1a) $\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle = \int_{-1}^{1} f(x)\,g(x)\,(1+x^2)\,dx$

$= \int_{-1}^{1} g(x)\,f(x)\,(1+x^2)\,dx$

$= \langle \underset{\sim}{g}, \underset{\sim}{f} \rangle$

$\langle \underset{\sim}{f} + \underset{\sim}{g}, \underset{\sim}{h} \rangle = \int_{-1}^{1} (f+g)(x)\,h(x)\,(1+x^2)\,dx$

$= \int_{-1}^{1} \big(f(x)+g(x)\big)\,h(x)\,(1+x^2)\,dx$

$= \int_{-1}^{1} f(x)\,h(x)\,(1+x^2)\,dx$

$+ \int_{-1}^{1} g(x)\,h(x)\,(1+x^2)\,dx$

$= \langle \underset{\sim}{f}, \underset{\sim}{h} \rangle + \langle \underset{\sim}{g}, \underset{\sim}{h} \rangle$

1a) $\langle k\underset{\sim}{f}, \underset{\sim}{g} \rangle = \int_{-1}^{1} k f(x) g(x) (1+x^2)\, dx$

$= k\int_{-1}^{1} f(x) g(x)(1+x^2)\, dx$

$= k\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle$

$\langle \underset{\sim}{f}, \underset{\sim}{f} \rangle = 0$

$\int_{-1}^{1} f(x) f(x) (1+x^2)\, dx = 0$

$\int_{-1}^{1} (f(x))^2 (1+x^2)\, dx = 0$

Using Lemma 1, since

$(f(x))^2(1+x^2)$ is continuous,

$(f(x))^2(1+x^2) > 0$ for all $x \in [-1,1]$

$\because (f(x))^2 > 0$ and $1+x^2 > 0$

for all $x \in [-1,1]$,

and $\int_{-1}^{1} (f(x))^2 (1+x^2)\, dx = 0$,

then $(f(x))^2(1+x^2) = 0$ for all $x \in [-1,1]$

1a) Since $1+x^2 > 0$ for all $x \in [-1,1]$

$\therefore (f(x))^2 = 0$ for $(f(x))^2(1+x^2) = 0$

$\therefore f(x) = 0$

$\therefore \underset{\sim}{f} = \underset{\sim}{0}$

When $\underset{\sim}{f} = \underset{\sim}{0}$,

$$\langle \underset{\sim}{f}, \underset{\sim}{f} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{0} \rangle$$
$$= \int_{-1}^{1} (0)^2 (1+x^2)\,dx$$
$$= 0$$

$\therefore \langle \underset{\sim}{f}, \underset{\sim}{g} \rangle$ is an inner product on V.

1b) Let $f(x) = x$,

$$\langle \underset{\sim}{f}, \underset{\sim}{f} \rangle = \int_{-1}^{1} f(x) f(x) x\,dx$$
$$= \int_{-1}^{1} x^3\,dx$$
$$= \frac{1}{4} x^4 \Big|_{-1}^{1}$$
$$= 0$$

$\therefore \langle \underset{\sim}{f}, \underset{\sim}{g} \rangle$ is not an inner product on V.

2) Let $V = C([0,1])$ with $\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle = \int_0^1 f(x)g(x)\,dx$

a) By the Cauchy-Schwarz inequality:

$$|\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle| \leq \|\underset{\sim}{f}\| \cdot \|\underset{\sim}{g}\|$$

$$|\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle|^2 \leq (\|\underset{\sim}{f}\| \cdot \|\underset{\sim}{g}\|)^2$$

$$\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle^2 \leq \|\underset{\sim}{f}\|^2 \cdot \|\underset{\sim}{g}\|^2$$

$$\left(\int_0^1 f(x)g(x)\,dx\right)^2 \leq \int_0^1 f(x)^2\,dx \cdot \int_0^1 g(x)^2\,dx$$

(shown)

b) By the triangle inequality:

$$\|\underset{\sim}{f} + \underset{\sim}{g}\| \leq \|\underset{\sim}{f}\| + \|\underset{\sim}{g}\|$$

$$\left(\int_0^1 (f+g)(x)^2\,dx\right)^{\frac{1}{2}} \leq \left(\int_0^1 f(x)^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 g(x)^2\,dx\right)^{\frac{1}{2}}$$

$$\left(\int_0^1 [f(x)+g(x)]^2\,dx\right)^{\frac{1}{2}} \leq \left(\int_0^1 f(x)^2\,dx\right)^{\frac{1}{2}} + \left(\int_0^1 g(x)^2\,dx\right)^{\frac{1}{2}}$$

