Math Module 2B Tutorial

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Contents

1 Question 1

1.1 (a)

$$\begin{aligned}
\frac{d}{dx}(af(x)+bg(x)) &= \lim_{h\to 0}\left[a\left(\frac{f(a+h)-f(a)}{h}\right)+b\left(\frac{g(a+h)-g(a)}{h}\right)\right] \\
&= \lim_{h\to 0} a\left(\frac{f(a+h)-f(a)}{h}\right)+\lim_{h\to 0} b\left(\frac{g(a+h)-g(a)}{h}\right) \\
&= a\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)+b\lim_{h\to 0}\left(\frac{g(a+h)-g(a)}{h}\right) \\
&= af'(x)+bg'(x)\ \textbf{(Proven)}
\end{aligned}$$

1.2 (b)

1.2.1 (i)

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h} \\
&= \lim_{h\to 0}\frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h} \\
&= \lim_{h\to 0}\frac{-(g(a+h)-g(a))}{h\cdot g(a+h)g(a)} \\
&= \lim_{h\to 0}\left(\frac{-(g(a+h)-g(a))}{h}\cdot\frac{1}{g(a+h)g(a)}\right) \\
&= \lim_{h\to 0}\frac{-(g(a+h)-g(a))}{h}\cdot\lim_{h\to 0}\frac{1}{g(a+h)g(a)} \\
&= -g'(x)\cdot\frac{1}{(g(x))^2} \\
&= \frac{-g'(x)}{(g(x))^2}\ \textbf{(Proven)}
\end{aligned}$$

1.2.2 (ii)

$$
\begin{aligned}
\frac{d}{dx}\frac{f(x)}{g(x)} &= f(x) \cdot \frac{1}{g(x)} \\
&= f'(x)\left(\frac{1}{g(x)}\right) + f(x)\left(\frac{-g'(x)}{(g(x))^2}\right) \\
&= \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x)}{(g(x))^2} + \frac{-f(x)g'(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} \ \textbf{(Proven)}
\end{aligned}
$$

2 Question 2

2.1 (a)

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{f(h) - 0}{h} \\
&= \lim_{h \to 0} \frac{f(h)}{h} \\
&= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} \\
&= \lim_{h \to 0} h \sin \frac{1}{x}
\end{aligned}
$$

By squeeze theorem:

$$
0 \leq \left| h \sin \frac{1}{h} - 0 \right| \leq |h|
$$

$$
\left| h \sin \frac{1}{h} \right| < |h|
$$

Hence:

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} h \sin \frac{1}{x} \\
&= 0
\end{aligned}
$$

2.2 (b)

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&= \lim_{h \to 0} \frac{f(h)}{h}
\end{aligned}
$$

Using the lemma:

$$
\begin{aligned}
\lim_{h \to 0} \frac{f(h)}{h} &= \lim_{n \to \infty} \frac{f(a_n)}{a_n} \\
&= \frac{a}{a} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\lim_{h \to 0} \frac{f(h)}{h} &= \lim_{n \to \infty} \frac{f(b_n)}{b_n} \\
&= \frac{0}{a} \\
&= 0 \neq 1
\end{aligned}
$$

Since:

$$
\lim_{n \to \infty} \frac{f(a_n)}{a_n} \neq \lim_{n \to \infty} \frac{f(b_n)}{b_n}
$$

$\lim_{h \to 0} \frac{f(h)}{h}$ doesn't exist and hence $f'(0)$ doesn't exist.

2.3 (c)

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{n\to\infty} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{n\to\infty} \frac{f(h)}{h} \end{aligned}$$

Using the lemma:

$$\begin{aligned} \lim_{h\to 0} \frac{f(h)}{h} &= \lim_{n\to\infty} \frac{f(a_n)}{a_n} \\ &= \frac{a^2}{a} \\ &= a \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0} \frac{f(h)}{h} &= \lim_{n\to\infty} \frac{f(b_n)}{b_n} \\ &= \frac{0}{b_n} \\ &= 0 \end{aligned}$$

When the limit point is 0, $a = 0$, hence:

$$\lim_{n\to\infty} \frac{f(a_n)}{a_n} = \lim_{n\to\infty} \frac{f(b_n)}{b_n} = 0$$

Thus, $f'(0) = 0$.

