Inductors Notes

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Contents

1 Definitions

1.1 Potential difference across an inductor

The potential difference across an inductor depends on the **rate of change of the current**.

$$V_{ab} = L\frac{dI}{dt}$$

Where:

- V_{ab} is the potential difference across the conductor.
- L is the self inductance of the conductor.
- $\frac{dI}{dt}$ is the rate of change of the current with respect to time.

1.2 Inductive reactance (SI Unit: Ω)

$$X_L = \omega L = 2\pi f L$$

Where:

- X_L is the inductive reactance.
- ω is the angular frequency.
- L is the inductance.
- f is the frequency.

1.3 Capacitive reactance (SI Unit: Ω)

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

Where:

- X_C is the capacitive reactance.
- ω is the angular frequency.
- C is the capacitance.
- f is the frequency.

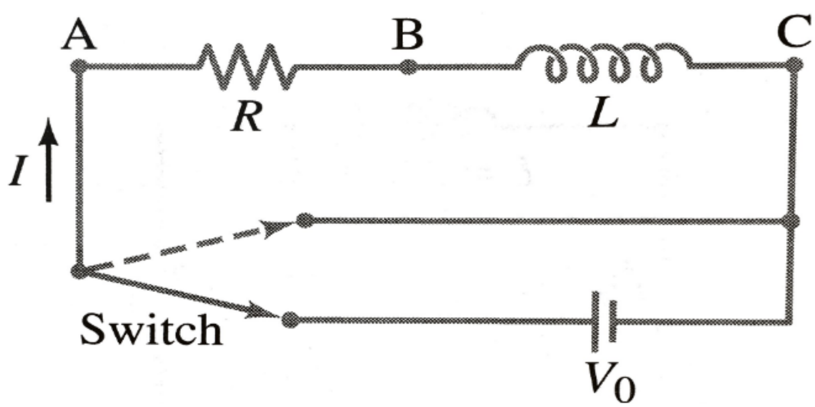
1.4 Resonance in A.C. circuits

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

Where:

- f_0 is the resonance frequency.
- ω_0 is the angular frequency.
- L is the impedance of the circuit.
- C is the capacitance of the circuit.

2 RL circuit



Consider the circuit shown above. At the instant the switch is closed, current starts to flow. The voltage or e.m.f induced across the inductor by this change in current from zero to some finite value opposes the change that induces it.

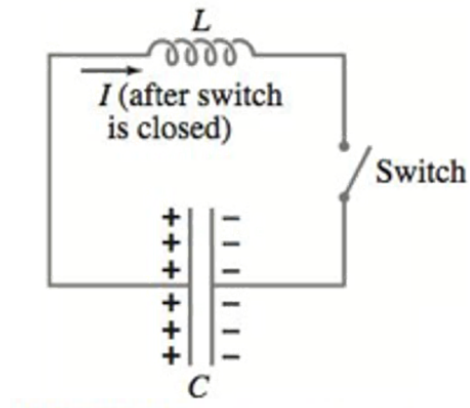
Derivation of $I(t)$. By Kirchhoff's voltage rule:

$$\begin{aligned}
V_0 - IR - L\frac{dI}{dt} &= 0 \\
L\frac{dI}{dt} &= V_0 - IR \\
\int_0^I \frac{dI}{V_0 - IR} &= \int_0^t \frac{dt}{L} \\
-\frac{1}{R}\left[\ln|V_0 - IR|\right]_0^I &= \frac{1}{L}[t]_0^t \\
-\frac{1}{R}\left(\ln|V_0 - IR| - \ln|V_0|\right) &= \frac{t}{L} \\
\ln|V_0 - IR| - \ln|V_0| &= -\frac{Rt}{L} \\
\ln\left|\frac{V_0 - IR}{V_0}\right| &= -\frac{Rt}{L} \\
\frac{V_0 - IR}{V_0} &= e^{-\frac{Rt}{L}} \\
V_0 - IR &= V_0 e^{-\frac{Rt}{L}} \\
IR &= V_0 - V_0 e^{-\frac{Rt}{L}} \\
I(t) &= \frac{V_0}{R}\left(1 - e^{-\frac{Rt}{L}}\right)
\end{aligned}$$

2.1 Application: surge arrestor

- If lightning strikes part of an electrical power transmission system, it causes a sudden spike in voltage that can damage the components of the system.
- To minimise these effects, large inductors are incorporated into the transmission system.
- These use the principle that an inductor opposes and suppresses any rapid changes in the current.

