1) $A^2 - 3A + I = \underset{\sim}{0}_{n\times n}$

$$A(3I - A) = I$$

$\therefore$ A is invertible as $A^{-1} = 3I - A$ (shown)

2a) Since A is invertible, A^{-1} exists.

$$A^{-1}A = A^0 = I \text{ (shown)}$$

$$(A^{-1})^{-1} = A^{-1(-1)} = A^1 = A \text{ (shown)}$$

b) $A^n (A^{-1})^n = A^n A^{-n} = A^0 = I$ (shown)

$$\begin{aligned}(A^n)^{-1} &= (AA^{n-1})^{-1} \\ &= (A^{n-1})^{-1} A^{-1} \\ &= (A^{-1})^{n-1} A^{-1} \quad \because (AB)^{-1} = B^{-1}A^{-1} \\ &= (A^{-1})^n \text{ (shown)}\end{aligned}$$

2c) $kA \cdot \frac{1}{k}A^{-1} = A^{0} = I$ (shown)

Since $\frac{1}{k}A^{-1}$ is the inverse of kA,

$$(kA)^{-1} = \frac{1}{k}A^{-1} \text{ (shown)}$$

3a) $$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$$

$$7A = \frac{1}{6-7}\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix}$$

$$7A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

$$A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$$

b) $$(I+2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$I + 2A = \frac{1}{6+1}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$2A = \frac{1}{7}\left(\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\right)$$

$$A = \frac{1}{14}\begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix}$$

4a) $$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 5 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & -5 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 & -5 \\ 0 & 0 & 1 & 1 & 1 & -5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -5 \\ 1 & 1 & -5 \end{bmatrix}$$

b) $$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 15 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 5 & 3 & 1 & 1 & 0 \\ 0 & 10 & 6 & -1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 5 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & -2 & 1 \end{array}\right]$$

A is singular.

5a) $\begin{vmatrix} 3 & 5 \\ -2 & 4 \end{vmatrix} = 3 \times 4 - 5 \times -2 = 22$

b) $\left|\begin{array}{ccc|cc} -2 & 1 & 4 & -2 & 1 \\ 3 & 5 & -7 & 3 & 5 \\ 1 & 6 & 2 & 1 & 6 \end{array}\right. = -20 - 7 + 72 - 20 - 84 - 6$

$= -65$

6a) $\begin{vmatrix} 3 & -17 & -3 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -30$

b) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 1 & 6 \end{vmatrix} = -12$

c) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$

7a) $\begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} = - \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -5$

b) $\begin{vmatrix} a & b & c \\ d & e & f \\ a+g & b+h & c+i \end{vmatrix} = -5$

c) $\begin{vmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ 3g & 3h & 3i \end{vmatrix} = 2(-1)(3) \times -5 = 30$

d) $\left|\begin{array}{ccc|cc} a & d & g & a & d \\ b & e & h & b & e \\ c & f & i & c & f \end{array}\right| = aei + dhc + gbf$

$- ceg - fha - ibd$

$\left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right| = aei + bfg + cdh$

$- gec - hfa - idb$

$= -5$

$= \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

8a) $$\begin{vmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -2$$

b) $$\begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ -2 & -7 & 0 & -4 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & -1 & 2 & 6 & 8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 1 & 5 & 3 \\ 0 & 1 & -2 & -6 & 8 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

9 i) $\begin{vmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = 2\left|\begin{array}{ccc|cc} 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 3 & 1 & 2 \end{array}\right| - \left|\begin{array}{ccc|cc} 1 & 3 & 3 & 1 & 3 \\ 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 3 & 1 & 2 \end{array}\right|$

$= 2(4 - 2 - 6) - (6 + 12 - 6 - 18)$

$= -2$

ii) $\begin{vmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = -2\left|\begin{array}{ccc|cc} 2 & 3 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 \end{array}\right|$

$+$

$+2\left|\begin{array}{ccc|cc} 2 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 \end{array}\right|$

$= -2(6 + 6 - 4 - 9) + 2(3 - 2 - 3)$

$= -2$

10) $\left|\begin{array}{ccc|cc} \cos\theta & \sin\theta & 0 & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta & 0 & -\sin\theta & \cos\theta \\ 0 & 0 & 1 & 0 & 0 \end{array}\right.$

$= \cos^2\theta + \sin^2\theta$

$= 1$

Since $\det A \neq 0$, A is invertible for all values of θ.