3 Question 3

3.1 (a)

Let $p = -n$:

$$x^p = \frac{1}{x^n}$$

Using quotient rule:

$$\begin{aligned}
\frac{d}{dx}x^p &= \frac{0(x) - 1(nx^{n-1})}{x^{n+2}} \\
&= \frac{-nx^{n-1}}{x^{2n}} \\
&= -nx^{n-1-2n} \\
&= -nx^{n-1-2n} \\
&= -nx^{-n-1} \\
&= px^{p-1} \quad (\because p = -n) \textbf{ (Shown)}
\end{aligned}$$

3.2 (b)

Using chain rule:

$$\begin{aligned}
\frac{d}{dx}x^{\frac{p}{q}} &= \frac{d}{dx}\left(x^{\frac{1}{q}}\right)^p \\
&= \frac{1}{q}x^{\frac{1}{q}-1} \cdot p\left(x^{\frac{1}{q}}\right)^{p-1} \\
&= \frac{p}{q}x^{\frac{1}{q}-1} \cdot x^{\frac{p-1}{q}} \\
&= \frac{p}{q}x^{\frac{1+p-1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-1} \textbf{ (Shown)}
\end{aligned}$$

4 Question 4

4.1 (a)

$$
\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\left[\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\cos x\frac{\cos h-1}{h}\cdot\frac{\cos h+1}{\cos h+1}-\sin x\frac{\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\cos x\frac{\cos^2 h-1}{h(\cos h+1)}-\sin x\frac{\sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\cos x\frac{-\sin^2 h}{h(\cos h+1)}-\sin x\frac{\sin h}{h}\right] \quad (\because \sin^2+\cos^2=1) \\
&= \lim_{h\to 0}\left[\cos x\cdot\frac{-\sin h}{h}\cdot\frac{\sin h}{\cos h+1}-\sin x\frac{\sin h}{h}\right] \\
&= \lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\left[\frac{-\sin h}{h}\frac{\sin h}{\cos h+1}\right]-\lim_{h\to 0}\frac{\sin x\sin h}{h} \\
&= 1\cdot\left(-1\cdot\frac{0}{2}\right)-1\sin x \quad \left(\because \lim_{x\to 0}\frac{\sin x}{x}=1\right) \\
&= 0-\sin x \\
&= -\sin x \ \textbf{(Proven)}
\end{aligned}
$$

4.2 (b)

Using the quotient rule:

$$
\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\frac{\sin x}{\cos x} \\
&= \frac{\cos x\cos x-\sin x\cdot -\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x+\sin^2}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \quad (\because \sin^2+\cos^2=1) \ \textbf{(Proven)}
\end{aligned}
$$

4.3 (c)

Let $y = \arccos x$:

$$y = \arccos x \quad \Leftrightarrow \quad x = \cos y, \quad y \in [0, \pi]$$

Hence:

$$x = \cos y$$

Differentiating both sides with respect to x:

$$1 = -\sin y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}} \quad (\because \sin^2 x + \cos^2 x = 1 \text{ and } y \in [0, \pi])$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}} \quad (\because x = \cos y)$$

Hence,

$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1 - x^2}} \textbf{ (Proven)}$$

5 Question 5

Writing down the rate of change as a mathematical expression:

$$\frac{dR_1}{dt} = 0.3$$

$$\frac{dR_2}{dt} = 0.2$$

Getting the value of R when $R_1 = 80\,\Omega$ and $R_2 = 100\,\Omega$:

$$\frac{1}{R} = \frac{1}{80} + \frac{1}{100}$$

$$R = \left(\frac{1}{80} + \frac{1}{100}\right)^{-1}$$

$$R = \frac{400}{9}\ \Omega$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Differentiating both sides with respect to t:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt}$$

$$\frac{1}{R^2}\frac{dR}{dt} = \frac{1}{R_1^2}\frac{dR_1}{dt} + \frac{1}{R_2^2}\frac{dR_2}{dt}$$

Substituting the values into the equation:

$$\frac{1}{\left(\frac{400}{9}\right)^2}\frac{dR}{dt} = \frac{1}{80^2}\cdot 0.3 + \frac{1}{100^2}\cdot 0.2$$

$$\frac{80}{160000}\frac{dR}{dt} = 0.000066875$$

$$\frac{dR}{dt} = 0.1320987654$$

$$\frac{dR}{dt} \approx 0.13\,\Omega/\text{s}$$

6 Question 6

Let f be the distance from the alien to the light and let s be the shadow's length.

