1a) $\underset{\sim}{F}(x, y) = (xy, 3y^2)$

$\underset{\sim}{r}(t) = (11t^4, t^3), \quad 0 \le t \le 1$

$r'(t) = (44t^3, 3t^2)$

$\int_C P\,dx + Q\,dy$

$= \int_C xy\,dx + 3y^2\,dy$

$= \int_0^1 11t^4(t^3)(44t^3) + 3(t^3)^2(3t^2)\,dt$

$= \int_0^1 484t^{10} + 9t^8\,dt$

$= \left[44t^{11} + t^9\right]_0^1$

$= 45$

1b) $\underset{\sim}{F}(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$

C is $y = 1 + x^2, \ -1 < x < 1$

Let $\underset{\sim}{F} = \nabla f$

$$f_x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{y}{\sqrt{x^2+y^2}}$$

Integrate f_x wrt x

$$\int f_x \, dx$$

$$= \int \frac{x}{\sqrt{x^2+y^2}} \, dx$$

$$= \frac{\frac{1}{2}\sqrt{x^2+y^2}}{\frac{1}{2}} + g(y)$$

$$= \sqrt{x^2+y^2} + g(y) - (1)$$

1b) Differentiate (1) wrt y

$$\frac{\partial}{\partial y}\left(\sqrt{x^2+y^2}+g(y)\right)$$

$$=\frac{2y}{\sqrt{x^2+y^2}}\left(\frac{1}{2}\right)+g'(y)$$

$$=\frac{y}{\sqrt{x^2+y^2}}+g'(y)-(2)$$

Comparing (2) with f_y,

$$g'(y)=0$$

Since $\underset{\sim}{F}=\nabla f$,

$\underset{\sim}{F}$ is conservative.

