# MA3005/MA3705 Control Theory

# Homework (Due on 9/11/25, 23:59)

## Instructions

A. Show all of your workings and drawings clearly.
B. Submit your homework as a pdf. The homework can be typed, written digitally or via pen and paper. Do ensure that the orientation of your submitted homework is upright.
C. You may use numerical software such as MATLAB for this homework. However, you have to sketch the root locus manually and not generate it via MATLAB or other software. This root locus sketch can be done via pen and paper, or digitally.

---

1. (a) Figure 1 shows gear 1 driving gear 2 and a cable drum, where gear 2 and the cable drum share the same rotating shaft. Gear 1, gear 2 and the cable drum have a radius of $3r$, $5r$ and $2r$, respectively. The moment of inertia of gear 1 is $J$ while the combined moment of inertia of gear 2 and the cable drum is $4J$. As the cable drum rotates, it will also hoist up a weight of mass $M$. All the angular and linear displacements of the components are shown in Figure 1. The rotary damping coefficient of gear 1 is $c$. The combined rotary damping coefficient of gear 2 and the drum is $2c$. The linear damping coefficient of the mass is $c$. To actuate this system, we have mounted a motor at gear 1 and the torque supplied by the motor is given as $\tau$. Consider the weight of the mass and assume zero initial conditions to derive the block diagram of the system where the displacement $x$ is the output. Express $\mathcal{L}(\tau) = T(s)$. (6 marks)

   (b) Determine and explain if the system is stable for output $x$. (3 marks)

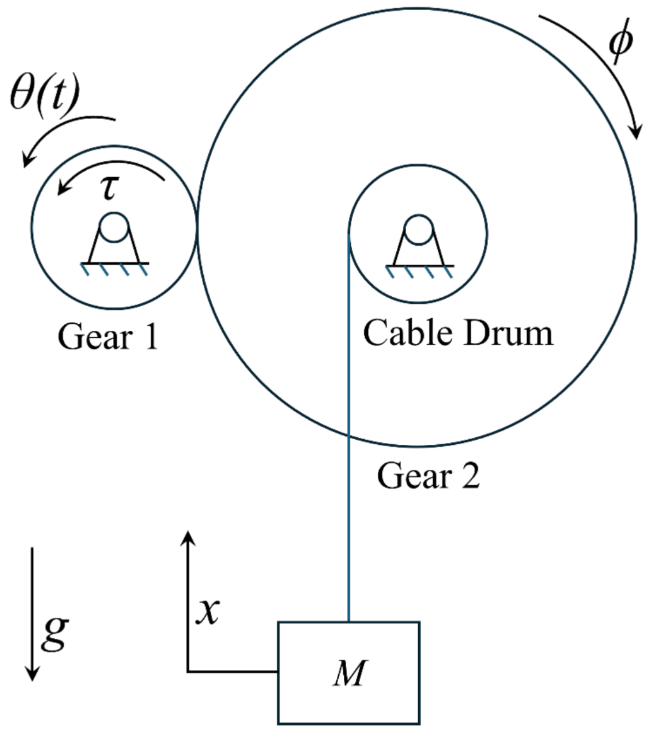
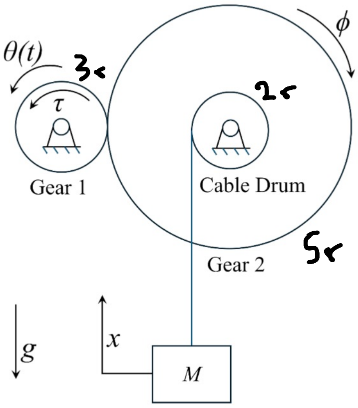
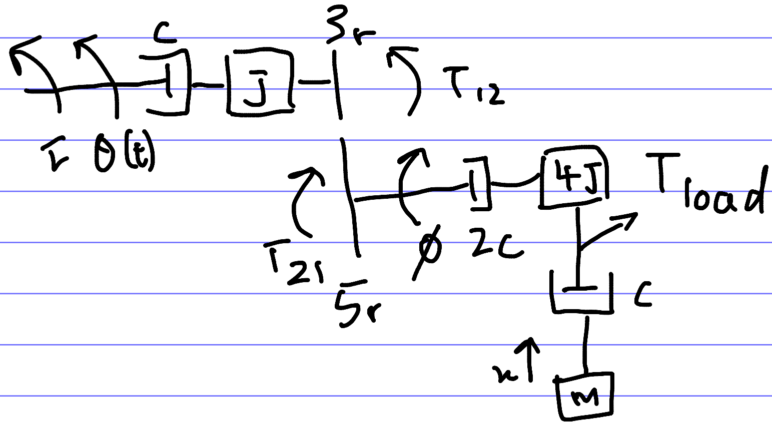



