Math Module 1B Cheat Sheet

Hankertrix

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Contents

1 Definitions

1.1 Distance between points

The distance between two points $x, y \in \mathbb{R}$ is $|x - y|$.

1.2 Limit points

For $A \subset \mathbb{R}$, a point $a \in \mathbb{R}$ is a **limit point** of A if for every $\delta > 0$, there exists a point xA such that $0 < |x - a| < \delta$.

1.3 Limit

For a function $f : A \to \mathbb{R}$, $A \subset \mathbb{R}$ with a as a limit point of A, f approaches a **limit** L if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\lim_{x \to a} f(x) = L$$

$$\Updownarrow$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta, \quad x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

1.4 Limits at infinity

Suppose f is defined on some interval (a, ∞). We say that $f(x)$ has a limit L as x approaches positive infinity, and write $\lim_{X \to +\infty} f(x) = L$, if for every $\varepsilon > 0$, there exists a number R such that:

$$x > R \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Likewise, for f defined on some interval $(-\infty, b)$, we say that $f(x)$ has a limit L as x approaches negative infinity, and write $\lim_{x \to -\infty} f(x) = L$, if for every $\varepsilon > 0$, there exists a number R such that:

$$x < R \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Limits at infinity follow the same limit laws as normal limits, so we can use limit laws to conclude that for any **positive** integer n, we also have:

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

When evaluating a limit at infinity, a common technique is to factor out the highest possible power.

1.5 Limit of a sequence

We say that a sequence (a_n) has the limit L and write $\lim_{n\to\infty} a_n = L$, if for every $\varepsilon > 0$, there exists a number N such that:

$$n > N \quad \Rightarrow \quad |a_n - L| < \varepsilon$$

The limits of sequences are evaluated with similar methods to other forms of limits.

2 Limit laws

Consider $f : A_1 \to \mathbb{R}, g : A_2 \to \mathbb{R}$. Suppose a is a limit point of $A_1 \cap A_2$, and $\lim_{x\to a} f(x) = l, \lim_{x\to a} g(x) = m$, then:

1. $$\begin{aligned} \lim_{x\to a}(Af(x) + Bg(x)) &= Al + Bm \\ &= A \cdot \lim_{x\to a} f(x) + B \cdot \lim_{x\to a} g(x) \end{aligned}$$

2. $$\begin{aligned} \lim_{x\to a}(f(x)g(x)) &= lm \\ &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \end{aligned}$$

3. $$\begin{aligned} \lim_{x\to a} \frac{f(x)}{g(x)} &= \frac{l}{m}, \text{ provided } m \neq 0 \\ &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \end{aligned}$$

4. $$\begin{aligned} \lim_{x\to a} \sqrt[n]{f(x)} &= \sqrt[n]{l}, \text{ provided } n \in \mathbb{N} \text{ and } l \geq 0 \text{ if } n \text{ is even} \\ &= \sqrt[n]{\lim_{x\to a} f(x)} \end{aligned}$$

5. L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \text{ when } \lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ and } g'(x) \neq 0$$

3 Squeeze Theorem

Suppose $f(x) \leq g(x) \leq h(x)$, for $x \in I \setminus \{a\}$, where I is some open interval containing the point a. Then:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \quad \Rightarrow \quad \lim_{x \to a} g(x) = L$$

3.1 Extremely useful result

For $f : A \to \mathbb{R}$, we have:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a} |f(x) - L| = 0$$

3.2 A useful lemma

For $0 < x < \frac{\pi}{2}$, we have:

$$x \cos^2 x < \sin x < x$$

If f and g are **even** functions such that $f(x) < g(x)$, for $x \in (0, a)$, then we also have:

$$f(x) < g(x), \text{ for } x \in (-a, 0)$$

4 Useful limits

$$\lim_{x \to 0} \sin x = 0$$

$$\lim_{x \to 0} \cos x = 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$