1a) $\det(\lambda I - A) = 0$

$$\left|\begin{array}{ccc|cc} \lambda-2 & 0 & 2 & \lambda-2 & 0 \\ 0 & \lambda-3 & 0 & 0 & \lambda-3 \\ 0 & 0 & \lambda-3 & 0 & \lambda-3 \end{array}\right| = 0$$

$$(\lambda-2)(\lambda-3)^2 = 0$$

$$\therefore \lambda = 2, \lambda = 3$$

For $\lambda = 2$,

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad s \in \mathbb{R}\setminus\{0\}$$

For $\lambda = 3$,

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + 2z = 0$$

$$x = -2z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}\setminus\{0\}$$

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

1b) $\det(\lambda I - A) = 0$

$$\left|\begin{matrix} \lambda+1 & -4 & 2 \\ 3 & \lambda-4 & 0 \\ 3 & -1 & \lambda-3 \end{matrix}\right| \begin{matrix} \lambda+1 & 4 \\ -3 & \lambda-4 \\ -3 & 1 \end{matrix} = 0$$

$(\lambda+1)(\lambda-4)(\lambda-3) - 6 - 6(\lambda-4) + 12(\lambda-3) = 0$

$(\lambda+1)(\lambda^2 - 3\lambda - 4\lambda + 12) - 6 - 6\lambda + 24 + 12\lambda - 36 = 0$

$(\lambda+1)(\lambda^2 - 7\lambda + 12) + 6\lambda - 18 = 0$

$\lambda^3 - 7\lambda^2 + 12\lambda + \lambda^2 - 7\lambda + 12 + 6\lambda - 18 = 0$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\therefore \lambda = 1, 2, 3$

For $\lambda = 1$,

$$\begin{bmatrix} 2 & -4 & 2 & 0 \\ 3 & -3 & 0 & 0 \\ 3 & -1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - y = 0 \rightarrow x = y$

$y - z = 0 \rightarrow y = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; s \in \mathbb{R} \setminus \{0\}$$

1b) For $\lambda = 2$,

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ 3 & -2 & 0 & 0 \\ 3 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -3 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 3 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3x - 2y = 0 \rightarrow 3x = 2y$$

$$y - z = 0 \rightarrow y = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}, \ s \in \mathbb{R} \setminus \{0\}$$

1b) For $\lambda = 3$,

$$\begin{bmatrix} 4 & -4 & 2 & 0 \\ 3 & -1 & 0 & 0 \\ 3 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -4 & 0 & 1 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$-4x + z = 0$ $\quad$ $3x - y = 0$

$z = 4x$ $\quad$ $3x = y$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

$$\therefore P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \quad P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

1c) $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda+2 & 0 & 0 & 0 \\ 0 & \lambda+2 & 0 & 0 \\ 0 & 0 & \lambda-3 & 0 \\ 0 & 0 & -1 & \lambda-3 \end{vmatrix} = 0$$

$$\lambda+2 \begin{vmatrix} \lambda+2 & 0 & 0 \\ 0 & \lambda-3 & 0 \\ 0 & -1 & \lambda-3 \end{vmatrix} \begin{matrix} \lambda+2 & 0 \\ 0 & \lambda-3 \\ 0 & -1 \end{matrix}$$

$$(\lambda+2)^2(\lambda-3)^2 = 0$$

$$\lambda = -2, 3$$

For $\lambda = -2$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & -1 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R} \setminus \{0\}$$

1c) For $\lambda = 3$,

$$\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

Since $\dim \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \neq 4$,

A is not diagonalisable.

