1a) $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} -1 & -1 \\ 8 & 8 \end{bmatrix}$

$AC = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$CA = \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 9 & 9 \\ -9 & -9 \end{bmatrix}$

1b) $A(B+C) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$\therefore A(B+C) = AB + AC$

c) $A(BC) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\left(\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}\right)$

$= \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$

1c) $(AB)C = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$$

$\therefore A(BC) = (AB)C$

d) $(AB)^T = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}^T$

$$= \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

$B^T A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

$\therefore (AB)^T = B^T A^T$

2) $$CB = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -2 \\ 0 & -4 \\ 8 & 6 \end{bmatrix}$$

$$B^T C = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 8 \\ -3 & 3 & 11 \end{bmatrix}$$

$$C\underset{\sim}{a} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 9 \\ 19 \end{bmatrix}$$

$$\underset{\sim}{d}C = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & 7 \end{bmatrix}$$

$$\underset{\sim}{a}\underset{\sim}{d} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 6 \\ 1 & 0 & 2 \\ 4 & 0 & 8 \end{bmatrix}$$

2) $\underset{\sim}{d}\underset{\sim}{a} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 11 \end{bmatrix}$

BC is not defined.

3) $S = \frac{1}{2}(A + A^T)$

$S^T = \left(\frac{1}{2}(A + A^T)\right)^T$

$= \frac{1}{2}(A + A^T)^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(A + A^T)$

$= S$ (shown)

3) $T = \frac{1}{2}(A - A^T)$

$$T^T = \left(\frac{1}{2}(A - A^T)\right)^T$$

$$= \frac{1}{2}(A - A^T)^T$$

$$= \frac{1}{2}(A^T - (A^T)^T)$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T)$$

$$= -T \text{ (shown)}$$

$$S + T = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}A + \cancel{\frac{1}{2}A^T} + \frac{1}{2}A - \cancel{\frac{1}{2}A^T}$$

$$= \frac{1}{2}A + \frac{1}{2}A$$

$$= A \text{ (shown)}$$

4) Assuming $AB = BA$,

For AB to be symmetric,

$$(AB)^T = AB$$

$$(AB)^T = B^T A^T$$

Since A and B are symmetric,

$$(AB)^T = BA \quad \because B^T = B \text{ and } A^T = A$$

Since $AB = BA$,

$$(AB)^T = AB$$

$\therefore AB$ is symmetric

Assuming AB is symmetric,

$$AB = (AB)^T$$

$$AB = B^T A^T$$

Since A and B are symmetric

$$AB = BA \quad \because B^T = B \text{ and } A^T = A$$

5a) $\begin{bmatrix} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 42 \\ -4 & 7 & 20 \\ 9 & 2 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 10 & 42 \\ 0 & 47 & 188 \\ 0 & -88 & -363 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 10 & 42 \\ 0 & 1 & 4 \\ 0 & 88 & 363 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 11 \end{bmatrix}$

No solution

b) $\begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ -3 & -17 & 1 & 2 & 2 \\ 4 & -17 & 8 & -5 & 2 \\ 0 & -5 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & -14 & 4 & 2 & 20 \\ 0 & -21 & 4 & -5 & -22 \\ 0 & -5 & -2 & 1 & 2 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 1 & 1 & 0 & 6 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & -21 & 4 & -5 & -22 \\ 0 & -5 & -2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{1}{7} & \frac{52}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & -21 & 4 & -5 & -22 \\ 0 & -5 & -2 & 1 & 2 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{1}{7} & \frac{52}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & -2 & -8 & -52 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{36}{7} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{1}{7} & \frac{52}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{10}{7} \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{36}{7} \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & -5 & -26 \\ 0 & 1 & 0 & 1 & 6 \\ 0 & 0 & 1 & 4 & 26 \\ 0 & 0 & 0 & 14 & 84 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 6 \end{bmatrix}$

$\therefore w = 4, x = 0, y = 2, z = 6$

5c) $\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ -5 & 2 & -5 & 4 & -5 \\ -3 & -4 & 7 & -2 & 7 \\ 2 & 3 & 1 & -11 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & -3 & 10 & -11 & 10 \\ 0 & -7 & 16 & -11 & 16 \\ 0 & 5 & -5 & -5 & -5 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & -1 & 3 & -3 & 3 \\ 0 & 1 & -\frac{10}{3} & \frac{11}{3} & -\frac{10}{3} \\ 0 & -7 & 16 & -11 & 16 \\ 0 & 5 & -5 & -5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{10}{3} & \frac{11}{3} & -\frac{10}{3} \\ 0 & 0 & -\frac{22}{3} & \frac{44}{3} & -\frac{22}{3} \\ 0 & 0 & \frac{35}{3} & -\frac{70}{3} & \frac{35}{3} \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{10}{3} & \frac{11}{3} & -\frac{10}{3} \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & \frac{35}{3} & -\frac{70}{3} & \frac{35}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore z \in \mathbb{R},\ y = 1 + 2z,\ x = 3z,\ w = 0$

5d) $\begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ -3 & -17 & 1 & 2 & 1 \\ 4 & -17 & 8 & -5 & 1 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & -14 & 4 & 2 & 10 \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 1 & 1 & 0 & 3 \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{5}{7} \\ 0 & -21 & 4 & -5 & -11 \\ 0 & -5 & -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{1}{7} & \frac{26}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{5}{7} \\ 0 & 0 & -2 & -8 & -26 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{18}{7} \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & \frac{9}{7} & \frac{1}{7} & \frac{26}{7} \\ 0 & 1 & -\frac{2}{7} & -\frac{1}{7} & -\frac{5}{7} \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & -\frac{24}{7} & \frac{2}{7} & -\frac{18}{7} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 & -13 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & 14 & 42 \end{bmatrix} \sim$

$\begin{bmatrix} 1 & 0 & 0 & -5 & -13 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 13 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

$\therefore p = 2, q = 0, r = 1, s = 3$