11a) $\left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_1 & x_2 \\ x_1^2 & x_2^2 & x_3^2 & x_1^2 & x_2^2 \end{array}\right. = x_2x_3^2 + x_3x_1^2 + x_1x_2^2 - x_1^2x_2 - x_2^2x_3 - x_3^2x_1$

$= x_3^2(x_2 - x_1) + x_1x_2(x_2 - x_1) + x_3(x_1^2 - x_2^2)$

$= (x_3^2 + x_1x_2)(x_2 - x_1) - x_3(x_2 + x_1)(x_2 - x_1)$

$= (x_3^2 + x_1x_2 - x_3(x_2 + x_1))(x_2 - x_1)$

$= (x_3^2 + x_1x_2 - x_3x_2 - x_3x_1)(x_2 - x_1)$

$= (x_3(x_3 - x_1) - x_2(x_3 - x_1))(x_2 - x_1)$

$= (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$ (shown)

11b) Let $B = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

For the equation above to have a unique solution, B must be invertible.

$$B \text{ is invertible} \Leftrightarrow \det B \neq 0$$

$$\det(B^T) = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$$

$$= \det B$$

$$\det B = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$$

$\therefore$ All x_1, x_2 and x_3 are distinct.

(12) $\det(AA^{-1}) = \det A \det(A^{-1})$

$$\det(I) = \det A \det(A^{-1})$$

$$1 = \det A \det(A^{-1})$$

$$\det(A^{-1}) = \frac{1}{\det A} \text{ (shown)}$$

(13) Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

$$\det(A+B) = \det\left(\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\right)$$

$$= \begin{vmatrix} 8 & 0 \\ 0 & 6 \end{vmatrix}$$

$$= 48$$

$$\det A + \det B = \begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix}$$

$$= 12 + 10$$

$$= 22 \neq 48$$

$$\therefore \det(A+B) \neq \det A + \det B$$

14) $\det(kA) = \det\left(\begin{bmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{bmatrix} A\right)$

$$= \det\begin{bmatrix} k & & 0 \\ & \ddots & \\ 0 & & k \end{bmatrix} \det A$$

$$= k^n \det A$$

15) $f(x,y) = (xy + x - y, \cos x + \sin y, \sin x + \cos y)$

$$f'(x,y) = \begin{bmatrix} y+1 & x-1 \\ -\sin x & \cos y \\ \cos x & -\sin y \end{bmatrix}$$

16) Since the partial derivatives $f'(x,y)$ are continuous everywhere due to being elementary functions, the linearisation is:

$$L(\underset{\sim}{x}) = f(0,0) + f'(0,0)\,(\underset{\sim}{x} - (0,0))$$

$$L(x,y) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & y \end{bmatrix}$$

$$L(x,y) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x-y \\ y \\ x \end{bmatrix}$$

$$L(x,y) = (x-y,\ y+1,\ x+1)$$

$$G_L = L(x,y) = (x-y,\ y+1,\ x+1)$$

17) $(x,y,z) = g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

$$g'(r,\theta,z) = \begin{bmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$dx\,dy\,dz = \left| \left| \begin{array}{ccc|cc} \cos\theta & -r\sin\theta & 0 & \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta & 0 & \sin\theta & r\cos\theta \\ 0 & 0 & 1 & 0 & 0 \end{array} \right. \right. dr\,d\theta\,dz$$

$$= \left| r\cos^2\theta + r\sin^2\theta \right| dr\,d\theta\,dz$$

$$= |r|\,dr\,d\theta\,dz$$

$$= r\,dr\,d\theta\,dz$$

17) $(x,y,z) = g(\rho,\theta,\varphi) = (\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)$

$$g'(\rho,\theta,\varphi) = \begin{bmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix}$$

$$dx\,dy\,dz = \left| \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \right| d\rho\,d\theta\,d\varphi$$