2a) $A^n = PD\cancel{P^{-1}P}D\cancel{P^{-1}P}DP^{-1} \dots$

$= PDIDIDI \dots P^{-1}$

$= PDDDD \dots P^{-1}$

$= PD^nP^{-1}$

b) $\det(\lambda I - A) = 0$

$$\left|\begin{matrix} \lambda+1 & -7 & 1 \\ 0 & \lambda-1 & 0 \\ 0 & -15 & \lambda+2 \end{matrix}\right| \begin{matrix} \lambda+1 & -7 \\ 0 & \lambda-1 \\ 0 & -15 \end{matrix} = 0$$

$(\lambda+1)(\lambda-1)(\lambda+2) = 0$

$\lambda = -2, -1, 1$

For $\lambda = -2$:

$$\begin{bmatrix} -1 & -7 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -15 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; s \in \mathbb{R} \setminus \{0\}$$

2b) For $\lambda = -1$:

$$\begin{bmatrix} 0 & -7 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -15 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$$

For $\lambda = 1$:

$$\begin{bmatrix} 2 & -7 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -15 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -7 & 1 & 0 \\ 0 & 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{5} & 0 \\ 0 & 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x - \frac{1}{5} z = 0 \qquad y - \frac{1}{5} z = 0$$

$$x = 5z \qquad y = 5z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2b)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \quad -1$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \quad -1$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -5 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \quad +5$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \sim \quad -5$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -5 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -5 & 1 \\ 0 & 1 & 0 & 1 & 4 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore P^{-1} = \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

2b) $\therefore A^{11} = P D^{11} P^{-1}$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} -2048 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2048 & -1 & 1 \\ 0 & 0 & 1 \\ -2048 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & -5 & 1 \\ 1 & 4 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

2a) Prove by induction:

1) Prove $P(0)$ is true

2) Prove $P(n)$ is true

$$\Downarrow$$

$P(n+1)$ is true

$$P(1): A = PDP^{-1}$$

Assume that $P(n)$ is true,

$$A^n = PD^n P^{-1}$$

$$\begin{aligned} P(n+1): A^{n+1} &= PD^n P^{-1} PDP^{-1} \\ &= PD^n I D P^{-1} \\ &= PD^{n+1} P^{-1} \end{aligned}$$

3) $\det(\lambda I - P) = 0$

$$\begin{vmatrix} \lambda - \frac{1}{2} & 0 \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = 0$$

$$\left(\lambda - \frac{1}{2}\right)(\lambda - 1) = 0$$

$$\lambda = \frac{1}{2}, 1$$

$$D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Let A be the matrix that diagonalises P.

For $\lambda = \frac{1}{2}$,

$$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

3) For $\lambda = 1$,

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

From 2(a), $P = ADA^{-1}$ and

$$P^n = AD^nA^{-1}$$

As $n \to \infty$, $D^n \to \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore P^n \to \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

3) $\therefore P^n \underset{\sim}{x}^{(0)} \to \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ a+b \end{bmatrix}$$

Since $a+b=1$,

$$P^n \underset{\sim}{x}^{(0)} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

4a) $\underset{\sim}{u}_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$\underset{\sim}{u}_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$F_{k+2} = aF_{k+1} + bF_k$

$F_{k+1} = cF_{k+1} + dF_k$

By inspection,

$a = 1, b = 1, c = 1, d = 0$

$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

4b) $\det(\lambda \underline{I} - A) = 0$

$$\begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{vmatrix} = 0$$

$$\lambda(\lambda - 1) - 1 = 0$$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda^2 - \lambda + \frac{1}{4} - 1 - \frac{1}{4} = 0$$

$$\left(\lambda - \frac{1}{2}\right)^2 = \frac{5}{4}$$

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

4b) For $\lambda = \frac{1+\sqrt{5}}{2}$

$$\begin{bmatrix} \frac{\sqrt{5}-1}{2} & -1 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -1+\left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 0 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & \frac{1+\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \frac{1+\sqrt{5}}{2} y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$$

4b) For $\lambda = \frac{1-\sqrt{5}}{2}$,

$$\begin{bmatrix} \frac{-\sqrt{5}-1}{2} & -1 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & \frac{1-\sqrt{5}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = \frac{1-\sqrt{5}}{2} y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

$$\therefore P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

Let $\sigma = \frac{1+\sqrt{5}}{2}$, $\mu = \frac{1-\sqrt{5}}{2}$,

$$P = \begin{bmatrix} \sigma & \mu \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \sigma & 0 \\ 0 & \mu \end{bmatrix}$$

$$P^{-1} = \frac{1}{\sigma - \mu} \begin{bmatrix} 1 & -\mu \\ -1 & \sigma \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\mu \\ -1 & \sigma \end{bmatrix}$$

