1(a) $B - I = AB$

$B - AB = I$

$(I - A)B = I$

$\therefore$ B has an inverse as

$(I - A)B = I$

$\therefore B^{-1} = I - A$

$C - A = AC$

$A = C - CA$

$A = C(I - A)$

$A = CB^{-1}$

$AB = C$

$$B - B^{-1} = I + AB - (I - A)$$
$$= \cancel{I} + AB - \cancel{I} + A$$
$$= C + A \text{ (Proven)}$$

1b) Let

$$\begin{array}{ccc|cc} 4-k & 3 & 1 & 4-k & 3 \\ 3 & -(4-k) & 7 & 3 & -(4-k) \\ 1 & 7 & -(6-k) & 1 & 7 \end{array}$$

$$= (4-k)^2(6-k) + 21 + 21 - \left(-(4-k) + 49(4-k) - 9(6-k)\right)$$

$$= (4-k)^2(6-k) + 42 + 4 - k - 196 + 49k + 54 - 9k$$

$$= (16 - 8k + k^2)(6-k) - 96 + 39k$$

$$= \cancel{96} - 16k - 48k + 8k^2 + 6k^2 - k^3 \cancel{-96} + 39k$$

$$= -k^3 + 14k^2 - 25k$$

$$= k(-k^2 + 14k - 25)$$

$$= k(k^2 - 14k + 25)$$

$\therefore$ $k = 0$

$k = 11.89897949 \approx 11.9$

$k = 2.101020514 \approx 2.10$

1b) For unique solutions,

$k \neq 0$

$k \neq 11.9$

$k \neq 2.10$

For no solutions other than the trivial solution,

$k = 11.9$

For infinite solutions,

$k = 2.10$

1c) $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

when $\underset{\sim}{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$,

$$\begin{bmatrix} a & -1 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

$$\begin{bmatrix} a-2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

$\therefore \lambda = 1,$

$a = 5$

$\det(A - \lambda I)$

$$\left|\begin{array}{ccc|cc} 5-\lambda & -1 & -1 & 5-\lambda & -1 \\ -1 & 4-\lambda & 0 & -1 & 4-\lambda \\ -1 & 0 & 4-\lambda & -1 & 0 \end{array}\right.$$

$= (5-\lambda)(4-\lambda)^2 - 4 + \lambda - 4 + \lambda$

$= (5-\lambda)(4-\lambda)^2 - 2(4-\lambda)$

$= (4-\lambda)((5-\lambda)(4-\lambda) - 2)$

1c) $\det(A) = (4-\lambda)\left((5-\lambda)(4-\lambda)-2\right)$

$$= (4-\lambda)(20-5\lambda-4\lambda+\lambda^2-2)$$

$$= (4-\lambda)(\lambda^2-9\lambda+18)$$

$$= (4-\lambda)(3-\lambda)(6-\lambda)$$

$\therefore \lambda = 3, 4, 6$

For $\lambda = 4$,

$$\begin{bmatrix} 5-4 & -1 & -1 & 0 \\ -1 & 4-4 & 0 & 0 \\ -1 & 0 & 4-4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = 0$$

$$y + z = 0$$

$$z = -y$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}$$

$\therefore$ the eigenvector is $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(c) when $\lambda = 6$,

$$\begin{bmatrix} 5-6 & -1 & -1 & 0 \\ -1 & 4-6 & 0 & 0 \\ -1 & 0 & 4-6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore x + 2z = 0$

$x = -2z$

$y - z = 0$

$y = z$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \ t \in \mathbb{R}$$

$\therefore$ the eigenvector is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

2ai) $a_0 = \frac{1}{2}\int_{\alpha}^{\alpha+2L} f(x)\,dx$

$$= \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\int_0^1 \sin(\pi x) + \frac{1}{2}\int_1^2 x\,dx$$

$$= \frac{1}{2}\left[\frac{-\cos(\pi x)}{\pi}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$

$$= -\frac{1}{2\pi}(-1-1) + \frac{1}{4}(4-1)$$

$$= \frac{1}{\pi} + \frac{3}{4}$$

$$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_0^2 f(x)\cos(n\pi x)\,dx$$

$$= \int_0^1 \sin(\pi x)\cos(n\pi x)\,dx + \int_1^2 x\cos(n\pi x)\,dx$$

$$= \frac{1}{2}\int_0^1 \sin(\pi x + n\pi x) + \sin(\pi x - n\pi x)\,dx$$

$$+ \cancel{\frac{1}{n\pi}\left[x\sin(n\pi x)\right]_1^2} - \frac{1}{n\pi}\int_1^2 \sin(n\pi x)\,dx$$

