Math Module 3A Tutorial

Hankertrix

September 25, 2023

Contents

1 Question 1

Let $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$.

$$f(x) = 2\sqrt{x} - 3 + \frac{1}{x}$$

Finding $f'(x)$:

$$\begin{aligned} f'(x) &= \frac{2}{\sqrt{x}} \cdot \frac{1}{2} - \frac{1}{x^2} \\ &= \frac{1}{\sqrt{x}} - \frac{1}{x^2} \end{aligned}$$

At $x = 1$:

$$\begin{aligned} f(1) &= 2\sqrt{1} - 3 + \frac{1}{1} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f'(1) &= \frac{1}{\sqrt{1}} - \frac{1}{1^2} \\ &= 0 \end{aligned}$$

This means that $x = 1$ is a stationary point.
At $x = 1^+$:

$$\begin{aligned} f'(1^+) &= \frac{1}{\sqrt{1^+} - \frac{1}{(1^+)^2}} \\ &> 0 \end{aligned}$$

Thus, when $x > 1$:

$$\begin{aligned} f(x) &> 0 \\ 2\sqrt{x} - (3 - \frac{1}{x}) &> 0 \\ 2\sqrt{x} &> 3 - \frac{1}{x} \textbf{ (Proven)} \end{aligned}$$

2 Question 2

$$|\arctan x - \arctan y| \leq |x - y|$$

$$\frac{|\arctan x - \arctan y|}{|x - y|} \leq 1$$

Let $f(x)$ be $\arctan x$:

$$f'(x) = \frac{1}{1 + x^2}$$

At an arbitrary point c:

$$\left| \frac{\arctan x - \arctan y}{x - y} \right| = |f'(c)|$$

$$\frac{|\arctan x - \arctan y|}{|x - y|} = \left| \frac{1}{1 + c^2} \right|$$

The maximum value of $\left| \frac{1}{1+c^2} \right|$ is 1, when $c = 0$. When $c > 0$, $\left| \frac{1}{1+c^2} \right| < 1$.
Hence:

$$\frac{|\arctan x - \arctan y|}{|x - y|} \leq 1 \textbf{ (Proven)}$$

3 Question 3

3.1 (a)

Proving the base cases:

When $r = 0$:

$$(fg)^{(0)} = \sum_{k=0}^{0} \binom{0}{k} f^{(k)} g^{(0-k)}$$
$$fg = \binom{0}{0} f^{(0)} g^{(0)}$$
$$fg = 1fg$$
$$fg = fg \textbf{ (Proven)}$$

When $r = 1$:

$$(fg)^{(1)} = \sum_{k=0}^{1} \binom{1}{k} f^{(k)} g^{(1-k)}$$
$$f'g + fg' = \binom{1}{0} f^{(0)} g^{(1-0)} + \binom{1}{1} f^{(1)} g^{(1-1)}$$
$$f'g + fg' = fg' + f'g$$
$$f'g + fg' = f'g + fg' \textbf{ (Proven)}$$

Assuming the equation holds for all $n \in \mathbb{R}$:

$$
\begin{aligned}
(fg)^{(n+1)} &= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= \sum_{k'=1}^{n} \binom{n}{k'-1} f^{(k')} g^{(n-(k'-1))} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} \\
&= \sum_{k'=1}^{n} \binom{n}{k'-1} f^{(k')} g^{(n-k'+1)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} \\
&= \sum_{k'=1}^{n} \binom{n}{k'-1} f^{(k')} g^{(n+1-k')} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} \\
&= \sum_{k'=1}^{n} \binom{n}{k'-1} f^{(k')} g^{(n+1-k')} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} + \binom{n}{0} f^{(0)} g^{(n+1-0)} \\
&= \sum_{k'=1}^{n} \binom{n}{k'-1} f^{(k')} g^{(n+1-k')} + \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n+1-k)} + fg^{(n+1)} \\
&= \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)} g^{(n+1-k)} + fg^{(n+1)} \\
&= \sum_{k=1}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} + fg^{(n+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)} \quad (\because fg^{(n+1)} = f^{(k)} g^{(n+1-k)} \text{ when } k = 0) \quad \textbf{(Proven)}
\end{aligned}
$$

