Math Module 6A Notes

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Contents

1 Definitions

1.1 Differentiability of a one variable function

$f(x)$ is **differentiable** at a if and only if the derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}$$

Furthermore, in this case the graph $y = f(x)$ has a **tangent line** at $x = a$ given by the equation:

$$y = f(a) + f'(a)(x - a)$$

1.2 Tangents of a surface

Consider a function $f(x, y)$. If the below partial derivatives exist, it means that the one variable functions $g(x) = f(x, b)$ and $h(y) = f(a, y)$ have tangent lines at $x = a, y = b$.

$$f_x(a, b) = \frac{d}{dx} f(x, b)|_{x=a}$$

$$f_y(a, b) = \frac{d}{dx} f(a, y)|_{x=b}$$

1.3 Tangent plane to a surface

A **tangent plane** should contain both tangent lines. The equation for the tangent plane, **if such a plane exists**, is:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

However, there is a tricky difference between the one variable case and several variables.

- In one variable, if the derivative $f'(a)$ exists, then the graph $z = f(x)$ has a tangent line at $x = a$ and it's given by:

  $$y = f(a) + f'(a)(x - a)$$

- In two variables, even if $f_x(a, b), f_y(a, b)$ both exist, the graph $z = f(x, y)$ **might still not have a tangent plane** at $(x, y) = (a, b)$. f might even fail to be continuous there.

1.3.1 Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

$$f(x,0) = \begin{cases} \frac{2x \cdot 0}{x^2+0^2} = 0 & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases} = 0 \text{ for all } x \tag{1}$$

$$\begin{aligned} f_x(0,0) &= \frac{d}{dx} f(x,0)|_{x=0} \\ &= \frac{d}{dx} 0|_{x=0} \quad \because (1) \\ &= 0 \end{aligned}$$

$$f(0,y) = \begin{cases} \frac{2 \cdot 0 \cdot y}{0^2+y^2} = 0 & \text{when } y \neq 0 \\ 0 & \text{when } y = 0 \end{cases} = 0 \text{ for all } y \tag{2}$$

$$\begin{aligned} f_y(0,0) &= \frac{d}{dy} f(0,y)|_{y=0} \\ &= \frac{d}{dy} 0|_{y=0} \quad \because (2) \\ &= 0 \end{aligned}$$

$$z = f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

At $(0,0)$:

$$z = f(0,0) + f_x(0,0)x + f_y(0,0)y = 0$$

1.4 Differentiability of a two variable function

Consider a function $f(x,y)$. Saying f is differentiable at (a,b), means the same as:

$$\frac{|f(x,y) - f(a,b) - f_x(a,b)(x-a) - f_y(a,b)(y-b)|}{||(x,y)-(a,b)||} \to 0, \text{ as } (x,y) \to (a,b)$$

If $f(x,y)$ is differentiable at (a,b), the graph $z = f(x,y)$ has a tangent plane at $(x,y) = (a,b)$, given by:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(x-b)$$

1.4.1 Example

Consider the function:

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

Is f differentiable at $(0,0)$? If so, what is the tangent plane at that point?

$$\begin{aligned} &\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0)|}{||(x,y)-(0,0)||} \\ &= \lim_{(x,y)\to(0,0)} \frac{|f(x,y) - 0 - 1(x-0) - 0(y-0)|}{||(x,y)-(0,0)||} \\ &= \lim_{(x,y)\to(0,0)} \frac{\left|\frac{x^3}{x^2+y^2} - x\right|}{\sqrt{x^2+y^2}} \end{aligned}$$

Approaching $(0,0)$ along $y = x$ gives us:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\left|\frac{x^3}{x^2+y^2} - x\right|}{\sqrt{x^2+y^2}} &= \frac{\left|\frac{x^3}{2x^2} - x\right|}{\sqrt{2x^2}} \\ &= \frac{\frac{1}{2}}{\sqrt{2}|x|} \\ &= \frac{1}{2\sqrt{2}} \neq 0 \end{aligned}$$

So, indeed, we do not have:

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - f_x(0,0)(x-0) - f_y(0,0)(y-0)|}{||(x,y)-(0,0)||} = 0$$

I.e. $f(x,y)$ is **not** differentiable at $(0,0)$.

