1a) Assuming A^{-1} and B^{-1} exist,

$$A = \frac{1}{2}(AB + B)$$

$$A = \frac{1}{2}(A + I)B$$

$$AB^{-1} = \frac{1}{2}(A + I)$$

$$A^{-1}AB^{-1} = \frac{1}{2}A^{-1}(A + I)$$

$$B^{-1} = \frac{1}{2}(I + A^{-1})$$

$$B^{-1}A = \frac{1}{2}(I + A^{-1})A$$

$$B^{-1}A = \frac{1}{2}(A + I)$$

$$\therefore AB^{-1} = B^{-1}A$$

$$A = B^{-1}AB$$

$$BA = AB \text{ (shown)}$$

(b) $\left|\begin{matrix} a^2 & 2 & -3 \\ 2a & 2 & 2 \\ 1 & 1 & 1 \end{matrix}\right| \begin{matrix} a^2 & 2 \\ 2a & 2 \\ 1 & 1 \end{matrix}$

$= \cancel{2a^2} + 4 - 6a + 6 - \cancel{2a^2} - 4a$

$= 10 - 10a$

$\therefore\ a = 1$ for non-unique solution

$a \neq 1$ for unique solution

When $a = 1$,

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 2 & 2 & b \\ 1 & 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 1 \\ 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & b-8 \end{bmatrix}$$

When $b \neq 8$, the system has no solutions

When $b = 8$, the system has infinitely many solutions.

1c) $\det(A - I\lambda)$

$$\left|\begin{array}{ccc|cc} a-6-\lambda & 1 & 0 & a-6-\lambda & 1 \\ -4 & 3-\lambda & 0 & -4 & 3-\lambda \\ 1 & 0 & 2-\lambda & 1 & 0 \end{array}\right.$$

$$= (a-6-\lambda)(3-\lambda)(2-\lambda) + 8 - 4\lambda$$

$$= (a-6-\lambda)(3-\lambda)(2-\lambda) + 4(2-\lambda)$$

$$= (2-\lambda)\left[(a-6-\lambda)(3-\lambda) + 4\right]$$

$$= (2-\lambda)\left[3a - a\lambda - 18 + 6\lambda - 3\lambda + \lambda^2 + 4\right]$$

$$= (2-\lambda)\left[\lambda^2 + (3-a)\lambda + (3a-14)\right]$$

when $\lambda = 1$,

$$1\left[1^2 + 3 - a + 3a - 14\right] = 0$$

$$2a - 10 = 0$$

$$a = 5$$

$$\therefore \det(A - \lambda I) = (2-\lambda)\left[\lambda^2 - 2\lambda + 1\right]$$

$$= (2-\lambda)(\lambda-1)^2$$

$$\therefore \lambda = 1, 2$$

when $\lambda = 1$,

$$\begin{bmatrix} -1-1 & 1 & 0 & 0 \\ -4 & 3-1 & 0 & 0 \\ 1 & 0 & 2-1 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 & 0 \\ -4 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + z = 0$$

$$x = -z$$

$$y + 2z = 0$$

$$y = -2z$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

when $\lambda = 2$,

$$\begin{bmatrix} -1-2 & 1 & 0 & 0 \\ -4 & 3-2 & 0 & 0 \\ 1 & 0 & 2-2 & 0 \end{bmatrix} \sim \begin{bmatrix} -3 & 1 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1c) $x = 0, \ y = 0$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ s \in \mathbb{R}$$