4b) From 2(a):

$$A^k = PD^kP^{-1}$$

$$= \begin{bmatrix} \sigma & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \mu \end{bmatrix}^k \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\mu \\ -1 & \sigma \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \sigma & \mu \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma^k & 0 \\ 0 & \mu^k \end{bmatrix} \begin{bmatrix} 1 & -\mu \\ -1 & \sigma \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \sigma^{k+1} & \mu^{k+1} \\ \sigma^k & \mu^k \end{bmatrix} \begin{bmatrix} 1 & -\mu \\ -1 & \sigma \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \sigma^{k+1} - \mu^{k+1} & -\mu\sigma^{k+1} + \sigma\mu^{k+1} \\ \sigma^k - \mu^k & -\mu\sigma^k + \sigma\mu^k \end{bmatrix}$$

$$A^k \underset{\sim}{u}_0 = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \sigma^{k+1} - \mu^{k+1} & -\mu\sigma^{k+1} + \sigma\mu^{k+1} \\ \sigma^k - \mu^k & -\mu\sigma^k + \sigma\mu^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \sigma^{k+1} - \mu^{k+1} \\ \sigma^k - \mu^k \end{bmatrix}$$

$$\therefore F_k = \frac{1}{\sqrt{5}} (\sigma^k - \mu^k)$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right]$$

5a) P is orthorgonal $\Rightarrow P^T = P^{-1}$

$$PP^T = I \qquad \det PP^T = \det I = 1$$

$$\det PP^T = 1$$

$$(\det P)^2 = 1$$

$$|\det P| = 1 \text{ (proven)}$$

5b) $(P\underset{\sim}{x}) \cdot (P\underset{\sim}{y})$

$$= (P\underset{\sim}{x})^T (P\underset{\sim}{y})$$

$$= \underset{\sim}{x}^T P^T P \underset{\sim}{y}$$

$$= \underset{\sim}{x}^T P^{-1} P \underset{\sim}{y}$$

$$= \underset{\sim}{x}^T I \underset{\sim}{y}$$

$$= \underset{\sim}{x}^T \underset{\sim}{y}$$

$$= \underset{\sim}{x} \cdot \underset{\sim}{y} \text{ (proven)}$$

5c) A being orthorgonally diagonalisable means that there exists an orthorgonal matrix P and a diagonal matrix D such that:

$$D = P^T A P$$

P being orthorgonal means $P^T = P^{-1}$, so

$$A = P D P^T$$

Since D is diagonal, $D^T = D$, hence

$$\begin{aligned} A^T &= (P D P^T)^T \\ &= P^{T^T} D^T P^T \\ &= P D P^T \\ &= A \end{aligned}$$

$\therefore$ A is symmetric.

6 a) $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{vmatrix} \begin{matrix} \lambda-2 & 1 \\ 1 & \lambda-2 \\ 1 & 1 \end{matrix} = 0$$

$$(\lambda-2)^3 + 2 - \lambda + 2 - \lambda + 2 - \lambda + 2 = 0$$

$$(\lambda^2 - 4\lambda + 4)(\lambda - 2) - 3\lambda + 8 = 0$$

$$\lambda^3 - 2\lambda^2 - 4\lambda^2 + 8\lambda + 4\lambda \cancel{-8} - 3\lambda \cancel{+8} = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda(\lambda - 3)^2 = 0$$

$$\therefore \lambda = 0, 3$$

6a) For $\lambda = 3$,

$$\begin{bmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x + y + z = 0$$
$$x = -y - z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}\setminus\{0\}$$

For $\lambda = 0$,

$$\begin{bmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x - y = 0 \qquad y - z = 0$$

$$x = y \qquad y = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R}\setminus\{0\}$$

