

MA3005 Control Theory

Notes

Hankertrix

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1 Definitions

1.1 Control theory

Control theory deals with the dynamic response of a system to commands or disturbances.

1.1.1 Classical

- Transfer function concept with analysis and design in the Laplace and frequency domain.
- More emphasis on physical understanding.

1.1.2 Modern

- State variable concept with emphasis on matrix algebra.
- Mathematical technique

1.1.3 Types of control systems

- Open-loop (O/L)
 - No feedback
- Closed-loop (C/L)
 - Has feedback
 - Error actuated

1.2 Control system

- A control system is a system in which some **physical quantities** are controlled by regulating certain energy inputs.
- A **system** is a group of physical components assembled to perform a specific function. The components may be electrical, mechanical, thermal, biomedical, chemical, pneumatic or any combination of the above.
- A system need not be limited to an ensemble of physical components. It can refer to abstract, dynamic phenomena such as economics (COE) or ecology systems.
- A physical quantity may be temperature, pressure, electrical voltage, mechanical position, velocity, pH value, liquid level, etc.
- There are many applications of control system including simple devices like air-conditioners, microwave, traffic light control to sophisticated systems like space vehicles, missile guidance, autopilot, robotic systems, power systems, assembly line, chemical processes, quality control, etc.

1.3 Open loop control (O/L)

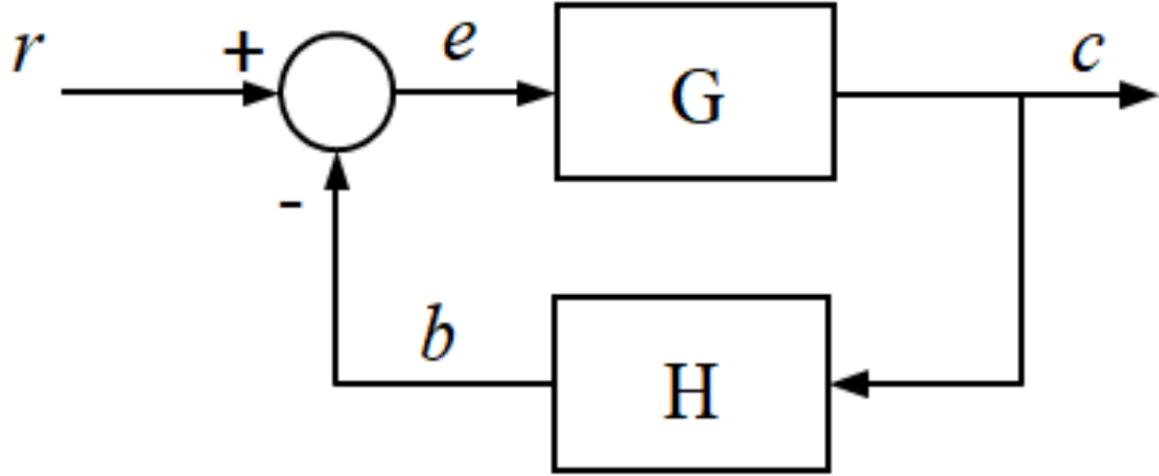
- The output has no effect on control action. Some examples are traffic lights, microwave, oven, dryers, toasters, etc. Requires careful calibration for good performance.
- Control variable adjusted to make output equal to input, but not readjusted to keep the two equal.
- Open loop control with certain safeguards are very common, such as guiding a process through a sequence of predetermined steps.

1.4 Closed loop control (C/L)

- There is an actuating signal, and hence the output depends on the input and the current output. Some examples are servo-controlled motor, missile, robots, etc.
- To improve performance, the operator adjusts the control variable based on observation of the system error, e.
- Feedback control system automates this action.
- The error is used to adjust the control variable by means of an actuator.

1.5 Control system equation

Consider a static system.



- Assume that G and H are static values, then:

$$e(t) = r(t) - b(t) = r(t) - Hc(t)$$

$$c(t) = Ge(t) = Gr(t) - GHc(t)$$

$$\frac{c(t)}{r(t)} = \frac{G}{1 + GH}$$

- Compare this to the open loop system where:

$$\frac{c(t)}{r(t)} = G$$

- Let $M := \frac{c(t)}{r(t)}$, the relation between $c(t)$ and $r(t)$

1.5.1 Effect on stability

- If $GH = -1$ then $\frac{c(t)}{r(t)} = \infty$
- Closed loop control system can become unstable if choices G, H are not chosen carefully.
- For the simple case above, the unstable condition of $GH = -1$ can be avoided using an appropriate value of H .
- However, in the systems that are being analysed, elements of G and H are functions of s (frequency) and proper handling is needed for such systems.
- Essentially, feedback can destabilise a system if it is not carefully designed.
- Open loop systems seldom have such problems, due to not having any feedback in the first place.

1.5.2 Effect on parameter sensitivity

- Defining a measure of sensitivity as S_G^M , which is the percentage change in M with respect to a percentage change in G .

$$S_G^M = \frac{\frac{\Delta M}{M}}{\frac{\Delta G}{G}} = \frac{\Delta M}{\Delta G} \frac{G}{M} \approx \frac{\partial M}{\partial G} \frac{G}{M}$$

- For a closed-loop system:

$$S_G^M = \frac{1}{1 + GH}$$

- For an open-loop system:

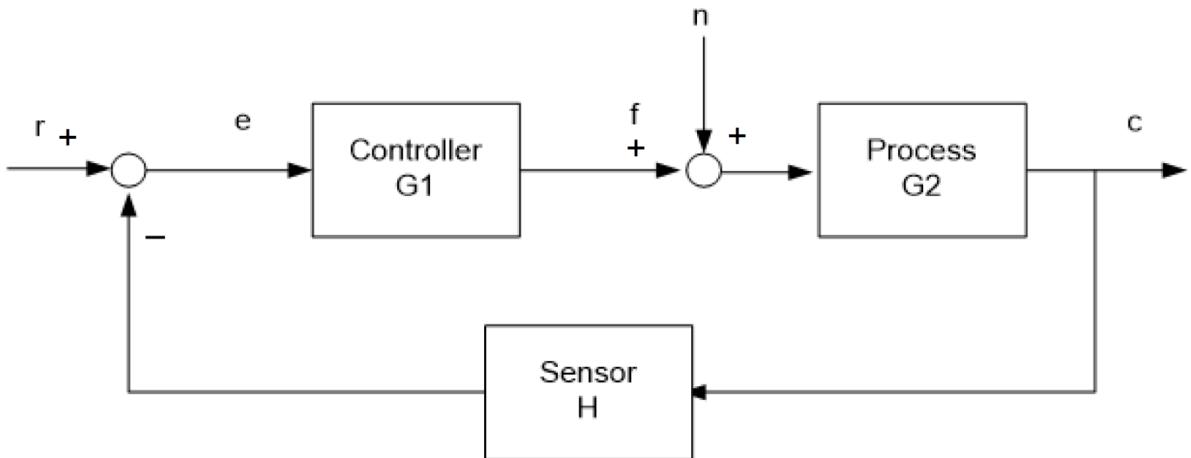
$$S_G^M = 1$$

- Implications:

- Open-loop systems “feel” the changes in G directly.
- Closed-loop systems can be made to be less sensitive to changes in G using an appropriately large value of H .

1.5.3 Effect of input noise.

Another significant advantage of feedback system is in its superior performance in the presence of disturbance or noise.



- Suppose $H = 0$ and $r = 0$, which is an open loop system with no feedback and zero reference, then:

$$c(t) = G_2 n(t), \quad \text{or} \quad \frac{c(t)}{n(t)} = G_2$$

- Suppose $H \neq 0$ but with $r = 0$, we have:

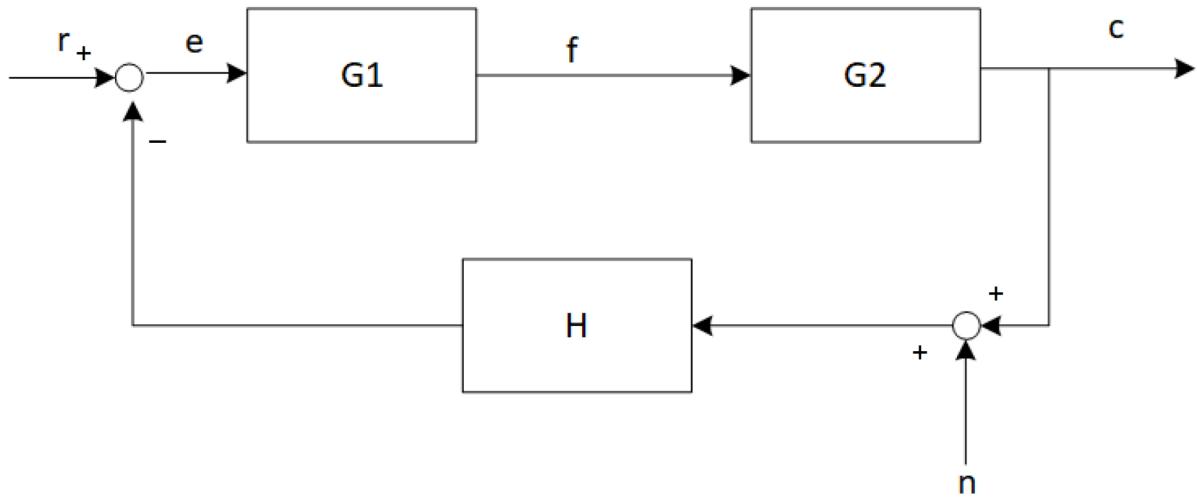
$$c(t) = G_2(f(t) + n(t)) = G_2(-G_1 H c(t) + n(t))$$

$$c(t) = \frac{G_2}{1 + G_1 G_2 H} n(t), \quad \text{or} \quad \frac{c(t)}{n(t)} = \frac{G_2}{1 + G_1 G_2 H}$$

- If H or G_1 is large, then $1 + G_1 G_2 H$ is large and $c(t)$ due to $n(t)$ is small.

1.5.4 Effect of sensor noise

However, feedback is not effective against sensor noise.



- Suppose $H \neq -$ but with $r = 0$, we have:

$$c(t) = G_2 G_1 e(t) = -G_2 G_1 H(n(t) + c(t))$$

$$\text{Or } c(t) = \frac{-G_2 G_1 H}{1 + G_2 G_1 H} n(t)$$

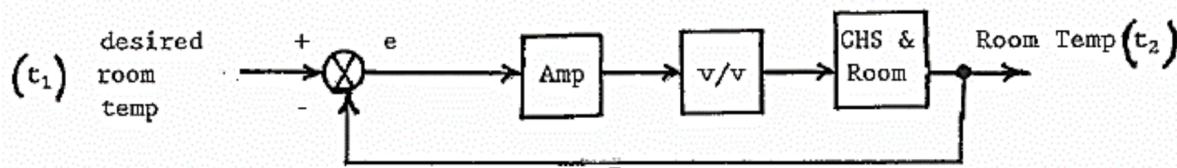
- If H is large, both numerator and denominator are large, and hence $n(t)$ is not attenuated by large H .

1.6 Feedback control systems

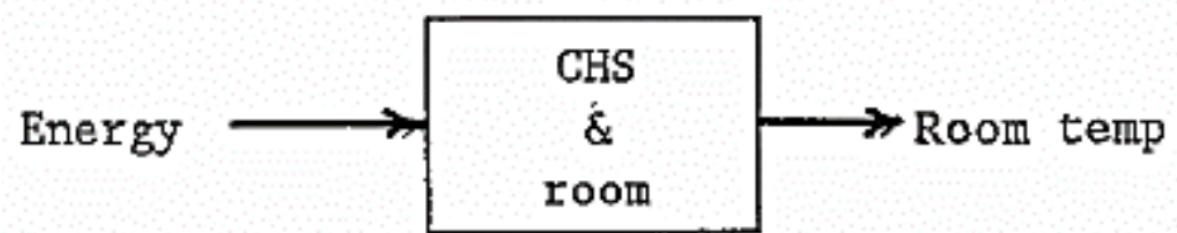
1.6.1 Regulator system

A regulator system's function is to maintain a constant output, despite unwanted disturbances to the system. The input is seldom changed.

- Open loop (O/L) system, which is manually controlled and has no thermostat

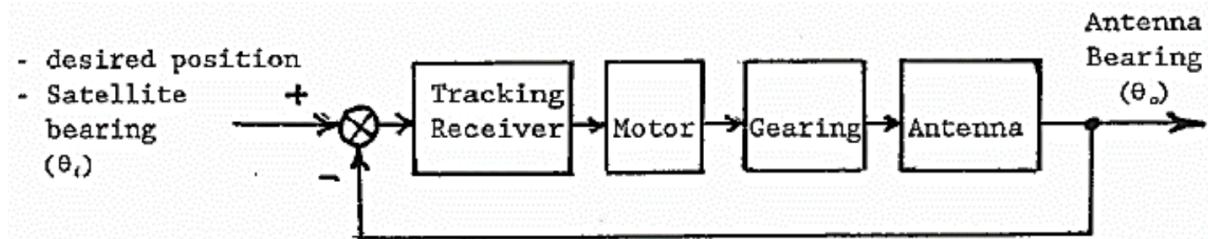


- Closed loop (C/L) system, which is automatic and requires a thermostat

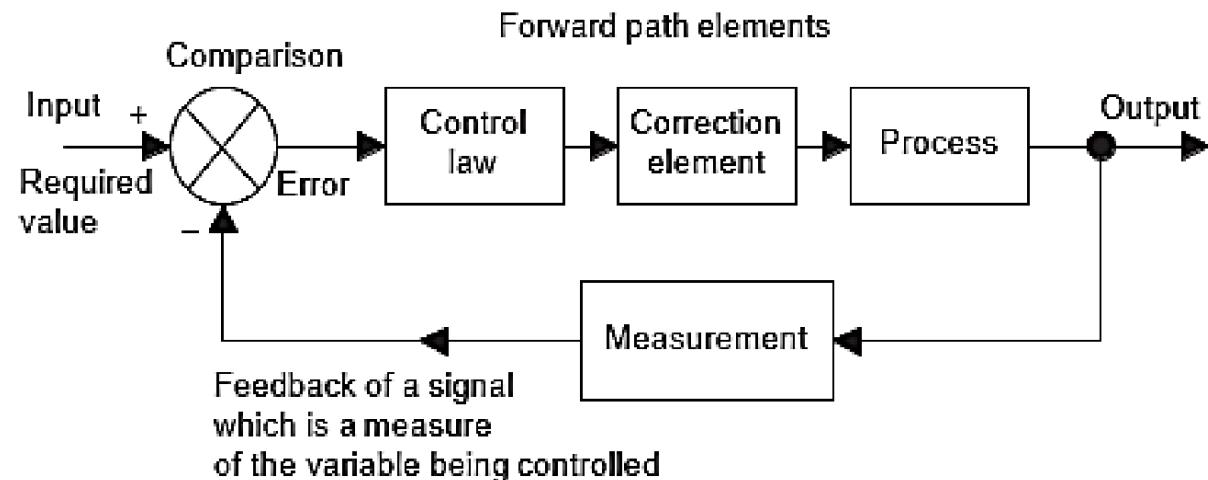


1.6.2 Follow-up system

A follow-up system keeps the output in close correspondence with the input, which is always changing.



1.6.3 Basic element of a control-loop system



1.7 Transfer function

The transfer function of a linear, time-invariant, differential equation system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input, under the assumption that all initial conditions are zero.

$$G(s) = \frac{\mathcal{L}\{\text{Output}\}}{\mathcal{L}\{\text{Input}\}}$$

1.8 Plant

- A plant is the combination of process and the actuator.
- A plant is often referred to with a transfer function (commonly in the s-domain) which indicates the relation between an input signal and the output signal of a system without feedback, commonly determined by the physical properties of the system.

1.9 Dynamic analysis

- Dynamic analysis is the determination of a response of a plant to commands, disturbances and changes in plant parameters.
- If the dynamic analysis is unsatisfactory, and the plant cannot be modified, it is necessary to select control elements needed to improve performance to an acceptable level.

1.10 Methods of analysis

1. Consider system performance in the time domain by measuring the output response for a given input, like step input, ramp input or sinusoidal input.
2. Analysis in the frequency domain. Output response to sinusoidal input is considered in steady state, with transients allowed to subside, before measurements are made. It is very common to work in the frequency domain.

1.11 Requirements of control theory

1. Stability.
2. Accuracy, which is specified in terms of errors. Errors can either be:
 - Steady state (s.s.), like due to static friction as the output cases to move.
 - Transient, which can reverse the direction or overshoot due to energy stored in inertia.
3. Speed of response.

All 3 requirements are incompatible with each other and a good design is a compromise of all 3 requirements.

- A highly accurate response would not be stable, since it would measure all the tiny fluctuations due to noise from the background.
- An extremely fast response time would likely result in inaccurate and unstable results, as it is difficult to bring the system to a steady state (critically damp the system) in an extremely short amount of time.

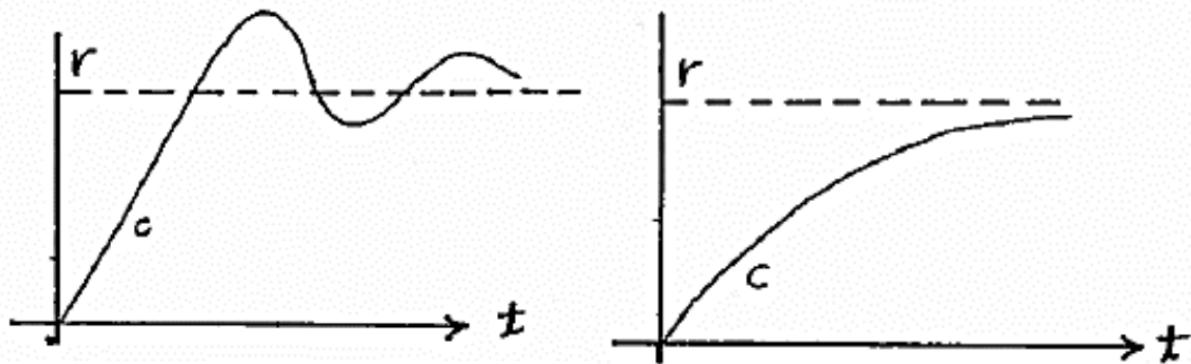
1.11.1 Example

If a system is subjected to a sudden change in input:

- Its transient period should be short, and its response should not be excessively oscillatory. In other words, the system should be critically damped.
- The steady state error must be small.

Transient response and steady state error characteristics can be improved using feedback (sensors) and the motivation for feedback.

- Reduce the effect of parameter variation.
- Reduce the effect of noise.
- Improve transient response characteristics.
- Reducing the steady state error.



1.12 Stationary or time invariant system

- The parameters of this system do not vary with time.
- The output is independent of time.
- The coefficient of the describing differential equation is constant.

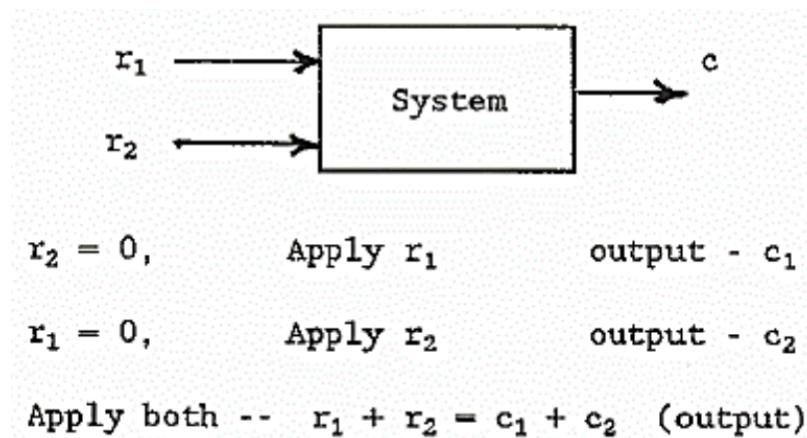
For example:

$$\ddot{y} + 4xy + 2x^2 = 3t \text{ (non-linear)}$$

1.13 Linear system

The principle of superposition holds for such a system, which means the inputs can be added together to obtain the output.

For example:



1.14 Lumped parameter system

- The physical characteristics of this system are assumed to concentrate in one or more lumps and thus independent of spatial distribution.
- For example:
 - Bodies are assumed to be rigid and treated as point mass.
 - Springs are regarded as massless and do not change with temperature.
 - Electrical leads are assumed to have no resistance.
- In contrast, when the parameters are distributed, the examples above become:
 - Bodies are elastic.
 - Springs have distributed mass.
 - Electrical leads have distributed resistance.
 - Temperature varies across a border.

1.15 Deterministic system or variable

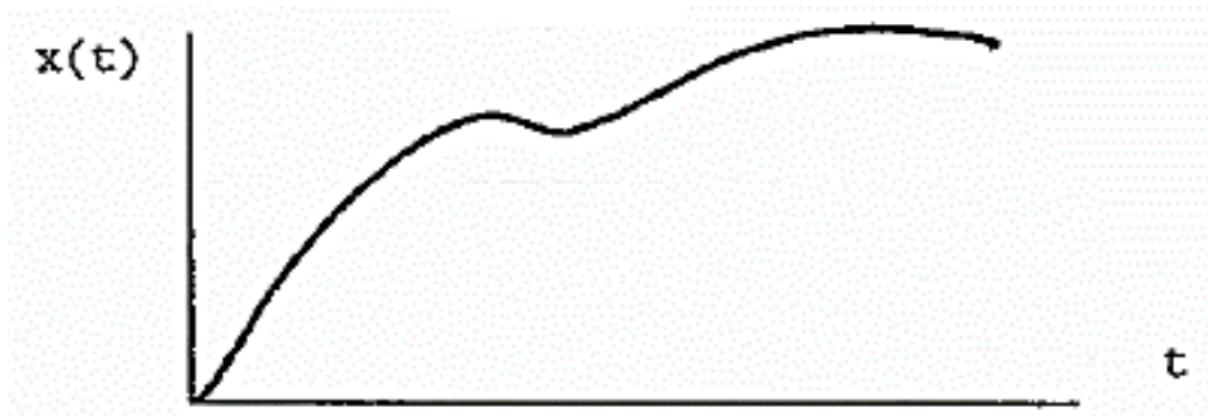
- A deterministic system or variable's future behaviour is predictable and repeatable within reasonable limits.
- Otherwise, system or variable is considered stochastic or random, and analysis of it is based on probability theory.

1.16 First order difference equations

A first order difference equation is a recursively defined sequence in the form:

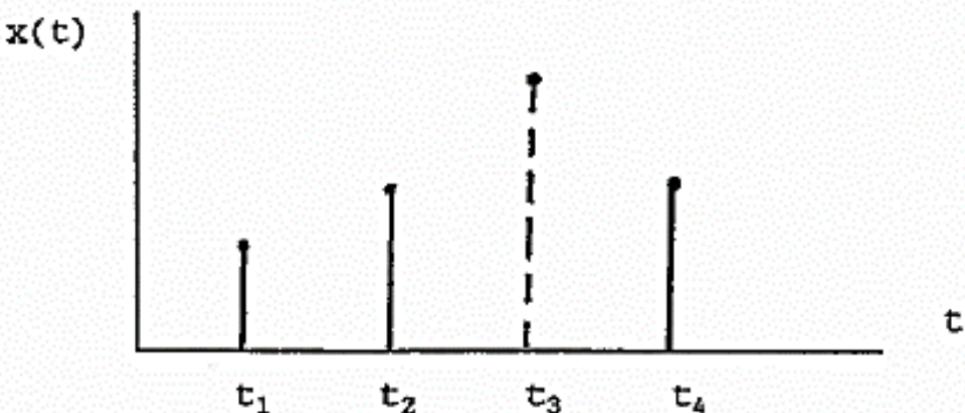
$$y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \dots$$

1.17 Continuous variable system



- All system variables are a continuous function of time.
- The describing equations are **differential** equations.

1.18 Discrete variable system



- A discrete variable system has one or more variables known only at a particular instant of time.
- The equations are **difference** equations.
- Such a system is usual for systems that sample data at regular time intervals.

1.19 Ways to classify systems

1.19.1 Static or dynamic systems

- Static systems are composed of simple linear gains or non-linear devices and are described by algebraic equations.
- Dynamic systems are described by differential or difference equations.

1.19.2 Continuous-time or discrete-time systems

Continuous-time dynamic systems are described by differential equations, and discrete-time dynamic systems are described by difference equations.

1.19.3 Linear or non-linear systems

- Linear dynamic systems are described by differential (or difference) equations having solutions that are linearly related to their inputs.
- Equations describing non-linear dynamic systems contain one or more non-linear terms.

Example:

$$\ddot{x} + 7\dot{x} + 3x^2 = 3\dot{y} + 3y$$

The equation above has an input in y and an output in x . It is non-linear due to the presence of the x^2 term.

1.19.4 Lumped or distributed parameters

Lumped-parameter, continuous-time dynamic systems are described by ordinary differential equations, and distributed parameter, continuous-time dynamic systems are described by partial differential equations.

Example:

$$\frac{\partial^2 y}{\partial t^2} - 10 \frac{\partial^2 y}{\partial x^2} = 0$$

The equation above is a distributed parameter system due to the difference equation above.

1.19.5 Time-varying (non-stationary) or time-invariant (stationary) systems

- Time-varying dynamic systems are described by differential (or difference) equations having one or more coefficients as functions of time.
- Time-invariant (constant-parameter) dynamic systems are described by differential (or difference) equations having only constant coefficients.

Example:

$$\ddot{x} + 5t\dot{x} + 6x = 5y$$

The equation above has an input in y and an output in x . It is a time-varying system due to the presence of the t term in the output.

1.19.6 Deterministic or stochastic systems

Deterministic systems have fixed (non-random) parameters and inputs, and stochastic systems have randomness in one or more parameters or inputs.

1.19.7 Multivariable or single variable systems

Multivariable systems have multiple variables in their input and while single variable systems only have one variable in their input.

Example:

$$\ddot{x} + 5\dot{x} + 3x = 2\dot{y} + 3y + 4z$$

The equation above has inputs in y and z and an output in x , hence it is a multivariable system.

1.20 Dynamic systems

Any dynamic system is the sum of first order dynamic systems.

1.21 Heaviside cover-up method (Heaviside theorem)

Heaviside cover-up method is a method to decompose a fraction into partial fractions.

Using the example of:

$$\frac{2s+1}{s(s-1)(s+1)}$$

1. Do the first step of regular partial fraction decomposition.

$$\frac{2s+1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$$

2. Pick one of the partial fractions, for example, $\frac{C}{s+1}$, and cover up its denominator. Equate it to the original fraction and also cover up the same term in the original fraction. By covering up the term, set its value to 0, and treat the term like it doesn't exist. Using $\frac{C}{s+1}$:

$$\begin{aligned} \frac{2s+1}{s(s-1) \boxed{s+1}} &= \boxed{\frac{C}{s+1}} \\ \frac{2s+1}{s(s-1) \blacksquare} &= \blacksquare \\ \frac{2s+1}{s(s-1)} \Big|_{s+1=0} &= C \\ \frac{2s+1}{s(s-1)} \Big|_{s=-1} &= C \\ \frac{2(-1)+1}{(-1)(-1-1)} &= C \\ \frac{-2+1}{(-1)(-2)} &= C \\ \frac{1}{2} &= C \\ C &= \frac{1}{2} \end{aligned}$$

1.21.1 Extension to multiple roots

The Heaviside cover-up method can be applied to multiple roots by factoring out the repeats and applying it to the remaining fraction. For example:

$$\begin{aligned} \frac{1}{(s+1)^2(s+2)} &= \frac{1}{s+1} \left(\frac{1}{(s+1)(s+2)} \right) && [\text{Apply Heaviside cover-up method}] \\ &= \frac{1}{s+1} \left(\frac{1}{s+1} - \frac{1}{s+2} \right) \\ &= \frac{1}{(s+1)^2} - \frac{1}{(s+1)(s+2)} && [\text{Apply Heaviside cover-up method}] \\ &= \frac{1}{(s+1)^2} - \left(\frac{1}{s+1} - \frac{1}{s+2} \right) \\ &= \frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s+2} \end{aligned}$$

1.21.2 Special case for roots of multiplicity 2, like $(s + 1)^2$

The Heaviside cover-up method can be easily used for roots of multiplicity 2 by multiplying the partial fraction by the denominator of the remaining unknown and setting the variable to infinity. For example:

$$\begin{aligned}\frac{1}{(s+1)^2(s+2)} &= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} && [\text{Apply Heaviside cover-up method}] \\ &= \frac{A}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{s+2}\end{aligned}$$

To find the remaining unknown (A), multiply the equation by the denominator of the remaining unknown (A):

$$\begin{aligned}\frac{1}{(s+1)^2(s+2)} &= \frac{A}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{s+2} \\ \frac{1}{(s+1)^2(s+2)} &= (s+1) \left(\frac{A}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{s+2} \right) \\ \frac{1}{(s+1)(s+2)} &= A - \frac{1}{(s+1)} + \frac{s+1}{s+2}\end{aligned}$$

Set $s = \infty$:

$$\begin{aligned}\frac{1}{(s+1)(s+2)} &= A - \frac{1}{(s+1)} + \frac{1}{(s+1)(s+2)} \\ \frac{1}{(\infty+1)(\infty+2)} &= A - \frac{1}{(\infty+1)} + \frac{\infty+1}{\infty+2} \\ 0 &= A - 0 + 1 \\ A &= -1 \\ \therefore \frac{1}{(s+1)(s+2)} &= -\frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{s+2} \\ \therefore \frac{1}{(s+1)(s+2)} &= \frac{1}{s+2} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\end{aligned}$$

1.22 SISO

SISO stands for single input, single output.

1.23 MIMO

- MIMO stands for multiple input, multiple output.
- A multiple input, multiple output (MIMO) system is equal to the sum of single input, single output systems (SISO), i.e.

$$\text{MIMO} = \text{Sum of SISO}$$

1.24 Poles

For a transfer function:

$$G(s) = \frac{N(s)}{D(s)}$$

The characteristic equation is:

$$D(s) = 0$$

The poles are defined as all values of s that satisfy the characteristic equation.

2 Laplace transform

Let $F(s)$ be the Laplace transform of $f(t)$, written as:

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty e^{-st} f(t) dt$$

The inverse Laplace transform is:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j} \left[\int_{c-j\infty}^{c+j\infty} F(s)e^{st} ds \right]$$

Where:

- $s = \sigma + j\omega$

Then:

$$\begin{aligned} \mathcal{L}\left\{\frac{df}{dt}\right\} &= sF(s) - f(0) \\ \mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} &= s^2F(s) - sf(0) - \frac{df}{dt}(0) \\ \mathcal{L}\left\{\frac{d^n f}{dt^n}\right\} &= s^n F(s) - \sum_{i=1}^n s^{n-i} \frac{d^{i-1}f}{dt^{i-1}} \end{aligned}$$

Laplace transform allows a differential equation in the time domain to be converted to an algebraic equation, which is then solved to obtain the solution in the Laplace domain, before converting back to the final solution in the time domain.

2.1 Property 1

$$\mathcal{L}\left[\int_0^t f(x) dx\right] = \frac{1}{s} \int_0^t f(x) dx \Big|_{t=0} + \frac{F(s)}{s}$$

2.2 Property 2

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = f(t)e^{st} \Big|_{t=0} + sF(s)$$

2.3 Property 3

$$\mathcal{L}[f(t)] = F(s)$$

$$\mathcal{L}[f(t)e^{-at}] = F(s+a)$$

2.4 Property 4

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

2.5 Shift theorems

- In automatic control systems, it is known as dead time.
- In the process industry, it is known as transport lag.

$$\mathcal{L}\{e^{at}f(t)\} = F(s+a)$$

$$\mathcal{L}\{f(t+a)\} = e^{as}F(s)$$

2.6 Convolution theorem

- The convolution theorem is the product of 2 Laplace transforms to form the Laplace transform of the convolution integral.
- τ is known as the dummy time variable.

$$\begin{aligned}\mathcal{L}^{-1}\{X(s)Y(s)\} &= \int_0^t x(t-\tau)y(\tau) d\tau \\ &= \int_0^t y(t-\tau)x(\tau) d\tau\end{aligned}$$

2.7 Final value theorem

The final value theorem is useful in determining steady state accuracy.

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

2.8 Initial value theorem

The initial value theorem is useful in the inverse Laplace transform when the initial condition is known to be zero.

$$\lim_{t \rightarrow 0+} f(t) = \lim_{s \rightarrow \infty} sF(s)$$

2.9 Laplace transform table

Time function: $f(t)$ (0 for $t < 0$)	Laplace transform: $F(s)$
A unit impulse	1
A unit step: constant = $u(t)$	$\frac{1}{s}$
A unit ramp: t	$\frac{1}{s^2}$
A ramp: at	$\frac{a}{s^2}$
$\frac{at^n}{n!}$	$\frac{a}{s^{n+1}}$
Exponential decay: e^{-at}	$\frac{1}{s+a}$
Exponential growth: $1 - e^{-at}$	$\frac{a}{s(s+a)}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t - \frac{1-e^{-at}}{a}$	$\frac{a}{s^2(s+a)}$
$e^{-at} - e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$
$(1-at)e^{-at}$	$\frac{s}{(s+a)^2}$
$1 - \frac{b}{b-a}e^{-at} + \frac{a}{b-a}e^{-bt}$	$\frac{ab}{s(s+a)(s+b)}$
$\frac{e^{-at}}{(b-a)(c-a)} + \frac{e^{-bt}}{(c-a)(a-b)} + \frac{e^{-ct}}{(a-c)(b-c)}$	$\frac{1}{(s+a)(s+b)(s+c)}$
A sine wave: $\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
A cosine wave: $\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$t \sin \omega t$	$\frac{2ws}{(s^2+\omega^2)^2}$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
A damped sine wave: $e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
A damped cosine wave: $e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega}{\sqrt{1-\zeta^2}} e^{-\zeta \omega t} \sin \omega \sqrt{1-\zeta^2} t$	$\frac{\omega^2}{s^2+2\zeta\omega s+\omega^2}$
$1 - \frac{1}{1-\zeta^2} e^{-\zeta \omega t} \sin(\omega \sqrt{1-\zeta^2} t + \phi), \cos \phi = \zeta$	$\frac{\omega^2}{s(s^2+2\zeta\omega s+\omega^2)}$

2.10 Inverse Laplace Transform for complex conjugate roots

Given a Laplace transform as a fraction:

$$\begin{aligned} Y(s) &= \frac{C}{[s - (a + jb)][s - (a - jb)](s - r_1)} \\ &= \frac{C}{(s^2 - 2as + a^2 + b^2)(s - r_1)} \\ &= \frac{Ks}{s^2 - 2as + a^2 + b^2} + \frac{K_1}{s - r_1} \end{aligned}$$

Where:

- C, K and K_1 are constants
- $(a \pm jb)$ are the complex conjugate roots
- $(s - r_1)$ is the remaining root

The general form of the inverse transformation for the above Laplace transform with complex conjugate roots ($Y(s)$) is:

$$y(t) = \frac{1}{b} |K(a + jb)| e^{at} \sin(bt + \alpha) + K_1 e^{r_1 t}$$

Where:

- α is the angle from the polar form of $K(a + jb)$

The response term is just the general form without the K_1 term, i.e.

$$\begin{aligned} y(t) &= \frac{1}{b} |K(a + jb)| e^{at} \sin(bt + \alpha) \\ &= \frac{1}{b} e^{at} |K(a + jb)| (\cos \alpha \sin bt + \sin \alpha \cos bt) \\ &= \frac{1}{b} e^{at} (A \sin bt + B \cos bt) \end{aligned}$$

Where:

- $A = |K(a + jb)| \cos \alpha$
- $B = |K(a + jb)| \sin \alpha$

2.10.1 Example

Given:

$$Y(s) = \frac{75}{(s^2 + 4s + 13)(s + 6)}$$

Since the $(s^2 + 4s + 13)$ term cannot be factorised into real roots, we make use of the above general form for complex conjugate roots:

$$\begin{aligned}s^2 + 4s + 13 &= [s - (a + jb)][s - (a - jb)] \\ s^2 + 4s + 13 &= s^2 - 2as + a^2 + b^2\end{aligned}$$

By comparing the coefficients:

$$\begin{aligned}-2a &= 4 & \text{and} & \quad a^2 + b^2 = 13 \\ a &= -2 \\ a^2 + b^2 &= 13 \\ (-2)^2 + b^2 &= 13 \quad \therefore a = -2 \\ b &= \sqrt{13 - 4} \\ b &= 3 \\ \therefore a &= -2 \quad \text{and} \quad b = 3\end{aligned}$$

Evaluating Ks using the Heaviside cover-up method:

$$\begin{aligned}\left. \frac{75}{(s^2 + 4s + 13)(s + 6)} \right|_{s=a+jb} &= \left. \frac{Ks}{s^2 + 4s + 13} \right|_{s=a+jb} \\ \left. \frac{75}{(s+6)} \right|_{s=a+jb} &= \left. \frac{Ks}{} \right|_{s=a+jb} \\ \left. \frac{75}{s+6} \right|_{s=a+jb} &= Ks|_{s=a+jb} \\ \left. \frac{75}{s+6} \right|_{s=-2+3j} &= Ks|_{s=-2+3j} \\ \frac{75}{-2+3j+6} &= K(-2+3j) \\ \frac{75}{4+3j} &= K(-2+3j) \\ K(-2+3j) &= \frac{75}{4+3j}\end{aligned}$$

Converting $4 + 3j$ to polar form:

$$\begin{aligned}
 |4 + 3j| &= \sqrt{4^2 + 3^2} \\
 &= \sqrt{25} \\
 &= 5 \\
 \alpha &= \arctan \frac{3}{4} \\
 &\approx 36.9^\circ \\
 \therefore 4 + 3j &= 5\angle 36.9^\circ
 \end{aligned}$$

Hence:

$$\begin{aligned}
 K(-2 + 3j) &= \frac{75}{5\angle 36.9^\circ} \\
 &= 15\angle -36.9^\circ \quad \because \frac{1}{i} = -i \\
 \therefore |K(-2 + 3j)| &= 15 \\
 \therefore \alpha &= \angle K(-2 + 3j) = -36.9^\circ
 \end{aligned}$$

Now that we have found the solution for the complex conjugate term, we need to find the value for r_1 . By comparing the coefficients:

$$\begin{aligned}
 s - r_1 &= s + 6 \\
 r_1 &= -6
 \end{aligned}$$

Using the Heaviside cover-up method to find K_1 :

$$\begin{aligned}
 \frac{75}{(s^2 + 4s + 13)(s + 6)} \Big|_{s+6=0} &= \frac{K_1}{s+6} \Big|_{s+6=0} \\
 \frac{75}{(s^2 + 4s + 13) \boxed{}} \Big|_{\{s=-6\}} &= \boxed{} \Big|_{s=-6} \\
 \frac{75}{s^2 + 4s + 13} \Big|_{s=-6} &= K_1 \Big|_{s=-6} \\
 \frac{75}{(-6)^2 + 4(-6) + 13} &= K_1 \\
 K_1 &= \frac{75}{25} \\
 K_1 &= 3
 \end{aligned}$$

Hence, the final equation is:

$$\begin{aligned}
y(t) &= \frac{1}{b} |K(a + jb)| e^{\{at\}} \sin(bt + \alpha) + K_1 e^{r_1 t} \\
&= \frac{1}{3} \cdot 15e^{-2t} \sin(3t - 36.9^\circ) + 3e^{-6t} \\
&= 5e^{-2t} \sin(3t - 36.9^\circ) + 3e^{-6t}
\end{aligned}$$

Using the relationship $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ in which $\alpha = 3t$ and $\beta = -36.9^\circ$ gives:

$$\begin{aligned}
y(t) &= 5e^{-2t} \sin(3t - 36.9^\circ) + 3e^{-6t} \\
&= 5e^{-2t} (\sin 3t \cos(-36.9^\circ) + \cos 3t \sin(-36.9^\circ)) + 3e^{-6t} \\
&= 5e^{-2t} \left(\frac{4}{5} \sin 3t + \frac{3}{5} \cos 3t \right) + 3e^{-6t} \\
&= e^{-2t} (4 \sin 3t + 3 \cos 3t) + 3e^{-6t}
\end{aligned}$$

Hence, the response term is:

$$e^{-2t} (4 \sin 3t + 3 \cos 3t)$$

2.10.2 Alternate method

Alternatively, we can also apply the standard partial fraction decomposition then evaluate the inverse Laplace Transform for each fraction. Using the same example:

$$Y(s) = \frac{75}{(s^2 + 4s + 13)(s + 6)}$$

Applying partial fraction decomposition:

$$\frac{75}{(s^2 + 4s + 13)(s + 6)} = \frac{A}{s + 6} + \frac{Bs + C}{s^2 + 4s + 13}$$

Using the Heaviside cover-up method to find A :

$$\begin{aligned} \frac{75}{(s^2 + 4s + 13)(s + 6)} \Big|_{s+6=0} &= \frac{A}{s + 6} \Big|_{s+6=0} \\ \frac{75}{(s^2 + 4s + 13) \boxed{}} \Big|_{s=-6} &= \boxed{A} \Big|_{s=-6} \\ \text{frac}\{75, s^2 + 4s + 13\} \Big|_{s=-6} &= A|_{s=-6} \\ \frac{75}{(-6)^2 + 4(-6) + 13} &= A \\ A &= \frac{75}{25} \\ A &= 3 \end{aligned}$$

Completing the square for the $Bs + C$ term:

$$\begin{aligned} \frac{Bs + C}{s^2 + 4s + 13} &= \frac{Bs + C}{s^2 + 4s + (\frac{4}{2})^2 - (\frac{4}{2})^2 + 13} \\ &= \frac{Bs + C}{s^2 + 4s + 2^2 - 2^2 + 13} \\ &= \frac{Bs + C}{s^2 + 4s + 2^2 + 9} \\ &= \frac{Bs + C}{(s + 2)^2 + 9} \\ &= \frac{Bs + C}{(s + 2)^2 + 3^2} \end{aligned}$$

Solving for the values of B and C :

$$\begin{aligned}
 \frac{75}{(s^2 + 4s + 13)(s + 6)} &= \frac{Bs + C}{(s + 2)^2 + 3^2} + \frac{3}{s + 6} \\
 \frac{75}{(s^2 + 4s + 13)(s + 6)} - \frac{3}{s + 6} &= \frac{Bs + C}{(s + 2)^2 + 3^2} + \frac{3}{s + 6} \\
 \frac{75}{(s^2 + 4s + 13)(s + 6)} - \frac{3(s^2 + 4s + 13)}{(s^2 + 4s + 13)(s + 6)} &= \frac{Bs + C}{(s + 2)^2 + 3^2} \\
 \frac{75 - 3(s^2 + 4s + 13)}{(s^2 + 4s + 13)(s + 6)} &= \frac{Bs + C}{(s + 2)^2 + 3^2} \\
 \frac{-3s^2 - 12s + 36}{(s^2 + 4s + 13)(s + 6)} &= \frac{Bs + C}{(s + 2)^2 + 3^2} \\
 \frac{-3s^2 - 12s + 36}{(s^2 + 4s + 13)(s + 6)} &= \frac{(Bs + C)(s + 6)}{((s + 2)^2 + 3^2)(s + 6)} \\
 \frac{3s^2 - 12s + 36}{(s^2 + 4s + 13)(s + 6)} &= \frac{(Bs^2 + 6Bs + Cs + 6C)}{((s + 2)^2 + 3^2)(s + 6)}
 \end{aligned}$$

By comparing the coefficients:

$$\begin{aligned}
 B &= -3 \quad \text{and} \quad 6B + C = -12 \\
 6(-3) + C &= -12 \\
 C &= 6
 \end{aligned}$$

Hence:

$$\begin{aligned}
 Y(s) &= \frac{75}{(s^2 + 4s + 13)(s + 6)} \\
 &= \frac{3}{s + 6} + \frac{-3s + 6}{(s + 2)^2 + 3^2} \\
 &= \frac{3}{s + 6} + \frac{-3s - 6 + 6 + 6}{(s + 2)^2 + 3^2} \\
 &= \frac{3}{s + 6} + \frac{-3s - 6 + 12}{(s + 2)^2 + 3^2} \\
 &= \frac{3}{s + 6} + \frac{-3(s + 2) + 4(3)}{(s + 2)^2 + 3^2} \\
 &= \frac{3}{s + 6} - 3\left(\frac{s + 2}{(s + 2)^2 + 3^2}\right) + 4\left(\frac{3}{(s + 2)^2 + 3^2}\right)
 \end{aligned}$$

Using the [Laplace transform table](#):

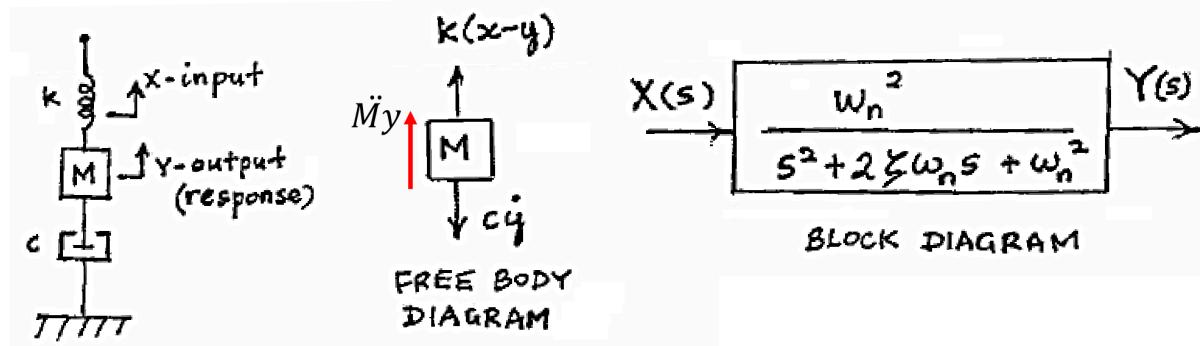
$$\begin{aligned}
 y(t) &= 3e^{-6t} - 3e^{-2t} \cos 3t + 4e^{-2t} \sin 3t \\
 &= e^{2t}(4 \sin 3t - 3 \cos 3t) + 3e^{-6t}
 \end{aligned}$$

2.11 Percentage overshoot

The percentage overshoot is given by:

$$\text{Percentage overshoot} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

2.12 Spring mass damper system



The inertial force $M\ddot{y}$ is equal to the external forces, i.e.

$$M\ddot{y} = k(x - y) - c\dot{y}$$

$$\therefore M\ddot{y} + c\dot{y} + ky = kx$$

Applying the Laplace transform on the function:

$$\begin{aligned}\text{Force} &= \mathcal{L}\{M\ddot{y} + c\dot{y} + ky\} = \mathcal{L}\{kx\} \\ &= Ms^2Y(s) + sY(s) + kY(s) = kX(s)\end{aligned}$$

Hence, the transfer function, $\frac{Y(s)}{X(s)}$, is:

$$\frac{Y(s)}{X(s)} = \frac{k}{Ms^2 + cs + k} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n^2}$$

Where:

- $\frac{k}{M} = \omega_n^2$
- $\frac{c}{M} = 2\zeta\omega_n$

2.13 Bounded signal

A bounded signal, $y(t)$ if there exists a finite constant μ , $\mu < \infty$, such that:

$$|y(t)| < \mu \text{ for all } t \geq 0$$

- Examples of bounded signals include $1(t)$, $\sin \omega t$, e^{-t} .
- Examples of unbounded signals are t , $t \sin \omega t$, e^t .

2.14 Bounded-input bounded-output (BIBO) stable

A system is bounded-input bounded-output (BIBO) stable if the output signal is bounded when the input signal is bounded.

The system $G(s)$ is BIBO stable if and only if all its poles are in the open left-half s -plane, or equivalently, the poles have negative real parts.

3 Classification of systems

3.1 Forward transference function ($KG(s)$)

$$\begin{aligned} KG(s) &= K \frac{K_G(s + a_1)(s + a_2)\dots(s^2 + bs + c)\dots}{s^l(s + d_1)(s + d_2)\dots(s^2 + es + f)\dots} \\ &= \frac{K \sum_{k=0}^m A_k s^k}{s^l \sum_{\{k=0\}} B_k s^k}, \quad n \geq m + 1 \end{aligned}$$

Where:

- n is the order of the system, which is the **highest** power of s in the **denominator**.
- $n - m$ is the rank of the system, which is the difference between the **highest** power of s in the **denominator** and the **numerator**.
- l is the class (or type number) of the system, which is the power of the **factor** s in the **denominator**.
- s^l is also known as the **integrator**.
- The **denominator** is also known as the **lag** term.
- The **numerator** is also known as the **lead** term.

3.1.1 Example

State the order, rank and type number of the systems with open-loop transfer function:

$$\begin{aligned} G(s) &= \frac{s + 2}{s^4 + 3s^3 + 3s^2 + s} \\ G(s) &= \frac{1}{s^3(s + 2)(s + 1)} \\ G(s) &= \frac{s^2 + s + 1}{(s + 2)(s^2 + s + 4)} \end{aligned}$$

1. For (1), the order is 4, the rank is $4 - 1 = 3$, and the type number is 1 since the denominator can be written as $s(s^3 + 3s^2 + 2s + 1)$.
2. For (2) the order is 5, the rank is 5 and the type number is 3 since s^3 is a factor.
3. For (3), the order is 3, the rank is 1, and the type number is 0.

3.2 Transfer function of closed-loop systems

The transfer function of a closed-loop system is given by:

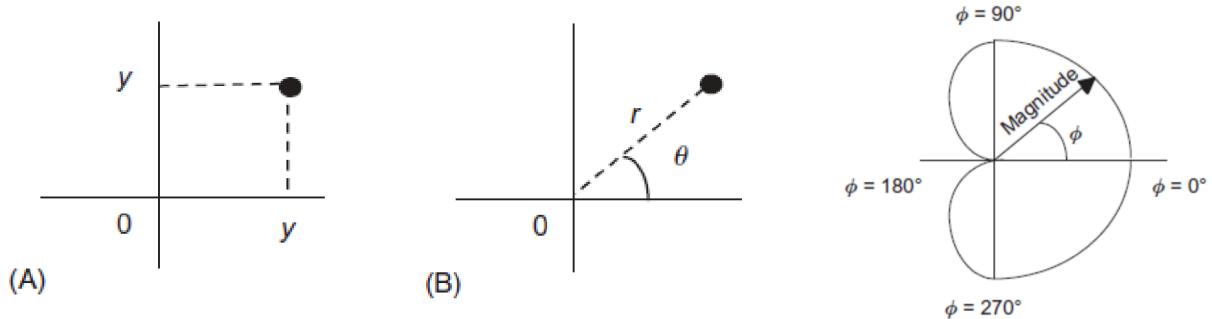
$$\frac{G(s)}{1 + G(s)H(s)}$$

Where:

- $G(s)$ is the transfer function of the forward path
- $H(s)$ is the transfer function of the feedback path
- $G(s)H(s)$ is the open-loop transfer function, which is the transfer function of a closed-loop system if the feedback path from the feedback element is broken.

3.3 Nyquist diagrams (polar plots)

The polar plot of the frequency response of a system is the line traced out as the frequency is changed from 0 to ∞ by the tips of the phasors (vectors) whose length represents the magnitude, like the amplitude gain of the system and the angle represents their phase ϕ .



When the polar plot of a transfer function touches -1 on the x -axis, the system becomes unstable.

3.3.1 Example 1

Determine the Nyquist diagram for a first-order system with an open-loop transfer function of:

$$\frac{1}{1 + \tau s}$$

The frequency response is:

$$s = j\omega$$

$$\frac{1}{1 + j\omega\tau} = \frac{1}{1 + j\omega\tau} \times \frac{1 - j\omega\tau}{1 - j\omega\tau} = \frac{1}{1 + \omega^2\tau^2} - j\frac{\omega\tau}{1 + \omega^2\tau^2}$$

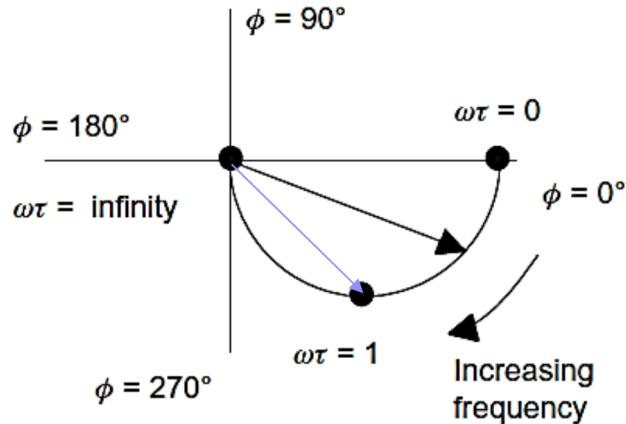
Hence, the magnitude is:

$$\text{Magnitude} = \frac{1}{\sqrt{1 + \omega^2 + \tau^2}}$$

And the phase is:

$$\text{Phase} = -\arctan \omega\tau$$

At zero frequency the magnitude is 1 and the phase is 0° . At infinite frequency the magnitude is zero and the phase is -90° . When $\omega\tau = 1$, the magnitude is $\frac{1}{\sqrt{2}}$ and the phase is -45° . Substitution of other values leads to the result shown below, which is the Nyquist diagram for a first-order system:



3.3.2 Example 2

Determine the Nyquist diagram for a system with the open-loop transfer function:

$$G(s) = \frac{s+2}{s+10}$$

Substituting $s = j\omega$:

$$\begin{aligned} G(j\omega) &= \frac{j\omega + 2}{j\omega(j\omega + 10)} \cdot \frac{-j\omega + 10}{-j\omega + 10} \\ &= \frac{\omega^2 + 20 + j(10\omega - 2\omega)}{j\omega(\omega^2 + 100)} \quad (\text{Rationalising the denominator}) \\ &= \frac{8\omega - j(\omega^2 + 20)}{\omega(\omega^2 + 100)} \end{aligned}$$

Hence, the polar coordinates are:

$$|G(j\omega)| = \frac{\sqrt{(\omega^2 + 20^2)^2 + 64\omega^2}}{\omega(\omega^2 + 100)}$$

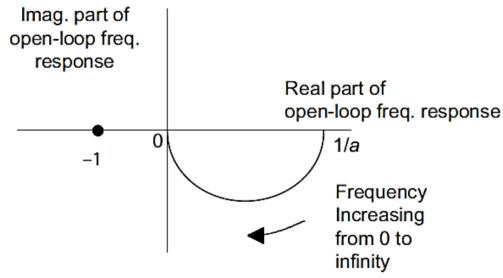
$$\angle G(j\omega) = \arctan \frac{\omega}{2} - 90^\circ - \arctan \frac{\omega}{10}$$

And the Cartesian coordinates are:

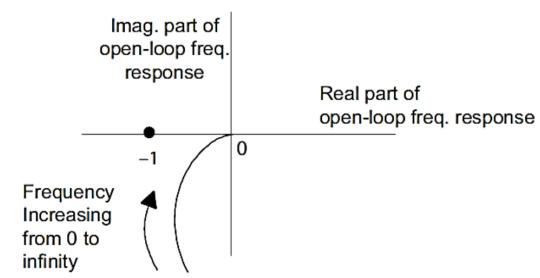
$$\text{Real}(G(j\omega)) = \frac{8}{\omega^2 + 100}$$

$$\text{Im}(G(j\omega)) = -\frac{\omega^2 + 20}{\omega(\omega^2 + 100)}$$

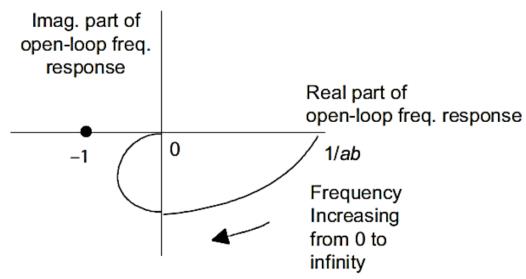
3.3.3 Nyquist diagram analysis



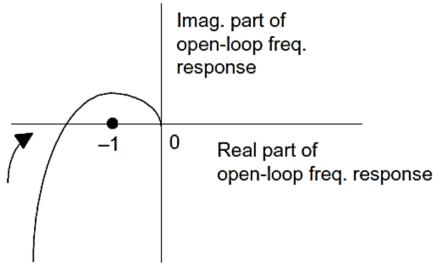
$$G(s)H(s) = K/(s+a), \text{ stable for all values of } K > 0.$$



$$G(s)G(s) = K/s(s+a), \text{ stable for all values of } K > 0.$$



$$G(s)H(s) = K/(s+a)(s+b), \text{ stable for all of } K > 0.$$



$G(s)H(s) = K/s(s+a)(s+b)$; this is values unstable with large K but can become stable if K is reduced, the point at which the plot crosses the axis being $-K/(a+b)$ and so stability is when $-K/(a+b) > -1$.

3.4 System types

3.4.1 System type 0

Systems of type 0 have no **integrator** term or s^l term in the denominator of the **transfer function**.

General forms:

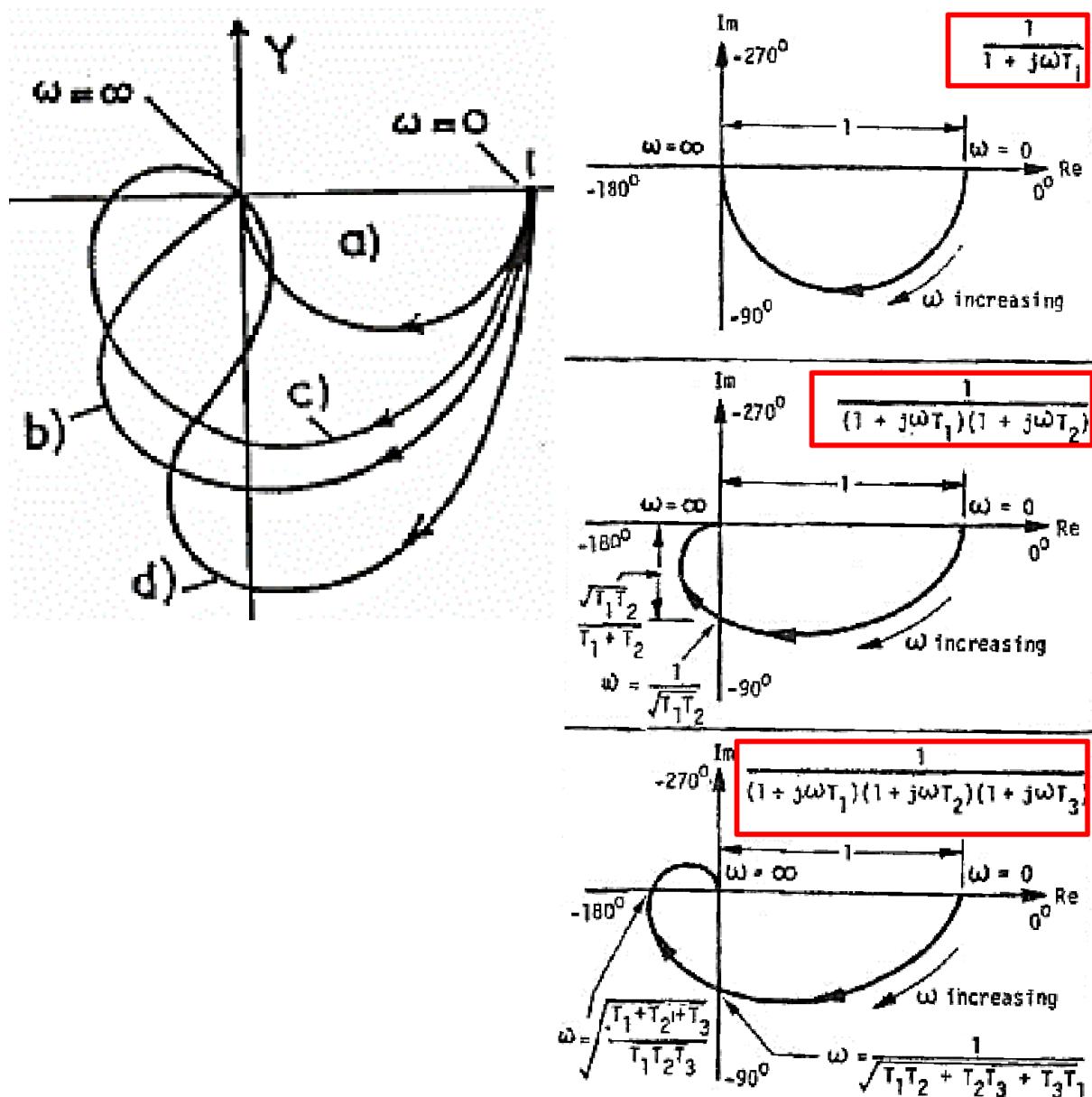
a) $G(s) = \frac{1}{1+sT}$

b) $G(s) = \frac{1}{1+bs+as^2}$

c) $G(s) = \frac{1}{(1+sT)(1+bs+as^2)}$

d) $G(s) = \frac{1+sT}{a+bs+as^2}$

Nyquist diagram of the functions above:



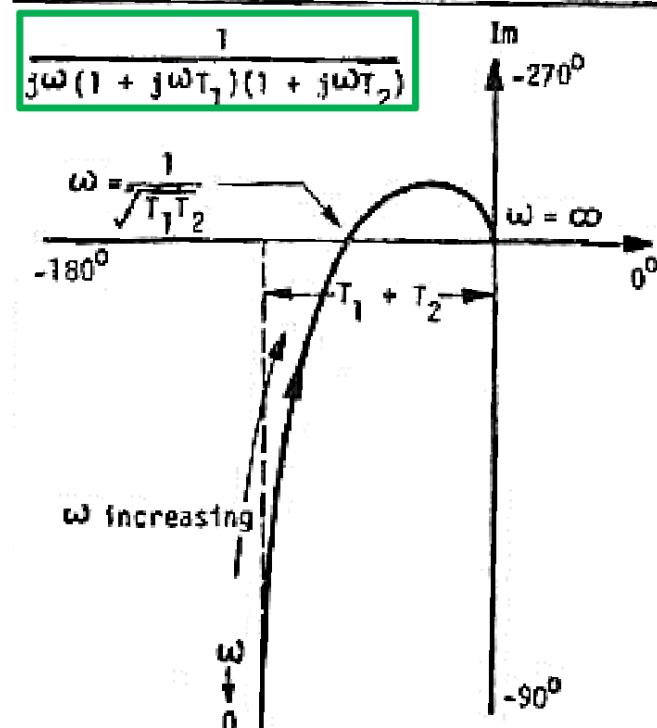
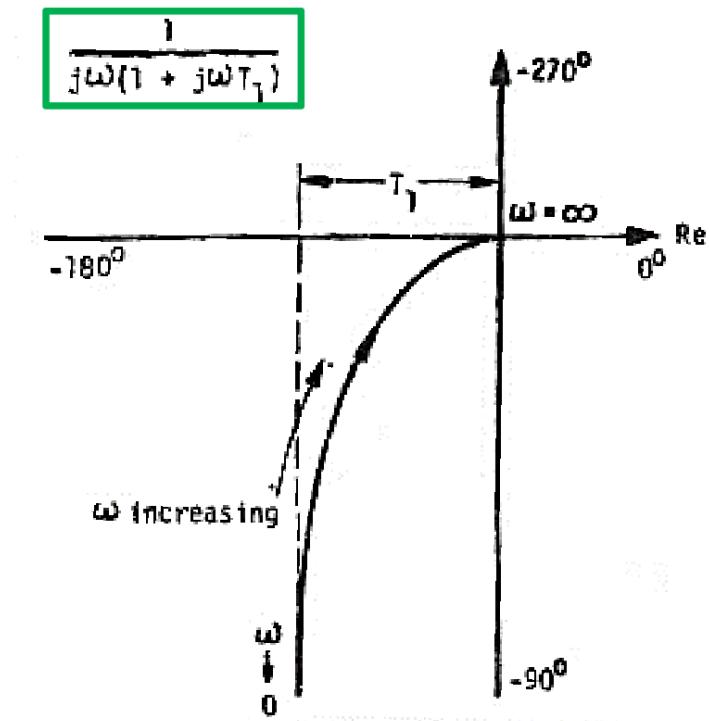
3.4.2 System type 1

Systems of type 1 have an **integrator** term of the power 1, or s^1 as the s^l term in the denominator of the **transfer function**.