1.5 The gradient vector

Consider $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, a point $\boldsymbol{a} \in A$, and suppose all partial derivatives $f_{x_k}(\boldsymbol{a}), k = 1, \ldots, n$ exist. The **gradient vector** (grad $f)(\boldsymbol{a})$ of f at $\boldsymbol{a}$ is the vector:

$$(\text{grad } f)(\boldsymbol{a}) = f_{x_1}(\boldsymbol{a}), f_{x_2}(\boldsymbol{a}), \ldots, f_{x_n}(\boldsymbol{a}) \in \mathbb{R}^n$$

1.5.1 Example

With $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f(x, y, z) = xy + yz + zx$$

We have:

$$(\text{grad } f)(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) = (y + z, x + z, y + x)$$

So, for example:

$$(\text{grad } f)(1, 2, 3) = (5, 4, 3)$$

1.5.2 Note

For $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, such that all partial derivatives exist on A, the gradient $(\text{grad } f)(\boldsymbol{x}) = (f_{x_1}(\boldsymbol{x}), f_{x_2}(\boldsymbol{x}), \ldots, f_{x_n}(\boldsymbol{x}))$ is a vector valued function on A, i.e. grad $f : A \to \mathbb{R}^n$. Such functions are also known as **vector fields**.

1.5.3 Rewriting the differentiability condition

With $\boldsymbol{a} = (a, b), \boldsymbol{x} = (x, y)$, we have:

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = (f_x(a, b), f_y(a, b)) \cdot (x - a, y - b) \quad = (\text{grad } f(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}))$$

Hence, the differentiability condition:

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - f(a, b) - f_x(a, b)(x - a) - f_y(a, b)(y - b)|}{||(x, y) - (a, b)||} = 0$$

Can be rewritten as:

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{a}) - (\text{grad } f)(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a})}{||\boldsymbol{x} - \boldsymbol{a}||} = 0$$

This expression also makes sense for a function f of n variables, regardless of n.

1.6 Vector field

For $A \subset \mathbb{R}^n$, a function $\boldsymbol{F} : A \to \mathbb{R}^n$ is called a **vector field** in $\mathbb{R}^n$.

1.6.1 Example

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by:

$$f(x, y) = x^2 + y^2$$

Then $(\text{grad } f) : \mathbb{R}^2 \to \mathbb{R}^2$ is given by:

$$F(x, y) = (\text{grad } f)(x, y) = (2x, 2y)$$

$$F(0, 0) = (0, 0)$$

$$F(1, 0) = (2, 0)$$

$$F(-1, -1) = (-2, -2)$$

1.7 Differentiability in n variables

For $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, saying that f is **differentiable** at $\boldsymbol{a} \in A$ means the same as:

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{|f(\boldsymbol{x}) - f(\boldsymbol{a}) - (\text{grad } f)(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a})}{||\boldsymbol{x} - \boldsymbol{a}||} = 0$$

1.7.1 Differentiability implies continuity

Consider $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, If f is differentiable at $\boldsymbol{a} \in A$, then f is continuous at $\boldsymbol{a}$.

1.7.2 A sufficient condition for differentiability

Consider $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$. If there exists $\delta > 0$ such that all partial derivatives of f are continuous on $\{x \in \mathbb{R}^n : ||\boldsymbol{x} - \boldsymbol{a} < \delta\}$, then f is differentiable at $\boldsymbol{a}$.

1.8 Tangents in n variables

For a function $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, differentiable at $\boldsymbol{a} \in A$, its **tangent space** at $\boldsymbol{x} = \boldsymbol{a}$ is the graph of the function:

$$T(\boldsymbol{x}) = f(\boldsymbol{a}) + (\text{grad } f)(\boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{a})$$

1.9 Chain rule

Consider $\boldsymbol{g} : A \to \mathbb{R}^n, A \subset \mathbb{R}, f : B \to \mathbb{R}, B \subset \mathbb{R}^n$. Suppose $\boldsymbol{g}$ is differentiable at $a \in A$ and suppose f is differentiable at $\boldsymbol{g}(a)$. Then:

$$\frac{d}{dt} f(\boldsymbol{g}(t))|_{t=a} = (\text{grad } f)(\boldsymbol{g}(a)) \cdot \boldsymbol{g}'(a)$$

Let's say $n = 2$, so with:

$$(x, y) = \boldsymbol{g}(t), \quad \text{and } z = f(x, y) = f(\boldsymbol{g}(t))$$

The theorem tells us that:

$$\begin{aligned}
\frac{dz}{dt} &= \frac{d}{dt} f(\boldsymbol{g}(t)) \\
&= (\text{grad } f)(\boldsymbol{g}(t)) \cdot \boldsymbol{g}'(t) \\
&= \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \\
&= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}
\end{aligned}$$