By Newton-Leibniz theorem,

$$\int_c \frac{x}{\sqrt{x^2+y^2}}\,dx+\frac{y}{\sqrt{x^2+y^2}}\,dy$$

$$=f(1,2)-f(-1,2)$$

$$=\sqrt{1^2+2^2}-\sqrt{(-1)^2+2^2}$$

$$=0$$

1c) $\underset{\sim}{F}(x,y) = \left(\frac{y^2}{1+x^2}, 2y\tan^{-1}x\right)$

$C = \underset{\sim}{r}(t) = (t^2, 2t), \ t \in [0,1]$

Let $\underset{\sim}{F} = \nabla f$

$$f_x = \frac{y^2}{1+x^2}$$

$$f_y = 2y\arctan(x)$$

$\int f_x \, dx$

$= \int \frac{y}{1+x^2} \, dx$

$= y\arctan(x) + g(y)$

$\frac{\partial}{\partial y}\left(y\arctan(x) + g(y)\right)$

$= \arctan(x) + g'(y) \quad -(1)$

Comparing (1) with f_y,

$$g'(y) = 2y\arctan x - \arctan x$$

$$g(y) = \int 2y\arctan x - \arctan x \, dy$$

$$= \frac{\cancel{2}y^2}{\cancel{2}}\arctan x - y\arctan x$$

$$= y^2\arctan x - y\arctan x$$

1c) $f = \cancel{y\arctan x} + y^2\arctan x - \cancel{y\arctan x}$

$= y^2 \arctan x$

Since $\underset{\sim}{F} = \nabla f$,

$\underset{\sim}{F}$ is conservative

$\therefore$ By Newton-Leibniz Theorem,

$\int_C \frac{y^2}{1+x^2}\,dx + 2y\tan^{-1}x\,dy$

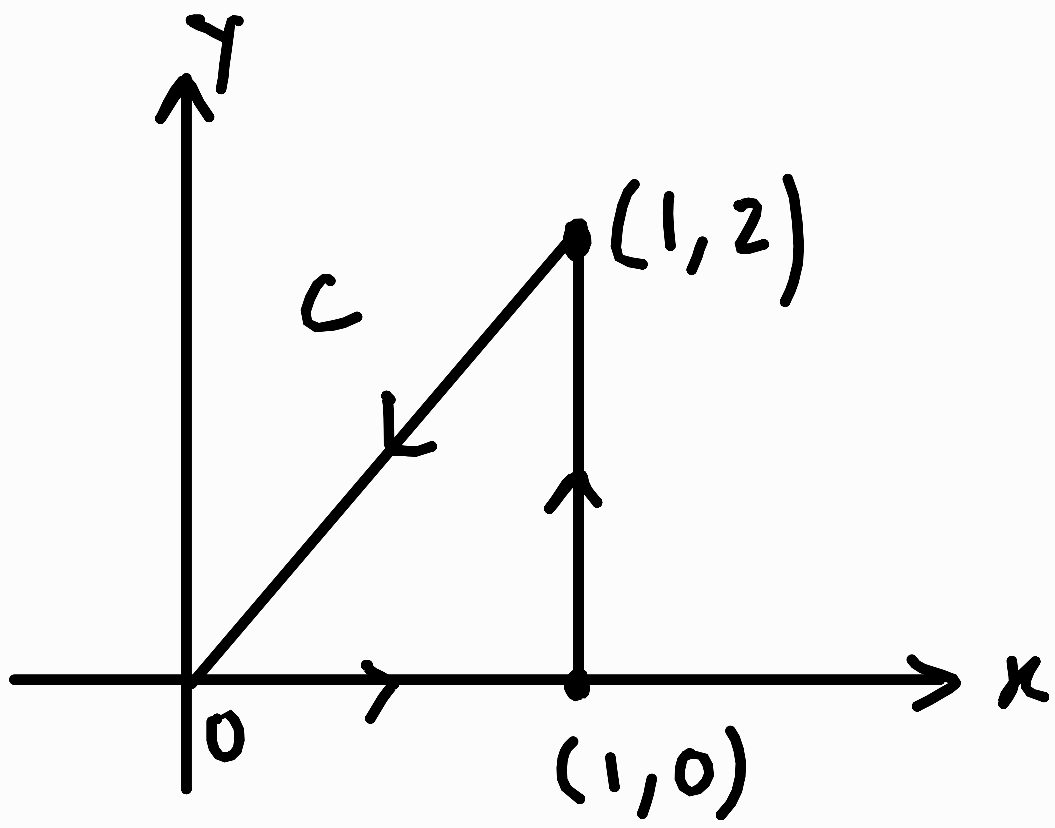
$= f(1,2) - f(0,0)$

$= 2^2\arctan(1) - 0$

$= \cancel{4}\left(\frac{\pi}{\cancel{4}}\right)$

$= \pi$

1d) $\underset{\sim}{F}(x,y) = (xy, x^2y^3)$



By Green's Theorem,

$$\int_{\partial D} P\,dx + Q\,dy$$

$$= \iint_D (Q_x - P_y)\,dx\,dy$$

$$= \iint_D (2xy^3 - x)\,dx\,dy$$

$$= \int_0^1 \left(\int_0^{2x} 2xy^3 - x\,dy \right) dx$$

$$= \int_0^1 \left(\frac{2x(2x)^4}{4} - x(2x) - 0 \right) dx$$

$$= \int_0^1 8x^5 - 2x^2\,dx$$

$$= \frac{8x^6}{6} - \frac{2x^3}{3} \Big|_0^1$$

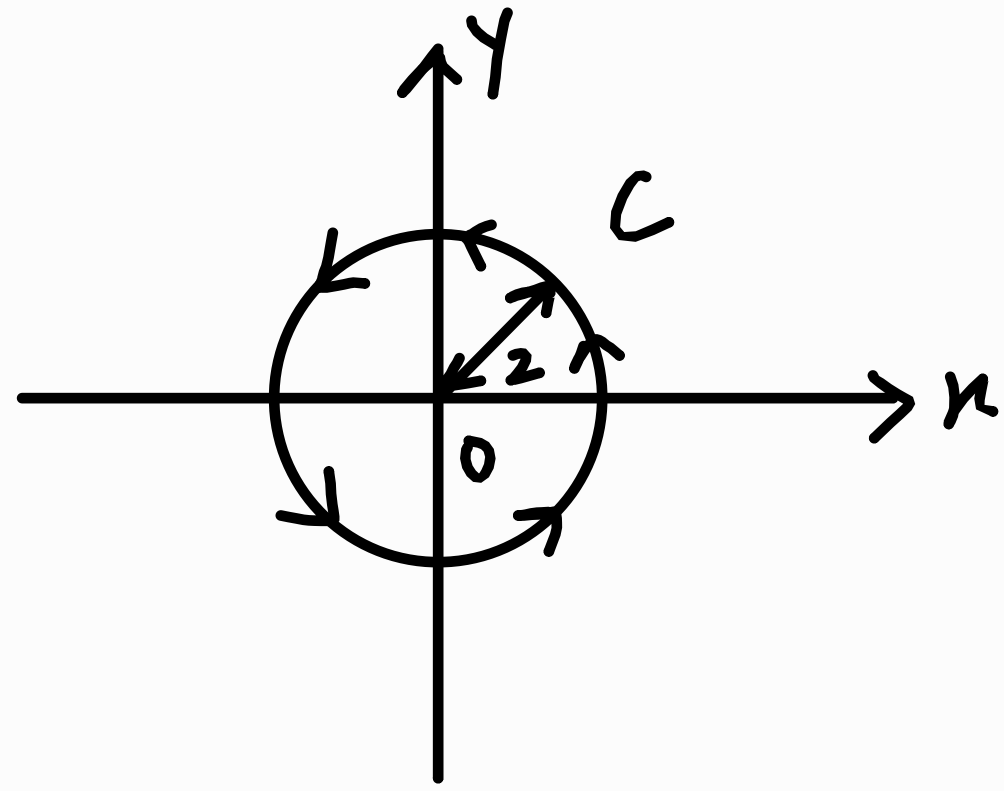
$$= \frac{8}{6} - \frac{2}{3}$$

$$= \frac{2}{3}$$

1e) $F(x,y) = (y^3, -x^3)$

$x^2+y^2=4$

$y = \pm\sqrt{4-x^2}$



By Green's Theorem,

$\int_C y^3\,dx - x^3\,dy$

$= \iint_D -3x^2 - 3y^2 \, dx\,dy$

$= \iint_D -3(x^2+y^2)\,dx\,dy$