2a) $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

$$= \sqrt{(2+b)^2 + (8t)^2 + (3b+2)^2} \, dt$$

$$= \sqrt{64t^2 + 4 + 4b + b^2 + 9b^2 + 12b + 4} \, dt$$

$$= \sqrt{64t^2 + 10b^2 + 16b + 8} \, dt$$

when $x = 2+b$,

$$2+b = 2t + bt$$

$$\therefore t = 1$$

when $x = 4+2b$,

$$4+2b = 2t + bt$$

$$\therefore t = 2$$

2a) $\int_C (8x + 6\sqrt{y} - 7z)\, ds$

$$= \int_1^2 (16t + 8bt + 12t - 21bt - 14t)\sqrt{64t^2 + 10b^2 + 16b + 8}\, dt$$

$$= \int_1^2 (14t - 13bt)\sqrt{64t^2 + 10b^2 + 16b + 8}\, dt$$

$$= (14 - 13b)\int_1^2 \frac{1}{128}(128t)\sqrt{64t^2 + 10b^2 + 16b + 8}\, dt$$

$$= \frac{14-13b}{128} \times \frac{2}{3}\left[(64t^2 + 10b^2 + 16b + 8)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{14-13b}{192}\left[(256 + 10b^2 + 16b + 8)^{\frac{3}{2}} - (64 + 10b^2 + 16b + 8)^{\frac{3}{2}}\right]$$

$$= \frac{14-13b}{192}\left[(264 + 10b^2 + 16b)^{\frac{3}{2}} - (72 + 10b^2 + 16b)^{\frac{3}{2}}\right]$$

2b) $\underset{\sim}{F} = \left(yz\cos(xyz)+2y,\ xz\cos(xyz)+2x,\ xy\cos(xyz)+1 \right)$

$$\frac{\partial \phi}{\partial x} = yz\cos(xyz)+2y$$

$$\phi = \int yz\cos(xyz)+2y\,dx$$

$$= \sin(xyz)+2xy+f(y,z)$$

$$\frac{\partial \phi}{\partial y} = xz\cos(xyz)+2x$$

$$\phi = \int xz\cos(xyz)+2x\,dy$$

$$= \sin(xyz)+2xy+f(x,z)$$

$$\frac{\partial \phi}{\partial z} = xy\cos(xyz)+1$$

$$\phi = \int xy\cos(xyz)+1\,dz$$

$$= \sin(xyz)+z+f(x,y)$$

By observation,

$$\phi = \sin(xyz)+2xy+z$$

2b) $W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$

$= \int_C \nabla \phi \cdot (dx, dy, dz)$

$= \int_C \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$

$= \int_C d\phi$

$= \left[\phi \right]_P^Q$

$= \left[\sin\left(1 \times 4 \times \frac{\pi}{8}\right) + 2(1)(4) + \frac{\pi}{8} - 0 - 0 - 2 \right]$

$= 1 + 8 + \frac{\pi}{8} - 2$

$= 7 + \frac{\pi}{8}$

2c) $z = x^2 + y^2$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = 2y$$

$$d\sigma = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

$$= \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$\iint_S \frac{xy + (ax)^2}{z} \sqrt{1+4x^2+4y^2}\, dA$$

$$= \iint_S \frac{xy + (ax)^2}{x^2+y^2} \sqrt{1+4x^2+4y^2}\, dA$$

$$= \iint_S \frac{\cancel{r^2}\cos\theta\sin\theta + a^2 r^2 \cos\theta}{\cancel{r^2}} \sqrt{1+4r^2}\, r\, dr\, d\theta$$

$$= \iint_S r\left(\cos\theta\sin\theta + a^2\cos^2\theta\right) \sqrt{1+4r^2}\, dr\, d\theta$$

$$= \frac{1}{2}\iint_S r\left(\sin 2\theta + a^2\cos^2\theta\right) \sqrt{1+4r^2}\, dr\, d\theta$$

$$= \frac{1}{2}\int_{r=2}^{3} \int_{\theta=0}^{\frac{\pi}{4}} \left(\sin 2\theta + a^2\cos^2\theta\right) r\sqrt{1+4r^2}\, dr\, d\theta$$

$$= \frac{1}{24}\int_0^{\frac{\pi}{4}} \left(\sin 2\theta + a^2\cos^2\theta\right) \left[(1+4r^2)^{\frac{3}{2}}\right]_2^3\, d\theta$$