1.9.1 Example

Let:

$$z = f(x, y) = x^2 y, \quad (x, y) = \boldsymbol{g}(t) = (\sin t, t^2)$$

By the chain rule:

$$\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
&= 2xy \cdot \cos t + x^2 \\
&= 2t^2 \sin t \cos t + 2t \sin^2 t
\end{aligned}$$

1.9.2 Chain rule as a procedure

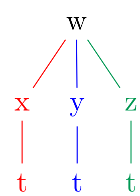
Let:

$$w = f(x, y, z), \quad (x, y, z) = \boldsymbol{g}(t)$$

By the chain rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

We can look at each term as a path in the tree below:



1.10 Laplace equation

Consider a function $f(x, y)$. The Laplace equation is:

$$f_{xx} + f_{yy} = 0$$

Or for a 3 variable function $f(x, y, z)$:

$$f_{xx} + f_{yy} + f_{zz} = 0$$

A function satisfying the Laplace equation is said to be **harmonic**.

1.11 Rate of change

For a real valued function $f(x, y)$:

$$f_x(a, b) = \frac{d}{dx}(x, b)|_{x=a} \text{ measures the rate of change of } f \text{ as } x \text{ increases}$$

$$f_y(a, b) = \frac{d}{dx}(a, y)|_{y=b} \text{ measures the rate of change of } f \text{ as } y \text{ increases}$$

We can rewrite the above as:

$$\begin{aligned} f_x(a, b) &= \frac{d}{dx}(x, b)|_{x=a} \\ &= \frac{d}{dt} f(a + t, b)|_{t=0} \\ &= \frac{d}{dt} f((a, b) + t(1, 0))|_{t=0} \end{aligned}$$

$$\begin{aligned} f_y(a, b) &= \frac{d}{dy}(a, y)|_{y=b} \\ &= \frac{d}{dt} f(a, b + t)|_{y=b} \\ &= \frac{d}{dt} f((a, b) + t(0, 1))|_{t=0} \end{aligned}$$

1.12 Directional derivative

Consider $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$, a point $\boldsymbol{a} \in A$, and a **unit vector** $\boldsymbol{u} \in \mathbb{R}^n$. The **directional derivative** $D_{\boldsymbol{u}} f(\boldsymbol{a})$ of f at $\boldsymbol{a}$ in the direction $\boldsymbol{u}$, provided the derivative exists, is defined as:

$$D_{\boldsymbol{u}} f(\boldsymbol{a}) = \frac{d}{dt} f(\boldsymbol{a} + t\boldsymbol{u})|_{t=0}$$

1.12.1 Example

For $f(x, y) = x^2 y$, find the directional derivative of f at $(2, 1)$ in the direction of $(1, 1)$.

A unit vector in the direction of $(1, 1)$ is:

$$\frac{1}{||(1,1)||}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \boldsymbol{u}$$

$$\begin{aligned}
D_{\boldsymbol{u}} f(2,1) &= \frac{d}{dt} f((2,1) + t\left(\frac{1}{\sqrt{2}}), \frac{1}{\sqrt{2}}\right)\Bigg|_{t=0} \\
&= \frac{d}{dt} f\left(2 + \frac{t}{\sqrt{2}}, 1 + \frac{t}{\sqrt{2}}\right)\Bigg|_{t=0} \\
&= \frac{d}{dt}\left(2 + \frac{t}{\sqrt{2}}\right)^2\left(1 + \frac{t}{\sqrt{2}}\right)\Bigg|_{t=0} \\
&= \frac{d}{dt}\left(2\left(2 + \frac{t}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}}\left(1 + \frac{t}{\sqrt{2}}\right) + \left(2 + \frac{t}{\sqrt{2}}\right)^2 \cdot \frac{1}{\sqrt{2}}\right)\Bigg|_{t=0} \\
&= 2 \cdot 2 \cdot \frac{1}{\sqrt{2}} \cdot 1 + 2^2 \cdot \frac{1}{\sqrt{2}} \\
&= \frac{8}{\sqrt{2}} \\
&= 4\sqrt{2}
\end{aligned}$$

1.13 Directional derivatives of differentiable functions

Consider $f : A \to \mathbb{R}, A \subset \mathbb{R}^n, \boldsymbol{a} \in A$, and a unit vector $\boldsymbol{u} \in \mathbb{R}^n$. If f is **differentiable** at $\boldsymbol{a}$, then:

$$D_{\boldsymbol{u}} f(\boldsymbol{a}) = (\text{grad } f)(\boldsymbol{a}) \cdot \boldsymbol{u}$$

1.13.1 Example

For $f(x,y) = x^2y$, find the directional derivative of f at $(2,1)$ in the direction of $(1,1)$.