$= -3\int_{r=0}^{2}\int_{\theta=0}^{2\pi} r^2\, r\, d\theta\, dr$

$= -3\int_{r=0}^{2} \left[r^3\theta\right]_0^{2\pi} dr$

$= -3\int_{r=0}^{2} 2\pi r^3\, dr$

$= -3\left[\frac{2\pi r^4}{4}\right]_0^2$

$= -3\left[\frac{2\pi(2)^4}{4} - 0\right]$

$= -24\pi$

2a) $\oint_{\partial D} -y\,dx + x\,dy$

$$Q_x = 1$$

$$P_y = -1$$

By Green's Theorem,

$$\oint_{\partial D} -y\,dx + x\,dy$$

$$= \iint_D 1-(-1)\,dx\,dy$$

$$= 2\iint_D dx\,dy$$

$$= 2A$$

2b) $\oint_{\partial D} -y\,dx + x\,dy$

$$x = \sin 2t$$
$$y = \sin t$$
$$\frac{dx}{dt} = 2\cos 2t$$
$$\frac{dy}{dt} = \cos t$$

$$\oint_{\partial D} -y\,dx + x\,dy$$
$$= \oint_0^{\pi} -\sin t(2\cos 2t) + \sin 2t(\cos t)\,dt$$
$$= \oint_0^{\pi} \sin 2t\cos t - 2\sin t\cos 2t\,dt$$
$$= \oint_0^{\pi} \cancel{2\sin t\cos^2 t} - 2\sin t(\cancel{\cos^2 t} - \sin^2 t)\,dt$$
$$= \oint_0^{\pi} 2\sin^3 t\,dt$$
$$= 2\oint_0^{\pi} (1-\cos^2 t)\sin t\,dt$$
$$= 2\oint_0^{\pi} \sin t - \sin t\cos^2 t\,dt$$
$$= 2\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi}$$
$$= 2\left[-(-1) + \frac{(-1)^3}{3} - \left(-1 + \frac{1}{3}\right)\right]$$
$$= 2\left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right]$$
$$= \frac{8}{3}$$

2b) Using the result in part (a)

$$\frac{8}{3} = 2A$$

$$A = \frac{4}{3}$$

3a) $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

$$Q_x = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$P_y = \frac{-x^2-y^2-(-y)(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

By Green's Theorem,

$$\oint \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \iint_D \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy$$

$$= \iint_D \frac{\cancel{y^2}\cancel{-x^2}\cancel{-y^2}\cancel{+x^2}}{(x^2+y^2)^2}\,dx\,dy$$

$$= 0$$

3b) $\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

$C = \underset{\sim}{r}(t) = (\varepsilon\cos t, \varepsilon\sin t),\ t \in [0, 2\pi]$