6a) $P' = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$D = P^T A P = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6b) $\det(\lambda I - A) = 0$

$$\left|\begin{matrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \lambda-\frac{1}{2} \end{matrix}\right| \begin{matrix} \lambda-1 & 0 \\ 0 & \lambda-\frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix} = 0$$

$$(\lambda-1)(\lambda-\tfrac{1}{2})^2 - \tfrac{1}{4}(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2-\lambda+\tfrac{1}{4}) - \tfrac{1}{4}\lambda + \tfrac{1}{4} = 0$$

$$\lambda^3 - \lambda^2 + \cancel{\tfrac{1}{4}\lambda} - \lambda^2 + \lambda - \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}\lambda} + \cancel{\tfrac{1}{4}} = 0$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda + 1) = 0$$

$$\lambda(\lambda-1)^2 = 0$$

$$\lambda = 0, 1$$

For $\lambda = 0$,

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x = 0 \qquad y - z = 0 \Rightarrow y = z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

6b) For $\lambda = 1$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y + z = 0$$
$$y = -z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad s, t \in \mathbb{R} \setminus \{0\}$$

$$\therefore P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = P^T A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6b) The linear transformation that in the standard basis is given by a multiplication with A, is in the basis

$$\underset{\sim}{v}_1 = \frac{1}{\sqrt{2}}(0,1,1), \quad \underset{\sim}{v}_2 = (1,0,0)$$
$$\underset{\sim}{v}_3 = \frac{1}{\sqrt{2}}(0,-1,1),$$

given by multiplication with D, which in this coordinate system is a projection on the x_2-x_3 plane. This is the plane through the origin that is orthorgonal to $\underset{\sim}{v}_1$, so in the standard coordinate system, we are projecting onto the plane $y+z=0$.

6c) This matrix is not symmetrical and heme not diagonalisable.

7a) $A = \begin{bmatrix} 5 & -2 \\ -2 & 8 \end{bmatrix}$

b) $\det(\lambda I - A) = 0$

$$\begin{vmatrix} \lambda-5 & 2 \\ 2 & \lambda-8 \end{vmatrix} = 0$$

$$(\lambda-5)(\lambda-8) - 4 = 0$$

$$\lambda^2 - 8\lambda - 5\lambda + 40 - 4 = 0$$

$$\lambda^2 - 13\lambda + 36 = 0$$

$$\lambda = 9, 4$$

For $\lambda = 9$,

$$\begin{bmatrix} \lambda-5 & 2 \\ 2 & \lambda-8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x = -\frac{1}{2}y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \end{bmatrix}, s \in \mathbb{R} \setminus \{0\}$$

7b) For $\lambda = 4$,

$$\begin{bmatrix} \lambda-5 & 2 \\ 2 & \lambda-8 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

