1a) Let $\underset{\sim}{x}_1 = (1,1)$, $\underset{\sim}{x}_2 = (-1,2)$, $\underset{\sim}{e}_1 = (1,0)$,

$\underset{\sim}{e}_2 = (0,1)$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$

$$\underset{\sim}{e}_1 = c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2$$

$$\underset{\sim}{e}_2 = c_3 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2$$

$$\left[\begin{array}{cc|c|c} 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 2 & 0 & 1 \\ 0 & -3 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c|c} 1 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\therefore c_1 = \frac{2}{3},\ c_2 = -\frac{1}{3},\ c_3 = \frac{1}{3},\ c_4 = \frac{1}{3}$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

$$= \frac{2}{3} T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - \frac{1}{3} T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

$$= \frac{2}{3}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

1a) $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\frac{1}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-1\\2\end{bmatrix}\right)$

$$= \frac{1}{3}\begin{bmatrix}1\\2\\5\end{bmatrix} + \frac{1}{3}\begin{bmatrix}-4\\1\\7\end{bmatrix}$$

$$= \begin{bmatrix}-1\\1\\4\end{bmatrix}$$

1b) $[T] = \left[\begin{bmatrix}2\\1\\1\end{bmatrix} \begin{bmatrix}-1\\1\\4\end{bmatrix}\right]$

2a) $T(\underset{\sim}{0} + \underset{\sim}{0}) = T(\underset{\sim}{0}) + T(\underset{\sim}{0})$

$$\underset{\sim}{0} = 2T(\underset{\sim}{0})$$

$$T(\underset{\sim}{0}) = \underset{\sim}{0} \text{ (shown)}$$

b) Since $\underset{\sim}{0} \in \ker T$, $\ker T \neq \emptyset$

Let $\underset{\sim}{u}, \underset{\sim}{v} \in \ker T$

$$T(\underset{\sim}{u} + \underset{\sim}{v}) = T(\underset{\sim}{u}) + T(\underset{\sim}{v})$$

$$= \underset{\sim}{0} + \underset{\sim}{0}$$

$$= \underset{\sim}{0} \in \ker T$$

2b) Let $k \in \mathbb{R}$, $\underset{\sim}{u} \in \ker T$

$$T(k\underset{\sim}{u}) = kT(\underset{\sim}{u})$$

$$= k\underset{\sim}{0}$$

$$= \underset{\sim}{0} \in \ker T$$

$\therefore$ $\ker T$ is a subspace of U

(shown)

c) T is one-to-one means T^{-1} exists.

$$\therefore (T^{-1} \circ T)(\underset{\sim}{x}) = \underset{\sim}{x}$$

$$T^{-1}(T(\underset{\sim}{x})) = \underset{\sim}{x}$$

$$T^{-1}(\underset{\sim}{0}) = \underset{\sim}{x}$$

$$\underset{\sim}{x} = \underset{\sim}{0}$$

$$\therefore \ker T = \{\underset{\sim}{0}\}$$

2c) $\ker T = \{\underset{\sim}{0}\}$

Let $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \in \ker T$,

$$T(\underset{\sim}{x}_1) = T(\underset{\sim}{x}_2) = \underset{\sim}{0}$$

Since $\ker T$ has only 1 element, $\underset{\sim}{0}$,

$$\underset{\sim}{x}_1 = \underset{\sim}{x}_2 = \underset{\sim}{0}$$

$$\therefore T(\underset{\sim}{x}_1) = T(\underset{\sim}{x}_2) \Rightarrow \underset{\sim}{x}_1 = \underset{\sim}{x}_2$$

$\therefore T$ is one-to-one.

Let $T(\underset{\sim}{x}) = \underset{\sim}{0}$,

$$T(\underset{\sim}{0}) = \underset{\sim}{0}$$

$$T(\underset{\sim}{x}) = T(\underset{\sim}{0})$$

Since T is one-to-one,

$$x = \underset{\sim}{0}$$

$$\therefore \ker T = \{\underset{\sim}{0}\}$$

2d)

Since B is linearly independent, the only solution to the equation below

$$k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + \dots + k_r\underset{\sim}{x}_r = \underset{\sim}{0}, \quad k_1, \dots, k_r \in \mathbb{R}$$

is

$$k_1 = k_2 = \dots = k_r = 0$$

Let $k_1 T(\underset{\sim}{x}_1) + k_2 T(\underset{\sim}{x}_2) + \dots + k_r T(x_r) = \underset{\sim}{0}$

$k_1, k_2, \dots, k_r \in \mathbb{R}$

$$T(k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + \dots + k_r\underset{\sim}{x}_r) = \underset{\sim}{0}$$

T is one-to-one, by part (c),

$$k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + \dots + k_r\underset{\sim}{x}_r = 0$$

Since S is linearly independent,

$$k_1 = k_2 = \dots = k_r = 0$$

$\therefore S'$ is linearly independent.

