1a) Parametrising C_1 with:

$$x = \cos t$$
$$y = \sin t$$

$$t \in [0, 2\pi]$$

$$dx = -\sin t \, dt$$
$$dy = \cos t \, dt$$

$$P = \frac{-\sin t}{\sqrt{\sin^2 t + \cos^2 t}}$$
$$= -\sin t$$

$$Q = \frac{\cos t}{\sqrt{\sin^2 t + \cos^2 t}}$$
$$= \cos t$$

1a) $\oint_{C_1} P\,dx + Q\,dy$

$$= \int_0^{2\pi} -\sin t(-\sin t) + \cos t(\cos t)\,dt$$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t$$

$$= \int_0^{2\pi} 1\,dt$$

$$= 2\pi$$

1b) Let the oriented boundary be ∂D and the shaded region be D.

$$\oint_{C_2} P\,dx - Q\,dy - \oint_{C_1} P\,dx - Q\,dy = \oint_{\partial D} P\,dx - Q\,dy$$

By Green's Theorem,

$$\oint_{\partial D} P\,dx - Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy$$

$$Q_x = \frac{\sqrt{x^2+y^2} - x\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{\cancel{x^2} + y^2 - \cancel{x^2}}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

1b) $P_y = -\dfrac{\sqrt{x^2+y^2} - y\dfrac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}}{x^2+y^2}$

$= -\dfrac{x^2+\cancel{y^2}-\cancel{y^2}}{(x^2+y^2)^{\frac{3}{2}}}$

$= \dfrac{-x^2}{(x^2+y^2)^{\frac{3}{2}}}$

$\iint_D (Q_x - P_y)\,dx\,dy = \iint_D \left(\dfrac{y^2}{(x^2+y^2)^{\frac{3}{2}}} - \dfrac{-x^2}{(x^2+y^2)^{\frac{3}{2}}}\right) dx\,dy$

$= \iint_D \dfrac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{\frac{3}{2}}\frac{1}{2}}}\,dx\,dy$

$= \iint_D \dfrac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

1b) $\iint_D \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$

$= \int_0^{2\pi} \int_0^{3+3\sin 3\theta} \frac{1}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}\, r\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^{3+3\sin 3\theta} \frac{1}{\cancel{r}}\, \cancel{r}\, dr\, d\theta$

$= \int_0^{2\pi} \int_1^{3+3\sin 3\theta} dr\, d\theta$

$= \int_0^{2\pi} 2+3\sin 3\theta\, d\theta$

$= 4\pi + 3\int_0^{2\pi} \sin 3\theta\, d\theta$

$= 4\pi + \cancel{3}\left[-\frac{\cos 3\theta}{\cancel{3}}\right]_0^{2\pi}$

$= 4\pi + (-1-(-1))$

$= 4\pi$

2a) $A = \frac{1}{4}\begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - \frac{5}{4} & -\frac{3}{4} \\ \frac{1}{4} & \lambda - \frac{1}{4} \end{vmatrix} = 0$$

$$\left(\lambda - \frac{5}{4}\right)\left(\lambda - \frac{1}{4}\right) + \frac{3}{16} = 0$$

$$\lambda^2 - \frac{1}{4}\lambda - \frac{5}{4}\lambda + \frac{5}{16} + \frac{3}{16} = 0$$

$$\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = 0$$

$$2\lambda^2 - 3\lambda + 1 = 0$$

$$\lambda = \frac{3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)}$$

$$= \frac{1}{2}, 1$$

2a) For $\lambda = \frac{1}{2}$,

$$\left[\begin{array}{ccc} \lambda - \frac{5}{4} & -\frac{3}{4} & 0 \\ \frac{1}{4} & \lambda - \frac{1}{4} & 0 \end{array}\right] \sim \left[\begin{array}{ccc} -\frac{3}{4} & -\frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} x + y = 0 \\ x = -y \end{array}$$

The eigenvectors are:

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \setminus \{0\}$$

For $\lambda = 1$,

$$\left[\begin{array}{ccc} \lambda - \frac{5}{4} & -\frac{3}{4} & 0 \\ \frac{1}{4} & \lambda - \frac{1}{4} & 0 \end{array}\right] \sim \left[\begin{array}{ccc} -\frac{1}{4} & -\frac{3}{4} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{c} x + 3y = 0 \\ x = -3y \end{array}$$

The eigenvectors are:

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \ t \in \mathbb{R} \setminus \{0\}$$