$$= \left| -\rho^2\cos^2\theta\sin^3\varphi - \rho^2\sin^2\theta\sin\varphi\cos^2\varphi - \rho^2\cos^2\theta\sin\varphi\cos^2\varphi - \rho^2\sin^2\theta\sin^3\varphi \right| d\rho\,d\theta\,d\varphi$$

$$= \left| -(\rho^2\sin^3\varphi) - (\rho^2\sin\varphi\cos^2\varphi) \right| d\rho\,d\theta\,d\varphi$$

$$= \left| -(\rho^2\sin\varphi(\sin^2\varphi + \cos^2\varphi) \right| d\rho\,d\theta\,d\varphi$$

$$= \left| -\rho^2\sin\varphi \right| d\rho\,d\theta\,d\varphi$$

$$= \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

18) $(x, y) = g(u, v) = (u^2, v)$

$$g'(u,v) = \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$$

$$= \begin{bmatrix} 2u & 0 \\ 0 & 1 \end{bmatrix}$$

$$dx\,dy = 1 \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} 1\,du\,dv = 2u\,du\,dv$$

$$\iint_D y e^{\sqrt{x}}\,dx\,dy = \int_{u=0}^{1} \int_{v=0}^{u} v e^{u} (2u)\,du\,dv$$

$$= 2\int_0^1 u e^u \left[\frac{v^2}{2}\right]_0^u du\,dv$$

$$= \int_0^1 u^3 e^u\,du$$

$$= 6 - 2e$$

19) $(x,y,z) = g(\rho,\theta,\varphi) = (a\rho\cos\theta\sin\varphi, b\rho\sin\theta\sin\varphi, c\rho\cos\varphi)$

$$g'(\rho,\theta,\varphi) = \begin{bmatrix} x_\rho & x_\theta & x_\varphi \\ y_\rho & y_\theta & y_\varphi \\ z_\rho & z_\theta & z_\varphi \end{bmatrix}$$

$$= \begin{bmatrix} a\cos\theta\sin\varphi & -a\rho\sin\theta\sin\varphi & a\rho\cos\theta\cos\varphi \\ b\sin\theta\sin\varphi & b\rho\cos\theta\sin\varphi & b\rho\sin\theta\cos\varphi \\ c\cos\varphi & 0 & -c\rho\sin\varphi \end{bmatrix}$$

$$dx\,dy\,dz = \left| \begin{vmatrix} a\cos\theta\sin\varphi & -a\rho\sin\theta\sin\varphi & a\rho\cos\theta\cos\varphi \\ b\sin\theta\sin\varphi & b\rho\cos\theta\sin\varphi & b\rho\sin\theta\cos\varphi \\ c\cos\varphi & 0 & -c\rho\sin\varphi \end{vmatrix} \right| d\rho\, d\theta\, d\varphi$$

$$= \left| -abc\rho^2\cos^2\theta\sin^3\varphi - abc\rho^2\sin^2\theta\sin\varphi\cos^2\varphi - abc\rho^2\cos^2\theta\sin\varphi\cos^2\varphi - abc\rho^2\sin^2\theta\sin^3\varphi \right| d\rho\, d\theta\, d\varphi$$

$$= abc\rho^2 \left| -(\sin^3\varphi) - (\sin\varphi\cos^2\varphi) \right| d\rho\, d\theta\, d\varphi$$

$$= abc\rho^2 \left| -\sin\varphi(\sin^2\varphi + \cos^2\varphi) \right| d\rho\, d\theta\, d\varphi$$

$$= abc\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$$

1a) $$\iiint dx\,dy\,dz = \int_{\rho=0}^{1}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} abc\,\rho^2 \sin\varphi \,d\rho\, d\theta\, d\varphi$$

$$= abc \int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \sin\varphi \left[\frac{\rho^3}{3}\right]_0^1 d\theta\, d\varphi$$

$$= \frac{abc}{3}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{\pi} \sin\varphi \, d\theta\, d\varphi$$

$$= \frac{abc}{3}\int_{0}^{2\pi} \left[-\cos\varphi\right]_0^{\pi} d\theta$$

$$= \frac{2abc}{3}\int_0^{2\pi} 1\, d\theta$$

$$= \frac{4\pi abc}{3}$$