Math Module 1A Tutorial

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Contents

1 Question 1

1.1 (a)

$$
\begin{aligned}
|A| &= 5 \\
|B| &= 4 \\
|A \cup B| &= 7 \\
|A \cap B| &= 2
\end{aligned}
$$

1.2 (b)

$$|A \cup B| = |A| + |B| - |A \cap B|$$

1.3 (c)

Let the set of 1^{st} year students registered at CNYSP be A, and the set of students registered at SPMS be B.

$$|A| = 50$$

$$|B| = 1102$$

$$|A \cup B| = 1138$$

The number of 1^{st} year students registered at SPMS and CNYSP together will be $|A \cap B|$. From (b):

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cap B| = |A| + |B| - |A \cup B|$$

$$|A \cap B| = 50 + 1102 - 1138$$

$$|A \cap B| = 14$$

Hence, there are 14 1^{st} year CNYSP students who also registered at SPMS.

1.4 (d)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

2 Question 2

$$|A \times B| = |A| \times |B|$$

3 Question 3

3.1 (a)

$$A = \{0\}$$
$$\mathcal{P}(A) = \{\emptyset, \{0\}\}$$
$$\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{\{0\}\}, \{\emptyset, \{0\}\}\}$$

3.2 (b)

$$2^n = 2^{|A|}$$

4 Question 4

4.1 (a)

$$\textit{not}\ (P \text{ or } Q) \Leftrightarrow (\textit{not}\ P) \text{ and } (\textit{not}\ Q)$$
$$\textit{not}\ (P \text{ and } Q) \Leftrightarrow (\textit{not}\ P) \text{ or } (\textit{not}\ Q)$$

4.2 (b)

$$\begin{aligned} P \Rightarrow Q &\Leftrightarrow \textit{not}\ (P \text{ and } (\textit{not}\ Q)) \\ &\Leftrightarrow (\textit{not}\ P) \text{ or } (\textit{not}\ (\textit{not}\ Q)) \\ &\Leftrightarrow (\textit{not}\ P) \text{ or } Q \end{aligned}$$

4.3 (c)

$$P \text{ and } Q \text{ or } ((\textit{not}\ P) \text{ and } (\textit{not}\ Q))$$

5 Question 5

5.1 (a)

$$(A \cup B)^c = (A^c \cap B^c)$$

5.2 (b)

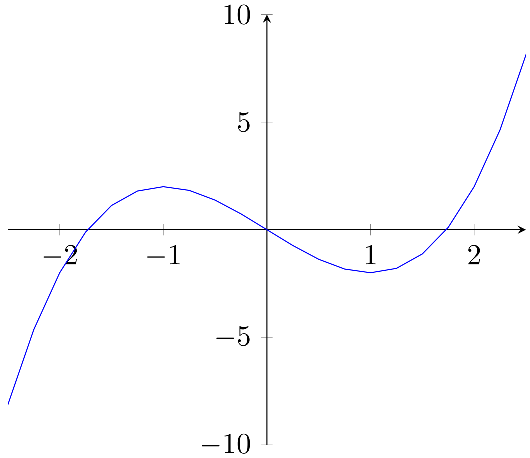
$$(A \cap B)^c = (A^c \cup B^c)$$

6 Question 6

6.1 (a)

For $f(x) \in \mathbb{R}$, $\sqrt(3x - x^3)$ must be real. Hence:

$$
\begin{gathered}
3x - x^3 \geq 0 \\
x(3 - x^2) \geq 0 \\
x(3 - x^2) \geq 0 \\
x(x^2 - 3) \leq 0 \\
x(x + \sqrt{3})(x - \sqrt{3}) \leq 0
\end{gathered}
$$



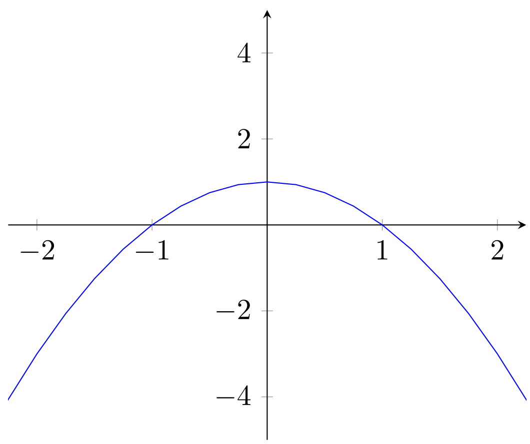
Thus, the domain of $f(x)$ is $(-\infty, -\sqrt{3}] \cup [0, \sqrt{3}]$.