Fig. 1. A hoisting system, where a motor torque of $\tau$ is generated at gear 1 to lift the mass.

1a)




Output of gear 1:

$$\tau(s) - T_{12}(s) = (Js^2 + cs)\,\theta(s)$$

$$T_{12}(s) = \tau(s) - (Js^2 + cs)\,\theta(s) \quad -(1)$$

Since gear 1 and 2 are meshing:

$$\frac{T_{21}}{T_{12}} = \frac{5\cancel{r}}{3\cancel{r}} \qquad \frac{\phi}{\theta} = -\frac{3r}{5r}$$

$$T_{21} = \frac{5}{3}T_{12} \quad -(2) \qquad \theta = -\frac{5}{3}\phi \quad -(3)$$

Output of gear 2:

$$T_{21}(s) - T_{load}(s) = (4Js^2 + 2cs)\,\phi(s)$$

$$T_{load}(s) = T_{21} - (4Js^2 + 2cs)\,\phi(s) \quad -(4)$$

Sub (2) into (4):

$$T_{load}(s) = \frac{5}{3}T_{12} - (4Js^2 + 2cs)\,\phi(s) \quad -(5)$$

Tension (P) of the string carrying the load:

$$P(s) - \frac{Mg}{s} = (Ms^2 + cs)\,X(s)$$

$$P(s) = (Ms^2 + cs)\,X(s) + \frac{Mg}{s} \quad -(6)$$

1a) $P(2r) = T_{load}(s)$

$$\therefore \left[(Ms^2+cs)X(s)+\frac{Mg}{s}\right](2r) = \frac{5}{3}T_{21} - (4Js^2+2cs)\phi(s)$$

$$2r\left[(Ms^2+cs)X(s)+\frac{Mg}{s}\right] = \frac{5}{3}T_{21} - (4Js^2+2cs)\phi(s) \quad -(7)$$

Sub (1) into (7):

$$2r\left[(Ms^2+cs)X(s)+\frac{Mg}{s}\right] = \frac{5}{3}\left[\tau(s)-(Js^2+cs)\theta(s)\right] - (4Js^2+2cs)\phi(s) \quad -(8)$$

Sub (3) into (8):

$$2r\left[(Ms^2+cs)X(s)+\frac{Mg}{s}\right] = \frac{5}{3}\left[\tau(s)-(Js^2+cs)\left(-\frac{5}{3}\right)\phi(s)\right] - (4Js^2+2cs)\phi(s)$$

Since $\phi = \frac{x}{2r}$:

$$2r\left[(Ms^2+cs)X(s)+\frac{Mg}{s}\right] = \frac{5}{3}\left[\tau(s)+\frac{5}{3}(Js^2+cs)\frac{X(s)}{2r}\right] - (4Js^2+2cs)\frac{X(s)}{2r}$$

$$\frac{2rMg}{s} + 2r(Ms^2+cs)X(s) = \frac{5}{3}\tau(s)+\frac{25}{9}(Js^2+cs)\frac{X(s)}{2r} - (4Js^2+2cs)\frac{X(s)}{2r}$$

$$\frac{2rMg}{s} + 2r(Ms^2+cs)X(s) = \frac{5}{3}\tau(s)+\left(-\frac{11}{9}Js^2+\frac{7}{9}cs\right)\frac{X(s)}{2r}$$

$$\frac{5}{3}\tau(s) = \left[\left(2rM+\frac{11}{9}\frac{1}{2r}J\right)s^2+\left(2r-\frac{7}{9}\frac{1}{2r}\right)cs\right]X(s) + \frac{2rMg}{s}$$