3a) $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x, y) \xrightarrow[\text{on the } y\text{-axis}]{\text{orthorgonal projection}} (0, y) \xrightarrow[\text{by factor } \frac{1}{2}]{\text{contraction}} (0, \tfrac{1}{2}y)$$

$$[T]\underset{\sim}{x} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}y \end{bmatrix}$$

$$\therefore [T] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

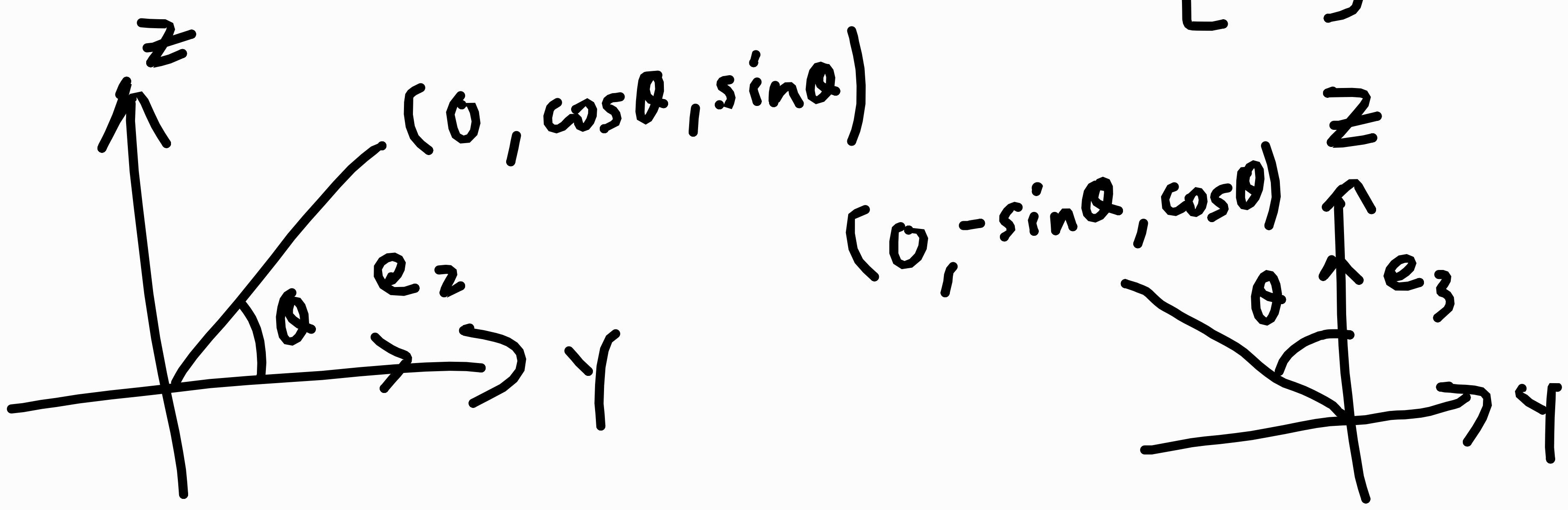
b) $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x, y) \xrightarrow[\text{Rotate } 90^\circ \text{ anti-clockwise}]{} (-y, x) \xrightarrow[\text{Reflect about } y = x]{} (x, -y)$$

$$[T]\underset{\sim}{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

$$\therefore [T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

4a) Let $\underset{\sim}{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\underset{\sim}{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\underset{\sim}{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



$$T(\underset{\sim}{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad T(\underset{\sim}{e}_2) = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}$$

$$T(\underset{\sim}{e}_3) = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

4b) Let $\underset{\sim}{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \underset{\sim}{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \underset{\sim}{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$T(\underset{\sim}{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \quad T(\underset{\sim}{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$$

$$T(e_3) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4c) $[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$$[T_2] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T] = [T_2][T_1]$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta\cos\theta & \sin^2\theta \\ \sin\theta & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

5a) $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ -4 & 5 & -8 \\ -3 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$[P] = \begin{bmatrix} 2 & - & 2 \\ -4 & 5 & -8 \\ -3 & 3 & -5 \end{bmatrix}$$

b) $[P]^2 = \begin{bmatrix} 2 & -1 & 2 \\ -4 & 5 & -8 \\ -3 & 3 & -5 \end{bmatrix}$

6) $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 1 & w_1 \\ 5 & -1 & 3 & w_2 \\ 4 & 1 & 2 & w_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 5w_1 \\ 0 & 9 & -2 & w_3 - 4w_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & w_1 \\ 0 & 9 & -2 & w_2 - 5w_1 \\ 0 & 0 & 0 & w_3 + w_1 - w_2 \end{bmatrix}$$

6) The above linear system is only consistent when $w_3 + w_1 - w_2 = 0$.