2ai) $$a_n = \frac{1}{2}\int_0^1 \sin((n+1)\pi x) + \sin((1-n)\pi x)\,dx$$

$$+ \frac{1}{(n\pi)^2}\Big[\cos(n\pi x)\Big]_1^2$$

$$= -\frac{1}{2}\left[\frac{\cos((n+1)\pi x)}{(n+1)\pi} + \frac{\cos((1-n)\pi x)}{(1-n)\pi}\right]_0^1$$

$$= -\frac{1}{2\pi}\left[\frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((1-n)\pi)}{1-n} - (1+1)\right]$$

$$+ \frac{1}{(n\pi)^2}\Big[1 - \cos(n\pi)\Big]$$

$$= \frac{1}{2\pi}\left[2 - \frac{\cos((n+1)\pi)}{n+1} - \frac{\cos((1-n)\pi)}{1-n}\right]$$

$$+ \frac{1}{(n\pi)^2}\Big[1 - \cos(n\pi)\Big]$$

2ai) $b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \sin(\pi x)\sin(n\pi x)\,dx + \int_1^2 x\sin(n\pi x)\,dx$$

$$= \frac{1}{2}\int_0^1 \cos((1-n)\pi x) - \cos((1+n)\pi x)\,dx -$$

$$\frac{1}{n\pi}\Big[x\cos(n\pi x)\Big]_1^2 + \frac{1}{n\pi}\int_1^2 \sin(n\pi x)\,dx$$

$$= \cancel{\frac{1}{2\pi}\left[\frac{\sin((1-n)\pi x)}{1-n} - \frac{\sin((1+n)\pi x)}{1+n}\right]_0^1}$$

$$- \frac{1}{n\pi}\Big[2 - \cos(n\pi)\Big]$$

$$= -\frac{1}{n\pi}\Big[2 - \cos(n\pi)\Big]$$

$$F.S = \frac{1}{\pi} + \frac{3}{4}$$

$$+ \sum_{n=1}^{\infty}\left[\frac{1}{2\pi}\left[2 - \frac{\cos((1+n)\pi x)}{1+n} - \frac{\cos((1-n)\pi x)}{1-n}\right]\cos(n\pi x)\right.$$

$$\left. - \frac{1}{n\pi}\Big[2 - \cos(n\pi)\Big]\sin(n\pi x)\right.$$

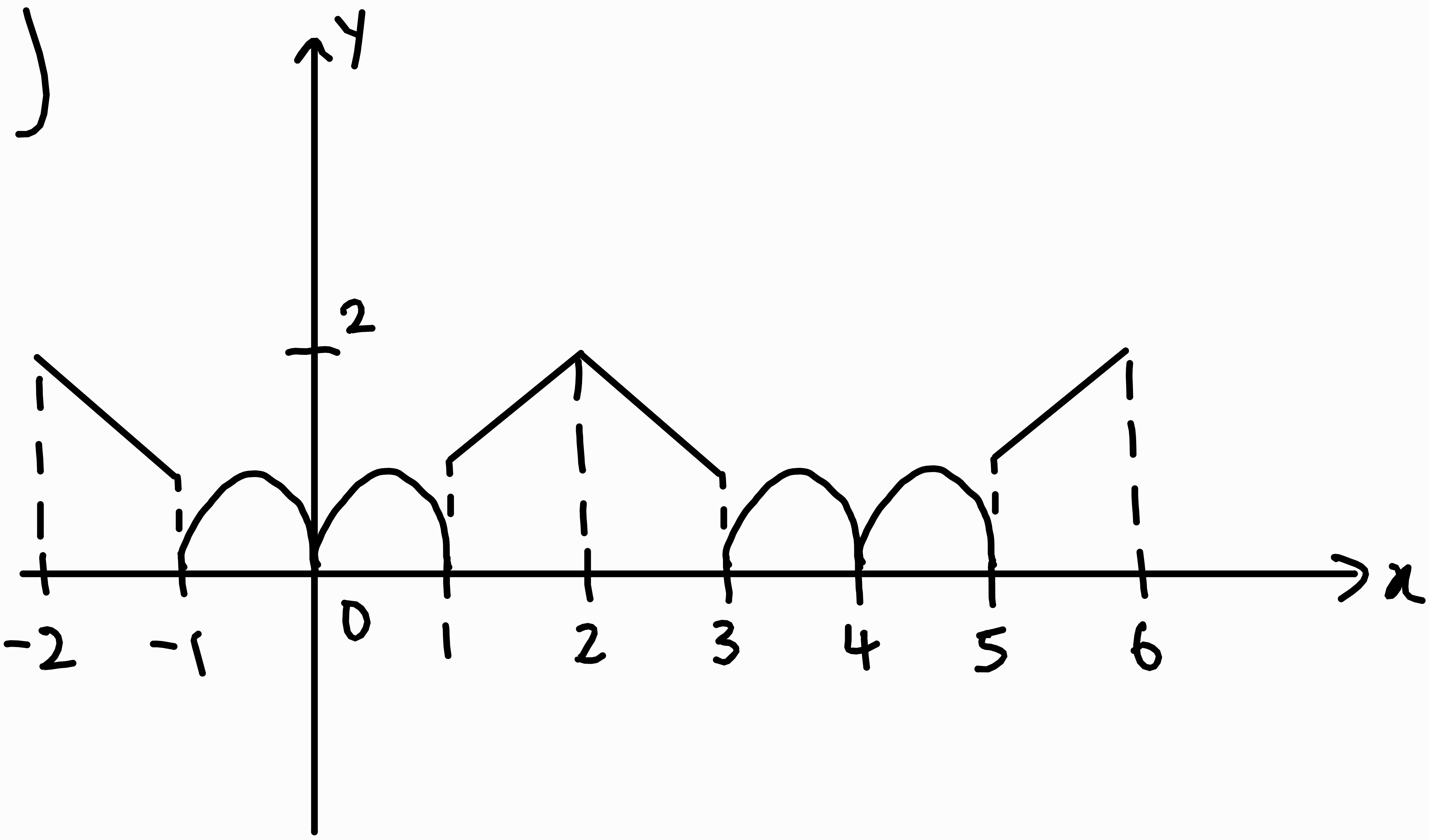
2aii) When $x=-1$, $f(x)$ is not continuous