(shown)

3a) $x+y+z=0$

$$z=-x-y$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, s,t \in \mathbb{R}$$

$\therefore$ a basis for W is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

b) Let $\underset{\sim}{v_1} = \frac{1}{\|(1,0,-1)\|}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$= \frac{1}{\sqrt{4+0+1}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Let $v_2' = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \text{proj}_{v_1}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \left\langle (0,1,-1), \frac{1}{\sqrt{5}}(1,0,-1) \right\rangle \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

Let $v_2 = \frac{1}{\|v_2'\|}\left(\frac{1}{5}\right)\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$

$$= \frac{1}{5}\left(\frac{1}{\sqrt{4\left(-\frac{1}{5}\right)^2 + 1^2 + \left(\frac{4}{5}\right)^2}}\right)\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{5}\left(\frac{1}{\sqrt{\frac{9}{5}}}\right)\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{\frac{15}{\sqrt{5}}}\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{\sqrt{5}}{15}\begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix}$$

4) By definition,

$$W^{\perp} = \{ \underset{\sim}{x} \in V : \langle \underset{\sim}{x}, \underset{\sim}{w} \rangle = 0 \text{ for all } \underset{\sim}{w} \in W \}$$

$\therefore$ when $\underset{\sim}{v} \in W^{\perp}$

$$\langle \underset{\sim}{v}, \underset{\sim}{x}_j \rangle = 0 \text{ for all } j = 1, \cdots, n$$

When $\langle \underset{\sim}{v}, \underset{\sim}{x}_j \rangle = 0$ for all $j = 1, \cdots, n$,

For any vector $\underset{\sim}{w} \in W$, by definition of W

$$\underset{\sim}{w} = c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + \dots + c_n \underset{\sim}{x}_n, \quad c_1, c_2, \dots, c_n \in \mathbb{R}$$

$$\begin{aligned}
\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle &= \langle \underset{\sim}{v}, c_1 \underset{\sim}{x}_1 + c_2 \underset{\sim}{x}_2 + \dots + c_n \underset{\sim}{x}_n \rangle \\
&= \langle \underset{\sim}{v}, c_1 \underset{\sim}{x}_1 \rangle + \langle \underset{\sim}{v}, c_2 \underset{\sim}{x}_2 \rangle + \dots + \langle \underset{\sim}{v}, c_n \underset{\sim}{x}_n \rangle \\
&= c_1 \langle \underset{\sim}{v}, \underset{\sim}{x}_1 \rangle + c_2 \langle \underset{\sim}{v}, \underset{\sim}{x}_2 \rangle + \dots + c_n \langle \underset{\sim}{v}, \underset{\sim}{x}_n \rangle \\
&= 0 + 0 + \dots + 0 \\
&\qquad \because \langle \underset{\sim}{v}, \underset{\sim}{x}_j \rangle = 0 \text{ for } j = 1, 2, \dots, n \\
&= 0
\end{aligned}$$

$\therefore$ by definition of $W^{\perp}$,

$\underset{\sim}{v} \in W^{\perp}$ (shown)

5) $a + b = 4$

$a + c = 2$

$a - b = 2$

$a - c = 3$

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$, $\underset{\sim}{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 3 \end{bmatrix}$

$$A^T A \underset{\sim}{x} = A^T \underset{\sim}{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} 11 \\ 2 \\ -1 \end{bmatrix}$$

$$\therefore y = \frac{11}{4} + \cos x - \frac{1}{2} \sin x$$

6a) $4a = 1$

$a - b + c = 1$

$4a + 2b + c = 1$

$c = 1$

Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\underset{\sim}{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$A^T A \underset{\sim}{x} = A^T \underset{\sim}{b}$$

$$\begin{bmatrix} 4 & 1 & 4 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 1 & -1 & 1 \\ 4 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} 4 & 1 & 4 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 33 & 7 & 5 \\ 7 & 5 & 1 \\ 5 & 1 & 3 \end{bmatrix} \underset{\sim}{x} = \begin{bmatrix} 9 \\ 1 \\ 3 \end{bmatrix}$$

