Math Module 6B Notes

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April 29, 2025

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1 Definitions

1.1 Riemann sum

For a function $f : [a, b] \to \mathbb{R}$, a Riemann sum is a sum:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad \Delta x_i = a_i - a_{i-1}$$

Points $a = a_0 < a_1 < \cdots < a_n = b$ form a **partition** of the interval $[a, b]$, and $x_i^* \in [a_{i-1}, a_i]$ are **sample points**. Further, let $\Delta x = \max\{\Delta x_i : i = 1, \ldots, n\}$. Suppose the limit of the Riemann sums exists and is independent of our choice of partitions or sample points. Then we say that f is integrable on $[a, b]$ and the limit below is the **Riemann integral** of f from a to b.

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\, dx$$

If $f(x) \geq 0$ on $[a, b]$, the Riemann sums approximate the area below the graph $y = f(x)$. The approximation improves as we take finer partitions of $[a, b]$.

1.2 Riemann integral

For a function $f : [a, b] \to \mathbb{R}$, a Riemann integral is the integral:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\, dx$$

1.3 Riemann sums in two variables

Now consider $f : R \to \mathbb{R}$, where R is a rectangle in $\mathbb{R}^2$, i.e.

$$R = [a, b] \times [c, d] = \{(x, y) : x \in [a, b], y \in [c, d]\}$$

By partitioning both $[a, b]$ and $[c, d]$:

$$a = x_0 < x_1 < \ldots < x_n = b$$

$$c = y_0 < y_1 < \ldots < y_n = d$$

We get a partition of R into smaller subrectangles:

$$R_{j,k} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$$

Each of the subrectangles has an area of:

$$\Delta A_{j,k} = \Delta x_j \Delta y_k = (x_j - x_{j-1})(y_k - y_{k-1})$$

In each subrectangle, choose a sample point (x_j^*, y_k^*) and form the **Riemann sum**:

$$\sum_{j=1}^{n} \sum_{k=1}^{m} f(x_j^*, y_k^*) \Delta x_j \Delta y_k$$

Where:

- Area $A_{j,k} = \Delta x_j \Delta y_k$
- $f(x_j^*, y_k^*) \Delta x_j \Delta y_k$ is the volume of a cuboid with base $R_{j,k}$ and height $f(x_i^*, y_k^*)$.

1.4 Double integral

Let:

$$\Delta x = \max_{j=1,\ldots,n} \Delta x_j, \quad \Delta y = \max_{k=1,\ldots,m} \Delta y_j$$

If the Riemann sums have a limit that is independent of our choice of sample points and partition, we say that $f(x, y)$ is integrable on R and the limit below is the double integral of f over R.

$$\iint_R f(x, y)\, dxdy$$

1.5 Double integral and volumes

If $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}$, then the Riemann sums below approximate the volume below the graph $z = f(x, y)$ above the xy-plane:

$$\sum_{j=1}^{n} \sum_{k=1}^{m} f(x_j^*, y_k^*) \Delta x_j \Delta y_k$$

As $(\Delta x, \Delta y) \to (0, 0)$, these approximations converge to the actual volume, i.e.

$$\iint_R f(x, y)\, dxdy = \text{"volume below the graph } z = f(x, y) \text{ but above the } xy\text{-plane."}$$

1.5.1 Example

Let $f : [-2, 2] \times [-2, 2] \to \mathbb{R}$ be given by:

$$f(x, y) = x^2 + y^2$$

Consider the volume between the surface $z = f(x, y)$ and the xy-plane. Taking finer partitions, the Riemann sums converge to the volume below the graph, i.e.

$$\text{Volume} = \iint_{[-2,2]\times[-2,2]} f(x, y)\, dxdy$$

1.6 Double integrals over non-rectangular regions

For $\iint_A f(x, y)\, dxdy$ where A is not rectangular:

1. Take a rectangle R that contains A.
2. Let:

$$g(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \in A \\ 0 & \text{for } (x, y) \notin A \end{cases}$$

