1a) Q being orthorgonal means

$$Q^T = Q^{-1}$$

b) Since A is symmetric, P exists.

$$\det(\lambda I - A) = 0$$

$$\left|\begin{matrix} \lambda-5 & 1 & -2 \\ 1 & \lambda-5 & -2 \\ -2 & -2 & \lambda-2 \end{matrix}\right|\begin{matrix} \lambda-5 & 1 \\ 1 & \lambda-5 \\ -2 & -2 \end{matrix} = 0$$

$$(\lambda-5)^2(\lambda-2) + 4 + 4 - 4(\lambda-5) - 4(\lambda-5) - \lambda + 2 = 0$$

$$(\lambda^2 - 10\lambda + 25)(\lambda-2) + 10 - 8(\lambda-5) - \lambda = 0$$

$$\lambda^3 - 2\lambda^2 - 10\lambda^2 + 20\lambda + 25\lambda \cancel{-50} \cancel{+10} - 8\lambda \cancel{+40} - \lambda = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda = 0$$

$$\lambda(\lambda^2 - 12\lambda + 36) = 0$$

$$\lambda(\lambda-6)^2 = 0$$

$\therefore \lambda = 0$ or $\lambda = 6$

1b) For $\lambda = 0$,

$$\begin{bmatrix} \lambda-5 & 1 & -2 & 0 \\ 1 & \lambda-5 & -2 & 0 \\ -2 & -2 & \lambda-2 & 0 \end{bmatrix} \sim \begin{bmatrix} -5 & 1 & -2 & 0 \\ 1 & -5 & -2 & 0 \\ -2 & -2 & -2 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -5 & -2 & 0 \\ 0 & -24 & -12 & 0 \\ 0 & -12 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - y = 0 \qquad 2y + z = 0$$
$$x = y \qquad z = -2y$$

$\therefore$ the eigenvectors are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \ s \in \mathbb{R}\setminus\{0\}$$

$$= t\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}, \ t \in \mathbb{R}\setminus\{0\}$$

1b) Normalising the vector:

$$\underset{\sim}{x}_1 = \frac{1}{\sqrt{1+1+4}}(-1,-1,2)$$

$$= \frac{1}{\sqrt{6}}(-1,-1,2)$$

For $\lambda = 6$,

$$\begin{bmatrix} \lambda-5 & 1 & -2 & 0 \\ 1 & \lambda-5 & -2 & 0 \\ -2 & -2 & \lambda-2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ -2 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + y - 2z = 0$$

$$x = 2z - y$$

$\therefore$ the eigenvectors are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R},$$

$$s = t \neq 0$$

1b) Normalising the vectors and getting an orthonormal basis:

$$\underset{\sim}{x}_2 = \frac{1}{\sqrt{1+1}}(-1,1,0)$$

$$= \frac{1}{\sqrt{2}}(-1,1,0)$$

Let $\underset{\sim}{y} = (2,0,1)$

$$\underset{\sim}{x}'_3 = \underset{\sim}{y} - \mathrm{proj}_{\underset{\sim}{x}_2}\underset{\sim}{y}$$

$$= (2,0,1) - \frac{1}{\sqrt{2}}(-1,1,0)\cdot(2,0,1)\cdot \frac{1}{\sqrt{2}}(-1,1,0)$$

$$= (2,0,1) \cancel{- \frac{1}{2}(-2)}(-1,1,0)$$

$$= (2,0,1) + (-1,1,0)$$

$$= (1,1,1)$$

$$\underset{\sim}{x}_3 = \frac{\underset{\sim}{x}'_3}{\|\underset{\sim}{x}'_3\|}$$

$$= \frac{1}{\sqrt{3}}(1,1,1)$$

1b) $\therefore$ The orthonormal basis

$$B = \left\{ \frac{1}{\sqrt{6}}(-1,-1,2), \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{3}}(1,1,1) \right\}$$

$$\therefore P = \begin{bmatrix} -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$D = P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

1c) Suppose Q is orthorgonal,

$$Q^T = Q^{-1}$$

$$\begin{aligned}
\|Q\underset{\sim}{x}\|^2 &= (Q\underset{\sim}{x}) \cdot (Q\underset{\sim}{x}) \\
&= (Q\underset{\sim}{x})^T \cdot (Q\underset{\sim}{x}) \\
&= (\underset{\sim}{x}^T Q^T) \cdot (Q\underset{\sim}{x}) \\
&= \underset{\sim}{x}^T Q^T Q \underset{\sim}{x} \\
&= \underset{\sim}{x}^T I \underset{\sim}{x} \\
&= \underset{\sim}{x}^T \underset{\sim}{x} \\
&= \|\underset{\sim}{x}\|^2
\end{aligned}$$

Since norms are non-negative,

$$\|Q\underset{\sim}{x}\|^2 = \|\underset{\sim}{x}\|^2$$

$$\Downarrow$$

$$\|Q\underset{\sim}{x}\| = \|\underset{\sim}{x}\| = \alpha$$

2) Finding an orthonormal basis for W:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}(1,0,10)$$

$$\mathbf{v}'_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1}\mathbf{x}_2$$

$$= (0,1,1,1) - \frac{1}{\sqrt{2}}(1,0,1,0)\cdot(0,1,1,1)\,\frac{1}{\sqrt{2}}(1,0,1,0)$$

$$= (0,1,1,1) - \frac{1}{2}(1)(1,0,1,0)$$

$$= \left(-\frac{1}{2}, 1, \frac{1}{2}, 1\right)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{\frac{1}{4}+1+\frac{1}{4}+1}}\left(-\frac{1}{2}, 1, \frac{1}{2}, 1\right)$$

$$= \sqrt{\frac{2}{5}}\sqrt{\frac{1}{4}}(-1,2,1,2)$$

$$= \frac{1}{\sqrt{10}}(-1,2,1,2)$$

$$\text{proj}_w \mathbf{b} = (\mathbf{b}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{b}\cdot\mathbf{v}_2)\mathbf{v}_2$$

$$= (1,1,1,1)\cdot\frac{1}{\sqrt{2}}(1,0,1,0)\,\frac{1}{\sqrt{2}}(1,0,1,0)$$

$$+ (1,1,1,1)\cdot\frac{1}{\sqrt{10}}(-1,2,1,2)\,\frac{1}{\sqrt{10}}(-1,2,1,2)$$

$$= \cancel{\frac{1}{2}}\cancel{(2)}(1,0,1,0) + \frac{1}{10}(4)(-1,2,1,2)$$

$$= \frac{1}{5}(3,4,7,4)$$

3) $$\lim_{x\to 0} \frac{\sin x - x\cos x}{(\ln(x+1))^2 + 2\cos x - 2}$$

$$= \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + o(x^5) - x\left(1 - \frac{x^2}{2} + o(x^4)\right)}{\left(x - \frac{x^2}{2} + o(x^3)\right)^2 + 2\left(1 - \frac{x^2}{2} + o(x^4)\right) - 2}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{3!} - \cancel{x} + \frac{x^3}{2} + o(x^5)}{\cancel{x^2} - x^3 + o(x^4) \cancel{+2} \cancel{-x^2} + o(x^4) \cancel{-2}}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3} + o(x^5)}{-x^3 + o(x^4)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3} + o(x^2)}{-1 + o(x)}$$

$$= -\frac{1}{3}$$