

Math Module 6B Cheat Sheet

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1 Definitions

1.1 Riemann sum

For a function $f : [a, b] \rightarrow \mathbb{R}$, a Riemann sum is a sum:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i, \quad \Delta x_i = a_i - a_{i-1}$$

Points $a = a_0 < a_1 < \cdots < a_n = b$ form a **partition** of the interval $[a, b]$, and $x_i^* \in [a_{i-1}, a_i]$ are **sample points**. Further, let $\Delta x = \max\{\Delta x_i : i = 1, \dots, n\}$. Suppose the limit of the Riemann sums exists and is independent of our choice of partitions or sample points. Then we say that f is integrable on $[a, b]$ and the limit below is the **Riemann integral** of f from a to b .

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx$$

If $f(x) \geq 0$ on $[a, b]$, the Riemann sums approximate the area below the graph $y = f(x)$. The approximation improves as we take finer partitions of $[a, b]$.

1.2 Riemann integral

For a function $f : [a, b] \rightarrow \mathbb{R}$, a Riemann integral is the integral:

$$\lim_{\Delta x \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i = \int_a^b f(x) dx$$

1.3 Riemann sums in two variables

Now consider $f : R \rightarrow \mathbb{R}$, where R is a rectangle in \mathbb{R}^2 , i.e.

$$R = [a, b] \times [c, d] = \{(x, y) : x \in [a, b], y \in [c, d]\}$$

By partitioning both $[a, b]$ and $[c, d]$:

$$a = x_0 < x_1 < \dots < x_n = b$$

$$c = y_0 < y_1 < \dots < y_m = d$$

We get a partition of R into smaller subrectangles:

$$R_{j,k} = [x_{j-1}, x_j] \times [y_{k-1}, y_k]$$

Each of the subrectangles has an area of:

$$\Delta A_{j,k} = \Delta x_j \Delta y_k = (x_j - x_{j-1})(y_k - y_{k-1})$$

In each subrectangle, choose a sample point (x_j^*, y_k^*) and form the **Riemann sum**:

$$\sum_{j=1}^n \sum_{k=1}^m f(x_j^*, y_k^*) \Delta x_j \Delta y_k$$

Where:

- Area $A_{j,k} = \Delta x_j \Delta y_k$
- $f(x_j^*, y_k^*) \Delta x_j \Delta y_k$ is the volume of a cuboid with base $R_{j,k}$ and height $f(x_j^*, y_k^*)$.

1.4 Double integral

Let:

$$\Delta x = \max_{j=1, \dots, n} \Delta x_j, \quad \Delta y = \max_{k=1, \dots, m} \Delta y_k$$

If the Riemann sums have a limit that is independent of our choice of sample points and partition, we say that $f(x, y)$ is integrable on R and the limit below is the double integral of f over R .

$$\iint_R f(x, y) \, dx \, dy$$

1.5 Double integral and volumes

If $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}$, then the Riemann sums below approximate the volume below the graph $z = f(x, y)$ above the xy -plane:

$$\sum_{j=1}^n \sum_{k=1}^m f(x_j^*, y_k^*) \Delta x_j \Delta y_k$$

As $(\Delta x, \Delta y) \rightarrow (0, 0)$, these approximations converge to the actual volume, i.e.

$$\iint_R f(x, y) \, dx dy = \text{"volume below the graph } z = f(x, y) \text{ but above the } xy\text{-plane."}$$

1.5.1 Example

Let $f : [-2, 2] \times [-2, 2] \rightarrow \mathbb{R}$ be given by:

$$f(x, y) = x^2 + y^2$$

Consider the volume between the surface $z = f(x, y)$ and the xy -plane. Taking finer partitions, the Riemann sums converge to the volume below the graph, i.e.

$$\text{Volume} = \iint_{[-2, 2] \times [-2, 2]} f(x, y) \, dx dy$$

1.6 Double integrals over non-rectangular regions

For $\iint_A f(x, y) \, dx dy$ where A is not rectangular:

1. Take a rectangle R that contains A .
2. Let:

$$g(x, y) = \begin{cases} f(x, y) & \text{for } (x, y) \in A \\ 0 & \text{for } (x, y) \notin A \end{cases}$$