$$\frac{5}{3}\tau(s) = 2r\left(\left[\left(M+\frac{11}{36r^2}J\right)s^2+\left(1-\frac{7}{36r^2}\right)cs\right]X(s)+\frac{Mg}{s}\right)$$

$$\frac{5}{6r}\tau(s) = \left[\left(M+\frac{11}{36r^2}J\right)s^2+\left(1-\frac{7}{36r^2}\right)cs\right]X(s)+\frac{Mg}{s}$$

$$\left[\left(M+\frac{11}{36r^2}J\right)s^2+\left(1-\frac{7}{36r^2}\right)cs\right]X(s) = \frac{5}{6r}\tau(s) - \frac{Mg}{s}$$

$$\tau(s) = \frac{6rs}{5}\left[\left(M+\frac{11}{36r^2}J\right)s+\left(1-\frac{7}{36r^2}\right)c\right]X(s) + \frac{6Mgr}{5s}$$

1a) Splitting the equation into its two inputs, $\tau(s)$ and $g$:

$\tau(s)$:

$$\frac{5}{6r}\tau(s) = \left[\left(M+\frac{11}{36r^2}J\right)s^2+\left(1-\frac{7}{36r^2}\right)cs\right]X(s)$$

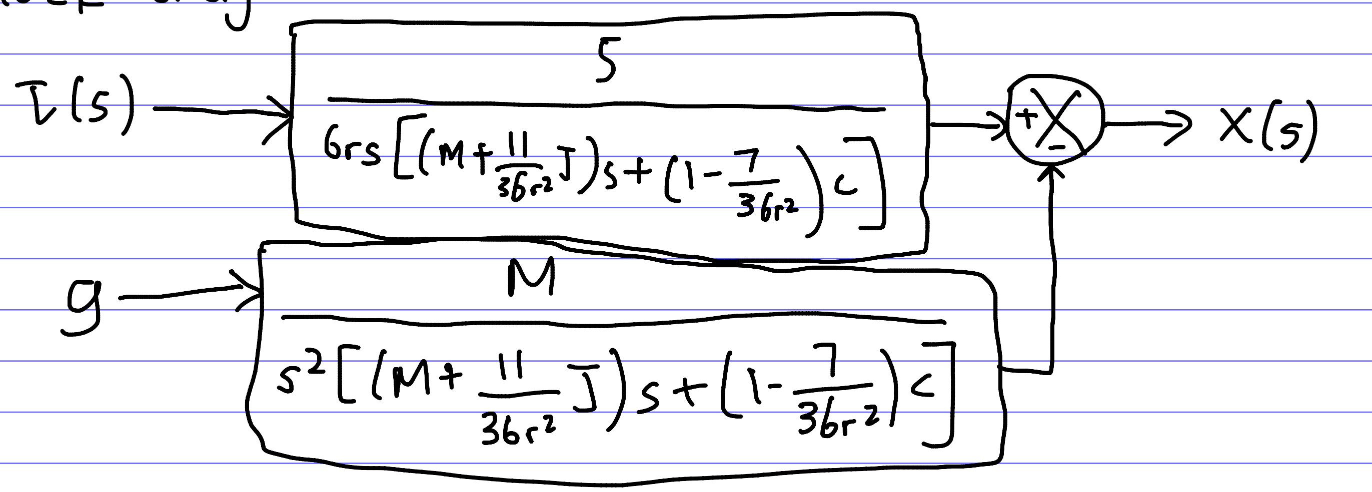
$$\frac{X(s)}{\tau(s)} = \frac{5}{6rs\left[\left(M+\frac{11}{36r^2}J\right)s+\left(1-\frac{7}{36r^2}\right)c\right]}$$ → Transfer function for $\tau(s)$ to $X(s)$

$g$:

$$-\frac{M}{s}g = \left[\left(M+\frac{11}{36r^2}J\right)s^2+\left(1-\frac{7}{36r^2}\right)cs\right]X(s)$$

$$\frac{X(s)}{g} = -\frac{M}{s^2\left[\left(M+\frac{11}{36r^2}J\right)s+\left(1-\frac{7}{36r^2}\right)c\right]}$$ → Transfer function for $g$ to $X(s)$

Block diagram:



1b) For input $\tau(s)$:

