# Math Module 4B Notes

Hankertrix

April 29, 2025

## Contents

# 1 Definitions

## 1.1 Differential equation (d.e or DE)

A differential equation is an equation involving one or more derivatives of an unknown function. The order of the highest derivative occurring in a differential equation is called the **order** of the differential equation.

### 1.1.1 Examples

$$\frac{dy}{dx} + x^2 = y$$

$$\frac{d^2y}{dx^2} = -k^2 y$$

$$\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^5 + \cos x = 0$$

$$\sin\left(\frac{dy}{dx}\right) + \arctan y = 1$$

The equations above are all differential equations in the unknown function $y(x)$.

## 1.2 Solution to a differential equation

Consider an $n$th order differential equation in the unknown $y$. A function $y(x)$ which is (at least) $n$ times differentiable on an interval $I$ is called a **solution** to the differential equation on $I$, if the substitution $y = y(x), y' = y'(x), \ldots, y^{(n)} = y^{(n)}(x)$ reduces the differential equation to an identity valid for all $x \in I$.

A solution to a differential equation is sometimes also called a **particular solution**.

The **general** solution to a differential equation is the collection of all (particular) solutions.

## 1.3 Initial-value problems (i.v.p)

An $n$th-order differential equation together with $n$ **initial conditions** of the form:

$$y(x_0) = y_0$$
$$y'(x_0) = y_0$$
$$\vdots$$
$$y^{(n-1)}(x_0) = y_{n-1}$$

Where $y_0, y_1, \ldots, y_{n-1}$ are constants, is called an **initial-value problem**. A solution to an initial value problem is a function that satisfies both the stated differential equations and all the initial conditions.

## 1.4 Separable differential equations

A first-order differential equation is called **separable** if it can be written in the form:

$$p(y)y' = q(x)$$

A separable differential equation is just a differential equation that we can separate $x$ and $y$.

If $p$ and $q$ are continuous, we can solve $p(y)y' = q(x)$ by integrating both sides with respect to $x$:

$$\int p(y)y'\,dx = \int q(x)\,dx$$
$$\int p(y)\,dy = \int q(x)\,dx$$

## 1.5 Linear differential equation

A differential equation that can be written in the form:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

Where $a_0, a_1, \ldots, a_n$ and $F$ are functions of $x$ only, is called a **linear** differential equation of order $n$.

Basically, the degree of all terms in $y$ is **at most one**.

### 1.5.1 Examples

$$I \cdot y'' + x^2y + (\sin x)y = e^x \tag{1}$$

$$xy''' + 4x^2y' - \frac{2}{1+x^2}y = 0 \tag{2}$$

The equations above ((1) and (2)) are **linear** differential equations.

$$y'' + x\sin(y') - xy = x^2 \tag{3}$$

$$y'' + x^2y' + y^2 = 0 \tag{4}$$

The equations above ((3) and (4)) are **nonlinear** differential equations.

## 1.6 Linear initial value problems

Let $a_1, a_2, \ldots, a_n, F$ be functions that are continuous on an interval $I$. Then, for any $x_0 \in I$, the initial-value problem below has a unique solution on $I$.

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x)$$

$$y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}x_0 = y_{n-1}$$

## 1.7 First-order linear differential equations

A differential equation that can be written in the form:

$$a(x)\frac{dy}{dx} + b(x)y = r(x)$$

Where $a(x), b(x),$ and $r(x)$ are functions defined on an interval $(a, b)$, is called a **first-order linear differential equation**.

If $a(x) \neq 0$ on $(a, b)$, we have the **standard form**:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Where $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{r(x)}{a(x)}$.

Let $P(x)$ be an antiderivative to $p(x)$ and multiply with $e^{P(x)}$ to get:

$$e^{P(x)}\frac{dy}{dx} + p(x)e^{P(x)}y = q(x)e^{P(x)}$$

$$\frac{d}{dx}\left(e^{P(x)}y\right) = q(x)e^{P(x)}$$

We can solve the problem by integration.

## 1.8 Homogeneous differential equation

Homogeneous just means that the differential equation is equal to 0.

### 1.9 Second-order linear differential equation

A **second-order linear** differential equation, has the form:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$$

Here, $a_0, a_1, a_2$ and $F$ are functions defined on an interval $I$. If $F(x) = 0$ for all $x \in I$, we say that the equation is **homogeneous**.

