1) curl(grad f)

$= \nabla \times (\text{grad } f)$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{array}$$

$= \cancel{\frac{\partial}{\partial y}\frac{\partial f}{\partial z}}\underset{\sim}{i} + \cancel{\frac{\partial}{\partial z}\frac{\partial f}{\partial x}}\underset{\sim}{j} + \cancel{\frac{\partial}{\partial x}\frac{\partial f}{\partial y}}\underset{\sim}{k} - \cancel{\frac{\partial f}{\partial x}\frac{\partial}{\partial y}}\underset{\sim}{k} - \cancel{\frac{\partial f}{\partial y}\frac{\partial}{\partial z}}\underset{\sim}{i} - \cancel{\frac{\partial f}{\partial z}\frac{\partial}{\partial x}}\underset{\sim}{j}$

$= \underset{\sim}{0}$ (shown)

div(curl $\underset{\sim}{V}$)

$= \nabla \cdot (\nabla \times \underset{\sim}{V})$

$= \nabla \cdot \left(\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (P, Q, R)\right)$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q & R & R & Q \end{array}$$

$= \nabla \cdot \left(\frac{\partial R}{\partial y}\underset{\sim}{i} + \frac{\partial P}{\partial z}\underset{\sim}{j} + \frac{\partial Q}{\partial x}\underset{\sim}{k} - \frac{\partial P}{\partial y}\underset{\sim}{k} - \frac{\partial Q}{\partial z}\underset{\sim}{i} - \frac{\partial R}{\partial x}\underset{\sim}{j}\right)$

$= \nabla \cdot \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\underset{\sim}{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\underset{\sim}{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\underset{\sim}{k}\right)$

1) $\operatorname{div}(\operatorname{curl} \underset{\sim}{V})$

$$= \nabla \cdot \left(\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \underset{\sim}{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \underset{\sim}{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underset{\sim}{k} \right)$$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \underset{\sim}{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \underset{\sim}{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \underset{\sim}{k} \right]$$

$$= \cancel{\frac{\partial^2 R}{\partial x \partial y}} - \cancel{\frac{\partial^2 Q}{\partial x \partial z}} + \cancel{\frac{\partial^2 P}{\partial y \partial z}} - \cancel{\frac{\partial^2 R}{\partial x \partial y}} + \cancel{\frac{\partial^2 Q}{\partial x \partial z}} - \cancel{\frac{\partial^2 P}{\partial y \partial z}}$$

$= 0$ (shown)

2) $\underset{\sim}{F}(x, y) = (P(x, y), Q(x, y))$

Since $\underset{\sim}{F}$ is conservative,

$$\underset{\sim}{F} = \operatorname{grad} f$$

$f_x(x, y) = P(x, y)$ — (1)

$f_y(x, y) = Q(x, y)$ — (2)

Differentiating (1) wrt y

$f_{xy}(x, y) = P_y(x, y)$ — (3)

Differentiating (2) wrt x,

$f_{yx}(x, y) = Q_x(x, y)$ — (4)

Since f, P and Q are continuous, using Clairaut's theorem,

$$f_{xy}(x, y) = f_{yx}(x, y)$$

$$P_y(x, y) = Q_x(x, y)$$

2) Since P and Q have continuous partial derivatives,

$$P(x,y) = Q(x,y) \text{ (shown)}$$

curl F

$= \nabla \times (P(x,y,z), Q(x,y,z), R(x,y,z))$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (P(x,y,z), Q(x,y,z), R(x,y,z))$

$$\begin{array}{ccccc} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} & \underset{\sim}{i} & \underset{\sim}{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q & R & R & Q \end{array}$$

$= (R_y - Q_z)\underset{\sim}{i} + (P_z - R_x)\underset{\sim}{j} + (Q_x - P_y)\underset{\sim}{k} \quad - (1)$

Since $\underset{\sim}{F}$ is conservative,

$$\underset{\sim}{F} = \text{grad } f$$

$f_x(x,y,z) = P(x,y,z) \quad - (2)$

$f_y(x,y,z) = Q(x,y,z) \quad - (3)$

$f_z(x,y,z) = R(x,y,z) \quad - (4)$

As P, Q and R have continuous partial derivatives,

Sub (2), (3), (4) into (1)

curl $\underset{\sim}{F} = (\cancel{f_{zy} - f_{yz}})\underset{\sim}{i} + (\cancel{f_{xz} - f_{zx}})\underset{\sim}{j} + (\cancel{f_{yx} - f_{xy}})\underset{\sim}{k}$

