Math Module 2A Notes

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Contents

1 Definitions

1.1 Diverging to positive infinity

Suppose a is a limit point of the domain A of f. If for every $M > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta, \quad x \in A \quad \Rightarrow \quad f(x) > M$$

We say that $f(x)$ **diverges to positive infinity as** x **approaches** a and write:

$$\lim_{x \to a} f(x) = +\infty$$

1.2 Diverging to negative infinity

Suppose a is a limit point of the domain A of f. If for every $M > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta, \quad x \in A \quad \Rightarrow \quad f(x) < -M$$

We say that $f(x)$ **diverges to negative infinity as** x **approaches** a and write:

$$\lim_{x \to a} f(x) = -\infty$$

1.3 Limit

Let $f : A \to \mathbb{R}$ be a function and let a be a limit point of A. If for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$0 < |x - a| < \delta, \quad x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

1.4 The restriction of a function

Consider $f : A \to \mathbb{R}$ and suppose $B \subset A$. The **restriction of** f **to** B, denoted by $f \restriction_B$, is the function with domain B given by:

$$f \restriction_B (x) = f(x), \text{ for } x \in B$$

1.5 Right-hand limit

Consider $f : A \to \mathbb{R}$. If the limit below exists:

$$\lim_{x \to a} f \restriction_{A \cap (a, \infty)} (x)$$

We call it the **right-hand limit** of f at a and denote it using:

$$\lim_{x \to a+} f(x)$$

1.6 Left-hand limit

Consider $f : A \to \mathbb{R}$. If the limit below exists:

$$\lim_{x \to a} f \restriction_{A \cap (-\infty, a)} (x)$$

We call it the **left-hand limit** of f at a and denote it using:

$$\lim_{x \to a-} f(x)$$

1.7 Heaviside function

$$H(x) = \begin{cases} 0; & \text{for } x < 0, \\ 1; & \text{for } x \geq 0. \end{cases}$$

1.8 Relation between limits and one-sided limits

Suppose that for some $\delta > 0$, $f(x)$ is defined on $(a - \delta, a) \cup (a, a + \delta)$. Then:

$$\lim_{x \to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to a+} f(x) = L = \lim_{x \to a-} f(x)$$

1.9 Continuous functions

A function $f : A \to \mathbb{R}$ is **continuous at** $a \in A$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$|x - a| < \delta, \quad x \in A \quad \Rightarrow \quad |f(x) - f(a)| < \varepsilon$$

If $B \subset A$ and f is continuous at every $a \in B$, we say that f is **continuous** on B.

We say that f is continuous if f is continuous on A.

1.9.1 Theorem

Consider a function $f : A \to \mathbb{R}$ and suppose $a \in A$ is also a **limit point of A**. Then f is continuous at a if and only if:

$$\lim_{x \to a} f(x) = f(a)$$

Do note that the criterion $\lim_{x\to a} f(x) = f(a)$ includes the condition that the limit $\lim_{x\to a} f(x)$ exists. If it doesn't, f is not continuous at a.

1.9.2 Notation

The set of functions continuous at a point a is denoted by $C(a)$. Hence, $f \in C(a)$ means that $f(x)$ is continuous at the point $a \in \mathbb{R}$. Similarly, $C(B)$ denotes the set of functions continuous on a set $B \subset \mathbb{R}$. Another example would be:

$$f \in C([0,1]) \quad \Leftrightarrow \quad f \text{ is a function that is continuous on } [0,1]$$

1.10 Composition rule

Given two functions $f(y)$ and $g(x)$, assume that:

1. $\lim_{x \to a} g(x) = L$
2. $f \in C(L)$

Then, we would have $\lim_{x\to a} f(g(x)) = f(L)$.

In other words, if f is continuous and f is continuous at $\lim_{x\to a} g(x)$:

$$\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x))$$

1.11 Elementary functions

Elementary functions are those functions $f(x)$ obtained from:

1. Constants and powers of x
2. Exponential, logarithm, trigonometric, inverse trigonometric functions

Through addition, subtraction, multiplication, division, power and composition. An example of composition is $f(g(x))$.