1. Let:

$$\iint_A f(x, y)\, dxdy = \iint_R g(x, y)\, dxdy$$

1.7 Fubini's theorem

If for some continuous g, h:

$$A = \{(x, y) : a \leq x \leq b, \quad g(x) \leq y \leq h(x)\}$$

Then for $f(x, y)$ continuous on A:

$$\iint_A f(x,y)\,dxdy = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x,y)\,dy \right) dx$$

If for some continuous g, h:

$$A = \{(x, y) : c \leq x \leq d, \quad g(y) \leq x \leq h(y)\}$$

Then for $f(x, y)$ continuous on A:

$$\iint_A f(x,y)\,dxdy = \int_{y=c}^{d} \left(\int_{x=g(y)}^{h(y)} f(x,y)\,dy \right) dx$$

1.7.1 Example

Let:

$$D = \{(x, y) : 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x}\}$$

Evaluate:

$$\iint_D x\,dxdy$$

$$\begin{aligned}
\iint_D x\,dxdy &= \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x}} x\,dy \right) dx \\
&= \int_0^1 x\sqrt{1-x}\,dx \\
&= -\frac{2}{3}(1-x)^{\frac{3}{2}} \cdot x \Big|_0^1 + \frac{2}{3}\int_0^1 (1-x)^{\frac{3}{2}}\,dx \\
&= -\frac{2}{3} \cdot \frac{2}{5}(1-x)^{\frac{5}{2}} \Big|_0^1 \\
&= \frac{4}{15}
\end{aligned}$$

Changing the order of integration:

$$\begin{aligned}
\iint_D x\,dxdy &= \int_0^1 \left(\int_{x=0}^{1-y^2} x\,dx \right) , dy \\
&= \int_0^1 \left[\frac{x^2}{2} \right]_{x=0}^{1-y^2} dy \\
&= \frac{1}{2} \int_0^1 (1-y^2)^2\,dy \\
&= \frac{1}{2} \int_0^1 (1-2y^2+y^4)\,dy \\
&= \frac{1}{2} \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1 \\
&= \frac{1}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) \\
&= \frac{1}{2} \cdot \frac{15 - 2\cdot 5 + 1 \cdot 3}{15} \\
&= \frac{4}{15}
\end{aligned}$$

1.8 Triple integrals

For a three variable function $f(x,y,z)$ and a region $Q \subset \mathbb{R}^3$, the triple integral below is defined and can be calculated using similar principles:

$$\iiint_Q f(x,y,z)\,dxdydz$$

1.8.1 Example 1

Evaluate:

$$\iiint_Q 6xy\,dxdydz$$

Where Q is the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $2x+y+z=4$.

$$
\begin{aligned}
\iiint_Q 6xy\,dxdydz &= \int_{x=0}^{2}\left(\int_{y=0}^{4-2x}\left(\int_{z=0}^{4-2x-y} dz\right) dy\right) dx \\
&= \int_0^2 \left(\int_0^{4-2x} 6xy(4-2x-y)\,dy\right) dx \\
&= \int_0^2 \left(\int_0^{4-2x} (24xy-12x^2-y-6xy^2)\,dy\right) dx \\
&= \int_0^2 \left[12xy^2-6x^2y^2-2xy^3\right]_{y=0}^{4-2x} dx \\
&= \int_0^2 \left(12x(4-2x)^2-6x^2(4-2x)^2-2x(4-2x)^3\right) dx \\
&= \int_0^2 \left(12x(4-2x)^2-6x^2(4-2x)^2-2x(4-2x)(4-2x)^2\right) dx \\
&= \int_0^2 \left(12x(4-2x)^2-6x^2(4-2x)^2-(8x-4x^2)(4-2x)^2\right) dx \\
&= \int_0^2 \left((12x-6x^2-8x+4x^2)(4-2x)^2\right) dx \\
&= \int_0^2 \left((4x-2x^2)(4-2x)^2\right) dx \\
&= \int_0^2 \left(x(4-2x)(4-2x)^2\right) dx \\
&= \int_0^2 \left(x(4-2x)^3\right) dx \\
&= \int_0^2 \left(8x(2-x)^3\right) dx \\
&= \left[8x\frac{-(2-x)^4}{4}\right]_0^2 + \int_0^2 \frac{-2(2-x)^4}{4}\cdot 8\,dx \\
&= \left[-2\cdot\frac{(2-x)^5}{5}\right]_0^2 \\
&= \left[0-\left(-2\cdot\frac{(2-0)^5}{5}\right)\right] \\
&= \frac{64}{5}
\end{aligned}
$$