The ratio of the smaller triangle created by the alien moving towards the light source to the bigger triangle created by the light source on the wall is $f : 12$.

Hence, getting the shadow's length in terms of the distance from the alien to the light source using similar triangles, i.e. s in terms of f:

$$\frac{f}{12} = \frac{2}{s}$$

$$\frac{s \cdot f}{12} = 2$$

$$s = \frac{24}{f}$$

Differentiating both sides with respect to t:

$$\frac{ds}{dt} = 24 \cdot -\frac{1}{f^2} \cdot \frac{df}{dt}$$

$$\frac{ds}{dt} = -\frac{24}{f^2}\frac{df}{dt}$$

Substituting $\frac{df}{dt} = 1.6$ and $f = 12 - 4 = 8$:

$$\frac{ds}{dt} = -\frac{24}{8^2} \cdot 1.6$$

$$\frac{ds}{dt} = -0.6$$

Hence, the length of his shadow on the building is decreasing at $0.6\,\mathrm{m\,s^{-1}}$.

7 Question 7

Finding the value of $\frac{d\theta}{dt}$:

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{8\pi}{60} \\ &= \frac{2\pi}{15}\end{aligned}$$

Finding the expression for x:

$$\begin{aligned}\tan\theta &= \frac{O}{A} \\ \tan\theta &= \frac{x}{3} \\ x &= 3\tan\theta\end{aligned}$$

Differentiating the expression for x with respect to θ:

$$\frac{dx}{d\theta} = 3\sec^2\theta$$

$$\frac{dx}{dt} = \frac{d\theta}{dt}\cdot\frac{dx}{d\theta}$$

Finding the value of θ when $x = 1$:

$$\begin{aligned}1 &= 3\tan\theta \\ \frac{1}{3} &= \tan\theta \\ \theta &= \arctan\frac{1}{3}\end{aligned}$$

Finding $\frac{dx}{dt}$:

$$\begin{aligned}\frac{dx}{dt} &= \frac{d\theta}{dt}\cdot\frac{dx}{d\theta} \\ &= \frac{2\pi}{15}\cdot\frac{3}{\cos^2\left(\arctan\frac{1}{3}\right)} \\ &= 1.396\,263\,402\,\mathrm{km\,s^{-1}} \\ &= 5026.548\,246\,\mathrm{km/h} \\ &\approx 5027\,\mathrm{km/h}\end{aligned}$$

8 Question 8

Let x be the distance PX. This means that $QX = 10 - x$:

Getting the length of the cable in water (l_w):

$$l_w = \sqrt{5^2 + x^2}$$

The length of the cable in land ($l_l = QX$)

$$l_l = 10 - x$$

The total cost would be (T):

$$\begin{aligned} T &= 5000\sqrt{5^2 + x^2} + 3000(10 - x) \\ &= 5000\sqrt{25 + x^2} + 30000 - 3000x \end{aligned}$$

Differentiating total cost with respect to x:

$$\begin{aligned} \frac{dT}{dx} &= 5000(2x) \cdot \frac{1}{2}(25 + x^2)^{-\frac{1}{2}} - 3000 \\ &= \frac{5000x}{\sqrt{25 + x^2}} - 3000 \end{aligned}$$

To find the stationary points, $\frac{dT}{dx} = 0$:

$$\begin{aligned} 3000 &= \frac{5000x}{\sqrt{25 + x^2}} \\ 3 &= \frac{5x}{\sqrt{25 + x^2}} \\ 9 &= \frac{25x^2}{25 + x^2} \\ 225 + 9x^2 &= 25x^2 \\ 16x^2 &= 225 \\ x^2 &= \frac{225}{16} \\ x &= 3.75 \quad (\because x > 0) \end{aligned}$$

When $x = 3.75^+$, $\frac{dT}{dx} > 0$ and when $x = 3.75^-$, $\frac{dT}{dx} < 0$, hence $x = 3.75$ is a minimum point. Thus, the point X should be $3.75\,\mathrm{m}$ from P.

9 Question 9

Let O be the centre of the swimming pool. Let the angle between OB and OC be θ.