6.2 (b)

For $f(x) \in \mathbb{R}$, $\sqrt{\frac{1+x}{1-x}}$ must be real. Hence:

$$\frac{1+x}{1-x} \geq 0$$

$$(1+x)(1-x) \geq 0,\ (1-x) \neq 0$$



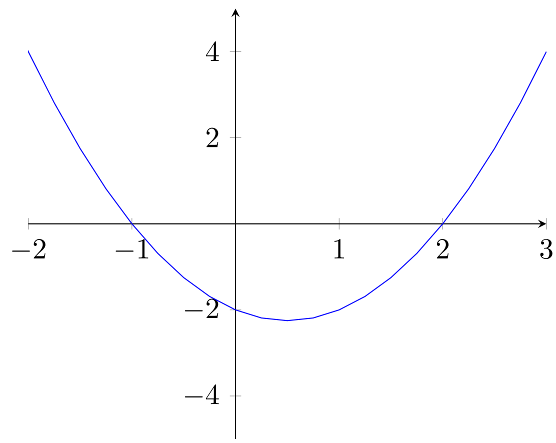
Thus, the domain of $f(x)$ is $[-1, 1)$.

7 Question 7

7.1 (a)

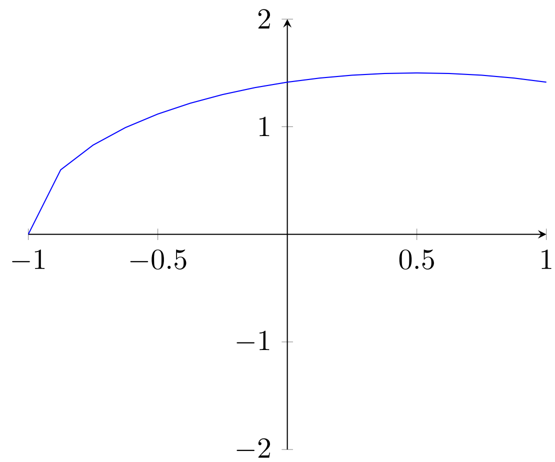
For $f(x) \in \mathbb{R}$, $\sqrt{2 + x - x^2}$ must be real. Hence:

$$2 + x - x^2 \geq 0$$
$$x^2 - x - 2 \leq 0$$
$$(x-2)(x+1) \leq 0$$



Hence, the domain of $f(x)$ is $[-1, 2]$.

Plotting the graph for $f(x)$ for $x \in [-2, 1]$:

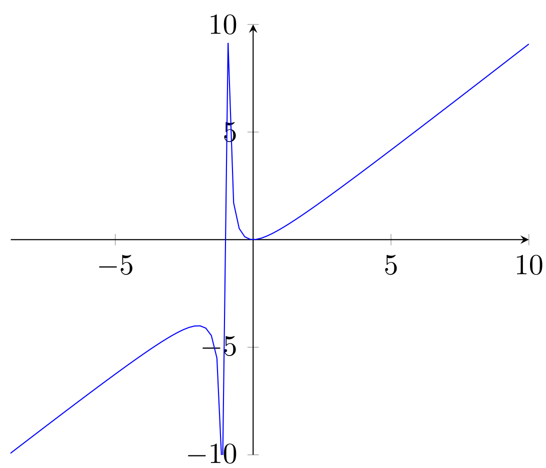


$$\text{The range of } f(x) \text{ is } \left[0, \frac{3}{2}\right].$$

7.2 (b)

For $f(x) \in \mathbb{R}$, since $x^2 > 0$, $(1+x)$ must be not be 0 for $f(x)$ to be real. Hence, the domain of $f(x)$ is $\mathbb{R} \setminus \{-1\}$.