2c) $\iint_S \frac{xy+(ax)^2}{z}\,d\sigma$

$$= \frac{1}{24}\int_0^{\frac{\pi}{4}} (\sin 2\theta + a^2\cos^2\theta)\left[(1+4r^2)^{\frac{3}{2}}\right]_2^3 d\theta$$

$$= \frac{1}{24}\left(37^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)\int_0^{\frac{\pi}{4}} \sin 2\theta + a^2 + a^2\cos 2\theta \, d\theta$$

$$= \frac{1}{24}\left(37^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)\left[-\frac{\cos 2\theta}{2} + a^2\theta + \frac{a^2\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

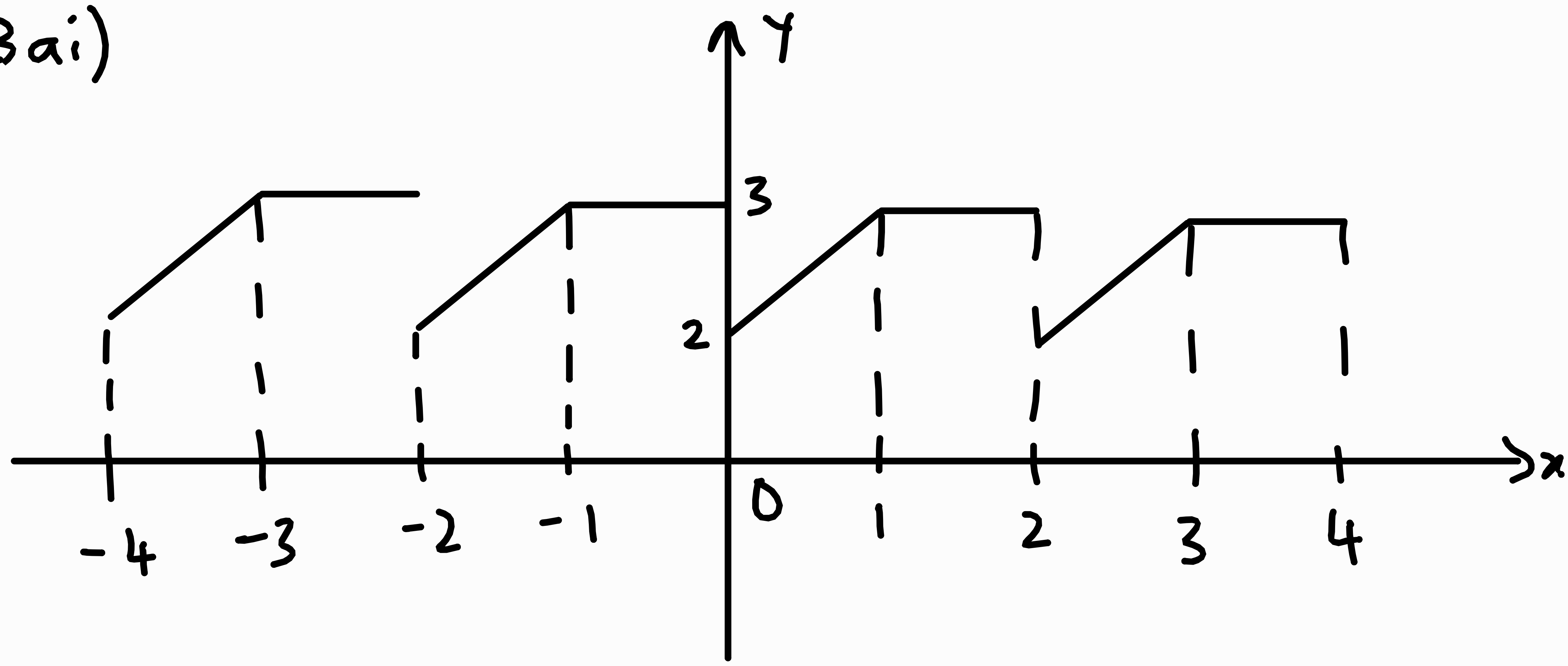
$$= \frac{1}{48}\left(37^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)\left[2a^2\theta + a^2\sin 2\theta - \cos 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{48}\left(37^{\frac{3}{2}} - 17^{\frac{3}{2}}\right)\left[\frac{\pi}{2}a^2 + a^2 - 1\right]$$