1.8.2 Example 2

Evaluate:

$$\iiint_Q \, dxdydz$$

Where Q is given by:

$$x^2 + y^2 \leq z \leq 1, \quad x \geq 0$$

$$\begin{aligned}
\iiint_Q \, dxdydz &= \int_{y=-1}^{1} \left(\int_{x=0}^{\sqrt{1-y^2}} \left(\int_{z=x^2+y^2} \, dz \right) \, dx \right) \, dy \\
&= \int_{-1}^{1} \left(\int_{0}^{\sqrt{1-y^2}} (1 - (x^2 + y^2)) \, dx \right) \, dy \\
&= \int_{-1}^{1} \left[x - \frac{x^3}{3} - y^2 x \right]_{x=0}^{\sqrt{1-y^2}} \, dy \\
&= \int_{-1}^{1} \left(\sqrt{1-y^2} - \frac{(1-y^2)^{\frac{3}{2}}}{3} - y^2\sqrt{1-y^2} \right) \, dy \\
&= \int_{-1}^{1} \left(\sqrt{1-y^2}(1-y^2) - \frac{(1-y^2)^{\frac{3}{2}}}{3} \right) \, dy \\
&= 2 \int_{0}^{1} \frac{2}{3}(1-y^2)^{\frac{3}{2}} \, dy
\end{aligned}$$

$$\begin{gathered}
y = \sin\theta \\
\sqrt{1-y^2} = \cos\theta \\
dy = \cos\theta \, d\theta \\
y = -1 \quad \Leftrightarrow \quad \theta = -\frac{\pi}{2} \\
y = 1 \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}
\end{gathered}$$

Substituting the above equations:

$$
\begin{aligned}
2\int_0^1 \frac{2}{3}(1-y^2)^{\frac{3}{2}}\,dy &= \frac{2}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2\,d\theta \\
&= \frac{4}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1}{4}+\cos 2\theta+\frac{\cos^2 2\theta}{4}\right)\,d\theta \\
&= \frac{4}{3}\left(\frac{\pi}{8}+\frac{1}{4}\int_0^{\frac{\pi}{2}} \cos^2 2\theta\,d\theta\right) \\
&= \frac{\pi}{6}+\frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 4\theta)\,d\theta \\
&= \frac{\pi}{6}+\frac{1}{3}\cdot\frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}
$$

1.9 Size of a region

Integrating the function **1**, always gives you the size of the region of integration:

$$
\begin{aligned}
\int_a^b 1\,dx = b-a &= \text{length of } [a,b] \\
\iint_R 1\,dxdy = \iint_R dA &= \text{area of } R \\
\iiint_Q 1\,dxdydz = \iiint_Q dV &= \text{volume of } Q
\end{aligned}
$$

Likewise, integrating a constant C gives you C times the size of the region.

1.10 Polar coordinates

A point $(x, y) \in \mathbb{R}^2$ can be represented by its **polar coordinates** (r, θ), where:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r \geq 0, \theta \in [0, 2\pi)$$
$$r^2 = \sqrt{x^2 + y^2}$$

1.10.1 Area element

In polar coordinates, the area element $dA = dxdy$ becomes $dA = r\,drd\theta$.

1.10.2 Example

Calculate:

$$\iint_D (x^2 + y^2)\,dxdy$$

Where:

$$D = \{(x, y) : x^2 + y^2 \leq 4, y \geq 0\}$$

$$\begin{aligned}
\iint_D (x^2 + y^2)\,dxdy &= \int_0^x \left(\int_0^2 r^3\,dr\right) d\theta \\
&= \int_0^\pi \left.\frac{r^4}{4}\right|_{r=0}^2 d\theta \\
&= \int_0^\pi 4\,d\theta \\
&= 4\pi
\end{aligned}$$

1.11 Cylindrical coordinates

A point $(x, y, z) \in \mathbb{R}^3$ can be represented by its cylindrical coordinates (r, θ, z), where:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad r \geq 0, \theta \in [0, 2\pi)$$

1.11.1 Volume element

In cylindrical coordinates, the volume element $dV = dxdydz$ becomes $dV = r\,drd\theta dz$, where r is the scaling factor.