General forms:

$$e) \quad G(s) = \frac{1}{s(1 + sT)}$$

$$f) \quad G(s) = \frac{1}{s(1 + bs + as^2)}$$

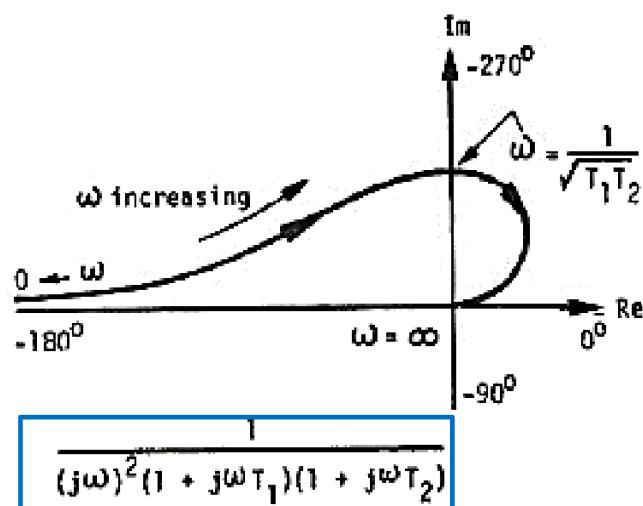
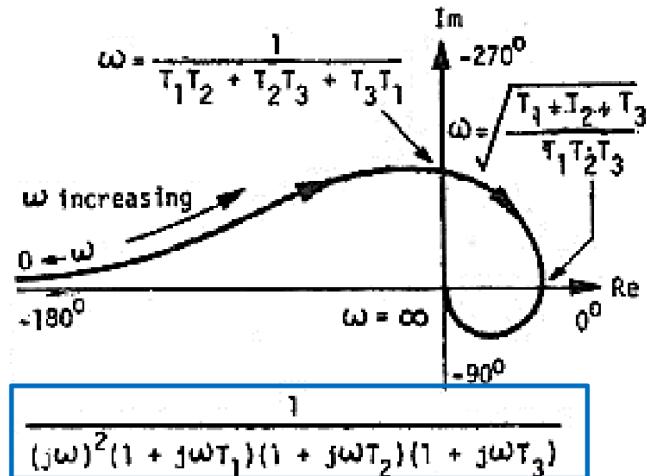


3.4.3 System type 2

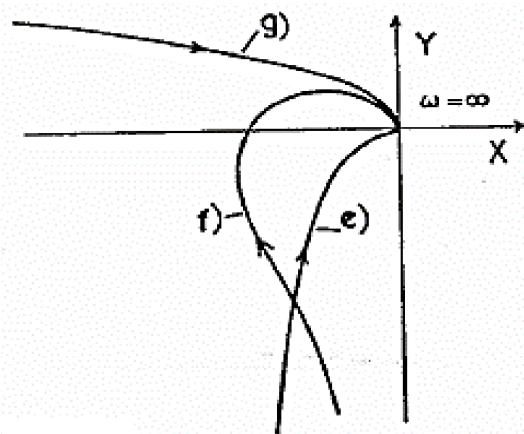
Systems of type 2 have an **integrator** term of the power 2, or s^2 as the s^l term in the denominator of the [transfer function](#).

General form:

$$g) G(s) = \frac{1}{s^2(1+sT)}$$



3.4.4 Nyquist diagram for systems type 1 and type 2



3.5 Examples of finding the magnitude and phase of an output

3.5.1 Example 1

Determine the magnitude and phase of the output from a system when subject to a sinusoidal input of $2 \sin 3t$ if it has a transfer function of:

$$G(s) = \frac{2}{s+1}$$

The frequency response function is obtained by replacing s with $j\omega$:

$$G(j\omega) = \frac{2}{j\omega + 1}$$

Multiplying the top and bottom of the equation by $(-j\omega + 1)$ gives:

$$\begin{aligned} G(j\omega) &= \frac{2}{j\omega + 1} \times \frac{-j\omega + 1}{-j\omega + 1} \\ &= \frac{-2j\omega + 2}{\omega^2 + 1} \\ &= \frac{2}{\omega^2 + 1} - j \frac{2\omega}{\omega^2 + 1} \end{aligned}$$

The magnitude of $a + jb$ is $\sqrt{a^2 + b^2}$, and hence for $G(j\omega)$:

$$\begin{aligned} |G(j\omega)| &= \sqrt{\frac{2^2}{(\omega^2 + 1)^2} + \frac{2^2\omega^2}{(\omega^2 + 1)^2}} \\ &= \frac{4}{\sqrt{\omega^2 + 1}} \end{aligned}$$

And the phase angle is given by:

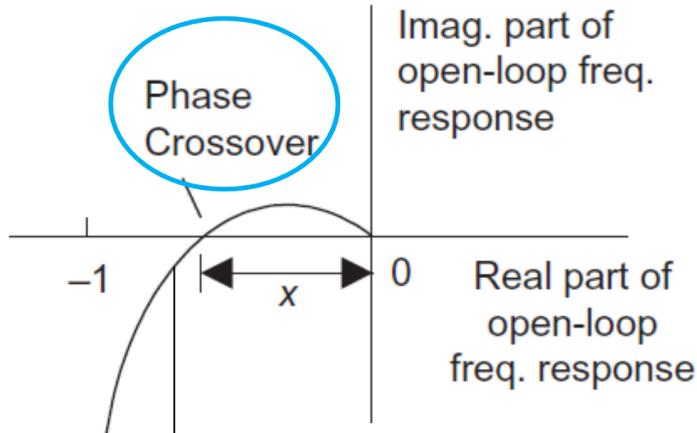
$$\tan \phi = -\omega$$

For the specified input, we have $\omega = 3 \text{ rad s}^{-1}$. The magnitude is thus:

$$|G(j\omega)| = \frac{2}{\sqrt{\omega^2 + 1}} = \frac{2}{\sqrt{3^2 + 1}} \approx 0.63$$

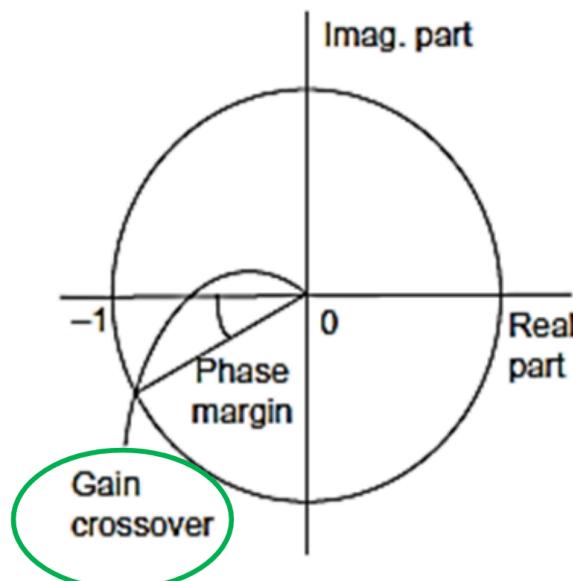
3.6 Gain margin

- The gain is the real value of the intersection of the polar plot of a system and the real axis or the x -axis of the polar plot.
- The intersection point is also known as the phase crossover.
- If x is the gain of a system, then the gain margin of a system is given by $\frac{1}{x}$, which is the margin a system has before becoming unstable.
- The gain margin is $20 \log \frac{1}{x}$ in decibels (dB).



3.7 Phase margin

- On the polar plot of a system, draw a circle of radius 1 from the origin.
- The intersection of the polar plot of the system and this circle is known as gain crossover.
- Draw a straight line to the origin from this intersection point, and the acute angle between the drawn straight line and the real axis or x -axis is called the phase margin.
- The obtuse angle is known as the phase angle.



3.8 Example of determining gain and phase margins

Determine the gain margin and the phase margin for a system that gave the following open-loop experimental frequency response data:

- At frequency 0.005 Hz, a gain of 1.00 and phase -120° .
- At frequency 0.010 Hz, a gain of 0.45 and phase -180° .

The phase crossover occurs at 0.010 Hz, and the gain at that point is 1.00, so the gain margin is:

$$\frac{1}{x} = \frac{1}{0.45} = 2.22$$

The phase margin is the number of degrees by which the phase angle is smaller than -180° at the gain crossover. The gain cross over is the frequency at which the open-loop gain first reaches the value 1 and so is 0.005 Hz. Thus, the phase margin is:

$$180^\circ - 120^\circ = 60^\circ$$

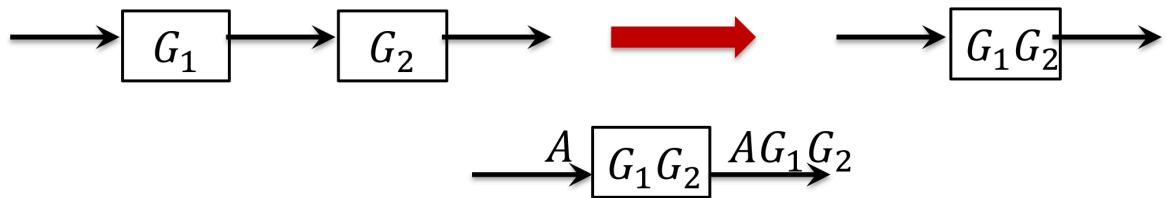
3.9 Criterion for stable systems

- A good stable control system usually has an open-loop gain significantly less than 1, typically about 0.4-0.5, when the phase shift is -180° and an open-loop phase shift of between -115° and 125° when the gain is 1.
- Such values give a slightly under-damped system which gives about 20-30% overshoot.

4 Block diagrams

4.1 General rules

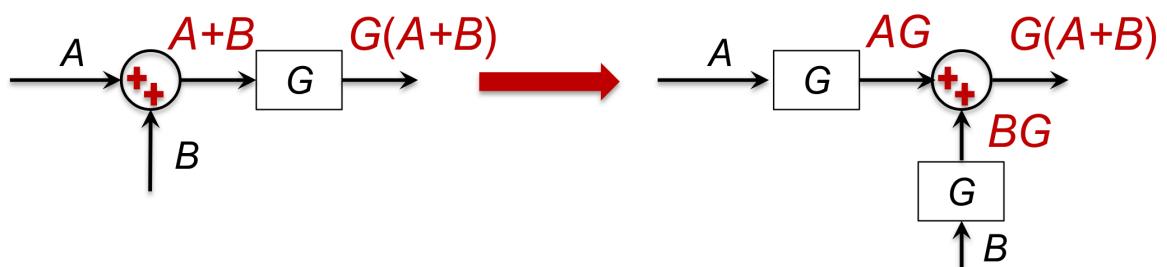
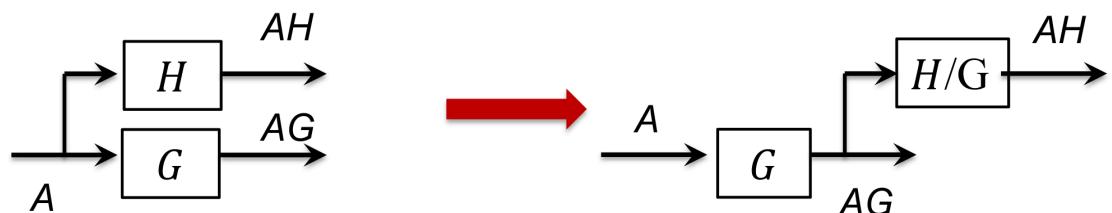
1. Multiplication



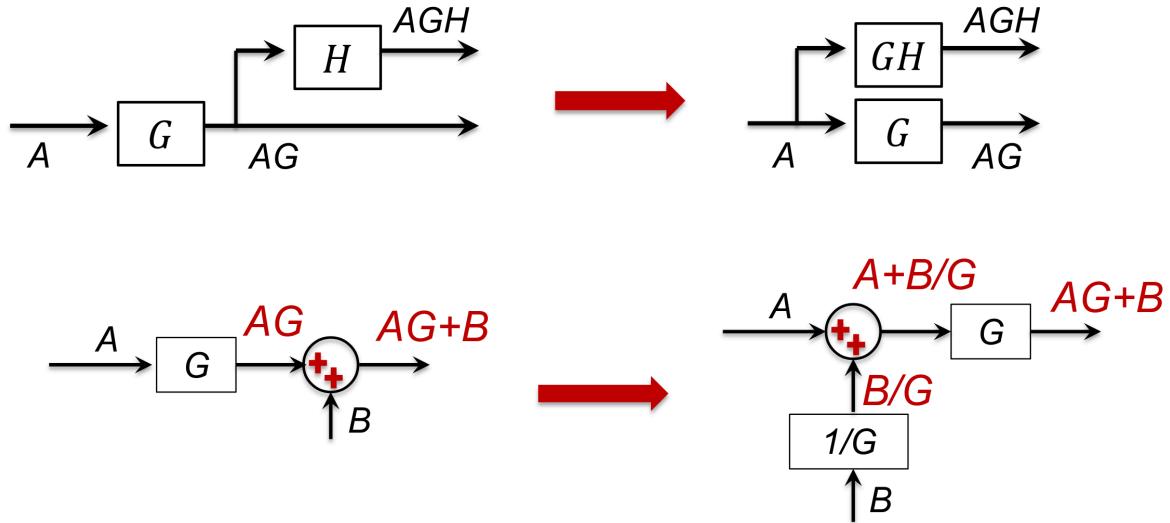
2. Summation



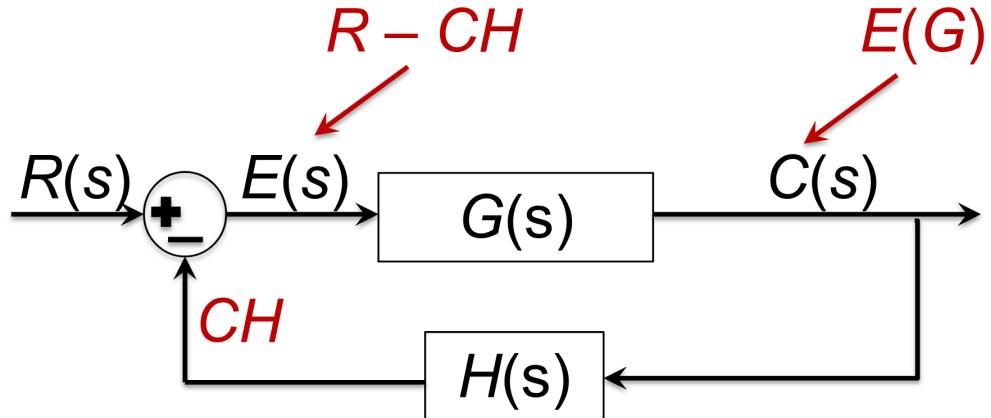
3. Pre-multiplication converted into post-multiplication



4. Post-multiplication converted into pre-multiplication



5. Closed-loop



To derive the closed-loop transfer function, we go along the signal path and follow the rules defined above.

$$(R - CH)G = C$$

$$RG - CHG = C$$

$$RG = C(1 + HG)$$

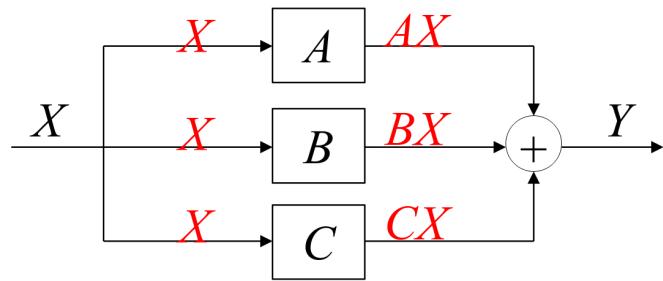
$$\frac{C}{R} = \frac{G}{1 + GH}$$

Where:

- G is the direct or forward transfer function
- H is the feedback transfer function
- GH is the loop transfer function or open-loop transfer function
- $\frac{C}{R}$ is the closed-loop transfer function
- $\frac{E}{R} = \frac{1}{1+GH}$ is the actuating signal ratio

4.1.1 Example 1

Establish the relationship between $Y(s)$ and $X(s)$:



$$Y = AX + BX + CX$$

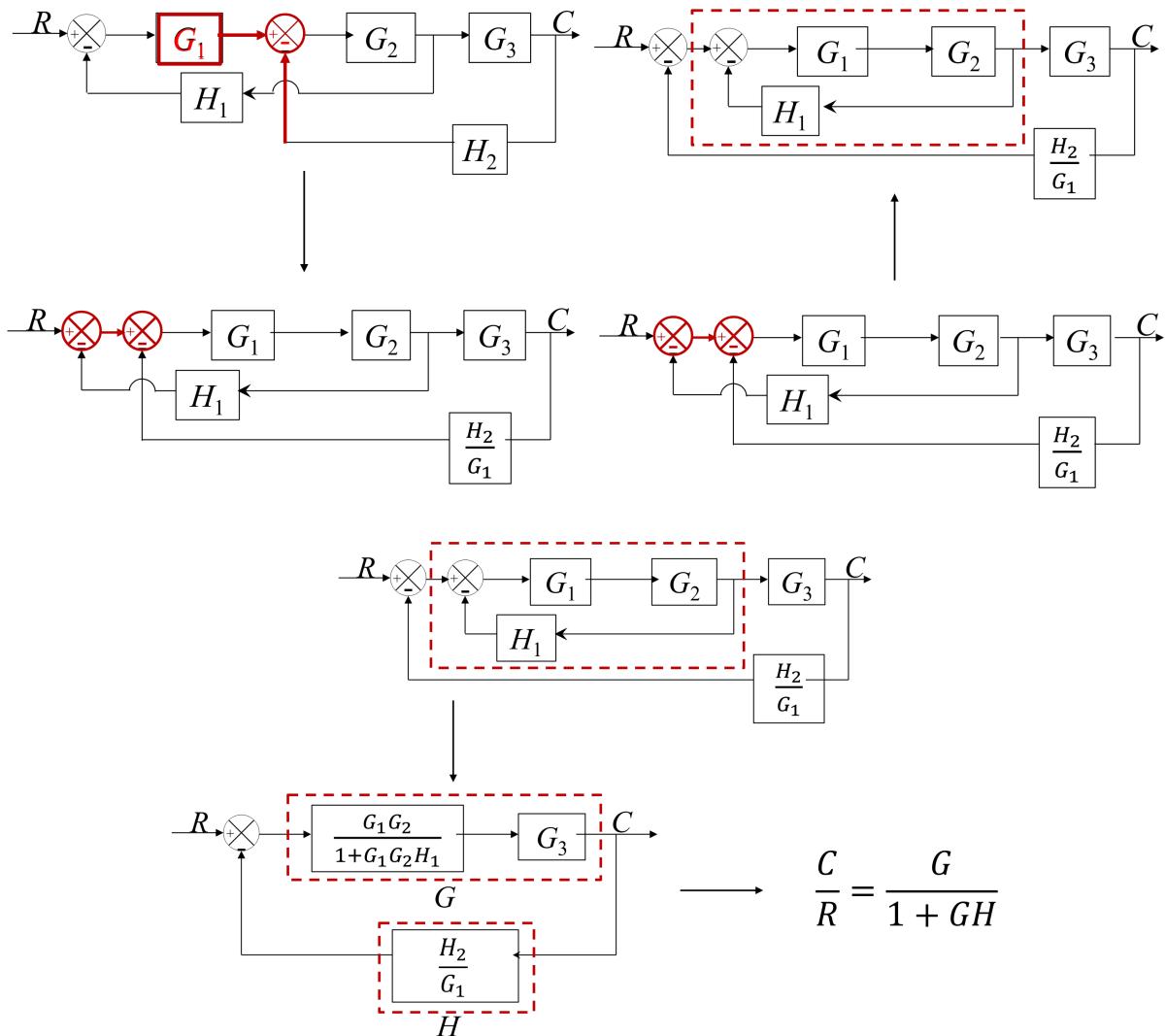
$$Y = (A + B + C)X$$

$$\frac{Y}{X} = A + B + C$$

$$\xrightarrow{X(s)} \boxed{A + B + C} \xrightarrow{Y(s)}$$

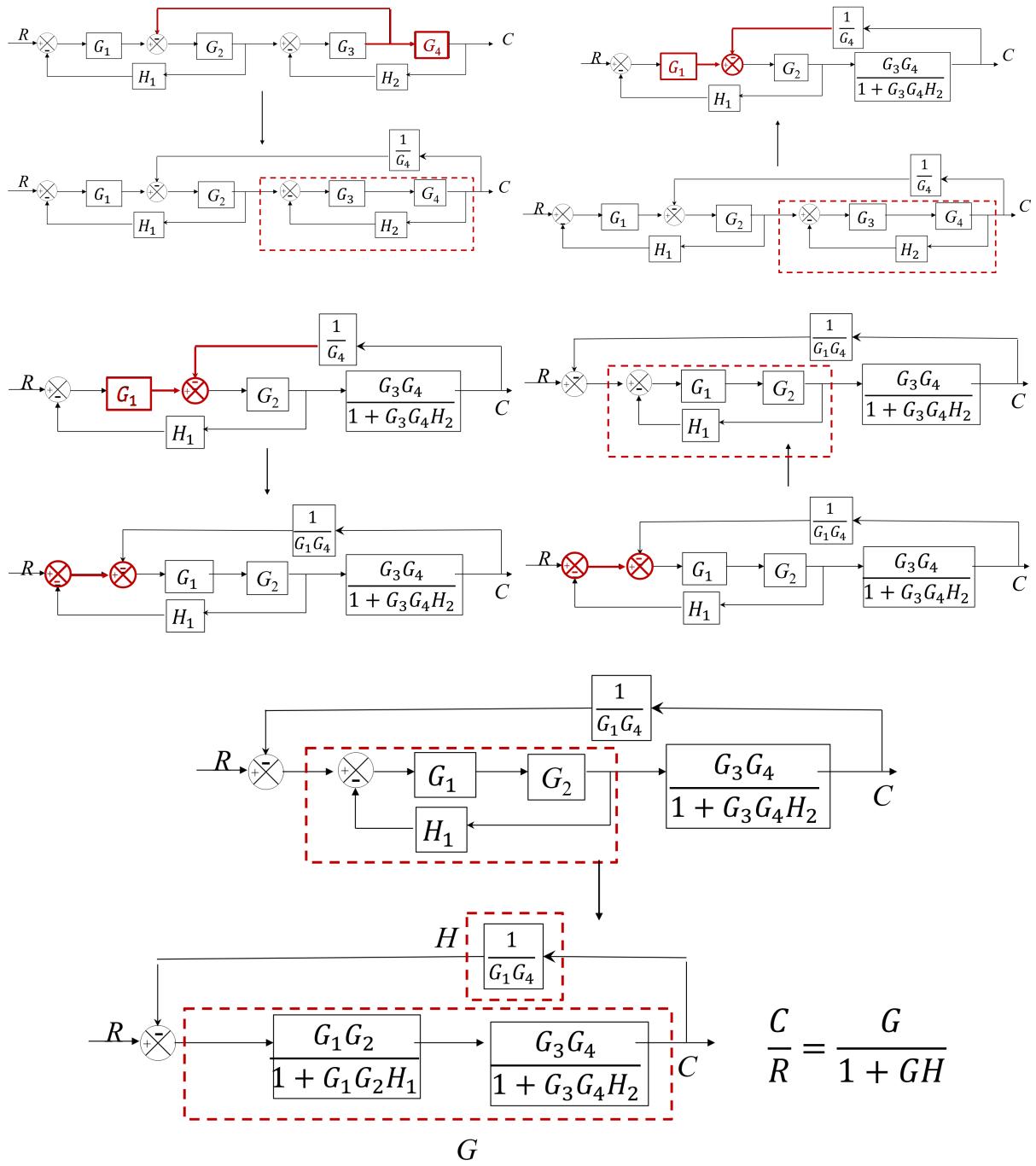
4.1.2 Example 2

Simplify the system and find $\frac{C}{R}$:



4.1.3 Example 3

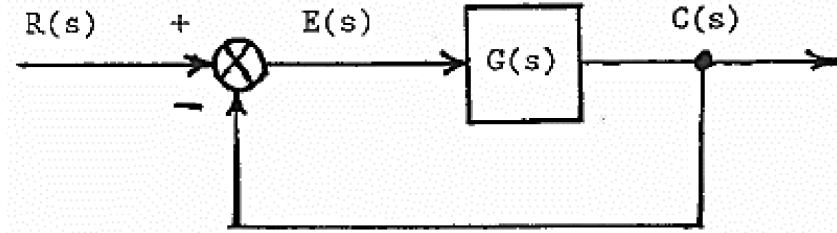
Simplify the system and find $\frac{C}{R}$:



4.2 Feedback types

4.2.1 Unity feedback

A system has unity feedback if there is no feedback transfer function, which means whatever is the output is exactly what the controller receives.



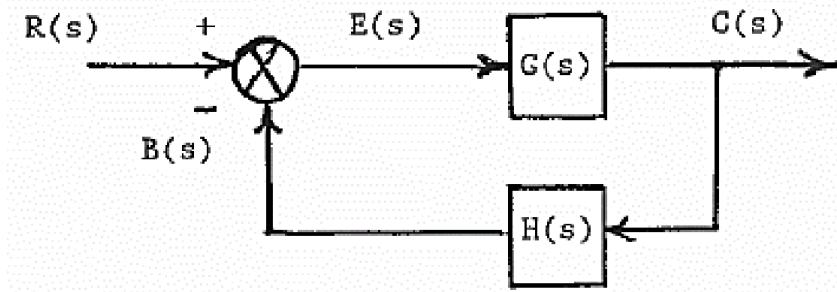
$$C(s) = G(s) \cdot E(s)$$

$$E(s) = R(s) - C(s)$$

$$\therefore C(s) = G(s)[R(s) - C(s)]$$

4.2.2 Non-unity feedback

A system has non-unity feedback if there is a feedback transfer function, which usually means the output has been amplified or diminished.



$$E(s) = R(s) - B(s)$$

$$B(s) = C(s)H(s)$$

$$C(s) = G(s)E(s)$$

$$\therefore \text{Closed loop transfer function} = \frac{G(s)}{1 + G(s)H(s)}$$

4.2.3 Example 1

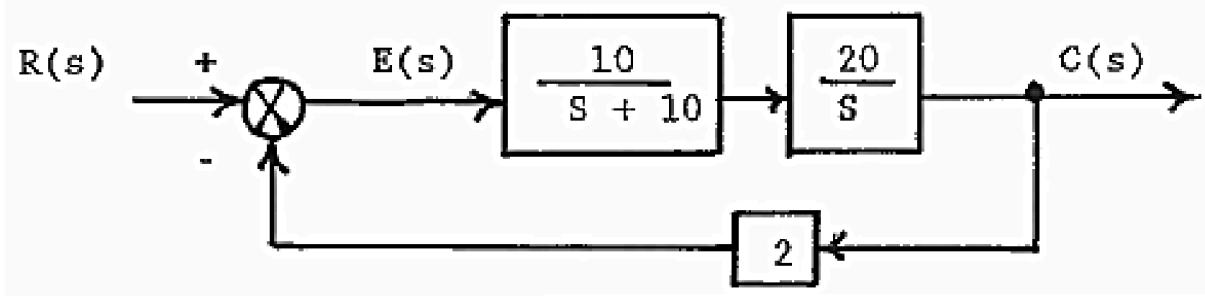
A negative feedback system has a forward path gain of 12 and a feedback path gain of 0.1. What is the overall gain of the system?

Using the non-unity feedback equation:

$$\text{System gain} = \frac{G}{1 + GH} = \frac{12}{1 + 0.1 \times 12} = 5.45$$

The overall gain is 5.45.

4.2.4 Example 2



$$G(s) = \frac{10}{S + 10} \cdot \frac{20}{S} = \frac{200}{S(S + 10)}$$

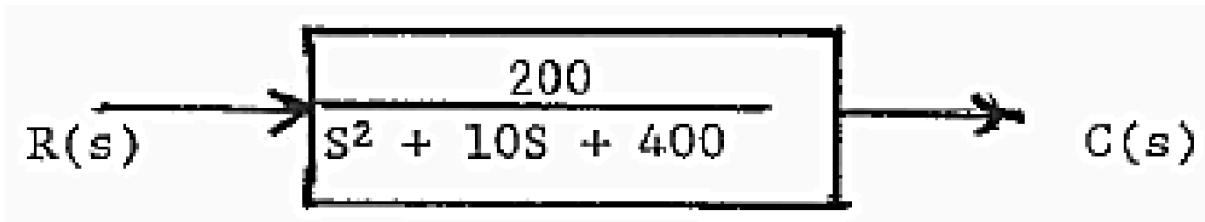
Open-loop transfer function (OLTF): $GH(s) = \frac{200}{S(S + 10)} \cdot 2 = \frac{400}{S(S + 10)}$

Closed-loop transfer function (CLTF): $\frac{C}{R}(S) = \frac{G(s)}{1 + \text{OLTF}} = \frac{200}{S^2 + 10S + 400}$

The characteristic equation is:

$$S^2 + 10S + 400 = 0$$

Reduced system:



5 Mason's gain formula

$$\frac{C(S)}{R(S)} = \sum_{k=1}^n \frac{T_k \Delta_k}{\Delta}$$

Where:

- n is the number of forward paths
- T_k is the gain (transmittance) of the k^{th} forward path
- Δ_k is the associated path factor
- Δ is the determinant of the signal flow graph

5.1 Obtaining the determinant of the signal flow graph (Δ)

Δ = Determinant of the signal flow graph

$$= 1 - [\text{Sum of gains of all the individual loops}]$$

$$+ [\text{Sum of the gain products of two non-touching loops}]$$

$$- [\text{Sum of the gain products of three non-touching loops}]$$

$$+ [\text{Sum of the gain products of four non-touching loops}]$$

+ ...

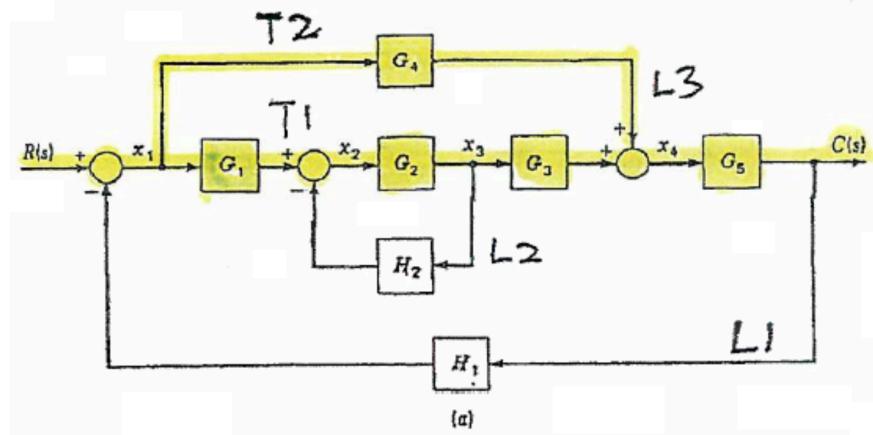
5.2 Obtaining the associated path factor (Δ_k)

Δ_k = The part of Δ not touching the k^{th} forward path

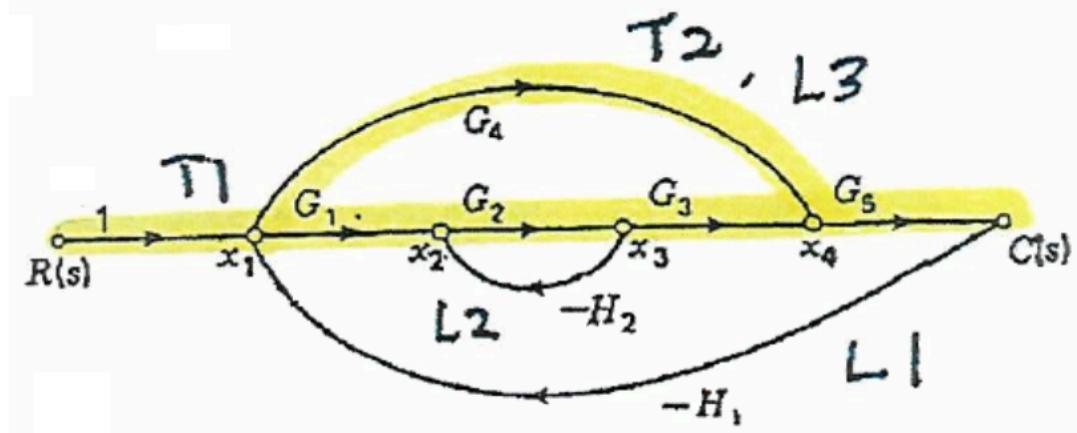
5.3 Example 1

The block diagram and corresponding signal flow graph for a more complex system is shown below. There are two forward paths from input $R(s)$ to output $C(s)$:

$$T_1 = G_1 G_2 G_3 G_5 \quad T_2 = G_4 G_5$$



Simplifying the diagram:



The gains for the three loops in the system are:

$$L_1 = -G_1 G_2 G_3 G_5 H_1 \quad L_2 = -G_2 H_2 \quad L_3 = -G_4 G_5 H_1$$

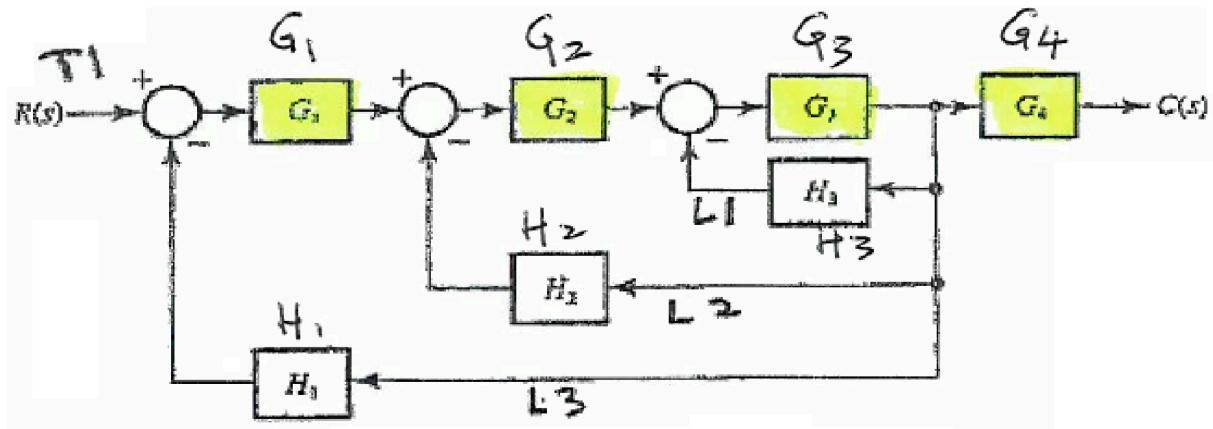
Because L_2 and L_3 are non-touching loops,

$$\Delta = 1 - (L_1 + L_2 + L_3) + L_2 L_3 = 1 + (G_1 G_2 G_3 G_5 H_1 + G_2 H_2 + G_4 G_5 H_1) + G_2 G_4 G_5 H_1 H_2$$

Because all three loops touch path T_1 , setting $L_1 = L_2 = L_3$ in the above expression for Δ gives $\Delta_1 = 1$. Because loops L_1 and L_2 touch path T_2 , setting $L_1 = L_3 = 0$ in the expression for Δ gives $\Delta_2 = 1 - L_2 = 1 + G_2 H_2$. Substitution of these results into Mason's gain formula yields:

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{T_1 \Delta_1 + T_2 \Delta_2}{\Delta} \\ &= \frac{G_1 G_2 G_3 G_5 + (G_4 G_5)(1 + G_2 H_2)}{1 + (G_1 G_2 G_3 + G_4 + G_2 G_4 H_2) G_5 H_1 + G_2 H_2} \end{aligned}$$

5.4 Example 2



- There is only one forward path from $R(s)$ to $C(s)$.
- 3 loops:

$$L_1 = -G_3 H_3 \quad L_2 = -G_2 G_3 H_2 \quad L_3 = -G_1 G_2 G_3 H_1$$

- All loops are touching each other, hence:

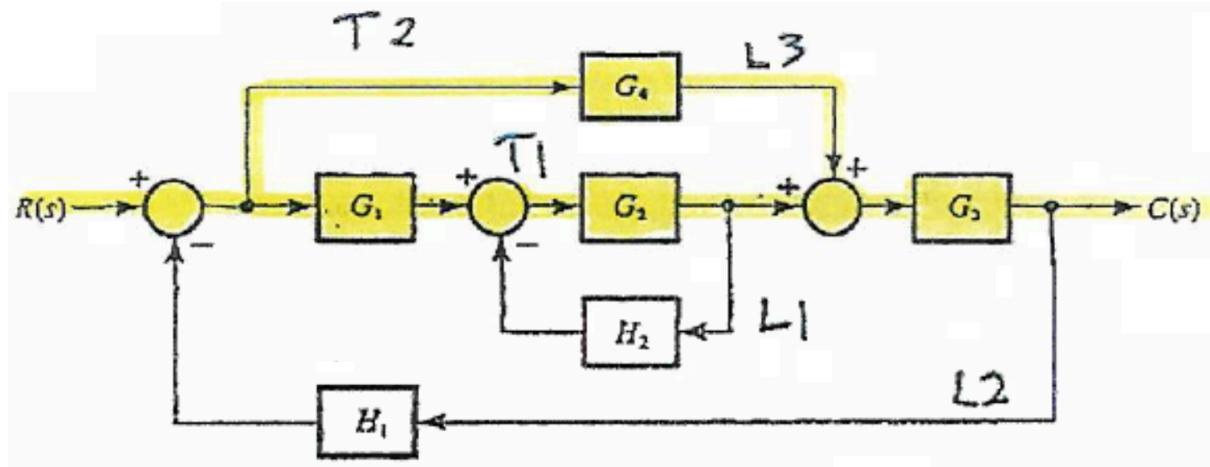
$$T_1 = G_1 G_2 G_3 G_4$$

$$\begin{aligned} \Delta &= 1 - (-G_3 H_3 - G_2 G_3 H_2 - G_1 G_2 G_3 H_1) \\ &= 1 + G_3 H_3 + G_2 G_3 H_2 + G_1 G_2 G_3 H_1 \end{aligned}$$

- Hence:

$$G(s) = \frac{T_1}{\Delta} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 H_3 + G_2 G_3 H_2 + G_1 G_2 G_3 H_1}$$

5.5 Example 3



There are two forward paths from $R(s)$ to $C(s)$, and:

$$T_1 = G_1 G_2 G_3$$

$$T_2 = G_4 G_3$$

There are 3 loops:

$$L_1 = -G_2 H_2 \quad L_2 = -G_1 G_2 G_3 H_1 \quad L_3 = -G_4 G_3 H_1$$

L_1 and L_3 are non-touching loops, so this gives us:

$$\Delta = 1 + (G_1 H_2 + G_1 G_2 G_3 H_1 + G_4 + G_3 H_1)$$

$$\Delta_1 = 1$$

$$\Delta_2 = 1 + G_2 H_2$$

Hence:

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3 + G_4 G_3 (1 + G_2 H_2)}{1 + G_2 H_2 + G_1 G_2 G_3 H_1 + G_4 G_3 H_1 + G_2 H_2 G_4 G_3 H_1}$$

6 Mathematical modelling

- Mathematical modelling is the first stage of any study, as mathematical modelling is needed to gain a good understanding of the system under investigation, like its purpose, specifications, main components, and its input and output.
- The simplicity of the model is an important consideration. Simplicity is necessary to also make changes in the mathematical model as convenient as possible.
- Simplicity is achieved by making assumptions about the system. It requires knowledge and understanding of the behaviour of the system to be controlled.
- Write the relevant equation of motion for the system.
 1. Newton's Law for mechanical motion.
 2. Kirchhoff's Law for electrical circuits.
 3. Mass and heat balance for chemical systems.
- In control engineering, these equations of motions, also called the mathematical model of a system, are usually set in differential equations.
- The mathematical model represents an approximation to the actual dynamics and steady state behaviour of the system.

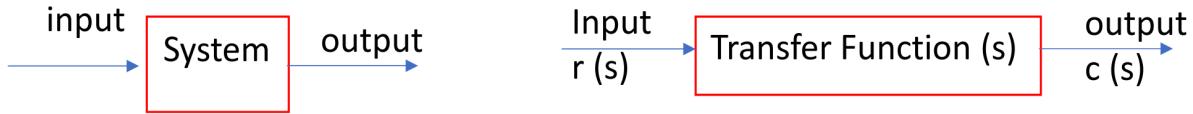
6.1 Fundamental steps to model the actual system

1. Conceptual model is first specified, the building block of the models are idealisation of events or phenomena occurring in real system being modelled blocks, like mass, viscous damping, spring stiffness, electrical resistance, capacitance, inductance, etc.
2. Relationship in the model expressed mathematically by appropriate physical laws to each building blocks and obtain necessary equation of motion.
3. Dynamic behaviour is obtained by solving model behaviour compared to actual or predicted system behaviour and adjustments are made if necessary to aid control system design.

6.2 When is a mathematical model useful?

1. When the system or the controller for the system doesn't exist at all, such as an automatic tracking system for satellites. The idea can be conveniently and cheaply designed and tested using the mathematical model rather than working with the actual system.
2. When the system is in complete operation, but does not work according to the specification. An example is the control of the water level in the boiler drum in a power station being unstable.

6.3 Transfer function (T.F.)



- If the input-output relationship of linear system is known, the characteristics of the system itself are also known.
- The input-output relationship in the Laplace domain is called the transfer function (T.F.)

$$\text{System} \rightarrow T.F.(s) = \frac{\text{Output}(s)}{\text{Input}} = \frac{C(s)}{R(s)}$$

- Usually, we assume that the component is linear and stationary, with zero initial conditions.
- For a physical system:

$$T.F.(s) = \frac{N(s)}{D(s)}$$

Usually $N(s)$ will be of a lower order than $D(s)$ as nature integrates rather than differentiates.

Where:

- $D(s)$ is the characteristic function.
 - It contains all the physical characteristics of the system.
 - If $D(s) = 0$, then it is known as the characteristic equation.
 - The roots of the characteristic equation determine the stability of the system and the general nature of the transient response to any input.
- $N(s)$ is a function of how the input enters the system.
 - It does not affect the absolute stability of the system, or the nature of transient modes of the system.
 - It determines the magnitude and sign of each transient mode and thus establishes the shape of the transient response as well as the value of the steady state output value.

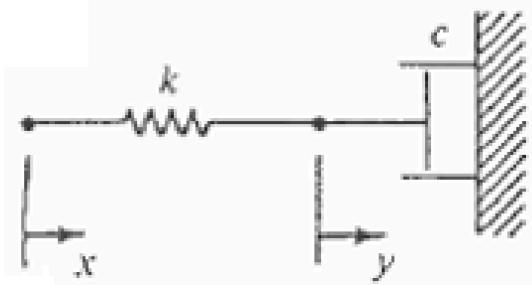
6.3.1 Ways to obtain transfer functions

1. Mathematical, which is to take the Laplace transform of differential equations of a system and solving for the transfer function with zero initial conditions.
2. Experimental, which is to apply known inputs like sine waves and step functions to systems and measure the output values to construct the transfer function.

A transfer function with a similar frequency can be obtained by replacing s in the transfer function with $j\omega$.

6.4 Modelling a spring mass damper system

6.4.1 Example 1



- x is the input in the system.
- Let the point above the label y be P .
- The spring pushes on the point P from the left with a force $k(x - y)$.
- The damper pushes on the point P from the right with a force $c\dot{y}$.
- At equilibrium, these forces cancel out and equate to 0.
- Hence, the system equation is:

$$k(x - y) = c\dot{y}$$

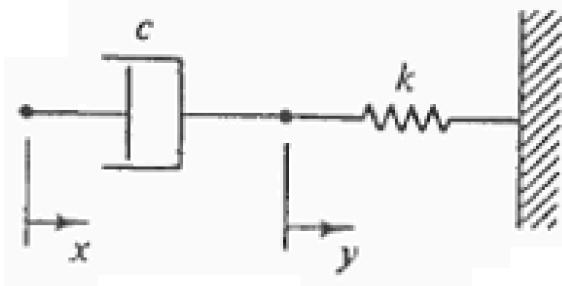
$$k(x - y) - c\dot{y} = 0$$

$$kx = c\dot{y} + ky$$

- Taking the Laplace transform of the system:

$$\frac{Y(s)}{X(s)} = \frac{k}{Cs + k}$$

6.4.2 Example 2



- x is the input in the system.
- Let the point above the label y be P .
- The damper pushes on the point P from the left with a force $c(\dot{x} - \dot{y})$.
- The spring pushes on the point P from the right with a force ky .
- At equilibrium, these forces cancel out and equate to 0.
- Hence, the system equation is:

$$c(\dot{x} - \dot{y}) - ky = 0$$

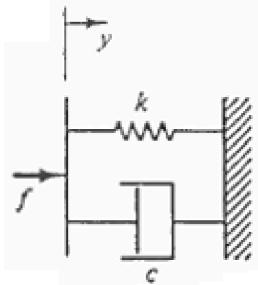
$$c(\dot{x} - \dot{y}) = ky$$

$$c\dot{x} = ky + \dot{y}$$

- Taking the Laplace transform of the system:

$$\frac{Y(s)}{X(s)} = \frac{Cs}{Cs + k}$$

6.4.3 Example 3



- f is the input in the system.
- Let the point below the label y be P .
- The spring pushes on the point P from the right with a force ky .
- The damper pushes on the point P from the right with a force $c\dot{y}$.
- At equilibrium, these forces cancel out and equate to 0.
- Hence, the system equation is:

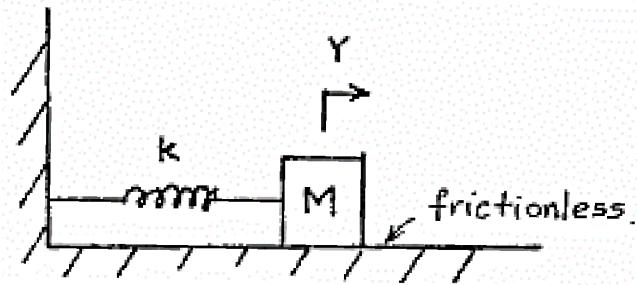
$$f - ky - c\dot{y} = 0$$

$$f = ky + c\dot{y}$$

- Taking the Laplace transform of the system:

$$\frac{Y(s)}{F(s)} = \frac{1}{Cs + k}$$

6.4.4 Example 4



Inertial forces = External forces

$$M \frac{d^2y}{dt^2} = \sum F_s$$

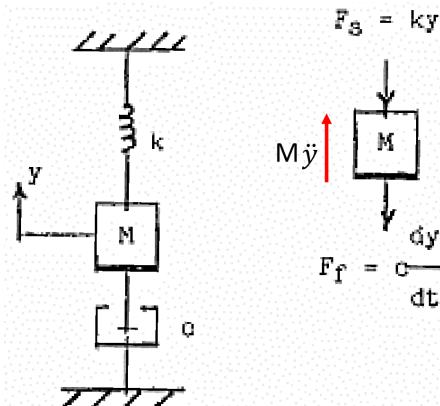
$$\sum F_s = ky$$

$$M \frac{d^2y}{dt^2} = -ky$$

$$M \frac{d^2y}{dt^2} + ky = 0$$

$$M\ddot{y} + ky = 0$$

6.4.5 Example 5



$$\text{Net force} = F - ky - c \frac{dy}{dt}$$

Then, applying Newton's second law, this force must be equal to ma , where a is the acceleration, and so:

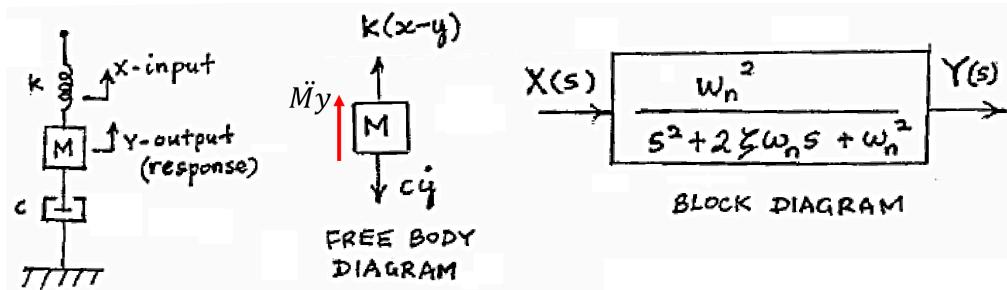
$$m \frac{d^2y}{dt^2} = F - ky - c \frac{dy}{dt}$$

The relationship between the input F to the system and the output y is thus described by the second-order differential equation:

$$m \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = F$$

The term *second-order* is used because the equation includes $\frac{d^2y}{dt^2}$ as its highest derivative.

6.4.6 Example 6



$$m\ddot{y} = k(x - y) - c\dot{y}$$

$$m\ddot{y} + c\dot{y} + ky = kx$$

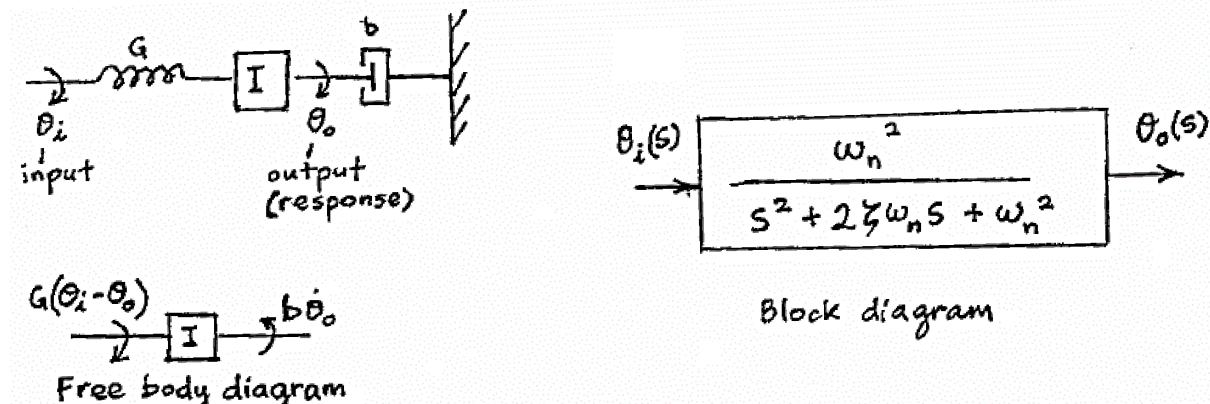
$$Ms^2Y(s) + CsY(s) + KY(s) = KX(s) = \text{Force}$$

$$\begin{aligned}\therefore \text{TF } \frac{Y(s)}{X(s)} &= \frac{K}{Ms^2 + Cs + k} \\ &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\end{aligned}$$

Where:

- $\frac{k}{m} = \omega_n^2$
- $\frac{c}{m} = 2\zeta\omega_n$

6.4.7 Example 7



Inertial forces = External forces

$$\begin{aligned} \sum T &= I\alpha \\ &= I\ddot{\theta}_0 \\ &= G(\theta_i - \theta_0) - b\dot{\theta}_0 \end{aligned}$$

$$\text{Torque} = I\ddot{\theta}_0 + b\dot{\theta}_0 + G\theta_0 = G\theta_i$$

$$Is^2\theta_0(s) + bs\theta_0(s) + G\theta_0(s) = G\theta_i(s)$$

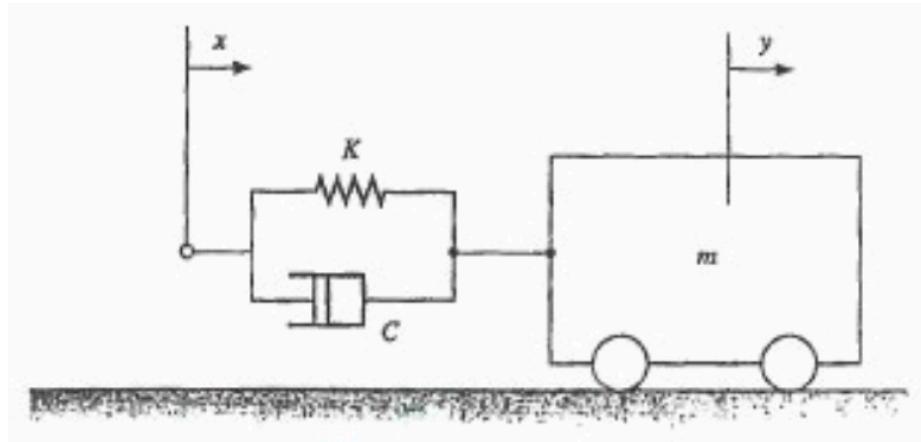
$$\begin{aligned} \therefore \text{TF: } \frac{\theta_0(s)}{\theta_i(s)} &= \frac{G}{Is^2 + bs + G} \\ &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n^2 + \omega_n^2} \end{aligned}$$

Where:

- $\frac{G}{I} = \omega_n^2$
- $\frac{b}{I} = 2\zeta\omega_n$

6.4.8 Example 8

For the mechanical system shown below, draw a free body diagram, assuming $y < x$. Determine the transfer function ($\frac{Y}{X}$) between the input displacement, x and the output displacement y .



$$k(x - y) + C(\dot{x} - \dot{y}) = M\ddot{y}$$

$$M\ddot{y} + c\dot{y} + ky = kx + c\dot{x}$$

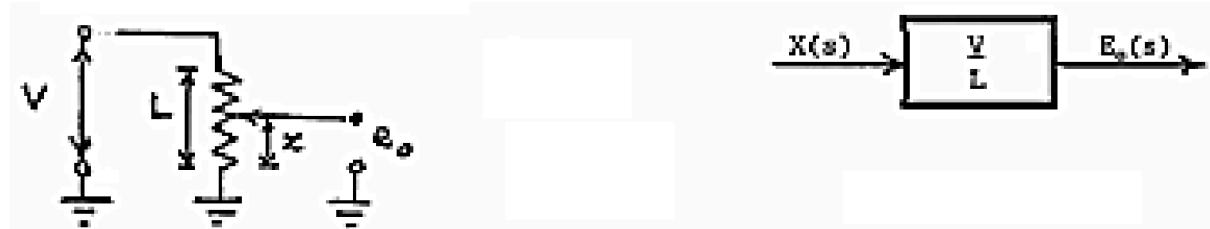
Applying the Laplace transform:

$$Y(s)[Ms^2 + Cs + K] = X(s)[K + Cs]$$

$$\frac{Y(s)}{X(s)} = \frac{k + Cs}{Ms^2 + Cs + K}$$

6.5 Typical transfer functions

6.5.1 Proportional block



$$\frac{e_0}{V} = \frac{x}{L}$$

The transfer function is constant, like the transducer used as a displacement.

6.5.2 Time constant block



The transfer function contains a time constant (1st order system).

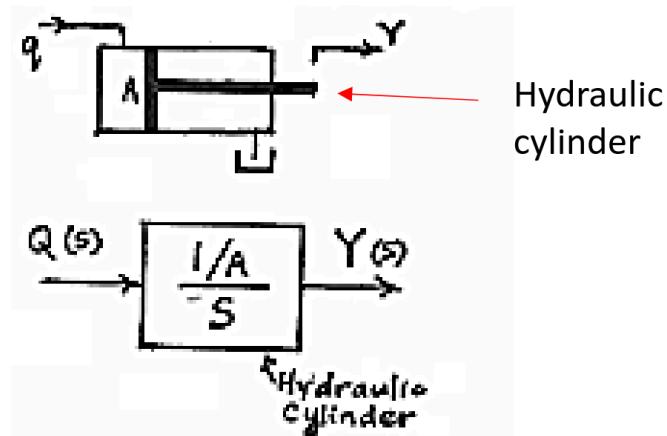
$$I\omega + b\omega = T$$

$$\text{Let } \frac{I}{b} = \tau$$

$$\tau s\Omega(s) + \Omega(s) = \frac{T(s)}{b}$$

$$\frac{\Omega(s)}{T(s)} = \frac{\frac{1}{b}}{1 + \tau s}$$

6.5.3 Integrating block



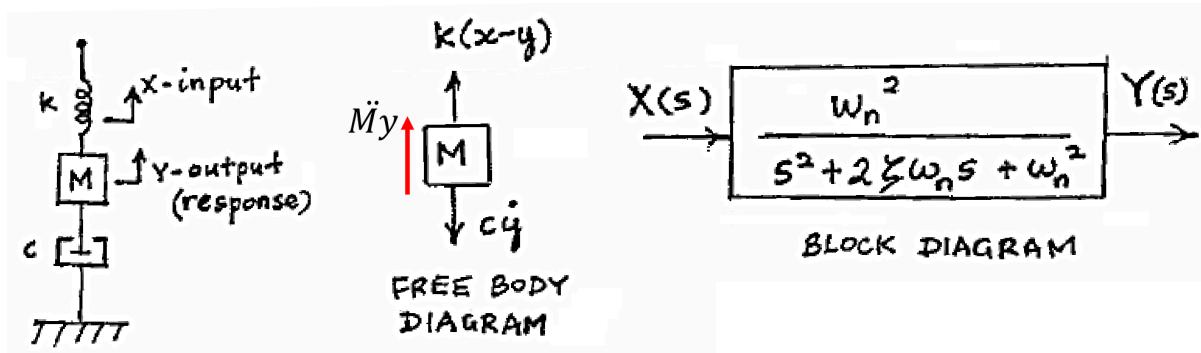
There is a constant input that produces an output that is a constant rate of change with respect to time.

$$\text{Response of cylinder: } Q = Ay$$

$$\therefore \frac{Y(s)}{Q(s)} = \frac{1}{As}$$

6.6 System types

6.6.1 Spring mass damper system



Inertial forces = External forces

$$m\ddot{y} = k(x - y) - c\dot{y}$$

$$m\ddot{y} + c\dot{y} = kx$$

6.6.2 Torsional system



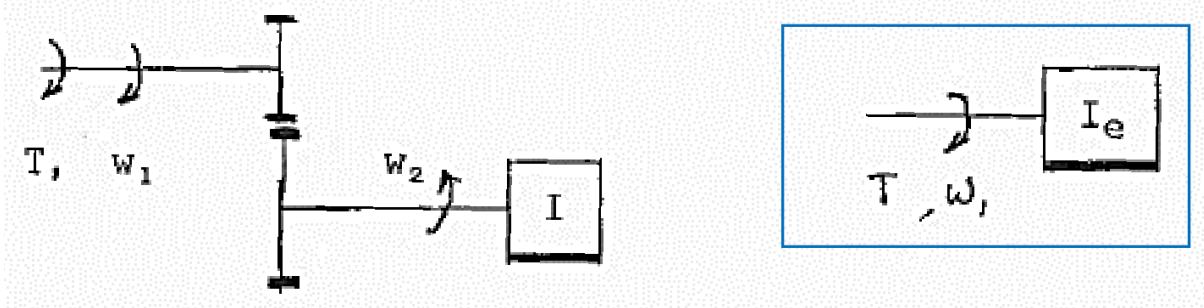
Inertial forces = External forces

$$\sum T = I\alpha$$

$$I\alpha = I\ddot{\theta}_0 = G(\theta_i - \theta_0) - b\dot{\theta}_0$$

$$\therefore I\ddot{\theta}_0 + b\dot{\theta}_0 + G\theta_0 = G\theta_i$$

6.6.3 System with gearing



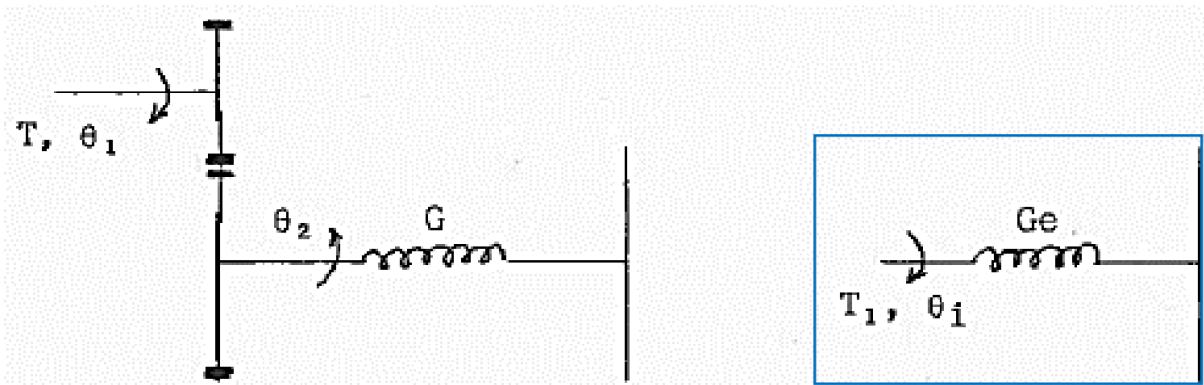
$$n = \frac{\text{Speed of the driving shaft}}{\text{Speed of the driven shaft}}$$

$$= \frac{\omega_1}{\omega_2}$$

Equating kinetic energy: $\frac{1}{2}I\omega_2^2 = \frac{1}{2}I_e\omega_1^2$

$$\therefore I_e = \frac{I\omega_2^2}{\omega_1^2} = \frac{I}{n^2}$$

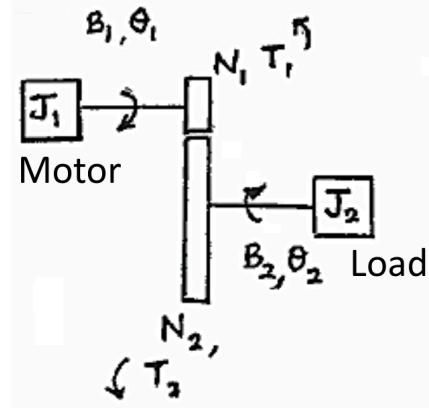
6.6.4 Gear driven spring system



Equating potential energy: $\frac{1}{2}G\theta_2^2 = \frac{1}{2}G\theta_1^2, \quad n = \frac{\theta_1}{\theta_2}$

$$G_e = \frac{G\theta_2^2}{\theta_1^2} = \frac{G}{n^2}$$

6.6.4.1 Reflecting from gear 2 to gear 1



$$\text{Inertia: } \left(\frac{N_1}{N_2}\right)^2 J_2$$

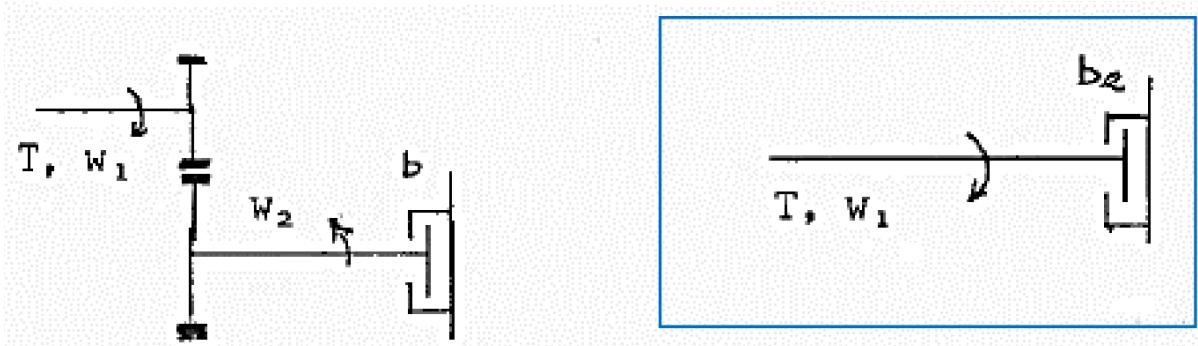
$$\text{Viscous friction: } \left(\frac{N_1}{N_2}\right)^2 B_2$$

$$\text{Torque: } \left(\frac{N_1}{N_2}\right) T_2$$

$$\text{Angular displacement: } \left(\frac{N_2}{N_1}\right) \theta_2$$

$$\text{Angular velocity: } \left(\frac{N_2}{N_1}\right) \omega_2$$

6.6.5 Gear driven with dashpot system



$$T = \omega b$$

$$P = b\omega^2$$

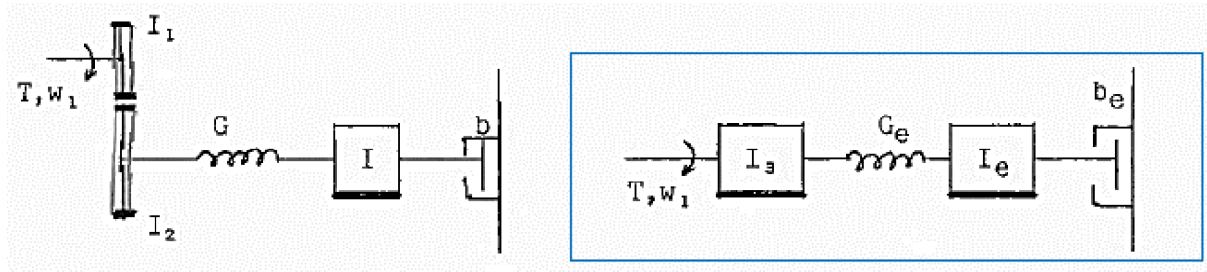
$$n = \frac{\omega_1}{\omega_2}$$

$$\text{Power} = \omega T$$

$$\therefore b\omega_2^2 = b_e \omega_1^2$$

$$b_e = \frac{b}{n^2}$$

6.6.6 Geared system



$$I_3 = I_1 + \frac{I_2}{n^2}$$

$$G_e = \frac{G}{n^2}$$

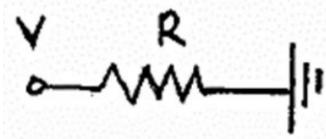
$$I_e = \frac{I}{n^2}$$

$$b_e = \frac{b}{n^2}$$

$$n = \frac{\text{Speed of the driving shaft}}{\text{Speed of the driven shaft}}$$

6.6.7 Electrical systems

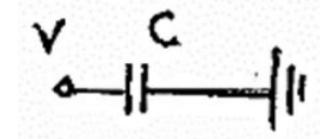
1. Resistance



$$V = iR$$

$$V = IZ, \quad Z = R$$

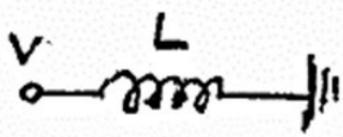
2. Capacitance



$$i = CV$$

$$V = IZ, \quad Z = \frac{1}{CS}$$

3. Inductance



$$V = Li, \quad V = IZ, \quad Z = LS$$

6.6.7.1 Loop method

Voltage equation: The sum of the voltage drop across a closed loop is 0.

$$V = iR + \frac{1}{C} \int i dt + Li$$

Since $q = \int i dt$, q = charge:

$$\therefore V = R \frac{dq}{dt} + \frac{q}{c} + L \frac{d^2q}{dt^2}$$

6.6.7.2 Node method

Current equation: The sum of currents at a circuit node or junction is 0.