$\underset{\sim}{r}'(t) = (-\varepsilon\sin t, \varepsilon\cos t)$

$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

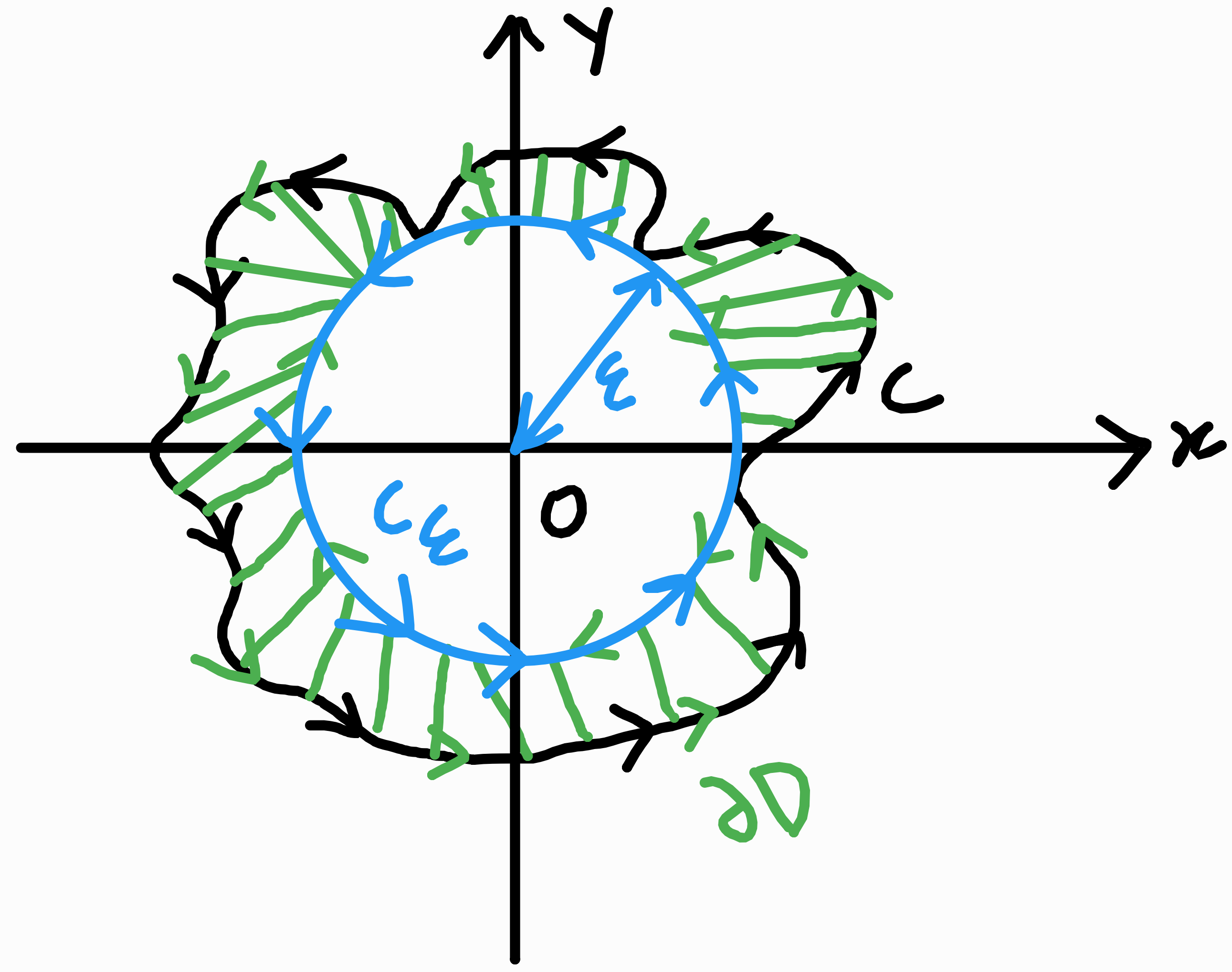
$= \oint_0^{2\pi} \frac{\cancel{-}\cancel{\varepsilon}\sin t}{\cancel{\varepsilon^2}}(\cancel{-}\cancel{\varepsilon}\sin t) + \frac{\cancel{\varepsilon}\cos t}{\cancel{\varepsilon^2}}(\cancel{\varepsilon}\cos t)\,dt$

$= \oint_0^{2\pi} \sin^2 t + \cos^2 t\,dt$

$= \oint_0^{2\pi} 1\,dt$

$= 2\pi$

3c)



For $\varepsilon > 0$, let C_ε be the anticlockwise oriented circle with radius ε centred at the origin. Take ε to be sufficiently small such that C_ε is contained the interior region of C and let D be the region whose oriented boundary is

$$\partial D = C - C_\varepsilon$$

$$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \oint_{\partial D + C_\varepsilon} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \oint_{\partial D} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy + \oint_{C_\varepsilon} \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

3c) Since $(0,0) \notin D$, by Green's Theorem,

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \iint_D \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)}\,dx\,dy + \oint_{C_\varepsilon} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

Using the answer in part (a) for

$$\iint \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)}\,dx\,dy$$

and the answer in part (b) for

$$\oint \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy,$$

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= 0 + 2\pi$$

$$= 2\pi$$

4) $\oint_C (y^3-3y)dx - x^3dy$

$$Q_x = -3x^2$$

$$P_y = 3y^2-3$$

Using Green's Theorem,

$\oint (y^3-3y)\,dx - x^3dy$

$= \iint_D -3x^2-(3y^2-3)\,dy\,dx$

$= 3\iint_D 1-x^2-y^2\,dy\,dx$

$= 3\iint_D 1-(x^2+y^2)\,dy\,dx$

$= 3\iint_D (1-r^2)\,r\,d\theta\,dr$

To get the maximum region, $0 \le r \le 1$, $0 \le \theta \le 2\pi$,

$\oint (y^3-3y)dx - x^3dy$

$= 3\int_{r=0}^{1}\int_{\theta=0}^{\pi} r-r^3\,d\theta\,dr$

$= 3\int_{r=0}^{1} r-r^3\,dr \int_{\theta=0}^{2\pi} d\theta$

$= 6\pi\int_0^1 r-r^3\,dr$

$= 6\pi\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1$

$= 6\pi\left[\frac{1}{2}-\frac{1}{4}\right]$

$= \frac{6\pi}{4} = \frac{3}{2}\pi$

5) $z = f(x,y)$, $(x,y) \in D$

$S : (x,y,z) = \underset{\sim}{r}(x,y) = (x, y, f(x,y))$

$\iint_S dS$

$= \iint_D \| \underset{\sim}{r}_x \times \underset{\sim}{r}_y \| \, dx\,dy$

$= \iint_D \| (1, 0, f_x(x,y)) \times (0, 1, f_y(x,y)) \| \, dx\,dy$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ 1 & 0 & f_x & 1 & 0 \\ 0 & 1 & f_y & 0 & 1 \end{array}$$