Let the distance he runs be D_r:

$$\begin{aligned} D_r &= \frac{40}{2} \cdot (\pi - \theta) \\ &= 20(\pi - \theta) \end{aligned}$$

Differentiating D_r with respect to t:

$$\begin{aligned} \frac{dD_r}{dt} &= -20 \cdot \frac{d\theta}{dt} \\ \frac{d\theta}{dt} &= -20 \cdot \left(\frac{dD_r}{dt}\right)^{-1} \end{aligned}$$

Let the distance he swims be D_s. Using the cosine rule:

$$\begin{aligned} D_s &= \sqrt{OB^2 + OC^2 - 2(OB)(OC)\cos\theta} \\ &= \sqrt{20^2 + 20^2 - 2(20)(20)\cos\theta} \\ &= \sqrt{800 - 800\cos\theta} \\ &= \sqrt{400(2 - 2\cos\theta)} \\ &= 20\sqrt{2 - 2\cos\theta} \\ &= 20\sqrt{2(1 - \cos\theta)} \\ &= 20\sqrt{2}\sqrt{1 - \cos\theta} \end{aligned}$$

Let the speed of the man swimming be the constant s. Since the man runs twice as fast as he swims, his running speed will be $2s$.

Dividing the distance by the speed to get the time for him running (t_r):

$$\begin{aligned} t_r &= \frac{20(\pi - \theta)}{2s} \\ &= \frac{10(\pi - \theta)}{s} \end{aligned}$$

Dividing the distance by the speed to get the time for him swimming (t_s):

$$t_s = \frac{20\sqrt{2}\sqrt{1-\cos\theta}}{s}$$

The total time taken (t) by the man would be:

$$\begin{aligned}
t &= t_r + t_s \\
&= \frac{10(\pi-\theta)}{s} + \frac{20\sqrt{2}\sqrt{1-\cos\theta}}{s} \\
&= \frac{10(\pi-\theta+2\sqrt{2}\sqrt{1-\cos\theta})}{s} \\
&= \frac{10}{s}\left(\pi-\theta+2\sqrt{2}\sqrt{1-\cos\theta}\right)
\end{aligned}$$

Differentiating t with respect to θ:

$$\begin{aligned}
\frac{dt}{d\theta} &= \frac{10}{s}\left(-1+2\sqrt{2}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{1-\cos\theta}}\cdot\sin\theta\right) \\
&= \frac{10}{s}\left(\frac{\sqrt{2}\sin\theta}{\sqrt{1-\cos\theta}}-1\right)
\end{aligned}$$

To find the stationary points, $\frac{dt}{d\theta}=0$:

$$\begin{aligned}
0 &= \frac{10}{s}\left(\frac{\sqrt{2}\sin\theta}{\sqrt{1-\cos\theta}}-1\right) \\
0 &= \frac{\sqrt{2}\sin\theta}{\sqrt{1-\cos\theta}}-1 \\
1 &= \frac{\sqrt{2}\sin\theta}{\sqrt{1-\cos\theta}} \\
\sqrt{1-\cos\theta} &= \sqrt{2}\sin\theta \\
1-\cos\theta &= 2\sin^2\theta \\
1-\cos\theta &= 2(1-\cos^2\theta) \\
1-\cos\theta &= 2-2\cos^2\theta \\
2\cos^2\theta-\cos\theta-1 &= 0 \\
(\cos\theta-1)(2\cos\theta+1) &= 0
\end{aligned}$$

Hence:

$$
\begin{aligned}
\cos\theta &= 1 && \text{or} && 2\cos\theta = -1 \\
\theta &= 0 && \text{or} && \cos\theta = -\frac{1}{2} \\
\theta &= 0 && \text{or} && \cos\theta = -\frac{1}{2} \\
\theta &= 0 && \text{or} && \theta = \frac{2\pi}{3}
\end{aligned}
$$

When $\theta = 0^+$, $\frac{dt}{d\theta} > 0$ and when $\theta = 0^-$, $\frac{dt}{d\theta} < 0$. Hence, $\theta = 0$ is a minimum point.

When $\theta = \left(\frac{2\pi}{3}\right)^+$, $\frac{dt}{d\theta} < 0$ and when $\theta = \left(\frac{2\pi}{3}\right)^-$, $\frac{dt}{d\theta} > 0$. Hence, $\theta = \frac{2\pi}{3}$ is a maximum point.

Since we want to minimise the total time taken, $\theta = 0$ and hence, the point C should be chosen to be at point B.