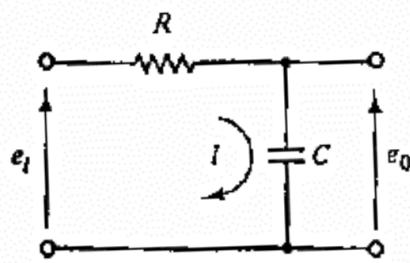
$$i = \frac{V}{R} + C \dot{V} + \frac{1}{L} \int V dt$$

Since $V = \frac{d\Phi}{dt}$, where Φ is magnetic flux:

$$\therefore i = \frac{d\Phi}{dt} + C \frac{d^2\Phi}{dt^2} + \frac{1}{L} \Phi$$

6.6.7.3 Common electrical circuits

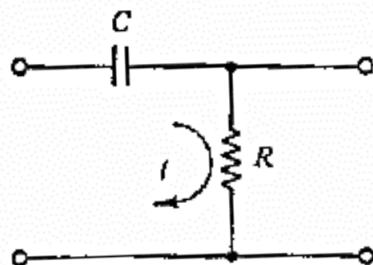
Simple lag circuit:



$$E_i = IZ = I(Z_r + Z_e), \quad Z_r = R, \quad Z_c = \frac{1}{Cs}$$

$$E_t = I \left(R + \frac{1}{Cs} \right), \quad E_0 = I \frac{1}{Cs}, \quad \therefore \frac{E_0}{E_i} = \frac{1}{RCs + 1}$$

Transient lead circuit:



$$E_i = I \left(R + \frac{1}{Cs} \right), \quad E_0 = IR, \quad \therefore \frac{E_0}{E_i} = \frac{RCs}{RCs + 1}$$

6.6.7.4 Electromechanical analogy

Force	Voltage	Current
M - Mass	L - Inductance	C - Capacitance
c - Viscous friction	R - Resistance	- Conductance
K - Stiffness	$\frac{1}{C}$ - Capacitance	$\frac{1}{L}$ - Inductance
x - Displacement	q - Charge	Φ - Magnetic flux
f - Force	v - Potential difference	i - Current
V - Velocity	i - current	v - Potential difference

6.6.8 Gear trains

Distance work done:

$$r_1 N_2 = r_2 N_1$$

Distance:

$$\theta_1 r_1 = \theta_2 r_2$$

Work done:

$$T_1 \theta_1 = T_2 \theta_2$$

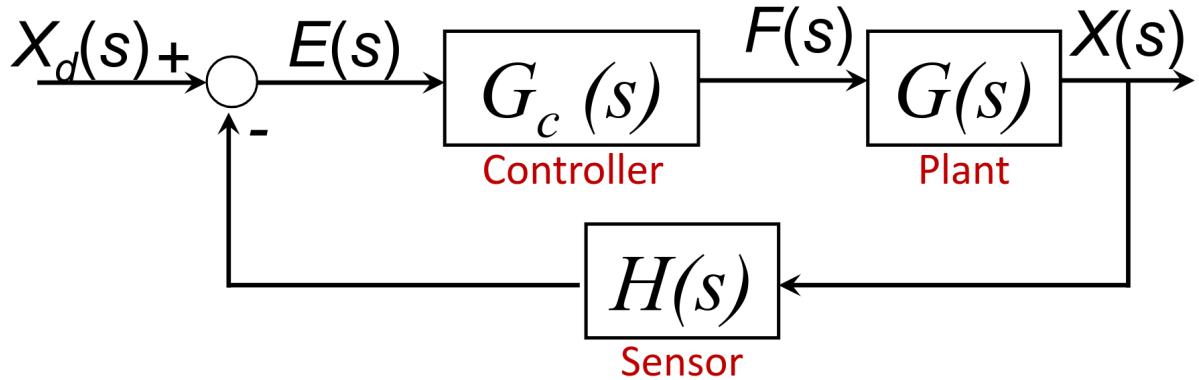
$$\therefore \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{1}{n}$$

Where:

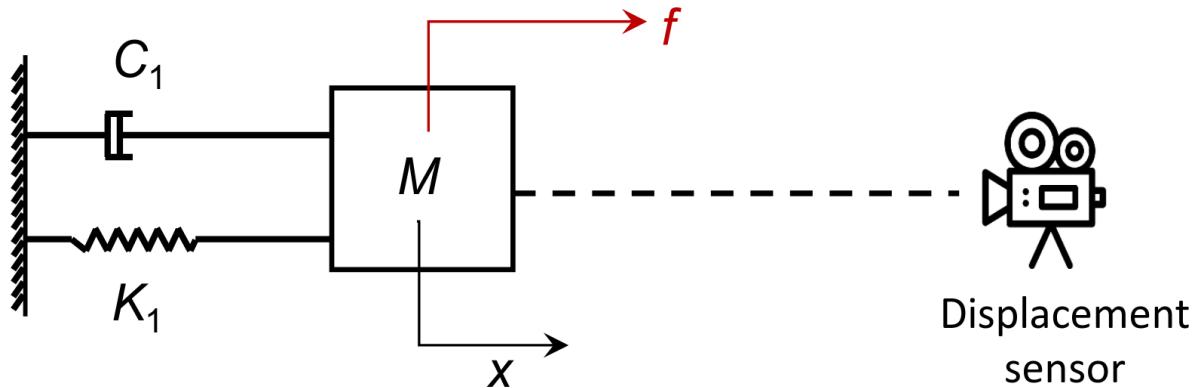
- N is the number of teeth
- r is the radius of the gear
- n is the gear ratio

6.7 Position control system

Physical system:

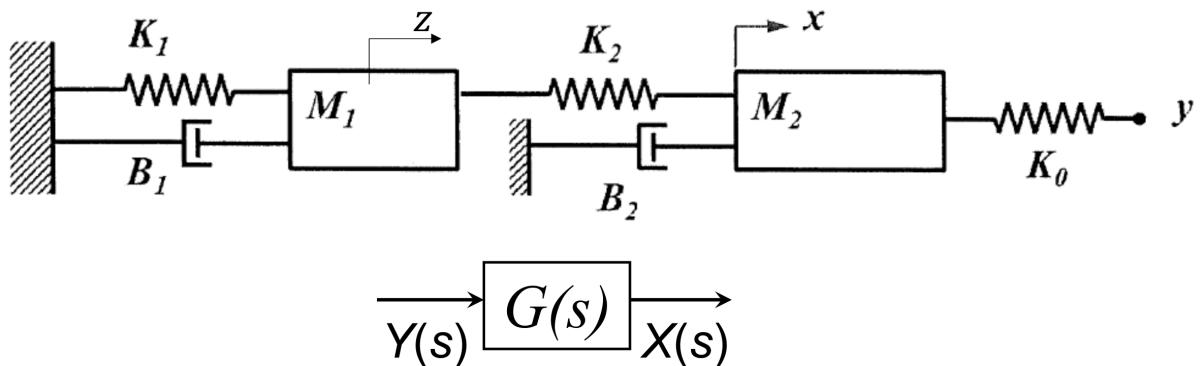


Block diagram:



6.8 Mechanical systems

Find $\frac{Y}{X}$.

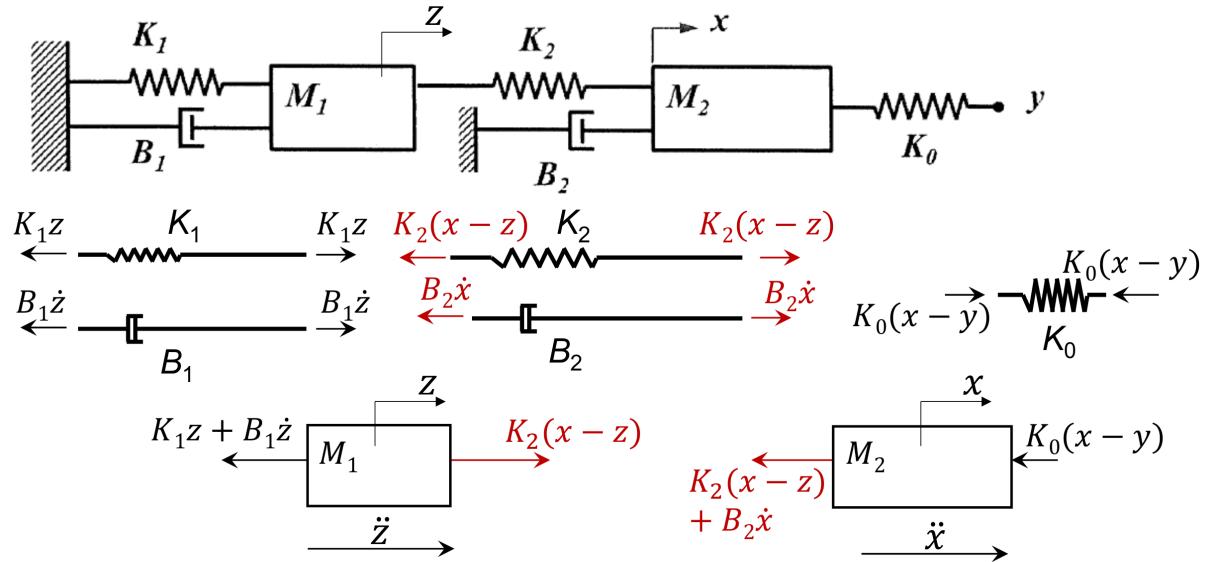


General steps:

1. Separate the springs, dampers and mass.
2. Analyse forces on springs and dampers.
3. Use free-body diagrams to write down the equation of motion.
4. Laplace transform.

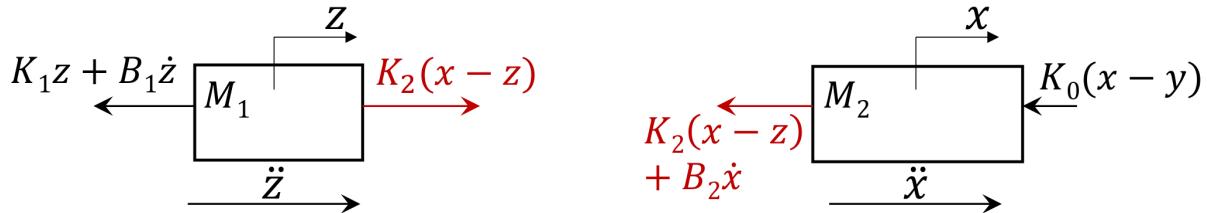
6.8.1 Steps 1-2

1. Separate the springs, dampers and mass.
2. Analyse forces on springs and dampers.



6.8.2 Steps 3-4 (Mass 1)

3. Use free-body diagrams to write down the equation of motion.



Using $f = ma$ on M_1 :

$$K_2(x - z) - K_1z - B_1\dot{z} = M_1\ddot{z}$$

$$K_2x = M_1\ddot{z} + B_1\dot{z} + (K_1 + K_2)z$$

4. Laplace transform:

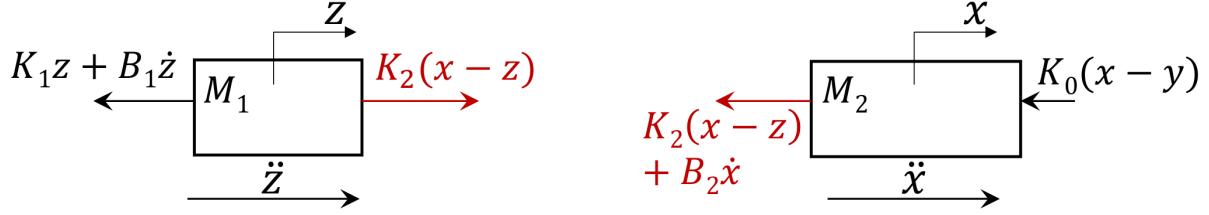
$$K_2X = M_1s^2Z + B_1sZ + (K_1 + K_2)Z$$

$$K_2X = (M_1s^2 + B_1s + (K_1 + K_2))Z$$

$$Z = \frac{K_2}{M_1s^2 + B_1s + (K_1 + K_2)}X \quad (1)$$

6.8.3 Steps 3-4 (Mass 2)

3. Use free-body diagrams to write down the equation of motion.



Using $f = ma$ on M_2 :

$$-K_2(x - z) - B_2 \dot{x} - K_0(x - y) = M_2 \ddot{x}$$

$$K_2 z + K_1 y = M_2 \ddot{x} + B_2 \dot{x} + (K_0 + K_2)x$$

4. Laplace transform:

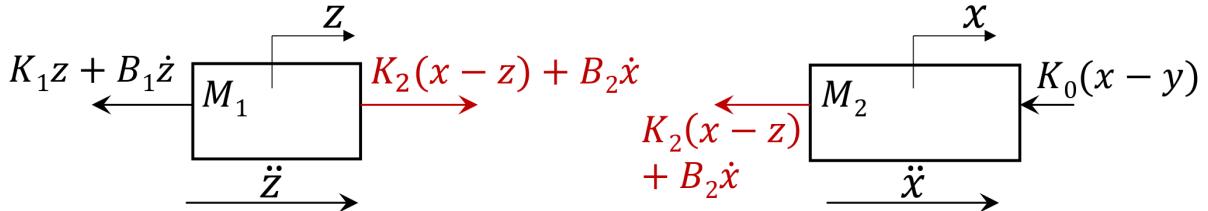
$$K_2 Z + K_0 Y = M_2 s^2 X + B_2 s X + (K_0 + K_2)X$$

$$K_2 Z + K_0 Y = (M_2 s^2 + B_2 s + (K_0 + K_2))X \quad (2)$$

6.8.4 Steps 3-4 (Solving $\frac{X}{Y}$)

3. Use free-body diagrams to write down the equation of motion.

4. Laplace transform.



$$K_2 \frac{K_2}{M_1 s^2 + B_1 s + K_1 + K_2} X + K_0 Y = M_2 s^2 + B_2 s + (K_0 + K_2)X$$

$$K_0 Y = \left(M_2 s^2 + B_2 s + K_0 + K_2 - \frac{K_2^2}{M_1 s^2 + B_1 s + K_1 + K_2} \right)$$

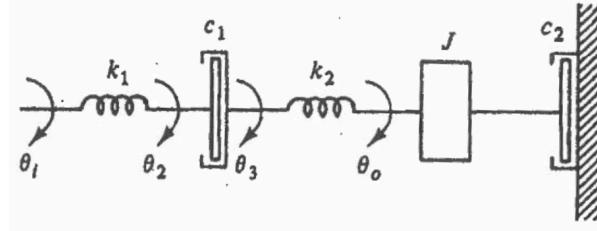
$$\frac{X}{Y} = \frac{K_0}{M_2 s^2 + B_2 s + K_0 + K_2 - \frac{K_2^2}{M_1 s^2 + B_1 s + K_1 + K_2}} = G(s)$$

6.9 Linear and rotary systems

	Springs	Linear	Rotary
Scenario 1 $\theta_2 > \theta_1$			
Scenario 2 $\theta_2 < \theta_1$			

6.9.1 Example

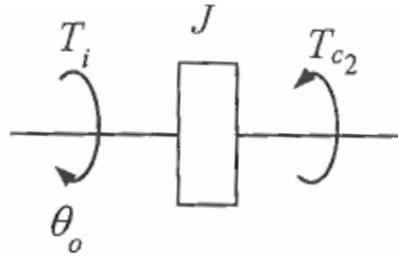
Write the differential equations and obtain the transfer function $\frac{\theta_o(s)}{\theta_i(s)}$ for the drive system shown below. In the diagram, the springs represent long shafts and damping effects due to bearings and shaft couplings are present.



Solution:

Equations are written in order, starting at the input. To help visualise signs, assume that $\theta_i > \theta_2, \dot{\theta}_2 > \dot{\theta}_3$, and $\theta_3 > \theta_o$. The shaft torsion torque $k_1(\theta_i - \theta_2)$ transmitted through the damper c_1 and spring k_2 accelerates inertia J and supplies the damping torque.

$$T_i = k_1(\theta_i - \theta_2) = c_1(\dot{\theta}_2 - \dot{\theta}_3) = k_2(\theta_3 - \theta_o) \quad (1)$$



From the free body diagram, the summation of torque at the inertia element J is:

$$T_i - c_2\dot{\theta}_o = J\ddot{\theta}_o \quad (2)$$

Applying the Laplace transform to both sides of equation (2) and re-arranging:

$$k_2\Theta_3(s) = (Js^2 + c_2s + k_2)\Theta_o(s)$$

$$\Rightarrow \Theta_3(s) = \frac{1}{k_2}(Js^2 + c_2s + k_2)\Theta_o(s)$$

Re-arranging the differential equations of equation (1) yield:

$$k_1\theta_i = c_1\dot{\theta}_2 + k_1\theta_2 - c_1\dot{\theta}_3$$

$$c_1\dot{\theta}_2 = c_1\dot{\theta}_3 + k_2\theta_3 - k_2\theta_o$$

Applying Laplace transform to the equations:

$$k_1\Theta_i(s) = (c_1s + k_1)\Theta_2(s) - c_1s\Theta_3(s)$$

$$c_1s\Theta_2(s) = (c_1s + k_2)\Theta_3(s) - k_2\Theta_o(s)$$

$$\Theta_2(s) = \frac{1}{c_1s}[(c_1s + k_2)\Theta_3(s) - k_2\Theta_o(s)]$$

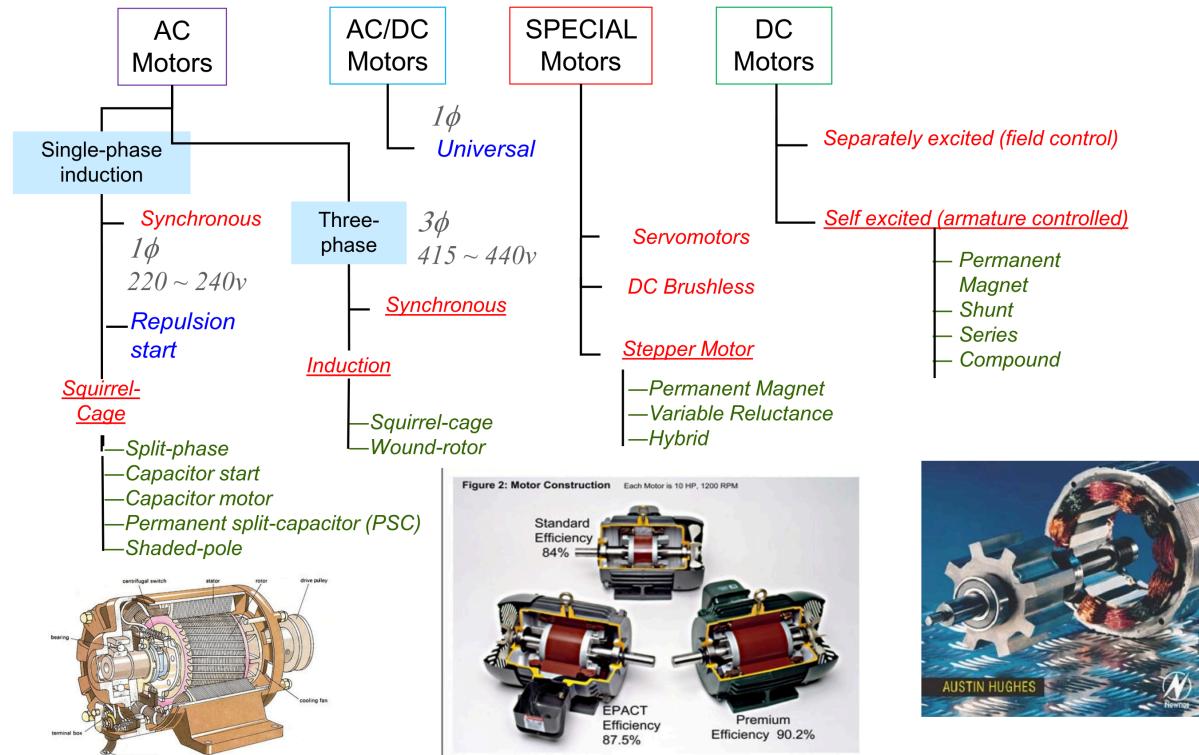
Substituting $\Theta_2(s)$ into the previous equation yields:

$$\begin{aligned}
k_1 \Theta_i(s) &= (c_1 s + k_1) \frac{1}{c_1 s} [(c_1 s + k_2) \Theta_3(s) - k_2 \Theta_o(s)] - c_1 s \Theta_3(s) \\
&= \frac{1}{c_1 s} \{ [(c_1 s + k_1)(c_1 s + k_2) - c^2 s^2] \Theta_3(s) - (c_1 s + k_1) k_2 \Theta_o(s) \} \\
&= \frac{1}{c_1 s} \{ [c_1 s(k_1 + k_2) + k_1 k_2] \Theta_3(s) - (c_1 s + k_1) k_2 \Theta_o(s) \} \\
\Rightarrow \Theta_3(s) &= \frac{c_1 k_1 s \Theta_i(s) + (c_1 s + k_1) k_2 \Theta_o(s)}{c_1 s(k_1 + k_2) + k_1 k_2}
\end{aligned}$$

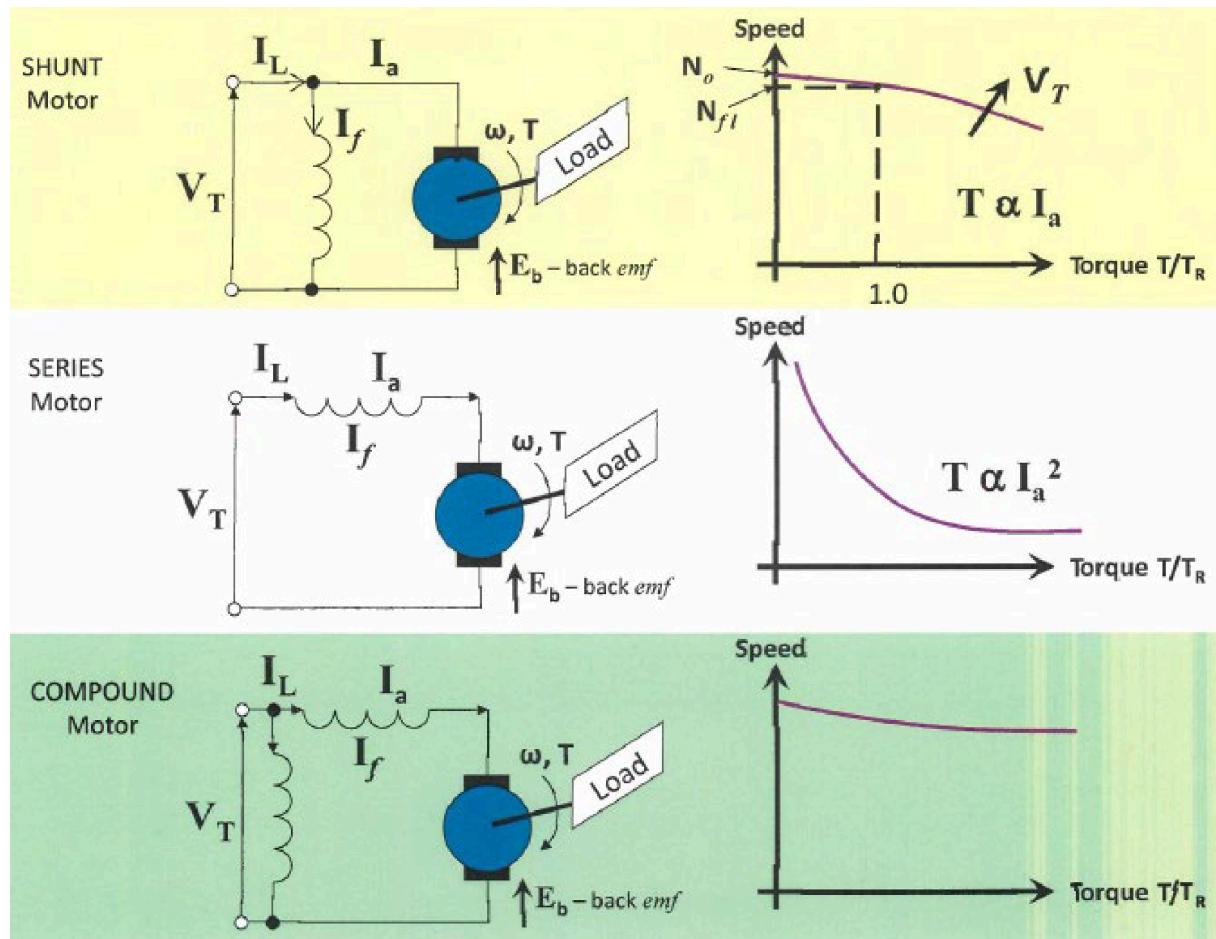
Equating the above with $\Theta_3(s)$ obtained earlier yields:

$$\begin{aligned}
\frac{J s^2 + c_2 s + k_2}{k_2} \Theta_o(s) &= \frac{c_1 k_1 s \Theta_i(s) + (c_1 s + k_1) k_2 \Theta_o(s)}{c_1 s(k_1 + k_2) + k_1 k_2} \\
\frac{c_1 k_1 k_2 s}{c_1 s(k_1 + k_2) + k_1 k_2} \Theta_i(s) &= \left[(J s^2 + c_2 s + k_2) - \frac{(c_1 s + k_1) k_2^2}{c_1 s(k_1 + k_2) + k_1 k_2} \right] \Theta_o(s) \\
\frac{\Theta_o(s)}{\Theta_i(s)} &= \frac{c_1 k_1 k_2}{J c_1 (k_1 + k_2) s^2 + [J k_1 k_2 + c_1 c_2 (k_1 + k_2)] s + k_1 k_2 (c_1 + c_2)}
\end{aligned}$$

6.10 Types of motors



6.10.1 Graphs



6.10.2 Shunt motors

$$V_T = E_b + I_a R_a$$

$$V_T = k\phi\omega + I_a R_a$$

$$\omega = \frac{V_T}{K\phi} = -\frac{I_a R_a}{K\phi} = \frac{V_T - I_a R_a}{K\phi}$$

$$\text{Speed regulation} = \frac{N_o - N_{fl}}{N_{fl}}$$

$$\text{Shunt motor : } T = k\phi I_a$$

$$\text{Starting current : } I_a = \frac{V_T - E_b}{R_a}$$

6.10.3 Permanent magnet DC motor

Voltage equation:

$$V = R_a I_a + K_e \Omega \quad (1)$$

Armature current:

$$I_a = \frac{V - K_e \Omega}{R_a} \quad (2)$$

Torque:

$$T = K_T I_a = \frac{K_T}{R_a} (V - K_e \Omega) \quad (3)$$

From equation (3),

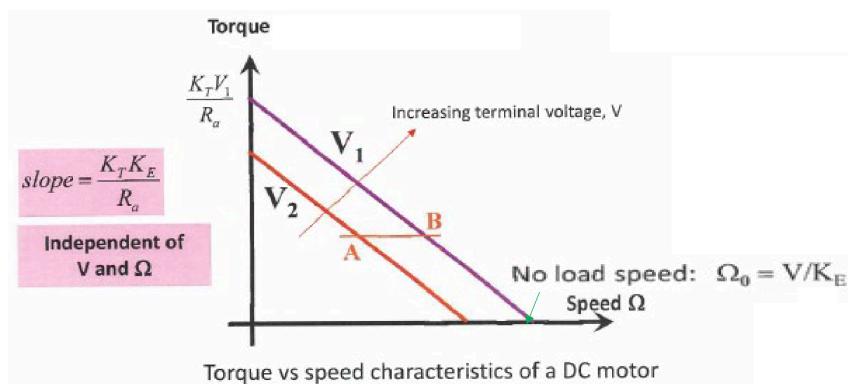
$$\text{Starting torque : } T_s = \frac{K_T V}{R_a} \quad (4)$$

$$\text{No load speed : } \Omega_0 = \frac{V}{K_e} \quad (5)$$

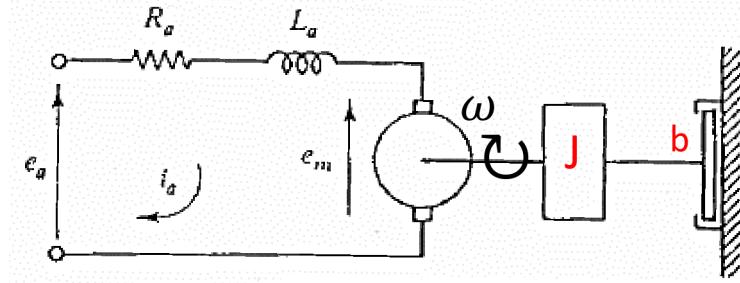
Using the above units:

$$K_T = \text{Torque constant (1 N m A}^{-1}\text{)}$$

$$K_e = \text{Back EMF constant (1 V s rad}^{-1}\text{)}$$



6.10.4 Armature controlled DC motor



$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

$$\therefore T(s) = s(Js + B)\theta(s)$$

Armature loop:

$$e_a = R_a i_a + L_a \dot{i}_a + e_m$$

$$\therefore E_a = (R_a + L_a s) I_a + E_m$$

Constant current source:

$$T_m = K_t i_a$$

$$T_m = K_t I_a$$

Eliminating I_a, E_m, T :

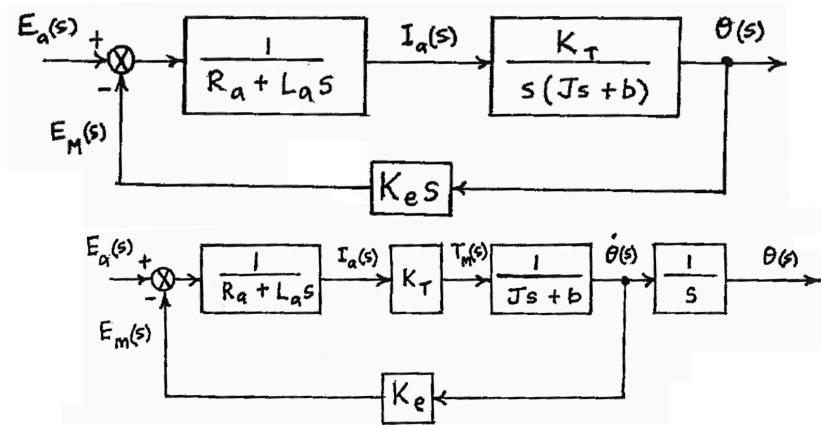
$$\frac{\theta}{E_a} = \frac{\frac{1}{K_e}}{s(T_a T_m S^2 + (T_m + \gamma T_a)s + \gamma + 1)}$$

Where:

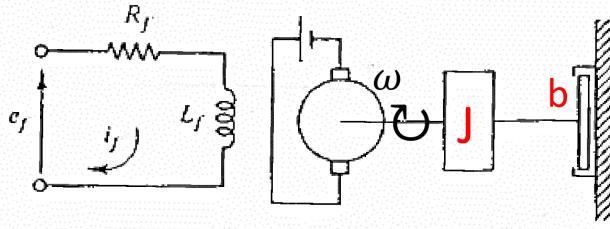
- $T_m = \frac{J R_a}{K_e K_t}$ is the motor time constant
- $T_a = \frac{L_a}{R_a}$ is the armature time constant
- $\gamma = \frac{B R_a}{K_e K_t}$ is the damping factor

Assuming armature inductance, L_a is small compared to armature resistance R_a :

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{R_a J_m}}{S \left[S + \frac{1}{J_m} \left(b_m + \frac{K_t K_e}{R_a} \right) \right]}$$



6.10.5 Field controlled DC motor



Generally, motor load is assumed to consist of an inertia and a damper with damping constant, B . Motor shaft position θ and motor torque are related.

$$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$

$$\therefore T(s) = s(Js + B)\theta(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{1}{s(Js + B)}$$

Equating loop field:

$$e_f = R_f I_f + L_f$$

$$E_f = (R_f + sL_f)I_f$$

With constant armature voltage, the developed motor torque, T , is proportional to the field current. $T = K_t I_f$. K_T is the motor torque constant.

Eliminating I_f and T :

$$\frac{\theta}{E_f} = \frac{\frac{K_t}{R_f B}}{s(T_m s + 1)(T_f s + 1)}$$

Where:

- $T_m = \frac{J}{B}$ is the motor time constant
- $T_f = \frac{L_f}{R_f}$ is the field time constant

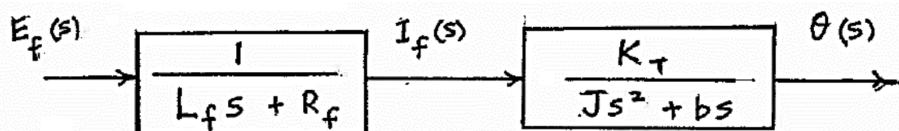
In the operating range of the motor:

$$T_f \ll T_m$$

$$\therefore \frac{\theta}{E_f} = \frac{\frac{K_t}{R_f B}}{s(T_m s + 1)}$$

The system type is $\frac{1}{s}$, so the motor is basically an integrator. For a constant input, θ_f has shaft angle θ which increases at a constant rate. Essentially:

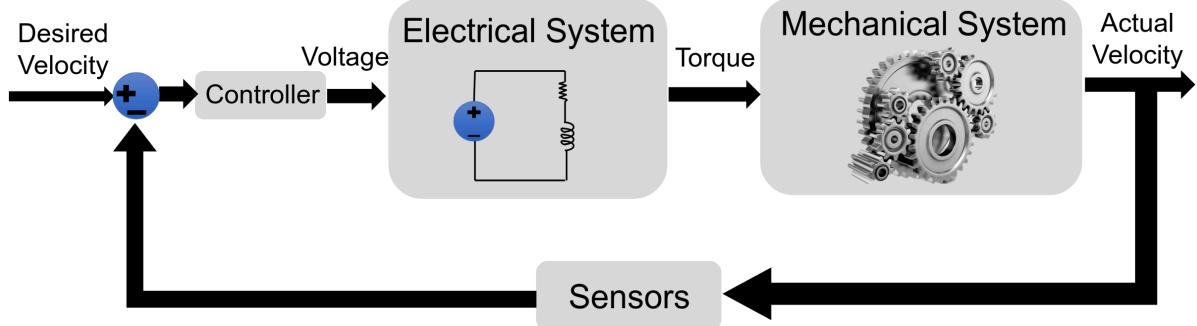
$$\theta \propto \int e_f$$



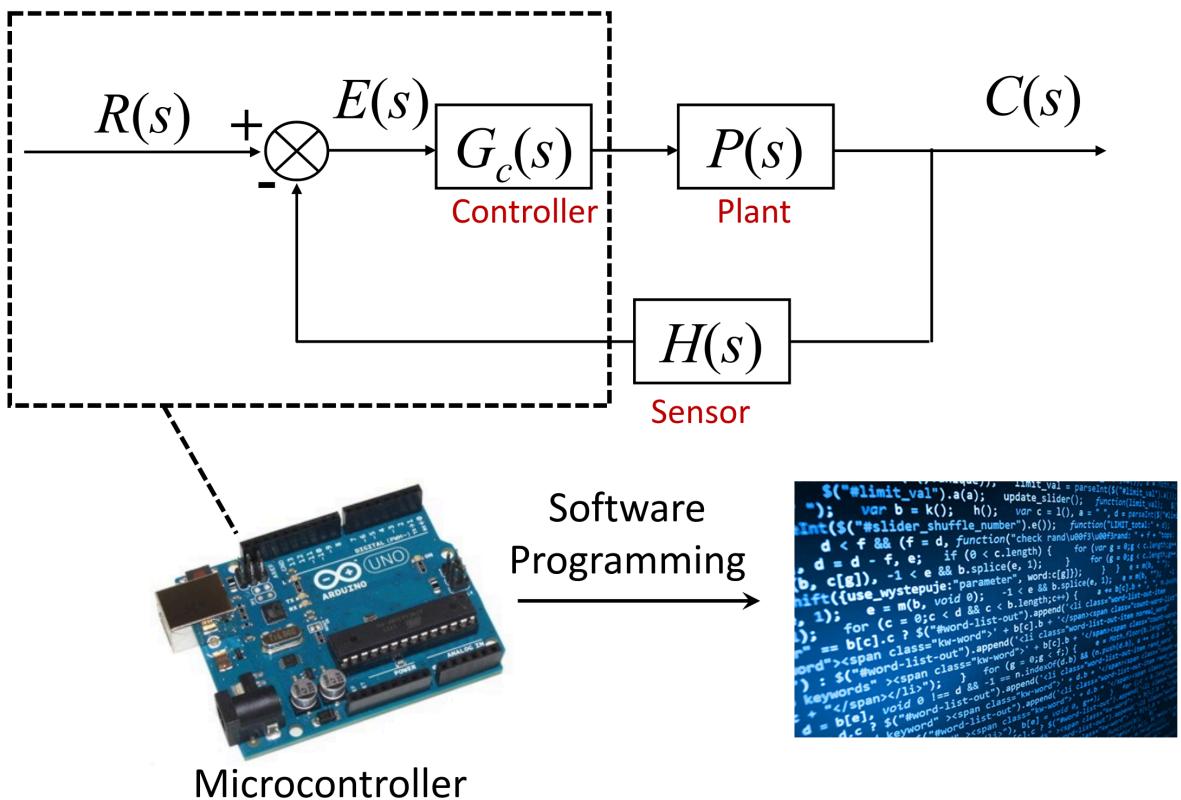
6.11 Motivation for mathematical modelling

1. A method to unify sub-systems across different disciplines.
 2. To regulate the output with respect to time t .

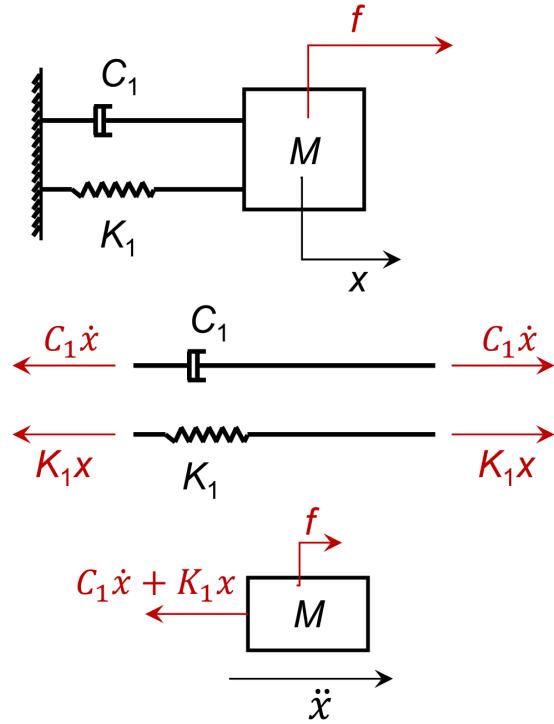
For example:



6.11.1 Physical system



6.11.2 Example



Using $f = ma$:

$$f - (C_1\dot{x} + K_1x) = M\ddot{x}$$

$$f = M\ddot{x} + C_1\dot{x} + K_1x$$

Laplace transform:

$$F(s) = Ms^2X + C_1sX + K_1X$$

$$F(s) = [Ms^2 + C_1s + K_1]X$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + C_1s + K_1} = G(s)$$

Assuming $M = 1 \text{ kg}$, $C_1 = 7 \text{ Ns m}^{-1}$, $K_1 = 6 \text{ N m}^{-1}$.

Solve $x(t)$ when $f(t) = 1 \text{ N}$.

Input in s-domain: $F(s) = \frac{1}{s}$

Solving $x(t)$:

$$X(s) = \frac{1}{Ms^2 + C_1s + K_1} F_s$$

$$X(s) = \frac{1}{s^2 + 7s + 6} \frac{1}{s}$$

$$X(s) = \frac{1}{s(s+1)(s+6)}$$

Partial fraction decomposition:

$$X(s) = \frac{1}{s(s+1)(s+6)}$$

$$X(s) = \frac{A}{s} + \frac{B}{s+1}, \frac{C}{s+6}$$

Comparing the numerator:

$$A(s+1)(s+6) + Bs(s+6) + Cs(s+1) = 1$$

Substitute $s = 0$:

$$A(1)(6) + 0 = 1 \rightarrow A = \frac{1}{6}$$

Substitute $s = -1$:

$$B(-1)(5) = 1 \rightarrow B = -\frac{1}{5}$$

Substitute $s = -6$:

$$C(-6)(-5) = 1 \rightarrow C = \frac{1}{30}$$

$$\therefore X(s) = \frac{1}{6s} + \frac{1}{5}(s+1) + \frac{1}{30(s+6)}$$

Laplace inverse:

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{1}{6} - \frac{1}{5}e^{-t} + \frac{1}{30}e^{-6t}$$

Recomputing the steady-state $x(x_{ss})$ with final value theorem:

$$\begin{aligned} x_{ss} &= \lim_{t \rightarrow \infty} x(t) = \lim_{(s \rightarrow s)X(s)} \\ &= \lim_{s \rightarrow 0} \frac{1}{s(s+1)(s+6)} \\ &= \lim_{s \rightarrow 0} \frac{1}{(1)(6)} = \frac{1}{6} \end{aligned}$$

This result is only valid when the system is stable.

7 Poles and stability

For a transfer function:

$$G(s) = \frac{N(s)}{D(s)}$$

The characteristic equation is:

$$D(s) = 0$$

The poles are defined as all values of s that satisfy the characteristic equation.

7.1 Stability analysis

- Intuitively, a stable system is one whose response does not “blow up” during its operation.
- Another intuitive understanding is the output of a system has “finite energy” if the input to the system has only “finite energy”.
- Since “energy” is related to the size or magnitude of a signal, this leads to the following stability concept, which is:

A system is bounded-input bounded output (BIBO) stable if the output signal is bounded when the input signal is bounded.

- In other words, if a stable system is driven by an input that doesn’t blow up, then the output doesn’t blow up either.
- On the other hand, if a system is unstable, then some bounded input produces an unbounded output.

7.1.1 Criteria for bounded-input bounded-output (BIBO) stability

- **Open** left-half of s -plane is the region to the left of the imaginary axis in the s -plane **excluding** the imaginary axis.
- **Closed** left-half of s -plane is the region to the left of the imaginary axis in the s -plane **including** the imaginary axis.
- **Open** right-half of s -plane is the region to the right of the imaginary axis in the s -plane **excluding** the imaginary axis.
- **Closed** left-half of s -plane is the region to the right of the imaginary axis in the s -plane **including** the imaginary axis.

The system $G(s)$ is BIBO stable if and only if all its poles are in the open left-half s -plane, or equivalently, the poles have negative real parts.

7.1.2 Example

Consider a simple case where the poles of the system are all distinct and that the system is driven by a unit step function, i.e.

$$r(t) = 1(t)$$

In this case, the output $C(s)$ is:

$$C(s) = \frac{N(s)}{D(s)} \frac{1}{s}$$

Performing partial fraction decomposition:

$$C(s) = \frac{1}{s} + \sum_{i=1}^n \frac{a_i}{s} + p_i$$

7.2 Transfer function poles

Poles dictate the homogeneous solutions and the transient response of a system.

7.2.1 Differential equation

$$M\ddot{x} + C\dot{x} + Kx = f$$

Homogeneous solution : $M\ddot{x} + C\dot{x} + Kx = 0$

$$Md^2 + Cd + K = 0$$

7.2.1.1 1st possibility

$$d = -1, 2$$

$$x = Ae^{-t} + Be^{2t}$$

7.2.1.2 2nd possibility

$$d = \pm 5j$$

$$x = A \cos(5t + \phi)$$

7.2.1.3 3rd possibility

$$d = -2 \pm 5j$$

$$x = Ae^{-2t} \cos(5t + \phi)$$

7.2.2 Laplace transform

$$[Ms^2 + Cs + K]X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + K}$$

Transfer function poles : $Ms^2 + Cs + K = 0$

7.2.2.1 1st possibility

$$s = -1, 2$$

$$x = Ae^{-t} + Be^{2t}$$

7.2.2.2 2nd possibility

$$s = \pm 5j$$

$$x = A \cos(5t + \phi)$$

7.2.2.3 3rd possibility

$$s = -2 \pm 5j$$

$$x = Ae^{-2t} \cos(5t + \phi)$$

7.3 Stable systems

Transfer function poles dictate stability.

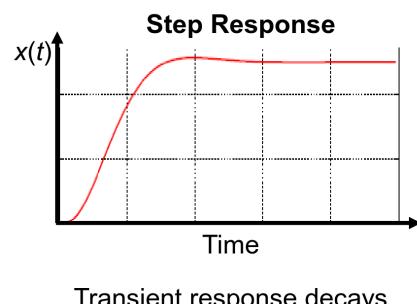
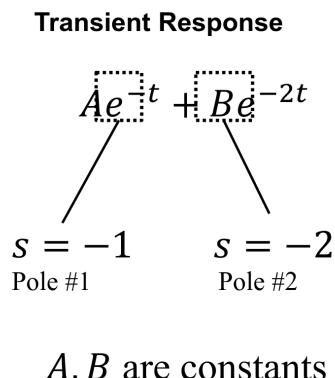
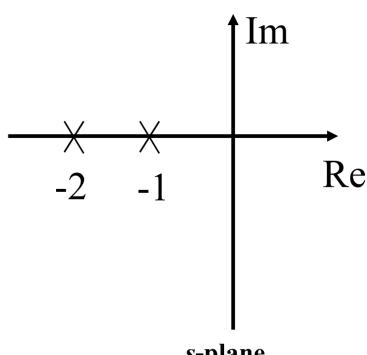
$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 3s + 2}$$

Transfer function poles:

$$s^2 + 3s + 2 = 0$$

$$(s + 1)(s + 2) = 0$$

$$s = -1, -2$$



→ **Stable system**

7.4 Partially-stable systems

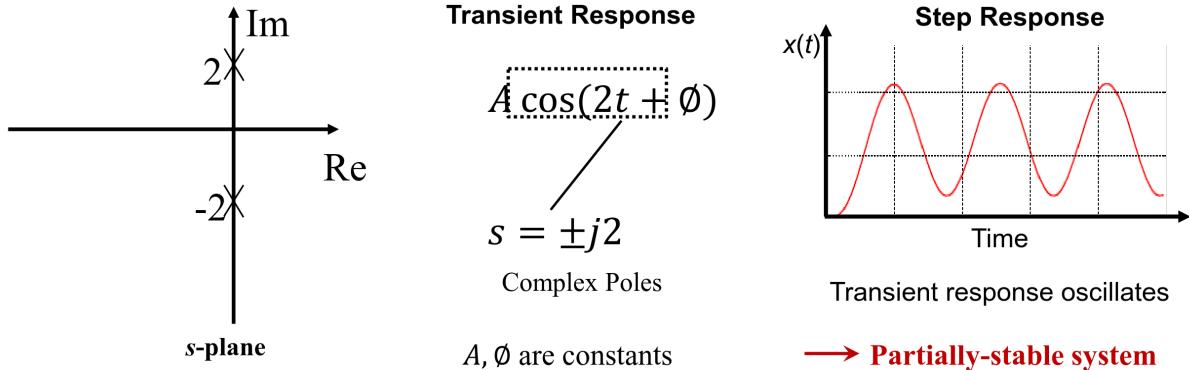
Transfer function poles dictate stability.

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 4}$$

Transfer function poles:

$$s^2 + 4 = 0$$

$$s = \pm 2j$$



7.5 Unstable systems

Transfer function poles dictate stability.

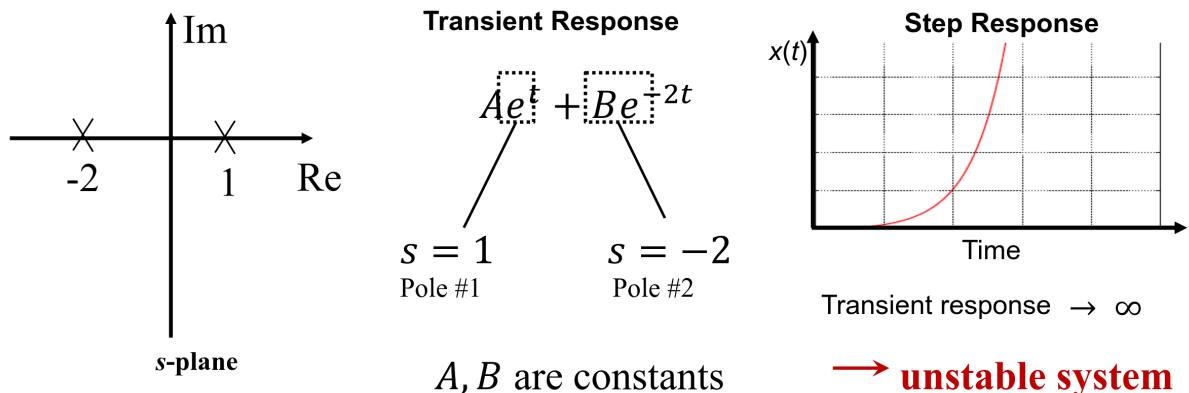
$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + s - 2}$$

Transfer function poles:

$$s^2 + s - 2 = 0$$

$$(s - 1)(s + 2) = 0$$

$$s = 1, -2$$



7.6 Types of poles

1. Real poles ($s = a$)

Transient response: Ae^{at}

2. Pure complex poles ($s = \pm bj$)

Transient response (always partially stable): $A \cos(bt + \phi)$

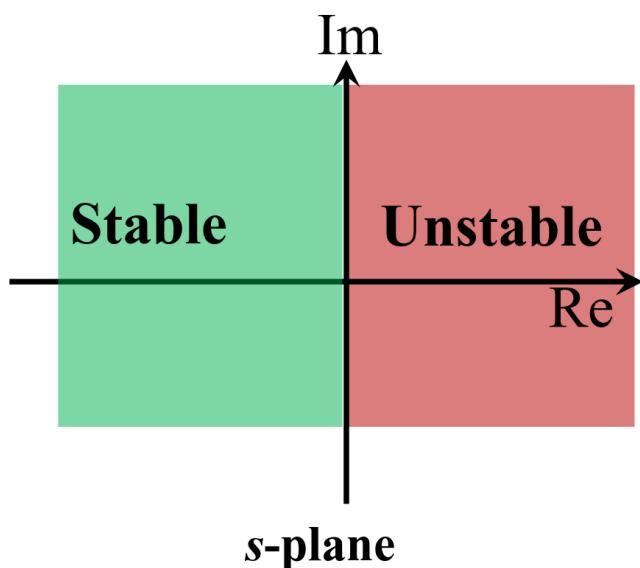
3. Complex poles with real parts ($s = a \pm bj$)

Transient response: $Ae^{at} \cos(bt + \phi)$

Where A and ϕ are constants.

Stable systems require $e^{at} \rightarrow 0$ when $t \rightarrow \infty$.

$$a < 0$$



7.7 Damping factor and stability

- In the design of a closed loop control system, the speed of response should be reasonably fast, while the percentage overshoot should not be too great and the settling time not too long.
- However, these are conflicting requirements, and hence the system design must compromise on the 3 factors.
- Usually, if the speed of the response and the peak overshoot are of highest importance, then the damping factor should be $\zeta = 0.4$.
- If settling time is the most important, a value of $\zeta = 0.7$ should be chosen.
- Thus, the damping factor chosen should lie within the range $0.4 < \zeta < 0.7$.

The shape of natural response is determined by the location of the poles of the transfer function.

$-e^{-2t}$ decays faster than e^{-t} . The point further to the left in the s -plane are associated with natural signals that decay faster than those associated with poles closer to the imaginary axis.

$$c(t) = c_{ss}(t) + c_{tr}$$

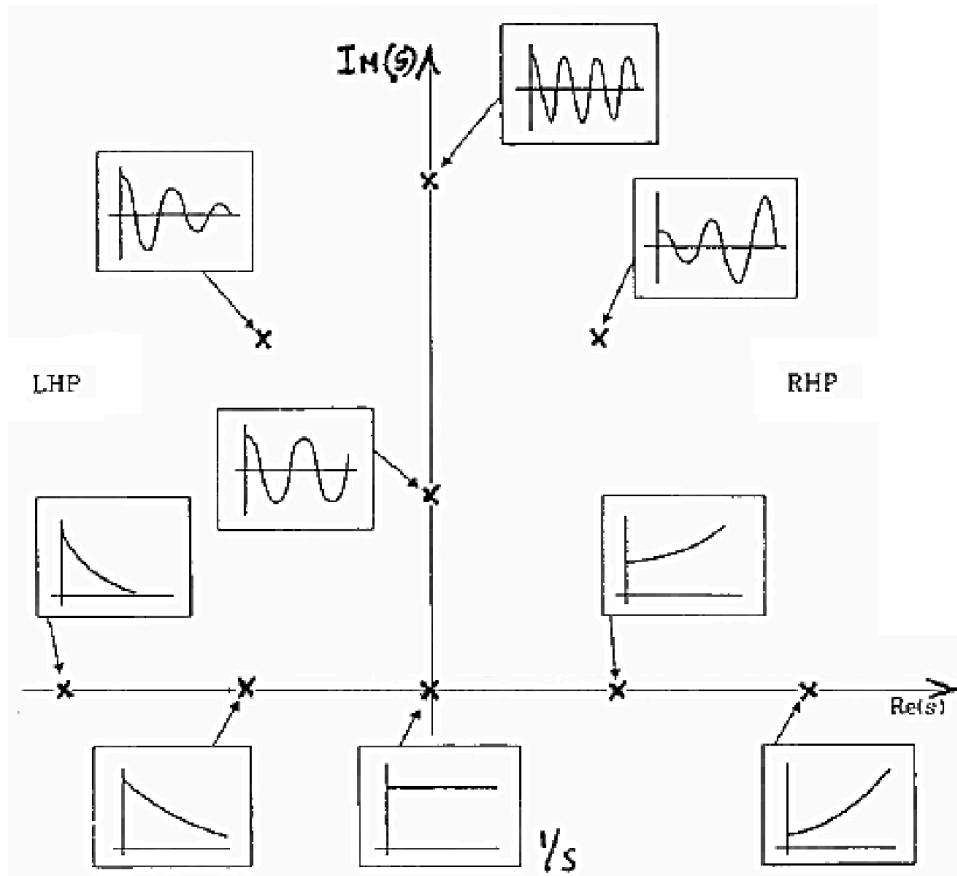
$$\lim c_{tr}(t) = 0 \quad (\text{stable})$$

$$G(s) = \frac{c(s)}{R(s)} = \frac{N(s)}{D(s)}$$

$$t \rightarrow \infty = \infty \quad (\text{unstable})$$

$$c_{tr}(s) = \frac{c_1}{s + r_1} + \frac{c_2}{s + r_2} + \frac{c_3}{s + r_3} + \dots$$

$$c_{tr}(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t} + c_3 e^{-r_3 t} + \dots$$



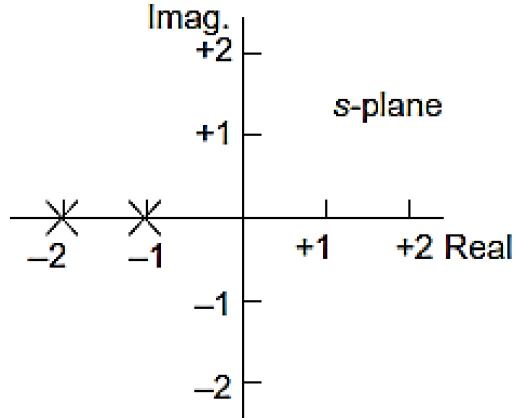
7.8 The s -plane

We can plot the positions of the poles on a graph with the real part of the pole value as the x -axis and the imaginary part as the y -axis. The resulting graph describes what is termed the s -plane.

7.8.1 Negative real roots

The graph below shows the s -plane for the transfer function:

$$G(s) = \frac{1}{(s+1)(s+2)}$$

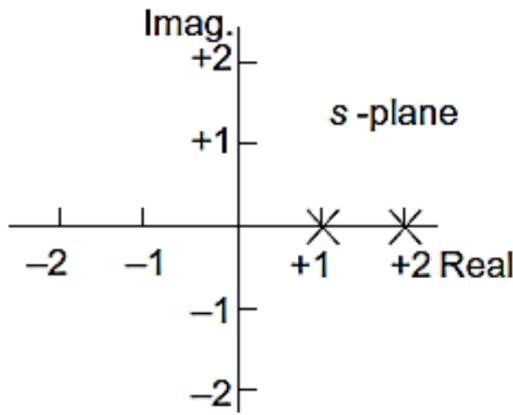


Since $s = -1$ and $s = -2$, there are no imaginary terms. The system is stable since the roots are negative and give rise to exponential terms that decrease with time.

