Math Module 3A Notes

Hankertrix

April 29, 2025

Contents

1 Definitions

1.1 Corollary

A corollary is a proposition that is inferred immediately from a proved proposition with little or no additional proof.

1.2 Lagrange's mean value theorem (MVT)

Suppose that:

1. f is continuous on a closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)

Then there is a point $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

For an illustration of this theorem, go to this link.

1.2.1 Corollaries

- If $f'(x) = 0$ on an interval, then f is constant on the interval.
- If $f' = g'$ on an interval, then $f = g + C$, where C is some constant.

1.3 Rolle's theorem

Suppose that:

1. f is continuous on a closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

1.4 Cauchy's mean value theorem

1. f and g are continuous on a closed interval $[a, b]$.

2. f and g are differentiable on the open interval (a, b).

Then there exists $c \in (a, b)$ such that:

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)]$$

With $g'(c) \neq 0$, $g(b) - g(a) \neq 0$, we get:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

With $g(x) = x$, we get Lagrange's mean value theorem.

1.5 Indeterminate forms

If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\left[\frac{0}{0}\right]$.

Likewise, if $\lim_{x \to a\pm} f(x) = \pm\infty$, $\lim_{x \to a\pm} g(x) = \pm\infty$, the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an **indeterminate form of type** $\left[\frac{\infty}{\infty}\right]$.

1.5.1 More examples

1. $\lim_{x \to 0} x \cdot \ln x$ produces the indeterminate form $[\infty \cdot 0]$.

2. $\lim_{x \to \infty} (\sqrt{n^2 + 2n} - n)$ produces the indeterminate form $[\infty - \infty]$.

3. $\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x$ produces the indeterminate form $[1^\infty]$.

1.6 L'Hôpital's Rule

If there exists $\delta > 0$ such that:

1. $f(x)$ and $g(x)$ are differentiable on $(a-\delta, a) \cup (a, a+\delta)$
2. $g'(x) \neq 0$ on $(a-\delta, a) \cup (a, a+\delta)$

   And also if:
   $$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$
   Then:
   $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
   This only applies if the right-hand limit exists.

1.6.1 The conditions, simplified

1. $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of type $\left[\frac{0}{0}\right]$
2. $\frac{f'(x)}{g'(x)}$ makes sense for x close to a, and has a limit as $x \to a$.

1.6.2 Variations

1. $\displaystyle\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$ if the right-hand side makes sense
2. $\displaystyle\lim_{x \to a-} \frac{f(x)}{g(x)} = \lim_{x \to a-} \frac{f'(x)}{g'(x)}$ if the right-hand side makes sense
3. $\displaystyle\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \lim_{x \to \pm\infty} \frac{f'(x)}{g'(x)}$ if the right-hand side makes sense

These rules **only apply** if the left-hand limit is an indeterminate form of the type $\left[\frac{0}{0}\right]$ or $\left[\frac{\infty}{\infty}\right]$

1.6.3 Example 1

$$\begin{aligned}\lim_{x \to 0} \frac{e^x = 1}{\sin x} &= \lim_{x \to 0} \frac{e^x}{\cos x} \\ &= \frac{1}{1} \\ &= 1\end{aligned}$$

1.6.4 Example 2

$$\begin{aligned}
\lim_{x\to+\infty} x^2 e^{-3x} &= \lim_{x\to+\infty} \frac{x^2}{e^{3x}} \\
&= \lim_{x\to+\infty} \frac{2x}{3e^{3x}} \\
&= \lim_{x\to+\infty} \frac{2}{9e^{3x}} \\
&= 0
\end{aligned}$$

1.6.5 Example 3

$$\begin{aligned}
\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x &= \lim_{x\to+\infty} e^{x\cdot\ln\left(1+\frac{1}{x}\right)} \\
&= e^{\lim_{x\to+\infty} x\cdot\ln\left(1+\frac{1}{x}\right)}
\end{aligned}$$

Finding the limit $\lim_{x\to+\infty} x\cdot\ln\left(1+\frac{1}{x}\right)$:

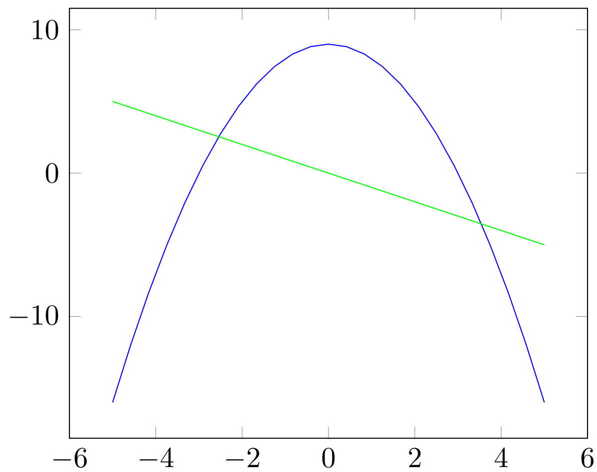
$$\begin{aligned}
\lim_{x\to+\infty} x\cdot\ln\left(1+\frac{1}{x}\right) &= \lim_{x\to+\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \\
&= \lim_{x\to+\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\left(\frac{-1}{x^2}\right)}{\frac{-1}{x^2}} \\
&= \lim_{x\to+\infty} \frac{1}{1+\frac{1}{x}} \\
&= \frac{1}{1+0} \\
&= 1
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x &= e^1 \\
&= e
\end{aligned}$$