3.2 (b)

$$
\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{kn!}{k(k-1)!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)(n-k)!} \\
&= \frac{kn!}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(k+n-k+1)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n+1-k)!} \\
&= \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k} \qquad \textbf{(Proven)}
\end{aligned}
$$

4 Question 4

$$g(x) = \begin{cases} x + \sin 2x, & \text{for } x \in (0, \frac{\pi}{2}) \\ x - \sin 2x, & \text{for } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

Differentiating g with respect to x:

$$g'(x) = \begin{cases} 1 + 2\cos 2x, & \text{for } x \in (0, \frac{\pi}{2}) \\ 1 - 2\cos 2x, & \text{for } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

Finding the stationary points:

$$g'(x) = 0$$

$$\begin{aligned} 1 + 2\cos 2x &= 0 \\ 2\cos 2x &= -1 \\ 2\cos 2x &= -1 \\ \cos 2x &= -\frac{1}{2} \\ 2x &= \frac{2\pi}{3} \\ x &= \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} 1 - 2\cos 2x &= 0 \\ -2\cos 2x &= -1 \\ 2\cos 2x &= 1 \\ \cos 2x &= \frac{1}{2} \\ 2x &= 2\pi - \frac{\pi}{3} \quad \left(\because x \in \left(\frac{\pi}{2}, \pi\right)\right) \\ 2x &= \frac{5}{3}\pi \\ x &= \frac{5\pi}{6} \end{aligned}$$

x	$\left(\frac{\pi}{3}\right)^-$	$\frac{\pi}{3}$	$\left(\frac{\pi}{3}\right)^+$
$g'(x)$	0^+	0	0^-
Shape	/	–	\

x	$\left(\frac{5\pi}{6}\right)^-$	$\frac{5\pi}{6}$	$\left(\frac{5\pi}{6}\right)^+$
$g'(x)$	0^+	0	0^-
Shape	/	–	\

Hence, both $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$ are maximum points.

Since f is g repeated over the entire domain of $x \in \mathbb{R}$, the maximum points will be:

$$x = \frac{\pi}{3} + k\pi \text{ and } x = \frac{5\pi}{6} + k\pi, \text{ where } k \in \mathbb{Z}$$

Thus, the part of the graph that will be increasing are in the intervals $(0 + k\pi, \frac{\pi}{3} + k\pi)$ and $(\frac{\pi}{2} + k\pi, \frac{5\pi}{6} + k\pi)$, where $k \in \mathbb{Z}$. The part of the graph that will be decreasing are in the intervals $(\frac{\pi}{3} + k\pi, \frac{\pi}{2} + k\pi)$ and $\frac{5\pi}{6} + k\pi, \pi + k\pi$, where $k \in \mathbb{Z}$.

5 Question 5

$$f(x) = \frac{x^2}{x^2+3}$$

Differentiating f with respect to x:

$$\begin{aligned} f'(x) &= \frac{(x^2+3)\cdot 2x - x^2 \cdot 2x}{(x^2+3)^2} \\ &= \frac{2x^3+6x-2x^3}{(x^2+3)^2} \\ &= \frac{6x}{(x^2+3)^2} \end{aligned}$$

Finding all the stationary points of f:

$$f'(x) = 0$$

$$\frac{6x}{(x^2+3)^2} = 0$$

$$6x = 0$$

$$x = 0$$

Hence, $x = 0$ is a stationary point. At $x = 0$:

x	0^-	0	0^+
$f'(x)$	0^-	0	0^+
Shape	$\backslash$	$-$	$/$

Thus, $x = 0$ is a local and global minimum.