7.8.2 Positive real roots

The graph below shows the s -plane for the transfer function:

$$G(s) = \frac{1}{(s-1)(s-2)}$$

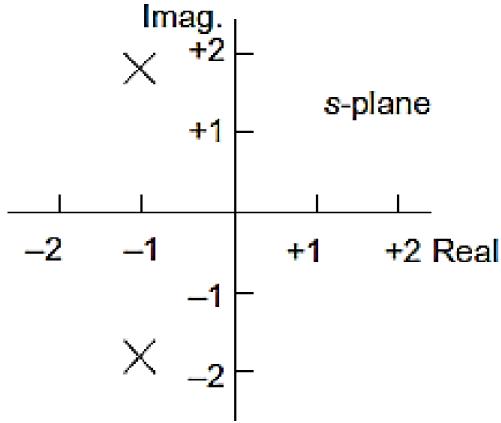


Since $s = +1$ and $s = +2$, there are no imaginary terms. The system is unstable since the roots are positive and give rise to exponential terms which increase with time.

7.8.3 Complex roots with positive real roots

The graph below shows the s -plane for the transfer function:

$$G(s) = \frac{1}{s^2 + 2s + 4}$$



The roots of the quadratic equation are:

$$s = \frac{1}{2}(-2 \pm \sqrt{4 - 16}) = -1 \pm \sqrt{3}j$$

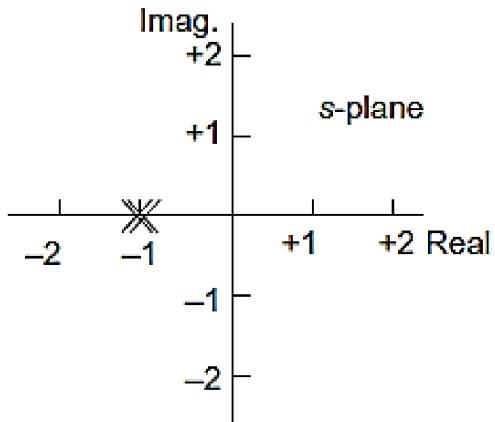
The system is under-damped but is stable. The system is stable because the real parts of the roots are negative.

7.8.4 Negative and equal real roots

The graph below shows the s -plane for the transfer function:

$$G(s) = \frac{1}{(s + 1)^2}$$

Since $s = -1$ only, there is critical damping. In general, we can state that a system is stable if all the system poles lie in the left half of the s -plane. The more negative the real part of the pole, the more rapidly the transient dies away. The larger the imaginary part of the pole, the higher the frequency of the oscillation. A system having a pole which has a positive real part is unstable.



7.9 Routh-Hurwitz stability criterion

The Routh-Hurwitz stability criterion is a simple way to check the **number of poles** that have **positive real parts**.

7.9.1 Steps

1. Form the roof array.
2. The roof array is formed by laying out all the powers of s in a table, with the powers of s being the rows, starting from the highest power of s at the top.
3. For the first 2 powers of s , fill the column next to that power of s with its coefficient, then skip a power of s and write down the next power of s in the next column, continuing until there are no more powers of s .
4. For example, if the power of s is 4, then the coefficient of s^4 is written down, followed by the coefficient of s^2 , and followed by s^0 , which is the constant term.
5. If the power of s is 3, then the coefficient of s^3 is written down, followed by the coefficient of s^1 , which is s .
6. Figure out the rest of the coefficients of the powers of s .
7. To do this, go to the bottom of the first numerical column and set that value as the anchor.
8. Multiply the anchor with the value that is both above and to the right of the current column, and subtract the multiplication of the value directly above the anchor and the value to the right of the current column.
9. Divide the result by the value of the anchor to obtain the coefficient for the power of s .
10. Move to the next column, but keep the anchor constant.
11. Repeat steps 8 and 9 to obtain the next coefficient for the same power of s .
12. Continue the same steps until the row is completed, then do the same thing for the next row, until the coefficients for all the powers of s have been filled.
13. If there is no change in sign (like positive to negative or negative to positive), that means there are no roots with positive real parts, and hence the system is stable.

Special cases:

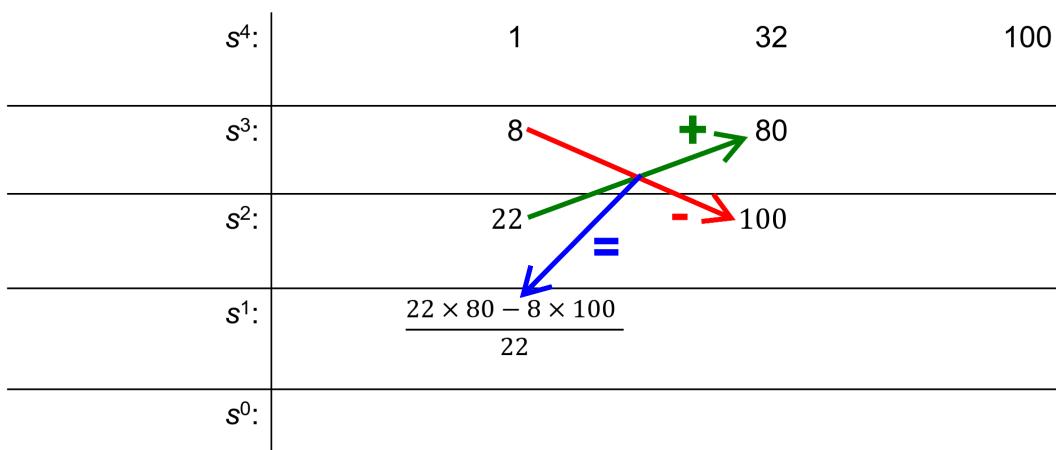
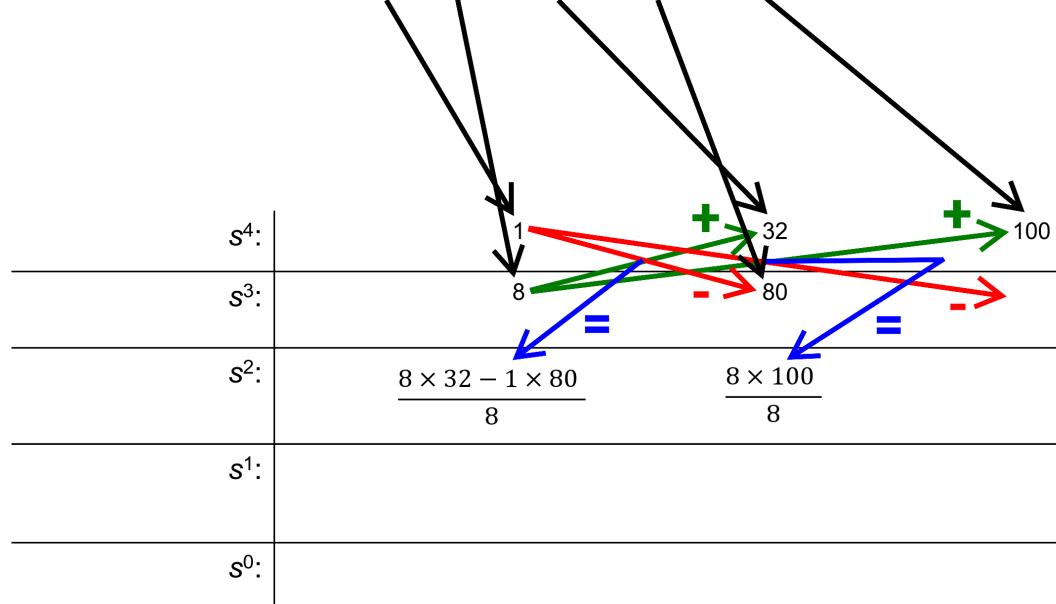
- If one of the elements in the first numerical column is 0, replace the zero with ε , which is a small positive number. In this case, the system is **partially stable** if there are **no** sign changes in the first numerical column, and **partially unstable** if there are **sign changes** in the first column.
- If an entire row of elements are zero, form an equation with the power of s above it and differentiate the equation with respect to s , then use the coefficients of the result to replace the 0s, then continue as normal. In this case, the system is **partially stable** if there are **no** sign changes in the first numerical column, and **partially unstable** if there are **sign changes** in the first column.

7.9.2 Example 1

Characteristic equation:

$$s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$$

Characteristics equation: $s^4 + 8s^3 + 32^2 + 80s + 100 = 0$



$s^4:$	+ 1	32	100
$s^3:$	+ 8	80	
$s^2:$	+ 22	100	
$s^1:$	+ 43.6		
$s^0:$	+ $\frac{43.6 \times 100}{43.6} = 100$		

Since there is no change in sign for the first numerical column, there are no roots with positive real parts and hence the system is stable.

7.9.3 Example of first special case

When one of the elements in the first numerical column is 0.

s^3	1	5
$s^2:$	1	5
$s^1:$	$\cancel{0} \varepsilon$	
$s^0:$	$\frac{5\varepsilon - 1(0)}{\varepsilon} = 5$	

Where ε is a small positive number.

Check for sign changes in the first numerical column, and the system is now either partially stable (**no** sign changes in the first numerical column) or partially unstable (**sign changes** in the first numerical column).

7.9.4 Example of the second special case

When there is an entire row of elements that are 0.

s^5	XX	XX	XX
$s^4:$	1	5	20
$s^3:$	0	0	
$s^2:$			

Form an equation with the coefficients of s in the row above (the row in red) to form the auxiliary polynomial, which is

$$P(s) = s^4 + 5s^2 + 20$$

Differentiate the auxiliary polynomial with respect to s :

$$\frac{dP(s)}{ds} = 4s^3 + 10s$$

Use the coefficients of $\frac{dP(s)}{ds}$ to replace the zero elements.

s^5	XX	XX	XX
$s^4:$	1	5	20
$s^3:$	4	10	
$s^2:$			

Continue as per normal.

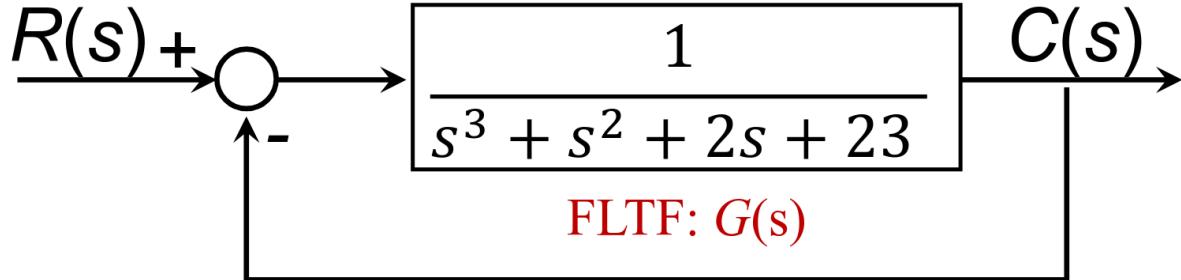
s^5	XX	XX	XX
$s^4:$	1	5	20
$s^3:$	4	-10	
$s^2:$	2.5	20	

Check for sign changes in the first numerical column, and the system is now either partially stable (**no** sign changes in the first numerical column) or partially unstable (**sign changes** in the first numerical column).

7.9.5 Example 2

Check the stability of a unity feedback system with a forward-loop transfer function of:

$$G(s) = \frac{1}{s^3 + s^2 + 2s + 23}$$



$$\begin{aligned} G_{CL} &= \frac{C}{R} = \frac{G}{1+G} \\ &= \frac{1}{s^3 + s^2 + 2s + 24} \end{aligned}$$

Characteristic equation:

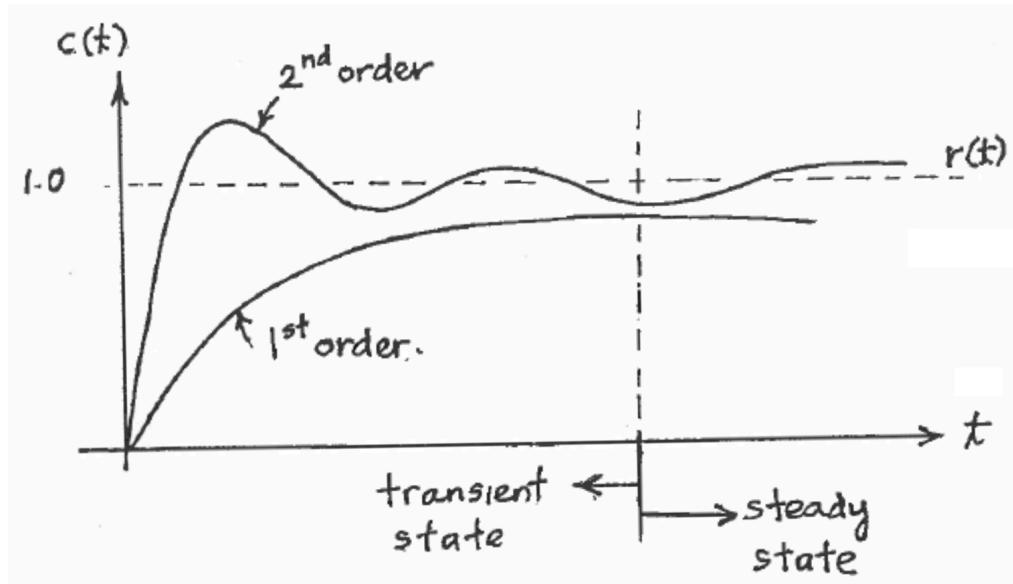
$$s^3 + s^2 + 2s + 24 = 0$$

s^3	1	2
$s^2:$	1	24
$s^1:$	$1 \times 2 - 1 \times 24$ 1	= -22
$s^0:$	24	

There are two sign changes in the first column, which means that there are two roots with positive real parts, and hence two unstable poles.

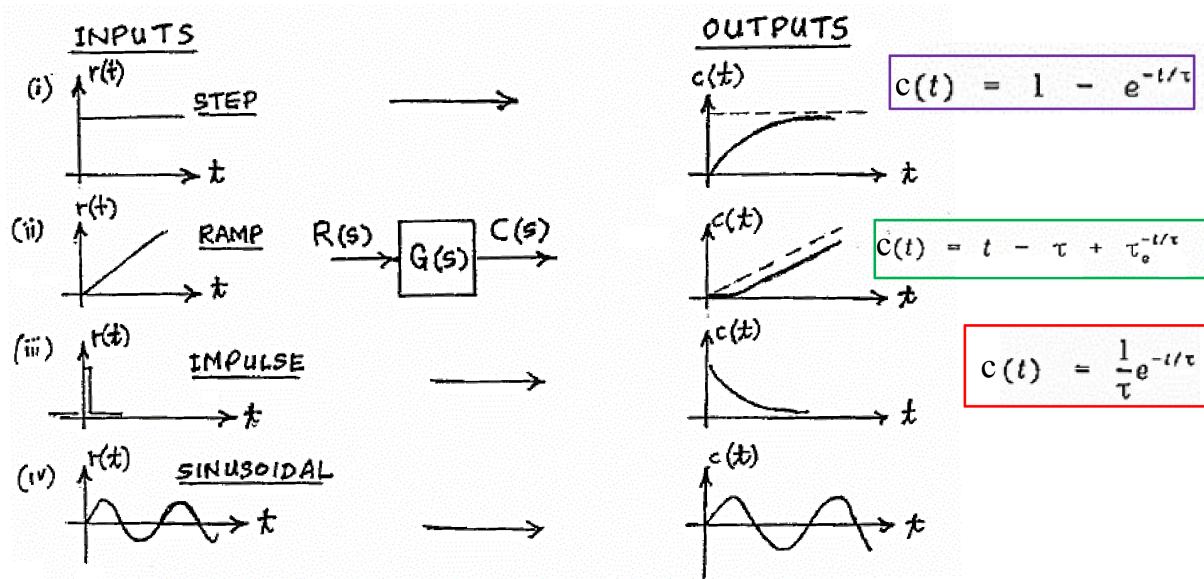
8 Time response

- Stable system
 - Transient response, c_{tr} , disappears with time.
 - Steady state, c_{ss} , measures the system accuracy.
- Unstable system
 - Only has transient response, c_{tr} .
- Transient response is always present in a system, due to effects like:
 - Inertia friction
 - Inductance
 - Capacitance

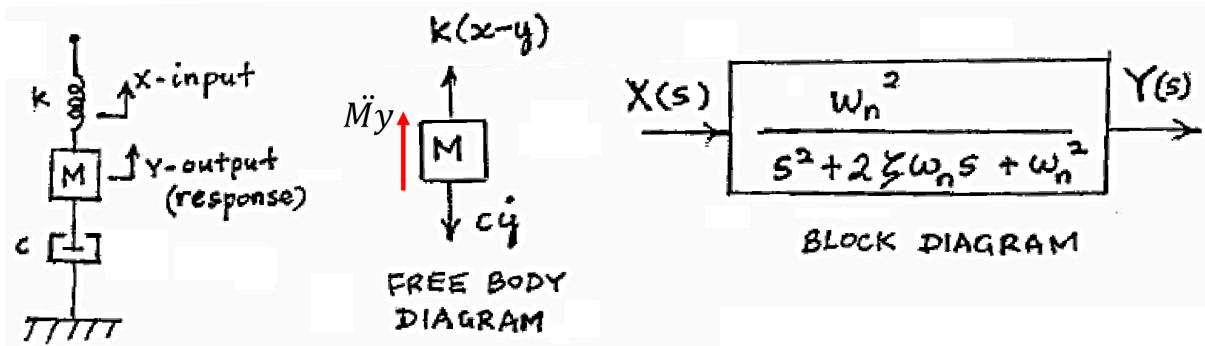


For a first-order system:

$$G(s) \rightarrow \frac{C}{R} = \frac{K}{1 + \tau s}$$



8.1 First order systems



At equilibrium:

$$c\dot{y} + ky = kx \quad (1)$$

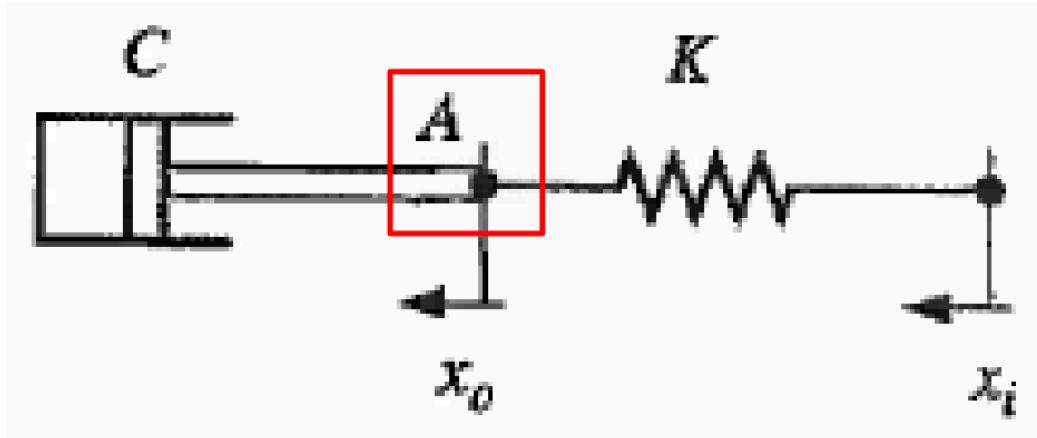
Laplace transform of (1) with zero initial conditions:

$$CSY(s) + KY(s) = KX(s)$$

$$\therefore \text{Transfer function : } G(s) = \frac{Y(s)}{X(s)} = \frac{k}{cs + k} = \frac{1}{1 + s\tau}$$

Where $\tau = \text{time constant} = \frac{C}{K}$.

8.1.1 Mechanical first-order systems



The input variable x_i and output variable x_o are related by the equation obtained by equating the forces at A :

$$K(x_i - x_o) = C\dot{x}_o$$

$$\frac{C}{K}\dot{x}_o + x_o = x_i$$

Taking the Laplace transform and assuming initial conditions to be zero:

$$X_o \left(s \frac{C}{K} + 1 \right) = X_i$$

$$\therefore \text{Transfer function : } \frac{X_o}{X_i} = \frac{1}{1 + \frac{C}{K}s} = \frac{1}{1 + \tau s}$$

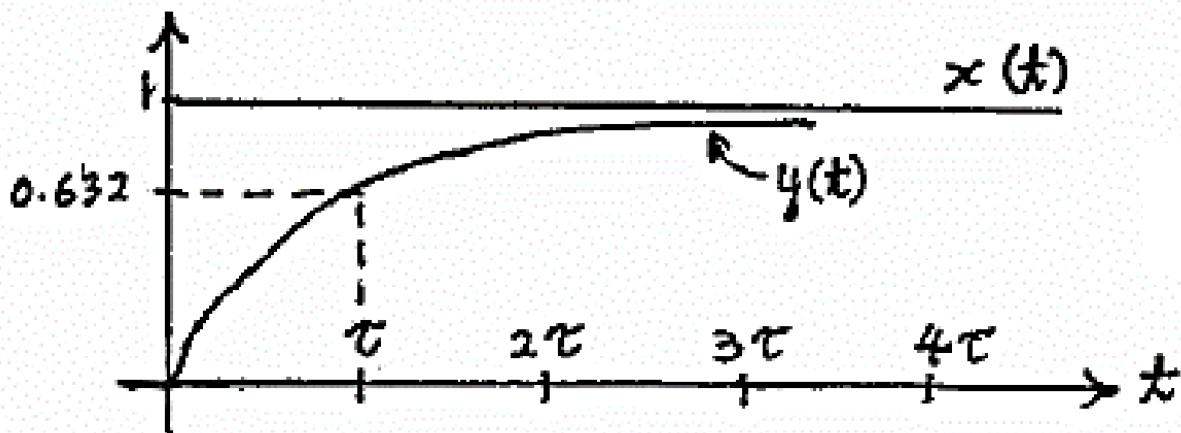
8.1.2 Unit step input

$$Y(s) = \frac{1}{1+s\tau} \cdot \frac{1}{s} = \frac{1}{s} - \frac{\tau}{1+\tau s} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = 1 - e^{-\frac{t}{\tau}}$$

Where:

- 1 is the steady state response.
- $e^{-\frac{t}{\tau}}$ is the transient response.



$$y(t) = \begin{cases} 1 - e^{-1} = 0.632 \\ 1 - e^{-2} = 0.865 \\ 1 - e^{-3} = 0.95 \\ 1 - e^{-4} = 0.982 \\ 1 - e^{-5} = 0.993 \\ \dots \end{cases}$$

The 2% settling criterion states that after 4 time constants, the output should be within 2% of the value at $t = \infty$.

The time constant is useful in studying the 1st order response system response time, and its units is in sec.

- Its value influences the response of the system.
- The smaller the time constant τ , the faster the system response.
- It is the negative reciprocal of the coefficient of t in exponential term e^{at} in transient responses, i.e.

$$a = -\frac{1}{\tau}$$

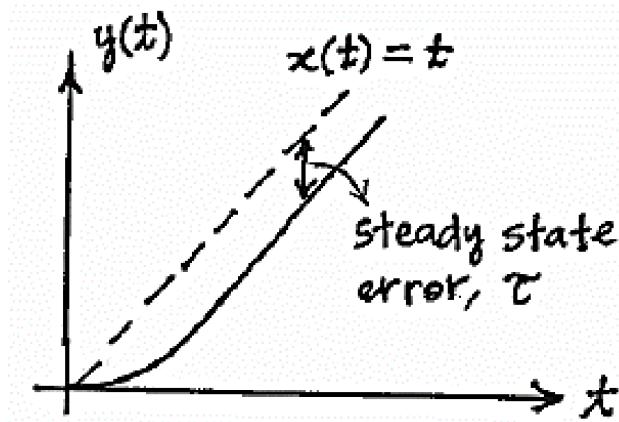
8.1.3 Unit ramp input

$$\begin{aligned}
 Y(s) &= \frac{1}{1+s\tau} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{1+\tau s} \\
 &= \frac{1}{s^2} - \tau \left(\frac{1}{s} \right) + \tau \left(\frac{1}{s + \frac{1}{\tau}} \right)
 \end{aligned}$$

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$$

Where:

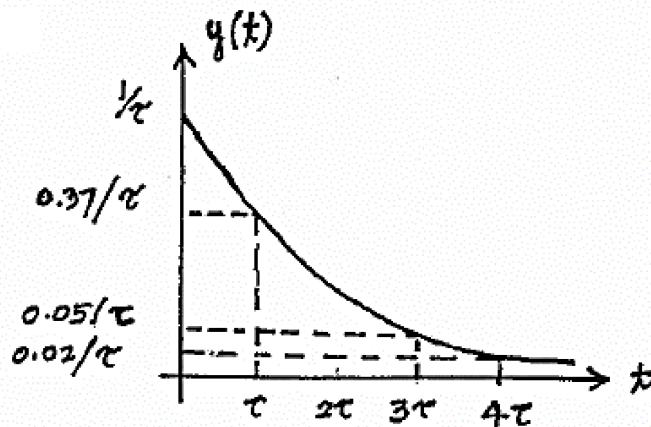
- $t - \tau$ is the steady state response.
- $\tau e^{-\frac{t}{\tau}}$ is the transient response, which decays to less than 2% of τ in time 4τ .



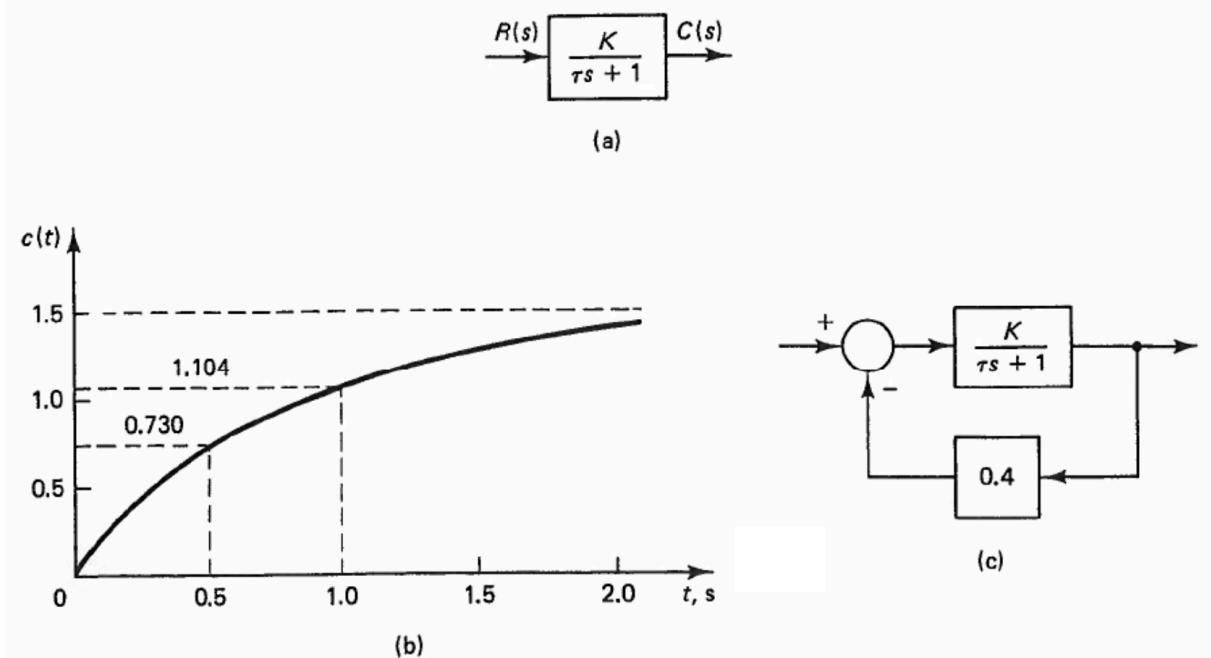
8.1.4 Unit impulse input

$$Y(s) = \frac{1}{1+s\tau} \cdot 1 = \frac{\frac{1}{\tau}}{s + \frac{1}{\tau}}$$

$$Y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}$$



8.1.5 Example 1



1. The first-order plant of Figure (a) has the unit step response given in Figure (b). Find the parameters of the transfer function.

$$c(t) = k \left(1 - e^{-\frac{t}{\tau}} \right)$$

$$c(0.5) = 0.730 = k \left(1 - e^{-\frac{0.5}{\tau}} \right)$$

$$c(1.0) = 1.104 = k \left(1 - e^{-\frac{1.0}{\tau}} \right)$$

$$\therefore \frac{1 - e^{-\frac{0.5}{\tau}}}{1 - e^{-\frac{1.0}{\tau}}} = \frac{0.730}{1.104}$$

Let $\gamma = e^{-\frac{0.5}{\tau}}$:

$$\therefore 1.104(1 - \gamma) = 0.730(1 - \gamma^2)$$

$$0.374 - 1.104\gamma + 0.73\gamma^2 = 0$$

$$\gamma = \frac{1.104 \pm \sqrt{1.21882 - 0.09208}}{2(0.73)} = 0.512$$

$$e^{-\frac{0.5}{\tau}} = 0.512$$

$$\tau = 0.746$$

$$k = \frac{0.730}{1 - 0.512} = 1.496$$

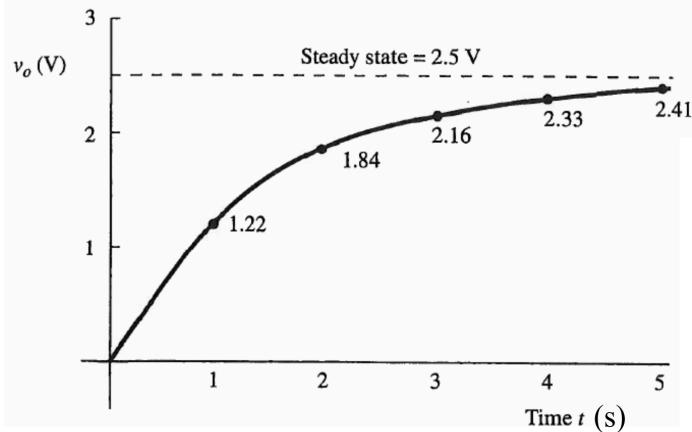
1. The plant is connected into the closed-loop system as shown in Figure (c). Sketch the unit step response of the closed-loop system.

$$\begin{aligned}
 T(s) &= \frac{\frac{K}{\tau s + 1}}{1 + 0.4 \frac{K}{\tau s + 1}} \\
 &= \frac{2}{s + 1.34 + 0.8} \\
 &= \frac{2}{s + 2.14} \\
 &= \frac{0.935}{0.467s + 1}
 \end{aligned}$$

$$\therefore k = 0.935, \quad \tau = 0.467s$$

8.1.6 Example 2

The graph below shows the experimentally obtained voltage output of an unknown system subjected to a step input of +10 V. Determine the transfer function of the system and locate its pole on the complex plane.



The system appears to be 1st order. Let it be where r and c are voltages. If r is a step of 10 V,

$$\lim_{t \rightarrow 0} c(t) = \lim_{s \rightarrow 0} sC(s) = \lim_{s \rightarrow 0} \frac{s(10a)}{s(1 + st)} = 10a = 2.5 \text{ V} \quad (\text{from graph})$$

Hence $a = 0.25$. Now consider T . The output should reach 63% of the steady state (1.575 V) in T seconds. From the graph, $T = 1.3 \text{ secs}$. So, we get:

$$\frac{C}{R} = \frac{0.25}{1 + 1.35}$$

$$C(t) = 2.5 \left(1 - e^{-\frac{t}{1.3}}\right) \text{ if } R = \frac{10}{s}$$

Comparing the results:

t	1	2	3	4
Predicted	1.34	1.96	2.25	2.38
Observed	1.22	1.84	2.16	2.33

8.1.7 General method

Find $c(t)$ when $r(t)$ is a step input with magnitude p .

$$r(t) = pu(t)$$

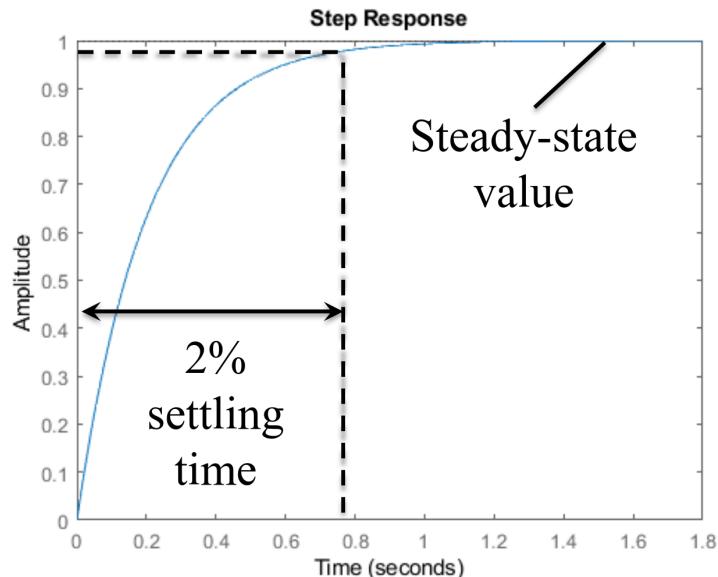
$$R(s) = \frac{p}{s}$$

$$C(s) = \left(\frac{p}{s}\right)\left(\frac{k}{\tau}s + 1\right) = \underbrace{\frac{A}{s}}_{\text{Steady-state output}} + \underbrace{\frac{B}{\tau s + 1}}_{\text{Transient output}}$$

Where A and B are residues.

$$\begin{aligned} C(s) &= \frac{A}{s} + B \\ &= \frac{pk}{s} - \frac{pk\tau}{\tau s + 1} \end{aligned}$$

$$c(t) = \underbrace{\frac{pk}{s}}_{\text{Steady-state output}} - \underbrace{pke^{-\frac{t}{\tau}}}_{\text{Transient output}}$$



When $p = k = 1$:

2% settling time : $t_s = 4\tau$

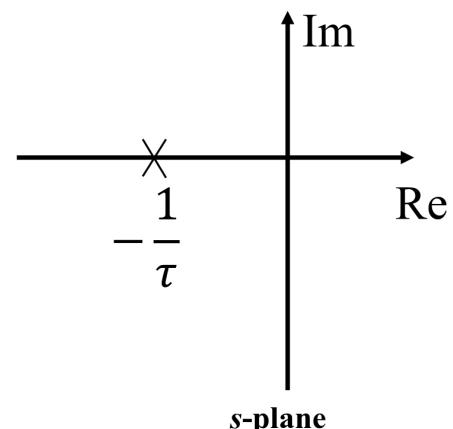
$$\text{DC gain of } G(s) = \lim_{s \rightarrow 0} \frac{k}{\tau s + 1} = k$$

$c_{ss} = pk$ (product of DC gain and magnitude of step)

$$\text{Poles of } G(s) : s = -\frac{1}{\tau}$$

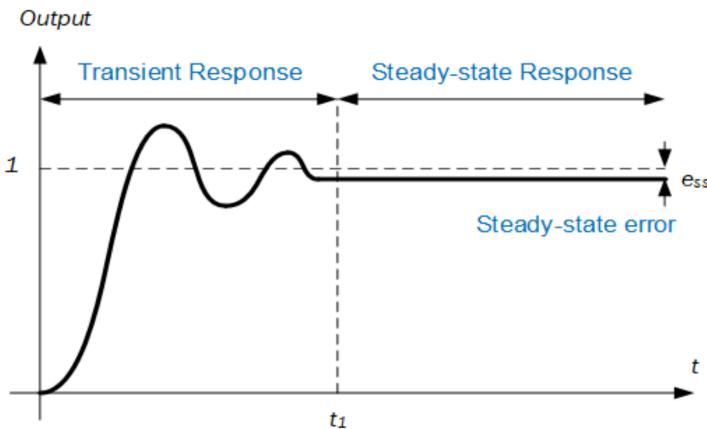
$$c_{tr}(t) = Ae^{-\frac{t}{\tau}}$$

$$c(t) = c_{ss} + c_{tr}$$



8.2 System response

- As systems are dynamical in nature, their performances are hard to measure or quantify.
- One accepted approach is to use a reference input to the system, and measure its performance in time domain analysis.
- Let $c(t)$ be the output of the system with input $r(t)$. The time response of a system consists of two parts, the transient and the steady state response.



- The responses can be written as:

$$c(t) = c_{tr}(t) + c_{ss}$$

Where:

- $c_{tr}(t)$ denotes the transient response
- $c_{ss}(t)$ denotes the steady state response

- The transient response is defined as the part of the time response that goes to zero as time becomes very large, or:

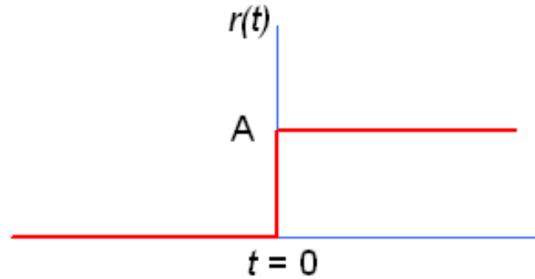
$$\lim_{t \rightarrow \infty} c_t(t) = 0$$

- The steady state response is the part of the total response that remains after the transient response has died out.
- All real control systems exhibit transient response to some extent before the steady state is reached, as inertia, mass, inductance, capacitance are unavoidable in physical systems. Because of these, the response of a typical control system cannot follow sudden changes in the input instantaneously and transients are usually observed.
- Hence, having a good transient response of a system is necessarily important, as it is a significant part of the dynamic behaviour of the system.
- It determines the deviation between the output response and the input (or the desired response) before the steady state is reached.
- The steady state response is also very important, as it determines where the system output response ends up at after a long time.
- If the steady state response of the output does not agree with the steady state of the input or desired response exactly, then a steady state error occurs.
- The design of a controller should include considerations of the transient and steady state performance specifications.

8.3 Test signals

8.3.1 Step input

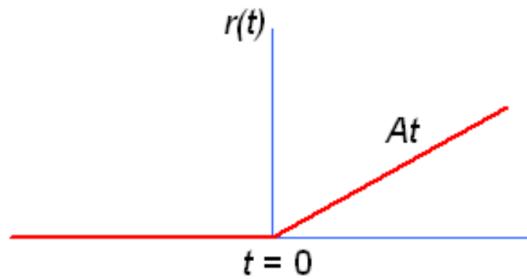
$$r(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- $R(s) = \frac{A}{s}$
- When $A = 1$, known as the unit step input, is very useful in studying changes in input signals.

8.3.2 Ramp input

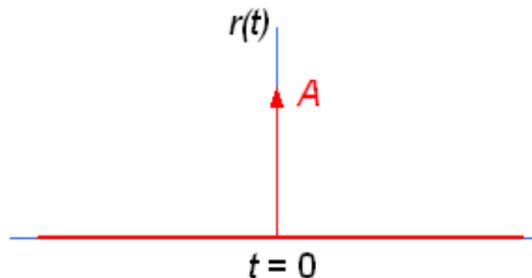
$$r(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- $R(s) = \frac{A}{s^2}$
- When $A = 1$, known as the unit ramp input, is useful in studying the system's ability to follow increasing input values.

8.3.3 Impulse input

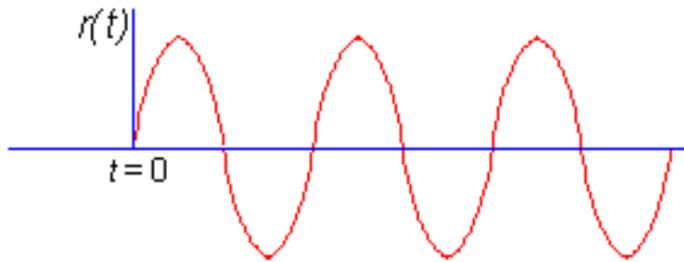
$$r(t) = A\delta$$



- $R(s) = A$
- When $A = 1$, known as the unit impulse input, is useful in studying sudden shock or impact and **system identification**.

8.3.4 Sinusoidal input

$$r(t) = \begin{cases} A \sin \omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- $R(s) = \frac{\omega}{s^2 + \omega^2}$
- When $A = 1$, known as the unit sinusoid input, is useful in studying the **frequency response** of a system.

8.4 Responses of first-order systems

For a first order system the transfer function is:

$$G(s) = \frac{1}{Ts + 1}$$

- The transfer function of a DC motor with armature voltage as input and angular velocity as output is:

$$\frac{\Omega(s)}{U(s)} = \frac{k_m}{\tau_m s + 1}$$

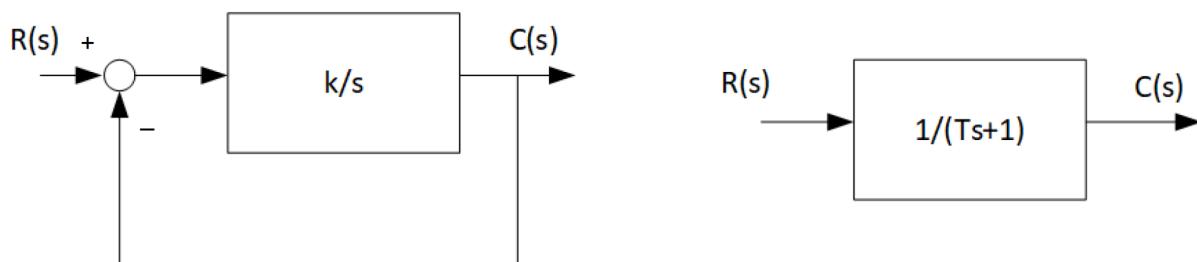
- The transfer function of a mechanical rotational system with torque as input and angular velocity as output:

$$\frac{\Omega(s)}{U(s)} = \frac{1}{Js + b}$$

- The transfer function for a thermal system:

$$\frac{\Theta_s(s)}{\Theta_e(s)} = \frac{R_1 + R_2}{R_1 + R_2 + CR_1R_2s}$$

In terms of block diagrams, the first-order systems can be represented in the following two forms:



Where $T, k > 0$.

8.4.1 Step input

$$R(s) = \frac{1}{s}$$

$$C(s) = \frac{1}{s(Ts + 1)} = \frac{1}{s} - \frac{T}{Ts + 1}$$

In time domain, it is:

$$c(t) = 1 - e^{-\frac{t}{T}}, \quad t \geq 0$$

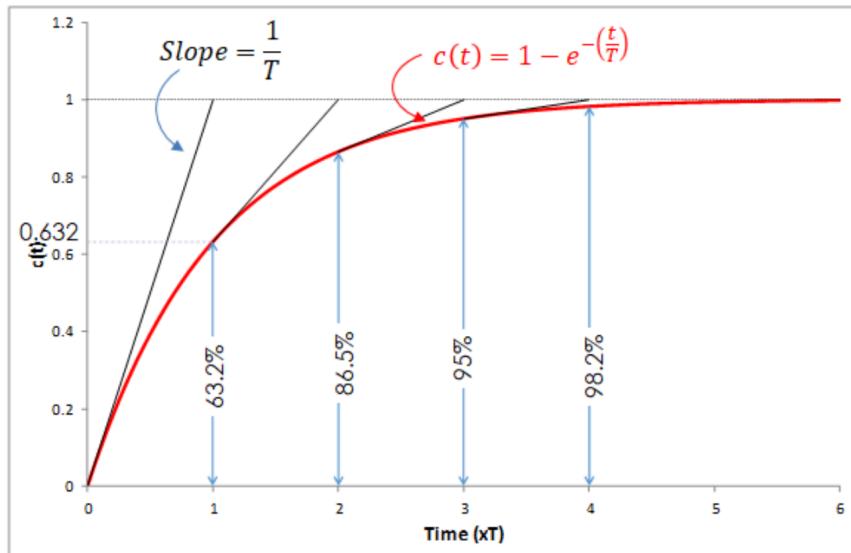
Note that:

$$c(0) = 1 - 1 = 0$$

$$c(T) = 1 - e^{-1} = 0.632$$

$$c(\infty) = 1$$

$$\left. \frac{dc(t)}{dt} \right|_{t=0} = \left. \frac{1}{T} e^{-\frac{t}{T}} \right|_{t=0} = \frac{1}{T}$$



- The smaller the T , the faster the response.
- For $t > 4T$, the response is within 2% of the final value.

8.4.2 Ramp input

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{1}{s^2(Ts + 1)} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts + 1}$$

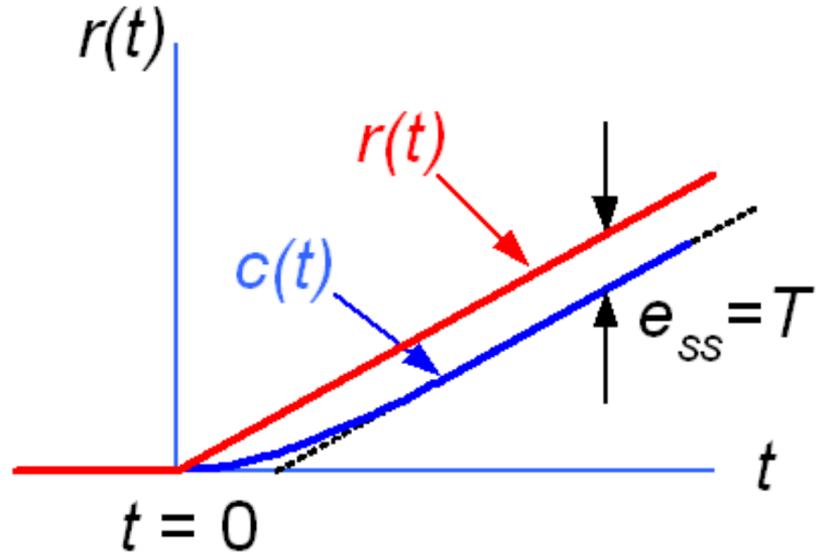
In time domain, it is:

$$c(t) = t - T + Te^{-\frac{t}{T}}, \quad t \geq 0$$

Note that:

$$c(0) = 0 - T + T = 0$$

$$c(\infty) = \infty$$



The error $e(t)$:

$$e(t) = r(t) - c(t) = T - Te^{-\frac{t}{T}} = T(1 - e^{-\frac{t}{T}})$$

The error at steady state is:

$$e(\infty) = T$$

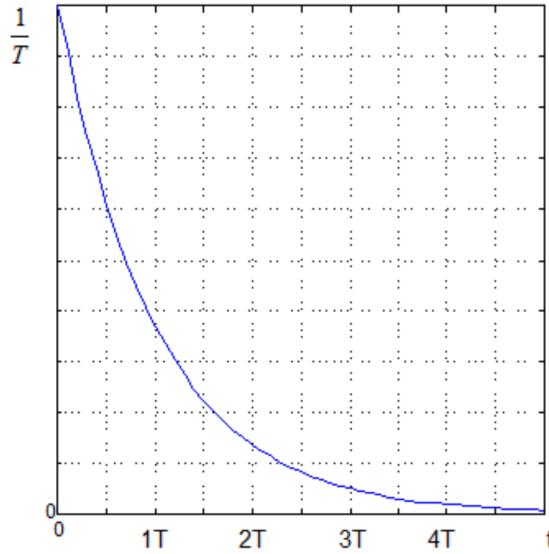
8.4.3 Impulse input

$$C(s) = \frac{1}{Ts + 1}$$

$$R(s) = \frac{1}{Ts + 1} \cdot 1$$

In time domain, it is:

$$c(t) = \frac{1}{T} e^{-\frac{t}{T}}, t \geq 0$$



8.4.4 Important property of the Laplace transform

- Suppose $c_1(t)$ is the output response corresponding to an input $r_1(t)$. Then if the input is changed to:

$$r_2(t) = \frac{dr_1(t)}{dt}$$

- Then the corresponding output $c_2(t)$ is given by:

$$c_2(t) = \frac{dc_1(t)}{dt}$$

- This property can be verified easily when placed in the Laplace domain. In this case, we have:

$$C_1(s) = G(s)R_1(s)$$

$$sC_1(s) = G(s)sR_1(s) = G(s)R_2(s)$$

The above implies that the output of the system with $R_2(s)$ as input is $sC_1(s)$. In time domain, this means output is $\frac{dc_1(t)}{dt}$ when input is $\frac{dr_1(t)}{dt}$.

- This fact manifests in the responses of first order system with ramp, step and impulse inputs. The step function is the derivative of the ramp function, and the impulse function is the derivative of the step input.

8.5 Responses of second-order systems

Second-order systems are represented as:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

They are represented by:

- The mechanical translational system with force as input and position as output:

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

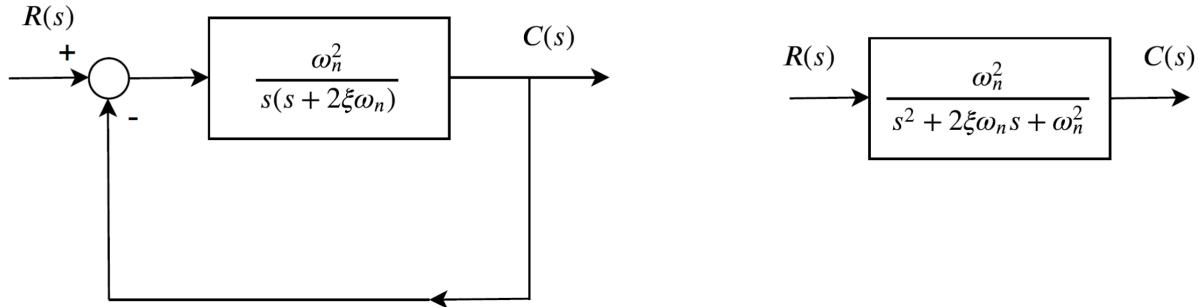
- The mechanical rotational system with torque as input and angular displacement as output:

$$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + bs + k}$$

- RLC circuit with voltage as input and capacitor voltage as output:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

In terms of block diagrams, a second-order system can be represented in the following two forms:



- There are two parameters that characterise the second order system:
 - ω_n , which is the natural frequency.
 - ζ , which is the damping ratio.
- In both block diagrams above, the transfer function between the input and the output is:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

8.5.1 Unit step input

$$R(s) = \frac{1}{s}$$

$$C(s) = G(s)R(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

- To obtain the steady state value, the final value theorem is used (assuming that the steady state exists):

$$\lim_{t \rightarrow \infty} c(t) = \lim_{s=0} sC(s) = \lim_{s=0} \frac{s\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = 1$$

- To obtain the expression of the transient response, expression of the poles of $G(s)$ is needed.
- The poles of $G(s)$ are the roots of the denominator polynomial, or the roots of:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

- Consider the roots of the above under different ranges of ζ .

1. $0 < \zeta < 1$ (underdamped case), the roots are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n \sqrt{1 - \zeta^2}$$

2. $\zeta = 0$ (undamped case), the roots are:

$$s_{1,2} = \pm j\omega_n$$

3. $\zeta = 1$ (critically damped case), the roots are:

$$s_{1,2} = -\omega_n$$

4. $\zeta > 1$ (overdamped case), the roots are:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

8.5.1.1 Underdamped case ($0 < \zeta < 1$)

The output is:

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_d^2} \cdot \frac{1}{s}$$

Where:

- $\omega_d = \omega_n \sqrt{1 - \zeta^2}$, known as the damped natural frequency.

Simplifying:

$$\begin{aligned} C(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \end{aligned}$$

Taking the inverse Laplace transform, we get:

$$\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t} \cos \omega_d t - e^{-\zeta\omega_n t} \frac{\zeta}{\sqrt{1 - \zeta^2}} \sin \omega_d t \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} (\zeta \sin \omega_d t + \sqrt{1 - \zeta^2} \cos \omega_d t) \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left(\omega_d t + \arctan \frac{\sqrt{1 - \zeta^2}}{\zeta} \right), \quad t \geq 0 \end{aligned}$$

8.5.1.2 Undamped case ($\zeta = 0$)

The output is

$$C(s) = \frac{\omega_n^2}{(s + j\omega_n)(s - j\omega_n)} \cdot \frac{1}{s} = \frac{\omega_n^2}{s^2 + \omega_n^2} \cdot \frac{1}{s} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$c(t) = 1 - \cos \omega_n t, \quad t \geq 0$$

8.5.1.3 Critically damped case ($\zeta = 1$)

$$C(s) = \frac{\omega_n^2}{(s + a)(s + b)} \cdot \frac{1}{s} = \frac{1}{s} + \frac{1}{a - b} \left(\frac{b}{s + a} - \frac{a}{s + b} \right)$$

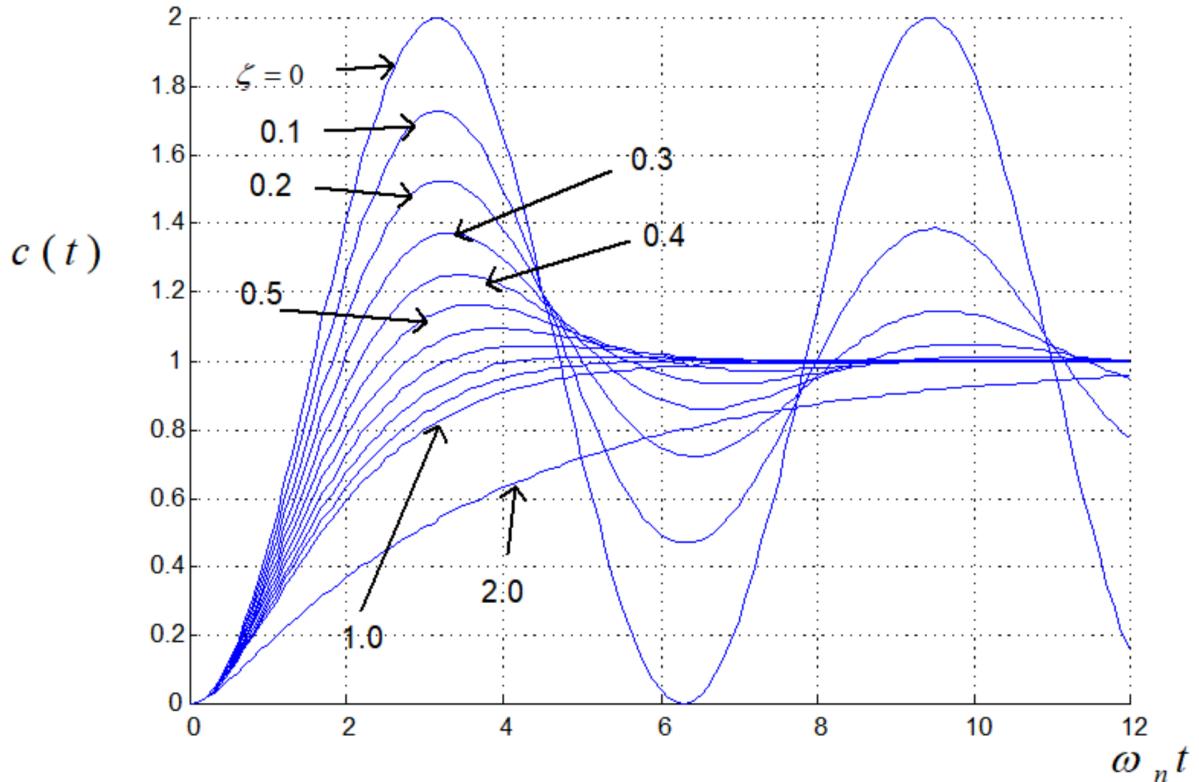
$$c(t) = 1 + \frac{1}{a - b} (be^{-at} - ae^{-bt}), \quad t \geq 0$$

Where:

- $a = \zeta\omega_n + \omega_n \sqrt{\zeta^2 - 1}$
- $b = \zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}$

8.5.1.4 Graph of the cases

Note that the x -axis is in units of $\omega_n t$.



Assuming that ω_n is fixed, a few notable points are:

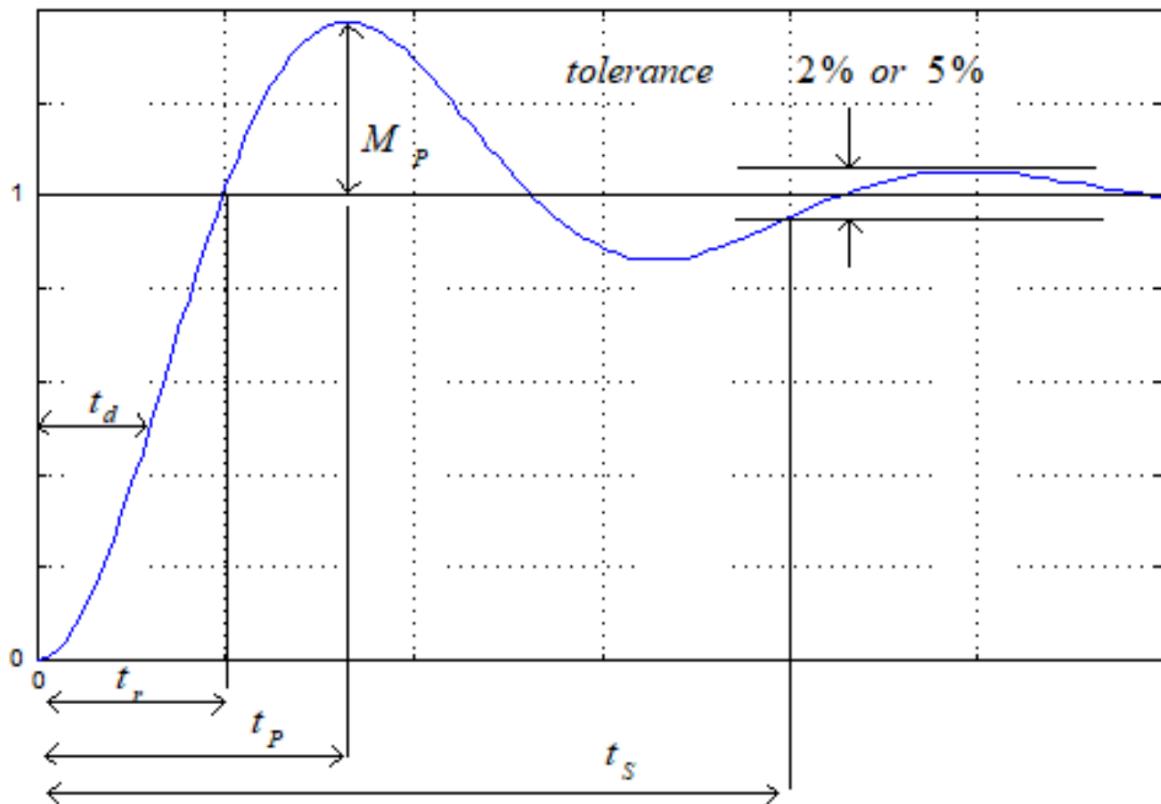
- Underdamped system with ζ between 0.5 and 0.8 gets close to the final value more rapidly than a critically damped or overdamped system.
- Among the systems responding without oscillation, a critically damped system exhibits the fastest response.
- An overdamped system is always sluggish in responding to any inputs.

8.6 Transient response

Systems with energy storage cannot respond instantaneously and will exhibit transient responses whenever they are subjected to inputs of disturbances.

- In many applications, the desired performance characteristics of control systems are specified in terms of time-domain quantities (both transient and steady state).
- As systems with energy storage cannot respond instantaneously to input demand, transient responses result.
- Frequently, the performance characteristics of a control system are specified in terms of transient response to a unit-step input since it is easy to generate and is sufficiently drastic.
- The transient response of a system to a unit-step input depends on the initial conditions. For purposes of comparison, a standard set of initial conditions are used.

8.6.1 Quantities in transient response



1. Delay time, t_d , is the time required for the response to reach half the final value for the very first time.
 2. Rise time, t_r , is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
 - For underdamped second-order systems, the 0% to 100% rise time is normally used.
 - For overdamped systems, the 10% to 90% rise time is commonly used.
 3. Peak time, t_p , is the time required for the response to reach the first peak of the overshoot.
 4. Maximum overshoot, M_p , is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot, which is defined by:
- $$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$
5. Settling time, t_s , which is the time required for the response curve to reach and stay within a range about the final value. The size of this range is specified by the absolute percentage of the final value, usually 2% or 5%. The settling time is related to the largest time constant of the control system. Which percentage error criterion to use is usually determined from the objectives of the system design in question.

8.6.2 Comments on transient response

- The time-domain specifications given are quite important since the responses in time are what are observed.
- Thus, the transient response should be satisfactory. Note that if values of t_d , t_r , t_p , M_p and t_s are specified, the shape of the response curve is virtually determined.
- Clearly, not all quantities are applicable for any given case. For example, t_p and M_p are not defined for an overdamped system, and t_s may not be reached if systems have steady-state errors to step input, etc.
- Save for special applications where oscillations cannot be tolerated, the transient response should be sufficiently fast and lightly damped (minimal overshoot). In the case of a second-order system, this means ζ is between 0.5 and 0.8. Small values of damping ratio (less than 0.4) yield excessive overshoot and large values of damping ratio (greater than 1), yields sluggish response.

8.6.3 Example: Underdamped second order system

$$\frac{X(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta < 1$$

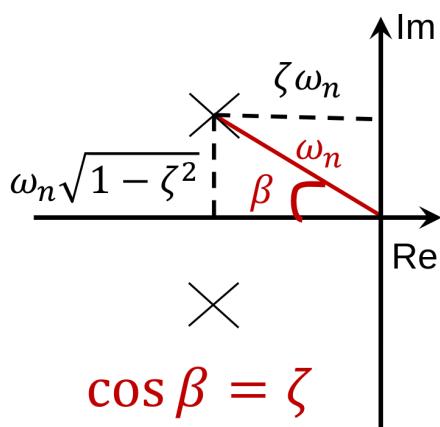
Poles:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$$

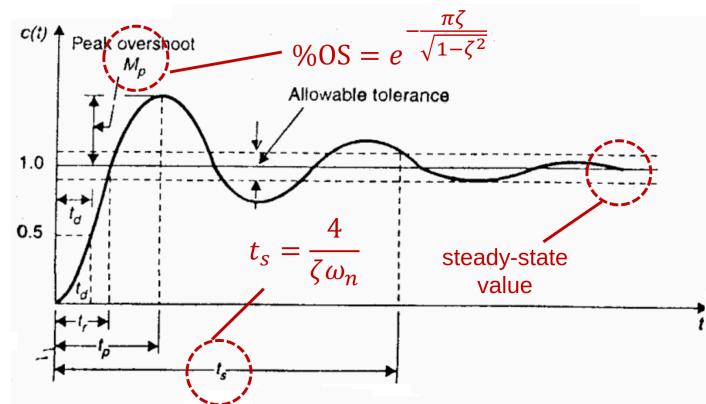
Transient response:

$$Ae^{-\zeta\omega_n t} \cos(\omega_n\sqrt{1 - \zeta^2}t + \phi)$$

S-plane:



Step response:



8.6.4 Transient quantities for second-order systems

8.6.4.1 Rise time (t_r)

In the special case of a standard second-order system with $0 \leq \zeta \leq 1$, the aforementioned quantities can be connected to the system parameters.

$$\begin{aligned} c(t) &= 1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t \right) \\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \left(\omega_d t + \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} \right) \end{aligned}$$

Note that t_r is defined by the smallest solution to $c(t_r) = 1$. Hence:

$$1 = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin \left(\omega_d t_r + \arctan \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

This implies that:

$$\sin(\omega_d t_r + \phi) = 0 \quad \because \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \neq 0$$

Simplifying:

$$\omega_d t_r + \phi = 0, \pi, 2\pi, \dots$$

For the first $t_r > 0$:

$$t_r = \frac{\pi - \phi}{\omega_d} = \frac{\pi - \arctan \frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_d}$$

8.6.4.2 Peak time (t_p)

Peak time t_p is defined by $\frac{dc(t)}{dt} \Big|_{t_p} = 0$.

$$\frac{dc(t)}{dt} \Big|_{t=t_p} = \frac{\omega_n e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} (-\zeta \sin(\omega_d t_p + \phi) + \sqrt{1-\zeta^2} \cos(\omega_d t_p + \phi)) = 0$$

This implies that:

$$\zeta \sin(\omega_d t_p + \phi) = \sqrt{1-\zeta^2} \cos(\omega_d t_p + \phi)$$

$$\tan(\omega_d t_p + \phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\omega_d t_p + \phi = \phi + n\pi$$

$$\text{or } t_p = \frac{\pi}{\omega_d}, \frac{2\pi}{\omega_d}, \dots$$

8.6.4.3 Maximum overshoot (M_p)

The maximum overshoot occurs at the peak time (assuming that there is an overshoot).