Any elementary function is **continuous** on its natural domain.

1.12 Intermediate value theorem

Assume that $f \in C([a,b]), f(a) \neq f(b)$ and y_0 is some real number between $f(a)$ and $f(b)$, like:

$$\min\{f(a), f(b)\} < y_0 < \max\{f(a), f(b)\}$$

Then there exists a $x_0 \in (a,b)$ such that $f(x_0) = y_0$.

f must be continuous on the whole **closed** interval $[a,b]$ for the conclusion to hold. For example, with $f : [0,1] \to \mathbb{R}$, given by:

$$f(x) = \begin{cases} 1 & \text{for } x = 0, \\ -1 & \text{for } x \in (0,1] \end{cases}$$

$f(x)$ is continuous on $(0,1]$ but not continuous on $[0,1]$. Since $y = 0$ is between $f(0)$ and $f(1)$, and there is no $x_0 \in (0,1)$ such that $f(x_0) = 0$, the conclusion fails.

1.13 Max/Min theorem

Assume that $f \in C([a,b])$. Then there are some points $x_m, x_M \in [a,b]$ such that for any $x \in [a,b]$, we have $f(x_m) \leq f(x) \leq f(x_M)$.

The result does not hold if we replace $[a,b]$ with (a,b). The function $f : (0,1) \to \mathbb{R},\ f(x) = x$ is a counterexample.

1.14 One-to-one functions

Consider a function $f : A \to \mathbb{R}$. We say that f is **one-to-one** or *injective* if for $x_1, x_2 \in A$:

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$$

Equivalently, we could say that $f : A \to \mathbb{R}$ is **one-to-one** if for $x_1, x_2 \in A$,

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$$

Note that if $f : A \to \mathbb{R}$ is one-to-one, then for every $y \in f(A)$ there exists **exactly one** $x \in A$, such that $f(x) = y$.

1.15 Inverse functions

Let $f : A \to \mathbb{R}$ be a one-to-one functions. The **inverse** function $f^{-1} : f(A) \to \mathbb{R}$ is defined by:

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y, \quad x \in A$$

This definition is usually expressed with the x and y swapped:

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x, \quad y \in A$$

Also, note that the $^{-1}$ in f^{-1} is **not** an exponent:

$$f^{-1}(x) \text{ does } \textbf{not} \text{ mean } \frac{1}{f(x)}$$

From the definition it also follows that if $f : A \to \mathbb{R}$ is one-to-one, then $g = f^{-1}$ if and only if:

1. $g(f(x)) = x, \quad$ for every $x \in A$
2. $f(g(y)) = y, \quad$ for every $y \in f(A)$

1.15.1 Example 1

The function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is given by $f(x) = \sin x$ is one-to-one. Its inverse is called arcsin. Hence:

$$\arcsin y = x \quad \Leftrightarrow \quad y = \sin x \text{ and } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

1.15.2 Example 2

The function $f : [0, \pi] \to \mathbb{R}$ is given by $f(x) = \cos x$ is one-to-one. Its inverse is called arccos. Hence:

$$\arccos y = x \quad \Leftrightarrow \quad y = \cos x \text{ and } x \in [0, \pi]$$

1.15.3 Example 3

The function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is given by $f(x) = \tan x$ is one-to-one. Its inverse is called arctan. Hence:

$$\arctan y = x \quad \Leftrightarrow \quad y = \tan x \text{ and } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

1.15.4 Comment on the examples

A lot of textbooks and calculators often use the notation $\sin^{-1},\ \cos^{-1},\ \tan^{-1}$ instead of $\arcsin,\ \arccos,\ \arctan$ respectively, but this is somewhat misleading since the 3 trigonometric functions themselves are not one-to-one and do not have an inverse.