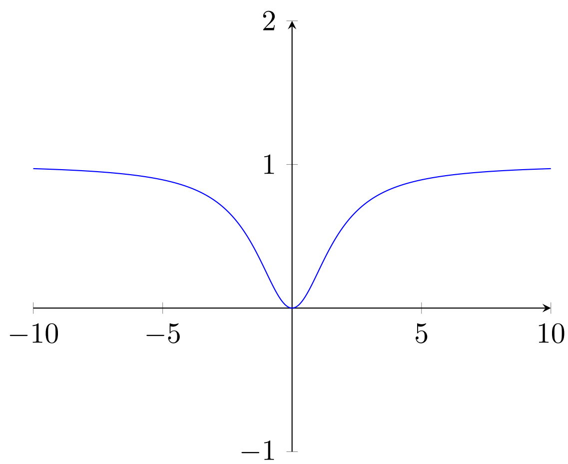
Since $x = 0$ is a global minimum, f is strictly decreasing in the interval $(-\infty, 0)$ and strictly increasing in the interval $(0, \infty)$.

Differentiating $f'(x)$ with respect to x:

$$\begin{aligned}
f''(x) &= \frac{6(x^2+3)^2 - 2(x^2+3)\cdot 2x \cdot 6x}{((x^2+3)^2)^2} \\
&= \frac{6(x^2+3)^2 - 24x^2(x^2+3)}{(x^2+3)^4} \\
&= \frac{(x^2+3)(6(x^2+3) - 24x^2)}{(x^2+3)^4} \\
&= \frac{6(x^2+3) - 24x^2}{(x^2+3)^3} \\
&= \frac{6x^2 + 18 - 24x^2}{(x^2+3)^3} \\
&= \frac{6x^2 - 24x^2 + 18}{(x^2+3)^3} \\
&= \frac{-18x^2 + 18}{(x^2+3)^3} \\
&= \frac{18(1-x^2)}{(x^2+3)^3}
\end{aligned}$$

Since $(x^2+3)^3 > 0$ for $x \in \mathbb{R}$, we can look at the numerator of $f''(x)$:

$18(1-x^2) > 0$ when $x^2 < 1$, which is when $-1 < x < 1$. Thus, f is concave up from $(-1, 1)$ and f is concave down elsewhere, which is on the intervals $(-\infty, -1)$ and $(1, \infty)$.



6 Question 6

Let θ be the angle of inclination of the ladder and $L(\theta)$ be the length of the ladder.

$$L(\theta) = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}, \quad \theta \in \left(0, \frac{\pi}{2}\right)$$

Differentiating $L(\theta)$ with respect to θ:

$$\begin{aligned} L'(\theta) &= \frac{1}{\cos^2\theta} \cdot \sin\theta + \frac{2}{\sin^2\theta} \cdot -\cos\theta \\ &= \frac{\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} \end{aligned}$$

Setting $L'(\theta)$ to 0 to find the stationary points:

$$\begin{gathered} L'(\theta) = 0 \\ \frac{\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} = 0 \\ \frac{\sin\theta}{\cos^2\theta} = \frac{2\cos\theta}{\sin^2\theta} \\ \frac{\sin\theta}{\cos^3\theta} = \frac{2}{\sin^2\theta} \\ \frac{\sin^3\theta}{\cos^3\theta} = 2 \\ \tan^3\theta = 2 \\ \tan\theta = 2^{\frac{1}{3}} \\ \theta = \arctan 2^{\frac{1}{3}} \end{gathered}$$

θ	$\left(\arctan 2^{\frac{1}{3}}\right)^-$	$\arctan 2^{\frac{1}{3}}$	$\left(\arctan 2^{\frac{1}{3}}\right)^+$
$L'(\theta)$	0^-	0	0^+
Shape	$\backslash$	$-$	$/$

Hence, $\arctan 2^{\frac{1}{3}}$ is a minimum point. The minimum length would be:

$$\begin{aligned} L(\arctan 2^{\frac{1}{3}}) &= \frac{1}{\cos\left(\arctan 2^{\frac{1}{3}}\right)} + \frac{2}{\sin\left(\arctan 2^{\frac{1}{3}}\right)} \\ &= 4.161938185 \\ &= 4.16\,\mathrm{m} \end{aligned}$$