$$F.S = \frac{1}{2}\left(\lim_{x\to -1^-} f(x) + \lim_{x\to -1^+} f(x)\right]$$

$$= \frac{1}{2}(0-1)$$

$$= -\frac{1}{2}$$

2bi)



2bii) Period is 4, so $L=2$,

$$a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$= \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{0}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{0}^{1} \sin(\pi x)\,dx + \frac{1}{2}\int_{1}^{2} x\,dx$$

$$a_n = \frac{1}{L}\int_{a}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{1} \sin(\pi x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$F.S = \frac{1}{2}\int_{0}^{1}\sin(\pi x)\,dx + \frac{1}{2}\int_{1}^{2} x\,dx$$
$$+ \sum_{n=1}^{\infty}\left[\int_{0}^{1}\sin(\pi x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx\right]\left(\cos\left(\frac{n\pi x}{2}\right)\right)$$

3a) $\underset{\sim}{a} = \underset{\sim}{b} \times \underset{\sim}{c}$, $\underset{\sim}{b} = \underset{\sim}{c} \times \underset{\sim}{a}$, $\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b}$

$\therefore$ $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$ and $\underset{\sim}{c}$

$\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{c}$

$\underset{\sim}{c}$ is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$

$$\underset{\sim}{a} = |\underset{\sim}{b}||\underset{\sim}{c}|\sin\theta = |\underset{\sim}{b}||\underset{\sim}{c}| \quad \because \theta = 90^\circ$$

$$\underset{\sim}{b} = |\underset{\sim}{c}||\underset{\sim}{a}|\sin\varphi = |\underset{\sim}{c}||\underset{\sim}{a}| \quad \because \varphi = 90^\circ$$

$$\underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{b}|\sin\alpha = |\underset{\sim}{a}||\underset{\sim}{b}| \quad \because \alpha = 90^\circ$$

$$|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{b}||\underset{\sim}{c}| \cdot |\underset{\sim}{b}||\underset{\sim}{c}|$$

$$= |\underset{\sim}{b}|^2|\underset{\sim}{c}|^2$$

$$\therefore |\underset{\sim}{a}| = |\underset{\sim}{b}||\underset{\sim}{c}|$$

$$|\underset{\sim}{b}|^2 = \underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{c}| \cdot |\underset{\sim}{a}||\underset{\sim}{c}|$$

$$= |\underset{\sim}{a}|^2|\underset{\sim}{c}|^2$$

$$\therefore |\underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{c}|$$

$$|\underset{\sim}{c}|^2 = \underset{\sim}{c} \cdot \underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{b}| \cdot |\underset{\sim}{a}||\underset{\sim}{b}|$$

$$= |\underset{\sim}{a}|^2|\underset{\sim}{b}|^2$$

$$\therefore |\underset{\sim}{c}| = |\underset{\sim}{a}||\underset{\sim}{b}|$$

3a) $|\underset{\sim}{a}| + |\underset{\sim}{b}| + |\underset{\sim}{c}|$

$= |\underset{\sim}{a}| + |\underset{\sim}{a}||\underset{\sim}{c}| + |\underset{\sim}{c}|$