Characteristic equation: $s\left[\left(M+\frac{11}{36r^2}J\right)s+\left(1-\frac{7}{36r^2}\right)c\right]=0$

$$s=0 \quad \text{or} \quad \left(M+\frac{11}{36r^2}J\right)s=-\left(1-\frac{7}{36r^2}\right)c$$

$$s=\frac{\frac{7}{36r^2}-1}{M+\frac{11}{36r^2}J}c$$

Let $N=\frac{7}{36r^2}-1$, $D=M+\frac{11}{36r^2}J$

Since $M, J, r$ and $c$ are always $>0$,
$D>0$

For $N<0$,

$$\frac{7}{36r^2}-1<0$$

$$\frac{7}{36r^2}<1$$

$$\frac{7}{36}<r^2$$

$$r>\sqrt{\frac{7}{36}}$$

$\therefore$ $s\leq 0$ when $r\geq\sqrt{\frac{7}{36}}$ and $s>0$ when $r<\sqrt{\frac{7}{36}}$

$\therefore$ The system is partially stable when $r\geq\sqrt{\frac{7}{36}}$, but becomes unstable when $r<\sqrt{\frac{7}{36}}$ with respect to input $\tau(s)$.

1b) For input g:

Characteristic equation: $s^2\left[\left(M+\frac{11}{36r^2}J\right)s+\left(1-\frac{7}{36r^2}\right)c\right]=0$

$s^2=0$ or $\left(M+\frac{11}{36r^2}J\right)s=-\left(1-\frac{7}{36r^2}\right)c$

$s=0$

$$s=\frac{\frac{7}{36r^2}-1}{M+\frac{11}{36r^2}J}c$$

Let $N=\frac{7}{36r^2}-1$, $D=M+\frac{11}{36r^2}J$

Since $M, J, r$ and $c$ are always $>0$,
$D>0$

For $N<0$,
$$\frac{7}{36r^2}-1<0$$
$$\frac{7}{36r^2}<1$$
$$\frac{7}{36}<r^2$$
$$r>\sqrt{\frac{7}{36}}$$

$\therefore$ $s\leq 0$ when $r\geq\sqrt{\frac{7}{36}}$ and $s>0$ when $r<\sqrt{\frac{7}{36}}$

$\therefore$ The system is partially stable when $r\geq\sqrt{\frac{7}{36}}$, but becomes unstable when $r<\sqrt{\frac{7}{36}}$ with respect to input g.

$\therefore$ Since the system is partially stable with respect to both $\tau(s)$ and g inputs when $r\geq\sqrt{\frac{7}{36}}$, but becomes unstable with respect to both $\tau(s)$ and g inputs when $r<\sqrt{\frac{7}{36}}$, the overall system is partially stable when $r\geq\sqrt{\frac{7}{36}}$, but becomes unstable when $r<\sqrt{\frac{7}{36}}$.

2. Draw the root locus for the unity feedback system shown in Fig. 2 by answering all the questions below.

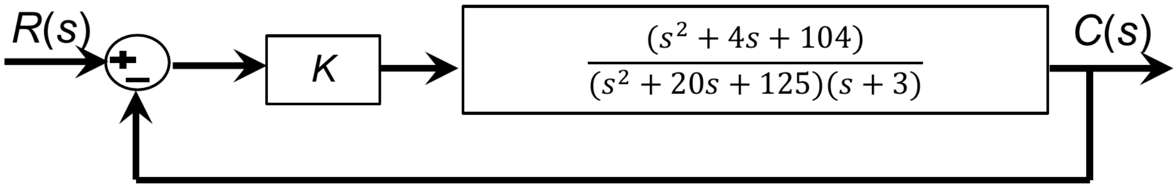



Fig. 2. A unity feedback system.

(a) Identify all the starting and ending points of the locus, including the infinity zeroes (if applicable). (3 marks)

(b) Find all the arrival angles. (2 marks)

(c) Find all the departure angles. (2 marks)

(d) Find all the break-in and break-out points (if applicable). (4 marks)

(e) Manually sketch the root locus. (5 marks)

2a) Open-loop poles:

Denominator = 0

$(s^2+20s+125)(s+3)=0$

$s=-10\pm5j$ or $s=-3$

Open-loop zeros:

Numerator = 0

$s^2+4s+104=0$

$s=-2\pm10j$

Since there are 3 open-loop poles and only 2 open-loop zeros, there is one open-loop zero at $\infty$.