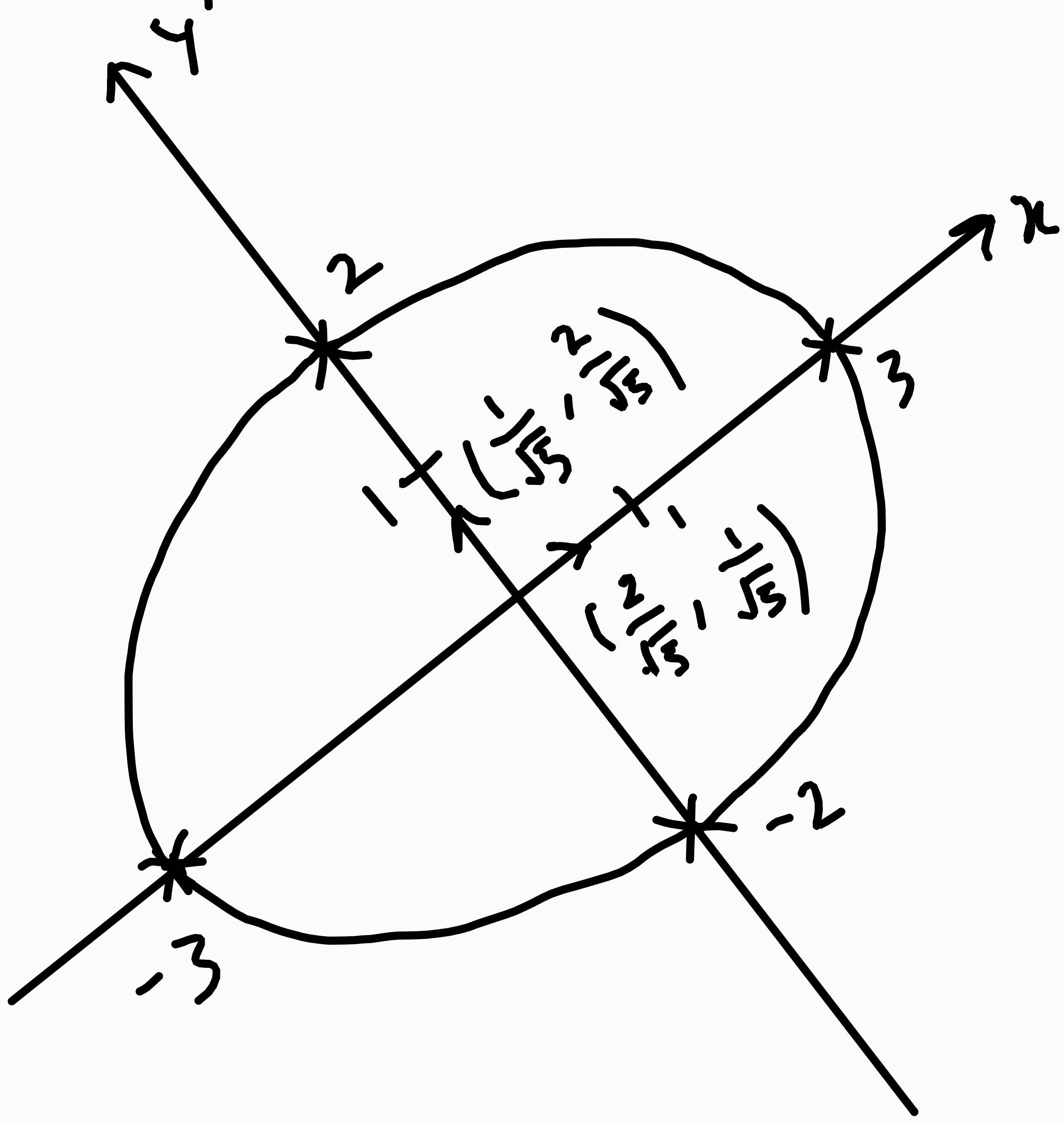
$$x = 2y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad s \in \mathbb{R} \setminus \{0\}$$

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$P^T A P = D = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$$

7c)



y'
x'
2
3
-3
-2
1
1
$\left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$
$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

8) Suppose λ is an eigenvalue of T,

$$\langle T(\underset{\sim}{x}), \underset{\sim}{x} \rangle = \langle \lambda \underset{\sim}{x}, \underset{\sim}{x} \rangle$$
$$= \lambda \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle$$

Since $\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \geq 0$, for $\langle T(\underset{\sim}{x}), \underset{\sim}{x} \rangle \geq 0$,

$$\lambda \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \geq 0$$
$$\therefore \lambda \geq 0$$

And for $\langle T(\underset{\sim}{x}), \underset{\sim}{x} \rangle \leq 0$,

$$\lambda \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle \leq 0$$
$$\therefore \lambda \leq 0$$

9) $\langle T(\underset{\sim}{x}), \underset{\sim}{y} \rangle = \langle \underset{\sim}{x}, T(\underset{\sim}{y}) \rangle$

Suppose $T(\underset{\sim}{x}_1) = \lambda_1 \underset{\sim}{x}_1$, $T(\underset{\sim}{x}_2) = \lambda_2 \underset{\sim}{x}_2$, $\lambda_1 \neq \lambda_2$,

Since they are different, one of the eigenvalues has to be non-zero. Without loss of generality, suppose $\lambda_1 \neq 0$.