1. Let:

$$\iint_A f(x, y) \, dx dy = \iint_R g(x, y) \, dx dy$$

1.7 Fubini's theorem

If for some continuous g, h :

$$A = \{(x, y) : a \leq x \leq b, \quad g(x) \leq y \leq h(x)\}$$

Then for $f(x, y)$ continuous on A :

$$\iint_A f(x, y) \, dx dy = \int_{x=a}^b \left(\int_{y=g(x)}^{h(x)} f(x, y) \, dy \right) dx$$

If for some continuous g, h :

$$A = \{(x, y) : c \leq x \leq d, \quad g(y) \leq x \leq h(y)\}$$

Then for $f(x, y)$ continuous on A :

$$\iint_A f(x, y) \, dx dy = \int_{y=c}^d \left(\int_{x=g(y)}^{h(y)} f(x, y) \, dy \right) dx$$

1.7.1 Example

Let:

$$D = \{(x, y) : 0 \leq x \leq 1, \quad 0 \leq y \leq \sqrt{1-x}\}$$

Evaluate:

$$\iint_D x \, dx dy$$

$$\begin{aligned} \iint_D x \, dx dy &= \int_{x=0}^1 \left(\int_{y=0}^{\sqrt{1-x}} x \, dy \right) dx \\ &= \int_0^1 x \sqrt{1-x} \, dx \\ &= -\frac{2}{3}(1-x)^{\frac{3}{2}} \cdot x \Big|_0^1 + \frac{2}{3} \int_0^1 (1-x)^{\frac{3}{2}} \, dx \\ &= -\frac{2}{3} \cdot \frac{2}{5}(1-x)^{\frac{5}{2}} \Big|_0^1 \\ &= \frac{4}{15} \end{aligned}$$

Changing the order of integration:

$$\begin{aligned}
 \iint_D x \, dx \, dy &= \int_0^1 \left(\int_{x=0}^{1-y^2} x \, dx \right) dy \\
 &= \int_0^1 \left[\frac{x^2}{2} \right]_{x=0}^{1-y^2} dy \\
 &= \frac{1}{2} \int_0^1 (1-y^2)^2 dy \\
 &= \frac{1}{2} \int_0^1 (1-2y^2+y^4) dy \\
 &= \frac{1}{2} \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1 \\
 &= \frac{1}{2} \left(1 - \frac{2}{3} + \frac{1}{5} \right) \\
 &= \frac{1}{2} \cdot \frac{15 - 2 \cdot 5 + 1 \cdot 3}{15} \\
 &= \frac{4}{15}
 \end{aligned}$$

1.8 Triple integrals

For a three variable function $f(x, y, z)$ and a region $Q \subset \mathbb{R}^3$, the triple integral below is defined and can be calculated using similar principles:

$$\iiint_Q f(x, y, z) \, dx \, dy \, dz$$

1.8.1 Example 1

Evaluate:

$$\iiint_Q 6xy \, dx \, dy \, dz$$

Where Q is the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $2x + y + z = 4$.

$$\begin{aligned}
\iiint_Q 6xy \, dx dy dz &= \int_{x=0}^2 \left(\int_{y=0}^{4-2x} \left(\int_{z=0}^{4-2x-y} dz \right) dy \right) dx \\
&= \int_0^2 \left(\int_0^{4-2x} 6xy(4-2x-y) dy \right) dx \\
&= \int_0^2 \left(\int_0^{4-2x} (24xy - 12x^2 - y - 6xy^2) dy \right) dx \\
&= \int_0^2 [12xy^2 - 6x^2y^2 - 2xy^3]_{y=0}^{4-2x} dx \\
&= \int_0^2 (12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)^3) dx \\
&= \int_0^2 (12x(4-2x)^2 - 6x^2(4-2x)^2 - 2x(4-2x)(4-2x)^2) dx \\
&= \int_0^2 (12x(4-2x)^2 - 6x^2(4-2x)^2 - (8x-4x^2)(4-2x)^2) dx \\
&= \int_0^2 ((12x-6x^2-8x+4x^2)(4-2x)^2) dx \\
&= \int_0^2 ((4x-2x^2)(4-2x)^2) dx \\
&= \int_0^2 (x(4-2x)(4-2x)^2) dx \\
&= \int_0^2 (x(4-2x)^3) dx \\
&= \int_0^2 (8x(2-x)^3) dx \\
&= \left[8x \frac{-(2-x)^4}{4} \right]_0^2 + \int_0^2 \frac{-2(2-x)^4}{4} \cdot 8 dx \\
&= \left[-2 \cdot \frac{(2-x)^5}{5} \right]_0^2 \\
&= \left[0 - \left(-2 \cdot \frac{(2-0)^5}{5} \right) \right] \\
&= \frac{64}{5}
\end{aligned}$$