Using the phase condition to obtain the asymptote angle:

$$\angle s = \frac{\pi(1+2q)}{n-m}$$

$$= \frac{\pi(1+2q)}{3-2}$$

$$= \pi$$

Obtaining the asymptote location:

$$\sigma_a = \frac{(-10-10-3)-(-2-2)}{3-2}$$

$$= -19$$

However, since the angle of the asymptote is $\pi$, the asymptote will stretch to $-\infty$.

2b) Arrival angles at point ☆:

| | $\Delta y$ | $\Delta x$ | Required angle |
|---|---|---|---|
| $z_1$ | — | — | $\theta$ |
| $z_2$ | 20 | 0 | $\frac{\pi}{2}$ |
| $p_1$ | 10 | 1 | $\arctan 10$ |
| $p_2$ | 5 | 8 | $\arctan\left(\frac{5}{8}\right)$ |
| $p_3$ | 15 | 8 | $\arctan\left(\frac{15}{8}\right)$ |

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_i) = \pm\pi$$

$$\theta + \frac{\pi}{2} - \arctan 10 - \arctan\left(\frac{5}{8}\right) - \arctan\left(\frac{15}{8}\right) = \pm\pi$$

$$\theta = -1.60182299 \text{ rad}$$
$$= -91.77769687^\circ$$
$$\approx -91.8^\circ$$

c) Departure angles at point □:

| | $\Delta y$ | $\Delta x$ | Required angle |
|---|---|---|---|
| $z_1$ | 5 | 8 | $\frac{3}{2}\pi - \arctan\left(\frac{8}{5}\right)$ |
| $z_2$ | 15 | 8 | $\frac{\pi}{2} + \arctan\left(\frac{8}{15}\right)$ |
| $p_1$ | 5 | 7 | $\frac{\pi}{2} + \arctan\left(\frac{7}{5}\right)$ |
| $p_2$ | — | — | $\varnothing$ |
| $p_3$ | 10 | 0 | $\frac{\pi}{2}$ |

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{j=1}^{n} \angle(s+p_i) = \pm\pi$$

$$\frac{3}{2}\pi - \arctan\left(\frac{8}{5}\right) + \frac{\pi}{2} + \arctan\left(\frac{8}{15}\right) - \frac{\pi}{2} - \arctan\left(\frac{7}{5}\right) - \varnothing - \frac{\pi}{2} = \pm\pi$$

$$\varnothing = -1.472786526 \text{ rad}$$
$$= -84.38445206$$
$$\approx -84.4^\circ$$

2d) $\dfrac{s^2+4s+104}{(s^2+20s+125)(s+3)} = \dfrac{s^2+4s+104}{s^3+20s^2+125s+3s^2+60s+375}$

$= \dfrac{s^2+4s+104}{s^3+23s^2+185s+375}$

$\dfrac{d}{dx}\left[\dfrac{s^2+4s+104}{s^3+23s^2+185s+375}\right] = \dfrac{1}{(s^3+23s^2+185s+375)^2}\Big[$

$(2s+4)(s^3+23s^2+185s+375)$
$-(3s^2+46s+185)(s^2+4s+104)\Big]$

$= \dfrac{1}{(s^3+23s^2+185s+375)^2}\Big[2s^4+46s^3$
$+370s^2+750s+4s^3+92s^2$
$+740s+1500-(3s^4+12s^3$
$+312s^2+46s^3+184s^2+4784s$
$+185s^2+740s+19240)\Big]$