$$\langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle = \left\langle \frac{1}{\lambda_1} T(\underset{\sim}{x}_1), \underset{\sim}{x}_2 \right\rangle$$

$$= \frac{1}{\lambda_1} \langle T(\underset{\sim}{x}_1), \underset{\sim}{x}_2 \rangle$$

Since $W \to V$ is Hermitian,

$$\frac{1}{\lambda_1} \langle T(\underset{\sim}{x}_1), \underset{\sim}{x}_2 \rangle = \frac{1}{\lambda_1} \langle \underset{\sim}{x}_1, T(\underset{\sim}{x}_2) \rangle$$

$$= \frac{1}{\lambda_1} \langle \underset{\sim}{x}_1, \lambda_2 \underset{\sim}{x}_2 \rangle$$

$$= \frac{\lambda_2}{\lambda_1} \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle$$

$$\langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle = \frac{\lambda_2}{\lambda_1} \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle$$

$$\left(1 - \frac{\lambda_2}{\lambda_1}\right) \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle = 0$$

Since $\lambda_2 \neq \lambda_1$, $\left(1 - \frac{\lambda_2}{\lambda_1}\right) \neq 0$,

$$\therefore \langle \underset{\sim}{x}_1, \underset{\sim}{x}_2 \rangle = 0$$

10 a) $A\underset{\sim}{x} \cdot \underset{\sim}{y} = (A\underset{\sim}{x})^T \underset{\sim}{y}$

$= \underset{\sim}{x}^T A^T \underset{\sim}{y}$

$= \underset{\sim}{x}^T A \underset{\sim}{y} \quad \because A^T = A$

$= \underset{\sim}{x} \cdot A\underset{\sim}{y}$ for all $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^n$

(shown)

b) Suppose A is symmetric. From part (a), $T(\underset{\sim}{x}) = A\underset{\sim}{x}$ is Hermitian.

$\therefore$ By Theorem 2, the eigenvectors of A corresponding to different eigenvalues are orthorgonal.

11a) $\langle T(f), f\rangle = \langle f'', f\rangle$

$$= \int_0^c f''(t) f(t)\,dt$$

$$= f'(t) f(t)\Big|_0^c - \int_0^c (f'(t))^2\,dt$$

$$= 0 - \int_0^c (f'(t))^2\,dt$$

$$= -\int_0^c (f'(t))^2\,dt$$

Since $f(t) \in \mathbb{R}$, $(f'(t))^2 \geq 0$

$\therefore \langle T(f), f\rangle \leq 0$, for all $f \in W$

Since $f \in C^2([0,c]) \subset C^1([0,c])$, the integrand $(f'(t))^2$ is a continuous non-negative function, so by the lemma, the only way for the integral to be zero is if $(f'(t))^2 = 0$ for all $t \in [0,c]$, $f(t) = C$ for some constant C.

Since $f(0) = f(c) = 0$, $C = 0$, so the only way for the integral to be 0 is if f is the zero function.

$\therefore \langle T(f), f\rangle < 0$ for all $f \in W$, $f \neq 0$

11a) $\therefore T$ is negative definite, and by Theorem 1, all eigenvalues of T are negative.

11b) The fact that all eigenvalues are negative, agrees with our findings from the lecture where we saw that the eigenvalues are

$$\lambda_k = -\frac{k^2\pi^2}{c^2}, \quad k \in \mathbb{Z}^+$$

11c) For $f, g \in W$,

$$\langle T(f), g \rangle = \langle f'', g \rangle \qquad uv - \int u\,dv$$

$$= \int_0^c \overset{u}{f''(t)}\,\overset{v}{g(t)}\,dt$$

$$= f'(t)g(t)\Big|_0^c - \int_0^c f'(t)g'(t)\,dt$$

$$= 0 - \int_0^c f'(t)g'(t)\,dt$$

$$= -\int_0^c \overset{u}{f(t)}\,\overset{v}{g'(t)}\,dt$$

$$= f(t)g'(t)\Big|_0^c + \int_0^c f(t)g''(t)\,dt$$

$$= 0 + \int_0^c f(t)g''(t)\,dt$$

$$= \int_0^c f(t)g''(t)\,dt$$

$$= \langle f, T(g) \rangle$$

11c) $\therefore T$ is Hermitian.