$= |\underset{\sim}{a}|\left(1 + |\underset{\sim}{c}| + \frac{|\underset{\sim}{c}|}{|\underset{\sim}{a}|}\right)$

$= |\underset{\sim}{a}|\left(1 + |\underset{\sim}{c}| + |\underset{\sim}{b}|\right)$

$= |\underset{\sim}{a}| + |\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{a}||\underset{\sim}{c}|$

By comparison, $|\underset{\sim}{a}| = 1$

$|\underset{\sim}{a}| + |\underset{\sim}{b}| + |\underset{\sim}{c}|$

$= |\underset{\sim}{b}||\underset{\sim}{c}| + |\underset{\sim}{b}| + |\underset{\sim}{c}|$

$= |\underset{\sim}{b}|\left(|\underset{\sim}{c}| + 1 + \frac{|\underset{\sim}{c}|}{|\underset{\sim}{b}|}\right)$

$= |\underset{\sim}{b}|\left(|\underset{\sim}{c}| + 1 + |\underset{\sim}{a}|\right)$

$= |\underset{\sim}{b}||\underset{\sim}{a}| + |\underset{\sim}{b}| + |\underset{\sim}{b}||\underset{\sim}{c}|$

By comparison, $|\underset{\sim}{b}| = 1$

$\therefore |\underset{\sim}{c}| = |\underset{\sim}{a}||\underset{\sim}{b}|$

$= 1 \times 1$

$= 1$

$\therefore |\underset{\sim}{a}| + |\underset{\sim}{b}| + |\underset{\sim}{c}| = 1 + 1 + 1$

$= 3$

3b) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$

$= \sqrt{(1+b)^2 + (2t)^2 + (b-2)^2}\ dt$

$= \sqrt{1+2b+b^2+4t^2+b^2-4b+4}\ dt$

$= \sqrt{4t^2+2b^2-2b+5}\ dt$

When $x = 2+2b$,

$t+bt = 2+2b$

$t = 2$

when $x = 4+4b$

$t+bt = 4+4b$

$t = 4$

3b) $\int_c (3x - 2\sqrt{y} + z)\, ds$

$$= \int_2^4 (3t + 3bt - 2t + bt - 2t)\sqrt{4t^2 + 2b^2 - 2b + 5}\, dt$$

$$= \int_2^4 (4bt - t)\sqrt{4t^2 + 2b^2 - 2b + 5}\, dt$$

$$= (4b-1)\int_2^4 t\sqrt{4t^2 + 2b^2 - 2b + 5}\, dt$$

$$= \frac{1}{8}(4b-1)\left[\frac{(4t^2 + 2b^2 - 2b + 5)^{\frac{3}{2}}}{\frac{3}{2}}\right]_2^4$$

$$= \frac{3}{16}(4b-1)\left[(4(4)^2 + 2b^2 - 2b + 5)^{\frac{3}{2}} - (4(2)^2 + 2b^2 - 2b + 5)^{\frac{3}{2}}\right]$$

$$= \frac{3}{16}(4b-1)\left[(69 + 2b^2 - 2b)^{\frac{3}{2}} - (21 + 2b^2 - 2b)^{\frac{3}{2}}\right]$$

3c) $\underset{\sim}{F} = (yz - y\sin(xy), xz - x\sin(xy), xy)$

$$\frac{\partial \phi}{\partial x} = \int yz - y\sin(xy)\,dx$$
$$= yzx + \cos(xy) + f(y,z)$$

$$\frac{\partial \phi}{\partial y} = \int xz - x\sin(xy)\,dy$$
$$= xzy + \cos(xy) + f(x,z)$$

$$\frac{\partial \phi}{\partial z} = \int xy\,dz$$
$$= xyz$$

By observation,

$$\phi = xyz + \cos(xy)$$

$$W = \int_C \underset{\sim}{F} \cdot d\underset{\sim}{r}$$
$$= \int_C \nabla\phi \cdot (dx, dy, dz)$$
$$= \int_C \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} \partial y + \frac{\partial \phi}{\partial z} dz$$
$$= \int_C d\phi$$

