1a) The domain of $\underset{\sim}{F}$ is:

$$x, y, z \in \mathbb{R}, \quad x = y = z \neq 0$$

b) $\operatorname{div} \underset{\sim}{F} = \nabla \cdot \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x, y, z)$

$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}(x, y, z)$$

$$= \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x(2x)\sqrt{x^2+y^2+z^2} \times \frac{3}{2}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - y(2y)\sqrt{x^2+y^2+z^2} \times \frac{3}{2}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - z(2z)\sqrt{x^2+y^2+z^2} \times \frac{3}{2}}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{\sqrt{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{\frac{3}{2}}}\left(\cancel{x^2} \cancel{+y^2} \cancel{+z^2} \cancel{-3x^2}\right.$$

$$\left.\cancel{+x^2} \cancel{+y^2} \cancel{+z^2} \cancel{-3y^2} \cancel{+x^2} \cancel{+y^2} \cancel{+z^2} \cancel{-3z^2}\right)$$

$$= 0$$

1c) $\iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S}$

$\underset{\sim}{u} = (x, y, z)$

$= \iint_{S_1} \underset{\sim}{F} \cdot \underset{\sim}{u} \, dS$

$= \iint_{S_1} \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} (x, y, z) \cdot (x, y, z) \, dS$

$= \iint_{S_1} \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dS$

$= \iint_{S_1} \frac{1}{\sqrt{x^2+y^2+z^2}} \, dS$

$= \iint_{S_1} \frac{1}{\|(x, y, z)\|} \, dS$

$= \iint_{S_1} 1 \, dS$

$=$ Area of S_1

$= 4\pi$

1d) Let Q be the region between S_1 and S_2.

Since S_1 is oriented towards Q and S_2 is oriented away from Q

$$\iint_{\partial Q} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} - \iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

Since $\underset{\sim}{F}$ and its derivatives are defined and continuous both on Q and on the boundary surfaces of S_1 and S_2,

By Gauss' Theorem,

$$\iint_{\partial Q} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} - \iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$\iiint_Q \operatorname{div} \underset{\sim}{F} \, dx\,dy\,dz = \iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} - \iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$0 = \iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} - \iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S}$$

$$\iint_{S_2} \underset{\sim}{F} \cdot d\underset{\sim}{S} = \iint_{S_1} \underset{\sim}{F} \cdot d\underset{\sim}{S} = 4\pi$$

2a) $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$

$$\begin{bmatrix} 7 & 1 & 0 & 0 \\ 1 & 7 & 0 & 0 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & -3 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 6 \\ 6 \end{bmatrix}$$

$$= 6 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore \underset{\sim}{x}$ is an eigenvector of A with eigenvalue 6.

2b) A is invertible means $\det A \neq 0$,

$$\begin{aligned} 1 &= \det I \\ &= \det A^{-1}A \\ &= \det A^{-1} \cdot \det A \end{aligned}$$

2b) $\det(\lambda I - AB) = \det A^{-1} \cdot \det(\lambda I - AB) \cdot \det A$

$$= \det(A^{-1}(\lambda I - AB)A)$$

$$= \det((A^{-1}\lambda I - A^{-1}AB)A)$$

$$= \det(A^{-1}\lambda I A - A^{-1}ABA)$$

$$= \det A^{-1} \cdot \det A \cdot \det(\lambda I) - \det(IBA)$$

$$= \det(\lambda I) - \det(BA)$$

$$= \det(\lambda I - BA)$$

$\therefore \det(\lambda I - AB) = 0 \iff \det(\lambda I - BA) = 0$

$\therefore$ AB and BA have the same eigenvalues.

3a) Let $\underset{\sim}{u} \in W$

$$k\underset{\sim}{u} = k(x, y)$$
$$= (kx, ky)$$

Since:

$$|x| = |y|$$
$$k|x| = k|y|$$
$$|kx| = |ky|$$
$$\therefore k\underset{\sim}{u} \in W$$

$\therefore$ W is closed under multiplication with scalar.

3b) Let $\underset{\sim}{u} = (2, -2)$,

$\underset{\sim}{v} = (1, 1)$

$$\underset{\sim}{u} + \underset{\sim}{v} = (2, -2) + (1, 1)$$
$$= (3, -1) \notin W \text{ as } |3| \neq |-1|$$

Since W is not closed under addition,

W is not a subspace of $\mathbb{R}^2$.