$= \iint_D \| \underset{\sim}{k} - f_x \underset{\sim}{i} - f_y \underset{\sim}{j} \| \, dx\,dy$

$= \iint_D \| (-f_x(x,y), -f_y(x,y), 1) \| \, dx\,dy$

$= \iint_D \sqrt{(-f_x(x,y))^2 + (-f_y(x,y))^2 + 1^2} \, dx\,dy$

$= \iint_D \sqrt{1 + (f_x(x,y))^2 + (f_y(x,y))^2} \, dx\,dy$ (shown)

6) $x = r\cos\theta$, $y = r\sin\theta$, $z = \theta$, $r \in [0,1]$, $\theta \in [0, 8\pi]$

$\underset{\sim}{r}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$

$\underset{\sim}{r}_r(r,\theta) = (\cos\theta, \sin\theta, 0)$

$\underset{\sim}{r}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 1)$

$\iint_S dS$

$$= \int_{r=0}^{1} \int_{\theta=0}^{8\pi} \left\| \underset{\sim}{r}_r \times \underset{\sim}{r}_\theta \right\| dr\, d\theta$$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \cos\theta & \sin\theta & 0 & \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta & 1 & -r\sin\theta & r\cos\theta \end{array}$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{8\pi} \left\| \sin\theta\, \underset{\sim}{i} + r\cos^2\theta\, \underset{\sim}{k} + r\sin^2\theta\, \underset{\sim}{k} - \cos\theta\, \underset{\sim}{j} \right\| dr\, d\theta$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{8\pi} \sqrt{\sin^2\theta + \cos^2\theta + r^2}\, dr\, d\theta$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{8\pi} \sqrt{1+r^2}\, dr\, d\theta$$

$$= \int_{r=0}^{1} \sqrt{1+r^2}\, dr \int_{\theta=0}^{8\pi} 1\, d\theta$$

$$= 8\pi \int_{r=0}^{1} \sqrt{1+r^2}\, dr \text{ (shown)}$$

7) The number dN of bacteria inhabiting a surface area element dS is:

$$dN = \sigma(x,y,z)\,dS = (z+1)\,dS$$

The total number of bacteria inhabiting S is:

$$N = \iint_S dN$$
$$= \iint_S (z+1)\,dS$$

$S: \mathbf{r}(\theta,\varphi) = (\cos\theta\sin\varphi, \sin\theta\cos\varphi, \cos\varphi),\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]$

$$\mathbf{r}_\theta(\theta,\varphi) = (-\sin\theta\sin\varphi, \cos\theta\sin\varphi, 0)$$
$$\mathbf{r}_\varphi(\theta,\varphi) = (\cos\theta\cos\varphi, \sin\theta\cos\varphi, -\sin\varphi)$$

$$\begin{array}{ccccc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ -\sin\theta\sin\varphi & \cos\theta\sin\varphi & 0 & -\sin\theta\sin\varphi & \cos\theta\sin\varphi \\ \cos\theta\cos\varphi & \sin\theta\cos\varphi & -\sin\varphi & \cos\theta\cos\varphi & \sin\theta\cos\varphi \end{array}$$

$$\mathbf{r}_\theta \times \mathbf{r}_\varphi = -\cos\theta\sin^2\varphi\,\mathbf{i} + 0 - \cancel{\sin^2\theta}\sin\varphi\cos\varphi\,\mathbf{k}$$
$$- \cancel{\cos^2\theta}\sin\varphi\cos\varphi\,\mathbf{k} - 0 + \sin\theta\sin^2\varphi\,\mathbf{j}$$
$$= (-\cos\theta\sin^2\varphi, \sin\theta\sin^2\varphi, -\sin\varphi\cos\varphi)$$

$$\|\mathbf{r}_\theta \times \mathbf{r}_\varphi\| = \sqrt{\cos^2\theta\sin^4\varphi + \sin^2\theta\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$
$$= \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi}$$
$$= \sqrt{\sin^2\varphi(\sin^2\varphi + \cos^2\varphi)}$$
$$= |\sin\varphi| = \sin\varphi \quad \because 0 \le \varphi \le \pi,\ 0 \le \sin\varphi \le 1$$

7) $N = \iint_S (z+1)\,dS$

$$= \int_{\varphi=0}^{\pi} \int_{\theta=0}^{2\pi} (\cos\varphi + 1)\sin\varphi \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} 1\,d\theta \int_0^{\pi} \cos\varphi\sin\varphi + \sin\varphi \, d\varphi$$

$$= 2\pi \left[\frac{-\cos^2\varphi}{2} - \cos\varphi \right]_0^{\pi}$$

$$= 2\pi \left(-\frac{\cos^2\pi}{2} - \cos\pi - \left(-\frac{\cos^2 0}{2} - \cos 0 \right) \right)$$

$$= 2\pi(2)$$

$$= 4\pi$$

Hence, the number of bacteria living on the surface is 4π million bacteria.