By Theorem 2, the eigenvectors corresponding to the different eigenvalues are orthorgonal.

The eigenvectors of T found in the lecture corresponding to the eigenvalues:

$$\lambda_k = -\frac{k^2\pi^2}{c^2}, k \in \mathbb{Z}^+$$

are the functions:

$$y_k(t) = C\sin(\sqrt{-\lambda}\, t)$$
$$= C\sin\left(-\frac{k^2\pi^2}{c}t\right), \quad C \neq 0$$

So, in view of the inner product, these being orthorgonal means:

$$\langle y_j, y_k \rangle = \int_0^c \sin\left(\frac{j\pi}{c}t\right)\sin\left(\frac{k\pi}{c}t\right)dt = 0,$$

for $j, k \in \mathbb{Z}^+, j \neq k$

12a) For $f \in W$,

$$\begin{aligned}
\langle T(f), f \rangle &= \langle f'', f \rangle \\
&= \int_0^c f''(t) f(t)\, dt \\
&= f'(t) f(t) \Big|_0^c - \int_0^c (f'(t))^2 dt \\
&= 0 - \int_0^c (f'(t))^2 dt \\
&= -\int_0^c (f'(t))^2 dt
\end{aligned}$$

Since $f'(t) \in \mathbb{R}$, $(f'(t))^2 \geq 0$

$\therefore \langle T(f), f \rangle \geq 0$, for all $f \in W$

$\therefore T$ is semidefinite.

However, with

$f(t) = 1$, for all $t \in [0, c]$

$f'(0) = f'(0) = 0$, hence $f \in W$, $f \neq 0$, but

$$\langle T(f), f \rangle = -\int_0^c (f'(t))^2 dt = 0$$

12a) $\therefore$ T is negative semi-definite but not negative definite, and hence all the eigenvalues of T are non-positive.

b) y is an eigenvector of T with eigenvalue λ if and only if

$$y \in W, \ y \neq 0, \ y'' = \lambda y$$

From part (a), since all the eigenvalues are non-positive, no non-trivial solutions exist for $\lambda > 0$.

For $\lambda = 0$,

$$y'' = 0$$

$$y(t) = Ct + D, \ C, D \in \mathbb{R}$$

Since $y'(0) = y'(c) = 0$, $C = 0$,

$\therefore$ The eigenvectors are $y(t) = D$, $D \neq 0$.

For $\lambda < 0$,

$$y'' - \lambda y = 0$$

$$y(t) = A\cos\left(\sqrt{-\lambda}\, t\right) + B\sin\left(\sqrt{-\lambda}\, t\right)$$

12 b) So $y'(t) = -A\sqrt{-\lambda}\sin(\sqrt{-\lambda}t) + B\sqrt{-\lambda}\cos(\sqrt{-\lambda}t)$

When $y'(0) = 0$,

$$B\sqrt{-\lambda} = 0$$

Since $\lambda < 0$, $B = 0$,

$$\therefore y'(t) = -A\sqrt{-\lambda}\sin(\sqrt{-\lambda}t)$$

When $y'(c) = 0$

$$-A\sqrt{-\lambda}\sin(\sqrt{-\lambda}c) = 0$$

Since we are looking for nontrivial solutions, $A \neq 0$,

$$\therefore \sin(\sqrt{-\lambda}c) = 0$$

$$\therefore \sqrt{-\lambda}c = k\pi, \ k \in \mathbb{Z}^+$$

$$\sqrt{-\lambda} = \frac{k\pi}{c}$$

$$\lambda = -\frac{k^2\pi^2}{c^2}$$

$\therefore$ the eigenvalues are: $\lambda = -\frac{k^2\pi^2}{c^2}$, $k \in \mathbb{Z}^+$ and the eigenvectors are:

$y_k(t) = A\cos(-\sqrt{\lambda_k}t) = A\cos(\frac{k\pi}{c}t)$, $A \neq 0$