1.7 Convexity

A function $f(x)$ is called **convex** (or it is said to **concave upward**) on an interval I if for all $a, b \in I$, the line segment joining the points $(a, f(a)), (b, f(b))$ lies above the graph of $f(x)$.

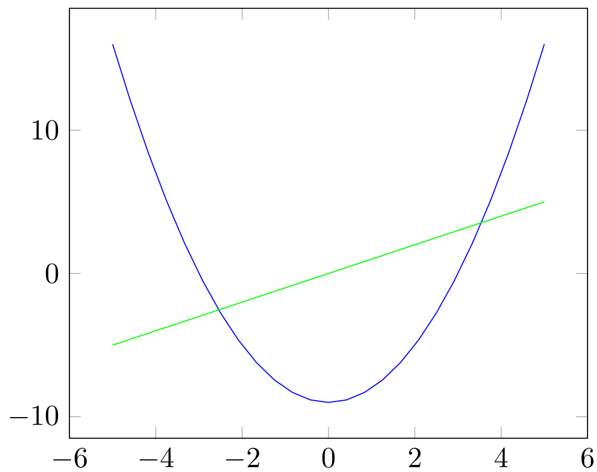


A function $f(x)$ is **convex** (or it is said to **concave upward**) on the interval I if and only if for any $a < x < b$ in I we have:

$$\frac{f(x) - f(a)}{x - a} < \frac{f(b) - f(a)}{b - a}$$

1.8 Concavity

A function $f(x)$ is called **concave** (or it is said to **concave downward**) on an interval I if for all $a, b \in I$, the line segment joining the points $(a, f(a)), (b, f(b))$ lies above the graph of $f(x)$.



A function $f(x)$ is **concave** (or it is said to **concave downward**) on the interval I if and only if for any $a < x < b$ in I we have:

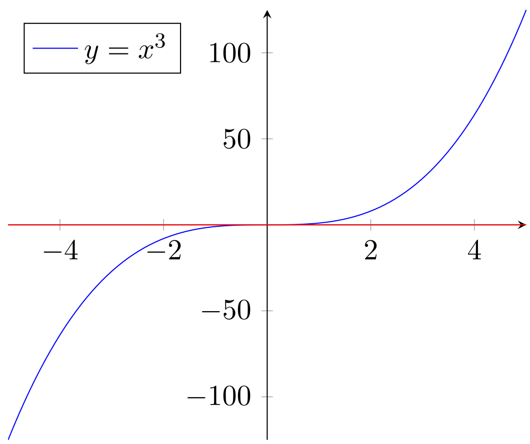
$$\frac{f(x) - f(a)}{x - a} > \frac{f(b) - f(a)}{b - a}$$

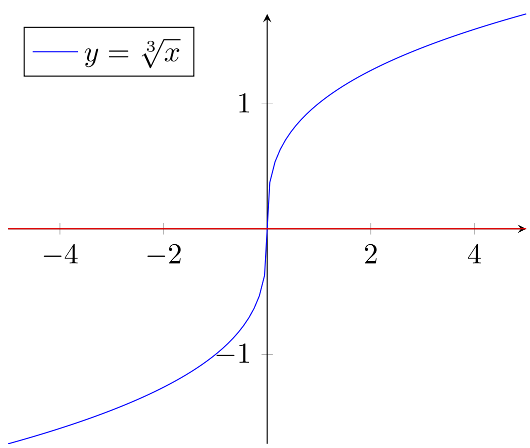
1.9 Inflection point

A point where the graph of a function has a tangent line and where the concavity changes, is called a **point of inflection** or an **inflection point**.

1.9.1 Example

Both $f(x) = x^3$ and $f(x) = \sqrt[3]{x}$ have a point of inflection at $x = 0$.





1.10 Second derivative

Given a function $f(x)$, its **second derivative** is the derivative of $f'(x)$.

