1a) $\operatorname{div} \underset{\sim}{F} = \nabla \cdot \underset{\sim}{F}$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$$

$$= \frac{\sqrt{x^2+y^2} - x\left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}\right)}{x^2+y^2} + \frac{\sqrt{x^2+y^2} - y\left(\frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}}\right)}{x^2+y^2}$$

$$= \frac{x^2+y^2 \cancel{-x^2} \cancel{+x^2} + \cancel{y^2} \cancel{-y^2}}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

(b) A horizontal surface will have unit normal vector $(0,0,1)$ or $(0,0,-1)$.
In either case, we get $\underset{\sim}{F} \cdot \underset{\sim}{U} = 0$ and the flux through $S$ is

$$\iint_S \underset{\sim}{F} \cdot \underset{\sim}{U} \, dS = 0$$

(c) A unit normal vector on the surface $S_1$ is $\underset{\sim}{U} = (x, y, 0)$

$\therefore$ the flux through $S_1$ is

$$\iint_{S_1} \underset{\sim}{F} \cdot \underset{\sim}{U} \, dS = \iint_{S_1} \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right) \cdot (x, y, 0) \, dS$$

$$= \iint_{S_1} \frac{x^2+y^2}{\sqrt{x^2+y^2}} \, dS$$

$$= \iint_{S_1} \sqrt{x^2+y^2} \, dS$$

$$= \iint_{S_1} 1 \, dS \qquad \because x^2+y^2 = 1$$

$$= \text{Area of } S_1 \qquad \therefore \sqrt{x^2+y^2} = \sqrt{1} = 1$$

$$= 2\pi(2) = 4\pi$$

1(d) Let $S_3$ be the horizontal surface

$$1 \leq \sqrt{x^2+y^2} \leq 2, \quad z=1,$$

with unit normal $\underset{\sim}{n} = (0,0,1)$

and $S_4$ the surface

$$1 \leq \sqrt{x^2+y^2} \leq 2, \quad z=-1,$$

with unit normal $\underset{\sim}{n} = (0,0,-1)$

Consider the region $Q \subset \mathbb{R}^3$ whose outward oriented boundary consists of $-S_1, S_2, S_3$ and $S_4$. By the divergence theorem we have

$$\iint_{-S_1+S_2+S_3+S_4} \underset{\sim}{F} \cdot \underset{\sim}{n} \, dS = \iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz$$

So,

$$\iint_{S_2} \underset{\sim}{F} \cdot \underset{\sim}{n} \, dS = \iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz + \iint_{S_1} \underset{\sim}{F} \cdot \underset{\sim}{n} \, dS$$

$$- \iint_{S_2+S_3} \underset{\sim}{F} \cdot \underset{\sim}{n} \, dS \quad - (1)$$

1d) By the result of part (b), the flux through $S_3$ and $S_4$ is $0$, and by part (c), the flux through $S_1$ is $4\pi$. From the result from (a):

$$\iiint_Q \operatorname{div} \underset{\sim}{F}\, dx\,dy\,dz = \iiint_Q \frac{1}{\sqrt{x^2+y^2}}\, dx\,dy\,dz$$

Using cylindrical coordinates,

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$dx\,dy\,dz = r\,dr\,d\theta\,dz$,

$$\int_{\theta=0}^{2\pi}\left(\int_{z=-1}^{1}\left(\int_{r=1}^{3-z^2}\frac{1}{\cancel{r}}\cdot\cancel{r}\,dr\right)dz\right)d\theta$$

$$= 2\pi\int_{-1}^{1}(3-z^2-1)\,dz$$

$$= 2\pi\int_{-1}^{1}(2-z^2)\,dz$$

$$= 2\pi\left[2z-\frac{z^3}{3}\right]_{-1}^{1}$$

$$= 2\pi\left(\frac{10}{3}\right) = \frac{20\pi}{3}$$

(d) Putting it all together in (1)

$$\iint_{S_2} \underset{\sim}{F} \cdot \underset{\sim}{U}\, dS = \frac{20\pi}{3} + 4\pi - 0$$

$$= \frac{32\pi}{3}$$

2a) $\det A = \left|\begin{array}{ccc|cc} 0 & 1 & a & 0 & 1 \\ 0 & a & 1 & 0 & a \\ 1 & 1 & 1 & 1 & 1 \end{array}\right.$

$$= 1 - a^2$$

2b) $A\underset{\sim}{x} = \underset{\sim}{b}$

A is invertible if and only if $\det A \neq 0$

$\therefore$ For $a \in \mathbb{R}, a \neq \pm 1$, the equation $A\underset{\sim}{x} = \underset{\sim}{b}$ has exactly one solution as A is invertible.

For $a = 1$,

$$\begin{bmatrix} 0 & 1 & a & a \\ 0 & a & 1 & a \\ 1 & 1 & 1 & a \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore A\underset{\sim}{x} = \underset{\sim}{b}$ when $a = 1$ has infinitely many solutions.

2b) For $a=-1$,

$$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ $A\underset{\sim}{x} = \underset{\sim}{b}$ when $a=-1$ has no solutions.

3a) S being linearly independent means that the only solution to the equation

$$k_1\underset{\sim}{x}_1 + k_2\underset{\sim}{x}_2 + \dots + k_r\underset{\sim}{x}_r = 0$$

is $k_1 = k_2 = \dots = k_r = 0$

b) Suppose S is linearly independent,

$$c_1(\underset{\sim}{x}_1 - \underset{\sim}{x}_2) + c_2(\underset{\sim}{x}_2 - x_3) + \dots + c_{r-1}(\underset{\sim}{x}_{r-1} - \underset{\sim}{x}_r) = 0$$

$$c_1\underset{\sim}{x}_1 + (c_2 - c_1)\underset{\sim}{x}_2 + (c_3 - c_2)\underset{\sim}{x}_3 + \dots + (c_{r-1} - c_{r-2})\underset{\sim}{x}_{r-1} - c_{r-1}\underset{\sim}{x}_r = 0$$

Since S is linearly independent,

$$c_1 = c_2 - c_1 = c_3 - c_2 = \dots = c_{r-1} - c_{r-2} = c_{r-1} = 0$$

$$\therefore c_1 = c_2 = c_3 = \dots = c_{r-1} = 0$$

$\therefore \{\underset{\sim}{x}_1 - \underset{\sim}{x}_2, \underset{\sim}{x}_2 - \underset{\sim}{x}_3, \dots, \underset{\sim}{x}_{r-1} - \underset{\sim}{x}_r\}$ is linearly independent.