Plotting the graph for $f(x), x \in \mathbb{R} \setminus \{-1\}$:



Hence, the range of $f(x)$ is $(-\infty, -4] \cup [0, \infty]$.

8 Question 8

8.1 (a)

For $f : \mathbb{R} \to \mathbb{R}$ to be both increasing and decreasing at the same time, it must satisfy the two conditions below:

1. $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
2. $x_1, x_2 \in A, x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

For f to satisfy the conditions $f(x_1) \leq f(x_2)$ and $f(x_1) \geq f(x_2)$, $f(x_1)$ must be equal to $f(x_2)$. Thus, f is constant.

8.2 (b)

For $f : \mathbb{R} \to \mathbb{R}$ to be even and odd at the same time, it must satisfy the two conditions below:

1. $x \in A \Rightarrow -x \in A$ and $f(-x) = -f(x)$.
2. $x \in A \Rightarrow -x \in A$ and $f(-x) = f(x)$.

Hence, $f(-x) = -f(x) = f(x)$

The only number that satisfies this condition is 0, thus $f(x) = 0$ for all $x \in \mathbb{R}$.

9 Question 9

9.1 (a)

For a function to be even, $f(-x) = f(x)$:

$$\begin{aligned} E(-x) &= \frac{1}{2}(f(-x) + f(x)) \\ &= \frac{1}{2}(f(x) + f(-x)) \\ &= E(x) \end{aligned}$$

Hence, $E(x)$ is even.

For a function to be odd, $f(-x) = -f(x)$:

$$\begin{aligned} O(-x) &= \frac{1}{2}(f(-x) - f(x)) \\ &= -\frac{1}{2}(f(x) - f(-x)) \\ &= -O(x) \end{aligned}$$

Hence, $O(x)$ is odd.

9.2 (b)

$$E(x) + O(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

$$E(x) + O(x) = \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x))$$

$$E(x) + O(x) = \frac{1}{2}(2f(x))$$

$$E(x) + O(x) = f(x) \rightarrow \text{ Proven}$$

9.3 (c)

Suppose $f(x) = e(x) + o(x)$ where $e(x)$ is even and $o(x)$ is odd.

$$f(-x) = e(-x) + o(-x)$$

Since $e(-x) = e(x)$ as $e(x)$ is even and $o(-x) = -o(x)$ as $o(x)$ is odd:

$$f(-x) = e(x) - o(x) \tag{1}$$

$$f(x) = e(x) + o(x) \tag{2}$$

Solving for (2) − (1):

$$\begin{aligned} f(-x) - f(x) &= e(x) - o(x) - (e(x) + o(x)) \\ f(-x) - f(x) &= -2o(x) \\ 2o(x) &= -f(-x) + f(x) \end{aligned}$$

$$\begin{aligned} o(x) &= \frac{1}{2}(f(x) - f(-x)) \qquad (3) \\ &= O(x) \end{aligned}$$

Substituting (3) into (2):

$$\begin{aligned} f(x) &= e(x) + \frac{1}{2}(f(x) - f(-x)) \\ e(x) &= f(x) - \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

$$\begin{aligned} e(x) &= \frac{1}{2}(f(x) + f(-x)) \\ &= E(x) \end{aligned}$$

Since $e(x) = E(x)$ and $o(x) = O(x)$, there is no other way to write $f(x)$ as a sum of odd and even functions.