3 LC circuit (without voltage source)



By Kirchhoff's voltage law:

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0 \tag{1}$$

The current comes from the decrease of the capacitor's charge:

$$I = -\frac{dQ}{dt} \tag{2}$$

Substituting (2) into (1):

$$-L \cdot -\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0 \tag{3}$$

Trying $Q = Q_0 \cos(\omega t + \phi)$ with (3):

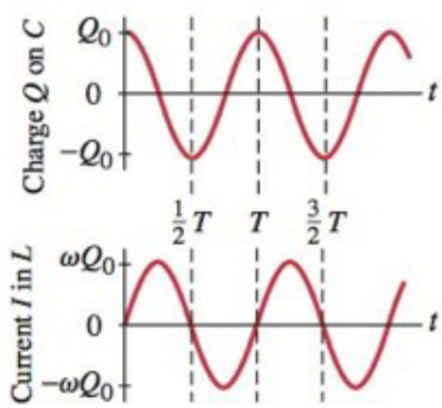
$$-\omega^2 Q_0 \cos(\omega t + \phi) + \frac{1}{LC} Q_0 \cos(\omega t + \phi) = 0$$

$$\left(-\omega^2 + \frac{1}{LC}\right) \cos(\omega t + \phi) = 0$$

$$\Longrightarrow = \left(-\omega^2 + \frac{1}{LC}\right) = 0$$

$$\omega = 2\pi f = \sqrt{\frac{1}{LC}}$$

3.1 Graph

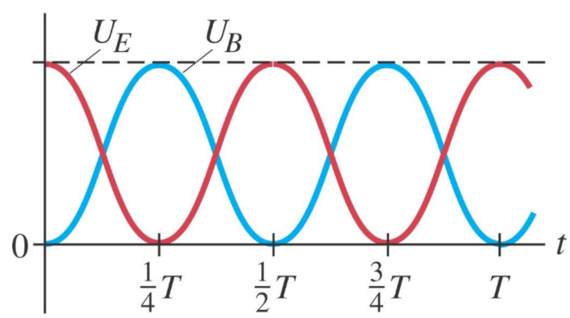


3.2 Total energy in the circuit

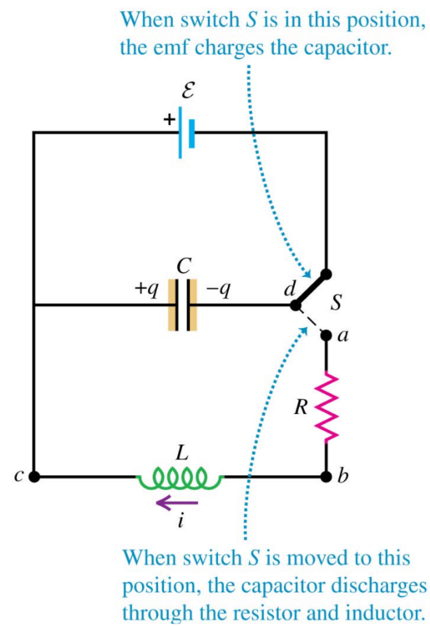
The total energy in the circuit is constant, it oscillates between the capacitor and the inductor.

$$
\begin{aligned}
U &= U_E + U_B \\
&= \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 \\
&= \frac{Q_0}{2C}\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] \\
&= \frac{Q_0^2}{2C}
\end{aligned}
$$

The frequency of energy oscillations is twice that of the frequency of charge and current oscillations.



4 RCL circuit (without voltage source)



Consider the circuit above. The e.m.f source charges the capacitor initially. When the switch is moved to the lower position, we have an inductor with inductance L and a resistor of resistance R connected in series across the terminals of a charged capacitor, forming an **RCL series circuit**. An RCL circuit exhibits damped harmonic motion if the resistance is not too large.

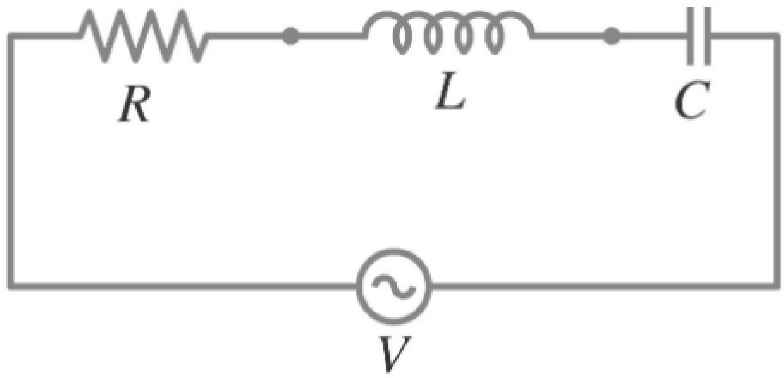
The charge as a function of time is a sinusoidal oscillation with an exponentially decaying amplitude, and angular frequency:

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

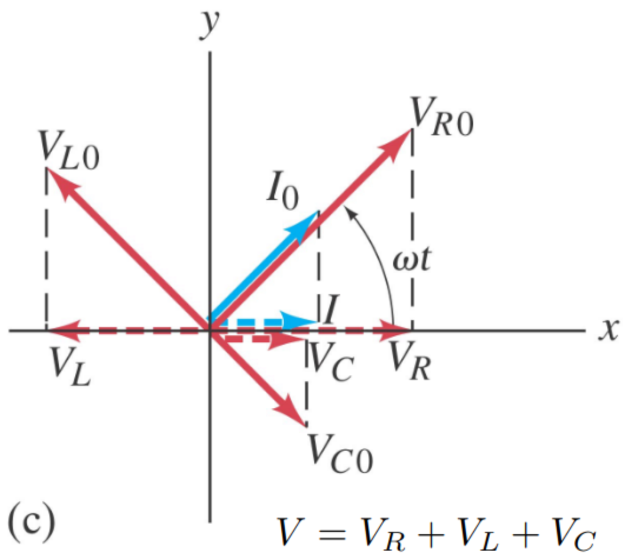
Where:

- ω' is the angular frequency of under damped oscillations in an L-R-C series circuit.
- L is the inductance of the circuit.
- C is the capacitance of the circuit.
- R is the resistance of the circuit.

5 RCL circuit



5.1 Phasor diagram analysis



The algebraic sum of voltages across each device at any point in time is equal to the source voltage V. **The algebraic sum is the sum of the projections of each phasor on the x-axis**. The current is the same throughout the series circuit.

5.2 Impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Where:

- Z is the impedance of the circuit.
- R is the resistance of the circuit.
- ω is the angular frequency.
- L is the inductance of the circuit.
- C is the capacitance of the circuit.

5.3 Current

$$I(t) = I_0 \cos \omega t$$

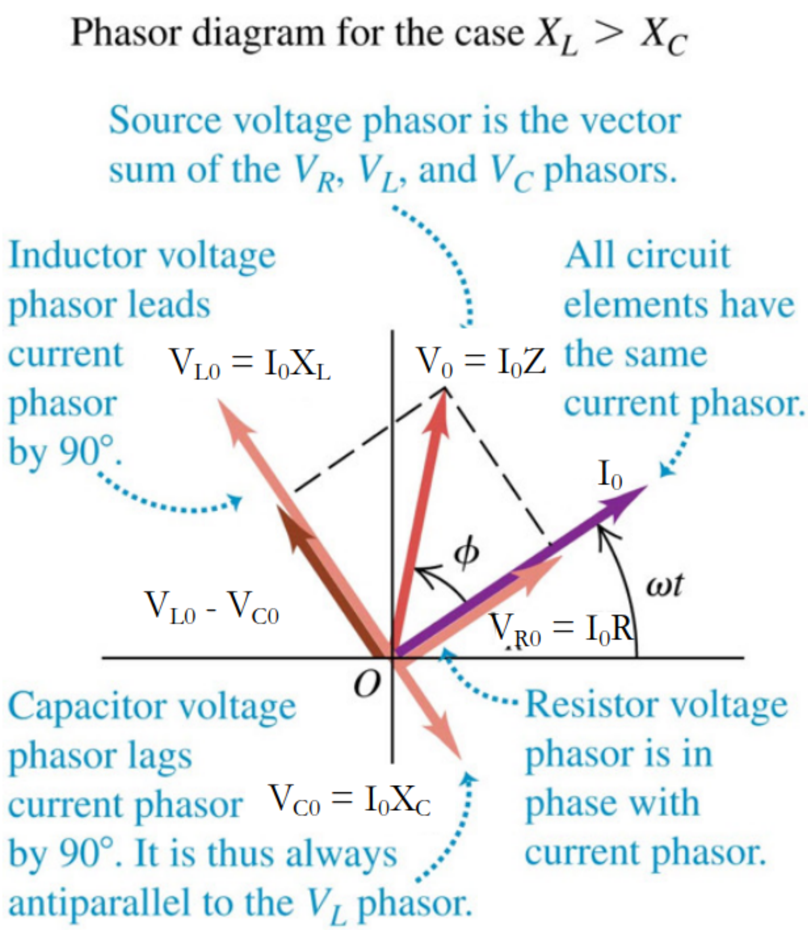
- $I(t)$ is the current.
- I_0 is the peak current.
- ω is the angular frequency.
- t is the time.

5.4 Voltage

$$V(t) = I_0 Z \cos(\omega t + \phi)$$

- $V(t)$ is the voltage
- I_0 is the peak current.
- Z is the impedance.
- ω is the angular frequency.
- t is the time.
- ϕ is the phase angle between the voltage and the current.