1.16 Inverse continuous functions on an interval

If $f : I \to \mathbb{R}$ is continuous and one-to-one, and if I is an interval, then f^{-1} is also continuous.

1.16.1 Example

The function $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ is given by $f(x) = \sin x$, is one-to-one and continuous.

Since the domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is an interval, the inverse function arcsin (with domain $[-1, 1]$) is also continuous.

By similar arguments, the arccos and arctan functions are also continuous.

2 Linking the limits of functions and sequences

For $f : A \to \mathbb{R}$, assume that $\lim_{x\to a} = L$. Then for any sequence (a_n) in A **satisfying both conditions:**

1. For each n, we have $a_n \neq a$
2. $\lim_{n\to\infty}$

We have $\lim_{n\to\infty} = L$.

2.1 Proof

Since $\lim_{x\to a} f(x) = L$, for any $\varepsilon > 0$, there is a $\delta > 0$ such that:

$$0 < |x - a| < \delta,\ x \in A \quad \Rightarrow \quad |f(x) - L| < \varepsilon$$

Since $a_n \to a,\ a_n \in A,\ a_n \neq a$, we can find N such that:

$$n > N \quad \Rightarrow \quad 0 < |a_n - a| < \delta \quad \Rightarrow \quad |f(a_n) - L| < \varepsilon$$

The proof is complete.

2.2 Contrapositive statement

Using the theorem we just proved, let's say that we have a function $f(x)$ and two sequences (a_n) and (b_n), such that:

$$a_n \neq a,\ b_n \neq a,\ \lim_{n\to\infty} a_n = \lim_{n\to\infty} = a,\ \lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$$

That means that there does not exist any $L \in \mathbb{R}$ such that:

$$\lim_{x\to a} f(x) = L$$

Hence, the limit $\lim_{x\to a} f(x)$ does not exist. Because if it exists, the theorem tells us that $\lim_{x\to\infty} f(a_n) = \lim_{x\to\infty} f(b_n) = L$.

2.2.1 Example

Let $f(x) = \sin \frac{1}{x}$. Note that:

$$0 = f(x) = \sin \frac{1}{x} \quad \Leftrightarrow \quad \frac{1}{x} = n\pi, \text{ where } n = \pm 1, \pm 2, \pm 3, \dots$$

$$1 = f(x) = \sin \frac{1}{x} \quad \Leftrightarrow \quad \frac{1}{x} = \frac{\pi}{2} + 2n\pi, \text{ where } n = 0, \pm 1, \pm 2, \dots$$

Let $a_n = \frac{1}{n\pi}$ for n = 1, 2, 3, …. We get $a_n \neq 0, a_n \to 0$ as $n \to \infty$. Hence:

$$f(a_n) = \sin \frac{1}{a_n} = \sin n\pi = 0$$

So $\lim_{n\to\infty} f(a_n) = 0$.

Let $a_n = \frac{1}{\frac{\pi}{2}+2n\pi}$ for n = 1, 2, 3, …. We get $b_n \neq 0, b_n \to 0$ as $n \to \infty$. Hence:

$$f(b_n) = \sin \frac{1}{b_n} = \sin \left(\frac{\pi}{2} + 2n\pi\right) = 1$$

So $\lim_{n\to\infty} f(b_n) = 1$.

Since $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$, $\lim_{x\to 0} f(x)$ does not exist.

3 Reasoning with limit laws

Suppose $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x) = L$.

Does $\lim_{x\to a}(f(x) + g(x))$ exist?

If for some $l \in \mathbb{R}$,

$$\lim_{x\to a}(f(x) + g(x)) = l$$

Then:

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}[f(x) + g(x) - g(x)] \\ &= l - L \\ &\neq \text{undefined}\end{aligned}$$

Since $\lim_{x\to a} f(x)$ does not exist, $\lim_{x\to a} f(x)$ cannot be equal to $l - L$, which is a contradiction.

Hence, $\lim_{x\to a}(f(x) + g(x))$ does not exist.