$$= 8.299891886a^2 - 3.228529541$$

$$\approx 8.3a^2 - 3.23$$

3ai)



aii) $$a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 2+x\,dx + \int_1^2 3\,dx\right]$$

$$= \frac{1}{2}\left[\left[2x+\frac{x^2}{2}\right]_0^1 + \left[3x\right]_1^2\right]$$

$$= \frac{5}{4} + \frac{3}{2}$$

$$= \frac{11}{4}$$

3aii) $a_n = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$= \int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 (2+x)\cos(n\pi x)\,dx + \int_1^2 3\cos(n\pi x)\,dx$$

$$= \int_0^1 2\cos(n\pi x) + x\cos(n\pi x)\,dx + \cancel{\frac{3}{n\pi}\left[\sin(n\pi x)\right]_1^2}$$

$$= \cancel{\frac{2}{n\pi}\left[\sin(n\pi x)\right]_0^1} + \frac{1}{(n\pi)^2}\left[\cos(n\pi x) + \cancel{n\pi x\sin(n\pi x)}\right]_0^1$$

$$= \frac{1}{(n\pi)^2}\left(\cos(n\pi) - 1\right)$$

$b_n = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 2\sin(n\pi x) + x\sin(n\pi x)\,dx + \int_1^2 3\sin(n\pi x)\,dx$$

$$= -\frac{2}{n\pi}\left[\cos(n\pi x)\right]_0^1 + \frac{1}{(n\pi)^2}\left[\cancel{\sin(n\pi x)} - n\pi x\cos(n\pi x)\right]_0^1$$

$$- \frac{3}{n\pi}\left[\cos(n\pi x)\right]_1^2$$

$$= -\frac{1}{(n\pi)^{\cancel{2}}}\cancel{n\pi}\cos(n\pi) + \frac{2}{n\pi}\left(1-\cos(n\pi)\right) - \frac{3}{n\pi}\left(1-\cos(n\pi)\right)$$

$$= -\frac{1}{n\pi}\left(\cancel{\cos(n\pi)} + \left(1 - \cancel{\cos(n\pi)}\right)\right)$$

$$= -\frac{1}{n\pi}$$

$$\therefore \text{F.S} = \frac{11}{4} + \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\left(\cos(n\pi) - 1\right)\cos(n\pi x) - \frac{1}{n\pi}\sin(n\pi x)$$

3aiii) At $x=1$, the graph is continuous, hence

$$F.S(x=1)=3$$

At $x=2$, the graph is not continuous, hence

$$F.S(x=2)=\frac{1}{2}\left(\lim_{x\to 2^-} f(x)+\lim_{x\to 2^+} f(x)\right)$$

$$=\frac{1}{2}(3+2)$$

$$=\frac{5}{2}$$

3b) $h(x)=x^{10}\sin(x)$

$$a_0=\frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi} x^{10}\sin(x)\,dx$$

$$=\frac{1}{\pi}\int_{0}^{\pi} x^{10}\sin(x)\,dx$$

$$a_n=\frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} x^{10}\sin(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$=\frac{2}{\pi}\int_{0}^{\pi} x^{10}\sin(x)\cos(nx)\,dx$$

$$F.S=\frac{1}{\pi}\int_0^{\pi} x^{10}\sin(x)dx+\frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi} x^{10}\sin(x)\cos(nx)dx\cos(nx)$$

4a) $f(t) = 4t^2\left(u(t-0) - u(t-2)\right) + 16\left(u(t-2) - u(t-3)\right)$
$+ 16e^{2t-6}u(t-3)$
$= 4u(t-0)t^2 - u(t-2)t^2 + 16\left(u(t-2) - u(t-3)\right)$
$+ 16u(t-3)e^{2t-6}$