1.8.2 Example 2

Evaluate:

$$\iiint_Q dx dy dz$$

Where Q is given by:

$$x^2 + y^2 \leq z \leq 1, \quad x \geq 0$$

$$\begin{aligned} \iiint_Q dx dy dz &= \int_{y=-1}^1 \left(\int_{x=0}^{\sqrt{1-y^2}} \left(\int_{z=x^2+y^2}^1 dz \right) dx \right) dy \\ &= \int_{-1}^1 \left(\int_0^{\sqrt{1-y^2}} (1 - (x^2 + y^2)) dx \right) dy \\ &= \int_{-1}^1 \left[x - \frac{x^3}{3} - y^2 x \right]_{x=0}^{\sqrt{1-y^2}} dy \\ &= \int_{-1}^1 \left(\sqrt{1-y^2} - \frac{(1-y^2)^{\frac{3}{2}}}{3} - y^2 \sqrt{1-y^2} \right) dy \\ &= \int_{-1}^1 \left(\sqrt{1-y^2}(1-y^2) - \frac{(1-y^2)^{\frac{3}{2}}}{3} \right) dy \\ &= 2 \int_0^1 \frac{2}{3} (1-y^2)^{\frac{3}{2}} dy \end{aligned}$$

$$y = \sin \theta$$

$$\sqrt{1-y^2} = \cos \theta$$

$$dy = \cos \theta d\theta$$

$$y = -1 \quad \Leftrightarrow \quad \theta = -\frac{\pi}{2}$$

$$y = 1 \quad \Leftrightarrow \quad \theta = \frac{\pi}{2}$$

Substituting the above equations:

$$\begin{aligned}
2 \int_0^1 \frac{2}{3} (1 - y^2)^{\frac{3}{2}} dy &= \frac{2}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \theta d\theta \\
&= \frac{4}{3} \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta \\
&= \frac{4}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta \\
&= \frac{4}{3} \int_0^{\frac{\pi}{2}} \left(\frac{1}{4} + \cos 2\theta + \frac{\cos^2 2\theta}{4} \right) d\theta \\
&= \frac{4}{3} \left(\frac{\pi}{8} + \frac{1}{4} \int_0^{\frac{\pi}{2}} \cos^2 2\theta d\theta \right) \\
&= \frac{\pi}{6} + \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 4\theta) d\theta \\
&= \frac{\pi}{6} + \frac{1}{3} \cdot \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

1.9 Size of a region

Integrating the function **1**, always gives you the size of the region of integration:

$$\begin{aligned}
\int_a^b 1 dx &= b - a = \text{length of } [a, b] \\
\iint_R 1 dxdy &= \iint_R dA = \text{area of } R \\
\iiint_Q 1 dxdydz &= \iiint_Q dV = \text{volume of } Q
\end{aligned}$$

Likewise, integrating a constant C gives you C times the size of the region.

1.10 Polar coordinates

A point $(x, y) \in \mathbb{R}^2$ can be represented by its **polar coordinates** (r, θ) , where:

$$\begin{aligned}x &= r \cos \theta, & y &= r \sin \theta, & r &\geq 0, \theta \in [0, 2\pi) \\r^2 &= \sqrt{x^2 + y^2}\end{aligned}$$

1.10.1 Area element

In polar coordinates, the area element $dA = dx dy$ becomes $dA = r dr d\theta$.

1.10.2 Example

Calculate:

$$\iint_D (x^2 + y^2) dx dy$$

Where:

$$D = \{(x, y) : x^2 + y^2 \leq 4, y \geq 0\}$$

$$\begin{aligned}\iint_D (x^2 + y^2) dx dy &= \int_0^\pi \left(\int_0^2 r^3 dr \right) d\theta \\&= \int_0^\pi \left. \frac{r^4}{4} \right|_{r=0}^2 d\theta \\&= \int_0^\pi 4 d\theta \\&= 4\pi\end{aligned}$$

1.11 Cylindrical coordinates

A point $(x, y, z) \in \mathbb{R}^3$ can be represented by its cylindrical coordinates (r, θ, z) , where:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad r \geq 0, \theta \in [0, 2\pi)$$

1.11.1 Volume element

In cylindrical coordinates, the volume element $dV = dx dy dz$ becomes $dV = r dr d\theta dz$, where r is the scaling factor.