1.11.2 Example

Evaluate

$$\iiint_Q dxdydz$$

Where the solid region Q, given by:

$$x^2 + y^2 \leq z \leq 1, x \geq 0$$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \\ x^2 + y^2 &= r^2 \\ dV = dxdydz &= r\,drd\theta dz \end{aligned}$$

$$\begin{aligned} \iiint_Q dxdydz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^1 \left(\int_{r^2}^1 r\,dz \right) dr \right) d\theta \\ &= \pi \int_0^1 \left(\int_{r^2}^1 r\,dz \right) dr \\ &= \pi \int_0^1 (r - r^3)\,dr \\ &= \pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{4} \right) \\ &= \frac{\pi}{4} \end{aligned}$$

1.12 Spherical coordinates

A point $(x, y, z) \in \mathbb{R}^3$ can be represented by its spherical coordinates (ρ, φ, θ), where:

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \cos\theta, \quad z = \rho \sin\varphi, \quad \rho \geq 0, \varphi \in [0, \pi], \theta \in [0, 2\pi)$$

Keeping ρ fixed while varying φ and θ gives us points on a sphere with radius ρ.

$$\varphi = \text{"latitude"}$$

$$\theta = \text{"longitude"}$$

In spherical coordinates, the volume element $dV = dxdydz$ becomes $dV = \rho^2 \sin\varphi \, d\rho d\varphi d\theta$, where $\rho^2 \sin\varphi$ is the scaling factor.

1.12.1 Example

Calculate the volume of a ball with radius R.

$$\begin{aligned}
\text{Volume} &= \iiint_Q dV \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^R \rho^2 \sin\varphi \, d\rho \right) d\varphi \right) d\theta \\
&= 2\pi \int_0^{\pi} \left(\sin\varphi \int_0^R \rho^2 \, d\rho \right) d\varphi \\
&= 2\pi \frac{R^3}{3} \int_0^{\pi} \sin\varphi \, d\varphi \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

2 Calculating a double integral

Consider a continuous $f : A \to \mathbb{R}$ where the region $A \subset \mathbb{R}^2$ has the form:

$$A = \{(x, y) : a \le x \le b, \quad g(x) \le y \le h(x)\}$$

Let's calculate:

$$\iint_A f(x, y)\, dxdy$$

For simplicity, suppose $f(x, y) \ge 0$ on A, so we can interpret $\iint_A f(x, y)\, dxdy$ as a volume:

$$A(x) = \int_{y=g(x)}^{h(x)} f(x, y)\, dy$$

$$\begin{aligned}
\iint_A f(x, y)\, dxdy &= \text{Volume} \\
&= \int_{\alpha=a}^{b} dV \\
&= \int_a^b A(x)\, dx \\
&= \int_a^b \left(\int_{y=g(x)}^{h(x)} f(x, y)\, dy \right) dx
\end{aligned}$$

Our usual approach leads us to the formula:

$$\iint_A f(x, y)\, dxdy = \int_{x=a}^{b} \left(\int_{y=g(x)}^{h(x)} f(x, y)\, dy \right) dx$$

The assumption $f(x, y) \ge 0$ is not necessary for the above to hold, it just made the illustration easier.

If the roles of x and y are reversed, we get an analogous result.

3 Double integrals heuristically

$$\begin{aligned}\iint_R dV &= \iint_R f(x,y)\,dA \\ &= \iint_R f(x,y)\,dxdy\end{aligned}$$

$$\text{Volume element} = dV = f(x,y)\,dA$$

$$\text{Area element} = dA = dxdy$$

4 Triple integrals heuristically

Let density= $\rho(x,y,z)$

$$\begin{aligned}\text{mass of Q} &= \iiint_Q dm \\ &= \iiint_Q \rho(x,y,z)\,dV \\ &= \iiint_Q \rho(x,y,z)\,dxdydz\end{aligned}$$

$$\text{Mass element} = dm = \rho(x,y,z)\,dV$$

$$\text{Volume element} = dV = dxdydz$$