8) Flux in general is: $\iint_S \underset{\sim}{F} \cdot \underset{\sim}{U}\, dS$

The flux of the unit normal field $\underset{\sim}{U}$ across S is:

$$\iint_S \underset{\sim}{U} \cdot \underset{\sim}{U}\, dS$$

$$= \iint_S \|\underset{\sim}{U}\|^2\, dS$$

$$= \iint_S 1\, dS \quad \because \underset{\sim}{U} \text{ is a unit vector}$$

$$= \text{The area of } S$$

9) $S: \underset{\sim}{r}(r,\theta) = (r\cos\theta, r\sin\theta, \theta),\ r \in [0,1],\ \theta = [0, 8\pi]$

$$\underset{\sim}{r}_r(r,\theta) = (\cos\theta, \sin\theta, 0)$$

$$\underset{\sim}{r}_\theta(r,\theta) = (-r\sin\theta, r\cos\theta, 1)$$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \cos\theta & \sin\theta & 0 & \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta & 1 & -r\sin\theta & r\cos\theta \end{array}$$

$$\underset{\sim}{r}_r \times \underset{\sim}{r}_\theta = \sin\theta\,\underset{\sim}{i} + 0 + r\cos^2\theta\,\underset{\sim}{k} + r\sin^2\theta\,\underset{\sim}{k} + 0 + \cos\theta\,\underset{\sim}{j}$$

$$= (\sin\theta, \cos\theta, r)$$

a) $$\int_{r=0}^{1} \int_{\theta=0}^{8\pi} (0,0,1)\cdot(\sin\theta, \cos\theta, r)\,dr\,d\theta$$

$$= \int_0^1 r\,dr \int_0^{8\pi} 1\,d\theta$$

$$= \left[\frac{r^2}{2}\right]_0^1 (8\pi)$$

$$= \frac{1}{2}(8\pi)$$

$$= 4\pi$$

9b) $$\int_{r=0}^{1}\int_{\theta=0}^{8\pi}(r\cos\theta, r\sin\theta, \theta)\cdot(\sin\theta, \cos\theta, r)\,dr\,d\theta$$

$$=\int_{r=0}^{1}\int_{\theta=0}^{8\pi} r\sin\theta\cos\theta + r\sin\theta\cos\theta + r\,\theta\,dr\,d\theta$$

$$=\int_{0}^{1} r\,dr\int_{0}^{8\pi} 2\sin\theta\cos\theta + \theta\,d\theta$$

$$=\left[\frac{r^2}{2}\right]_0^1\left[\frac{2\sin^2\theta}{2}+\frac{\theta^2}{2}\right]_0^{8\pi}$$

$$=\frac{1}{2}(32\pi^2)$$

$$=16\pi^2$$

10) $S: \underset{\sim}{r}(\theta, t) = (t\cos\theta, t\sin\theta, t)$, $\theta \in [0, 2\pi]$, $t \in [0,1]$

$$\underset{\sim}{r}_\theta(\theta, t) = (-t\sin\theta, t\cos\theta, 0)$$

$$\underset{\sim}{r}_t(\theta, t) = (\cos\theta, \sin\theta, 1)$$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ -t\sin\theta & t\cos\theta & 0 & -t\sin\theta & t\cos\theta \\ \cos\theta & \sin\theta & 1 & \cos\theta & \sin\theta \end{array}$$

$$\underset{\sim}{r}_\theta \times \underset{\sim}{r}_t = t\cos\theta\underset{\sim}{i} + 0 - t\sin^2\theta\underset{\sim}{k} - t\cos^2\theta\underset{\sim}{k} - 0 + t\sin\theta\underset{\sim}{j}$$

$$= (t\cos\theta, t\sin\theta, -t)$$

To have the normal pointing upwards,

$$\underset{\sim}{r}_\theta \times \underset{\sim}{r}_t = (-t\cos\theta, -t\sin\theta, t)$$

$$\|\underset{\sim}{r}_\theta \times \underset{\sim}{r}_t\| = \sqrt{t^2\cos^2\theta + t^2\sin^2\theta + t^2}$$

$$= \sqrt{2t^2}$$

$$= \sqrt{2}t$$

a) Surface area of $S = \iint_S dS$

$$= \int_{t=0}^{1} \int_{\theta=0}^{2\pi} \sqrt{2}t \, dt \, d\theta$$

$$= \int_0^1 \sqrt{2}t \, dt \int_0^{2\pi} 1 \, d\theta$$

$$= 2\pi\left[\frac{\sqrt{2}t}{2}\right]_0^1$$

$$= \cancel{2}\pi \frac{\sqrt{2}}{\cancel{2}}$$

$$= \sqrt{2}\pi$$

10b) Flux of $\underset{\sim}{F} = \iint_S \underset{\sim}{F} \cdot \underset{\sim}{n} \, dS$