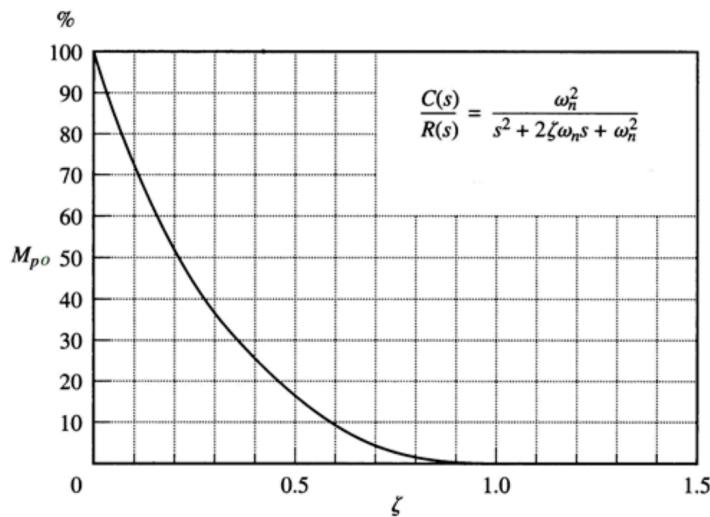
$$\begin{aligned} M_p &= c(t_p) - 1 \\ &= -e^{-\zeta \omega_n t_p} \cos \omega_d t_p + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t_p \\ &= e^{-\zeta \omega_n (\frac{\pi}{\omega_d})} \left(\cos \pi + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \pi \right) \end{aligned}$$

This implies that:

$$M_p = e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}}$$

In terms of percentage:

$$e^{-\frac{\zeta \pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

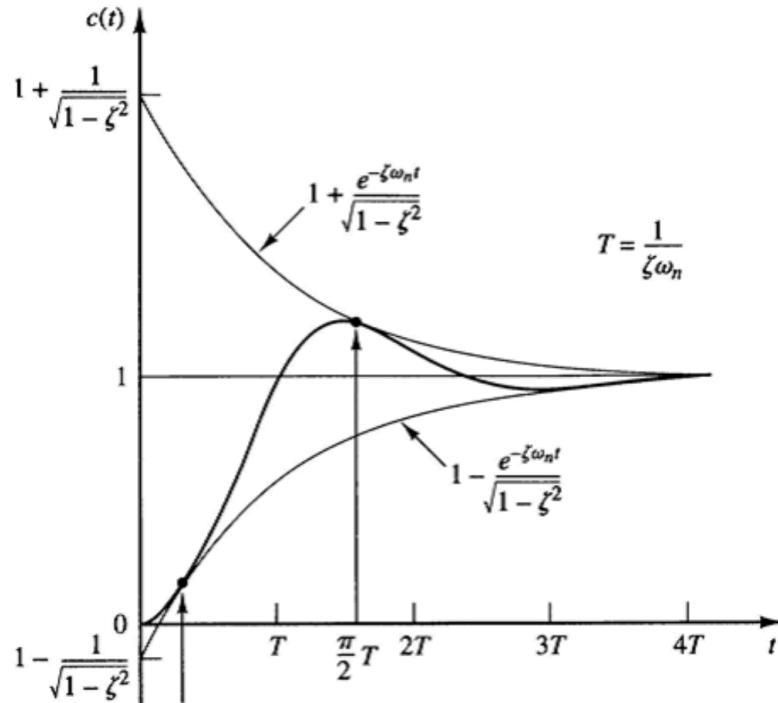


8.6.4.4 Settling time (t_s)

The settling time t_s can be obtained by considering the envelope of the response, $c(t)$

- Specifically, note that $1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$ contains the response of $c(t)$ for all values of time t .
- The speed of decay of the envelope curves depends on the value of the time constant $T = \frac{1}{\zeta\omega_n}$.
- Using the results of the first-order system, t_s for $\pm 2\%$ and $\pm 5\%$ tolerant band are approximated by:

$$t_s = 4T = \frac{4}{\zeta\omega_n}, \quad \text{and} \quad t_s = 3T - \frac{3}{\zeta\omega_n}$$



- Note that the settling time is inversely proportional to the product of the damping ratio and the undamped natural frequency of the system.
- Since the value of ζ is usually determined from the requirement of permissible maximum overshoot, t_s can be determined by ω_n .
- This means that the duration of the transient period may be varied, without changing the maximum overshoot.
- ω_n must be large for rapid response.
- To limit the maximum overshoot and to make the settling time small, the damping ratio ζ should not be too small.
- Note that if ζ is between 0.4 and 0.8, then the maximum percent overshoot for the step response is between 25% and 2.5%.

8.6.4.5 Summary

Consider the unit-step response of the second-order system defined by:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where:

- ω_n is the undamped natural frequency
- ζ is the damping ratio

Damped natural frequency, ω_d :

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Let:

$$\phi = \arctan \frac{\omega_d}{\zeta\omega_n}$$

Rise time, t_r :

$$t_r = \frac{\pi - \phi}{\omega_d}$$

Peak time, t_p :

$$t_p = \frac{\pi}{\omega_d}$$

Maximum overshoot, M_p :

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

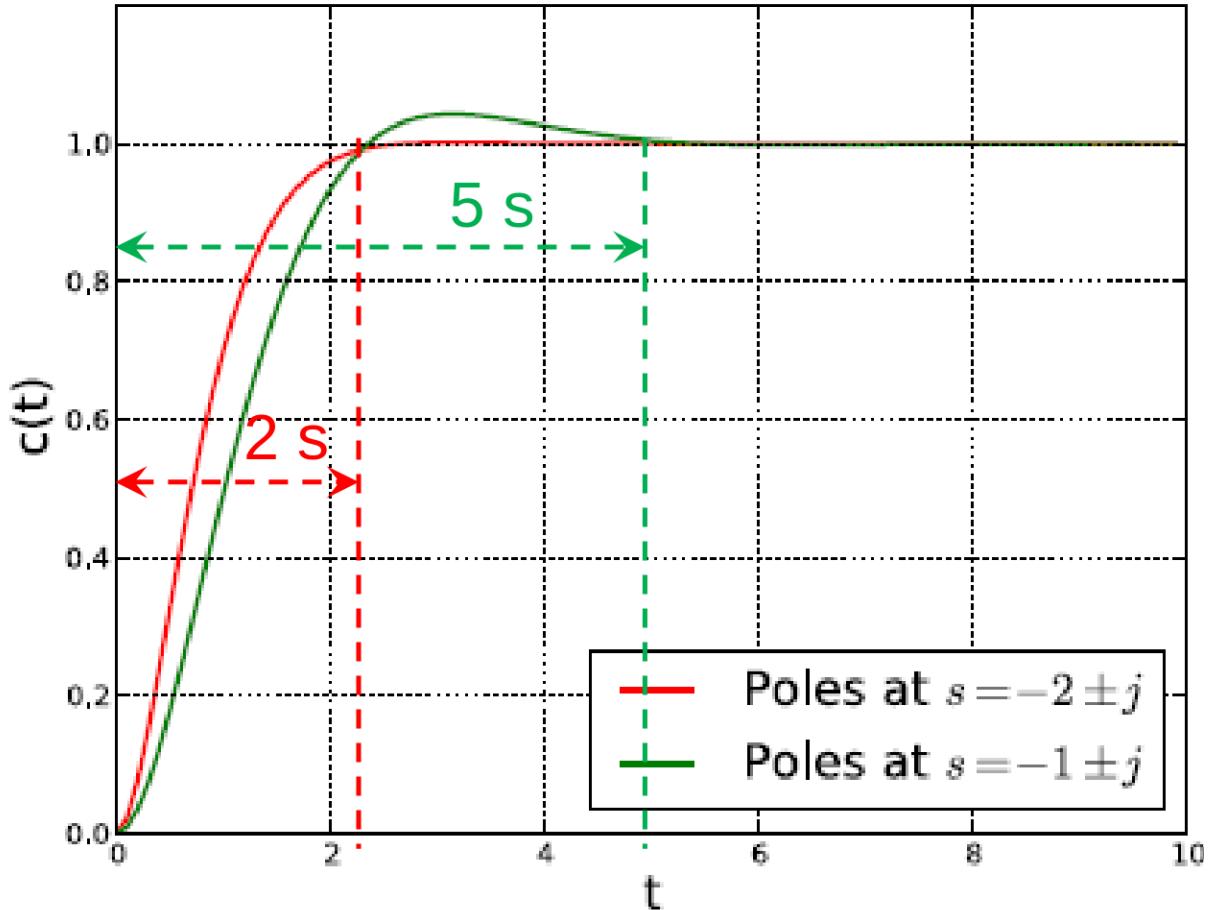
Settling time, t_s :

$$t_s = 3T = \frac{3}{\zeta\omega_n} \quad \text{for } 5\% \text{ criterion}$$

$$t_s = 4T = \frac{4}{\zeta\omega_n} \quad \text{for } 2\% \text{ criterion}$$

8.7 Dominant poles

Dominant poles are the rightmost poles (slowest).



System 1 has two poles:

$$s = -2 \pm j$$

$$\text{Transient response : } Ae^{-2t} = \cos(t + \phi)$$

System 2 has two poles:

$$s = -1 \pm j$$

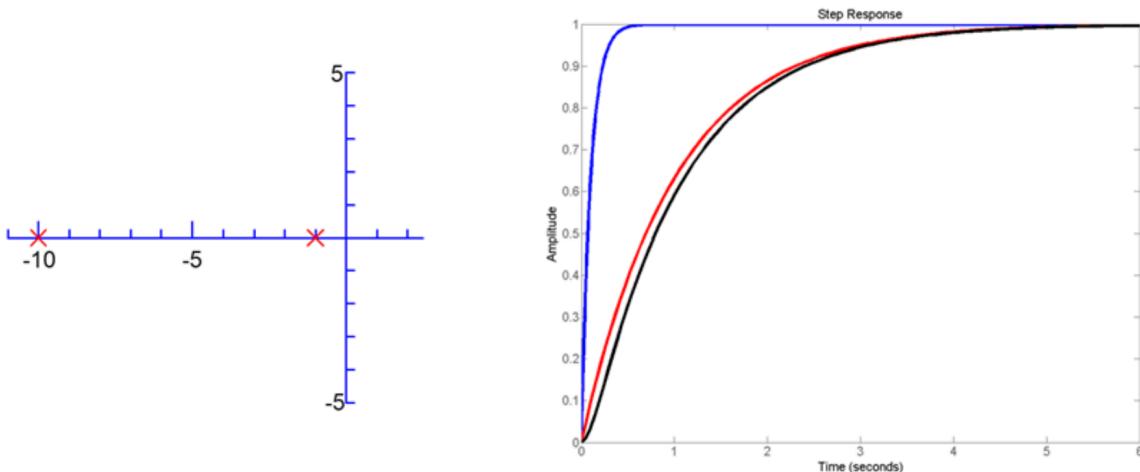
$$\text{Transient response : } Ae^{-t} = \cos(t + \phi)$$

System 2 is slower.

8.8 Responses of higher-order systems

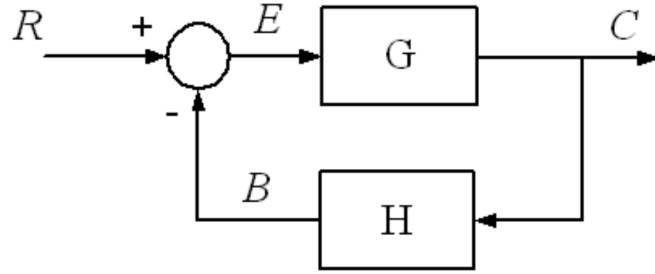
- Higher order systems can be decomposed into multiple first and second order systems.
- Expect some deviations from the first and second order systems.
- The location of the poles of the transfer function in the s -plane has great effects on the transient response of the system. So it is important to sort out the poles that have dominant effect on the transient response, known as the dominant poles.
- The poles that are close to the imaginary axis in the left half of the s -plane give rise to transient responses that will decay relatively slowly, whereas the poles that are far away from the imaginary axis (relative to the dominant poles) correspond to fast decaying time response.
- To see this effect, take

$$G(s) = \frac{10}{(s + 1)(s + 10)}$$



- Hence, the transient response will be dominated by the dominant poles, which are also known as the slowest poles.
- The steady-state value is equal to the product of the DC gain of the transfer function and the magnitude of the input step.

9 Steady state analysis of systems



- Consider the feedback control system shown above where the loop transfer function $G(s)H(s)$ is written as:

$$G(s)H(s) = \frac{K(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)}$$

- The system is called type 0, type 1, type 2, type 3, etc. If $N = 0, N = 1, N = 2, N = 3$, etc.
- Note that the error is for a closed-loop system type defined by the open loop transfer function or the loop transfer function.

9.1 Error analysis

- Note that:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \text{ and } \frac{E(s)}{C(s)} = \frac{1}{G(s)}$$

- Hence, the error signal is given by:

$$E(s) = \frac{R(s)}{1 + G(s)H(s)}$$

- The steady state error can be found using the final-value theorem:

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s=0} sE(s) = \lim_{s=0} \frac{sR(s)}{1 + G(s)H(s)}$$

- The final value theorem can be applied only if all the roots of the polynomial in the denominator of $\frac{sR(s)}{1 + G(s)H(s)}$ have negative real parts.

9.1.1 Step input ($R(s) = \frac{1}{s}$)

- Consider the case where $R(s) = \frac{1}{s}$, or $r(t) = 1$ for $t \geq 0$. Then:

$$e_{ss} = \lim_{s=0} sE(s) = \lim_{s=0} \frac{s^{\frac{1}{s}}}{1 + G(s)H(s)} = \frac{1}{1 + G(0)H(0)} = \frac{1}{1 + K_p}$$

Where $K_p := \lim_{s=0} G(s)H(s) = G(0)H(0)$ is the step-error constant.

- For type 0 systems:

$$K_p = \lim_{s=0} \frac{K(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^0(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)} = K$$

- For type 1 systems or higher ($N \geq 1$):

$$K_p = \lim_{s=0} \frac{K(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)} = \infty$$

- In summary, for step input:

$$e_{ss} = \frac{1}{1 + K_p}, \quad \text{for type 0 systems}$$

$$e_{ss} = 0, \quad \text{for type 1 systems or higher}$$

9.1.2 Ramp input ($R(s) = \frac{1}{s^2}$)

- Consider the case where $R(s) = \frac{1}{s^2}$, or $r(t) = t$ for $t \geq 0$. Then:

$$e_{ss} = \lim_{s=0} sE(s) = \lim_{s=0} \frac{s^{\frac{1}{s^2}}}{1 + G(s)H(s)} = \frac{1}{s + sG(0)H(0)} = \frac{1}{K_v}$$

Where $K_v := \lim_{s=0} sG(s)H(s)$ is the ramp-error constant.

- For type 0 systems:

$$K_v = \lim_{s=0} \frac{sK(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^0(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)} = 0 \leq \frac{1}{s+1}$$

- For type 1 systems:

$$K_v = \lim_{s=0} \frac{sK(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^1(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)} = K$$

- For type 2 systems or higher ($N \geq 2$):

$$K_v = \lim_{s=0} \frac{sK(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)} = \infty$$

- In summary, for ramp input:

$$e_{ss} = \infty, \quad \text{for type 0 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 1 systems}$$

$$e_{ss} = 0, \quad \text{for type 2 systems or higher}$$

9.1.3 Parabolic input ($R(s) = \frac{1}{s^3}$)

- Consider the case where $R(s) = \frac{1}{s^3}$, or $r(t) = 1$ for $t \geq 0$. Then:

$$e_{ss} = \lim_{s=0} sE(s) = \lim_{s=0} \frac{s \frac{1}{s^3}}{1 + G(s)H(s)} = \frac{1}{s^2 + s^2 G(0)H(0)} = \frac{1}{K_a}$$

Where $K_a := \lim_{s=0} s^2 G(s)H(s)$ is the parabolic-error constant.

- For type 0 and type 1 systems ($N = 0, 1$):

$$K_a = \lim_{s=0} \frac{s^2 K(\tau_a s + 1)(\tau_b s + 1) \dots (\tau_m s + 1)}{s^{\textcolor{red}{N}} (\tau_1 s + 1)(\tau_2 s + 1) \dots (\tau_p s + 1)} = \textcolor{red}{0}$$

- For type 2 systems:

$$K_a = \lim_{s=0} \frac{s^2 K(\tau_a s + 1)(\tau_b s + 1) \dots (\tau_m s + 1)}{s^{\textcolor{red}{2}} (\tau_1 s + 1)(\tau_2 s + 1) \dots (\tau_p s + 1)} = \textcolor{red}{K}$$

- For type 3 systems or higher ($N \geq 3$):

$$K_a = \lim_{s=0} \frac{s^2 K(\tau_a s + 1)(\tau_b s + 1) \dots (\tau_m s + 1)}{s^{\textcolor{red}{N}} (\tau_1 s + 1)(\tau_2 s + 1) \dots (\tau_p s + 1)} = \infty$$

- In summary, for ramp input:

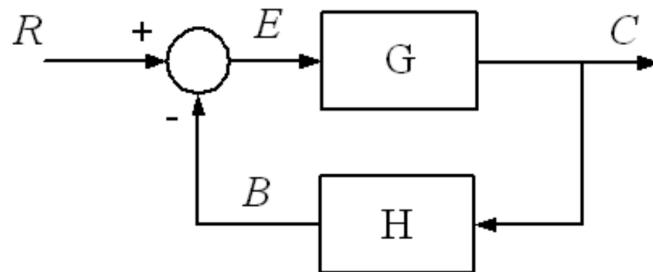
$$e_{ss} = \infty, \quad \text{for type 0 and type 1 systems}$$

$$e_{ss} = \frac{1}{K}, \quad \text{for type 2 systems}$$

$$e_{ss} = 0, \quad \text{for type 3 systems or higher}$$

9.1.4 Summary

Given a closed-loop feedback system:



The above system has an open loop transfer function given by:

$$G(s)H(s) = \frac{K(\tau_a s + 1)(\tau_b s + 1)\dots(\tau_m s + 1)}{s^N(\tau_1 s + 1)(\tau_2 s + 1)\dots(\tau_p s + 1)}$$

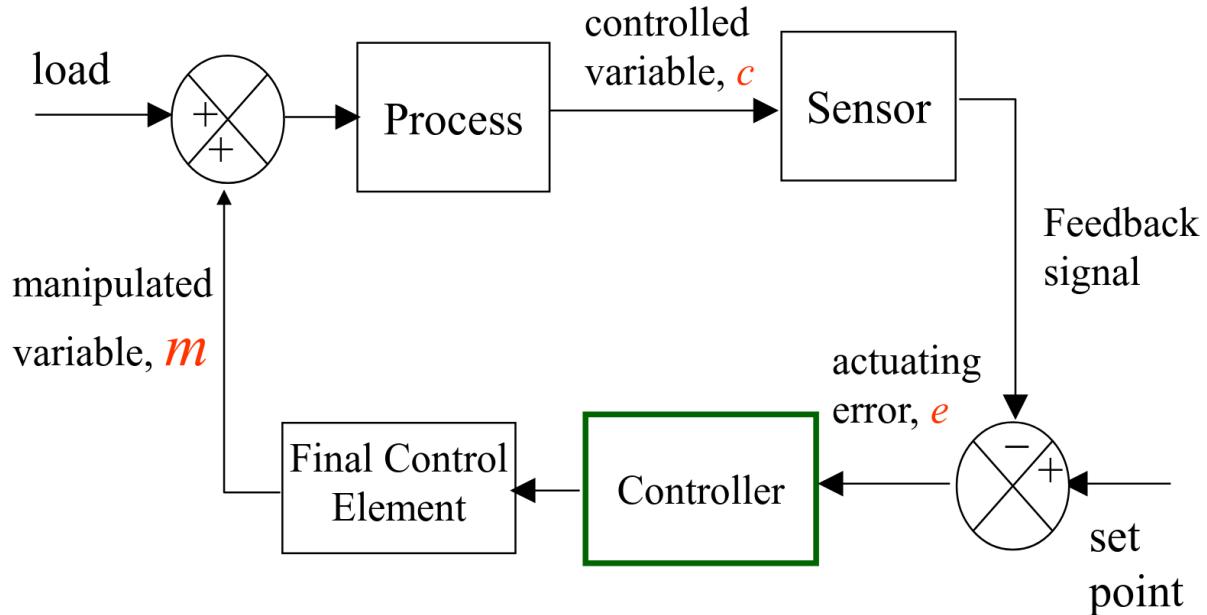
The error table for different system types and input functions are:

Type of system	Step input ($\frac{a}{s}$)	Ramp input ($\frac{a}{s^2}$)	Parabolic input ($\frac{a}{s^3}$)
Type 0	$\frac{1}{1+K}$	∞	∞
Type 1	0	$\frac{1}{K}$	∞
Type 2	0	0	$\frac{1}{K}$
Type 3	0	0	0

10 Process controllers

- Single loop process controllers:
 - ▶ On-off
 - ▶ Proportional
 - ▶ Integral (also called reset and anti-reset windup)
 - ▶ Derivative (also called rate)
- Traditionally, they are analogue and pneumatic but today, most of them would be digital.
- PLC stands for programmable logic controllers.
- DCS stands for distributed control systems.
- PCS stands for process automation systems, which combine distributed control systems (DCS) and programmable logic controllers (PLC).

10.1 Control loops



- Process, which refers to the system, machine or plant under control.
- Sensors or transducers.
- Controller, e.g. on or off, analogue or digital.
- Final control elements or actuators, like motors and control valves.
 - Pumps, conveyors and blowers, which manipulate the flow of material and energy into and out of the process.
 - Automatic control valves, which adjust the flow of material and can also change the process pressure, level and temperature.
 - Propellers and thrusters, which manipulate the speed, position and orientation of ships and rigs.
 - Motors (AC, DC, stepper), which manipulate mechanical position and the speed of machines.

10.2 Instrumentation and control

All branches of engineering and science depend on instrumentation. It extends human sensing ability by measuring more accurately, rapidly, and over greater distances. The types of measurement systems are:

- Monitoring of process and operation.
- Control of process and operation.
- Experimental engineering analysis (theoretical and experimental).

10.3 Bridge circuits

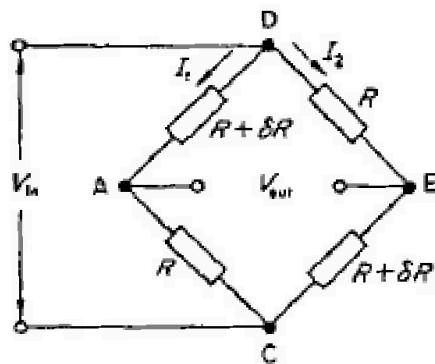
- Bridge circuits are commonly used as variable conversion elements in measuring systems and produce outputs in the form of voltage levels which changes as measured physical quantity changes in value.
- They provide an accurate method of measuring resistance, inductance and capacitance values.
- The bridge is excited by:
 - DC voltage for resistance measurements.
 - AC voltage for inductance and capacitance measurements.

10.3.1 Bridge types

- Null bridge type
 - Mainly employed for calibration purposes.
 - Balancing bridge by using unknown adjustments.
 - Null points measured with highly sensitive galvanometers.
- Deflection bridge type
 - Used with closed loop control systems.
 - Used to determine the magnitude of unbalance from the meter reading.
 - For static inputs, it uses a ordinary meter or galvanometer to measure deflection.
 - For dynamic signals, it uses an oscilloscope.
- An important factor in determining the type to use is the bridge sensitivity.
 - For a large change in resistance, use a series arrangement.
 - For a small change in resistance, use a shunt arrangement.

Types of electrical bridge circuits:

Bridge Type	Bridge Features
Voltage-sensitive bridge	Readout instrument does not “load” bridge. That means it requires no current, like an electronic voltmeter or CRO.
Current-sensitive bridge	Readout requires current, like a low-impedance indicator such as a simple galvanometer.
Null balance bridge	Adjustment is required to maintain balance. This becomes the source of the readout, like a manually adjusted strain indicator.
Deflection bridge	The readout is a deviation of the bridge output from the initial balance, e.g. as required by CRO.
AC bridge	Alternating current or voltage is used.
DC bridge	Direct current or voltage is used.
Constant voltage	Voltage input to bridge remains constant, like a battery or voltage-regulated power supply.
Constant current	Current input to bridge remains constant regardless of bridge unbalance, like a current-regulated power supply.
Resistance bridge	The bridge arms are made up of “pure” resistance elements.
Impedance bridge	The bridge arms may include reactance elements.



Unbalanced bridge with two variable arms

10.4 Amplifiers

- Increase low level signals from transducers, like thermocouples, strain gauge bridge outputs, to a level sufficient for recording with voltage measuring instruments.
- Linear integrated circuits, which are built miniature amplifier circuits which can be used directly in a number of applications, often with the addition of only a few passive external components (integrated circuit technology).
- In most applications of integrated circuits, a high gain amplifier forms the internal amplifier to which passive circuit elements are connected to provide an overall amplifier circuit. An example of this are operational amplifiers.
- Operational amplifiers are DC differential amplifiers with associated components that together operate upon a direct voltage or current.

10.4.1 Inverting amplifiers:

- Summing amplifiers, which are the basis of analogue to digital and digital to analogue converters.
- Integrating amplifiers which are used for low-pass filters.
- Differentiating amplifiers which are used for high-pass filters.

10.4.2 Differential amplifiers:

- These are also called instrumentation amplifiers.
- They are used to handle spurious noise, temporary drifts and power supply ripples.
- They are also used to handle math operations.

10.4.3 Non-inverting amplifier:

- They are used as unity gain buffer amplifiers to deal with high input impedance, also known as a voltage follower.
- They are also used to deal with the limitations of operational amplifiers, like input bias, offset current, offset voltage, common mode rejection ratio and slewing rate.

10.5 Data acquisition

10.5.1 Feedback type

The feedback type uses a closed loop arrangement.

1. Ramp type
 - These can only count up.
 - They are the simplest and can only convert one bit at a time.
2. Tracking type
 - These can count up and down.
 - They do not require sampling and holding (S/H).
3. Successive approximation
 - These require sampling and holding (S/H) to digitise rapidly changing analogue signals.
 - The conversion time depends on the number of bits (resolution).
 - For 8 bits, the time taken is $0.8 \mu\text{sec}$.
4. Simultaneous (parallel)
 - These use flash encoding, making them the fastest.
 - They also do not require sampling and holding (S/H).

10.5.2 Integrating type

- The integrating type gives repeatable results despite noise by averaging noise components.
- They have a longer conversion time, which is a disadvantage.
- However, they have higher accuracy, do not require sampling and holding (S/H), and are ideal for slow varying signals.
- Types:
 1. Dual slope, an old technique, is used for low-speed and low-cost applications.
 2. Quantised feedback, a new technique, uses the charge balancing technique.

10.6 Analogue to digital and digital to analogue conversion

The output from the transducer is usually analogue, which varies linearly and continuously with the quantity being measured. With larger and more complex instrumentation system, it is often useful to convert this analogue signals to digital signals so that it can be analysed or processed with computers or microprocessors.

10.6.1 Digital to analogue conversion

1. Binary weighted resistor network, which requires a very large range of resistance.
2. R-2R ladder network.

10.6.2 Analogue to digital conversion

1. Components consist of comparators, number of registers, control logic and digital to analogue converters.
2. Sampling (Shannon sampling theorem), which is an operation performed by a sample and hold device (S/H).
3. Quantisation
4. Encoding

10.6.3 Converter characteristics

The operating principles and philosophy of the analogue to digital converters have now been described without much emphasis on limiting constraints or possible errors.

10.6.3.1 Word length and resolution

The smallest quantum is one least significant bit, which quantified in voltage (or current) terms as:

$$1 \text{ LSB} = \frac{\text{Voltage range}}{2^n}$$

Where n is the converter word length.

The resolution of an analogue-to-digital converter with a word length of 8 bits and an analogue signal input range of 10 V is:

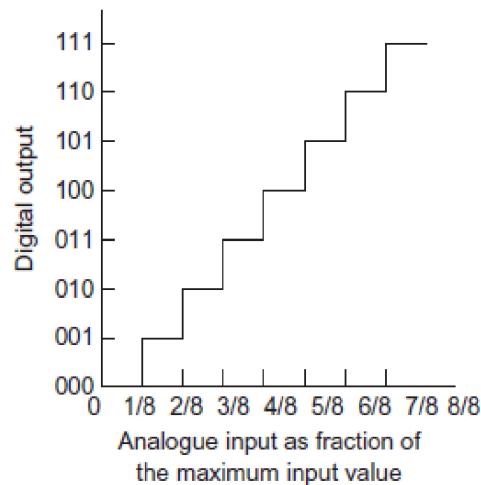
$$\text{Resolution} = \frac{10}{2^8} = 0.39 \text{ V} = 39 \text{ mV}$$

10.6.3.2 Full scale range (FSR)

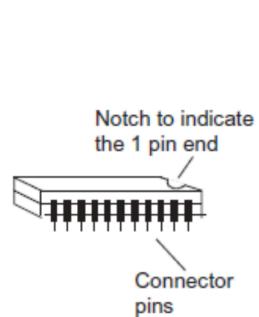
The full scale range (FSR) is the maximum positive value (it can be negative for a negative unipolar converter) the converter represents with all “ones” in its register. It is given by:

$$\text{FSR} = \text{Voltage range} \times \frac{2^n - 1}{2^n}$$

10.6.4 Analogue to digital converter

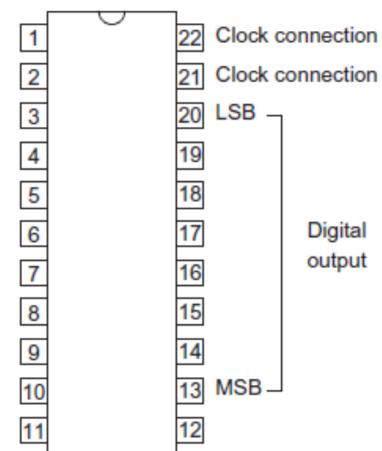


$2^{n-1}, \dots, 2^3, 2^2, 2^1, 2^0$
 ↑ ↑
 Most significant Least significant
 bit (MSB) bit (LSB)

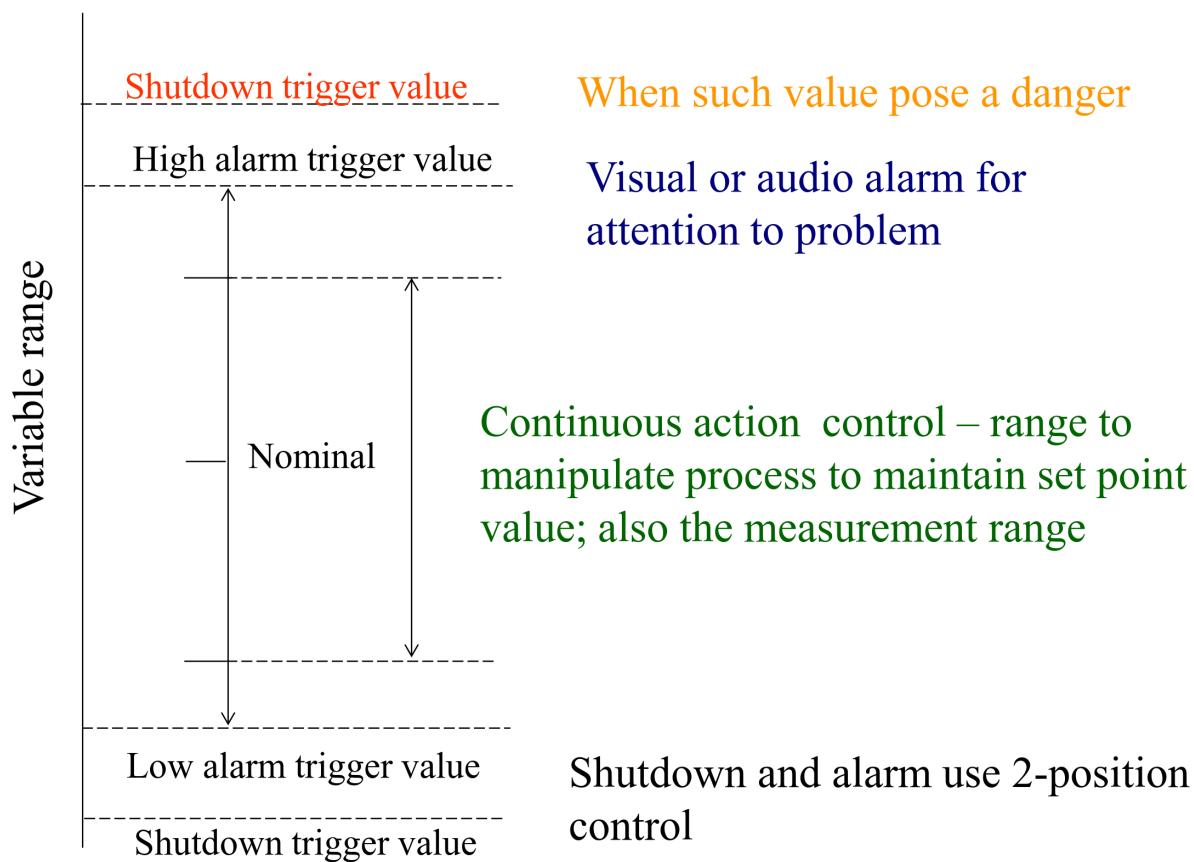


Connections for controlling the circuit

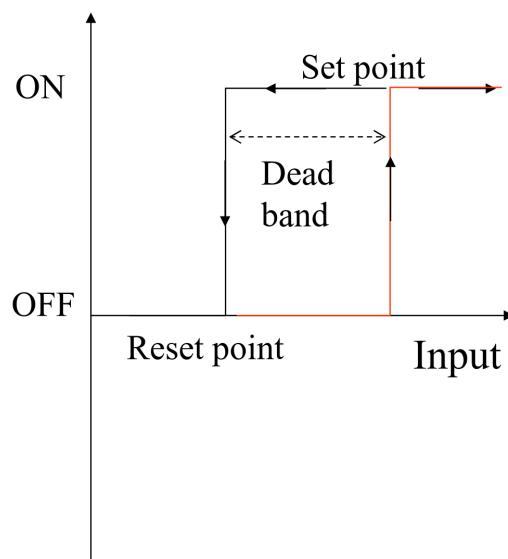
- Controls access
- Output enable
- Start conversion
- Status
- Common voltage
- Digital ground
- Analogue in
- Analogue ground
- Reference voltage trim
- Reference voltage out
- Reference voltage in



10.7 Controlled variable range



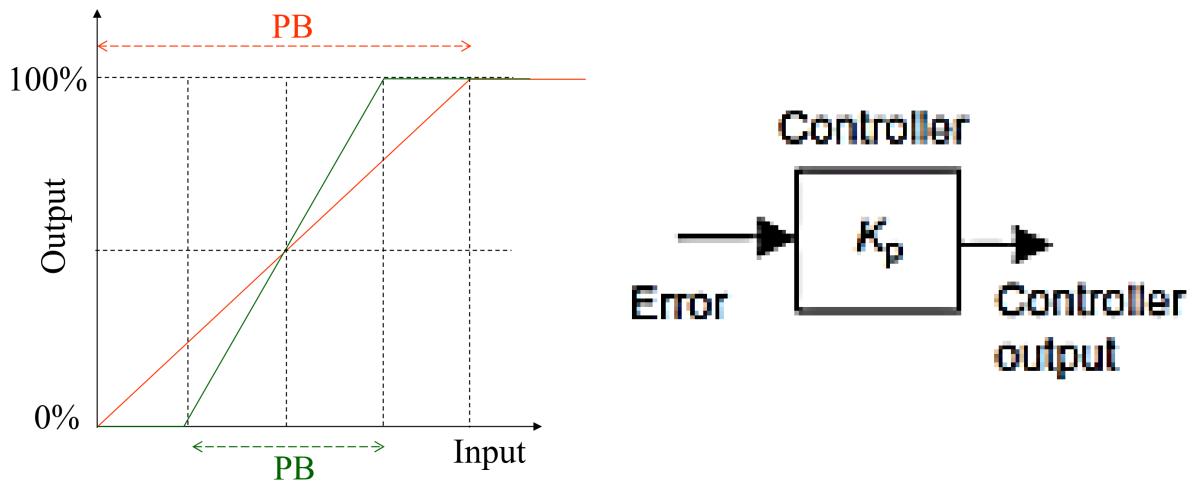
10.8 Two position action control



- On increasing input, at set point, the controller is activated.
- Controller is deactivated only when the input decreases to the reset point.
- Adjustable dead band, which is a narrow band that can cause system oscillation.
- An example of such a control system is the temperature control of a HVAC system.

10.9 Proportional control

The proportional band (PB), is the input range where proportional control is effective.



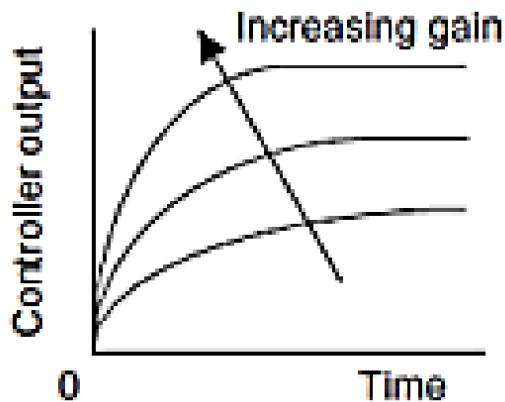
- With proportional control, the controller produces a control action that is proportional to the error.
- There is a constant gain K_p acting on the error signal e and so:

$$\text{Controller output} = K_p e$$

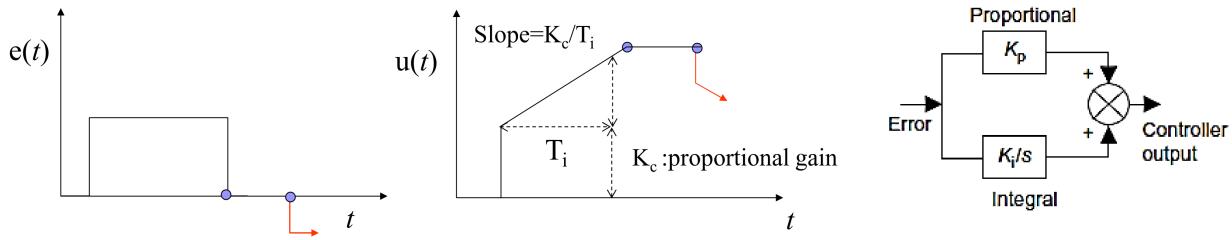
- Increasing the gain speeds up the response to changes in inputs, but does not eliminate the input offset.

$$PB = \frac{\text{Percentage change in input}}{\text{Percentage change in output}} \times 100\%$$

$$K_c = \frac{1}{PB}$$



10.10 Proportional with integral control



- With proportional with integral (PI) control, the proportional element is augmented with an additional element which gives an output proportional to the integral of the error with time.
- The proportional element has an input of the error e and an output of $K_p e$.
- The integral element has an input of e and an output which is proportional to the integral of the error with time, i.e.

$$K_i \int e \, dt$$

Where K_i is the integrating gain.

- Thus the controller output:

$$\text{Controller output } K_p e + K_i \int e \, dt$$

- In terms of the Laplace transform:

$$\text{Controller output } (s) = \left(K_p + \frac{K_i}{s} \right) E(s)$$

- This can be written as:

$$\text{Controller output } (s) = \frac{K_p}{s} \left(s + \frac{1}{T_i} \right) E(s)$$

Where $T_i = \frac{K_p}{K_i}$ and is called the integral time constant.

- The presence of integral control eliminates steady-state offsets and this is generally an important feature required in a control system.

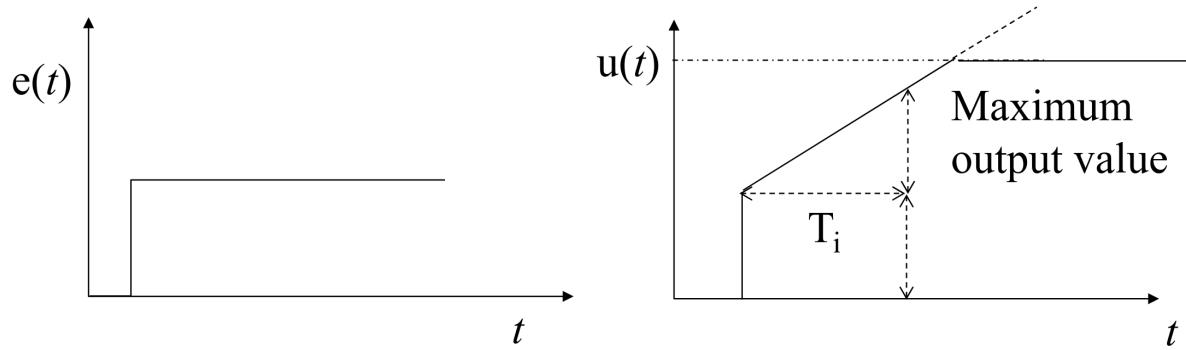
$$u(t) = K_c \left[e(t) + \frac{1}{T_i} \int_0^t e(t) \, dt \right]$$

It is used to eliminate steady state error:

- Proportional control has an immediate response.
- Integral control continues even when $e(t)$ becomes 0, because:

$$\dot{u}(t) \propto e(t)$$

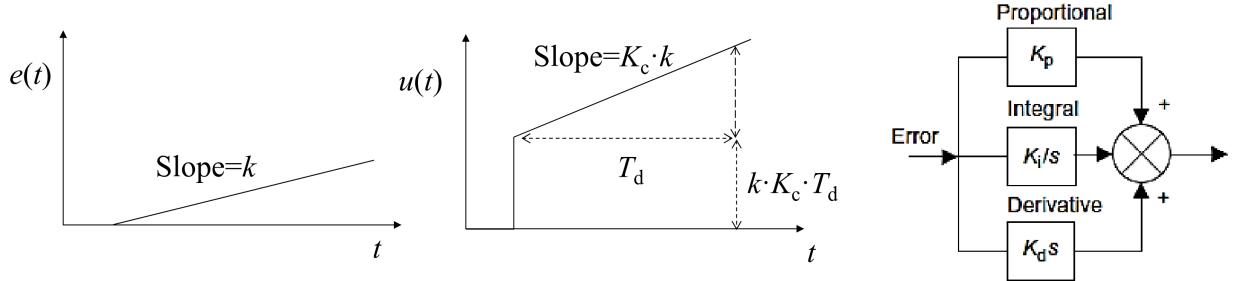
10.10.1 Reset windup problem with integral action



- $u(t)$ will drop only when it causes $e(t)$ to change sign.
- When $e(t)$ changes sign frequently, we get oscillations.
- If the error is sustained, $u(t)$ may reach maximum value before the error changes sign and remains at maximum.
- It can occur after large set-point change, when there is large sustained load disturbance or during start-up.

To overcome the problem, the integral output is usually prevented from reaching its maximum limit by turning off the integral action.

10.11 Proportional with derivative control



$$u(t) = K_c \left[e(t) + T_d \frac{de(t)}{dt} \right]$$

- With derivative control, the controller produces a control action that is proportional to the rate at which the error is changing.
- Derivative control is not used alone but always in conjunction with proportional control and, often, also integral control.
- The proportional element has an input of the error e and an output of $K_p e$.
- The derivative element has an input of e and an output which is proportional to the derivative of the error with time, i.e.

$$K_d \frac{de}{dt}$$

Where K_d is the derivative gain.

- Thus the controller output is:

$$\text{Controller output} = K_p e + K_d \frac{de}{dt}$$

- In terms of the Laplace transform, we have:

$$\text{Controller output } (s) = (K_p + K_d s) E(s)$$

- This can be written as:

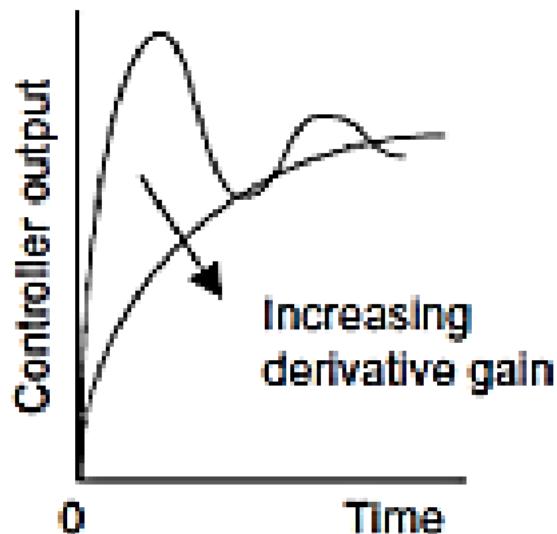
$$\text{Controller output } (s) = K_p (1 + T_d s) E(s)$$

Where $T_d = \frac{K_d}{K_p}$ and is called the derivative time constant.

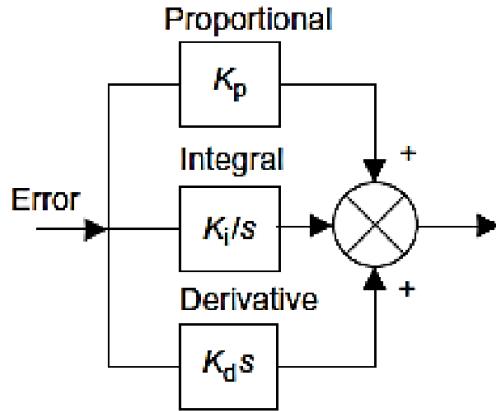
- Derivative control has the controller effectively anticipating the way an error signal is growing and responding as the error signal begins to change.
- A problem with this is that noise can lead to quite large responses.
- Adding derivative control to proportional control still leaves the output steady-state offset.
- Changing the amount of derivative control in a closed-loop system will change the damping ratio since increasing K_d increases the damping ratio.
- The proportional gain follows $e(t)$.
- The derivative control has immediate response as:

$$u(t) \propto \dot{e}(t)$$

- The graph to the right shows that increasing the derivative gain increases the damping ratio, for a step input.



10.12 PID controller



- PID stands for proportional with integral and derivative control.
- A PID controller is also known as a three-term controller.
- The controller output is:

$$\text{Controller output} = K_p e + K_i \int e dt + K_d \frac{de}{dt}$$

- Taking the Laplace transform gives:

$$\text{Controller output } (s) = K_p \left(1 + \frac{K_i}{K_p s} + \frac{K_d}{K_p} s \right) E(s)$$

- And so:

$$\text{Controller output } (s) = K_p \left(1 + \frac{1}{T_i s} + T_d s \right) E(s)$$

- Processes and equipment are usually maintained at nominal or steady state value that is not zero.
- Error is the deviation of controlled variable from the set point.

$$u(t) = K_c \left[e(t) + \frac{1}{T_i} \int_0^t e(t) dt + T_d \frac{de(t)}{dt} \right] + \bar{u}$$

Where:

- \bar{u} is the bias value, which is equal to the nominal or steady state value.
- $u(t) = \bar{u}$ when $e(t) = 0$

10.12.1 Digital PID controller

Position:

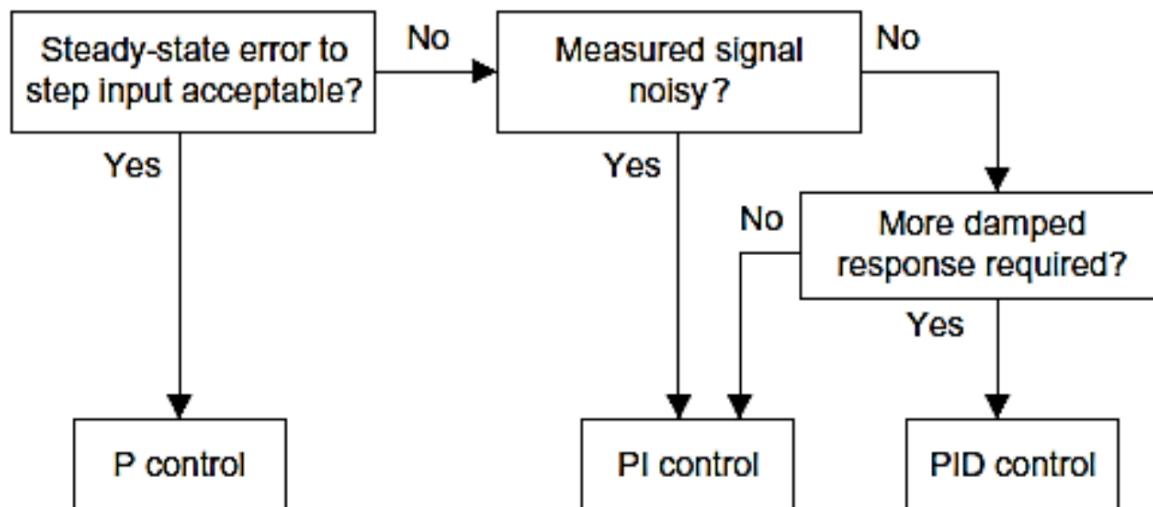
$$u(t_i) = K_c \left[e(t_i) + \frac{1}{T_i} \int_0^{t_i} e(t) dt + T_d \frac{de(t_i)}{dt} \right] \quad (1)$$

$$u(t_{i-1}) = K_c \left[e(t_{i-1}) + \frac{1}{T_i} \int_0^{t_{i-1}} e(t_{i-1}) dt + T_d \frac{de(t_{i-1})}{dt} \right] \quad (2)$$

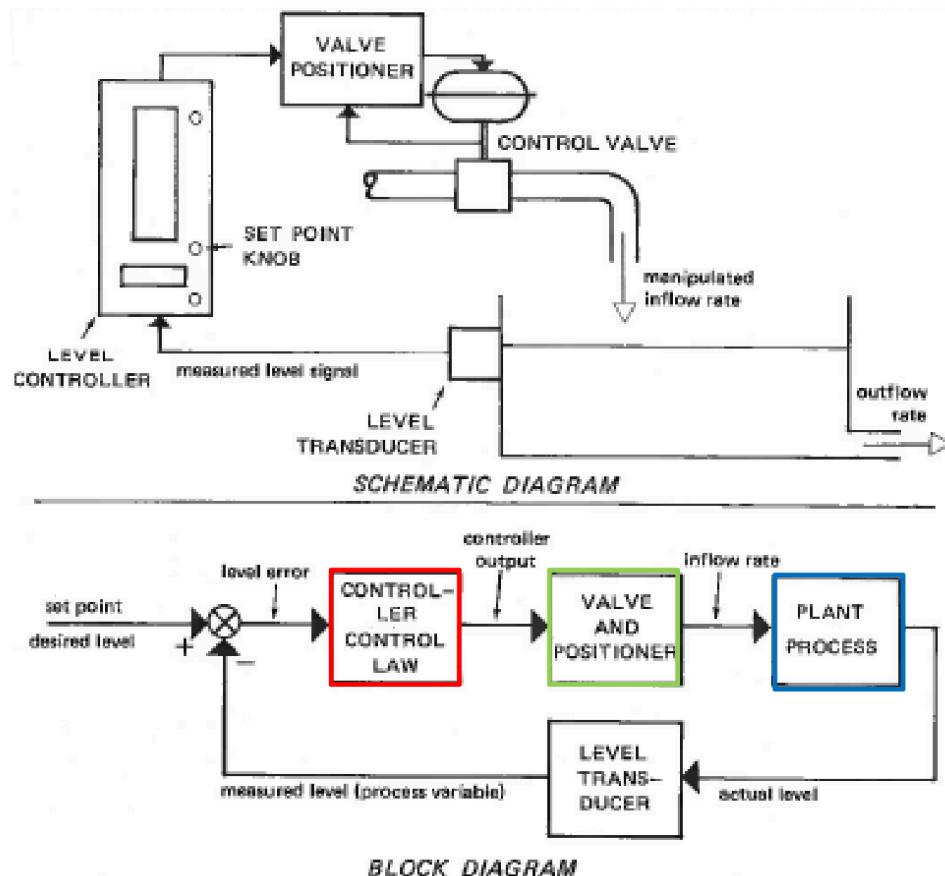
Let $\Delta t = t_i - t_{i-1}$, $\Delta u_i = u(t_i) - u(t_{i-1})$, and velocity is:

$$(1) - (2) : \Delta u = K_c \left[\Delta e_i + \left(e_i \frac{\Delta T}{T_i} \right) + \frac{T_d}{\Delta T} (\Delta e_i - \Delta e_{i-1}) \right]$$

10.13 Selecting a controller



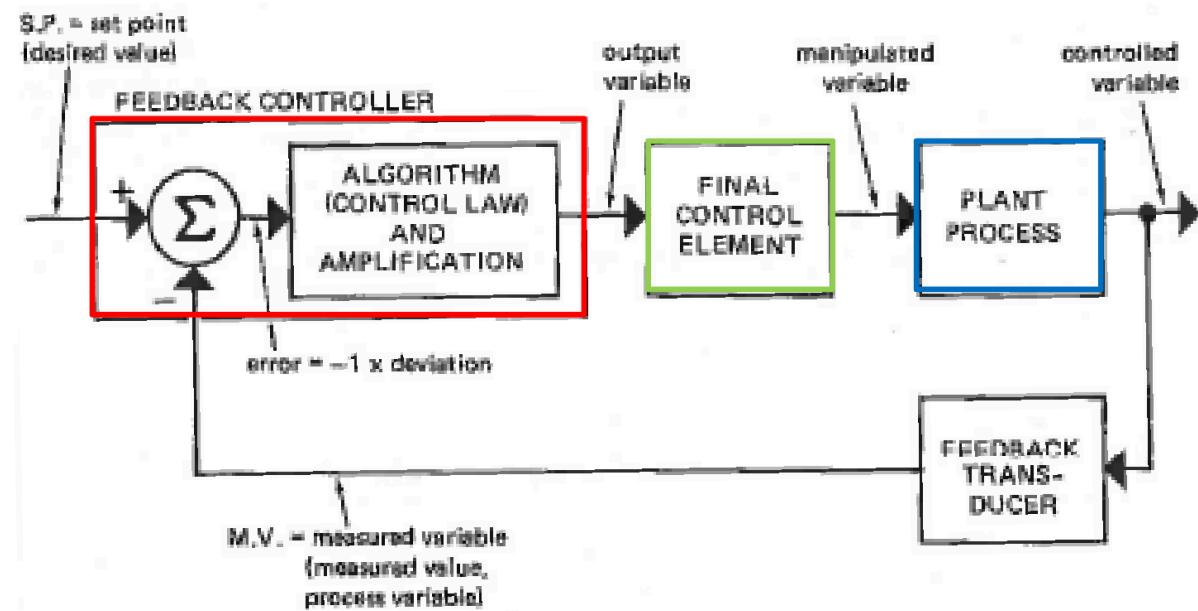
10.14 Pneumatic controller



The functions of the pneumatic controller are:

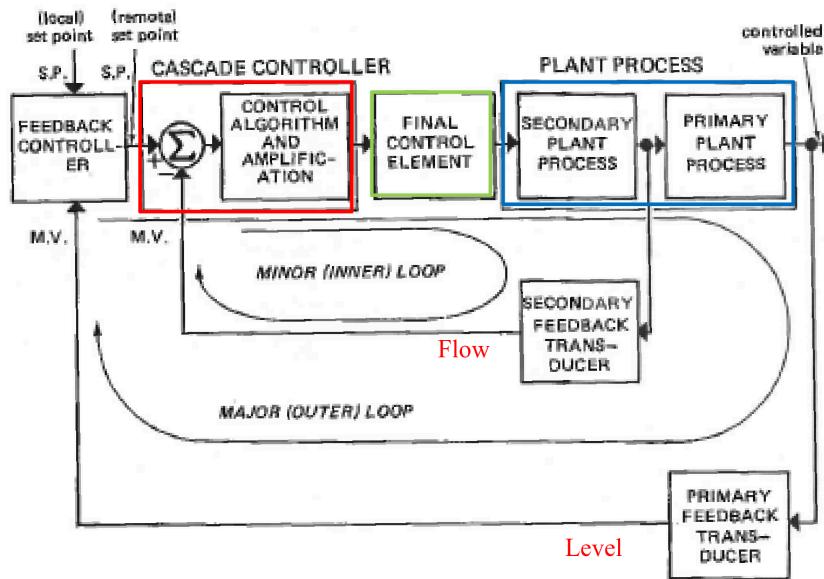
- To enable the set point signal to be generated.
- To receive the feedback signal representing the measured level.
- To generate an error signal by comparing the above two signals.
- To amplify the error signal and to incorporate dynamic terms, in generating the controller output signal.

10.15 Feedback controller

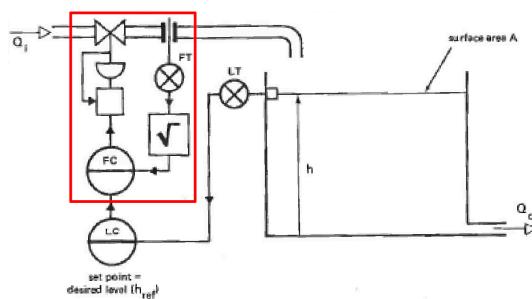


A very simple example of the use of a feedback controller would be for the control of the level of liquid in a storage tank, by manipulating the inflow rate, with outflow rate being a wild variable.

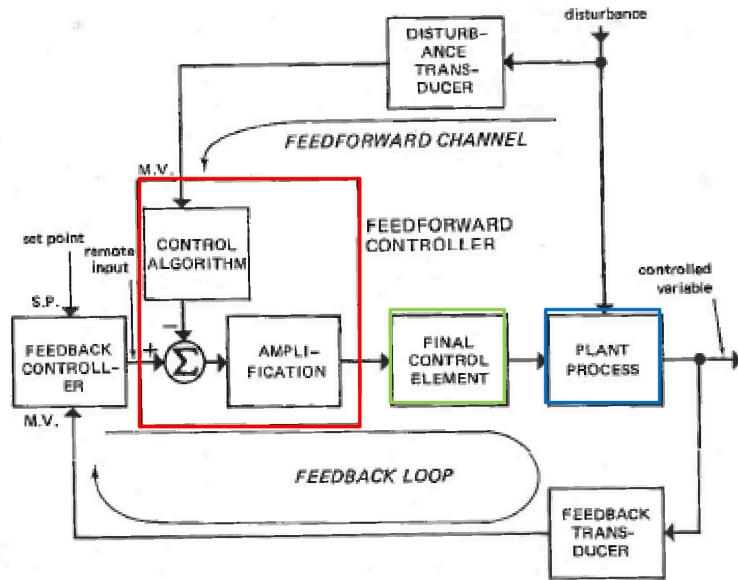
10.16 Cascade controller



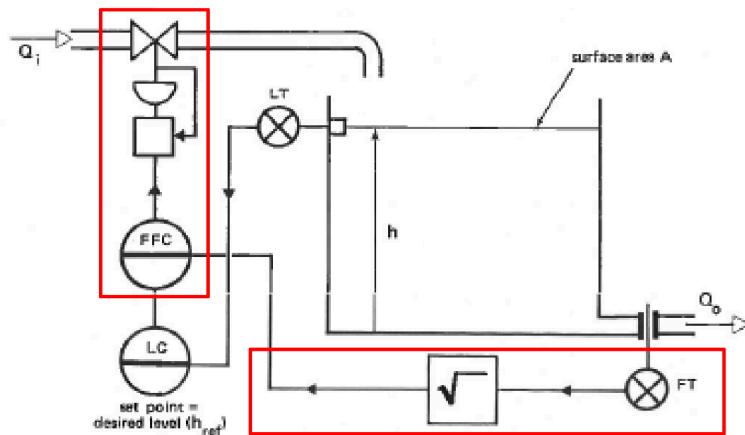
- The cascade controller is very similar to the feedback controller, only that the output from the feedback controller becomes the (variable) set point of the cascade controller, so that this set point would not be manually adjustable.
- This set point is known as a “remote” set point, as opposed to a “local set point”, which is set manually.
- The configuration represents a dual loop system, with the outer (major) loop having greater authority than the inner (minor) loop.
- The purpose of the inner loop will relate to one or more of the following factors, provided that the gain of the loop is made sufficiently high:
 - The closed inner loop, on its own, will have a faster transient response than that of the secondary section of the process. This means the speed of the response of the outer loop will be increased.
 - The closed inner loop will be relatively insensitive to changes in the properties of the secondary section of the process, so that this loop becomes an element having stationary properties, as far as the outer loop is concerned.
 - The closed inner loop will be relatively linear in its behaviour, despite possible departure from linearity in the characteristics of the secondary section of the process, with the result that the linearity of the outer loop will be enhanced.
 - Parasitic disturbances occurring within the secondary section of the plant will have a minimal effect on the output of the primary section of the plant process.
- A typical use of a cascade controller is to introduce a minor feedback loop, in which the inflow rate is measured and fed to a flow controller.