Unit vector in the direction of $(1,1)$ is $\boldsymbol{u} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

$$f_x(x,y) = 2xy, \quad f_y(x,y) = x^2 \text{ are both continuous.}$$

Hence, f is differentiable.

$$\begin{aligned} D_{\boldsymbol{u}}f(2,1) &= (\text{grad } f)(2,1) \cdot \boldsymbol{u} \\ &= (4,4) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\ &= \frac{8}{\sqrt{2}} \\ &= 4\sqrt{2} \end{aligned}$$

1.14 Maximum/minimum and zero directional derivative

Suppose $f(x_1, \ldots, x_m)$ be differentiable at $\boldsymbol{a} \in \mathbb{R}^n$, and suppose $(\text{grad } f)(\boldsymbol{a}) \neq \mathbf{0}$. Consider the directional derivative $D_{\boldsymbol{u}}f(\boldsymbol{a})$ for different unit vector $\boldsymbol{u}$.

1. The maximum value of $D_{\boldsymbol{u}}f(\boldsymbol{a})$ is $||\text{grad } f(\boldsymbol{a})||$ and is attained when $\boldsymbol{u}$ points in the direction of grad $f(\boldsymbol{a})$.

2. The minimum value of $D_{\boldsymbol{u}}f(\boldsymbol{a})$ is $||-\text{grad } f(\boldsymbol{a})||$ and is attained when $\boldsymbol{u}$ points in the opposite direction of grad $f(\boldsymbol{a})$.

3. $D_{\boldsymbol{u}}f(\boldsymbol{a}) = 0$ if and only if $\boldsymbol{u}$ is orthogonal to grad $f(\boldsymbol{a})$.

1.15 Tangent space

Suppose $f : A \to \mathbb{R}, A \subset \mathbb{R}^n$ is a differentiable at $\boldsymbol{a}$ and with $(\text{grad } f)(\boldsymbol{a}) \neq \mathbf{0}$. Let $c = f(\boldsymbol{a})$ and let S be the level set:

$$S = \{\boldsymbol{x} \in A : f(\boldsymbol{x}) = c\}$$

With the **tangent space** of S at $\boldsymbol{a}$, we mean the set:

$$T = \{\boldsymbol{x} \in \mathbb{R}^n : (\text{grad } f)(\boldsymbol{a})(\boldsymbol{x} - \boldsymbol{a}) = 0\}$$

1.15.1 Example

Let S be the surface given by the equation:

$$x^3 + y^2 - z^2 = 0$$

And let $\boldsymbol{a} = (2, 1, 3)$. Note that $\boldsymbol{a} \in S$. What is the tangent space for S at $\boldsymbol{a}$?

Let:

$$f(x, y, z) = x^3 + y^2 - z^2$$

S is a level surface of f:

$$S : f(x, y, z) = 0$$

Hence, a normal vector for S at $\boldsymbol{a}$ is (grad f)($\boldsymbol{a}$):

$$(\text{grad } f)(x, y, z) = (3x^2, 2y, -2z)$$

So (grad$f$)$(2, 1, 3) = (12, 2, -6)$. An equation for the tangent space (tangent plane) at $\boldsymbol{a}$ is:

$$\begin{aligned} 12(x-2) + 2(y-1) - 6(z-3) &= 0 \\ 6x + y - 3z &= 4 \end{aligned}$$

2 Property of tangents in one variable

In **one variable**, let's look at the size of the **absolute error** compared to $|x - a|$ when we approximate $y = f(x)$ with its tangent line:

$$y = f(a) + f'(a)(x - a)$$

We get:

$$\begin{aligned} \frac{\text{absolute error}}{|x-a|} &= \frac{|f(x) - f(a) - f'(a)(x-a)|}{|x-a|} \\ &= \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| \to 0, \text{ as } x \to a \end{aligned}$$

2.1 Generalising to two variables

Consider a function $f(x, y)$ of **two** variables. We approximate it near $(x, y) = (a, b)$ with the plane:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

For this plane to really be a **tangent plane**, we need the same behaviour:

$$\frac{\text{absolute error}}{||(x, y) - (a, b)||} \to 0, \text{ as } (x, y) \to (a, b)$$

I.e.

$$\frac{|f(x, y) - f(a, b) - f_x(a, b)(x - a) - f_y(a, b)(y - b)|}{||(x, y) - (a, b)||} \to 0, \text{ as } (x, y) \to (a, b)$$