$= 0$ (shown by Clairaut's theorem)

3a) Vector pointing outwards from the origin: (x, y, z)

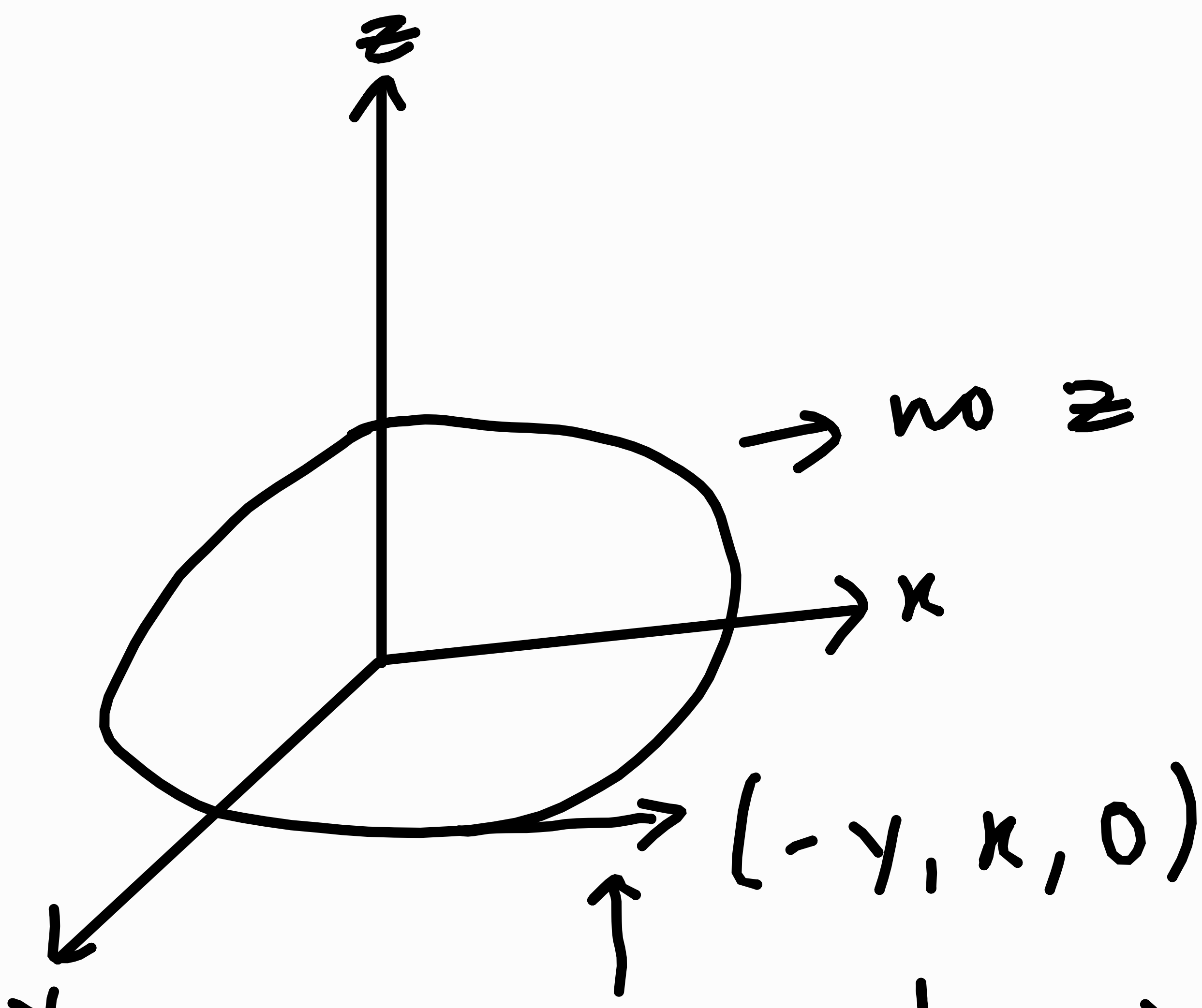
$$\underset{\sim}{E} = \|\underset{\sim}{E}\| \underset{\sim}{u}$$