10.17 Feedforward controller

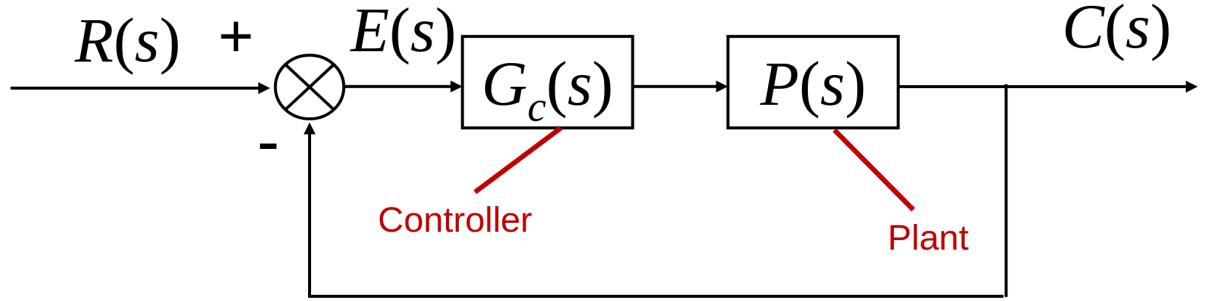


- The feedback controller has to attempt to hold the liquid level at the set point value, despite changes in the wild variable, the outflow rate Q_o .
- If, in a practical case, the controller cannot achieve the desired accuracy, then the situation can be relieved by instrumenting the wild variable and incorporating a feedforward controller, as shown below:



11 System types

11.1 Closed-loop system

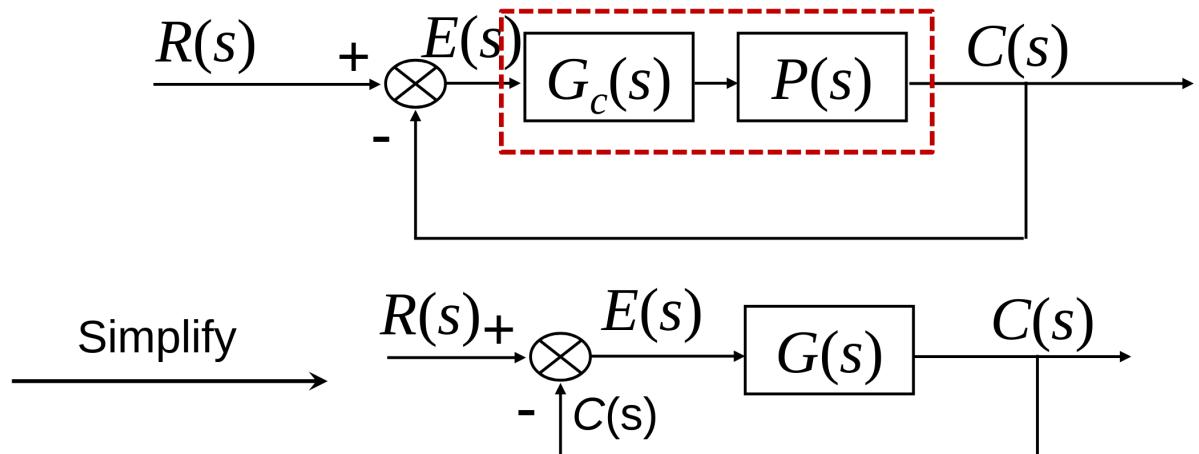


Objective: Design $G_c(s)$ to achieve the **desired steady-state errors and poles** for closed-loop transfer function.

To make $c(t)$ stable, control its:

1. Steady-state error
2. Transient response

11.2 Unity feedback systems



$G(s)$ is the Open loop transfer function.

$$E = R - C$$

$$C = GE$$

$$C = G(R - C)$$

$$C(1 + G) = GR$$

$$\frac{C}{R} = \frac{G}{1 + G}$$

11.3 System Type

$G(s)$ is the open-loop transfer function for unity feedback system.

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1) \cdots (T_m s + 1)}{S^N(T_1 s + 1)(T_2 s + 1) \cdots (T_p s + 1)}$$

- $G(s)$ is a Type N system, where N can be 0, 1, 2, etc.
- The system type is the power of s in the denominator.

When N increases, and is used to make a closed-loop system, the **steady state error decreases**, which is **desirable**, but **stability also decreases**, which is **undesirable**.

Hence, N is a **trade-off** between steady-state error and stability.

11.4 Steady-state error (e_{ss})

$$\text{Transfer function : } \frac{C}{R} = \frac{G}{1+G}$$

$$E = R - C$$

$$E = R - \frac{C}{R}R$$

$$E = R \left(1 - \frac{G}{1+G} \right)$$

$$E(s) = \frac{R}{1+G}$$

Computing the steady-state error e_{ss} via the final value theorem:

$$\begin{aligned} e_{ss} &= \lim_{t \rightarrow \infty} e(t) \\ &= \lim_{s \rightarrow 0} sE(s) \\ &= \lim_{s \rightarrow 0} \frac{sR(s)}{1+G(s)} \end{aligned}$$

11.5 Step input error (e_{ss})

$$\text{General formula : } e_{ss} = \lim_{s \rightarrow 0} \frac{sR}{1 + G}$$

$$\text{For step input : } R = \frac{1}{s} \rightarrow e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \cdot \frac{1}{s} = \frac{1}{1 + G(s=0)}$$

We define K_p to be the **static position error constant**, such that:

$$K_p = \lim_{s \rightarrow 0} G(s) = G(s=0)$$

Using this definition, the step input error (e_{ss}) can be expressed as:

$$e_{ss} = \frac{1}{1 + K_p}$$

11.5.1 K_p vs system types

11.5.1.1 Type 0 systems

$$\begin{aligned} K_p &= \lim_{s \rightarrow 0} \frac{K(T_a s + 1)(T_b s + 1)\dots}{(T_1 s + 1)(T_2 s + 1)\dots c} = K \\ \rightarrow e_{ss} &= \frac{1}{1 + K_p} = \frac{1}{1 + K} \quad (\text{Non-zero error}) \end{aligned}$$

11.5.1.2 Type 1 or higher systems

$$\begin{aligned} K_p &= \lim_{s \rightarrow 0} \frac{K(T_a s + 1)(T_b s + 1)\dots}{s^N (T_1 s + 1)(T_2 s + 1)\dots c} \rightarrow \infty, \quad \text{for } N \geq 1 \\ \rightarrow e_{ss} &= \frac{1}{1 + K_p} \\ \rightarrow e_{ss} &= 0 \quad (\text{Steady-state error eliminated}) \end{aligned}$$

11.5.1.3 Summary

For **step inputs**, the steady state error e_{ss} :

1. Type 0 systems:

$$e_{ss} = \frac{1}{1 + K} \quad (\text{Non-zero value})$$

e_{ss} decreases as K increases.

2. Type 1 or higher systems

$$e_{ss} = 0 \quad (\text{Steady-state error eliminated})$$

Requires at least **one integrator** in the forward path.

11.6 Ramp input error (e_{ss})

$$\text{General formula : } e_{ss} = \lim_{s \rightarrow 0} \frac{sR}{1 + G}$$

$$\text{For ramp input : } R = \frac{1}{s^2} \rightarrow e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \cdot \frac{1}{s^2} = \frac{1}{\lim_{s \rightarrow 0} sG}$$

We define K_v to be the **static velocity error constant**, such that:

$$K_v = \lim_{s \rightarrow 0} sG(s)$$

Using this definition, the ramp input error (e_{ss}) can be expressed as:

$$e_{ss} = \frac{1}{1 + K_v}$$

11.6.1 K_v vs system types

11.6.1.1 Type 0 systems

$$\begin{aligned} K_v &= \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1)\dots}{(T_1 s + 1)(T_2 s + 1)\dots c} = 0 \\ \rightarrow e_{ss} &= \frac{1}{K_v} \rightarrow \infty \quad (\text{Can't follow input}) \end{aligned}$$

11.6.1.2 Type 1 systems

$$\begin{aligned} K_v &= \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1)\dots}{s(T_1 s + 1)(T_2 s + 1)\dots c} = K \\ \rightarrow e_{ss} &= \frac{1}{K} \quad (\text{Non-zero-error}) \end{aligned}$$

11.6.1.3 Type 2 and higher systems

$$\begin{aligned} K_v &= \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1)\dots}{s^N(T_1 s + 1)(T_2 s + 1)\dots c} \rightarrow \infty, \quad \text{for } N \geq 2 \\ \rightarrow e_{ss} &= 0 \quad (\text{Steady-state error eliminated}) \end{aligned}$$

11.6.1.4 Summary

For **ramp inputs**, the steady-state error e_{ss} :

- Type 0 systems:

$$e_{ss} = 0 \quad (\text{Can't follow ramp input})$$

- Type 1 systems:

$$e_{ss} = \frac{1}{K_v} \quad (\text{Non-zero value})$$

e_{ss} decreases as K_v increases.

- Type 2 or higher systems:

$$e_{ss} = 0 \quad (\text{Steady-state error eliminated})$$

11.7 Unit parabolic input error (e_{ss})

$$\text{General formula : } e_{ss} = \lim_{s \rightarrow 0} \frac{sR}{1 + G}$$

$$\text{For ramp input : } R = \frac{1}{s^3} \rightarrow e_{ss} = \lim_{s \rightarrow 0} \frac{s}{1 + G} \cdot \frac{1}{s^3} = \frac{1}{\lim_{s \rightarrow 0} s^2 G}$$

We define K_a to be the **static acceleration error constant**, such that:

$$K_v = \lim_{s \rightarrow 0} s^2 G(s)$$

Using this definition, the parabolic input error (e_{ss}) can be expressed as:

$$e_{ss} = \frac{1}{1 + K_a}$$

11.7.1 K_a vs system types

11.7.1.1 Type 0 systems

$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \dots}{(T_1 s + 1)(T_2 s + 1) \dots c} = 0$$

$$\rightarrow e_{ss} = \frac{1}{K_v} \rightarrow \infty \quad (\text{Can't follow input})$$

11.7.1.2 Type 1 systems

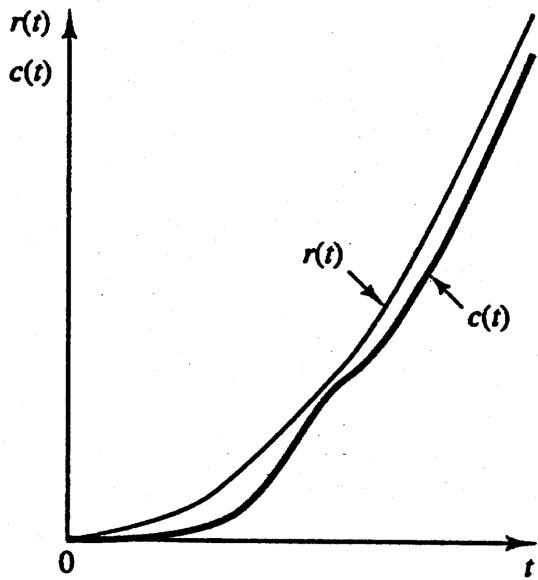
$$K_a = \lim_{s \rightarrow 0} \frac{s^2 K (T_a s + 1)(T_b s + 1) \dots}{s(T_1 s + 1)(T_2 s + 1) \dots c} = 0$$

$$\rightarrow e_{ss} = \frac{1}{K_a} \rightarrow \infty \quad (\text{Can't follow input})$$

11.7.1.3 Type 2 systems

$$K_a = \lim_{s \rightarrow 0} \frac{s K (T_a s + 1)(T_b s + 1) \dots}{s^2 (T_1 s + 1)(T_2 s + 1) \dots c} = K$$

$$\rightarrow e_{ss} = \frac{1}{K} \quad (\text{Non-zero-error})$$



11.7.1.4 Type 3 and higher systems

$$K_a = \lim_{s \rightarrow 0} \frac{sK(T_a s + 1)(T_b s + 1)\dots}{s^N(T_1 s + 1)(T_2 s + 1)\dots c} \rightarrow \infty, \quad \text{for } N \geq 3$$

$\rightarrow e_{ss} = 0$ (Steady-state error eliminated)

11.7.1.5 Summary

For **parabolic inputs**, the steady-state error e_{ss} :

1. Type 0 and 1 systems:

$$e_{ss} \rightarrow \infty \text{ (Can't follow parabolic input)}$$

2. Type 2 systems:

$$e_{ss} = \frac{1}{K_a} \text{ (Non-zero value)}$$

e_{ss} decreases as K_a increases.

3. Type 3 or higher systems:

$$e_{ss} = 0 \text{ (Steady-state error eliminated)}$$

11.8 Steady-state error (e_{ss}) summary

System Types	Steady-state Error (e_{ss})		
	Unit step input	Unit ramp input	Unit parabolic input
Type 0	$\frac{1}{1 + K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$
Type 3	0	0	0

The table above increases in system “level” going down, and increases in input “level” going right.

- Input “level” **higher than** system “level”:

$$e_{ss} \rightarrow \infty$$

- Input “level” **equal to** system “level”:

$$e_{ss} \text{ is non-zero}$$

- Input “level” **lower than** system “level”:

$$e_{ss} = 0$$

Thus:

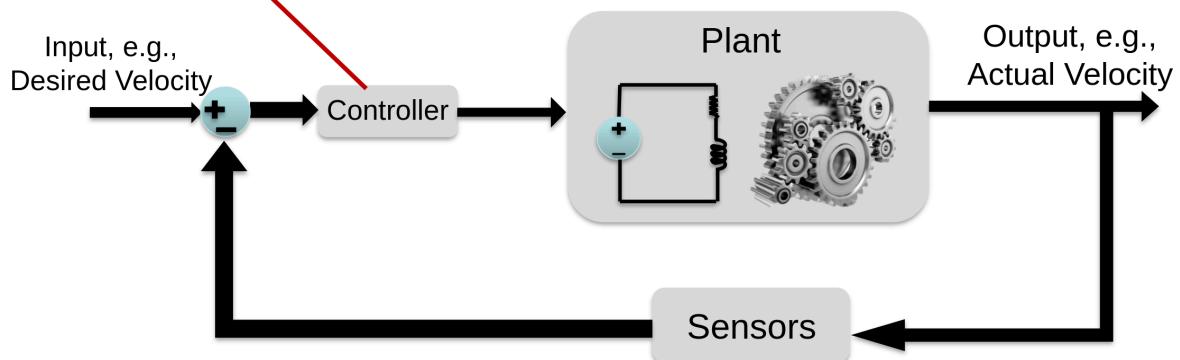
- **Steady-state errors** are related to K_p , K_v and K_a .
- **Increasing system type** requires an **integrator** in the forward path, but this will have a **destabilising effect**.
- Designing a **stable-system** with **more than two integrators** in the forward path is **generally difficult**.

12 Root locus

12.1 Closed loop system

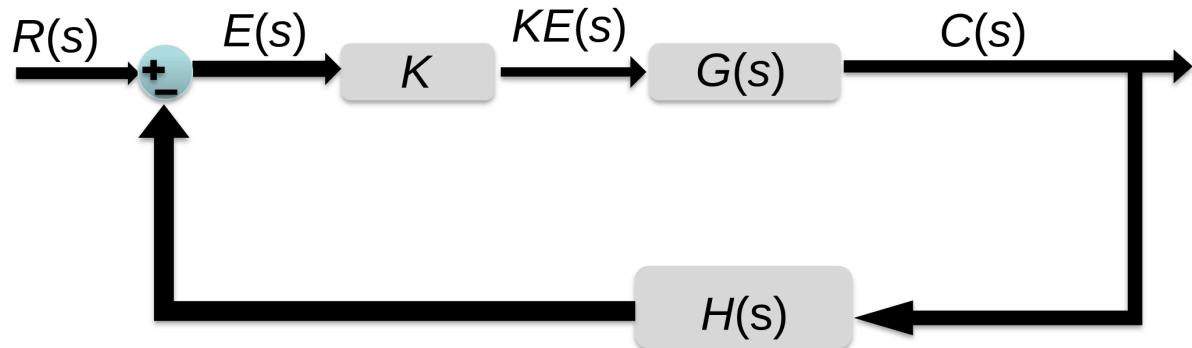
Purpose: Create desired transient and steady-state response

Changing the system's pole locations



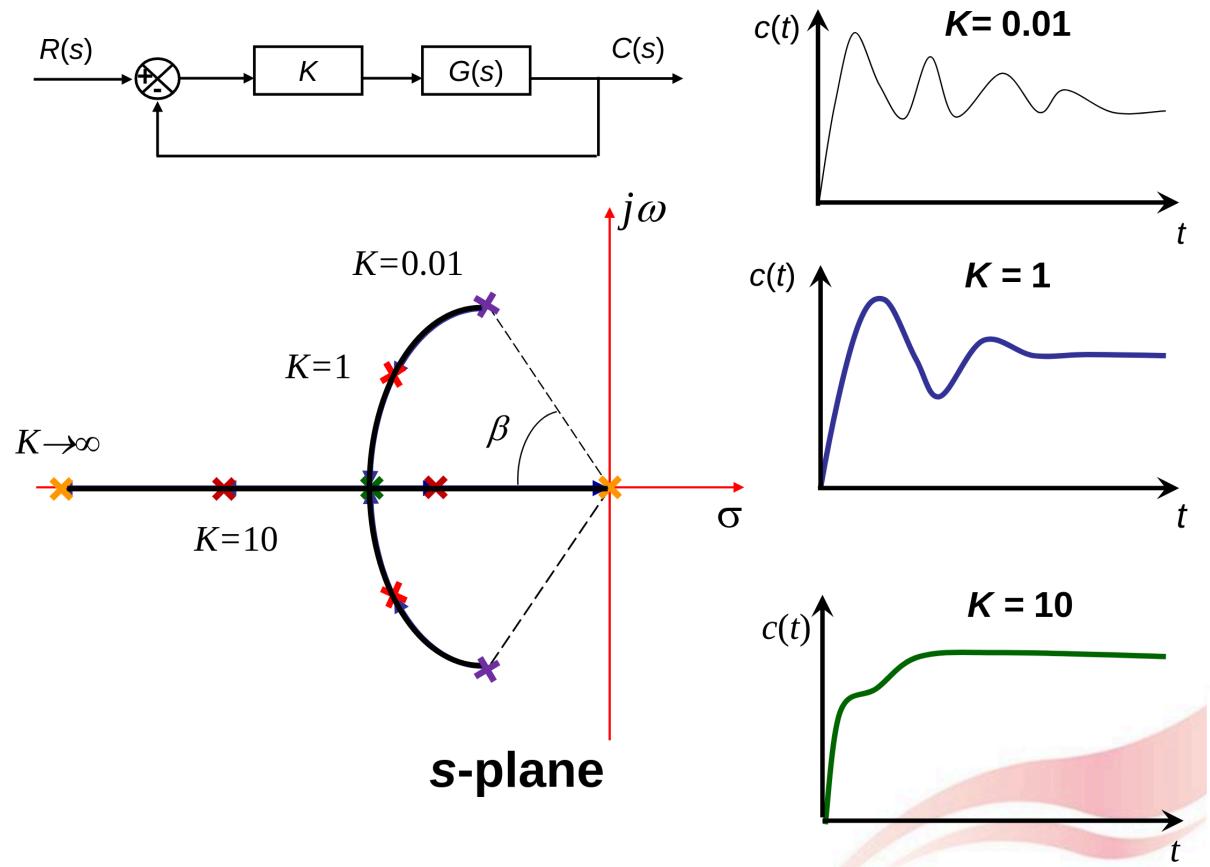
12.2 Root Locus

Using a proportional controller, like controller = K :



The root locus is the trajectory of the closed-loop system's poles as K varies from zero to infinity.

12.2.1 Illustration



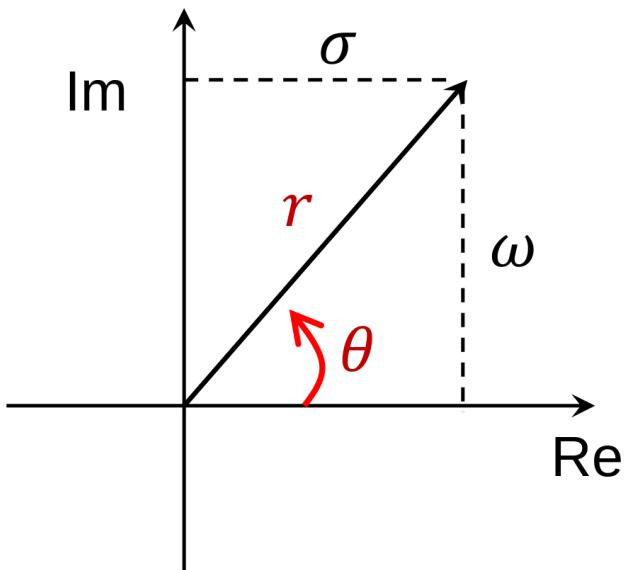
12.3 Recap on complex numbers

$$\sigma + j\omega = re^{j\theta}$$

Where:

- $r = \sqrt{\sigma^2 + \omega^2}$
- $\theta = \arctan \frac{\omega}{\sigma}$

$$\begin{aligned} \frac{\sigma_1 + j\omega_1}{\sigma_2 + j\omega_2} &= \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} \\ &= \underbrace{\frac{r_1}{r_2}}_{\text{Magnitude}} e^{\underbrace{j(\theta_1 - \theta_2)}_{\text{Phase}}} \end{aligned}$$



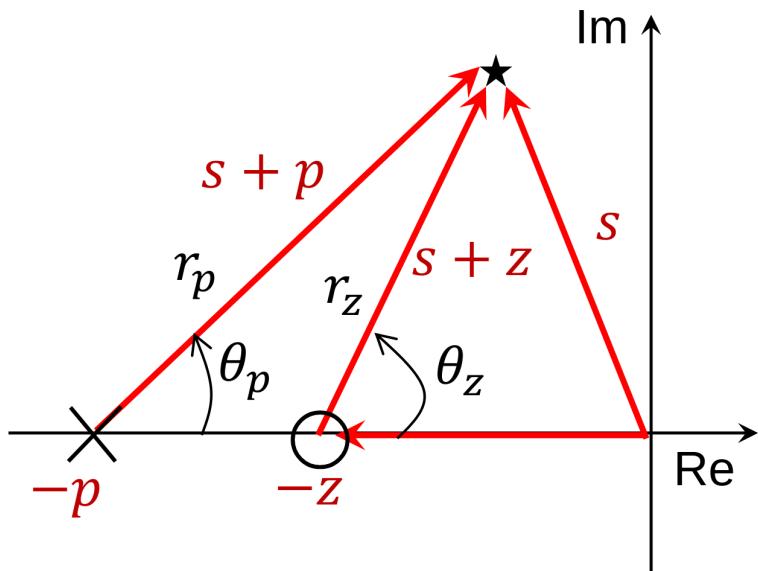
$$s + z = r_z e^{j\theta_z}$$

Where $r_z = |s + z|$.

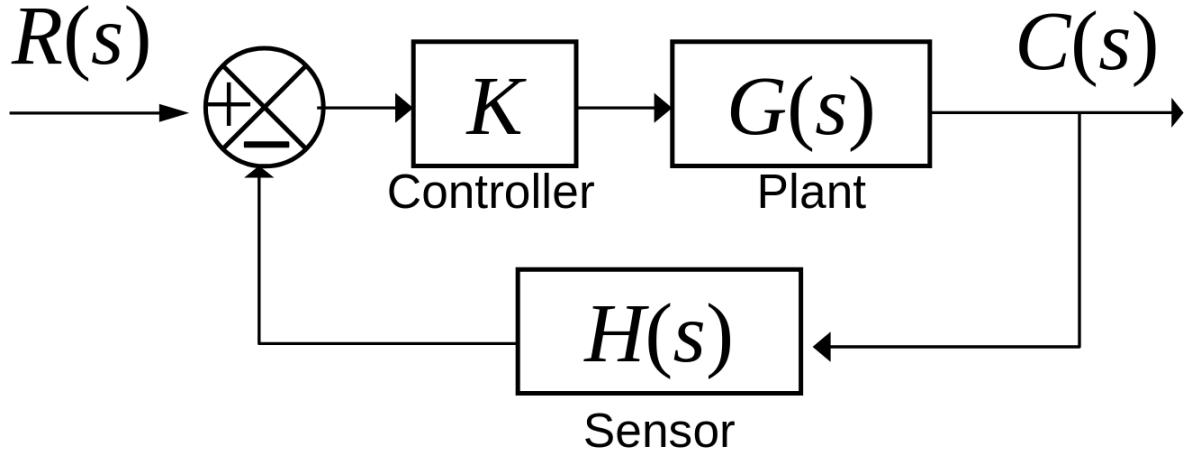
$$s + p = r_p e^{j\theta_p}$$

Where $r_p = |s + p|$.

$$\frac{s + z}{s + p} = \underbrace{\frac{r_z}{r_p}}_{\text{Magnitude}} e^{\underbrace{j(\theta_z - \theta_p)}_{\text{Phase}}}$$



12.4 Characteristic equation (CE)



$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Characteristic equation (CE):

$$1 + KG(s)H(s) = 0$$

For a given K , any s that satisfies the characteristic equation will be a closed-loop pole.

When K varies, the location of the closed-loop poles changes. The root locus represents the trajectory of the closed-loop poles.

12.5 Required conditions

Characteristic equation (CE):

$$1 + KG(s)H(s) = 0$$

$$K \frac{N(s)}{D(s)} = K \frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = 1e^{\pm j(\pi \pm 2q\pi)}$$

$$K \frac{N(s)}{D(s)} = 1e^{\pm j(\pi \pm 2q\pi)}$$

$$\left[K \left| \frac{N(s)}{D(s)} \right| \right] e^{j(\angle N - \angle D)} = 1e^{\pm j(\pi \pm 2q\pi)}$$

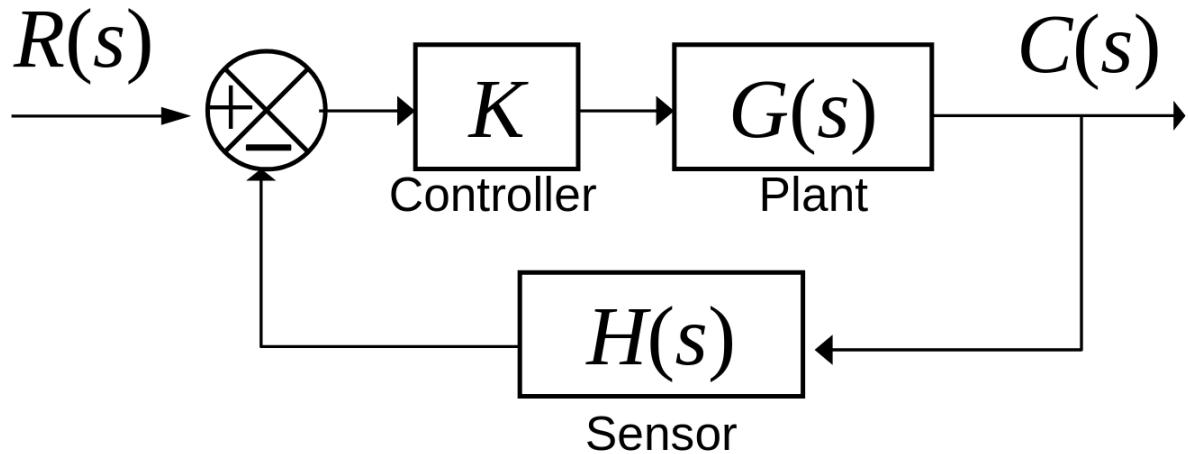
Magnitude condition:

$$K \left| \frac{N(s)}{D(s)} \right| = 1$$

Phase condition:

$$\begin{aligned} \angle N - \angle D &= -\pi - 2q\pi \\ \angle N - \angle D &= -(1 + 2q)\pi \end{aligned}$$

12.6 Starting and ending points



Characteristic equation (CE):

$$\text{CE} : 1 + K \frac{N(s)}{D(s)} = 0$$

$$D(s) + KN(s) = 0$$

Smallest K , like $K = 0$:

$$\text{CE} : D(s) + KN(s) = 0 \rightarrow D(s) = 0$$

Solution shows that **closed-loop poles** are equal to **open-loop poles**.

Largest K , like $K \rightarrow \infty$:

$$\text{CE} : D(s) + KN(s) = 0 \rightarrow N(s) = 0$$

Solution shows that **closed-loop poles** are equal to **open-loop zeros**.

12.7 Infinite open-loop zeros

$$1 + K \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

$$K \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 1e^{-j\pi(1+2q)}$$

In general, when $m \leq n$, some **closed-loop poles** end at infinite open-loop zeros.

$$\lim_{s \rightarrow \infty} K \left(\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \right) = 1e^{(-j\pi(1+2q))}$$

$$\lim_{s \rightarrow \infty} K \left(\frac{s^m}{s^n} \right) = 1e^{-j\pi(1+2q)}$$

$$\lim_{s \rightarrow \infty} K(s)^{m-n} = 1e^{-j\pi(1+2q)}$$

$$\lim_{s \rightarrow \infty} K(|s| e^{j\angle s})^{m-n} = 1e^{-j\pi(1+2q)}$$

Phase condition:

Magnitude condition: $(m-n)\angle s = -\pi(1+2q)$

$$K(|s|)^{m-n} = 1 \quad \angle s = \frac{\pi(1+2q)}{n-m}$$

12.8 Physical interpretation

Phase condition at $s \rightarrow \infty$:

$$\angle s = \frac{\pi(1+2q)}{n-m}, \quad q = 0, \dots, n-m-1$$

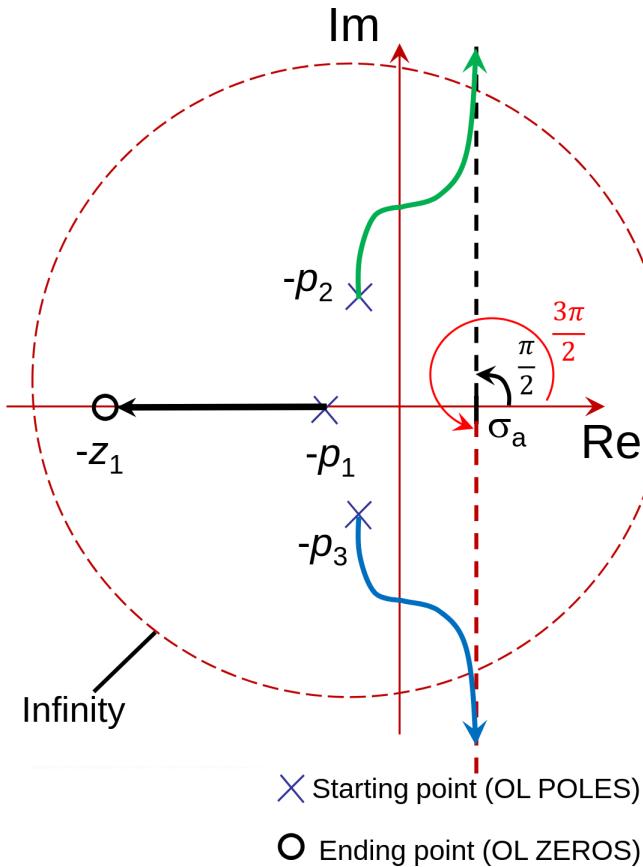
$\angle s$ represents the **asymptote angle**.

It is required to substitute q with $|n-m|$ number of integer values.

The asymptote location on the real axis is:

$$\sigma_a = \frac{\sum_i^n (-p_i) - \sum_j^m (-z_j)}{n-m}$$

12.8.1 Example



$$1 + K \frac{s + z_1}{(s + p_1)(s + p_2)(s + p_3)} = 0$$

$$m = 1, \quad n = 3$$

Asymptote formula:

$$\sigma_a = \frac{\sum_i^n (-p_i) - \sum_j^m (-z_j)}{n - m}, \quad \angle s = \frac{\pi(1 + 2q)}{n - m}$$

$$m = 1, \quad n = 3$$

$$\sigma_a = \frac{\sum_i^3 (-p_i) + z_1}{3 - 1}, \quad \angle s = \frac{\pi(1 + 2q)}{3 - 1}$$

$$\angle s = \frac{\pi}{2} + q\pi, \quad q = 0, 1$$

$$\angle s = \frac{\pi}{2}, \frac{3\pi}{2}$$

12.9 Locus on the real axis

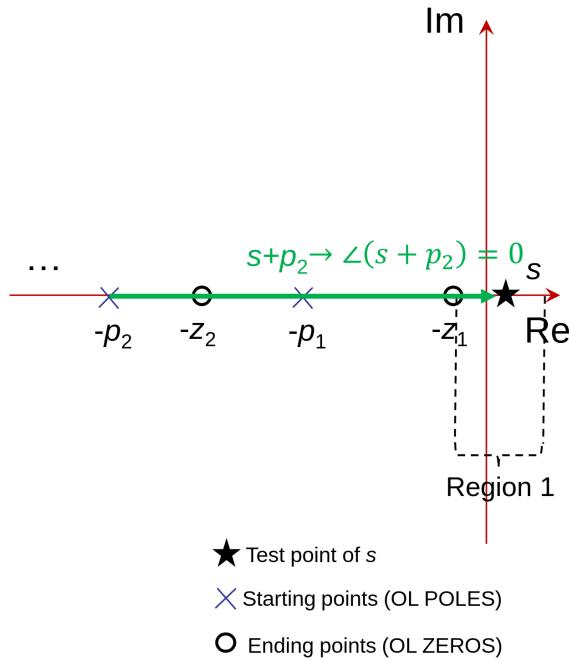
Phase condition:

$$\angle \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \pm\pi$$

$$\underbrace{\sum_{i=1}^m \angle(s+z_i)}_{\text{Angle contributions for all open-loop zeros}} - \underbrace{\sum_{j=1}^n \angle(s+p_j)}_{\text{Angle contributions from all open-loop poles}} = \pm\pi$$

Ignore complex open-loop poles or zeros. Conjugate pairs cancel their angle contributions.

12.9.1 Region 1



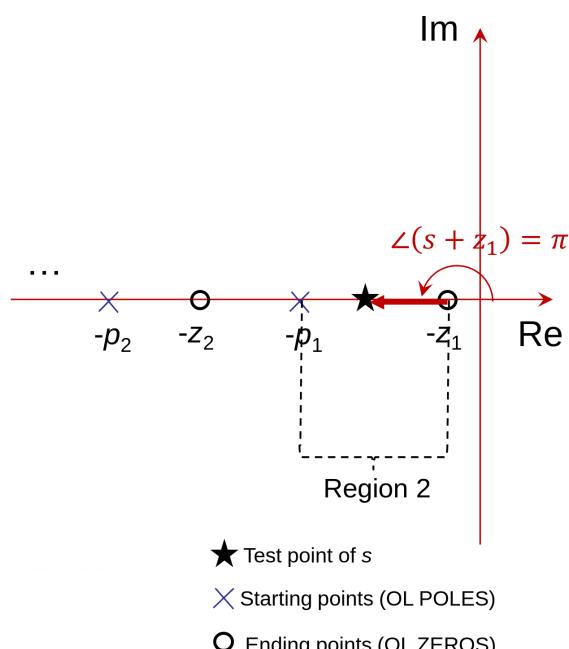
Phase condition:

$$\angle \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \pm\pi$$

$$\angle(s+z_i) = \angle(s+p_j) = 0$$

For region 1, the left hand side angles are 0, which is not equal to $\pm\pi$, hence there is **no locus** in region 1.

12.9.2 Region 2



Phase condition:

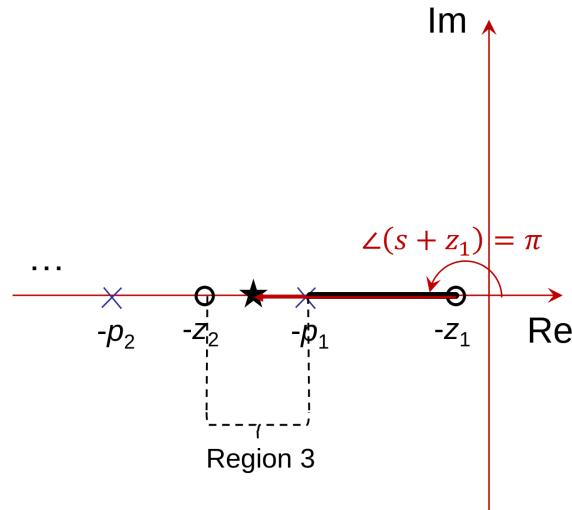
$$\angle \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \pm\pi$$

For region 2, vectors for all the \times and \circ that are on the left of \star produce:

$$\angle(s+z_2) = \angle(s+p_1) = \angle(s+p_2) = 0$$

Only vector $\angle(s+z_1) = \pi$, so for region 2, the left hand side angles are π , hence there is a **locus** in region 2.

12.9.3 Region 3



★ Test point of s

✗ Starting points (OL POLES)

○ Ending points (OL ZEROS)

Phase condition:

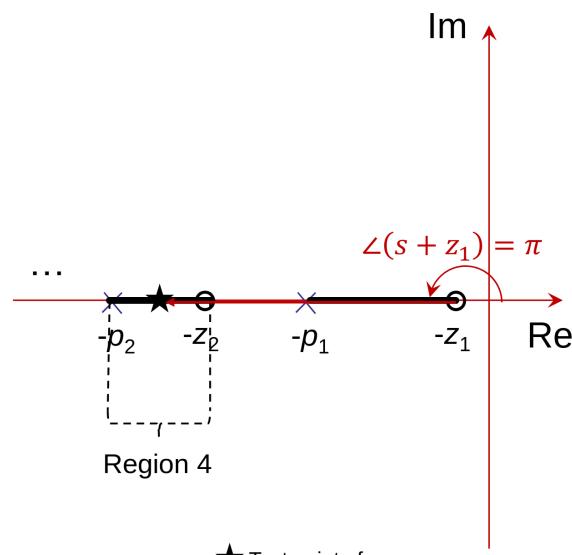
$$\angle \frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = \pm\pi$$

2 vectors on the right of * produce angles of π :

$$\angle(s + z_1) = \angle(s + p_1) = \pi$$

Net phase on the left hand side angles is 0, which is not equal to $\pm\pi$, hence there is **no locus** in region 3.

12.9.4 Region 4



★ Test point of s

✗ Starting points (OL POLES)

○ Ending points (OL ZEROS)

Phase condition:

$$\angle \frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = \pm\pi$$

3 vectors on the right of * produce angles of π :

$$\angle(s + z_2) = \angle(s + z_1) = \angle(s + p_1) = \pi$$

Net phase on the left hand side angles is $\pm\pi$, hence there is a **locus** in region 4.

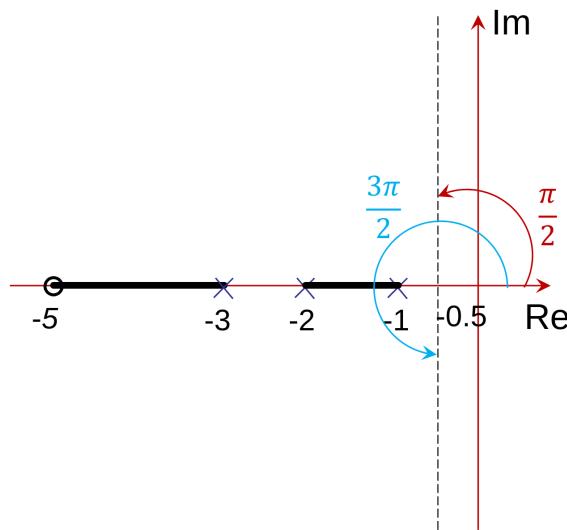
12.9.5 Overall

Phase condition:

$$\angle \frac{(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)} = \pm\pi$$

Regions with **an odd number** of ✗ and ○ that are **on the right of *** will satisfy the phase condition.

12.10 Break-out points



\times Starting point (OL POLES)

\circ Ending point (OL ZEROS)

Using the characteristic equation:

$$\text{CE} : 1 + K \frac{s+5}{(s+1)(s+2)(s+3)} = 0$$

Finding the asymptote:

$$\text{Angle} : \angle s = \frac{\pi(1+2q)}{3-1}$$

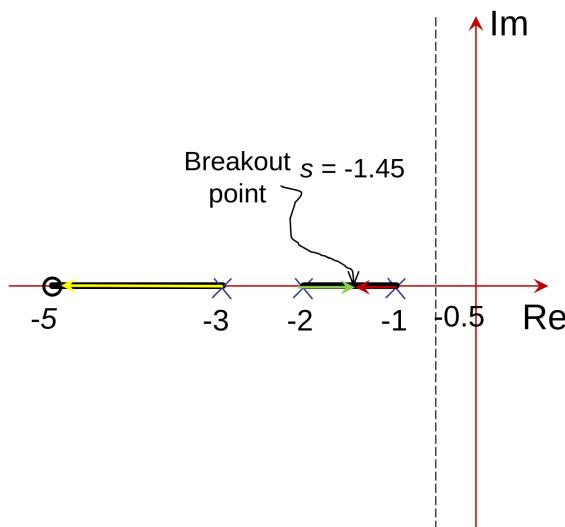
$$\angle s = \frac{\pi}{2}, 3\frac{\pi}{2}, \quad q = 0, 1$$

$$\text{Location} : \sigma_a = \frac{(-1-2-3)-(-5)}{3-1} = -0.5$$

$$K = -\frac{(s+1)(s+2)(s+3)}{s+5}$$

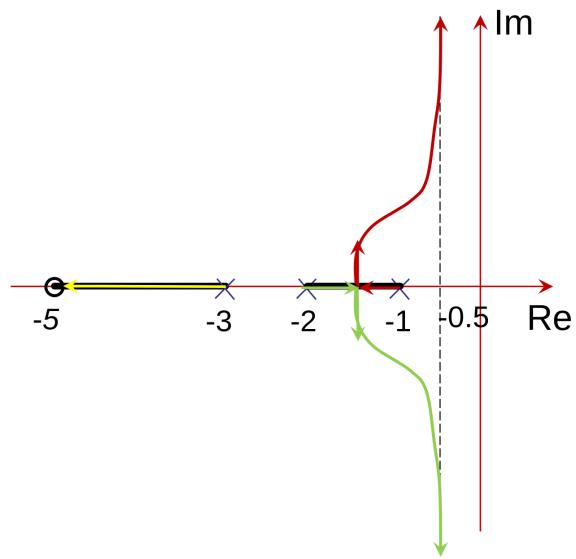
$$\frac{dK}{ds} = 0 \quad (\text{max point})$$

$$\rightarrow s = -1.45, \quad \underbrace{-2.62, -6.42}_{\text{rejected because they are not on locus}}$$



\times Starting point (OL POLES)

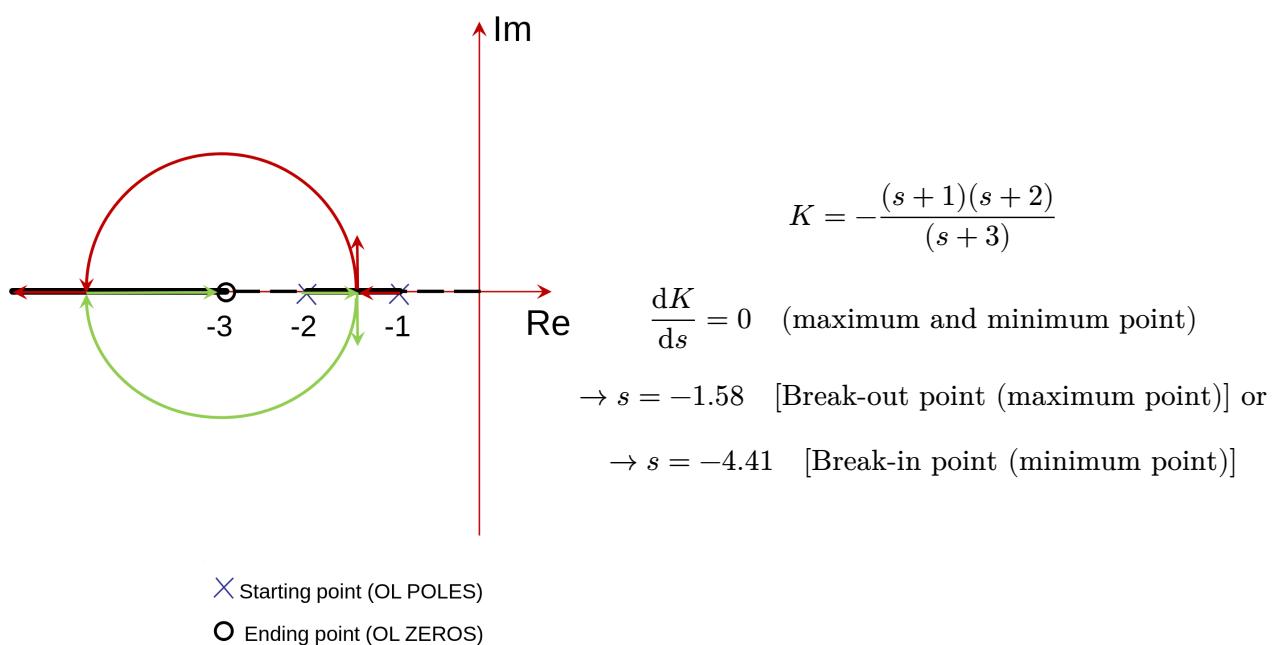
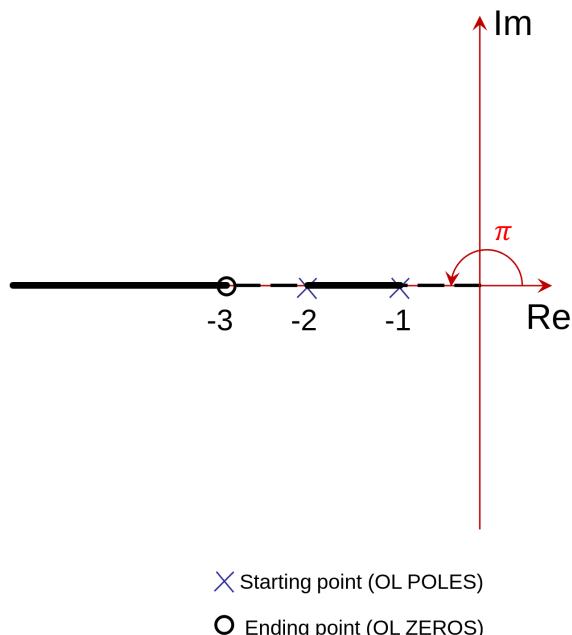
\circ Ending point (OL ZEROS)



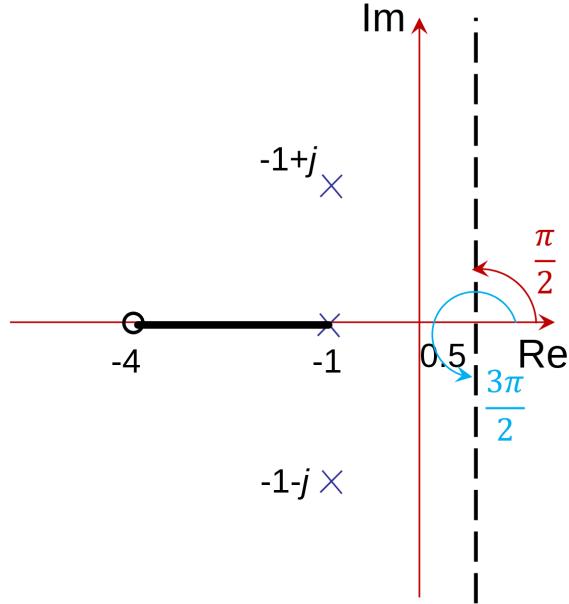
\times Starting point (OL POLES)

\circ Ending point (OL ZEROS)

12.11 Break-in points



12.12 Departure angles



Using the characteristic equation:

$$\text{CE : } 1 + K \frac{s+4}{(s+1)(s+1+j)(s+1-j)} = 0$$

Asymptote:

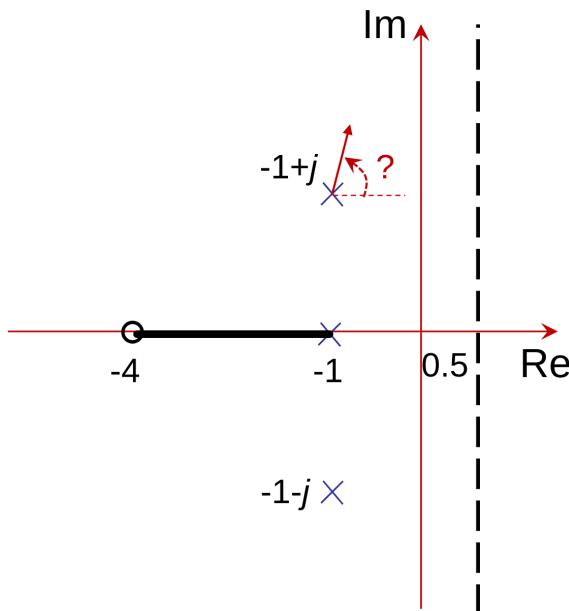
$$\text{Angle : } \angle s = \frac{\pi(1+2q)}{3-1}$$

$$\angle s = \frac{\pi}{2}, 3\frac{\pi}{2}, \quad q = 0, 1$$

$$\text{Location : } \sigma_a = \frac{(-1 - 1 - 1) - (-4)}{3 - 1} = 0.5$$

✗ Starting point (OL POLES)

○ Ending point (OL ZEROS)



$$K = -\frac{(s+1)(s+1+j)(s+1-j)}{s+4}$$

$$\frac{dK}{ds} = 0$$

No break-out points.

✗ Starting point (OL POLES)

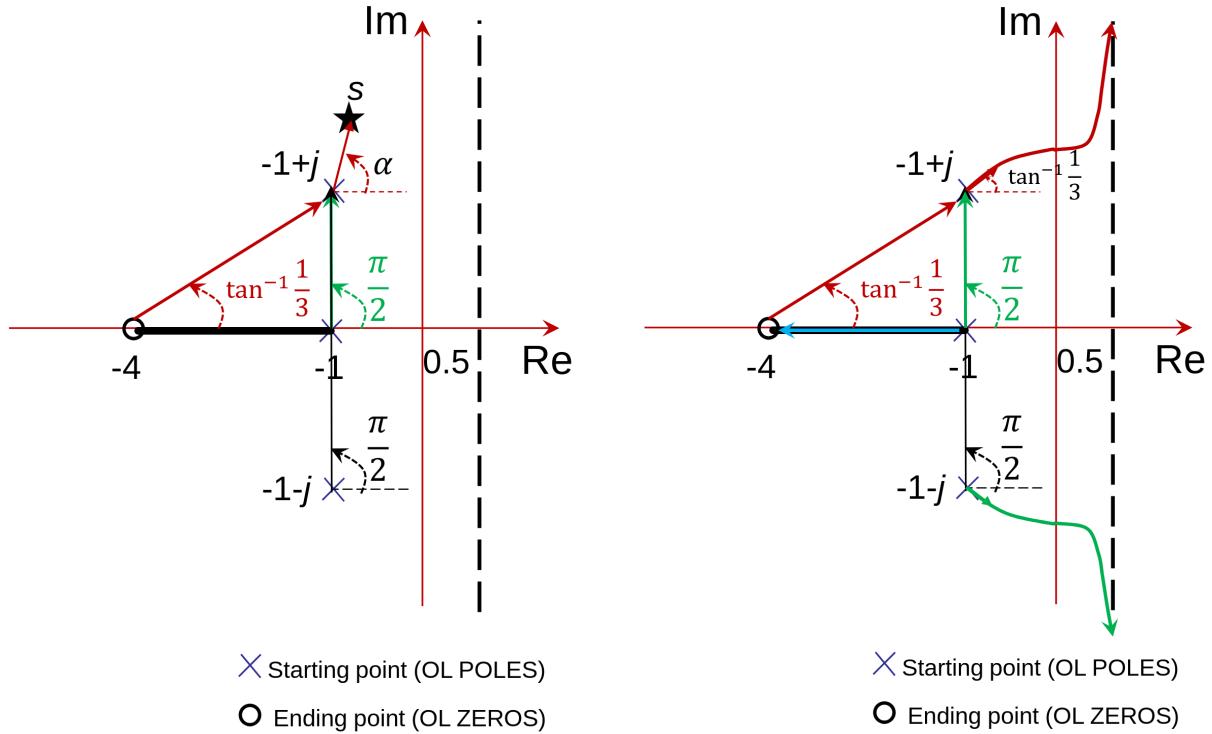
○ Ending point (OL ZEROS)

$$\text{CE} : 1 + K \frac{s+4}{(s+1)(s+1+j)(s+1-j)} = 0$$

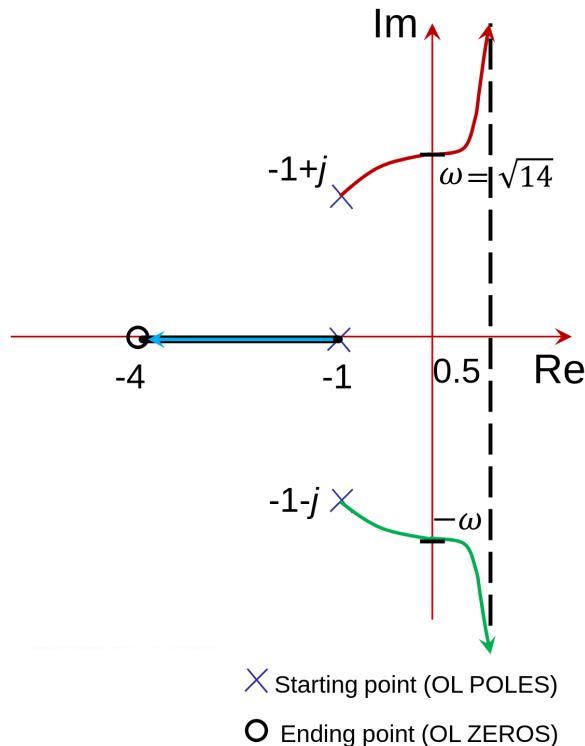
$$\text{Phase} : \sum_{i=1}^{m=1} \angle(s+z_i) - \sum_{j=1}^{n=3} \angle(s+p_j) = \pm\pi$$

$$\arctan\left(\frac{1}{3}\right) - \frac{\pi}{2} - \frac{\pi}{2} - \alpha = -\pi$$

$$\alpha = \arctan\left(\frac{1}{3}\right)$$



12.13 Intersection of imaginary axis



Using the characteristic equation (CE):

$$\text{CE : } 1 + K \frac{s+4}{(s+1)(s+1+j)(s+1-j)} = 0$$

$$s^3 + 3s^2 + (4+K)s + (2+4K) = 0$$

Substituting $s = j\omega$:

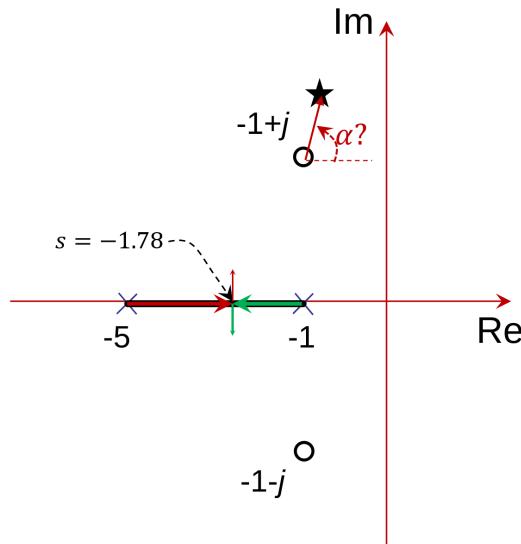
$$(j\omega)^3 + 3(j\omega)^2 + (4+K)j\omega + (2+4K) = 0$$

$$j\omega^3 + 3\omega^2 + (4+K)j\omega + (2+4K) = 0$$

$$\underbrace{(-3\omega^2 + (2+4K))}_{\text{Real part}} + \underbrace{j((4+K)\omega - \omega^3)}_{\text{Imaginary part}} = 0$$

$$K = 10, \quad \omega = \sqrt{14}$$

12.14 Arrival angles



Using the characteristic equation:

$$\text{CE} : 1 + K \frac{(s+1+j)(s+1-j)}{(s+1)(s+5)} = 0$$

$$K = -\frac{(s+1)(s+5)}{(s+1-j)(s+1+j)}$$

$$\frac{dK}{ds} = 0$$

$$\rightarrow s = -1.78 \quad (\text{maximum point})$$

$s = -1.78$ is a **break-out point**.

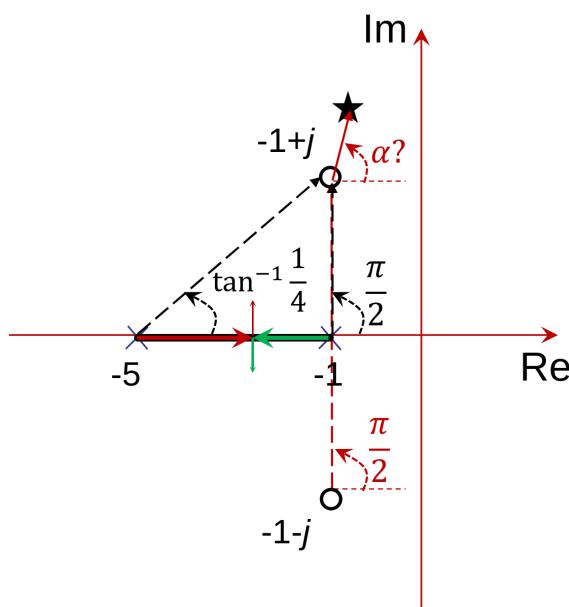
\times Starting point (OL POLES)

\circ Ending point (OL ZEROS)

$$\text{Phase : } \sum_{i=1}^m \angle(s+z_i) - \sum_{j=1}^n \angle(s+p_j) = \pm\pi$$

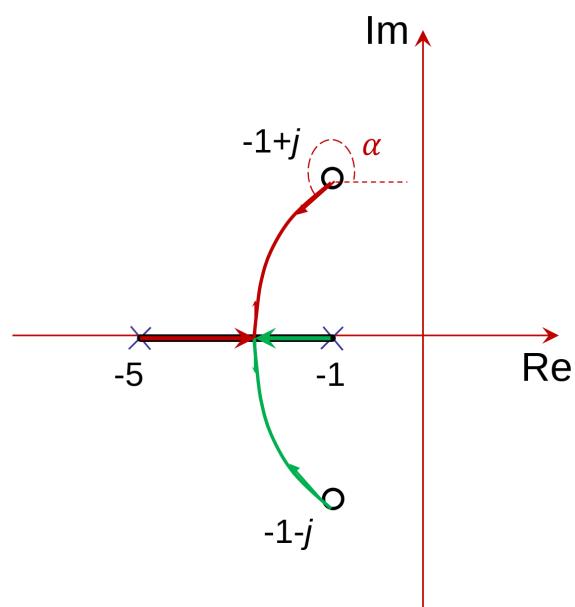
$$\frac{\pi}{2} + \alpha - \frac{\pi}{2} - \arctan\left(\frac{1}{4}\right) = \pi$$

$$\alpha = \pi + \arctan\left(\frac{1}{4}\right)$$



\times Starting point (OL POLES)

\circ Ending point (OL ZEROS)



\times Starting point (OL POLES)

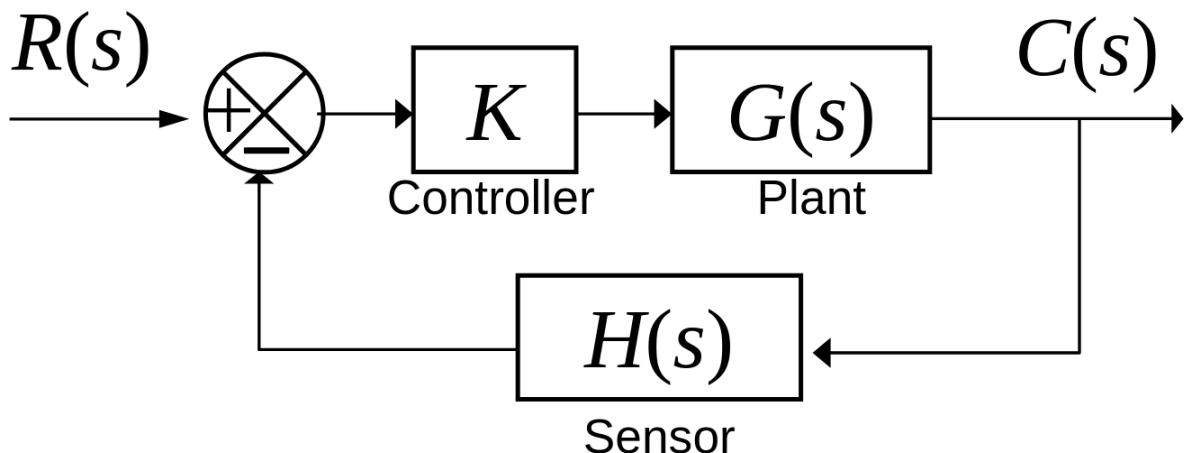
\circ Ending point (OL ZEROS)

12.15 Summary

1. Starting and ending points of the locus
 - a. Starting points - **open loop poles**
 - b. Ending points - **open loop zeroes**
2. Asymptotes (**infinite open-loop zeros**)
 - a. Angles
 - b. Location
3. Locus on real axis
4. Break-in and break-out points
 - a. Break-in (**minimum K**)
 - b. Break-out (**maximum K**)
5. Departure and arrival angles
6. Imaginary axis intersection

13 Root locus controller design

13.1 Root locus



$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Characteristic equation (CE):

$$\text{CE} : 1 + KG(s)H(s) = 0$$

Root locus shows how the closed-loop poles change for $0 < K < \infty$. We can also use the **root locus** technique if the controller is not equal to K .