$$= \int_{t=0}^{1} \int_{\theta=0}^{2\pi} (t\cos\theta, t\sin\theta, 0) \cdot (-t\cos\theta, -t\sin\theta, t) \, dt \, d\theta$$

$$= \int_{t=0}^{1} \int_{\theta=0}^{2\pi} -t^2\cos^2\theta - t^2\sin^2\theta \, dt \, d\theta$$

$$= \int_{t=0}^{1} \int_{\theta=0}^{2\pi} -t^2(\cos^2\theta + \sin^2\theta) \, dt \, d\theta$$

$$= -\int_0^1 t^2 \, dt \int_0^{2\pi} 1 \, d\theta$$

$$= -\left[\frac{t^3}{3}\right]_0^1 (2\pi)$$

$$= -\frac{1}{3}(2\pi)$$

$$= -\frac{2}{3}\pi$$

11) $\oint_C y^3\,dx - x^3\,dy + z^3\,dz$

$$\underset{\sim}{F} = (y^3, -x^3, z^3)$$

$$\text{curl } \underset{\sim}{F} = \nabla \times \underset{\sim}{F}$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (y^3, -x^3, z^3)$$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^3 & -x^3 & z^3 & y^3 & -x^3 \end{array}$$

$$= \cancel{\frac{\partial}{\partial y} z^3}\,\underset{\sim}{i} + \cancel{\frac{\partial}{\partial z} y^3}\,\underset{\sim}{j} - \frac{\partial}{\partial x} x^3 \underset{\sim}{k} - \frac{\partial}{\partial y} y^3 \underset{\sim}{k} + \cancel{\frac{\partial}{\partial z} x^3}\,\underset{\sim}{i} - \cancel{\frac{\partial}{\partial x} z^3}\,\underset{\sim}{j}$$

$$= (0, 0, -3x^2 - 3y^2)$$

S being part of the plane $x + y + z = 2$,

can be parametrised as:

$$(x, y, z) = \underset{\sim}{r}(x, y) = (x, y, 2 - x - y),\ x \in [0, 2],\ y \in [0, 2]$$

$$\underset{\sim}{r}_x(x, y) = (1, 0, -1)$$
$$\underset{\sim}{r}_y(x, y) = (0, 1, -1)$$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

$$\underset{\sim}{r}_x \times \underset{\sim}{r}_y = \underset{\sim}{k} + \underset{\sim}{i} + \underset{\sim}{j}$$
$$= (1, 1, 1)$$

11) By Stoke's Theorem,

$$\oint_C y^3\,dx - x^3\,dy + z^3\,dz$$

$$= \iint_S \operatorname{curl}(y^3, -x^3, z^3) \cdot d\underset{\sim}{S}$$

$$= \iint_S (0, 0, -3x^2 - 3y^2) \cdot (1, 1, 1)\,dx\,dy$$

$$= \iint_S -3x^2 - 3y^2\,dx\,dy$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} -3r^2 r\,d\theta\,dr$$

$$= \int_0^2 -3r^3\,dr \int_0^{2\pi} 1\,d\theta$$

$$= -3\left[\frac{r^4}{4}\right]_0^2 (2\pi)$$

$$= -6\pi\left(\frac{2^4}{4}\right)$$

$$= -24\pi$$

12) $\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$

$$\underset{\sim}{F} = (y+2z,\ x+2z,\ x+2y)$$

Let S be the part of the plane that is cut out by the sphere.

Unit normal vector to S, $\underset{\sim}{u} = \frac{1}{3}(1,2,2)$

$$\operatorname{curl}\underset{\sim}{F} = \nabla \times (y+2z,\ x+2z,\ x+2y)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (y+2z,\ x+2z,\ x+2y)$$

$$\begin{array}{ccccc}
\underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\
y+2z & x+2z & x+2y & y+2z & x+2z
\end{array}$$

$$= \frac{\partial}{\partial y}(x+2y)\underset{\sim}{i} + \frac{\partial}{\partial z}(\cancel{y}+2z)\underset{\sim}{j} + \frac{\partial}{\partial x}(x+\cancel{2z})\underset{\sim}{k}$$

$$- \frac{\partial}{\partial y}(y+\cancel{2z})\underset{\sim}{k} - \frac{\partial}{\partial z}(\cancel{x}+2z) - \frac{\partial}{\partial x}(x+\cancel{2y})\underset{\sim}{j}$$

$$= \cancel{2\underset{\sim}{i}} + 2\underset{\sim}{j} + \cancel{\underset{\sim}{k}} \cancel{-\underset{\sim}{k}} \cancel{-2\underset{\sim}{i}} - \underset{\sim}{j}$$

$$= (0,1,0)$$

$$\operatorname{curl}\underset{\sim}{F}\cdot\underset{\sim}{u} = \frac{1}{3}(1,2,2)\cdot(0,1,0)$$