2a) $P = \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

$$D = P^{-1}AP = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

2b) $\underset{\sim}{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underset{\sim}{x}_k = A\underset{\sim}{x}_{k-1}$ for $x \in \mathbb{Z}^+$

$$\lim_{k \to \infty} \underset{\sim}{x}_k = \lim_{k \to \infty} A^k x_0$$

$$= PD\cancel{P^{-1}P}DP^{-1} \dots x_0$$

$$= \lim_{k \to \infty} PD^kP^{-1} x_0$$

$$= \lim_{k \to \infty} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^k \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

3a) $x_1+x_2+x_3+x_4=0$

$$x_1=-x_2-x_3-x_4$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -r-s-t \\ r \\ s \\ t \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$r,s,t \in \mathbb{R}$

Let $B = \{(-1,1,0,0),(-1,0,1,0),(-1,0,0,1)\}$,

By inspection, $W = \operatorname{span} B$ and B is linearly independent.

$\therefore$ B is a basis for W.

$\therefore \dim W = 3$

3b) $W^{\perp} = \{\underset{\sim}{x} \in V : \langle \underset{\sim}{x}, \underset{\sim}{w} \rangle = 0 \text{ for all } \underset{\sim}{w} \in W\}$

$= \operatorname{span}\{(1,1,1,1)\}$

$\therefore \dim W^{\perp} = 1$

3c) $\underset{\sim}{x} = (1,2,3,2)$

$$\text{proj}_{W^\perp}\underset{\sim}{x} = \frac{(1,2,3,2)\cdot(1,1,1,1)}{\|(1,1,1,1)\|^2}(1,1,1,1)$$

$$= \frac{8}{4}(1,1,1,1)$$

$$= (2,2,2,2)$$

Sime $\underset{\sim}{x} = \text{proj}_W\underset{\sim}{x} + \text{proj}_{W^\perp}\underset{\sim}{x}$,

$$(1,2,3,2) = (2,2,2,2) + \text{proj}_{W^\perp}\underset{\sim}{x}$$

$$\text{proj}_{W^\perp}\underset{\sim}{x} = (-1,0,1,0)$$

4a) By definition, A is symmetric if and only if

$$A^T = A$$

$$A = Y^T Y$$

$$A^T = (Y^T Y)^T$$

$$A^T = Y^T Y$$

$$A^T = A$$

$\therefore A$ is symmetric.

4bi) $$q(\underset{\sim}{x}) = \underset{\sim}{x}^T Y^T Y \underset{\sim}{x}$$

$$= (Y\underset{\sim}{x})^T Y \underset{\sim}{x}$$

$$= (Y\underset{\sim}{x}) \cdot (Y\underset{\sim}{x})$$

$$= \|Y\underset{\sim}{x}\|^2 \geq 0$$

$\therefore q$ is positive semidefinite.

ii) No, as if Y is the zero matrix, $q(\underset{\sim}{x})$ is 0 for non-zero $\underset{\sim}{x}$.

4c) If Y is invertible, $Y\underset{\sim}{x} = \underset{\sim}{0}$ only if $\underset{\sim}{x} = \underset{\sim}{0}$.

$\therefore$ for (bi), $q(\underset{\sim}{x})$ is only 0 if and only if $\underset{\sim}{x} = \underset{\sim}{0}$.

$\therefore$ q is positive definite.

5a) $$f(x) = x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{n}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n+1}}{n}, \quad x \in [-1, 1]$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$\frac{f^{(2023)}(0)}{2023!} x^{2023} = (-1)^{1011-1} \frac{x^{2(1011)+1}}{1011}$$

$$\frac{f^{(2023)}(0)}{2023!} \cancel{x^{2023}} = \frac{1}{1011} \cancel{x^{2023}}$$

$$f^{(2023)}(0) = \frac{2023!}{1011}$$

5b)

$$\lim_{x\to 0} \frac{\cos x - x^{-2}\ln(1+x^2)}{(\sin x)^4}$$

$$= \lim_{x\to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{4!} + O(x^6) - x^{-2}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + O(x^8)\right)}{\left(x + O(x^3)\right)^4}$$

$$= \lim_{x\to 0} \frac{\cancel{1} \cancel{-\frac{x^2}{2}} + \frac{\cancel{x^4}^{\,1}}{4!} + O(x^6) \cancel{-1} \cancel{+\frac{x^2}{2}} - \frac{\cancel{x^4}^{\,1}}{3} + O(x^6)}{\cancel{x^4}\left(1 + O(x^2)\right)^4}$$

$$= \lim_{x\to 0} \frac{\frac{1}{4!} - \frac{1}{3} + O(x^2)}{\left(1 + O(x^2)\right)^4}$$

$$= \frac{-\frac{7}{24}}{1}$$

$$= -\frac{7}{24}$$