$\therefore$ the range of T is not all of $\mathbb{R}^3$.

$\therefore$ A vector in $\mathbb{R}^3$ that is not in the range is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

7a) $$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\left| \begin{array}{ccc|cc} \lambda-4 & 0 & -1 & \lambda-4 & 0 \\ 2 & \lambda-1 & 0 & 2 & \lambda-1 \\ 2 & 0 & \lambda-1 & 2 & 0 \end{array} \right. = 0$$

$$-2(\lambda-1) - (\lambda-1)^2(\lambda-4) = 0$$

$$(1-\lambda)\left[2 + (\lambda-1)(\lambda-4)\right] = 0$$

7a) $$(1-\lambda)\left[\lambda^2-4\lambda-\lambda+4+2\right]=0$$

$$(1-\lambda)(\lambda^2-5\lambda+6)=0$$

$\therefore \lambda=1$ or $\lambda=\dfrac{5\pm\sqrt{25-4\times 6}}{2(1)}$

$$=\frac{5\pm 1}{2}$$

$$=2 \text{ or } 3$$

$\therefore \lambda = 1, 2, 3$

$$\begin{bmatrix} \lambda-4 & 0 & -1 \\ 2 & \lambda-1 & 0 \\ 2 & 0 & \lambda-1 \end{bmatrix}$$

For $\lambda=1$,

$$\begin{bmatrix} -3 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{3} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7a) The solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\}$$

For $\lambda = 2$,

$$\begin{bmatrix} -2 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + \frac{1}{2}z = 0 \qquad y - z = 0$$

$$z = -2x \qquad y = z$$

The solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\}$$

7a) For $\lambda = 3$,

$$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x + z = 0 \qquad y - z = 0$$
$$x = -z \qquad y = z$$

The solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\}$$

7b) $$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix} \right) = 0$$

$$\left| \begin{array}{ccc|cc} \lambda+2 & 0 & -1 & \lambda+2 & 0 \\ 6 & \lambda+2 & 0 & 6 & \lambda+2 \\ -19 & -5 & \lambda+4 & -19 & -5 \end{array} \right| = 0$$

$$19(\lambda+2) - (\lambda+2)^2(\lambda+4) - 30 = 0$$
$$19\lambda + 38 - (\lambda^2 + 4\lambda + 4)(\lambda+4) - 30 = 0$$
$$19\lambda + 8 - (\lambda^3 + 4\lambda^2 + 4\lambda^2 + 16\lambda + 4\lambda + 16) = 0$$
$$\lambda^3 + 8\lambda^2 + \lambda + 8 = 0$$

7b) $\therefore \lambda = -8$ as $\lambda \in \mathbb{R}$

When $\lambda = -8$,

$$\begin{bmatrix} -6 & 0 & -1 & 0 \\ 6 & -6 & 0 & 0 \\ -19 & -5 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 6 & -6 & 0 & 0 \\ 0 & -6 & -1 & 0 \\ -19 & -5 & -4 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & -24 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x - y = 0$

$y = x$

$6y - z = 0$

$z = -6y$

The solutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -6 \end{bmatrix}, \quad t \in \mathbb{R} \setminus \{0\}$$

7c) $\lambda = 0$

With $\lambda = 0$, the solutions are:

$$(x, y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

8a) $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = u_1 v_1 + 2 u_2 v_2$

$= v_1 u_1 + 2 v_2 u_2$

$= \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle$

$\langle \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{w} \rangle = (u_1 + v_1) w_1 + 2(u_2 + v_2) w_2$

$= u_1 w_1 + v_1 w_1 + 2 u_2 w_2 + 2 v_2 w_2$

$= u_1 w_1 + 2 u_2 w_2 + v_1 w_1 + 2 v_2 w_2$

$= \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

$\langle k \underset{\sim}{u}, \underset{\sim}{v} \rangle = k u_1 v_1 + 2 k u_2 v_2$

$= k(u_1 v_1 + 2 u_2 v_2)$

$= k \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = u_1 u_1 + 2 u_2 u_2$

$= u_1^2 + 2 u_2^2 \geq 0$

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 0$

$u_1^2 + 2u_2^2 = 0$

$u_1^2 = -2u_2^2$

For $u_1, u_2 > 0$, $u_1^2 = -2u_2^2$ is impossible as $u_1^2 > 0$ while $-2u_2^2 < 0$.