10 Question 10

10.1 (a)

Recurrent formula:

$$v(n) = a_0 = 1, a_1 = 2, a_n = 2a_{n-1}$$

Explicit formula:

$$v(n) = 2^n$$

10.2 (b)

Let $b(n)$ be the number of humans that were bitten on night n.

$$b(1) = v(0) = 1$$

$$\begin{aligned} v(1) &= 1 + b(1) \\ &= 1 + v(0) \\ &= 1 + 1 \\ &= 2 \end{aligned}$$

For $n \geq 2$:

$$b(n) = v(n-2)$$

$$v(n) = v(n-1) + b(n)$$

$$v(n) = v(n-1) + v(n-2)$$

Recurrent formula:

$$v(n) = a_0 = 1, a_1 = 2, a_n = a_{n-1} + a_{n-2}$$

10.3 Bonus

Proving the base cases:

$$\begin{aligned} v(0) &= \frac{(\frac{1+\sqrt{5}}{2})^{0+2} - (\frac{1-\sqrt{5}}{2})^{0+2}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

$$\begin{aligned} v(1) &= \frac{(\frac{1+\sqrt{5}}{2})^{1+2} - (\frac{1-\sqrt{5}}{2})^{1+2}}{\sqrt{5}} \\ &= 2 \end{aligned}$$

Let $n \in \mathbb{Z}^+$:

$$\begin{aligned} v(n) &= \frac{(\frac{1+\sqrt{5}}{2})^{n+2} - (\frac{1-\sqrt{5}}{2})^{n+2}}{\sqrt{5}} \\ v(n-1) &= \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned} v(n) + v(n-1) &= \frac{(\frac{1+\sqrt{5}}{2})^{n+2} - (\frac{1-\sqrt{5}}{2})^{n+2}}{\sqrt{5}} + \frac{(\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2})^{n+2} - (\frac{1-\sqrt{5}}{2})^{n+2} + (\frac{1+\sqrt{5}}{2})^{n+1} - (\frac{1-\sqrt{5}}{2})^{n+1}}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2})^{n+1}(\frac{1+\sqrt{5}}{2} + 1) + (\frac{1-\sqrt{5}}{2})^{n+1}(-\frac{1-\sqrt{5}}{2} - 1)}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2})^{n+1}(\frac{1+\sqrt{5}}{2} + 1) - (\frac{1-\sqrt{5}}{2})^{n+1}(\frac{1+\sqrt{5}}{2} + 1)}{\sqrt{5}} \end{aligned}$$

Let $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$:

$$v(n) + v(n-1) = \frac{a^{n+1}(a+1) - b^{n+1}(b+1)}{\sqrt{5}} \tag{1}$$

Getting the value of $a+1$ and a^2:

$$\begin{aligned}a+1 &= \frac{1+\sqrt{5}}{2}+1\\ &= \frac{3+\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}a^2 &= \left(\frac{1+\sqrt{5}}{2}\right)^2\\ &= \frac{1+2\sqrt{5}+5}{4}\\ &= \frac{6+2\sqrt{5}}{4}\\ &= \frac{3+\sqrt{5}}{2}\\ &= a+1\end{aligned}$$

Hence,

$$a+1=a^2 \tag{2}$$

Getting the value of $b+1$ and b^2:

$$\begin{aligned} b+1 &= \frac{1-\sqrt{5}}{2}+1 \\ &= \frac{3-\sqrt{5}}{2} \end{aligned}$$

$$\begin{aligned} b^2 &= \left(\frac{1-\sqrt{5}}{2}\right)^2 \\ &= \frac{1-2\sqrt{5}+5}{4} \\ &= \frac{6-2\sqrt{5}}{4} \\ &= \frac{3-\sqrt{5}}{2} \\ &= b+1 \end{aligned}$$

Hence,

$$b+1 = b^2 \tag{3}$$

Substituting (2) and (3) into (1):

$$\begin{aligned} v(n)+v(n-1) &= \frac{a^{n+1}(a^2)-b^{b+1}(b^2)}{\sqrt{5}} \\ &= \frac{a^{n+3}-b^{n+3}}{\sqrt{5}} \\ &= \frac{(\frac{1+\sqrt{5}}{2})^{n+3}-(\frac{1-\sqrt{5}}{2})^{n+3}}{\sqrt{5}} \\ &= v(n+1) \rightarrow \textbf{ Proven by induction} \end{aligned}$$

Thus,

$$v(n) = \frac{(\frac{1+\sqrt{5}}{2})^{n+2}-(\frac{1-\sqrt{5}}{2})^{n+2}}{\sqrt{5}}$$