$= \dfrac{1}{(s^3+23s^2+185s+375)^2}\Big[-s^4-8s^3-219s^2$
$-4034s-17740\Big]$

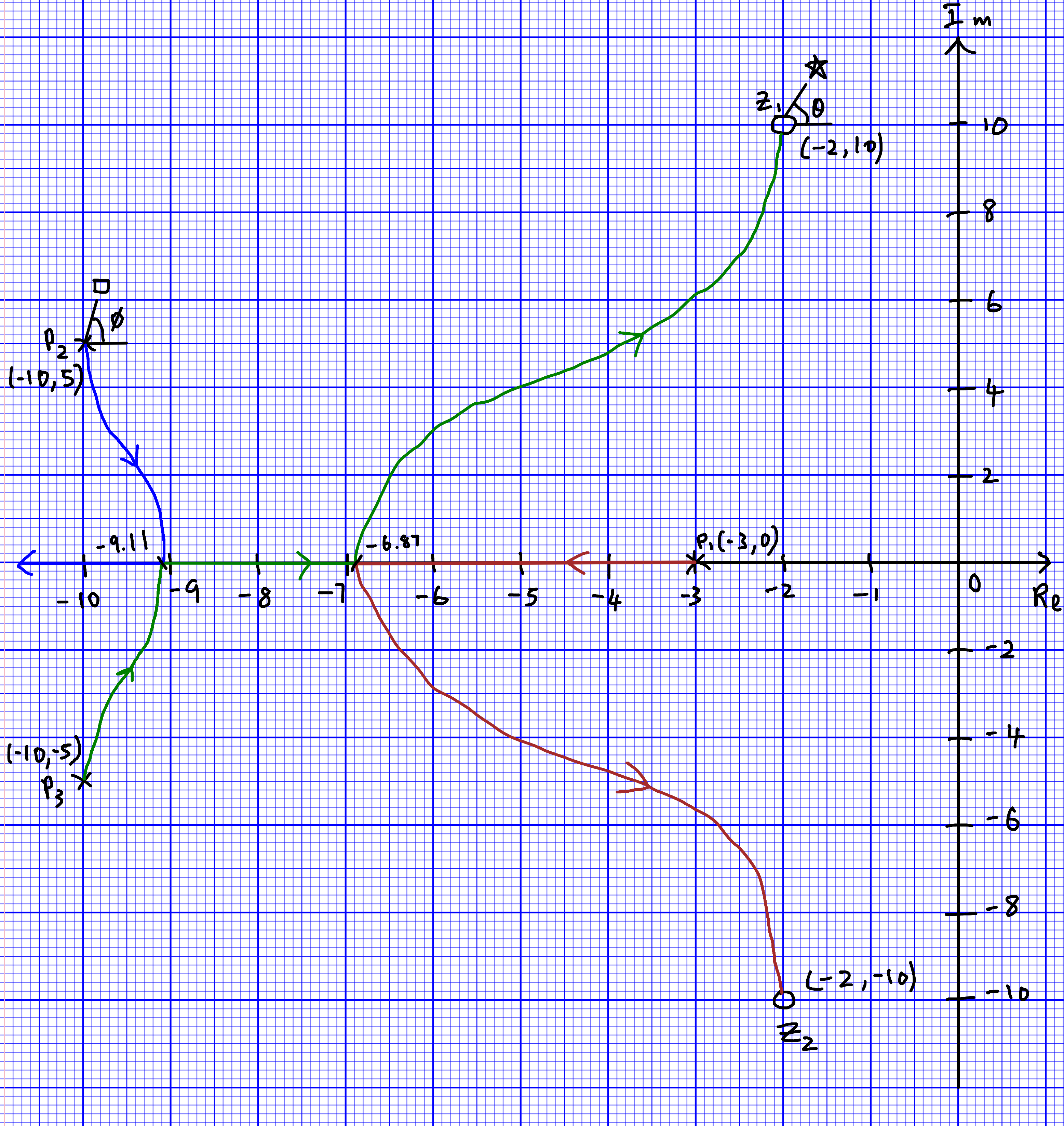
$\dfrac{d}{dx}\left[\dfrac{s^2+4s+104}{s^3+23s^2+185s+375}\right] = 0$

$\cancel{-\dfrac{1}{(s^3+23s^2+185s+375)^2}}(s^4+8s^3+219s^2+4034s+17740)=0$

$s^4+8s^3+219s^2+4034s+17740=0$

Solving the equation:

$s=-9.10517915$ or $s=-6.86553102$
$\approx -9.11$ $\approx -6.87$


Im
Re
Z₁
θ
(-2, 10)
P₂
ϕ
(-10, 5)
(-10, -5)
P₃
-9.11
-6.87
P₁ (-3, 0)
(-2, -10)
Z₂
-10
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
10
8
6
4
2
-2
-4
-6
-8
-10