Let $g(t-a) = t^2$
$g(t) = (t+a)^2$
$g(t) = t^2 + 2at + a^2$

Let $h(t-a) = e^{2t-6}$
$h(t) = e^{2t+2a+6}$

$$\mathcal{L}\{f(t)\} = \frac{8}{s^3} - 4\mathcal{L}\{u(t-2)[t^2+4t+4]\} + \frac{16e^{-2s}}{s}$$
$$- \frac{16e^{-3s}}{s} + \frac{16e^{-3s}}{s-2}$$
$$= \frac{8}{s^3} - 4e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right) + \frac{16e^{-2s}}{s}$$
$$- \frac{16e^{-3s}}{s} + \frac{16e^{-3s}}{s-2}$$
$$= \frac{8}{s^3}\left(1 - e^{-2s}\right)\left(1 + 2s + 2s^2\right) + \frac{16}{s}\left[e^{-2s} - e^{-3s}\right]$$
$$+ \frac{16e^{-3s}}{s-2}$$

4a) For $\mathcal{L}\{f(t)\}$ to exist, $s > 2$.

bi) $$p(s) = \ln(\sqrt{s^2+25}) - \ln(\sqrt{s^2-25})$$
$$= \frac{1}{2}(\ln(s^2+25) - \ln(s^2-25))$$
$$p'(s) = \frac{1}{2}\left[\frac{2s}{s^2+25} - \frac{2s}{s^2-25}\right]$$
$$(-1)p'(s) = \frac{s}{s^2-25} - \frac{s}{s^2+25}$$
$$\mathcal{L}^{-1}\{-p'(s)\} = \cosh(5t) - \cos(5t)$$
$$\therefore t\,p(t) = \cosh(5t) - \cos(5t)$$
$$p(t) = \frac{1}{t}(\cosh(5t) - \cos(5t))$$

bii) $$s(s^2+36)p(s) = 1$$
$$p(s) = \frac{1}{s(s^2+6^2)}$$
$$\mathcal{L}\{p(t)\} = \frac{1}{s}\left(\frac{1}{s^2+6^2}\right)$$
$$\mathcal{L}\{p(t)\} = \frac{1}{s}\left(\frac{1}{6}\right)\left(\frac{6}{s^2+6^2}\right)$$
$$\mathcal{L}\{p(t)\} = \frac{1}{s}\left(\frac{1}{6}\right)\mathcal{L}\{\sin 6t\}$$
$$\mathcal{L}\{p(t)\} = \frac{1}{6}\mathcal{L}\left\{\int_0^t \sin 6\tau \, d\tau\right\}$$
$$p(t) = -\frac{1}{6}\left(\frac{1}{6}\right)\left[\cos 6\tau\right]_0^t$$
$$p(t) = -\frac{1}{36}(\cos 6t - 1)$$
$$= \frac{1}{36}(1 - \cos 6t)$$

4c) $y'' - 4y' + 4y = 0$

Let $Y = \mathcal{L}\{y\}$,

$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$

$= s^2Y - 3s - 7$

$\mathcal{L}\{y'\} = sY - y(0)$

$= sY - 3$

$\mathcal{L}\{y'' - 4y' + 4y\} = s^2Y - 3s - 7 - 4(sY - 3) + 4Y$

$= s^2Y - 4sY - 3s + 4Y + 5$

$= Y(s^2 - 4s + 4) - 3s + 5$

$\mathcal{L}\{y'' - 4y' + 4y\} = 0$

$Y(s^2 - 4s + 4) - 3s + 5 = 0$

$$Y = \frac{3s - 5}{s^2 - 4s + 4}$$

$$Y = \frac{3s - 5}{(s-2)^2}$$

$$Y = \frac{3(s-2) + 1}{(s-2)^2}$$

$$Y = \frac{3}{s-2} + \frac{1}{(s-2)^2}$$

4c) $y(t) = 3e^{2t} + te^{2t}$

$= e^{2t}(3+t)$