1.11 Higher order derivatives

Given a function $f(x)$, its n-th derivative is:

$$\underbrace{((f')' \cdots)'}_{n \text{ differentiations}}$$

1.11.1 Standard notation

- f'' for the second derivative
- f''' for the third derivative
- f^{iv} for the fourth derivative
- $f^{(n)}$ for the n-th derivative
- $C^n(A) = \{f : f^{(n)} \text{ exists and is continuous on } A\}$

Note that:

$$C(A) \subset C'(A) \subset C''(a) \subset \ldots \subset C^{\infty} A$$

1.12 Vertical asymptote

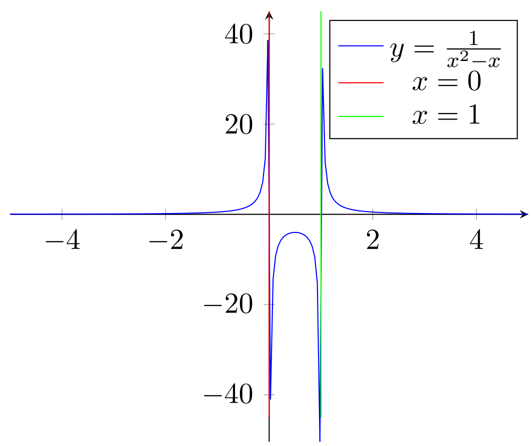
The graph of $y = f(x)$ has a **vertical asymptote** at $x = a$ if:

$$\lim_{x \to a-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a+} f(x) = \pm\infty$$

1.12.1 Example

$$f(x) = \frac{1}{x^2 - x}$$

The graph of $f(x)$ has vertical asymptotes at $x = 0$ and $x = 1$.



1.13 Horizontal asymptote

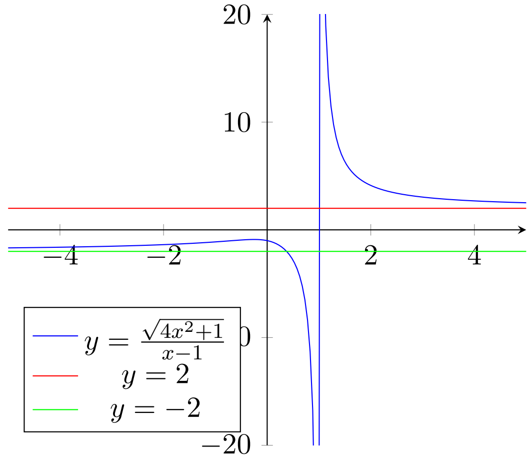
The graph of $y = f(x)$ has a **horizontal asymptote** $y = L$ if:

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to +\infty} f(x) = L$$

1.13.1 Example

$$f(x) = \frac{\sqrt{4x^2 + 1}}{x - 1}$$

The graph of $f(x)$ has a horizontal asymptote $y = 2$ and another horizontal asymptote $y = -2$.



1.14 Oblique asymptote

The straight line $y = ax+b, (a \neq 0)$, is an **oblique asymptote** of the graph of $y = f(x)$ if:

$$\lim_{x\to-\infty}(f(x) - (ax+b)) = 0 \quad \text{or} \quad \lim_{x\to+\infty}(f(x) - (ax+b)) = 0$$

1.14.1 Example

Find the oblique asymptote of:

$$f(x) = \frac{x^3}{x^2+x+1}$$

Long divide x^3 by $x - 1$:

$$f(x) = \frac{x^3}{x^2+x+1} = x - 1 + \frac{1}{x^2+x+1}$$

So:

$$f(x) - (x-1) = \frac{1}{x^2+x+1} \to 0 \text{ as } x \to \pm\infty$$

Hence, $y = x - 1$ is an oblique asymptote for $f(x)$.

2 Relationship between the derivative and monotonicity

Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then:

- If $f'(x) > 0$ on (a, b), then f is strictly increasing on $[a, b]$.
- If $f'(x) \geq 0$ on (a, b), then f is increasing on $[a, b]$.
- If $f'(x) < 0$ on (a, b), then f is strictly decreasing on $[a, b]$.
- If $f'(x) \leq 0$ on (a, b), then f is decreasing on $[a, b]$.

3 Standard limits

The following equations hold for any numbers $p > 0$ and $\varepsilon > 0$:

1. $$\lim_{x \to +\infty} \frac{x^p}{e^{\varepsilon x}}$$
2. $$\lim_{x \to +\infty} \frac{(\ln x)^p}{x^\varepsilon}$$

Rule of thumb:

- Exponentials beat powers
- Powers beat logarithms

4 Second derivative and concavity

1. If $f''(x) > 0$ on an interval I, then f is **convex** (or is said to **concave upward**) on I (positive means happy face).
2. If $f''(x) < 0$ on an interval I, then f is **concave** (or is said to **concave downward**) on I (negative means sad face).
3. If a is an inflection point for f, then either $f''(a)$ does not exist, or $f''(a) = 0$.

5 Second derivative and extreme points

Suppose $f \in C^2(I)$, where I is some open interval containing a, and suppose $f'(a) = 0$. We have:

1. If $f''(a) > 0$, then a is a point of local minimum.
2. If $f''(a) < 0$, then a is a point of local maximum.

Note that if $f''(a) = 0$, we get no information. $x = a$ might be a local maximum or minimum or neither.