1.11.2 Example

Evaluate

$$\iiint_Q dx dy dz$$

Where the solid region Q , given by:

$$x^2 + y^2 \leq z \leq 1, x \geq 0$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$x^2 + y^2 = r^2$$

$$dV = dx dy dz = r dr d\theta dz$$

$$\begin{aligned} \iiint_Q dx dy dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^1 \left(\int_{r^2}^1 r dz \right) dr \right) d\theta \\ &= \pi \int_0^1 \left(\int_{r^2}^1 r dz \right) dr \\ &= \pi \int_0^1 (r - r^3) dr \\ &= \pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{4} \right) \\ &= \frac{\pi}{4} \end{aligned}$$

1.12 Spherical coordinates

A point $(x, y, z) \in \mathbb{R}^3$ can be represented by its spherical coordinates (ρ, φ, θ) , where:

$$x = \rho \sin \varphi \cos \theta, \quad y = \rho \sin \varphi \sin \theta, \quad z = \rho \cos \varphi, \quad \rho \geq 0, \varphi \in [0, \pi], \theta \in [0, 2\pi)$$

Keeping ρ fixed while varying φ and θ gives us points on a sphere with radius ρ .

$$\varphi = \text{"latitude"}$$

$$\theta = \text{"longitude"}$$

In spherical coordinates, the volume element $dV = dx dy dz$ becomes $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$, where $\rho^2 \sin \varphi$ is the scaling factor.

1.12.1 Example

Calculate the volume of a ball with radius R .

$$\begin{aligned} \text{Volume} &= \iiint_Q dV \\ &= \int_0^{2\pi} \left(\int_0^\pi \left(\int_0^R \rho^2 \sin \varphi d\rho \right) d\varphi \right) d\theta \\ &= 2\pi \int_0^\pi \left(\sin \varphi \int_0^R \rho^2 d\rho \right) d\varphi \\ &= 2\pi \frac{R^3}{3} \int_0^\pi \sin \varphi d\varphi \\ &= \frac{4\pi R^3}{3} \end{aligned}$$

2 Calculating a double integral

Consider a continuous $f : A \rightarrow \mathbb{R}$ where the region $A \subset \mathbb{R}^2$ has the form:

$$A = \{(x, y) : a \leq x \leq b, \quad g(x) \leq y \leq h(x)\}$$

Let's calculate:

$$\iint_A f(x, y) \, dx dy$$

For simplicity, suppose $f(x, y) \geq 0$ on A , so we can interpret $\iint_A f(x, y) \, dx dy$ as a volume:

$$A(x) = \int_{y=g(x)}^{h(x)} f(x, y) \, dy$$

$$\begin{aligned} \iint_A f(x, y) \, dx dy &= \text{Volume} \\ &= \int_{x=a}^b dV \\ &= \int_a^b A(x) \, dx \\ &= \int_a^b \left(\int_{y=g(x)}^{h(x)} f(x, y) \, dy \right) dx \end{aligned}$$

Our usual approach leads us to the formula:

$$\iint_A f(x, y) \, dx dy = \int_{x=a}^b \left(\int_{y=g(x)}^{h(x)} f(x, y) \, dy \right) dx$$

The assumption $f(x, y) \geq 0$ is not necessary for the above to hold, it just made the illustration easier.

If the roles of x and y are reversed, we get an analogous result.

3 Double integrals heuristically

$$\begin{aligned}\iint_R dV &= \iint_R f(x, y) dA \\ &= \iint_R f(x, y) dx dy\end{aligned}$$

$$\text{Volume element} = dV = f(x, y) dA$$

$$\text{Area element} = dA = dx dy$$

4 Triple integrals heuristically

Let density = $\rho(x, y, z)$

$$\begin{aligned}\text{mass of } Q &= \iiint_Q dm \\ &= \iiint_Q \rho(x, y, z) dV \\ &= \iiint_Q \rho(x, y, z) dx dy dz\end{aligned}$$

$$\text{Mass element} = dm = \rho(x, y, z) dV$$

$$\text{Volume element} = dV = dx dy dz$$