitive to bending strain.
- It can compensate for axial strain and temperature.

### 21.8.3 Full bridge example

1. Configuration 1:



$$S_{G1} = S^a + S^b + S^T$$

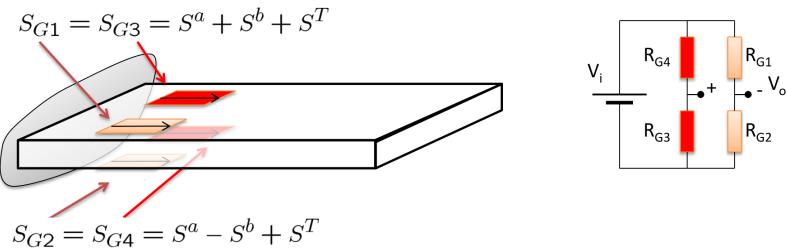
$$S_{G2} = S^a - S^b + S^T$$

$$S_{d1} = S^{d2} = S^T$$

$$\frac{dV_o}{V_i} = \frac{1}{4}\mathcal{G}(S_{G1} - S_{d1} + S_{G2} - S_{d2}) = \frac{1}{2}GS^a$$

- The configuration above can sense axial strain.
- It also can compensate for temperature and bending.

2. Configuration 2:



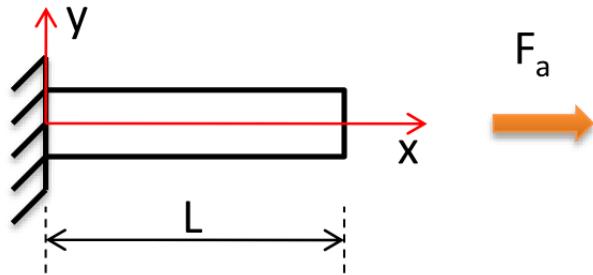
$$\frac{dV_o}{V_i} = \frac{1}{4}\mathcal{G}(S_{G1} - S_{G2} + S_{G3} - S_{G4}) = GS^b$$

- The configuration above can sense bending strain, with maximum bridge sensitivity.
- It also can compensate for temperature and axial strain.

#### 21.8.4 Cantilever beams example

Cantilever beams have longitudinal strain due to:

- Axial loading ( $F_a$ ):

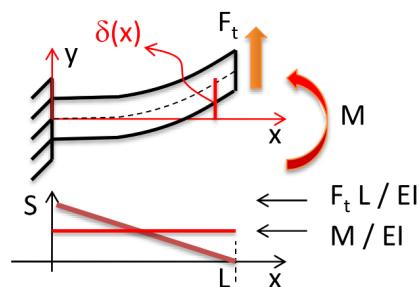


$$S_a = \frac{F_a}{EA}$$

Where:

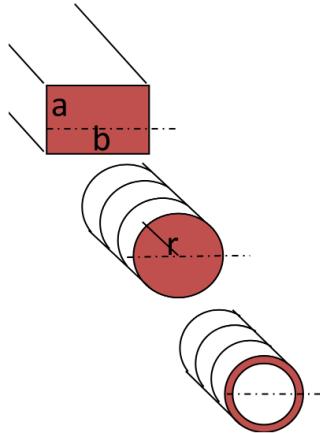
- $L$ : Length
- $t$ : Thickness
- $A$ : Cross-sectional area
- $E$ : Young's modulus

- Bending ( $M, F_t$ ):



$$S_b = -\frac{M + F_t(L - x)}{EI} \frac{t}{2}$$

### 21.8.5 Area moment of inertia ( $I$ )



$$I = \frac{ab^3}{12}$$

$$I = \frac{\pi r^4}{4}$$

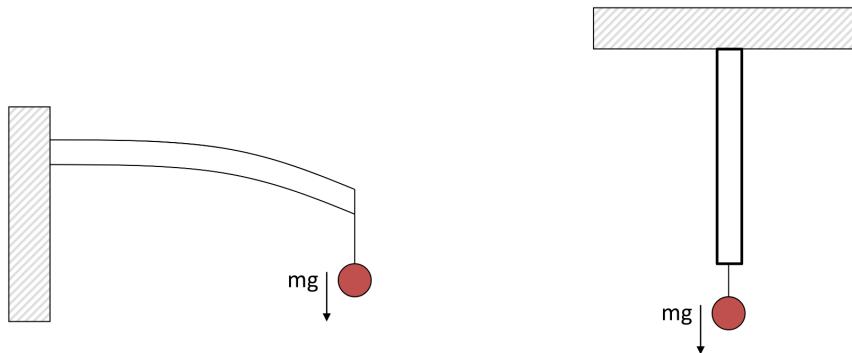
$$I = \frac{\pi(r_{max}^4 - r_{min}^4)}{4}$$

### 21.9 Measuring forces

- You are given 2 identical strain-gauges, where:

$$\frac{dR}{R} = GS, \quad \text{where } G = 2$$

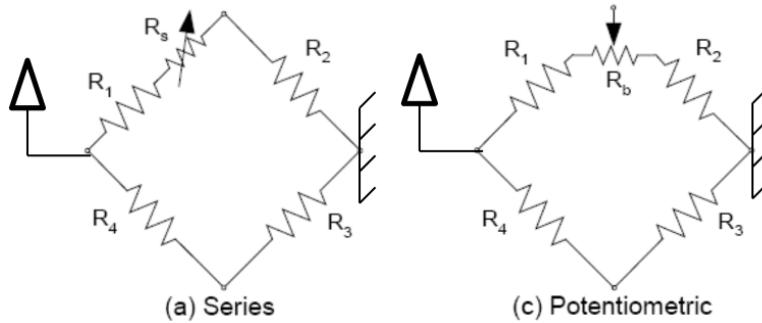
- Where and how would you place them, along the beam, to measure the weight of a mass  $m$  and maximise sensitivity as well as compensate for temperature changes?



## 21.10 Bridge balancing

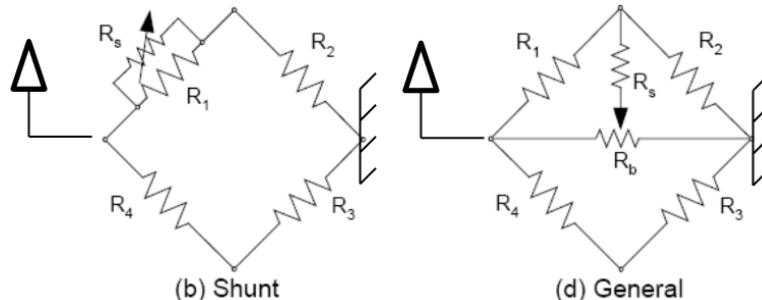
In reality, the bridge is never balanced when stress is applied, hence there is a need to re-establish balance by modifying arm resistors:

$$R_1 R_3 = R_2 R_4$$



The above two circuits require:

- Very low resistance resistors, which is not practical.
- In-series switches or contacts, which results in unreliable extra resistance being added.



The above two circuits are the most suitable, as:

- Much larger resistors can be used.
- The structure of the bridge is not modified, which allows for parallel insertion.
- (d) is more general as it balances both sides.