3c) $W = \int_c d\phi$

$$= [\phi]_P^Q$$

$$= \left[\cancel{4} \times \frac{\pi}{\cancel{4}} \times 3 + \cos\left(\cancel{4} \times \frac{\pi}{\cancel{4}}\right) - (0 + 1)\right]$$

$$= 3\pi - 2$$

4a) $f(t) = u(t-0)(t+1)^2 e^{-t} - u(t-1)(t+1)^2 e^{-t}$

Let $g(t-a) = (t+1)^2 e^{-t}$

$$g(t) = (t+a+1)^2 e^{-t-a}$$

when $a = 0$,

$$g(t-0) = (t+1)^2 e^{-t}$$

$$= (t^2 + 2t + 1) e^{-t}$$

$$\mathcal{L}\{g(t-0)\} = \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

when $a = 1$,

$$g(t-1) = (t+2)^2 e^{-t-1}$$

$$= (t^2 + 4t + 4) e^{-t-1}$$

4a) $\mathcal{L}\{g(t-1)\} = \frac{1}{e}\left[\frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{s+1}\right]$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-0)g(t-0)\} - \mathcal{L}\{u(t-1)g(t-1)\}$

$$= \frac{2}{(s+3)^3} + \frac{2}{(s+1)^2} + \frac{1}{s+1}$$

$$- e^{-(s+1)}\left(\frac{2}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4}{s+1}\right)$$

$$= \frac{2(1-e^{-s-1})}{(s+1)^3} + \frac{2(1-2e^{-s-1})}{(s+1)^2} + \frac{1-4e^{-s-1}}{s+1}$$

4b) Let $\mathcal{L}\{f(t)\} = \dfrac{5s^2+11s-5}{(s-2)(5s^2+6s+5)}$

$$= \frac{1}{s-2} + \frac{5s-10}{(s-2)(5s^2+6s+5)}$$

$$= \frac{1}{s-2} + \frac{5}{5s^2+6s+5}$$

$$= \frac{1}{s-2} + \frac{1}{s^2+1.2s+1}$$

$$= \frac{1}{s-2} + \frac{1}{(s+0.6)^2+0.8^2}$$

$$= \frac{1}{s-2} + \frac{1}{0.8}\left(\frac{0.8}{(s+0.6)^2+0.8^2}\right)$$

$\therefore f(t) = e^{2t} + 1.25e^{-0.6t}\sin(0.8t)$

4c) $\mathcal{L}\{y''+y'-12y\}$

$$= s^2\mathcal{L}\{y\} - \cancel{sy(0)} - \cancel{y'(0)} + s\mathcal{L}\{y\} - \cancel{y(0)}$$
$$-12\mathcal{L}\{y\}$$

$$= s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} - 12\mathcal{L}\{y\}$$

$$12\mathcal{L}\{u(t-0)-u(t-1)\} = s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} - 12\mathcal{L}\{y\}$$

$$\frac{12}{s} - \frac{12e^{-s}}{s} = s^2\mathcal{L}\{y\} + s\mathcal{L}\{y\} - 12\mathcal{L}\{y\}$$

$$12(1-e^{-s}) = \mathcal{L}\{y\}s(s^2+s-12)$$

$$\mathcal{L}\{y\} = \frac{12(1-e^{-s})}{s(s-3)(s+4)}$$

$$\frac{1}{12}\mathcal{L}\{y\} = \frac{1}{s(s-3)(s+4)} - \frac{e^{-s}}{s(s-3)(s+4)}$$

$$\frac{1}{12}\mathcal{L}\{y\} = -\frac{1}{12s} + \frac{1}{21(s-3)} + \frac{1}{28(s+4)} - e^{-s}\Big[$$
$$-\frac{1}{12s} + \frac{1}{21(s-3)} + \frac{1}{28(s+4)}\Big]$$

$$= \mathcal{L}\Big\{-\frac{1}{12} + \frac{1}{21}e^{3t} + \frac{1}{28}e^{-4t} + \frac{1}{12}u(t-1)$$
$$- \frac{1}{21}u(t-1)e^{3t-3} - \frac{1}{28}u(t-1)e^{-4t+4}\Big]$$

4c) $\therefore y(t) = -1 + \frac{4}{7}e^{3t} + \frac{3}{7}e^{-4t}$

$+ u(t-1) - \frac{4}{7}u(t-1)e^{3t-3}$

$- \frac{3}{7}u(t-1)e^{-4t+4}$