1a) Since A is symmetric, Q can be found.

$$\det(\lambda I - 3A) = 0$$

$$\left| \begin{array}{ccc|cc} \lambda-1 & -1 & -1 & \lambda-1 & -1 \\ -1 & \lambda-1 & -1 & -1 & \lambda-1 \\ -1 & -1 & \lambda-1 & -1 & -1 \end{array} \right. = 0$$

$$(\lambda-1)^3 \cancel{-1} \cancel{-1} - \lambda \cancel{+1} - \lambda \cancel{+1} - \lambda + 1 = 0$$

$$(\lambda^2 - 2\lambda + 1)(\lambda - 1) - 3\lambda + 1 = 0$$

$$\lambda^3 - \lambda^2 - 2\lambda^2 \cancel{+2\lambda} \cancel{+\lambda} \cancel{-1} - \cancel{3\lambda} \cancel{+1} = 0$$

$$\lambda^3 - 3\lambda^2 = 0$$

$$\lambda^2(\lambda - 3) = 0$$

$$\therefore \lambda = 0 \text{ or } \lambda = 3$$

$\therefore$ the eigenvalues of $3A$ are 0 and 3

1a) For $\lambda=0$,

$$\begin{bmatrix} \lambda-1 & -1 & -1 & 0 \\ -1 & \lambda-1 & -1 & 0 \\ -1 & -1 & \lambda-1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x+y+z=0$$
$$x=-y-z$$

$\therefore$ the corresponding eigenvectors of $3A$ are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ s,t\in\mathbb{R}, s\neq t\neq 0$$

For $\lambda=3$,

$$\begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x-z=0 \qquad y-z=0$$
$$x=z \qquad y=z$$

$\therefore$ the corresponding eigenvectors of $3A$ are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ s\in\mathbb{R}\setminus\{0\}$$

1a) Finding an orthornormal basis for the first eigenspace:

$$\underset{\sim}{x}_1 = \frac{1}{\sqrt{2}}(-1,1,0)$$

$$\underset{\sim}{x}_2' = (-1,0,1) - \text{proj}_{\underset{\sim}{x}_1}(-1,0,1)$$

$$= (-1,0,1) - \frac{1}{\sqrt{2}}(-1,0,1)\cdot(-1,1,0)\cdot\frac{1}{\sqrt{2}}(-1,1,0)$$

$$= (-1,0,1) - \frac{1}{2}(1)(-1,1,0)$$

$$= \left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$\underset{\sim}{x}_2 = \frac{\underset{\sim}{x}_2'}{\|\underset{\sim}{x}_2'\|}$$

$$= \frac{1}{\sqrt{\frac{1}{2}^2+\frac{1}{2}^2+1^2}}\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$$

$$= \sqrt{\frac{2}{3}} \times \frac{1}{2}(-1,-1,2)$$

$$= \frac{1}{\sqrt{6}}(-1,-1,2)$$

$\therefore$ the basis $B_1 = \left\{\frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(-1,-1,2)\right\}$

Finding an ON-basis for the second eigenspace

$$B_2 = \left\{\frac{1}{\sqrt{3}}(1,1,1)\right\}$$

1a) $\therefore Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}$

$\therefore D = Q^T A Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) Let the new basis be:

$$B' = \left\{ \frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,1,0), \frac{1}{\sqrt{6}}(-1,-1,2) \right\}$$

$T(\underset{\sim}{x})$ is the change of basis from the standard basis $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ to the new basis B', and is the orthorgonal projection onto the first coordinate axis, which is parallel to $\frac{1}{\sqrt{3}}(1,1,1)$ in the standard coordinate axis.

2a) $\underset{\sim}{x}$ and $\underset{\sim}{y}$ being orthogonal means

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 0$$

b) $$\|\underset{\sim}{x} + \underset{\sim}{y}\|^2 = \langle \underset{\sim}{x}, \underset{\sim}{x} \rangle + 2\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle + \langle \underset{\sim}{y}, \underset{\sim}{y} \rangle$$

$$= \|\underset{\sim}{x}\|^2 + 2(0) + \|\underset{\sim}{y}\|^2$$

$$\because \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = 0$$

$$= \|\underset{\sim}{x}\|^2 + \|\underset{\sim}{y}\|^2 \text{ (proven)}$$

c) For $\underset{\sim}{w}, \underset{\sim}{w}' \in W$, $\underset{\sim}{w} \neq w'$, $(\underset{\sim}{v} - \underset{\sim}{w}) \in W^\perp$,

$\|\underset{\sim}{w} - \underset{\sim}{w}'\| > 0$, by properties of norm,

$\underset{\sim}{w} - \underset{\sim}{w}' \in W$, since W is closed under addition and multiplication.

Hence, $\underset{\sim}{v} - \underset{\sim}{w}$ and $\underset{\sim}{w} - \underset{\sim}{w}'$ are orthorgonal and part (b) tells us:

$$\|\underset{\sim}{v} - \underset{\sim}{w}'\|^2 = \|\underset{\sim}{v} - \underset{\sim}{w} + \underset{\sim}{w} - \underset{\sim}{w}'\|^2$$

$$= \|\underset{\sim}{v} - \underset{\sim}{w}\|^2 + \|\underset{\sim}{w} - \underset{\sim}{w}'\|^2$$

$$> \|\underset{\sim}{v} - \underset{\sim}{w}\|^2$$ as norms are non-negative

3a) There exists constants $C, \delta > 0$ such that

$$|f(x) - x| < C|\underset{\sim}{x}^3| \text{ for } x \in (-\delta, \delta)$$

b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\lim_{x \to 0} \frac{\cos(x^2) - e^{x^4}}{\cos(5x^2) - 1}$$

$$= \lim_{x \to 0} \frac{\cancel{1} - \frac{x^4}{2!} + \frac{x^8}{4!} + O(x^{12}) - (\cancel{1} + x^4 + \frac{x^8}{2!} + O(x^{12}))}{\cancel{1} - \frac{25x^4}{2!} + \frac{125x^8}{4!} + O(x^{12}) - \cancel{1}}$$

$$= \lim_{x \to 0} \frac{-\frac{3}{\cancel{2}} \cancel{x^4} + O(x^8)}{-\frac{25}{\cancel{2}} \cancel{x^4} + O(x^8)}$$

$$= \frac{3}{25}$$