1(a) Since A is symmetric, Q exists.

$$\det(\lambda I - 3A) = 0$$

$$\left|\begin{matrix} \lambda-1 & -1 & -1 & \lambda-1 & -1 \\ -1 & \lambda-1 & -1 & -1 & \lambda-1 \\ -1 & -1 & \lambda-1 & -1 & -1 \end{matrix}\right. = 0$$

$$(\lambda-1)^3 - 1 - 1 - (\lambda-1) - (\lambda-1) - (\lambda-1) = 0$$

$$(\lambda^2 - 2\lambda + 1)(\lambda - 1) - 2 - 3(\lambda - 1) = 0$$

$$\lambda^3 - \lambda^2 - 2\lambda^2 \cancel{+2\lambda + \lambda - 1 - 2 - 3\lambda + 3} = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

$$\therefore \lambda = 3 \text{ or } \lambda = 0$$

For $\lambda = 3$,

$$\begin{bmatrix} \lambda-1 & -1 & -1 & 0 \\ -1 & \lambda-1 & -1 & 0 \\ -1 & -1 & \lambda-1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1a) $\quad x - z = 0 \qquad y - z = 0$

$\qquad x = z \qquad y = z$

$\therefore$ the eigenvectors of $3A$ when $\lambda = 3$ are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, s \in \mathbb{R}\setminus\{0\}$$

For $\lambda = 0$,

$$\left[\begin{array}{ccc|c} \lambda-1 & -1 & -1 & 0 \\ -1 & \lambda-1 & -1 & 0 \\ -1 & -1 & \lambda-1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + y + z = 0$$

$$x = -y - z$$

$\therefore$ the eigenvectors of $3A$ when $\lambda = 0$ are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}, s = t \neq 0$$

(a) Finding the orthornormal basis:

$$\underset{\sim}{x}_1 = \frac{1}{\sqrt{3}}(1,1,1)$$

$$\underset{\sim}{x}_2 = \frac{1}{\sqrt{2}}(-1,1,0)$$

Let $\underset{\sim}{y} = (-1,0,1)$

$$\underset{\sim}{x}_3' = \underset{\sim}{y} - \mathrm{proj}_{\underset{\sim}{x}_2} \underset{\sim}{y}$$

$$= (-1,0,1) - \frac{1}{\sqrt{2}}(-1,1,0)\cdot(-1,0,1)\cdot\frac{1}{\sqrt{2}}(-1,1,0)$$

$$= (-1,0,1) - \frac{1}{2}(1)(-1,1,0)$$

$$= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$\underset{\sim}{x}_3 = \frac{\underset{\sim}{x}_3'}{\|\underset{\sim}{x}_3'\|}$$

$$= \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \frac{1}{\sqrt{\frac{3}{2}}}\left(\frac{1}{2}\right)(-1,-1,2)$$

$$= \sqrt{\frac{2}{3}}\sqrt{\frac{1}{4}}(-1,-1,2)$$

$$= \frac{1}{\sqrt{6}}(-1,-1,2)$$

1a) $\therefore$ $Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$

$$D = Q^T A Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b) Let $B = \left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(-1,-1,2) \right\}$

T is the change of basis from the standard basis to the basis B and orthorgonally projects the vector onto the first coordinate axis, which is parallel to $(1,1,1)$ in the standard basis.

2a) $\underset{\sim}{x}$ and $\underset{\sim}{y}$ being orthorgonal means

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 0$$

b) Let $B = \{ \underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n \}$

Since B is linearly independent,

$$\underset{\sim}{x}_j \neq \underset{\sim}{0} \text{ for all } j = 1, 2, \dots, n$$

By the properties of norm,

$$\| \underset{\sim}{x}_j \| > 0 \text{ for all } j = 1, 2, \dots, n \quad - (1)$$

Since B is an orthorgonal basis,

$$\langle \underset{\sim}{x}_j, \underset{\sim}{x}_k \rangle = 0 \text{ for } j \neq k$$

Since B spans V,

$$\underset{\sim}{x} = \sum_{j=1}^{n} c_j \underset{\sim}{x}_j \text{ for some } c_1, c_2, \dots, c_n \in \mathbb{R}$$

2b) $\langle \underset{\sim}{x}, \underset{\sim}{x}_k \rangle = \left\langle \sum_{j=1}^{n} c_j \underset{\sim}{x}_j, \underset{\sim}{x}_k \right\rangle$

$$= c_j \langle \underset{\sim}{x}_k, \underset{\sim}{x}_k \rangle$$

$$= c_j \|\underset{\sim}{x}_k\|^2, k = 1, 2, \dots, n$$

Since (1),

$$\langle \underset{\sim}{x}, \underset{\sim}{x}_k \rangle = c_j \|\underset{\sim}{x}_k\|^2$$

$$c_j = \frac{\langle \underset{\sim}{x}, \underset{\sim}{x}_k \rangle}{\|\underset{\sim}{x}_k\|^2}$$

$\therefore \underset{\sim}{x} = \sum_{j=1}^{n} \frac{\langle \underset{\sim}{x}, \underset{\sim}{x}_j \rangle}{\|\underset{\sim}{x}_j\|^2}$ (Proven)

3a) It means that there exists constants $C, \delta > 0$, such that:

$$|f(x) - x| \leq C|x^3| \text{ for } x \in (-\delta, \delta)$$

b) $$\lim_{x \to 0} \frac{\cos(x^2) - e^{x^4} + \frac{3}{2}x^4}{\sin(x^8)}$$

$$= \lim_{x \to 0} \frac{\cancel{1} - \cancel{\frac{x^4}{2!}} + \frac{x^8}{4!} - (\cancel{1} + \cancel{x^4} + \frac{x^8}{2}) + \cancel{\frac{3}{2}x^4} + O(x^{12})}{x^8 + O(x^{16})}$$

$$= \lim_{x \to 0} \frac{\frac{x^8}{4!} - \frac{x^8}{2} + O(x^{12})}{x^8 + O(x^{16})}$$

$$= \lim_{x \to 0} \frac{\cancel{x^8}\left[\left(-\frac{11}{24}\right) + O(x^4)\right]}{\cancel{x^8}(1 + O(x^8))}$$

$$= \lim_{x \to 0} \frac{-\frac{11}{24} + O(x^4)}{1 + O(x^8)}$$

$$= -\frac{11}{24}$$