$$= \frac{2}{3}$$

12) By Stoke's Theorem,

$$\oint_C (y+2z)\,dx + (x+2z)\,dy + (x+2y)\,dz$$

$$= \iint_S \text{curl}\, \underset{\sim}{F} \cdot \underset{\sim}{n}\, dS$$

$$= \iint_S \frac{2}{3}\, dS$$

$$= \frac{2}{3}(\text{The area of } S)$$

$$= \frac{2}{3}(\pi (1)^2)$$

$$= \frac{2}{3}\pi$$

13) Since $\underset{\sim}{F}$ has continuous partial derivatives,

$$\operatorname{div}(\operatorname{curl}\underset{\sim}{F}) = 0$$

$$\text{Flux} = \iint_S \underset{\sim}{V} \cdot d\underset{\sim}{S}$$

$$= \iint_S \operatorname{curl}\underset{\sim}{F} \cdot d\underset{\sim}{S}$$

By Gauss' Theorem,

$$= \iiint_Q \operatorname{div}(\operatorname{curl}\underset{\sim}{F})\,dx\,dy\,dz$$

$$= \iiint_Q 0\,dx\,dy\,dz$$

$$= 0$$

14a) $F(x,y,z) = (0, y, -z)$

By Gauss' Theorem,

$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$

$= \iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz$

$= \iiint_Q \nabla \cdot \underset{\sim}{F} \, dx\,dy\,dz$

$= \iiint_Q \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (0, y, -z) \, dx\,dy\,dz$

$= \iiint_Q 0 \, dx\,dy\,dz$

$= 0$

14b) $\underset{\sim}{F}(x,y,z) = (3xy^2, xe^z, z^3)$

By Gauss' Theorem,

$$\iint_S \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$= \iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz$$

$$= \iiint_Q \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (3xy^2, xe^z, z^3)\,dx\,dy\,dz$$

$$= \iiint_Q 3y^2 + 3z^2 \,dx\,dy\,dz$$

$$= \iiint_Q y^2 + z^2 \,dx\,dy\,dz$$

$$= 3\int_{r=0}^{1} \int_{\theta=0}^{2\pi} r^2 r \,d\theta\,dr \int_{-1}^{2} 1\,dx$$

$$= 3\int_0^1 r^3\,dr \int_0^{2\pi} 1\,d\theta\,(3)$$

$$= 18\pi \left[\frac{r^4}{4}\right]_0^1$$

$$= \frac{9}{2}\pi$$

14ci) $\underset{\sim}{F}(x, y, z) = (xy, yz, zx)$

$$\iint_S F \cdot dS$$

$$= \iint_S \underset{\sim}{F} \cdot \underset{\sim}{u}\, dS$$

$$= \iint_S (xy, yz, zx) \cdot (0, 0, 1)\, dS$$

$$= \iint_S zx\, dS$$

$$= \int_{z=0}^{0} \int_{x=0}^{2} zx\, dS$$

$$= \int_2^0 0\, dS$$

$$= 0$$

14cii) $\underset{\sim}{F}(x, y, z) = (xy, yz, zx)$

By Gauss' Theorem,

$$\iint_S \underset{\sim}{F} \cdot dS$$

$$= \iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz$$

$$= \iiint_Q \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (xy, yz, zx) \, dx\,dy\,dz$$

$$= \iiint_Q y + z + x \, dx\,dy\,dz$$

$$= \iiint_Q x + y + z \, dx\,dy\,dz$$

$$= \iint_D \int_{z=0}^{4-x^2-y^2} x + y + z \, dx\,dy\,dz$$

$$= \iint_D \left[(x+y)z + \frac{z^2}{2}\right]_0^{4-x^2-y^2} dx\,dy$$

$$= \iint_D (x+y)(4-x^2-y^2) + \frac{(4-x^2-y^2)^2}{2} \, dx\,dy$$

$$= \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \left[(r\cos\theta + r\sin\theta)(4-r^2) + \frac{(4-r^2)^2}{2}\right] r \, dr\,d\theta$$

$$= \int_{r=0}^{2} \left[r^2(\sin\theta - \cos\theta)(4-r^2) + \frac{r(4-r^2)^2\theta}{2}\right]_0^{2\pi} dr$$

$$= \int_0^2 \cancel{r^2(4-r^2)(\sin 2\pi - \cos 2\pi - (\sin 0 - \cos 0))} + \pi r(4-r^2)^2 \, dr$$

$$= \int_0^2 \pi r(16 - 8r^2 + 4r^4) \, dr$$

$$= \pi \int_0^2 16r - 8r^3 + r^5 \, dr$$

$$= \pi \left[\frac{16}{2}r^2 - \frac{8}{4}r^4 + \frac{1}{6}r^6\right]_0^2$$

$$= \frac{32}{3}\pi$$

14c iii) Let S_1 be the surface in part (i), S_2 be the surface in part (ii).

$$S = S_2 - S_1$$

$$= \frac{32}{3}\pi - 0$$

$$= \frac{32}{3}\pi$$