7 Question 7

Let the width of the wooden beam be w and the height of the wooden beam be h. Let the stiffness of the wooden beam be s.

$$s = kwh^3, \quad \text{where } k \text{ is an arbitrary constant}$$

By Pythagoras' Theorem:

$$R^2 = \left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2$$

$$\frac{w}{2} = \sqrt{R^2 - \frac{h^2}{4}}$$

$$w = 2\sqrt{R^2 - \frac{h^2}{4}}$$

Hence,

$$\begin{aligned} s &= 2kh^3\sqrt{R^2 - \frac{h^2}{4}}, \quad h \in (0, 2R) \\ &= Ch^3\sqrt{R^2 - \frac{h^2}{4}}, \quad \text{where } C = 2k \end{aligned}$$

Differentiating s with respect to h:

$$\begin{aligned}
\frac{ds}{dh} &= 3Ch^2\sqrt{R^2 - \frac{h^2}{4}} + Ch^3\frac{1}{\sqrt{R^2 - \frac{h^2}{4}}} \cdot -\frac{2h}{4} \cdot \frac{1}{2} \\
&= 3Ch^2\sqrt{R^2 - \frac{h^2}{4}} - \frac{Ch^4}{4\sqrt{R^2 - \frac{h^2}{4}}} \\
&= \frac{12Ch^2\left(R^2 - \frac{h^2}{4}\right) - Ch^4}{4\sqrt{R^2 - \frac{h^2}{4}}} \\
&= \frac{12Ch^2R^2 - \frac{12Ch^4}{4} - Ch^4}{4\sqrt{R^2 - \frac{h^2}{4}}} \\
&= \frac{12Ch^2R^2 - 4Ch^4}{4\sqrt{R^2 - \frac{h^2}{4}}} \\
&= \frac{3Ch^2R^2 - 4Ch^4}{\sqrt{R^2 - \frac{h^2}{4}}} \\
&= \frac{Ch^2(3R^2 - Ch^2)}{\sqrt{R^2 - \frac{h^2}{4}}}
\end{aligned}$$

Finding the stationary points by setting $\frac{ds}{dh}$ to 0:

$$\begin{gathered}
\frac{ds}{dh} = 0 \\
\frac{Ch^2(3R^2 - Ch^2)}{\sqrt{R^2 - \frac{h^2}{4}}} = 0 \\
Ch^2(3R^2 - Ch^2) = 0 \\
3Ch^2R^2 = Ch^4 \\
h^2 = 3R^2 \\
h = \sqrt{3}R
\end{gathered}$$

We see that $f'(x)$ exists everywhere on $(0, 2R)$ and the only $x \in (0, 2R)$ where $f'(x) = 0$ is $x = \sqrt{3}R$. We also note that:

$$\lim_{x\to 0} f(x) = \lim_{x\to 2R} f(x) = 0, \quad \text{when } f\left(\sqrt{3}R\right) \text{ is positive}$$

This makes $x\sqrt{3}R$ the global maximum on $(0, 2R)$ and the corresponding width is:

$$
\begin{aligned}
w &= 2\sqrt{R^2 - \frac{h^2}{4}} \\
&= 2\sqrt{R^2 - \frac{\left(\sqrt{3}R\right)^2}{4}} \\
&= 2\sqrt{R^2 - \frac{3R^2}{4}} \\
&= 2\sqrt{\frac{1}{4}R^2} \\
&= 2 \cdot \frac{1}{2}R^2 \\
&= R
\end{aligned}
$$

Hence, the stiffness of the beam is the maximum when the height is $\sqrt{3}R$ and the width is R.

8 Question 8

Considering the two points P_1 and P_2, which are both distance 1 away from the plane interface separating the two mediums. Let x be the horizontal distance from point P_1 to the vertical line from which the angles θ_1 and θ_2 are drawn. Let d be the distance between the point P_1 and P_2. Furthermore, let d_1 and d_2 be the separation be the distance travelled by light in the 2 mediums.