If $a_0(x) \neq 0$ on $I$, dividing gives the **standard form**:

$$y'' + p(x)y' + q(x)y = f(x)$$

#### 1.9.1 Theorem

For a **homogeneous** linear differential equation:

$$a(x)y'' + b(x)y' + c(x)y = 0$$

If $y_1(x)$ and $y_2(x)$ are two solutions on the interval $I$, then any **linear combination** below is also a solution on $I$:

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

Where $C_1, C_2$ are constants.

The linearity principle holds **only** for differential equations that are **both homogeneous** and **linear**. The result is stated above for a second-order linear homogeneous equation, but the analogous result holds for $n$th order linear homogeneous equations.

## 1.10 Linearly dependent functions

Two functions defined on the interval $I$ are said to be **linearly dependent** on $I$, if one function is a **scalar (constant) multiple** of another function.

### 1.10.1 Example

$$y_1(x) = \cos 2x$$
$$y_2(x) = 3(1 - 2\sin^2 x)$$

$$\begin{aligned} y_2(x) &= 3(1 - 2\sin^2 x) \\ &= 3\left(1 - 2 \cdot \frac{1 - \cos 2x}{2}\right) \\ &= 3(1 - (1 - \cos 2x)) \\ &= 3\cos 2x \\ &= 3y_1(x) \end{aligned}$$

Since $y_2(x) = 3y_1(x), y_1(x)$ and $y_2(x)$ are **linearly dependent**.

## 1.11 Linearly independent functions

Two functions defined on the interval $I$ are said to be **linearly independent** if one function is **not** a scalar (constant) multiple of another function.

### 1.11.1 Example

$$y_1(x) = e^x$$
$$y_2(x) = xe^x$$

Since neither $e^x$ nor $xe^x$ is a constant multiple of the other, the two functions are **linearly independent**.

### 1.11.2 Theorem

Let $I$ be an interval and consider the equations:

$$y'' + p(x)y' + q(x)y = f(x) \tag{3}$$

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

Let $y_1(x), y_2(x)$ be **linearly independent** solutions of (4) and $y_p(x)$ a solution of (3) on $I$. Then:

- The general solution of (4) on $I$ is:

$$y(x) = C_1y_1(x) + C_2y_2(x), \quad C_1, C_2 \in \mathbb{R}$$

- The general solution to (3) on $I$ is:

$$y(x) = C_1y_1(x) + C_2y_2(x) + y_p(x), \quad C_1, C_2 \in \mathbb{R}$$

So, the **general solution** to the non-homogeneous differential equation $y'' + p(x)y' + q(x)y = f(x)$ is of the form:

$$y(x) = y_c(x) + y_p(x)$$

Where $y_c(x) = C_1y_1(x)+C_2y_2(x)$ is the general solution to the associated homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0$$

And $y_p$ is a particular solution to:

$$y'' + p(x)y' + q(x)y = f(x)$$

### 1.12 Characteristic equation

Consider a homogeneous linear differential equation of order 2 with **constant coefficients**.

$$ay'' + by' + cy = 0$$

Where $a, b, c \in \mathbb{R}, a \neq 0$.

Since, in order to satisfy the equation, we want constant multiples of $y$ and its derivatives to cancel, we should look for solutions of this differential equation of the form $y(x) = e^{rx}$ (since derivatives of $y$ are constant multiples of $y$). Trying this:

With $y(x) = e^{rx}$, we get:

$$y'(x) = re^{rx},\ y''(x) = r^2 e^{rx}$$

Substituting the above equation into the differential equation $ay'' + by' + cy = 0$, we get:

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0$$

Since $e^{rx} > 0$, the equation is satisfied only if:

$$ar^2 + br + c = 0$$

The quadratic equation above is called the **characteristic equation** for the differential equation above.

## 2 Solving first-order differential equations (FODEs)

In order to solve first-order differential equations of the form:

$$\frac{dy}{dx} + p(x)y = q(x) \tag{1}$$

We will use a method called the integrating factor. Introducing an integrating factor called $\mu$ to the equation above:

$$\mu\frac{dy}{dx} + \mu p(x)y = \mu q(x)$$

Considering the product rule to simplify the above equation:

$$\frac{d}{dx}(uv) = u\frac{du}{dx} + v\frac{du}{dx}$$

Comparing this with the left-hand side of the original first-order differential equation, we have:

$$u = \mu \tag{2}$$

$$\frac{du}{dx} = \frac{dy}{dx} \tag{3}$$

$$v = y \tag{4}$$

$$\frac{du}{dx} = \mu p(x) \tag{5}$$

Equations (2) and (5) yield:

$$\frac{d\mu}{dx} = \frac{du}{dx} = \mu p(x)$$

Solving this with the separation of variables:

$$\frac{d\mu}{dx} = \mu p(x)$$

$$\frac{1}{\mu}\frac{d\mu}{dx} = p(x)$$

$$\int \frac{1}{\mu}\, d\mu = \int p(x)\, dx$$

$$\ln|\mu| = \int p(x)\, dx$$

$$\mu = e^{\int p(x)\, dx} \tag{6}$$

Substituting (6) into (1):

$$e^{\int p(x)\,dx}\frac{dy}{dx}+e^{\int p(x)\,dx}y=e^{\int p(x)\,dx}q(x)$$

Using equations $(2),(3),(4),(5)$ and $(6)$ with the product rule:

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right)=e^{\int p(x)\,dx}q(x)$$

Integrating both sides with respect to $x$, we get:

$$e^{\int p(x)\,dx}y=\int q(x)e^{\int p(x)\,dx}\,dx$$

In summary, reduce the given first-order differential equation into the form $\frac{dy}{dx}+p(x)y=q(x)$, then find the integrating factor with $\mu=e^{\int p(x)\,dx}$ and multiply every term by it. Apply the product rule to obtain $\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right)=e^{\int p(x)\,dx}q(x)$. Then integrate both sides with respect to $x$ and solve for $y$.

# 3 Solving second-order differential equations (SODEs)

## 3.1 Solving linear homogeneous second-order differential equations

To solve linear (degree of all terms is at most 1) homogeneous (the equation is equal to 0) second-order differential equations of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

We are finding the **general solution** of $y$. First, we need to identify the **characteristic or auxiliary equation** of the second-order differential equation. It is given by:

$$am^2+bm+c=0$$

Using the quadratic formula,

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We now have three cases for the different types of roots.

### 3.1.1 Case 1: Roots are real and distinct ($b^2 - 4ac > 0$)

The general solution is:

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Where $C_1$ and $C_2$ are constants to be found.

### 3.1.2 Case 2: Roots are real and equal ($b^2 - 4ac = 0$)

The general solution is:

$$y = (C_1 + C_2 x) e^{mx}$$

### 3.1.3 Case 3: Roots are complex ($b^2 - 4ac < 0$)

The general solution is:

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)), \quad m = \alpha + \beta i$$

## 3.2 Solving linear non-homogeneous second-order differential equations

To solve linear non-homogeneous second-order differential equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

We must first find the **complimentary function**, which is the function $y = q(x)$. When this function is substituted into the second-order differential equation, the right-hand side is 0 (similar to the **general solution** of a linear homogeneous second-order differential equation). After which, we must find the **particular solution**, which is the function $y = p(x)$. When this function is substituted into the second-order differential equation, it gives us $f(x)$. Finally, the general solution to the linear non-homogeneous second-order differential equation is given by:

$$y = \textbf{Complimentary function} + \textbf{Particular solution}$$

To find the particular solution, we must consider 3 cases.

### 3.2.1 Case 1: $f(x)$ is a polynomial of degree $n, f(x) = a_0 + a_1 x + \ldots + a_n x^n$

The particular solution is a polynomial with degree equal to the degree of $f(x)$.

$$p(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n$$

### 3.2.2 Case 2: $f(x) = (c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)e^{kx},\ c_n \in \mathbb{R}$

1. The complimentary function does not have $e^{kx}$

   The particular solution is:

$$p(x) = (c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)e^{kx}$$

2. The complimentary function has $e^{kx}$ but not $xe^{kx}$

   The particular solution is:

$$p(x) = x(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)e^{kx}$$

3. The complimentary function has $e^{kx}$ and $xe^{kx}$

   The particular solution is:

$$p(x) = x^2(c_0 + c_1x + c_2x^2 + \ldots + c_nx^n)e^{kx}$$

4. The complimentary function is $q(x) = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x))$

   The particular solution is:

$$p(x) = pe^{kx}$$

### 3.2.3 Case 3: $f(x) = k\cos(ax), k\sin(ax)$ or $k\cos(ax) + r\sin(ax)$

1. The complimentary function does not have $A\cos(ax) + B\sin(ax)$

   The particular solution is:

$$p(x) = p\cos(ax) + q\sin(ax)$$

2. The complimentary function has $A\cos(ax) + B\sin(ax)$

   The particular solution is:

$$p(x) = x(p\cos(ax) + q\sin(ax))$$

### 3.2.4 After the particular solution is found

Once we find the particular solution, we must find its first and second derivatives, $p'(x)$ and $p''(x)$. After which, we substitute them into the original second-order differential equation to find the constants $p$ and $q$. And now, the full general solution to the linear non-homogeneous second-order differential equation is:

$$y = q(x) + p(x)$$