Hence, for $\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 0$, $\underset{\sim}{u} = \underset{\sim}{0}$

8a) when $\underset{\sim}{u} = \underset{\sim}{0}$

$$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{0} \rangle$$
$$= 0(0) + 2(0,0)$$
$$= 0$$

$\therefore \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$ is an inner product on V.

8b) $\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = u_1 u_1 - 2u_2 u_2$

$$= u_1^2 - 2u_2^2 \ngeq 0 \text{ for } u_1, u_2 \in \mathbb{R}$$

$\therefore \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$ is not an inner product of V.

c) $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + u_2 v_2$

$$= v_1 u_1 - v_1 u_2 - v_2 u_1 + v_2 u_2$$
$$= \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle$$

$\langle \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{w} \rangle = (u_1 + v_1) w_1 - (u_1 + v_1) w_2$
$- (u_2 + v_2) w_1 + (u_2 + v_2) w_2$
$= u_1 w_1 + v_1 w_1 - u_1 w_2 - v_1 w_2$
$- u_2 w_1 - v_2 w_1 + u_2 w_2 + v_2 w_2$
$= u_1 w_1 - u_1 w_2 - u_2 w_1 + u_2 w_2$
$+ v_1 w_1 - v_1 w_2 - v_2 w_1 + v_2 w_2$
$= \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

8c) $\langle k\underset{\sim}{u}, \underset{\sim}{v} \rangle = ku_1v_1 - ku_1v_2 - ku_2v_1 + ku_2v_2$

$= k(u_1v_2 - u_1v_2 - u_2v_1 + u_2v_2)$

$= k\langle u, \underset{\sim}{v} \rangle$

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = u_1u_1 - u_1u_2 - u_2u_1 + u_2u_2$

$= u_1^2 - 2u_1u_2 + u_2^2 \geq 0$

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 0$

$u_1^2 - 2u_1u_2 + u_2^2 = 0$

$u_1^2 + u_2^2 = 2u_1u_2$

$\therefore \langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 0$ if $u_1^2 + u_2^2 = 2u_1u_2$,

Like when $u_1, u_2 = 1$,

$1^2 + 1^2 = 2(1)(1)$

$2 = 2$

$\therefore \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$ is not an inner product of V.

8d) $\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle = 2u_1v_1 - u_1v_2 - u_2v_1 + 2u_2v_2$

$= 2v_1u_1 - v_1u_2 - v_2u_1 + 2v_2u_2$

$= \langle \underset{\sim}{v}, \underset{\sim}{u} \rangle$

$\langle \underset{\sim}{u} + \underset{\sim}{v}, \underset{\sim}{w} \rangle = 2(u_1 + v_1)w_1 - (u_1 + v_1)w_2$

$- (u_2 + v_2)w_1 + 2(u_2 + v_2)w_2$

$= 2u_1w_1 + 2v_1w_1 - u_1w_2 - v_1w_2$

$- u_2w_1 - v_2w_1 + 2u_2w_2 + 2v_2w_2$

$= 2u_1w_1 - u_1w_2 - u_2w_1 + 2u_2w_2$

$+ 2v_1w_1 - v_1w_2 - v_2w_1 + 2v_2w_2$

$= \langle \underset{\sim}{u}, \underset{\sim}{w} \rangle + \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$

$\langle k\underset{\sim}{u}, \underset{\sim}{v} \rangle = 2ku_1v_1 - ku_1w_2 - ku_2w_1 + 2ku_2w_2$

$= k(2u_1v_1 - u_1w_2 - u_2w_1 + 2u_2w_2)$

$= k\langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 2u_1u_1 - u_1u_2 - u_2u_1 + 2u_2u_2$

$= 2u_1^2 - 2u_1u_2 + 2u_2^2$

$= u_1^2 - 2u_1u_2 + u_2^2 + u_1^2 + u_2^2$

$= (u_1 - u_2)^2 + u_1^2 + u_2^2 \geq 0$

8d) $\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = 0$

$$(u_1 - u_2)^2 + u_1^2 + u_2^2 = 0$$

The only way for $(u_1 - u_2)^2 + u_1^2 + u_2^2 = 0$ is for $u_1 = u_2 = 0$.