Using Pythagoras' Theorem:

$$d_1^2 = 1 + x^2$$

$$d_2^2 = 1 + (d - x)^2$$

The time taken for the light to travel would be given by:

$$\begin{aligned} T(x) &= \frac{d_1}{v_1} + \frac{d_2}{v_2} \\ &= \frac{\sqrt{1 + x^2}}{v_1} + \frac{\sqrt{1 + (d - x)^2}}{v_2} \\ &= \frac{\sqrt{1 + x^2}}{v_1} + \frac{\sqrt{1 + d^2 - 2dx + x^2}}{v_2} \end{aligned}$$

Differentiating with respect to x:

$$\begin{aligned} T'(x) &= \frac{2x}{v_1\sqrt{1 + x^2}} \cdot \frac{1}{2} + \frac{-2d + 2x}{v_2\sqrt{1 + (d - x)^2}} \cdot \frac{1}{2} \\ &= \frac{x}{v_1\sqrt{1 + x^2}} + \frac{x - d}{v_2\sqrt{1 + (d - x)^2}} \end{aligned}$$

Finding the stationary points by setting $T'(x) = 0$:

$$T'(x) = 0$$
$$\frac{x}{v_1\sqrt{1+x^2}} + \frac{x-d}{v_2\sqrt{1+(d-x)^2}} = 0$$
$$\frac{x}{v_1\sqrt{1+x^2}} = -\frac{x-d}{v_2\sqrt{1+(d-x)^2}}$$
$$\frac{x}{v_1\sqrt{1+x^2}} = \frac{d-x}{v_2\sqrt{1+(d-x)^2}}$$
$$xv_2\sqrt{1+(d-x)^2} = (d-x)v_1\sqrt{1+x^2}$$
$$v_2\frac{x}{\sqrt{1+x^2}} = v_1\frac{d-x}{1+(d-x)^2}$$

Since $\sin\theta_1 = \frac{x}{\sqrt{1+x^2}}$ and $\sin\theta_2 = \frac{d-x}{\sqrt{1+(d-x)^2}}$:

$$v_2\sin\theta_1 = v_1\sin\theta_2$$
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \quad \textbf{(Shown)}$$

Differentiating $T'(x)$ with respect to x:

$$\begin{aligned}
T''(x) &= \frac{1}{v_1}\frac{\sqrt{1+x^2} - x\frac{1}{\sqrt{1+x^2}}\cdot\frac{1}{2}\cdot 2x}{1+x^2} + \frac{1}{v_2}\frac{\sqrt{1+(d-x)^2} - (d-x)\frac{1}{\sqrt{1+(d-x)^2}}\cdot\frac{1}{2}\cdot(2x-2d)}{1+(d-x)^2} \\
&= \frac{1}{v_1}\frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2} + \frac{1}{v_2}\frac{\sqrt{1+(d-x)^2} - \frac{(x-d)(d-x)}{\sqrt{1+(d-x)^2}}}{1+(d-x)^2} \\
&= \frac{1}{v_1}\frac{\frac{1+x^2-x^2}{\sqrt{1+x^2}}}{1+x^2} + \frac{1}{v_2}\frac{\frac{1+(d-x)^2-(x-d)(d-x)}{\sqrt{1+(d-x)^2}}}{1+(d-x)^2} \\
&= \frac{1}{v_1(1+x^2)^{\frac{3}{2}}} + \frac{1+(d-x)(d-x-(x-d))}{v_2(1+(d-x)^2)^{\frac{3}{2}}} \\
&= \frac{1}{v_1(1+x^2)^{\frac{3}{2}}} + \frac{1}{v_2(1+(d-x)^2)^{\frac{3}{2}}} > 0 \text{ for } x \in \mathbb{R}
\end{aligned}$$

Since the second derivative of T is always positive, and we only have 1 critical point for T, that means the critical point is a global minimum. Hence:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \quad \text{is minimum}$$