5.5 Phase angle between the voltage and the current



$$\phi = \arctan\left(\frac{X_L - X_C}{R}\right)$$

Where:

- ϕ is the phase angle between the voltage and the current.
- X_L is the inductive reactance.
- X_C is the capacitive reactance.
- R is the resistance of the circuit.

6 A.C. circuit components

6.1 Resistor

6.1.1 Current

The current through a resistor is **in phase** with the voltage. The root-mean-square current is an average measure of the current.

$$I_{rms} = \frac{I_0}{\sqrt{2}}$$

Where:

- I_{rms} is the root-mean-square current.
- I_0 is the peak current.

6.2 Inductor

6.2.1 Voltage

The voltage across an inductor is given by:

$$\begin{aligned} V &= L\frac{dI}{dt} \\ &= -\omega L I_0 \sin \omega t \\ &= \omega L I_0 \cos\left(\omega t + \frac{\pi}{2}\right) \\ &= V_0 \cos\left(\omega t + \frac{\pi}{2}\right) \end{aligned}$$

Where:

- L is the inductance of the inductor.
- $\frac{dI}{dt}$ is the rate of change of current.
- V is the voltage.
- I_0 is the peak current.
- ω is the angular frequency.
- t is the time.
- V_0 is the peak voltage.

6.2.2 Current

The current through an inductor **lags** the voltage by 90°.

$$I_0 = \frac{V_0}{\omega L}$$

Where:

- I_0 is the peak current.
- V_0 is the peak voltage.
- ω is the angular frequency.
- L is the inductance of the inductor.

6.3 Capacitor

6.3.1 Voltage

The voltage across a capacitor is given by:

$$\begin{aligned}
V &= \frac{Q}{C} \\
&= \frac{1}{C}\int I_0 \cos\omega t \, dt \\
&= \frac{I_0}{\omega C}\sin\omega t \\
&= \frac{I_0}{\omega C}\cos\left(\omega t - \frac{\pi}{2}\right) \\
&= V_0 \cos\left(\omega t - \frac{\pi}{2}\right)
\end{aligned}$$

Where:

- V is the voltage.
- Q is the charge held in the capacitor.
- C is the capacitance of the capacitor.
- I_0 is the peak current.
- ω is the angular frequency.
- t is the time.
- V_0 is the peak voltage.

6.3.2 Current

The current through a capacitor **leads** the voltage by 90°.

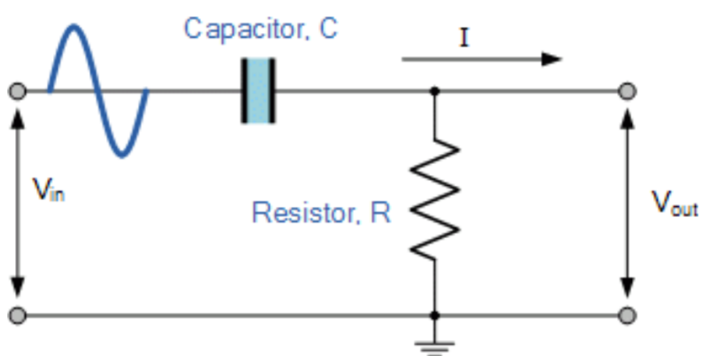
$$I_0 = V_0 \omega C$$

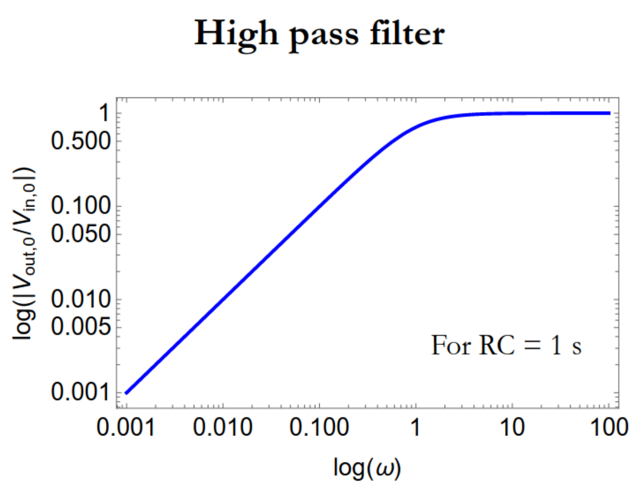
Where:

- I_0 is the peak current.
- V_0 is the peak voltage.
- ω is the angular frequency.
- C is the capacitance of the capacitor.

6.4 High pass filter

A high pass filter is a filter that filters out low frequencies. Think of the name as high frequencies passing through the filter unhindered.





6.5 Low pass filter

A low pass filter is a filter that filters out high frequencies. Think of the name as low frequencies passing through the filter unhindered.