$33a + 7b + 5c = 9$

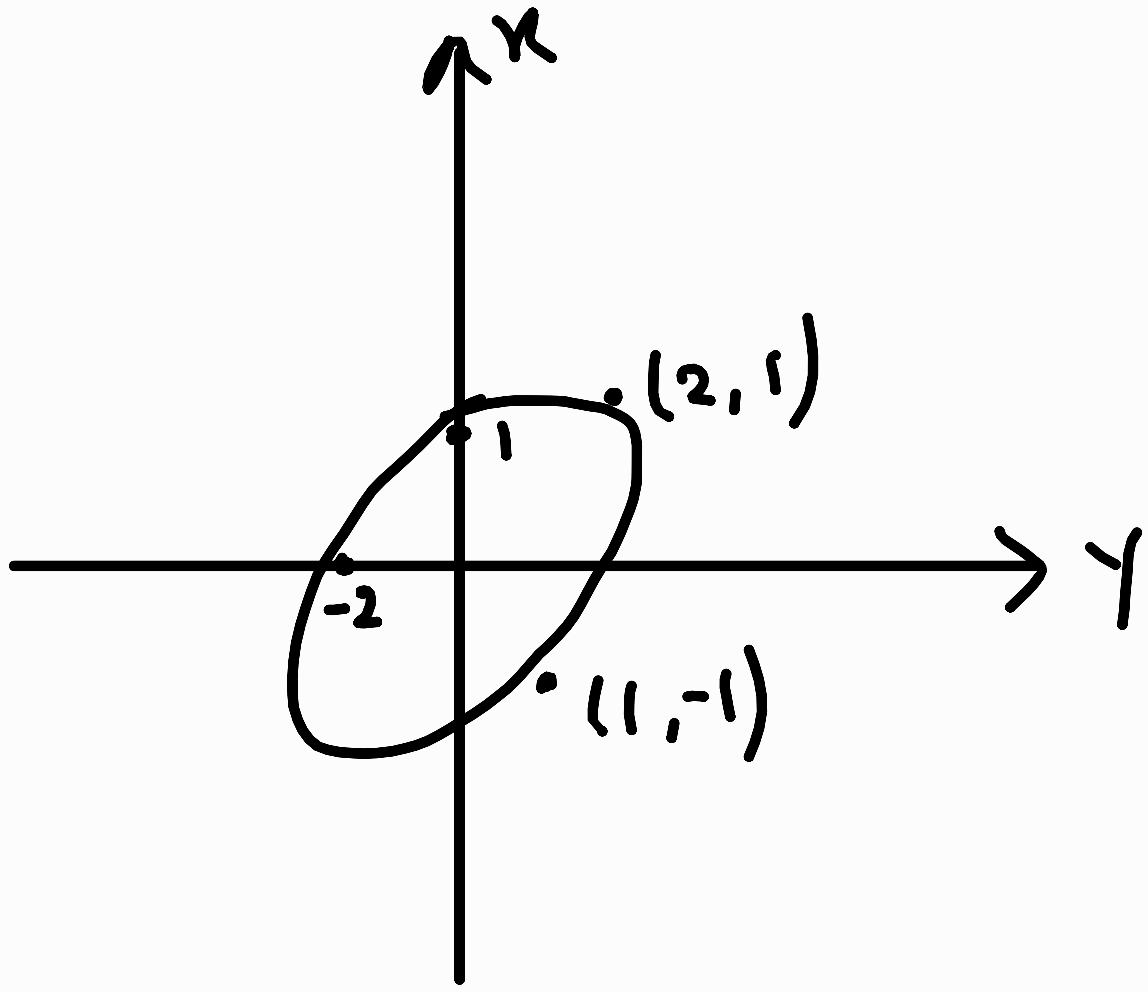
$7a + 5b + c = 1$

$5a + b + 3c = 3$

b) $a = \frac{14}{65}$, $b = -\frac{16}{65}$, $c = \frac{47}{65}$

$a \approx 0.22$, $b \approx -0.25$, $c \approx 0.72$

6b)



7) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, \ldots, n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, \ldots, n$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx$$

$$= \frac{1}{\pi}\Big[-x\Big]_{-\pi}^{0} + \frac{1}{\pi}\Big[x\Big]_{0}^{\pi}$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx$$

$$= 0 + 0$$

$$= 0$$

7) $b_n = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_{0}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1}}{n} - \left(-\frac{1}{n}\right)\right]$$

$$= \frac{2}{n\pi}\left((-1)^{n+1}+1\right)$$

$$= \frac{2}{n\pi}\left(1+(-1)^{n+1}\right)$$

The fourier series for f is:

$$\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1+(-1)^{n+1}\right)\sin nx = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \dots$$

8a) All vectors in P_2 can be represented as a linear combination of the vectors in the basis B.

b) $q_0 = 1$

$$q_1' = x - \langle x, 1 \rangle (1)$$

$$= x - \left(\int_0^1 x \, dx\right)(1)$$

$$= x - \frac{1}{2}$$

$$= \frac{1}{2}(2x-1)$$

$$q_1 = \frac{1}{\|q_1'\|}\left(\frac{1}{2}\right)(2x-1)$$

$$= \left(\langle q_1', q_1' \rangle\right)^{-\frac{1}{2}}\left(\frac{1}{2}\right)(2x-1)$$

$$= \frac{1}{2}\left(\int_0^1 \left(x-\frac{1}{2}\right)^2 dx\right)^{-\frac{1}{2}}(2x-1)$$

$$= \frac{1}{2}\left(\int_0^1 x^2 - x + \frac{1}{4}\right)^{-\frac{1}{2}}(2x-1)$$

$$= \frac{1}{2}\left(\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{4}x\right]_0^1\right)^{-\frac{1}{2}}(2x-1)$$

$$= \frac{1}{2}\left(\frac{1}{12}\right)^{-\frac{1}{2}}(2x-1)$$

$$= \sqrt{\frac{12}{4}}(2x-1)$$

$$= \sqrt{3}(2x-1)$$

$$q_2' = x^2 - \langle x^2, q_1 \rangle x - \langle x^2, q_0 \rangle 1$$

$$= x^2 - \left(\int_0^1 x^2 \left(\sqrt{3}(2x-1)\right) dx\right) \sqrt{3}(2x-1) - \int_0^1 x^2 \, dx$$

$$= x^2 - \frac{\sqrt{3}}{6} \sqrt{3}(2x-1) - \frac{1}{3}$$

$$= x^2 - \frac{1}{2}(2x-1) - \frac{1}{3}$$

$$= x^2 - x + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{6}\left(6x^2 - 6x + 1\right)$$

$$q_2 = \frac{1}{\|q_2'\|} \left(\frac{1}{6}\right)\left(6x^2 - 6x + 1\right)$$

$$= \frac{1}{6\sqrt{\frac{1}{6}\int_0^1 (6x^2-6x+1)^2 dx}} \left(6x^2 - 6x + 1\right)$$

$$= \frac{1}{6\sqrt{\frac{1}{6^2}\int_0^1 (6x^2-6x+1)^2 dx}} \left(6x^2 - 6x + 1\right)$$

$$= \frac{1}{\sqrt{\cancel{\frac{36}{36}}\left(\frac{1}{5}\right)}} \left(6x^2 - 6x + 1\right)$$

$$= \sqrt{5}\left(6x^2 - 6x + 1\right)$$

8c) $\text{proj}_{P_2} f = \langle f, p_0 \rangle p_0 + \langle f, p_1 \rangle p_1 + \langle f, p_2 \rangle p_2$

$$= \int_0^1 e^x \, dx + \sqrt{3}(2x-1)\int_0^1 e^x \sqrt{3}(2x-1)\,dx$$

$$+ \sqrt{5}(6x^2-6x+1)\int_0^1 e^x \sqrt{5}(6x^2-6x+1)\,dx$$

Using GC,

$$= (210e - 570)x^2 + (588 - 216e)x + 39e - 105$$

8d) By the general theory, for $\int_0^1 (p(x)-e^x)^2 dx$ to be the smallest, we have $p(x) = \text{proj}_{P_2}(e^x)$

$$\int_0^1 (p(x)-e^x)^2 dx = \langle p-f, p-f \rangle$$

$$= \|p-f\|^2$$

$$= \|(210e-570)x^2 + (588-216e)x + 39e - 105 - e^x\|^2$$

$$= \int_0^1 \left[(210e-570)x^2 + (588-216e)x + 39e - 105 - e^x\right]^2 dx$$

By GC,

$$= -\frac{497}{2}e^2 + 1350e - \frac{3667}{2}$$