$$= \|\underset{\sim}{E}\| \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{q}{4\pi\varepsilon_0\left(\sqrt{x^2+y^2+z^2}\right)^2} \cdot \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{q(x,y,z)}{4\pi\varepsilon_0(x^2+y^2+z^2)^{\frac{3}{2}}}$$

b)



required vector is a tangent

$$\underset{\sim}{B} = \|\underset{\sim}{B}\| \underset{\sim}{u}$$

$$= \|\underset{\sim}{B}\| \frac{(-y,x,0)}{\sqrt{y^2+x^2}}$$

$$= \frac{\mu_0 I}{2\pi\sqrt{x^2+y^2}} \cdot \frac{(-y,x,0)}{\sqrt{x^2+y^2}}$$

$$= \frac{\mu_0 I(-y,x,0)}{2\pi(x^2+y^2)}$$

4) $y = f(x),\ x \in [a,b]$

Parametrise $y=f(x)$ with:

$(x,y) = r(t) = (x, f(x)),\ x \in [a,b]$

$$ds = \|r'(t)\| dx$$
$$= \|(1, f'(x))\| dx$$
$$= \sqrt{1+(f'(x))^2}\, dx$$

$\int_c dx = \int_a^b \sqrt{1+(f'(x))^2}\, dx$ (shown)

5) $\int_c f_{avg}\, ds = \int_c f(x)\, ds$

$$f_{avg} \int_c ds = \int_c f(x)\, ds$$
$$f_{avg} = \frac{\int_c f(x)\, ds}{\int_c ds}$$
$$f_{avg} = \frac{1}{\text{length}(c)} \int_c f(x)\, ds$$

6) $x^2+y^2=ax$

$x^2-ax+y^2=0$

$x^2-ax+\left(\frac{a}{2}\right)^2+y^2-\left(\frac{a}{2}\right)^2=0$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

$\downarrow$

circle centred at $\left(\frac{a}{2},0\right)$ with radius $\frac{a}{2}$

$\left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2$

$y=\pm\sqrt{\left(\frac{a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}$

Let $x=\left(\frac{a}{2}\right)\cos t+\frac{a}{2}$, $y=\left(\frac{a}{2}\right)\sin t$

$(x,y)=\underset{\sim}{r}(t), =\left(\frac{a}{2}\cos t+\frac{a}{2}, \frac{a}{2}\sin t\right), t\in[0,2\pi]$

$ds=\|r'(t)\|dt$

$=\left\|\left(-\frac{a\sin t}{2}, \frac{a\cos t}{2}\right)\right\|dt$

$=\sqrt{\left(\frac{a\sin t}{2}\right)^2+\left(\frac{a\cos t}{2}\right)^2}\,dt$

$=\sqrt{\frac{a^2}{4}\sin^2 t+\frac{a^2}{4}\cos^2 t}\,dt$

$=\sqrt{\frac{a^2}{4}}\,dt$

$=\left|\frac{a}{2}\right|dt$

6) $\int_c \sqrt{x^2+y^2}\, ds$

$$= \int_0^{2\pi} \sqrt{\left(\frac{a}{2}\cos t + \frac{a}{2}\right)^2 + \left(\frac{a}{2}\sin t\right)^2}\ \left|\frac{a}{2}\right| dt$$

$$= \int_0^{2\pi} \sqrt{\frac{a^2}{4}\cancel{\cos^2 t} + \frac{a^2}{2}\cos t + \frac{a^2}{4} + \frac{a^2}{4}\cancel{\sin^2 t}}\ \left|\frac{a}{2}\right| dt$$

$$= \int_0^{2\pi} \sqrt{\frac{a^2}{2} + \frac{a^2}{2}\cos t}\ \left|\frac{a}{2}\right| dt$$

$$= \int_0^{2\pi} \sqrt{\frac{a^2}{4}(2) + \frac{a^2}{2}(2\cos t)}\ \left|\frac{a}{2}\right| dt$$

$$= \int_0^{2\pi} \left|\frac{a}{2}\right| \sqrt{2+2\cos t}\ \left|\frac{a}{2}\right| dt$$