When $\underset{\sim}{u} = \underset{\sim}{0}$,

$$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{0} \rangle$$
$$= 2(0)(0) - 0(0) - 0(0) + 2(0)(0)$$
$$= 0$$

$\therefore \langle \underset{\sim}{u}, \underset{\sim}{v} \rangle$ is a inner product of V.

9) If $\underset{\sim}{0} \in S$, then S is linearly dependent.

Let $\underset{\sim}{0} = \lambda_1 \underset{\sim}{v}_1 + \lambda_2 \underset{\sim}{v}_2 + \dots + \lambda_r \underset{\sim}{v}_r$, $\lambda_1, \lambda_2, \lambda_r \in \mathbb{R}$

For any $1 \leq i \leq r$,

$$\langle \underset{\sim}{0}, \underset{\sim}{v}_i \rangle = \langle \lambda_1 \underset{\sim}{v}_1 + \dots + \lambda_r \underset{\sim}{v}_r, \underset{\sim}{v}_i \rangle$$

$$= \sum_{j=1}^{r} \lambda_j \langle \underset{\sim}{v}_j, \underset{\sim}{v}_i \rangle$$

$= \lambda_i \langle \underset{\sim}{v}_i, \underset{\sim}{v}_i \rangle$ as S is an orthorgonal set

Since $\underset{\sim}{0} \notin S_1$, $\underset{\sim}{v}_i \neq \underset{\sim}{0}$. So $\langle \underset{\sim}{v}_i, \underset{\sim}{v}_i \rangle \neq 0$, so $\lambda_i = 0$.

$$\therefore \lambda_1 = \lambda_2 = \dots = \lambda_r = 0$$

$\therefore S$ is linearly independent.

10) Let $\underset{\sim}{v}$ be a vector in the inner product space with the property that

$$\langle \underset{\sim}{v}, \underset{\sim}{v}_k \rangle = 0 \text{ for all } k = 1, 2, \ldots, n$$

Let $\underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \ldots + c_n \underset{\sim}{v}_n$, $c_1, c_2, \ldots, c_n \in \mathbb{R}$

$$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{v}, c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 + \ldots + c_n \underset{\sim}{v}_n \rangle$$

$$= c_1 \langle \underset{\sim}{v}, \underset{\sim}{v}_1 \rangle + c_2 \langle \underset{\sim}{v}, \underset{\sim}{v}_2 \rangle + \ldots + c_n \langle \underset{\sim}{v}, \underset{\sim}{v}_n \rangle$$

$$= c_1 0 + c_2 0 + \ldots + c_n 0$$

$$= 0$$

By property 4 of inner product spaces, for

$$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = 0$$

$$\underset{\sim}{v} = \underset{\sim}{0}$$

$\therefore$ $\underset{\sim}{0}$ is the only vector that is orthorgonal to all the basis vectors.

11) $\underset{\sim}{w} = \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1 + \langle \underset{\sim}{w}, \underset{\sim}{v}_2 \rangle \underset{\sim}{v}_2 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle \underset{\sim}{v}_n$

$$\|\underset{\sim}{w}\|^2 = \langle \underset{\sim}{w}, \underset{\sim}{w} \rangle$$

$$= \langle \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle \underset{\sim}{v}_n, \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle \underset{\sim}{v}_n \rangle$$

$$= \langle \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1, \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle \underset{\sim}{v}_1 \rangle + \dots + \langle \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle v_n, \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle \underset{\sim}{v}_n \rangle$$

$$= \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle^2 \langle \underset{\sim}{v}_1, \underset{\sim}{v}_1 \rangle + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle^2 \langle \underset{\sim}{v}_n, \underset{\sim}{v}_n \rangle$$

$$= \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle^2 \|\underset{\sim}{v}_1\|^2 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle^2 \|\underset{\sim}{v}_n\|^2$$

$$= \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle^2 (1)^2 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle^2 (1)^2$$

$\because \|\underset{\sim}{v}_k\| = 1$ for $k = 1, 2, \dots, n$

$$= \langle \underset{\sim}{w}, \underset{\sim}{v}_1 \rangle^2 + \langle \underset{\sim}{w}, \underset{\sim}{v}_2 \rangle^2 + \dots + \langle \underset{\sim}{w}, \underset{\sim}{v}_n \rangle^2$$

(shown)