9 Question 9

9.1 (a)

$$\begin{aligned}
\lim_{x\to 1} \frac{\ln x}{x-1} &= \lim_{x\to 1} \frac{\frac{1}{x}}{1} \\
&= \frac{\frac{1}{1}}{1} \\
&= 1
\end{aligned}$$

9.2 (b)

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - x}{x^3} &= \lim_{x\to 0} \frac{\frac{1}{\cos^2 x} - 1}{3x^2} \\
&= \lim_{x\to 0} \frac{\frac{-2(-\sin x)}{\cos^3 x}}{6x} \\
&= \lim_{x\to 0} \frac{\sin x}{3x\cos^3 x} \\
&= \lim_{x\to 0} \frac{\cos x}{3x\cos^2 x \cdot -\sin x \cdot 3 + 3\cos^3 x} \\
&= \lim_{x\to 0} \frac{\cos x}{3\cos^3 x - 9x\cos^2 x \sin x} \\
&= \frac{1}{3(1)^3 - 0} \\
&= \frac{1}{3}
\end{aligned}$$

9.3 (c)

Using limit laws:

$$\begin{aligned}
\lim_{x\to \pi} \frac{\sin x}{1-\cos x} &= \frac{0}{1-(-1)} \\
&= 0
\end{aligned}$$

9.4 (d)

$$\begin{aligned}
\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} &= \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} \\
&= \lim_{x\to 1} \frac{a}{b} x^{a-1-(b-1)} \\
&= \lim_{x\to 1} \frac{a}{b} x^{a-b} \\
&= \frac{a}{b}
\end{aligned}$$

9.5 (e)

We have

$$\begin{aligned}
\lim_{x\to\infty} x \ln\left(1 + \frac{a}{x}\right) &= \lim_{x\to\infty} \left(\frac{\ln(1+\frac{a}{x})}{\frac{1}{x}}\right) \\
&= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}} \cdot \frac{-a}{x^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty} \frac{a}{1+\frac{a}{x}} \\
&= a
\end{aligned}$$

Hence:

$$\begin{aligned}
\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} &= \lim_{x\to\infty} e^{bx\ln(1+\frac{a}{x})} \\
&= e^{\lim_{x\to\infty} bx \ln\left(1+\frac{a}{x}\right)} \\
&= e^{ba} \\
&= e^{ab}
\end{aligned}$$

10 Question 10

For $\varepsilon, p > 0$, we have:

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^p}{x^\varepsilon} &= \lim_{x\to\infty} \left(\frac{\ln x}{x^{\frac{\varepsilon}{p}}}\right)^p \\
&= \lim_{x\to\infty} \left(\frac{\frac{1}{x}}{\frac{\varepsilon}{p} x^{\frac{\varepsilon}{p}-1}}\right)^p \\
&= \lim_{x\to\infty} \left(\frac{1}{\frac{\varepsilon}{p} x^{\frac{\varepsilon}{p}}}\right)^p \\
&= 0^p \\
&= 0 \quad \textbf{(Shown)}
\end{aligned}$$

11 Question 11

By definition, f is concave upwards on I if and only if for all $a, b \in I$, the line segment joining the points $(a, f(a)), (b, f(b))$ lies above the graph of $f(x)$.

Since the line segment joining the two points is the graph of the function:

$$l(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

We see that f is concave up on I if and only if for every $a < x < b$, we have $l(x) > f(x)$, or:

$$f(a) + \frac{f(b) - f(a)}{b - a}(x - a) > f(x)$$

Since $x - a > 0$, we have:

$$\begin{aligned}
f(a) + \frac{f(b) - f(a)}{b - a}(x - a) &> f(x) \\
\frac{f(b) - f(a)}{b - a}(x - a) &> f(x) - f(a) \\
\frac{f(b) - f(a)}{b - a} &> \frac{f(x) - f(a)}{x - a} \quad \textbf{(Shown)}
\end{aligned}$$