13.2 Basic controllers

13.2.1 PD controller

Time domain:

$$a = K_p e + K_D \frac{de}{dt}$$

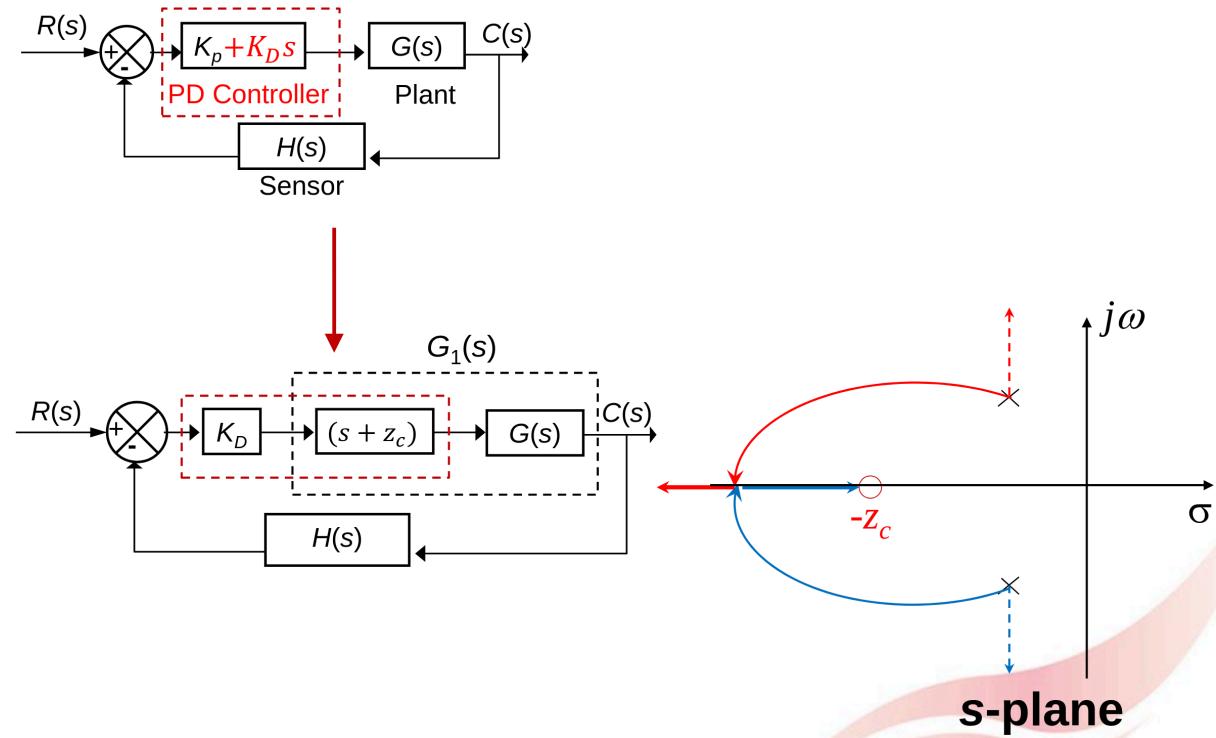
Laplace domain:

$$\begin{aligned} A &= K_p E + K_D s E \\ &= E \underbrace{(K_p + K_D s)}_{G_c \text{ for PD}} \end{aligned}$$

Equation:

$$K_D s + K_p = K_D \left(s + \underbrace{\frac{K_p}{K_D}}_{z_c} \right)$$

K_D and z_c change closed-loop poles.



13.2.2 PI controller

Time domain:

$$a = K_p e + K_I \int_0^t e \, dt$$

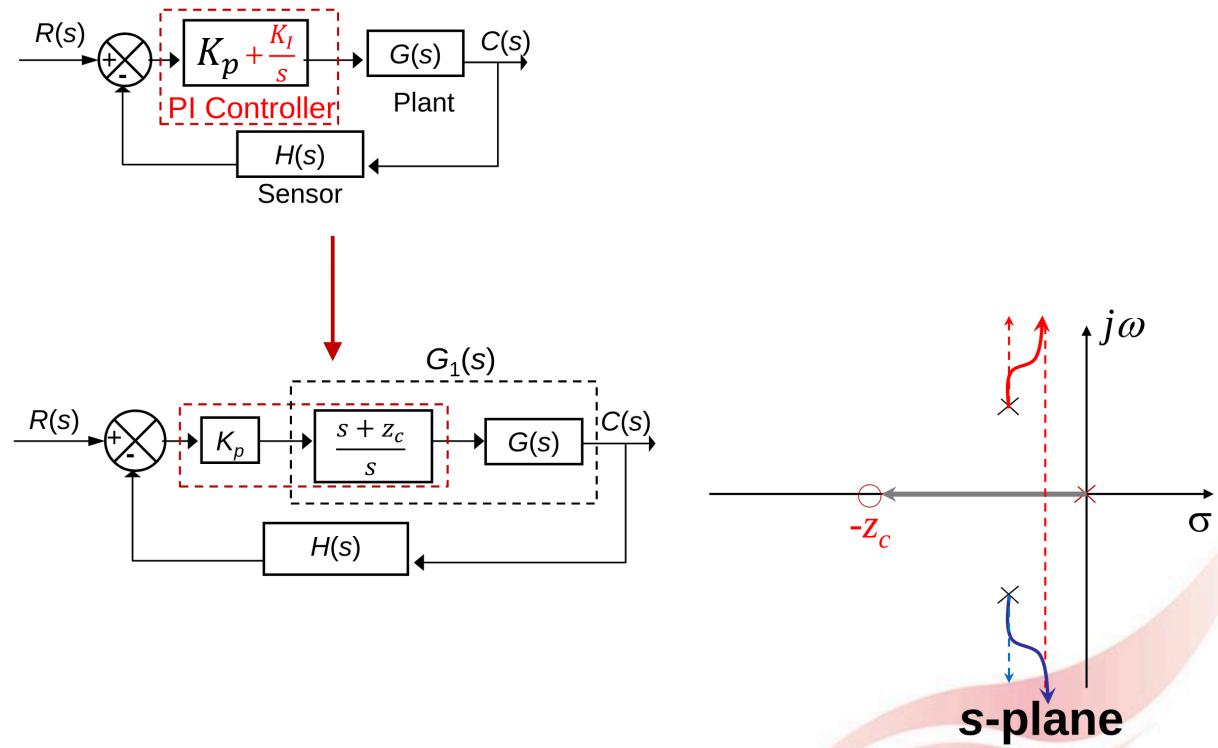
Laplace domain:

$$\begin{aligned} A &= K_p E + \frac{K_I}{s} E \\ &= E \underbrace{\left(K_p + \frac{K_I}{s} \right)}_{G_c \text{ for PI}} \end{aligned}$$

Equation:

$$K_p + \frac{K_I}{s} = K_p \frac{\left(s + \frac{K_I}{K_p} \right)}{s}$$

PI controller eliminates steady-state error.



13.2.3 Comparison

PD controller:

Increase speed and stability.

Deals with the **transient response**.

PI controller:

Eliminates steady-state error e_{ss} .

Deals with the **steady-state response**.

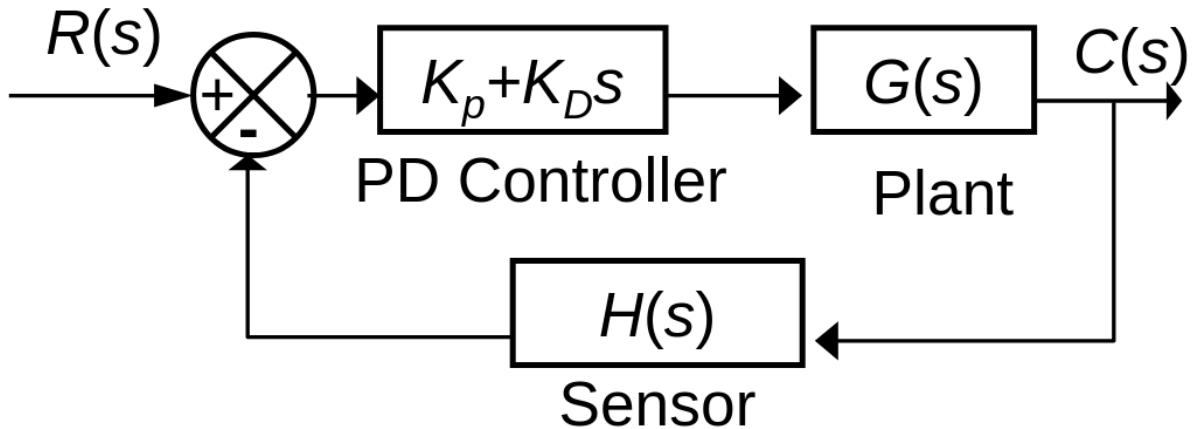
Properties \ Controllers	K_p Proportional	$K_D s$ Derivative	$\frac{K_I}{s}$ Integral
Stability	↓	↑	↓
Steady-state error (e_{ss})	↑	—	↑
Speed	↑	↑	↓
Disturbance rejection	↑	—	↓
Noise rejection	—	↓	↑

Legend:

- ↑ Good
- — Neutral
- ↓ Bad

13.3 Improving transient response

13.3.1 Design of PD controllers

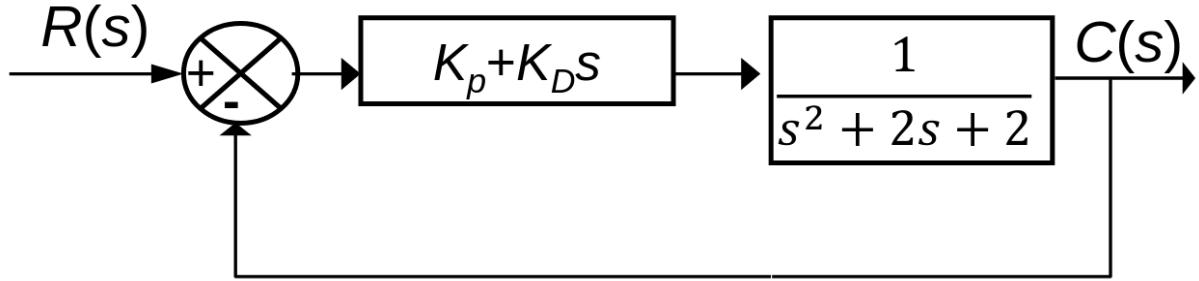


- Closed loop poles define the transient response.
- The aim is to find K_p and K_D to achieve the desired closed-loop poles.

13.3.2 General steps

1. Determine the **desired closed-loop poles**.
2. Express the **characteristic equation** of the **closed-loop transfer function** in terms of K_p and K_D .
3. **Substitute** the poles into the **characteristic equation**.
4. Solve the **real and imaginary parts** of the characteristic equation to obtain the **required K_p and K_D** .

13.3.3 Example



13.3.3.1 Step 1

Determine the **desired closed-loop poles**.

Criterion 1: 10% overshoot

Criterion 2: 2% settling time

$$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.1 \quad \frac{4}{\zeta\omega_n} = 0.5$$

$$-\frac{\pi\zeta}{\sqrt{1-\zeta^2}} = \ln 0.1 \quad \frac{4}{0.59\omega_n} = 0.5$$

$$\zeta = \sqrt{\frac{(\ln 0.1)^2}{(\ln 0.1)^2 + \pi^2}} = 0.59 \quad \omega_n = \frac{4}{(0.59)(0.5)} = 13.5$$

Desired poles:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \approx -8 \pm 11j$$

13.3.3.2 Step 2

Express the **characteristic equation** of the **closed-loop transfer function** in terms of K_p and K_D .

$$G_{cl} = \frac{\frac{K_p + K_D s}{s^2 + 2s + 2}}{1 + \frac{K_p + K_D s}{s^2 + 2s + 2}} = \frac{K_p + K_D s}{s^2 + 2s + 2 + K_p + K_D s}$$

$$G_{cl} = \frac{K_p + K_D s}{s^2 + (2 + K_D)s + (2 + K_p)}$$

Characteristic equation:

$$s^2 + (2 + K_D)s + (2 + K_p) = 0$$

13.3.3.3 Step 3

Substitute the poles into the characteristic equation:

$$\text{Desired poles : } s = -8 \pm 11j$$

$$\text{Characteristic equation : } s^2 + (2 + K_D)s + (2 + K_p) = 0$$

$$(-8 + 11j)^2 + (2 + K_D)(-8 + 11j) + (2 + K_p) = 0$$

$$(-71 + K_p - 8K_D) + j(11K_D - 154) = 0$$

$$\text{Real part : } -71 + K_p - 8K_D = 0 \quad (1)$$

$$\text{Complex part } 11K_D - 154 = 0 \quad (2)$$

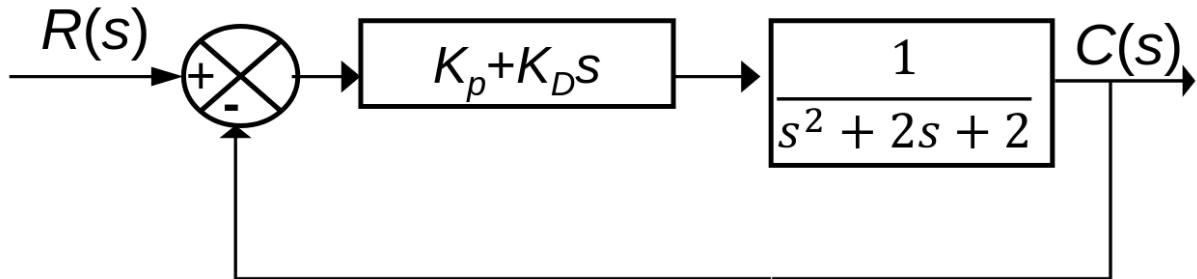
13.3.3.4 Step 4

Solve the **real and imaginary parts** of the characteristic equation:

$$K_D = 14, \quad K_p = 183$$

$$K_p + K_Ds = 183 + 14s = 14(s + 13.1)$$

13.3.3.5 Root locus



$$K_p + K_Ds = 14(s + 13.1)$$

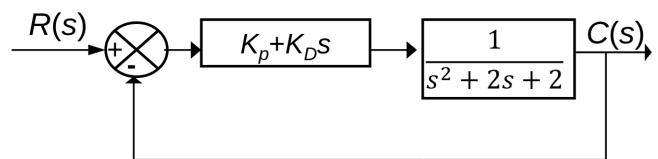
$$\text{Desired poles : } s = -8 \pm 11j$$

P controller characteristic equation:

$$1 + \frac{K_p}{s^2 + 2s + 2} = 0$$

PD controller characteristic equation:

$$1 + \frac{K_D(s + 13.1)}{s^2 + 2s + 2} = 0$$



13.3.4 Drawbacks of PD controllers

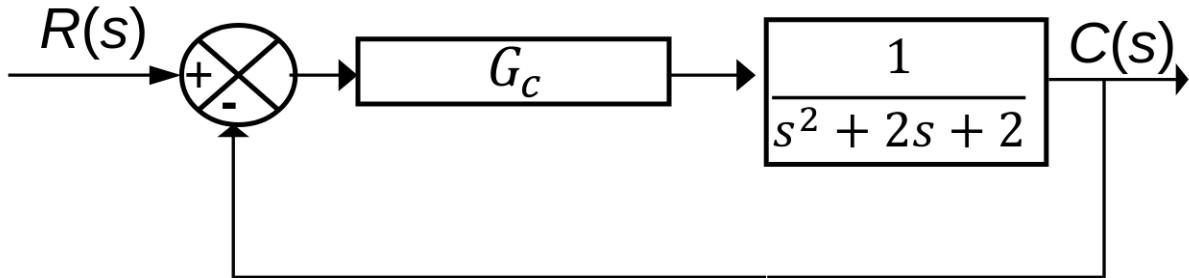
- Cannot be realised with passive components.
- Sensitive to high frequency noise.

$$\text{PD controller : } G_c = K_D s + K_p = K_D(s + z_c)$$

$$\text{Noise amplification : } \lim_{\omega \rightarrow \infty} |G_c(s = j\omega)|$$

Alternative controller: **Lead compensator**

13.3.5 Lead compensator

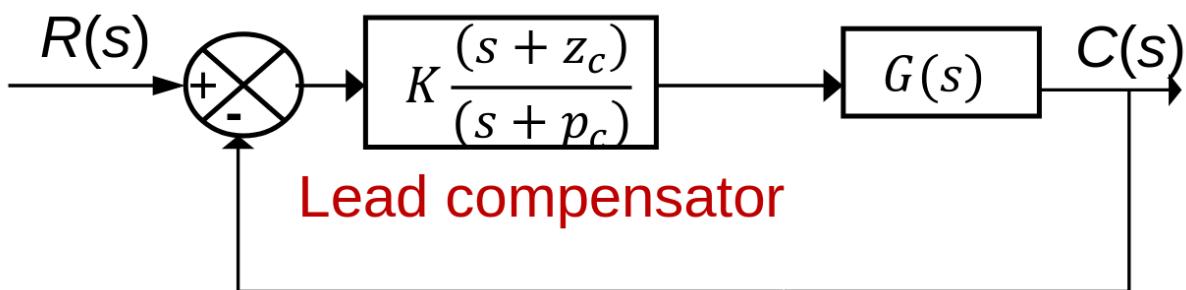


$$G_c = K \frac{s + z_c}{s + p_c}$$

$$\begin{aligned} \text{Noise amplification : } & \lim_{\omega \rightarrow \infty} |G_c(s = j\omega)| \\ &= \lim_{\omega \rightarrow \infty} \left| \frac{K(j\omega + z_c)}{(j\omega + p_c)} \right| \\ &= \lim_{\omega \rightarrow \infty} \left| \frac{K(j\omega)}{j\omega} \right| \\ &= K \end{aligned}$$

The lead compensator has finite amplification, which is **better than the PD controller**.

13.3.6 Designing the lead compensator



3 unknowns:

$$K, z_c, p_c$$

$$p_c > z_c$$

13.3.7 General steps

1. Design the desired transient response with a PD controller.

$$K_D s + K_p = K_D(s + z'_c)$$

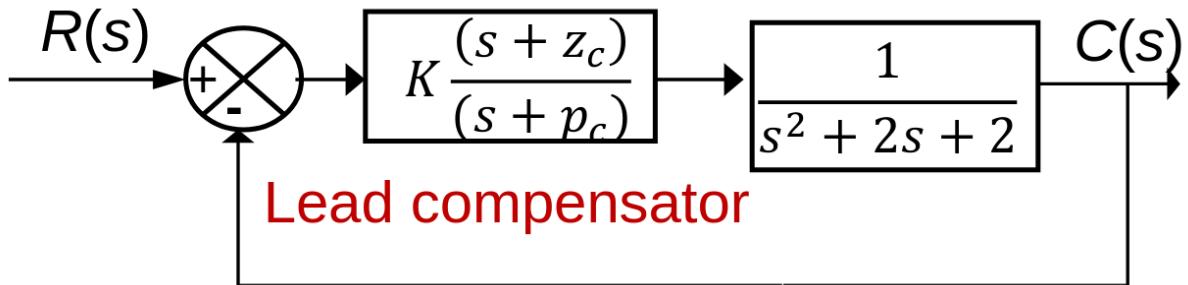
2. Using the obtained PD controller, specify a value of z_c such that:

$$z_c < z'_c$$

3. Use the characteristic equation to determine:

$$K, p_c$$

13.3.8 Example



Desired poles:

$$s = -8 \pm 11j$$

13.3.8.1 Step 1

Design the desired transient response with a PD controller:

$$K_D s + K_p = 14(s + 13.1) = K_D(s + z'_c)$$

13.3.8.2 Step 2

Use the obtained PD controller to specify a value of z_c such that:

$$z_c < z'_c$$

$$z_c = 10$$

$$\text{Lead compensator : } K \frac{s + 10}{s + p_c}$$

13.3.8.3 Step 3

Use the characteristic equation to determine K and p_c

$$\text{Characteristic equation : } 1 + G_c G = 0$$

$$1 + K \frac{s+10}{s+p_c} \left(\frac{1}{s^2 + 2s + 2} \right) = 0$$

$$(s+p_c)(s^2 + 2s + 2) + K(s+10) = 0$$

$$s^3 + (2+p_c)s^2 + (2+2p_c+K)s + (10K+2p_c) = 0$$

Substituting $s = -8 + 11j$ into the equation:

$$(2K - 71p_c + 2262) + j(-154p_c + 451 + 11K) = 0$$

$$\text{Real part : } 2K - 71p_c + 2262 = 0$$

$$\text{Imaginary part : } -154p_c + 451 + 11K = 0$$

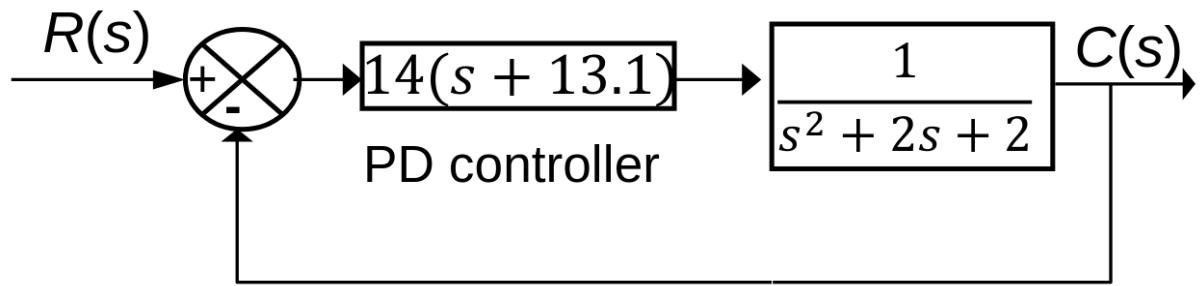
Solving:

$$K = 669$$

$$p_c = 50.7$$

$$G_c = 669 \frac{s+10}{s+50.7}$$

13.3.8.4 S-plane for PD controller



Desired poles:

$$s = -8 \pm 11j$$

PD controller characteristic equation:

$$1 + \frac{14(s + 13.1)}{s^2 + 2s + 2} = 0$$

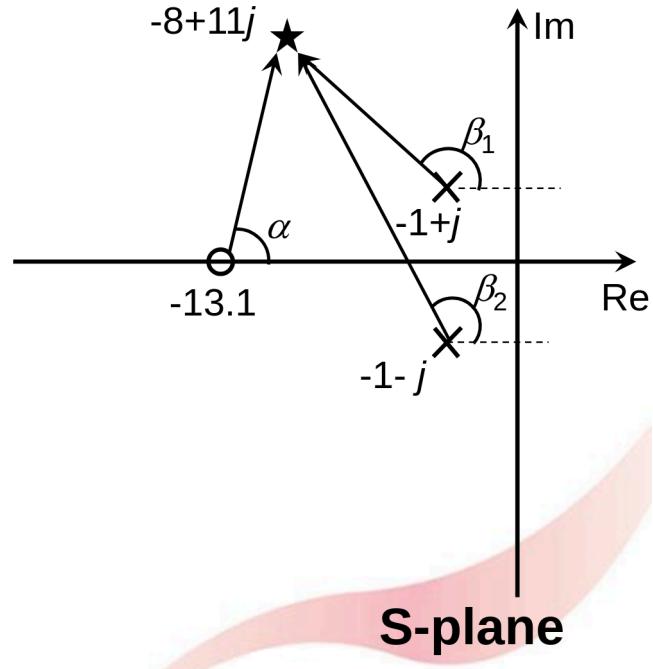
$$14 \frac{s + 13.1}{s^2 + 2s + 2} = 1e^{j\pi}$$

$$\angle \frac{s + 13.1}{s^2 + 2s + 2} = \pi$$

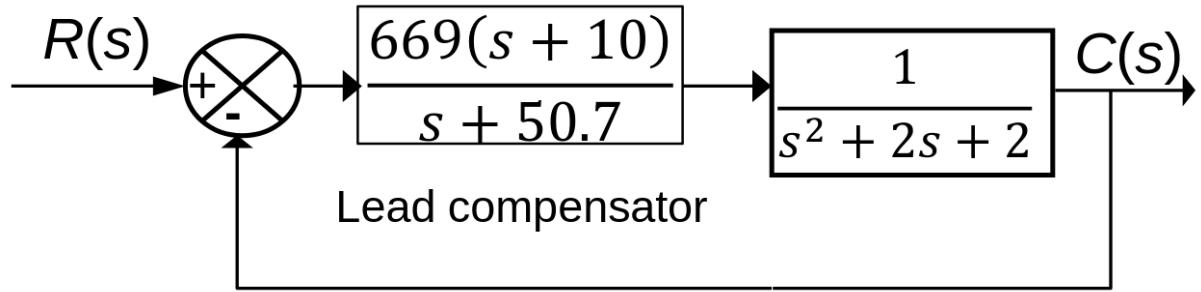
$$\angle(s + 13.1) - \angle(s^2 + 2s + 2) = \pi$$

Phase condition:

$$\alpha - \beta_1 - \beta_2 = \pi$$



13.3.8.5 S-plane for lead compensator



Desired poles:

$$s = -8 \pm 11j$$

Lead compensator characteristic equation:

$$1 + \left(\frac{669(s+10)}{s+50.7} \right) \left(\frac{1}{s^2+2s+2} \right) = 0$$

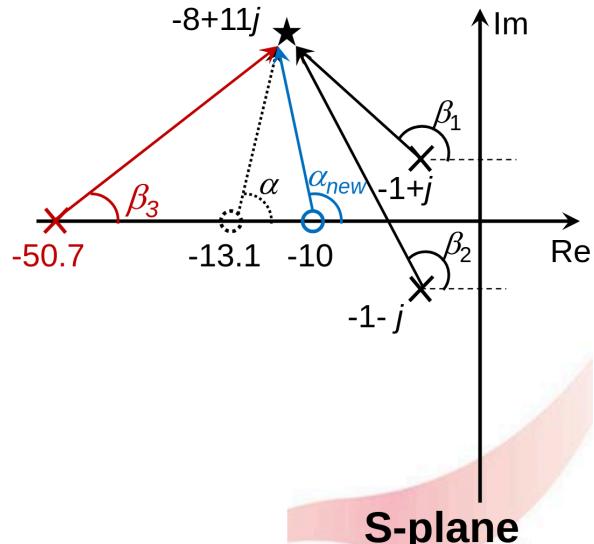
$$\angle \frac{s+10}{s+50.7} = \pi$$

$$\angle(s+10) - \angle(s^2+2s+2) - s + 50.7 = \pi$$

$$\alpha_{\text{new}} > \alpha$$

Phase condition:

$$\alpha_{\text{new}} - \beta_1 - \beta_2 - \beta_3 = \pi$$



13.4 Improving steady-state response

13.4.1 Design of PI controllers

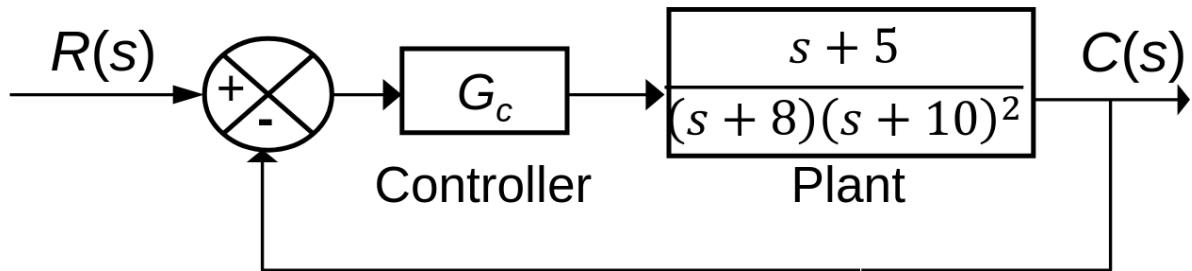
- The aim is to upgrade the system type.
- A PI controller can be implemented if having a transient response is fine.

Note that:

$$\begin{aligned} K_p + \frac{K_I}{s} &= \frac{K_p \left(s + \frac{K_I}{K_p} \right)}{s} \\ &= K_p \underbrace{\frac{s + z_c}{s}}_{\text{Pole-zero cancellation}}, \quad \text{if } z_c \text{ is small} \end{aligned}$$

The transient response is slightly affected.

13.4.1.1 Example: Root locus

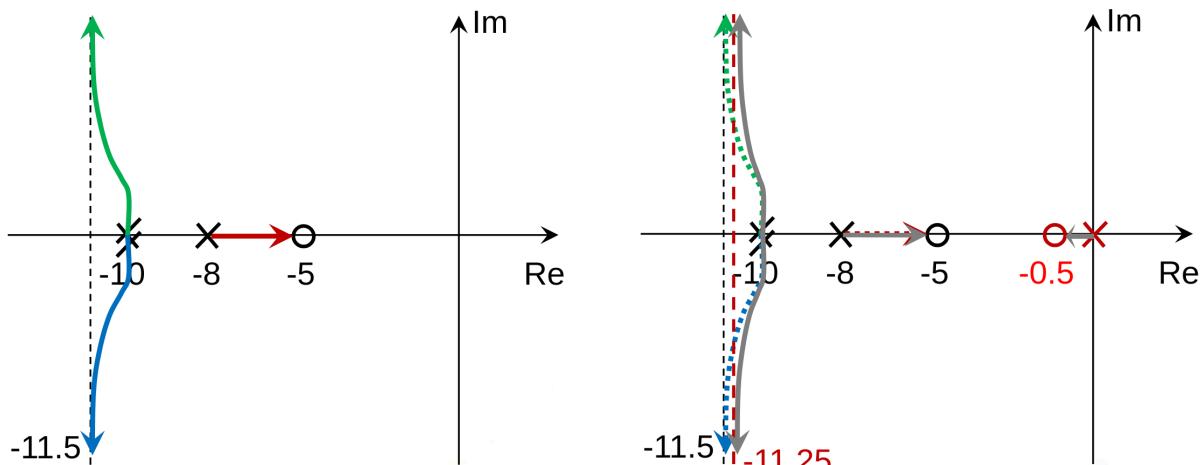


Characteristic equation for K_p controller:

$$1 + K_p \left(\frac{s + 5}{(s + 8)(s + 10)^2} \right) = 0$$

Characteristic equation for K_I controller:

$$1 + K_p \underbrace{\left(\frac{s + 0.5}{s} \right)}_{\text{PI Controller}} \left(\frac{s + 5}{(s + 8)(s + 10)^2} \right) = 0$$

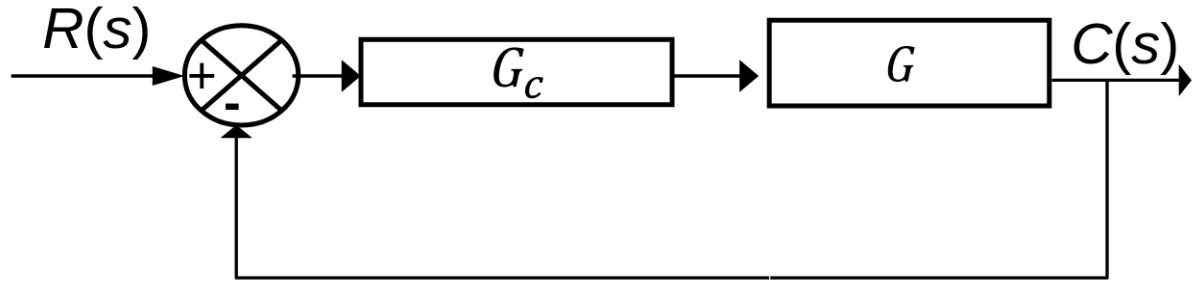


The root locus does not change much.

13.4.2 Drawbacks of PI controllers

- Cannot be realised with passive components.
- Alternative controller: **Lag compensator**.

13.4.3 Lag compensator



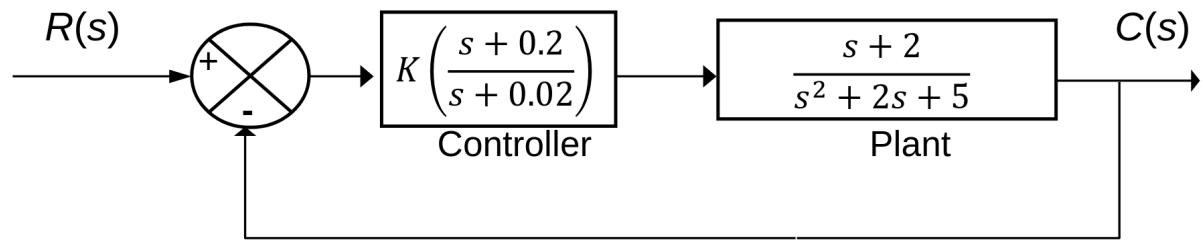
The objective is to enhance position error constant K_p .

$$\begin{aligned}
 K_p &= \lim_{s \rightarrow 0} G_c G \\
 &= \lim_{s \rightarrow 0} K \frac{s + z_c}{s + p_c} G_s \\
 &= K \frac{z_c}{p_c} G(0)
 \end{aligned}$$

$$e_{ss} = \frac{1}{1 + K_p}$$

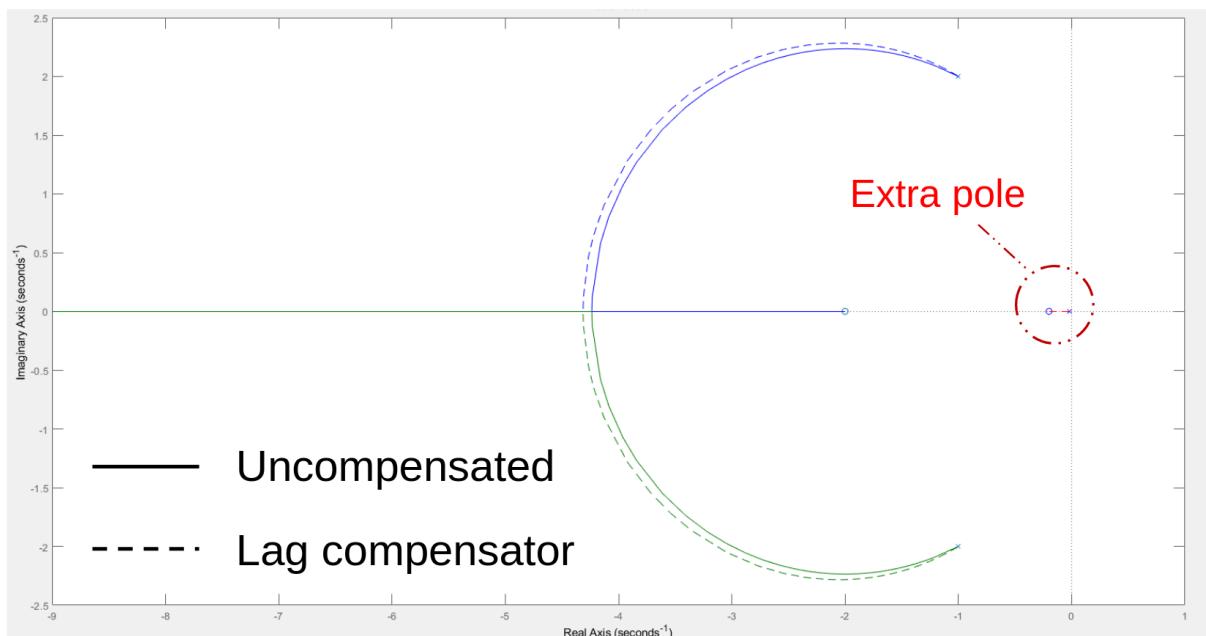
As the $K \frac{z_c}{p_c}$ factor increases, the value of K_p increases and the steady-state error (e_{ss}) decreases. The error reduces, but is not fully eliminated.

13.4.3.1 Example: Root locus



$$e_{ss} = \frac{1}{1 + K_p}$$

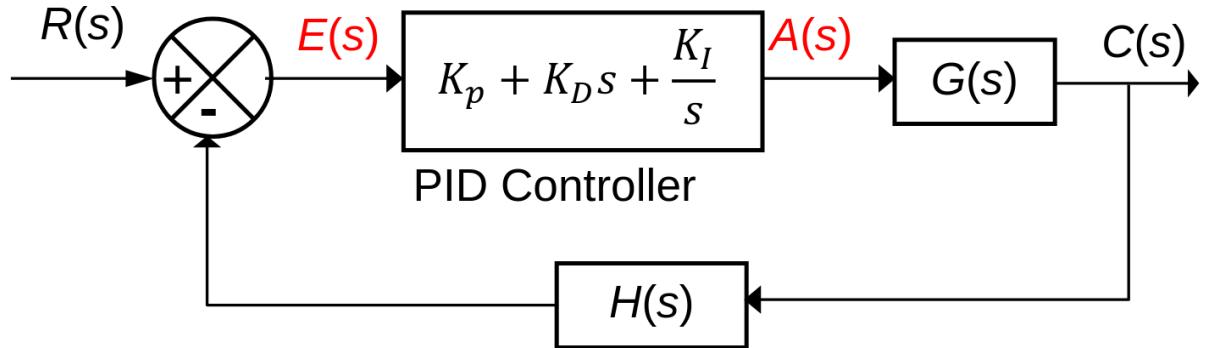
When the K_p increases, the steady-state error (e_{ss}) decreases.



The root locus does not change much.

13.5 Improving transient and steady-state response

13.5.1 PID controllers



Time domain:

$$a = K_p e + K_D \frac{de}{dt} + K_I \int_0^t e dt$$

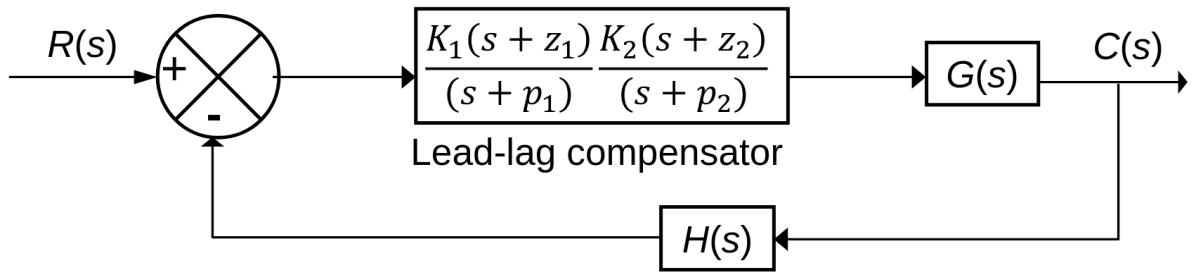
S-plane domain:

$$\begin{aligned} \text{PID : } K_p + K_D s + \frac{K_I}{s} &= \frac{K_D s^2 + K_p s + K_I}{s} \\ &= \frac{K_D (s + z_1)(s + z_2)}{s} \\ &= K_D \underbrace{(s + z_1)}_{\text{PD controller}} \underbrace{\frac{s + z_2}{s}}_{\text{PI controller}} \end{aligned}$$

Steps:

1. Use the PD controller to tune the transient response.
2. Use the PI controller to eliminate steady-state error.

13.5.2 Lead-lag compensators



Lead-lag compensator:

$$\underbrace{\left(\frac{K_1(s + z_1)}{s + p_1} \right)}_{\substack{\text{Lead compensator} \\ z_1 < p_1}} \underbrace{\left(\frac{K_2(s + z_2)}{s + p_2} \right)}_{\substack{\text{Lag compensator} \\ p_2 < z_2 \ll 1}}$$

Steps:

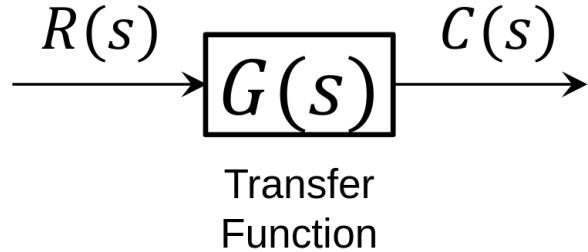
1. Design the lead compensator to tune the transient response.
2. Use the lag compensator to reduce steady-state error.

13.6 Summary

	Active	Passive
Transient Response	PD	Lead
Steady-state Response	PI	Lag
Transient and steady-state Response	PID	Lead-lag

14 Bode plots

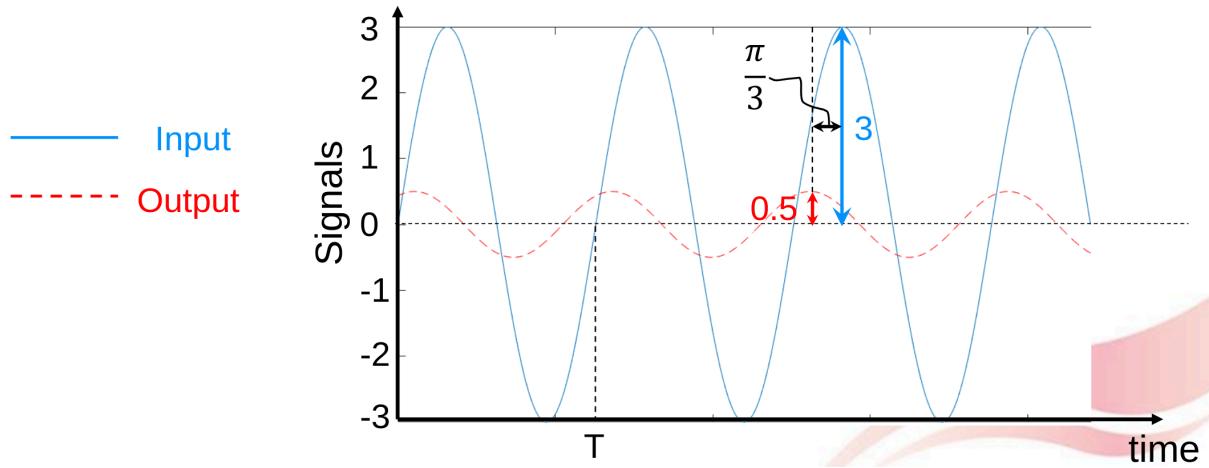
14.1 Harmonic inputs



Time domain:

$$\text{Input} : r = r_0 \cos(\omega t)$$

$$\text{Steady-state} : c = \underbrace{(r_0 |G(s = j\omega)|)}_{\text{Magnitude changed by a factor of } |G(s=j\omega)|} \cos \left(\omega t + \underbrace{\angle G(s = j\omega)}_{\text{Phase shift by a factor of } \angle G(s=j\omega)} \right)$$



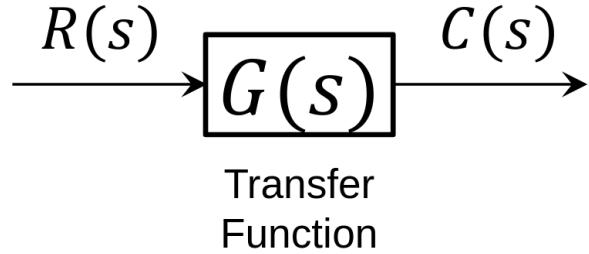
14.1.1 Example

$$\text{Input} : r(t) = r_0 \cos(10t)$$

$$\begin{aligned} G(s = 10j) &= \frac{2}{10j + 2} \\ &= \frac{2}{\sqrt{104} e^{j(\arctan 5)}} \\ &= \frac{2}{\sqrt{104}} e^{-j(\arctan 5)} \end{aligned}$$

$$c_{ss}(t) = r_0 \underbrace{|G(s = j\omega)|}_{\frac{2}{\sqrt{104}}} \cos \left(10t + \underbrace{\angle G(s = j\omega)}_{-\arctan 5} \right)$$

14.2 Bode plot

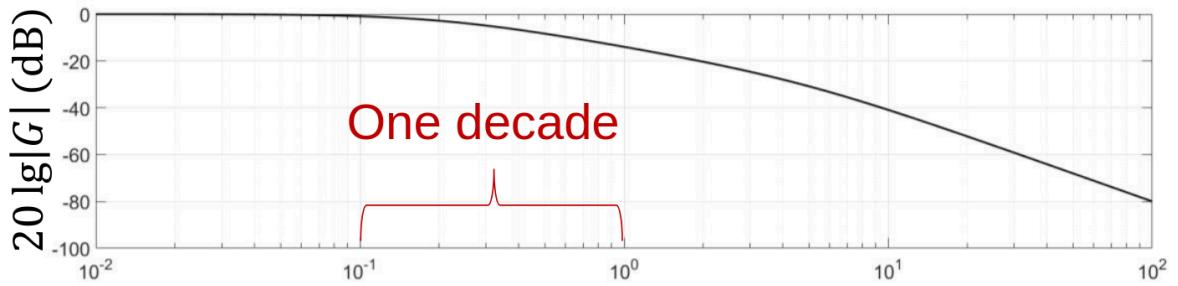


The bode plot shows the frequency response of the transfer function and generates two logarithmic scale plots:

1. Magnitude plot

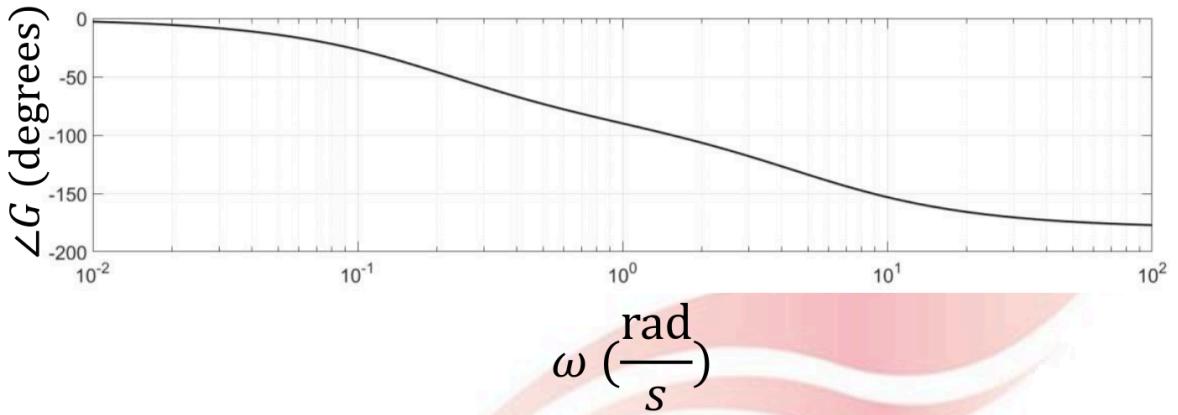
$$20 \log_{10}(|G(s = j\omega)|) \text{ against } \lg \omega$$

x-axis is $\lg \omega$.



2. Phase plot

$$\angle G(s = j\omega) \text{ against } \lg \omega$$



14.2.1 Motivation

1. Together with Fourier series analysis it can compute the output of periodic inputs.

For example, $r = 2 \sin(3t) + 6 \cos(100t + \pi)$

Output : $c = 2 |G(s = 3j)| \sin(3t + \angle G(s = 3j)) + 6|G(s = 100j)| \cos(100t + \pi + \angle G(s = 100))$

2. For analysis:

- Resonance
- Filtering

3. For designing closed-loop systems:

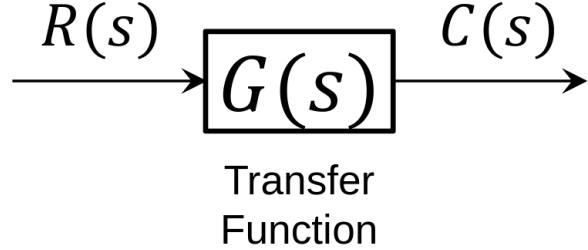
- **Poles**
- **Steady-state response**

4. Reverse engineering:

- **Identifying the transfer function** of the system

14.3 Superposition principle

The objective is to plot two log-scale plots.



$$G(s) = K \frac{\prod_{i=1}^n (s + z_i)}{\prod_{j=1}^m (s + p_j)}$$

Frequency response:

$$\begin{aligned} G(s = j\omega) &= K \frac{\prod_{i=1}^n (s + z_i)}{\prod_{j=1}^m (s + p_j)} \\ &= K \frac{\prod_{i=1}^n \left(\sqrt{\omega^2 + z_i^2} e^{j \arctan(\frac{\omega}{z_i})} \right)}{\prod_{j=1}^m \left(\sqrt{\omega^2 + p_j^2} e^{j \arctan(\frac{\omega}{p_j})} \right)} \\ &= K \underbrace{\frac{\prod_{i=1}^n \left(\sqrt{\omega^2 + z_i^2} \right)}{\prod_{j=1}^m \left(\sqrt{\omega^2 + p_j^2} \right)}}_{|G(s=j\omega)|} e^{\overbrace{j \left(\sum_{i=1}^n \arctan\left(\frac{\omega}{z_i}\right) - \sum_{j=1}^m \arctan\left(\frac{\omega}{p_j}\right) \right)}^{\angle G(s=j\omega)}} \end{aligned}$$

Transfer function:

$$G(s) = K \frac{\prod_{i=1}^n (s + z_i)}{\prod_{j=1}^m (s + p_j)}$$

$\angle G$, phase:

$$\sum_{i=1}^n \arctan\left(\frac{\omega}{z_i}\right) - \sum_{j=1}^m \arctan\left(\frac{\omega}{p_j}\right)$$

The above equation is a **linear combination of angles**.

$|G|$, magnitude:

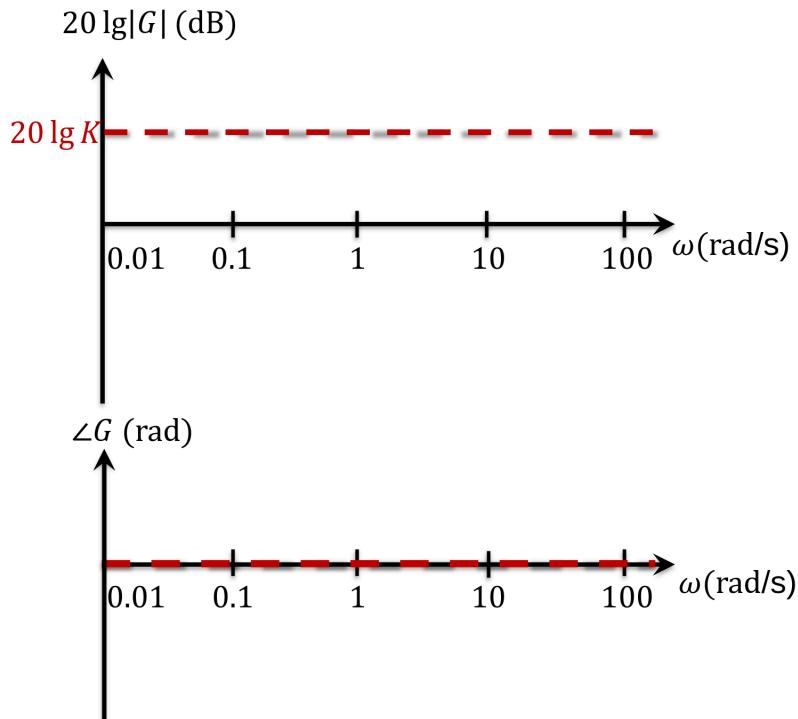
$$\begin{aligned} K \frac{\prod_{i=1}^n \left(\sqrt{\omega^2 + z_i^2} \right)}{\prod_{j=1}^m \left(\sqrt{\omega^2 + p_j^2} \right)} &\xrightarrow{\text{Log-scale}} 20 \lg \left(K \frac{\prod_{i=1}^n \left(\sqrt{\omega^2 + z_i^2} \right)}{\prod_{j=1}^m \left(\sqrt{\omega^2 + p_j^2} \right)} \right) \\ 20 \lg |G| &= 20 \lg(K) + \sum_{i=1}^n 20 \lg \left(\sqrt{\omega^2 + z_i^2} \right) - \sum_{j=1}^m 20 \lg \left(\omega^2 + p_j^2 \right) \end{aligned}$$

The above equation is a **linear combination of magnitudes**.

14.4 Principal components

1. Constants (K)
2. Differentiator (s)
3. Integrator ($\frac{1}{s}$)
4. First order systems ($\frac{a}{s+a}$)
5. Second order systems ($\frac{\omega_n^2}{s+2\zeta\omega_n s+\omega_n^2}$)

14.5 Plotting constants



$$G(s) = K$$

$$G(s = j\omega) = Ke^{j0}$$

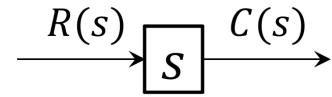
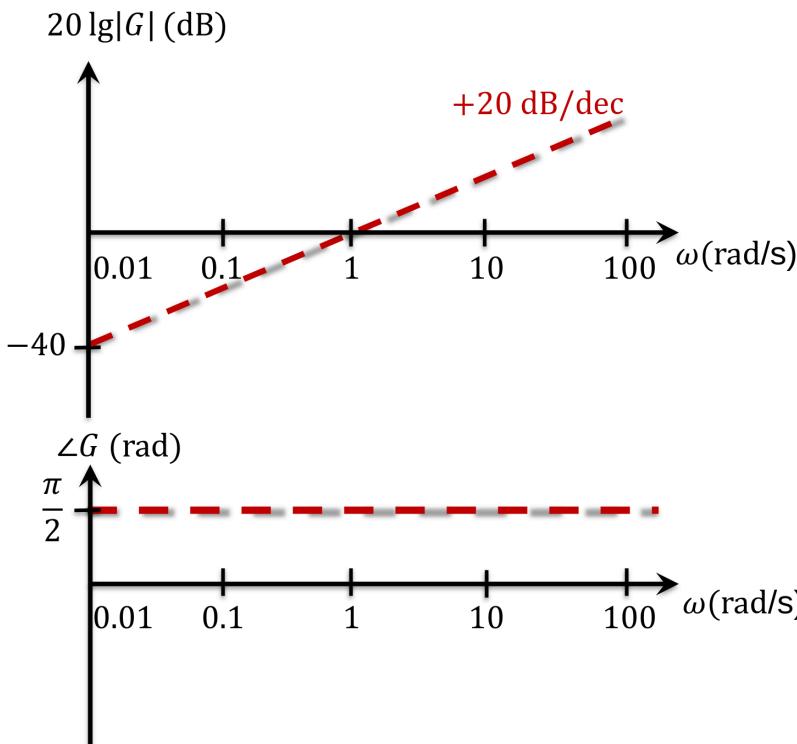
Magnitude:

$$20 \lg|G(j\omega)| = 20 \lg K$$

Phase:

$$\angle G(j\omega) = 0$$

14.6 Plotting differentiator



$$G(s) = s$$

$$G(s = j\omega) = j\omega$$

$$G(s = j\omega) = \omega e^{j\frac{\pi}{2}}$$

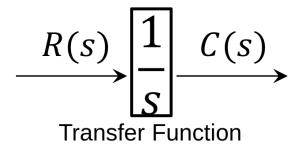
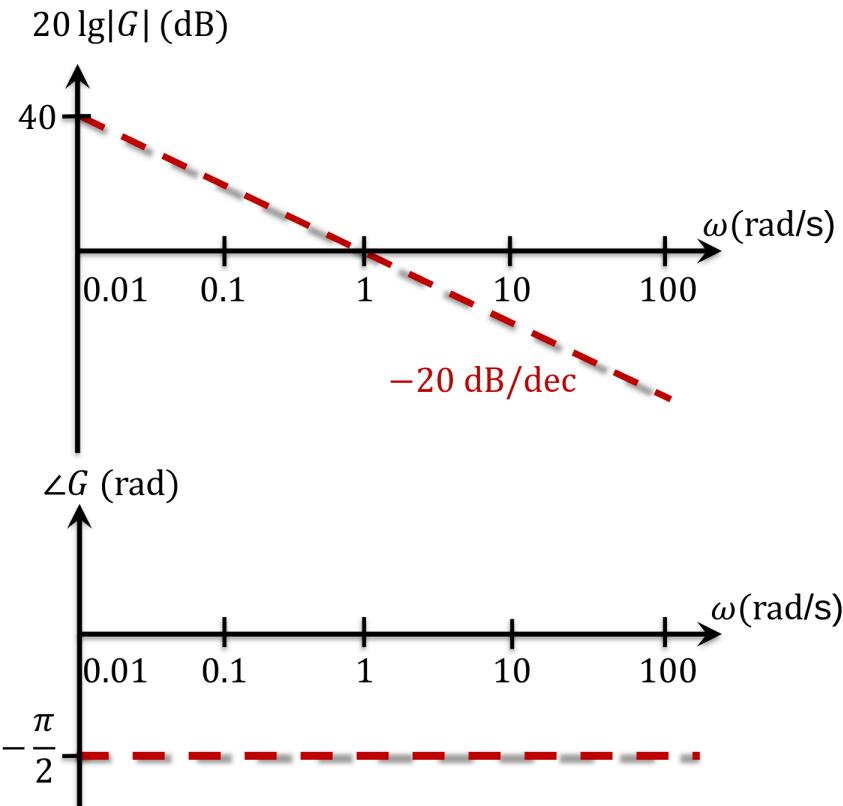
Magnitude:

$$20 \lg|G(j\omega)| = 20 \lg \omega$$

Phase:

$$\angle G(j\omega) = \frac{\pi}{2}$$

14.7 Plotting integrator



$$G(s) = \frac{1}{s}$$

$$G(s = j\omega) = \frac{1}{j\omega}$$

$$G(s = j\omega) = \frac{1}{\omega} e^{j - \frac{\pi}{2}}$$

Magnitude:

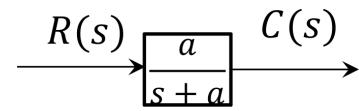
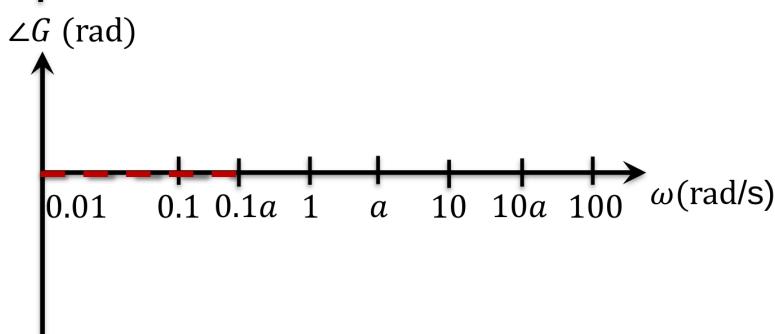
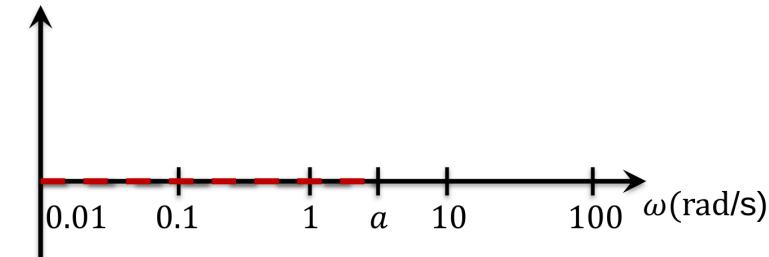
$$20 \lg|G(j\omega)| = -20 \lg \omega$$

Phase:

$$\angle G(j\omega) = -\frac{\pi}{2}$$

14.8 Plotting first order pole

$20 \lg|G|$ (dB)



$$G(s = j\omega) = \frac{a}{j\omega + a}$$

When $\omega \ll a$:

$$G(s = j\omega) = \frac{a}{a} = 1 e^{j0}$$

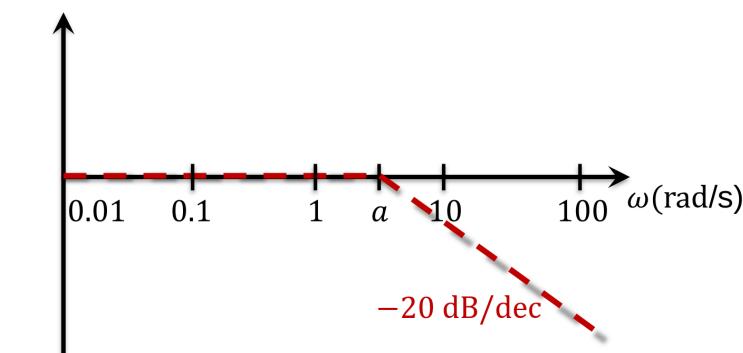
Magnitude:

$$20 \lg|G(j\omega)| = 0$$

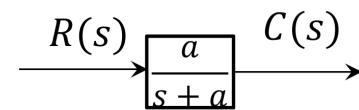
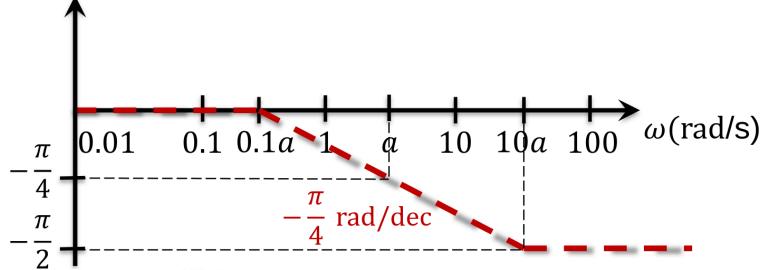
Phase:

$$\angle G(j\omega) = 0$$

$20 \lg|G|$ (dB)



$\angle G$ (rad)



$$G(s = j\omega) = \frac{a}{j\omega + a}$$

When $\omega \gg a$:

$$G(s = j\omega) = \frac{a}{j\omega} = \frac{a}{\omega} e^{j(-\frac{\pi}{2})}$$

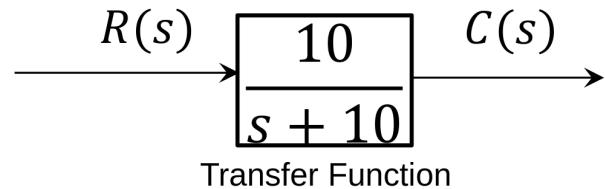
Magnitude:

$$\begin{aligned} 20 \lg|G(j\omega)| &= 20 \lg \frac{a}{\omega} \\ &= 20 \lg a - 20 \lg \omega \end{aligned}$$