$$= \int_0^{2\pi} \frac{a^2}{4}\sqrt{2+2\cos t}\, dt$$

$$= \frac{a^2}{4}\int_0^{2\pi} \sqrt{4\cos^2\left(\frac{t}{2}\right)}\, dt$$

$$= \frac{a^2}{4}\int_0^{2\pi} 2\left|\cos\frac{t}{2}\right| dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} \left|\cos\frac{t}{2}\right| dt$$

$$= \frac{a^2}{2}\left[\int_0^{\pi} \cos\frac{t}{2}\, dt - \int_{\pi}^{2\pi} \cos\frac{t}{2}\, dt\right]$$

$$= \frac{a^2}{2}\left[\frac{1}{\frac{1}{2}}\sin\frac{t}{2}\, dt\Big|_0^{\pi} - \frac{1}{\frac{1}{2}}\sin\frac{t}{2}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{a^2}{2}\left[2\sin\frac{t}{2}\Big|_0^{\pi} - 2\sin\frac{t}{2}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{a^2}{2}\left[2\sin\frac{\pi}{2} - 2\sin\frac{0}{2} - 2\sin\frac{2\pi}{2} + 2\sin\frac{\pi}{2}\right]$$

$$= \frac{a^2}{2}(4) = 2a^2$$

7a) $y = x^2, x \in [-1, 2]$

Let $y = f(x)$

$(x, y) = \underset{\sim}{r}(x) = (x, f(x)) = (x, x^2), t \in [-1, 2]$

$\frac{d}{dx}(x, x^2) = (1, 2x)$

$\int_c 2xy\,dx + \left(\frac{3}{2}x + y\right)dy$

$= \int_{-1}^{2} 2x(x^2)\,dx + \left(\frac{3}{2}x + x^2\right)2x\,dx$

$= \int_{-1}^{2} 2x^3 + 3x^2 + 2x^3\,dx$

$= \int_{-1}^{2} 4x^3 + 3x^2\,dx$

$= \frac{\cancel{4}x^4}{\cancel{4}} + \frac{\cancel{3}}{\cancel{3}}x^3 \Big|_{-1}^{2}$

$= x^4 + x^3 \Big|_{-1}^{2}$

$= 2^4 + 2^3 - \left[(-1)^4 + (-1)^3\right]$

$= 24$

7b) $\int_C \frac{2x\,dx}{x^2+y^2} + \frac{2y\,dy}{x^2+y^2}$

$F = \text{grad} f$

$f_x = \frac{2x}{x^2+y^2} \quad -(1)$

$f_y = \frac{2y}{x^2+y^2} \quad -(2)$

$f(x,y) = \int \frac{2x}{x^2+y^2}\,dx$

$= \ln|x^2+y^2| + g(x)$

$f_y(x,y) = \frac{2y}{x^2+y^2} + g'(x) \quad -(3)$

Comparing (3) and (2),

$g'(x) = 0$

$\therefore$ The vector field $\underset{\sim}{F} = \left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$ is conservative.

$\therefore \underset{\sim}{F} = \text{grad}\left(\ln|x^2+y^2|\right)$

7b) Using Newton-Leibniz Theorem,

$$\int_C \frac{2x\,dx}{x^2+y^2} + \frac{2y\,dy}{x^2+y^2}$$

$= f(\text{End point of } C) - f(\text{start point of } C)$

Sub $x = t\cos t$,
$y = t\sin t$,
$t \in [\pi, 2\pi]$

$$\int_C \frac{2x\,dx}{x^2+y^2} + \frac{2y\,dy}{x^2+y^2}$$

$$= \ln\left|t^2\cos^2 t + t^2\sin^2 t\right| \Big|_{\pi}^{2\pi}$$

$$= \ln\left|(2\pi)^2\cos^2(2\pi) + (2\pi)^2\sin^2(2\pi)\right| - \ln\left|\pi^2\cos^2\pi + \pi^2\sin^2\pi\right|$$

$$= \ln|4\pi^2| - \ln|\pi^2|$$

$$= \ln\left|\frac{4\cancel{\pi^2}}{\cancel{\pi^2}}\right|$$

$$= \ln 4$$

$$= 2\ln 2$$