Phase:

$$\angle G(j\omega) = -\frac{\pi}{2}$$

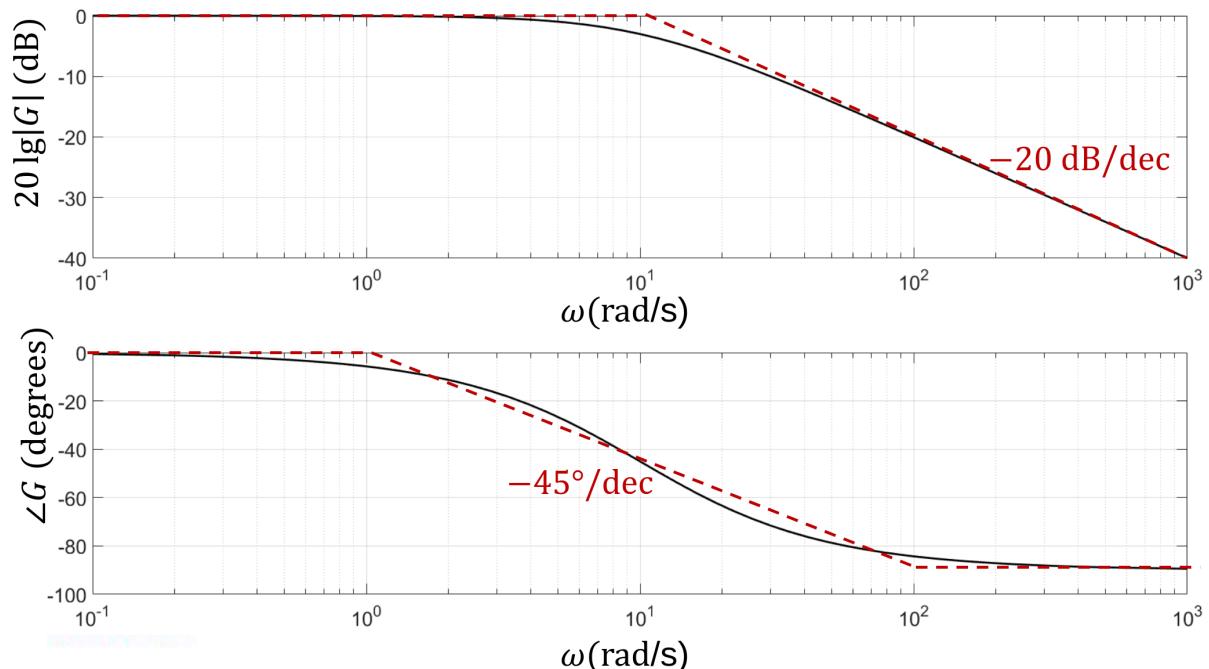
14.8.1 Example



$$a = 10$$

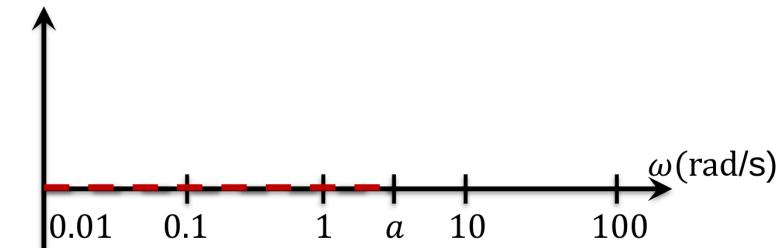
Corner frequencies:

$$\omega = 1, 10, 100$$

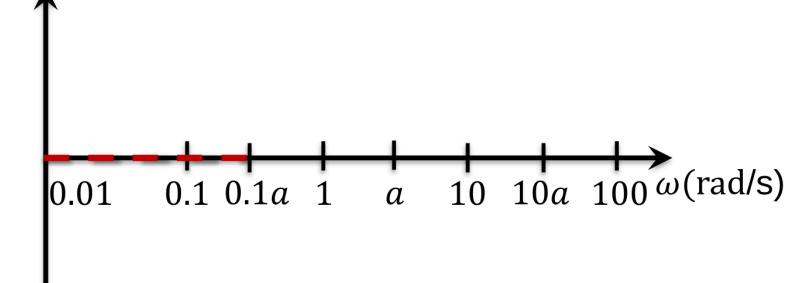


14.9 Plotting first order zero

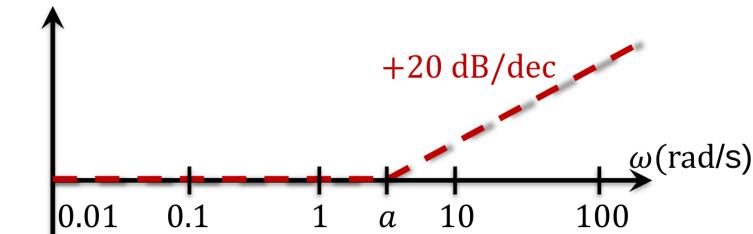
$20 \lg|G|$ (dB)



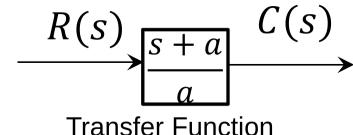
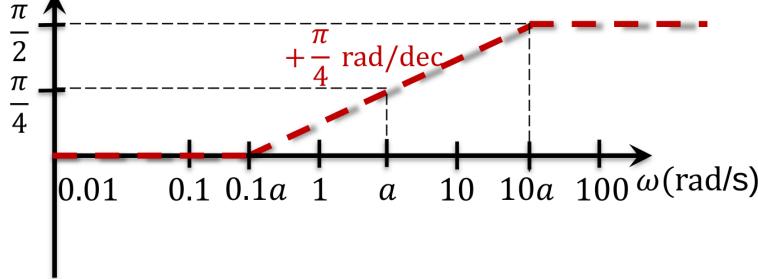
$\angle G$ (rad)



$20 \lg|G|$ (dB)



$\angle G$ (rad)



$$G(s = j\omega) = \frac{a}{j\omega + a}$$

When $\omega \ll a$:

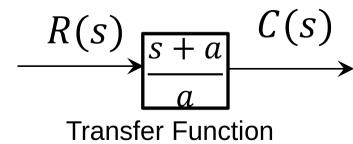
$$G(s = j\omega) = \frac{a}{a} = 1e^{j0}$$

Magnitude:

$$20 \lg|G(j\omega)| = 0$$

Phase:

$$\angle G(j\omega) = 0$$



$$G(s = j\omega) = \frac{a}{j\omega + a}$$

When $\omega \gg a$:

$$G(s = j\omega) = \frac{a}{j\omega} = \frac{a}{\omega} e^{j(\frac{\pi}{2})}$$

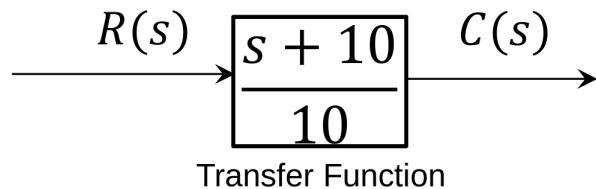
Magnitude:

$$\begin{aligned} 20 \lg|G(j\omega)| &= 20 \lg \frac{a}{\omega} \\ &= 20 \lg a - 20 \lg \omega \end{aligned}$$

Phase:

$$\angle G(j\omega) = \frac{\pi}{2}$$

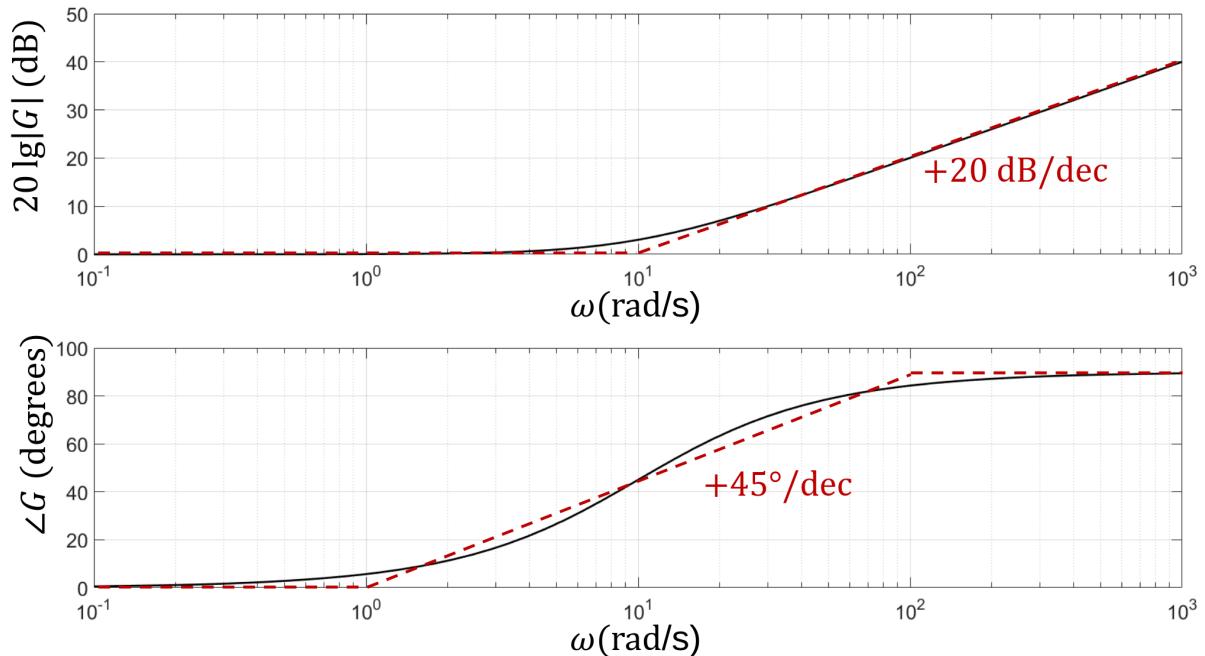
14.9.1 Example



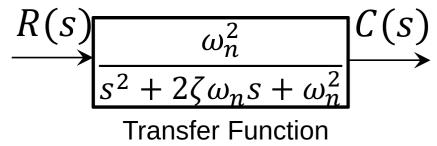
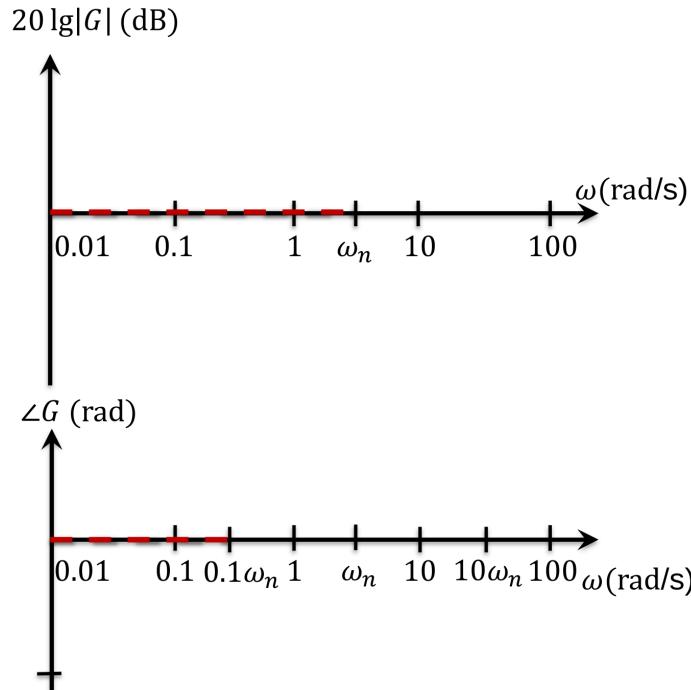
$$a = 10$$

Corner frequencies:

$$\omega = 1, 10, 100$$



14.10 Plotting second order pole



$$G(s = j\omega) = \frac{\omega_n^2}{(-\omega^2) + j2\zeta\omega_n\omega + \omega_n^2}$$

When $\omega \ll \omega_n$:

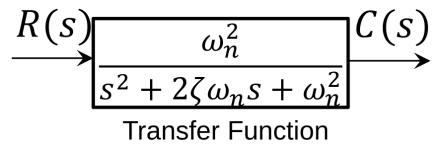
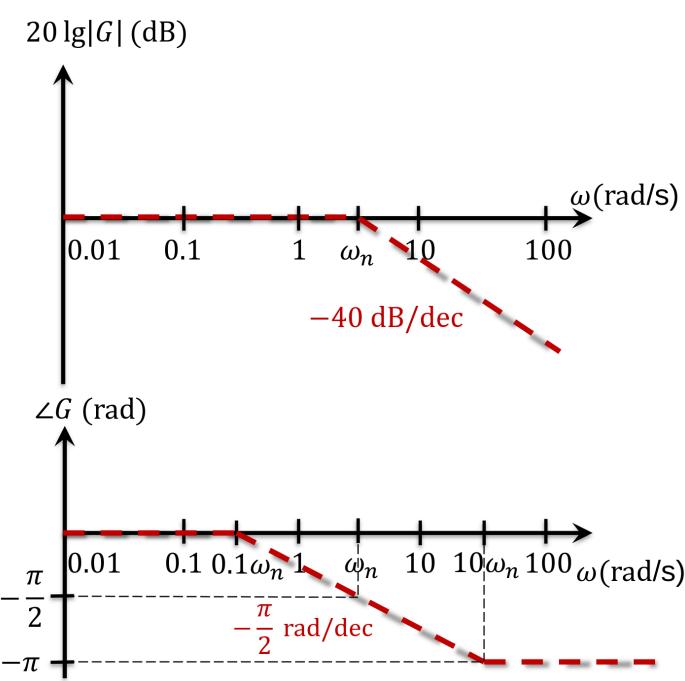
$$G(s = j\omega) = \frac{\omega_n^2}{\omega_n^2} = 1e^{j0}$$

Magnitude:

$$20 \lg|G(j\omega)| = 0$$

Phase:

$$\angle G(j\omega) = 0$$



$$G(s = j\omega) = \frac{\omega_n^2}{(-\omega^2) + j2\zeta\omega_n\omega + \omega_n^2}$$

When $\omega \gg \omega_n$:

$$G(s = j\omega) = \frac{\omega_n^2}{-\omega^2} = \frac{\omega_n^2}{\omega^2} e^{j(-\pi)}$$

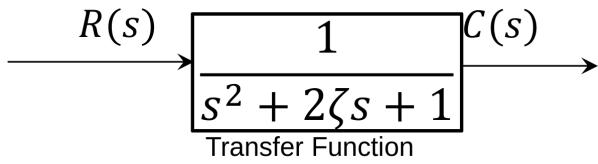
Magnitude:

$$\begin{aligned} 20 \lg|G(j\omega)| &= 20 \lg \frac{\omega_n^2}{\omega^2} \\ &= 40 \lg \omega_n - 40 \lg \omega \end{aligned}$$

Phase:

$$\angle G(j\omega) = -\pi$$

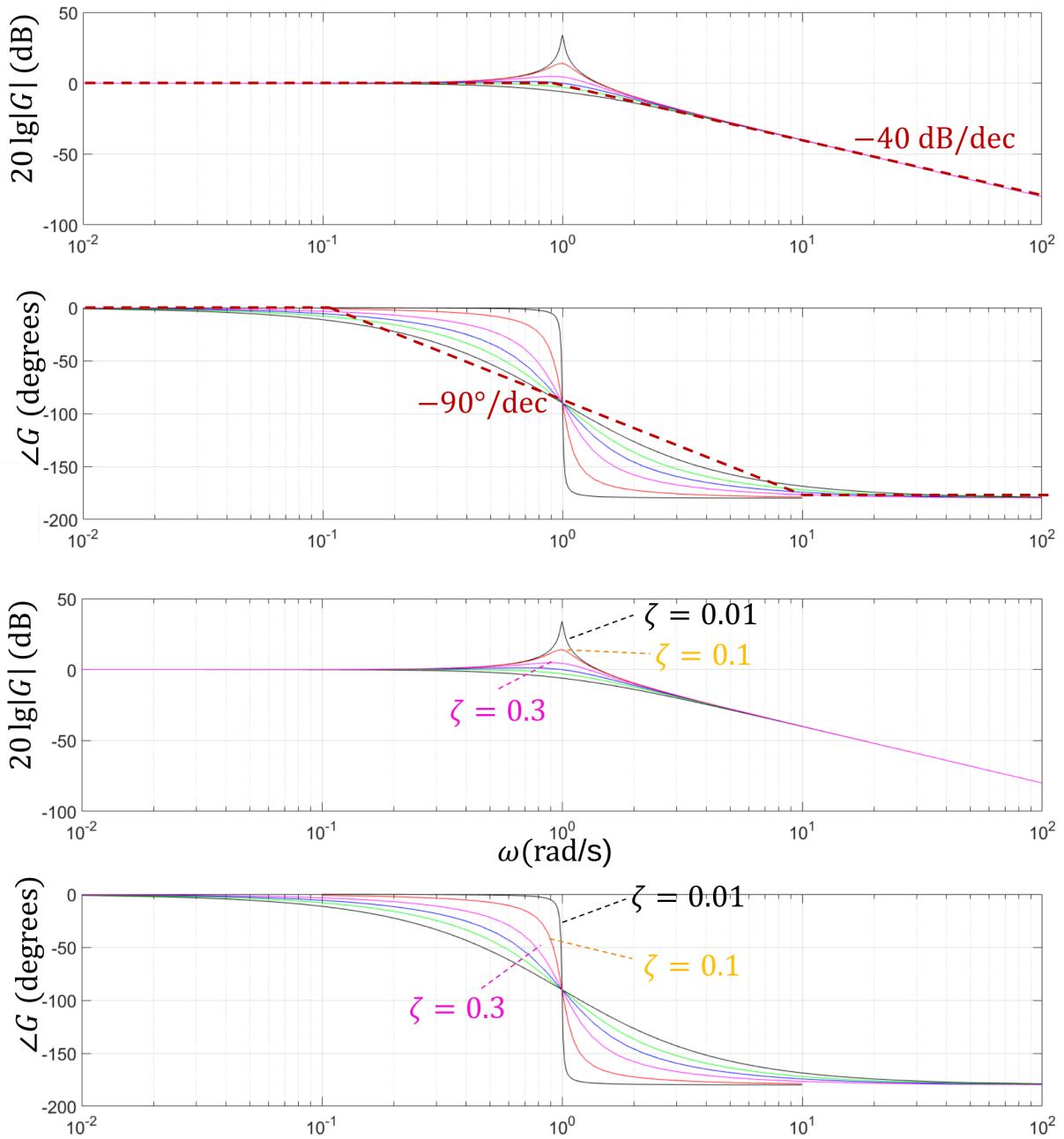
14.10.1 Example

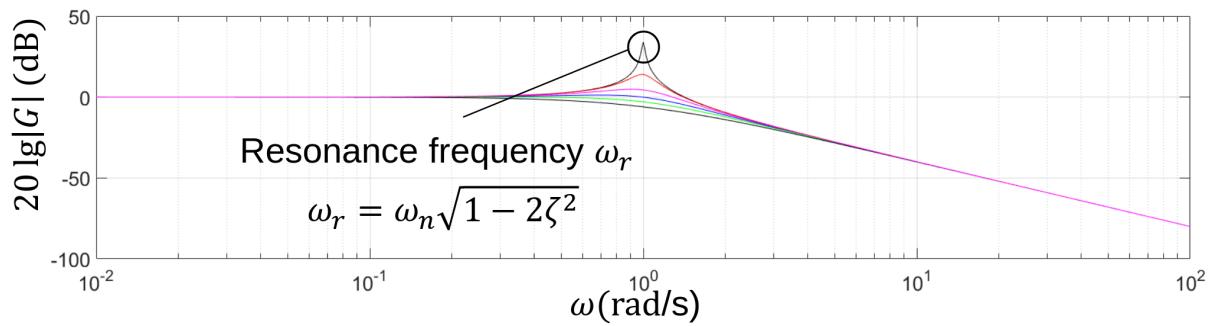


$$\omega_n = 1$$

Corner frequencies:

$$\omega = 0.1, 1, 10$$



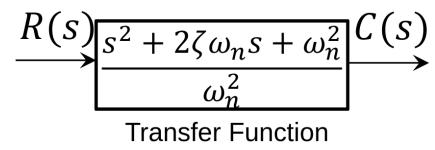
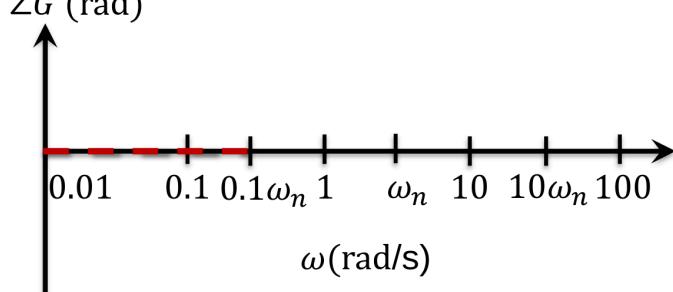
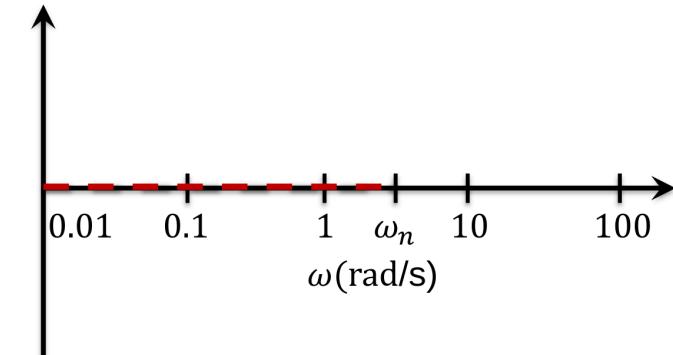


Resonance frequency ω_r :

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2}$$

14.11 Plotting second order zero

$20 \lg|G|$ (dB)



$$G(s = j\omega) = \frac{(-\omega^2) + j2\zeta\omega_n\omega + \omega_n^2}{\omega_n^2}$$

When $\omega \ll \omega_n$:

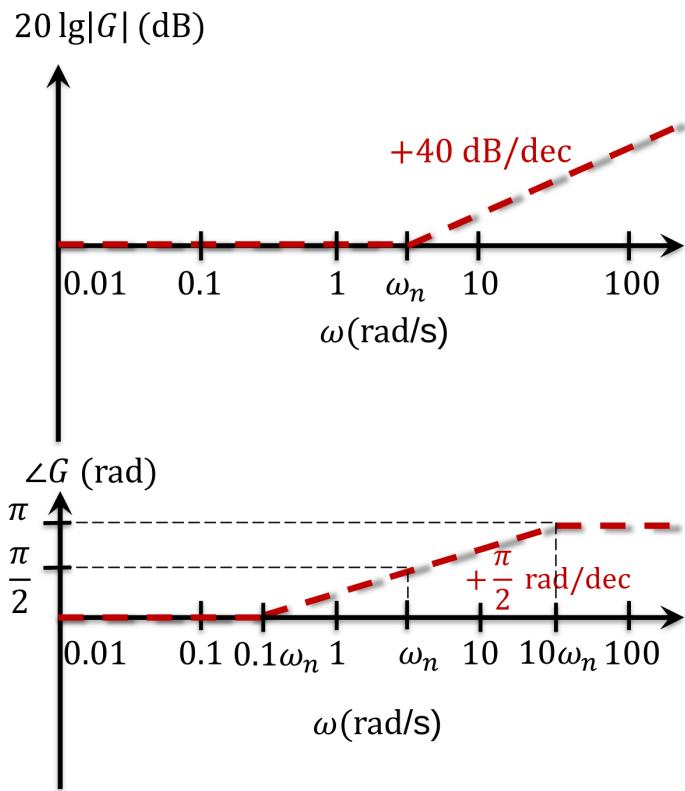
$$G(s = j\omega) = \frac{\omega_n^2}{\omega_n^2} = 1e^{j0}$$

Magnitude:

$$20 \lg|G(j\omega)| = 0$$

Phase:

$$\angle G(j\omega) = 0$$



$$\frac{R(s)}{C(s)} = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{\omega_n^2}$$

Transfer Function

$$G(s = j\omega) = \frac{(-\omega^2) + j2\zeta\omega_n\omega + \omega_n^2}{\omega_n^2}$$

When $\omega \gg \omega_n$:

$$G(s = j\omega) = -\frac{\omega^2}{\omega_n^2} = \frac{\omega^2}{\omega_n^2} e^{j\pi}$$

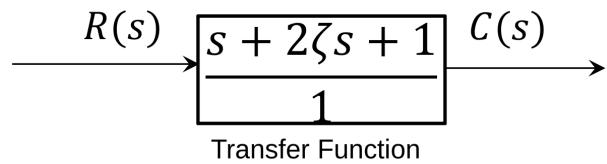
Magnitude:

$$\begin{aligned} 20 \lg|G(j\omega)| &= 20 \lg \frac{\omega^2}{\omega_n^2} \\ &= 40 \lg \omega - 40 \lg \omega_n \end{aligned}$$

Phase:

$$\angle G(j\omega) = \pi$$

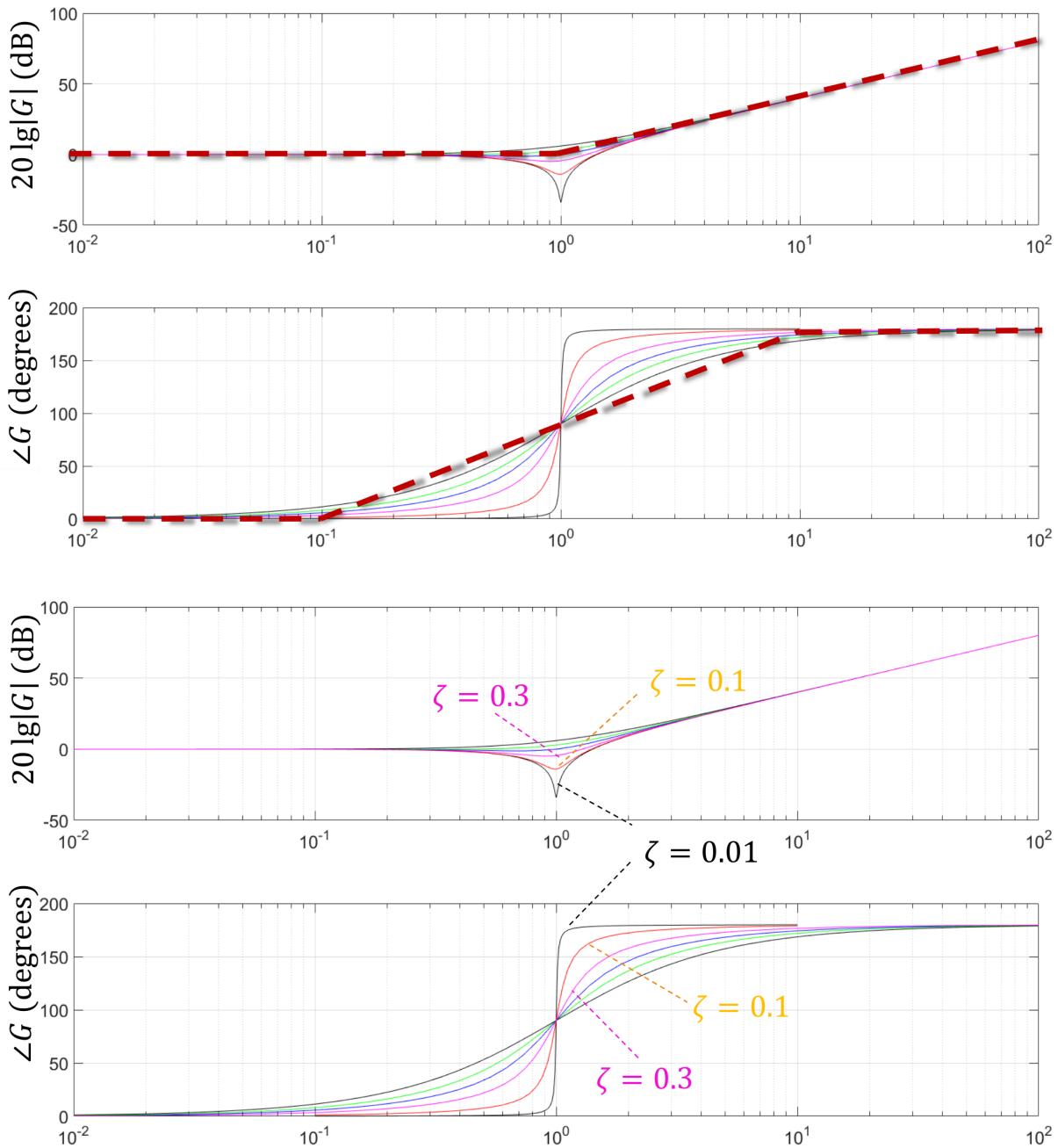
14.11.1 Example



$$\omega_n = 1$$

Corner frequencies:

$$\omega = 0.1, 1, 10$$



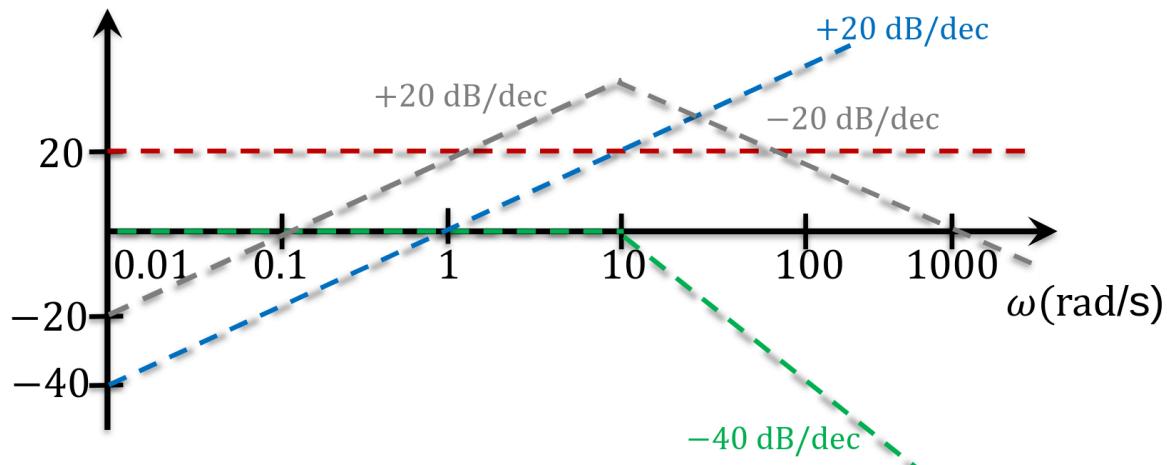
14.12 Complex function

$$G = \frac{1000s}{s^2 + s + 100}$$

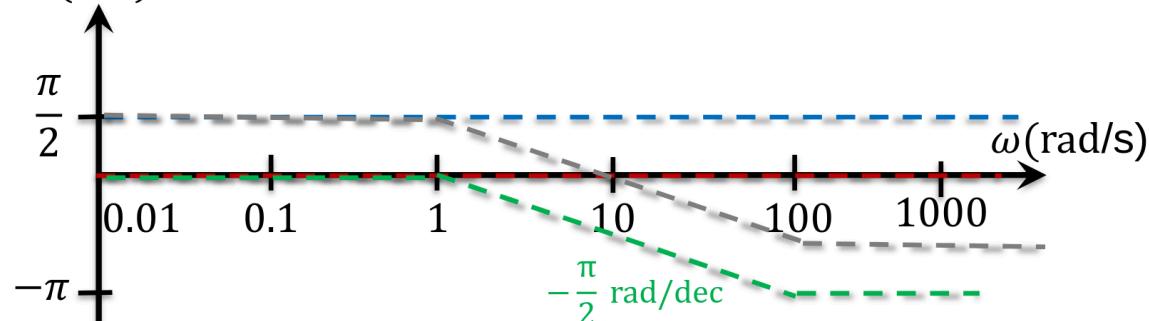
$$G = \frac{1000}{100} (s) \left(\frac{100}{s^2 + s + 100} \right)$$

14.12.1 Asymptotes

$20 \lg|G|$ (dB)



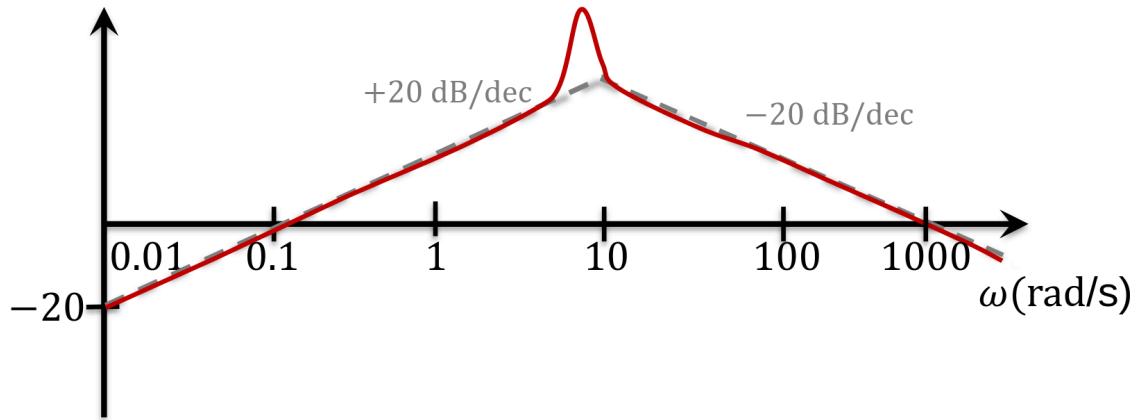
$\angle G$ (rad)



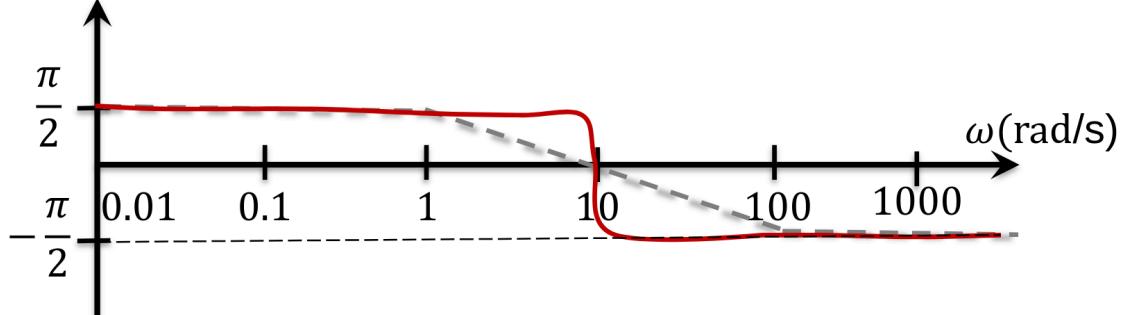
14.12.2 Sketch

$$G = \frac{1000}{100} (s) \left(\frac{100}{s^2 + s + 100} \right)$$

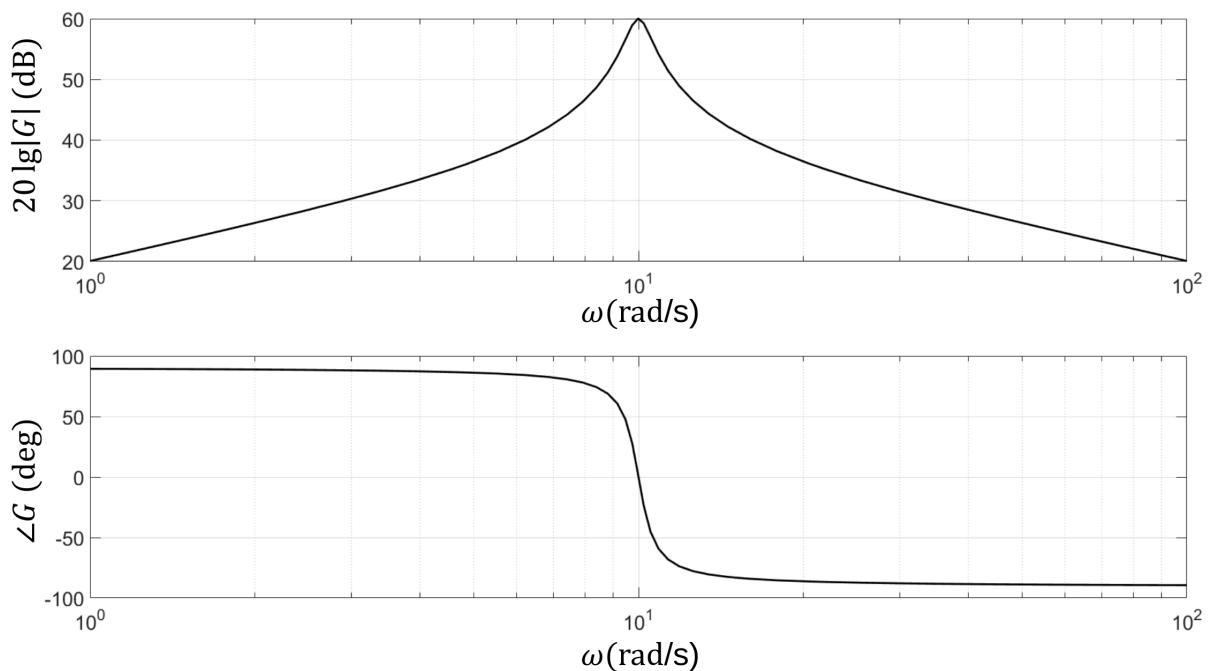
$20 \lg|G|$ (dB)



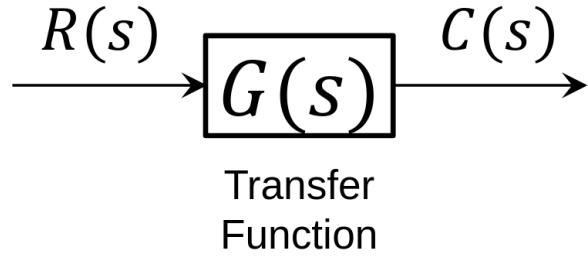
$\angle G$ (rad)



14.12.3 MATLAB plot



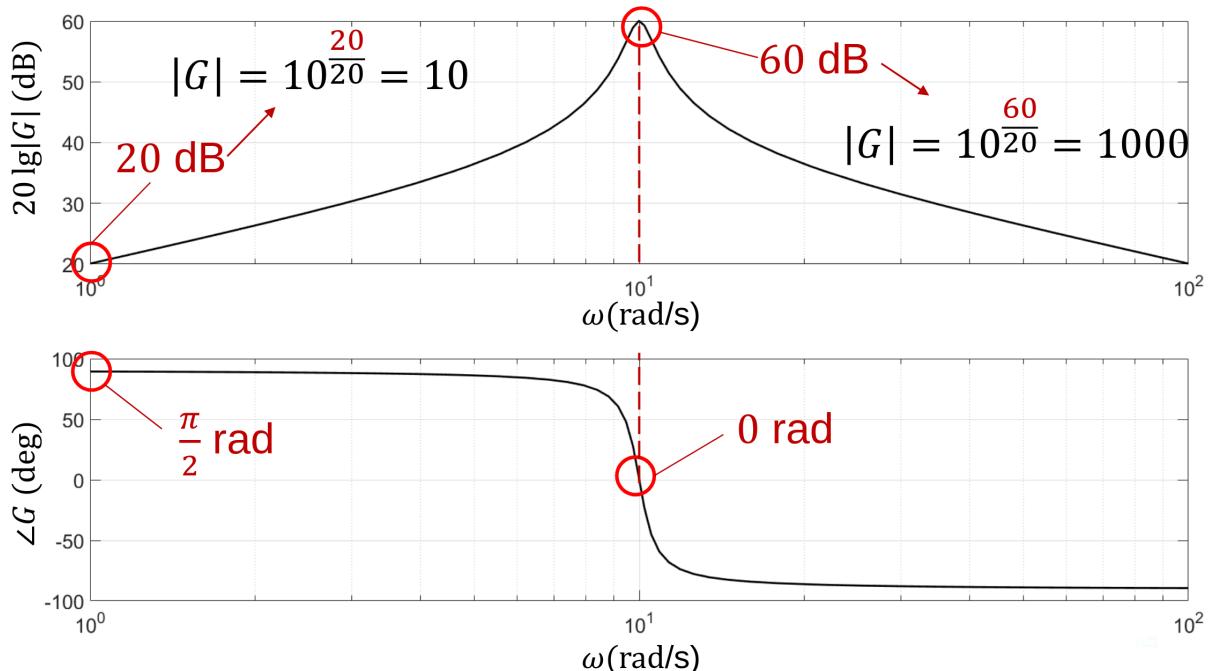
14.12.4 Reading the bode plot



Compute the output if the input is:

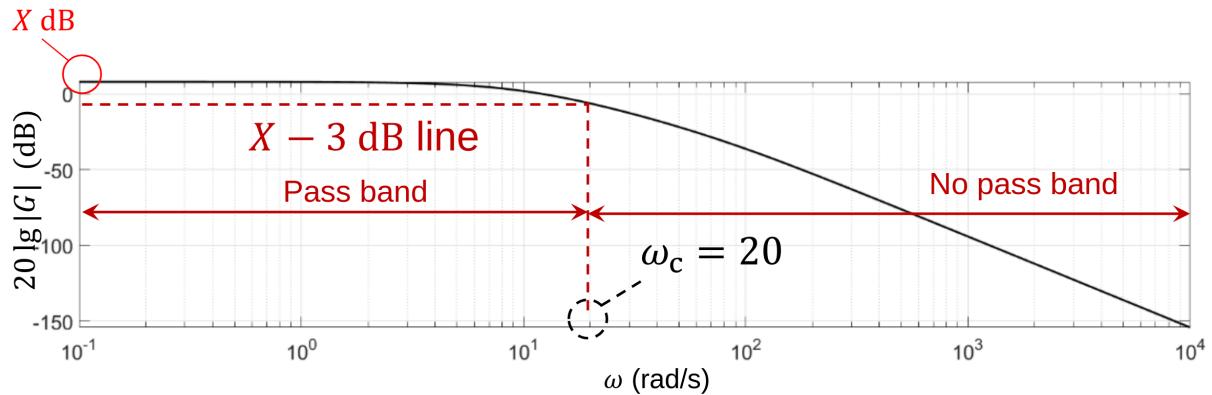
$$r(t) = 2 \cos t + 5 \cos 10t$$

$$c(t) = 2|G(s = j)| \cos(t + \angle G(s = j)) + 5|G(s = 10j)| \cos(10t + \angle G(s = 10j))$$



$$\begin{aligned} c(t) &= 2 \underbrace{|G(s = j)|}_{10} \cos \left(t + \underbrace{\angle G(s = j)}_{\frac{\pi}{2}} \right) + 5 \underbrace{|G(s = 10j)|}_{1000} \cos \left(10t + \underbrace{\angle G(s = 10j)}_0 \right) \\ \therefore c(t) &= 20 \cos \left(t + \frac{\pi}{2} \right) + 5000 \cos(10t) \end{aligned}$$

14.13 Low-pass filter



Input Signal

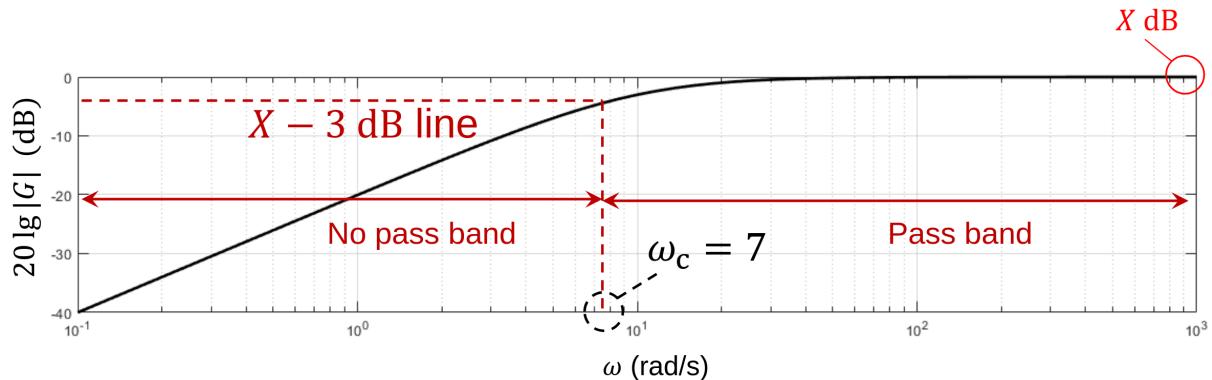
$$r(t) = \cos(\omega t) \quad \rightarrow \quad c(t) = |G(s = j\omega)| \cos(\omega t + \angle G)$$

Output Signal

Relationship between $|G(s = j\omega)|$ and the cut-off frequency (ω_c):

$$|G(s = j\omega)| = \begin{cases} \ll |G(s = j\omega_{\text{low}})|, & \omega \geq \omega_c \\ \sim |G(s = j\omega_{\text{low}})|, & \omega < \omega_c \end{cases}$$

14.14 High-pass filter



Input Signal

$$r(t) = \cos(\omega t) \quad \rightarrow \quad c(t) = |G(s = j\omega)| \cos(\omega t + \angle G)$$

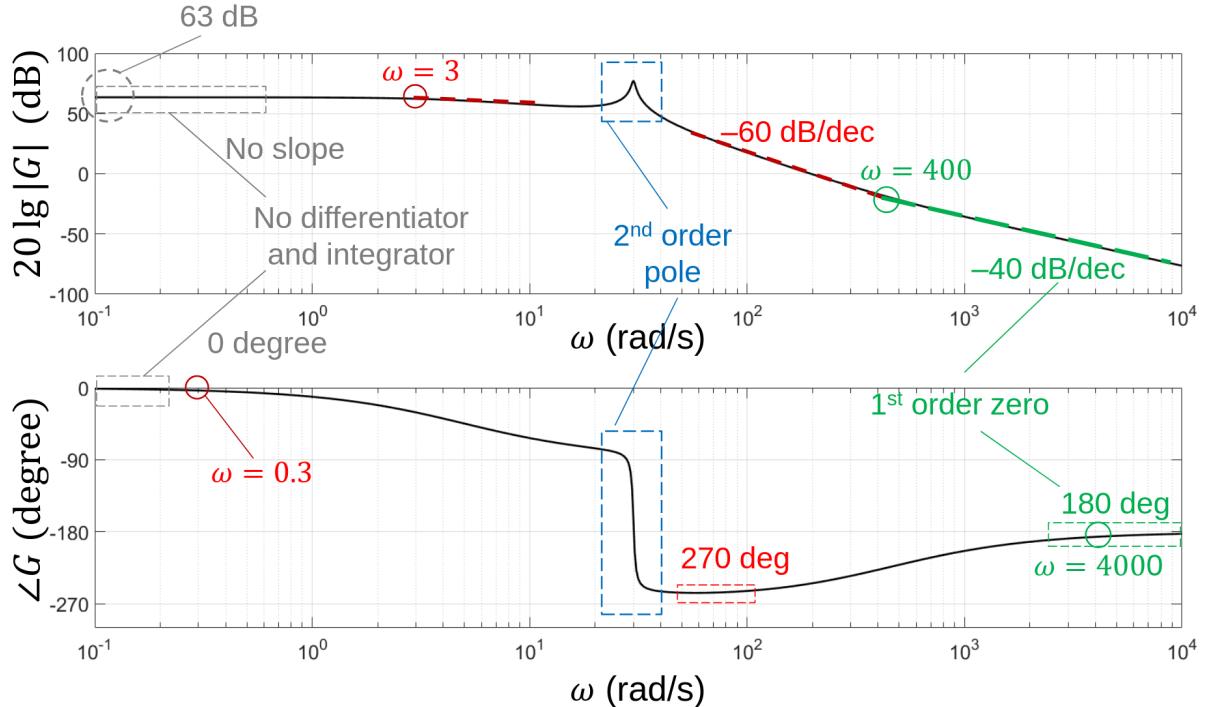
Output Signal

Relationship between $|G(s = j\omega)|$ and the cut-off frequency (ω_c):

$$|G(s = j\omega)| = \begin{cases} \sim |G(s = j\omega_{\text{high}})|, & \omega \geq \omega_c \\ \ll |G(s = j\omega_{\text{high}})|, & \omega < \omega_c \end{cases}$$

14.15 Reverse engineering

Reverse engineering's aim is to use bode plots to approximate the transfer function.



Estimated transfer function:

$$G(s) = \frac{30^2}{s^2 + 2(0.01)(30)s + 30^2} \left(\frac{3}{s+3} \right) \left(\frac{s+400}{400} \right) K$$

$$G(s) = \frac{900K}{s^2 + 0.6s + 900} \left(\frac{3}{s+3} \right) \left(\frac{s+400}{400} \right)$$

DC gain from plot is 63.5 dB.

Hence:

$$20 \lg \left| \lim_{s \rightarrow 0} G(s) \right| = 63.5$$

$$\left| \lim_{s \rightarrow 0} G(s) \right| = 10^{\frac{63.5}{20}}$$

$$\frac{900K}{900} \left(\frac{3}{3} \right) \left(\frac{400}{400} \right) = 10^{\frac{63.5}{20}}$$

$$K = 10^{\frac{63.5}{20}} = 1500$$

Estimated

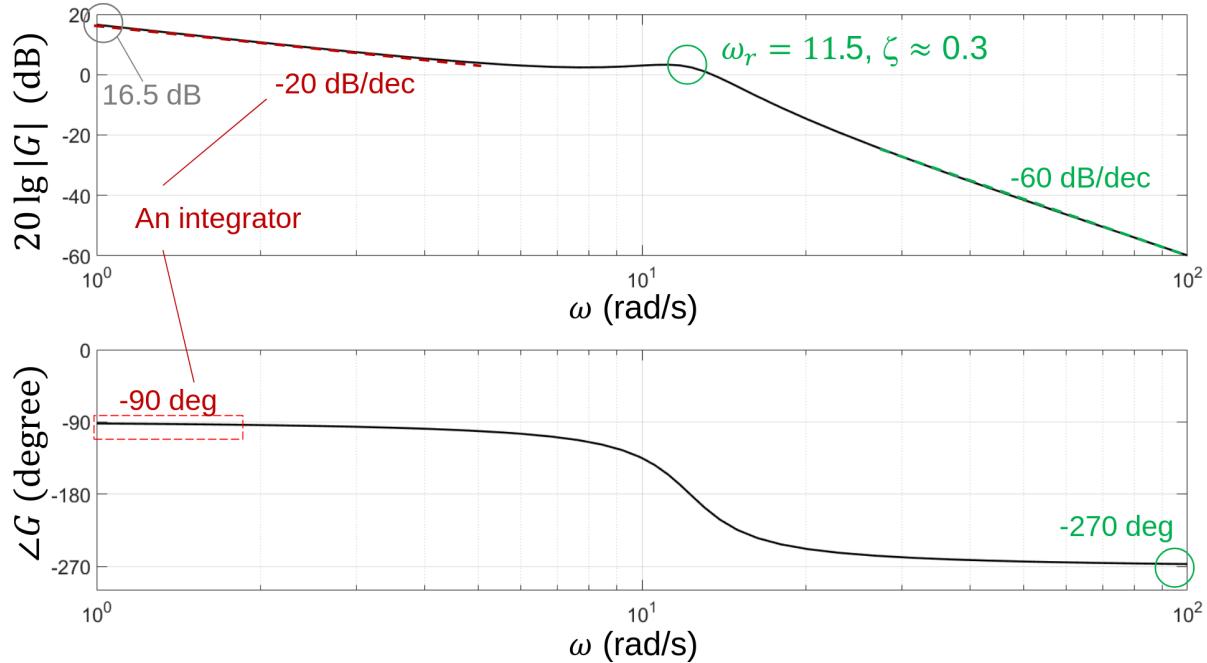
$$G(s) = \frac{10125(s+400)}{(s^2 + 0.6s + 900)(s+3)}$$

Actual

$$G(s) = \frac{15000(s+450)}{(s^2 + s + 900)(s+5)}$$

14.15.1 Example

Use the bode plots to approximate the transfer function:



Estimated transfer function:

$$G(s) = \frac{1}{s} (G_1) K$$

Low frequency gain = 16.5 dB

$$\omega_r = 11.5 \text{ rad}, \quad \zeta \approx 0.3$$

Resonance frequency analysis:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} = 11.5$$

$$\omega_n = 12.7$$

$$G_1 = \frac{12.7^2}{s^2 + 2(0.3)(12.7)s + 12.7^2}$$

Low frequency gain analysis:

$$20 \lg \left| \lim_{s \rightarrow j} G(s) \right| = 16.5 \quad \rightarrow \quad \left| \lim_{s \rightarrow j} G(s) \right| = 10^{\frac{16.5}{20}}$$

$$\left| \left(\frac{K}{j} \right) \left(\frac{161}{-1 + 7.6j + 161} \right) \right| = 10^{\frac{16.5}{20}}$$

$$K = 6.6$$

Estimated

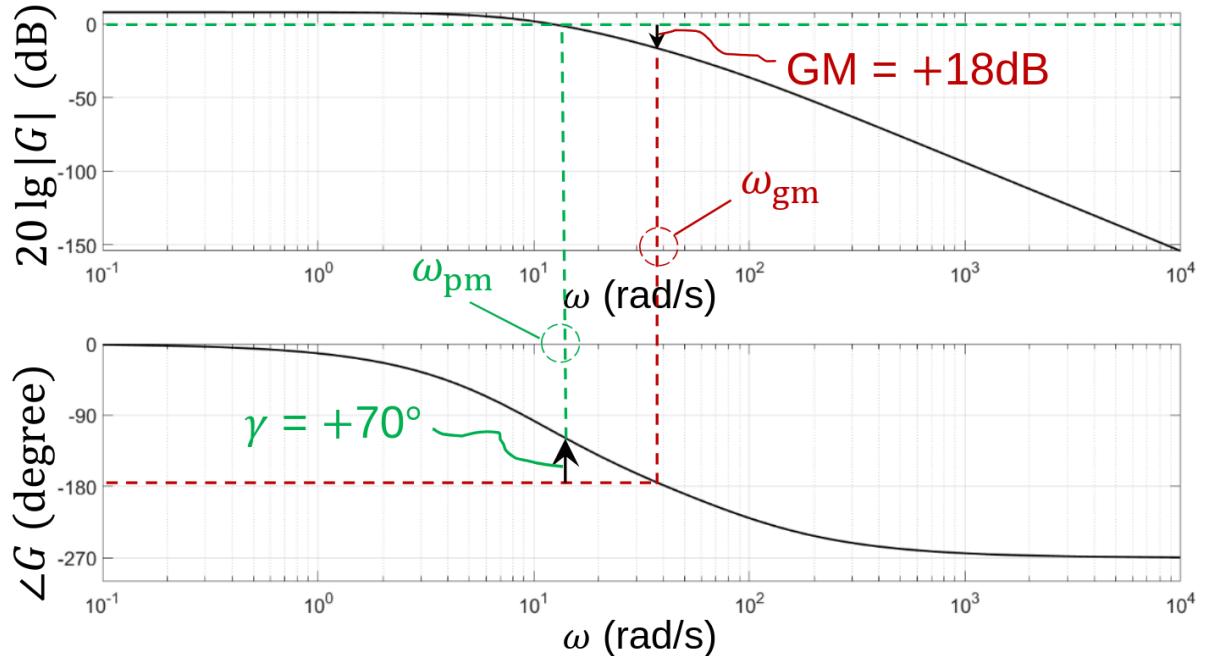
$$G(s) = \frac{1065}{(s^2 + 7.6s + 161)s} \quad G(s) = \frac{1000}{(s^2 + 5s + 150)s}$$

Actual

14.16 Closed-loop response

The bode plot of G (open-loop transfer function) also reveals the response of the **unity feedback system**.

14.17 Gain and phase margins



- **Gain margin (Stable G_{OL})**

1. Occurs at the frequency when $\angle G = -\pi$ rad (ω_{gm})
2. **Gain margin (GM)** is given by:

$$GM = 0 - |G(s = j\omega_{gm})|_{\text{dB}}$$

- **Phase margin (Stable G_{OL})**

1. Occurs at the frequency when $|G| = 0$ dB (ω_{pm})
2. **Phase margin (γ)** is given by:

$$\gamma = \angle G(s = j\omega_{pm} - (-\pi))$$

- **Stability**

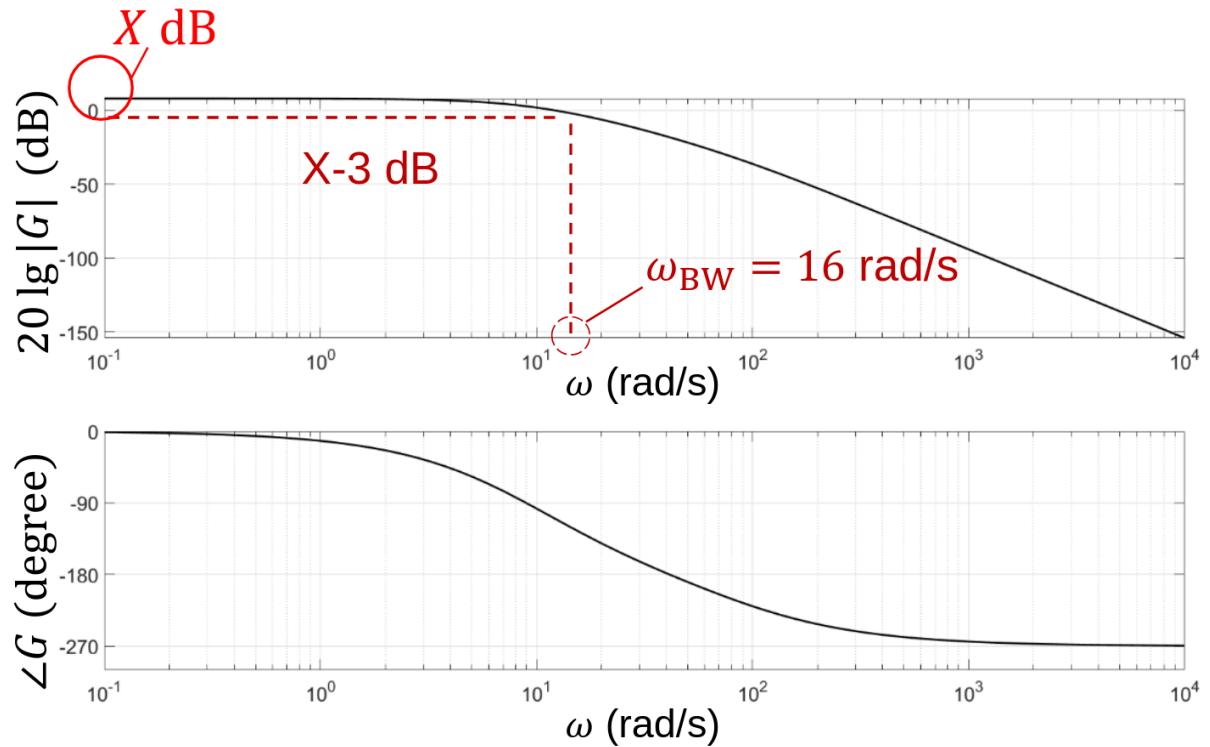
Unity feedback is stable if and only if:

$$\gamma > 0 \text{ and } GM > 0$$

- **Closed-loop ζ_{cl}**

γ increasing implies that ζ_{cl} is increasing too.

14.18 Bandwidth



- **Bandwidth:**

The frequency when $|G| = X - 3$ dB. It is denoted as ω_{BW} .

- **Closed loop $\omega_{n,cl}$:**

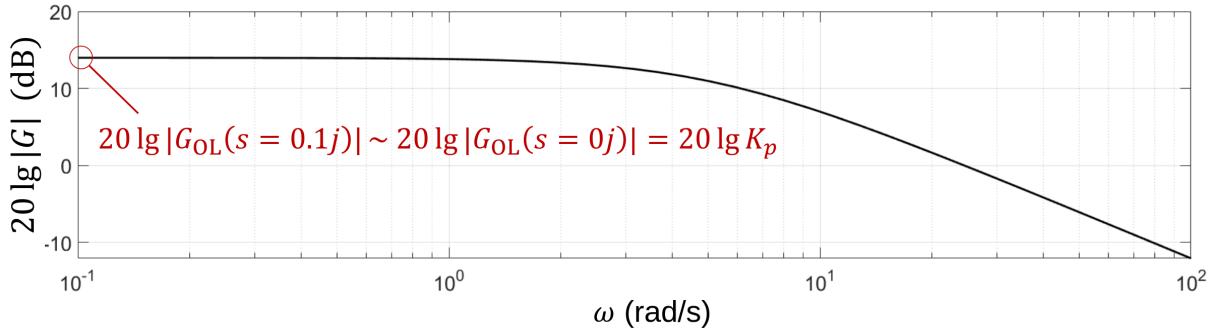
ω_{BW} increasing implies that $\omega_{n,cl}$ is increasing too.

- **Closed-loop poles:**

$$s_{1,2} = -\zeta_{cl}\omega_{n,cl} \pm j\omega_{n,cl}\sqrt{1 - \zeta_{cl}^2}$$

14.19 Steady-state errors

14.19.1 Type 0 systems



Steady-state error e_{ss} (step inputs):

$$\frac{1}{1 + K_p}$$

Static position error constant:

$$K_p = \lim_{s \rightarrow 0} G_{OL}$$

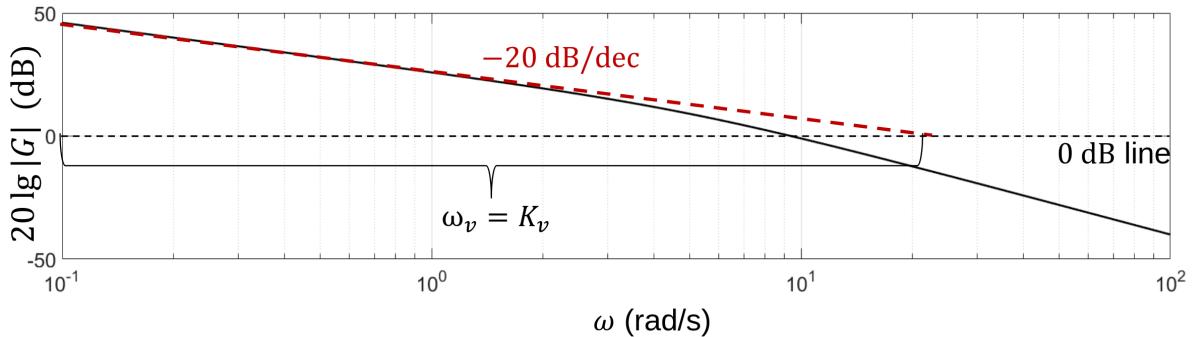
For example:

$$14 \text{ dB} = 20 \lg K_p$$

$$K_p = 10^{\frac{14}{20}} = 5$$

$$e_{ss} = \frac{1}{1 + 5} = \frac{1}{6}$$

14.19.2 Type 1 systems



Steady-state error e_{ss} (ramp inputs):

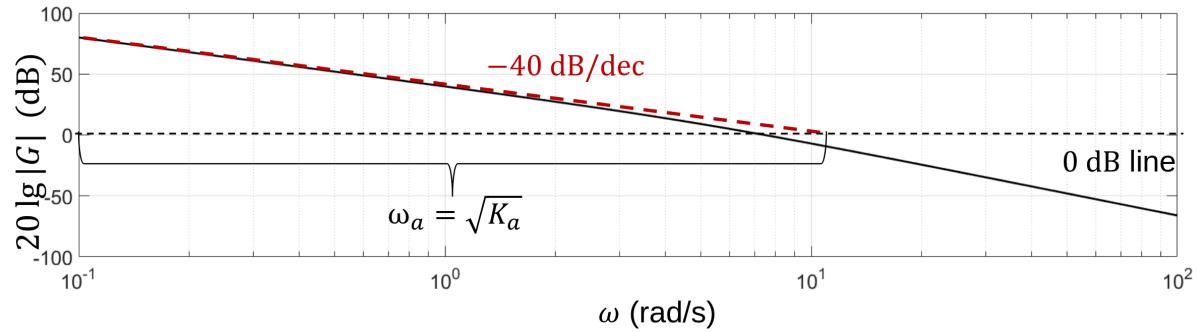
$$\frac{1}{K_v}$$

For example:

$$K_v = \omega_v = 21$$

$$e_{ss} = \frac{1}{21}$$

14.19.3 Type 2 systems



Steady-state error e_{ss} (parabolic inputs):

$$\frac{1}{K_a}$$

For example:

$$\omega_a = 10.8$$

$$K_a = 116$$

$$e_{ss} = \frac{1}{116}$$

14.20 Summary

1. Bode plot is the frequency response of the system.
2. Sketching techniques
 - a) Superposition principle
 - b) Sketch the principal components
 - c) Combine all the components together
3. Low-pass and high-pass filters
4. Reverse engineering
5. Unity feedback systems based on open-loop bode plots
 - a) Stability
 - b) Transient response
 - c) Steady-state errors