Math Module 3B Notes

Hankertrix

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1 Definitions

1.1 Riemann sum

A Riemann sum is the sum of all the areas of the rectangles under a curve.

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

The limit of a Riemann sum is the area under a curve as the maximum width of a rectangle approaches 0 and the number of rectangles approaches infinity. Hence, the area under the curve is:

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

1.2 Riemann integral

Given a function $f : [a, b] \to \mathbb{R}$, a Riemann sum is:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i, \quad \Delta x_i = a_i - a_{i-1}$$

Points $a = a_0 < a_1 < \cdots < a_n = b$ form a **partition** of the interval $[a, b]$, and x_i^* are **sample points**. Further, let $\Delta x = max\{\Delta x_i : i = 1, \ldots, n\}$. Suppose the limit of the Riemann sums **exists** and is independent of our choice of partition or sample points. Then we say that f is integrable on $[a, b]$ and the limit below is called the **Riemann integral** of f from a to b.

$$\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\, dx$$

Also, for $a = b$ and $a > b$, we define:

$$\int_a^a f(x)\, dx = 0, \quad \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

1.3 Antiderivatives

Given a function $f(x)$, any function satisfying $F' = f$ is called an **antiderivative** (or primitive function) to the function f.

If $F' = G' = f$ on an interval I, then $F(x) = G(x) + C$ on I for some constants C. This means that on an interval, different antiderivatives of a function can only differ by a constant.

1.4 Improper integrals

If $f(x)$ is unbounded on $(a, b]$, but integrable (and hence bounded) on $[c, b]$ for every $c > a$, put:

$$\int_a^b f(x)\,dx = \lim_{c \to a+} \int_c^b f(t)\,dt$$

The left-hand side of the equation above is called an **improper integral**, and if the limit on the right exists, we say that the improper integral **converges**. Otherwise, we say that it **diverges**.

Similarly, we can consider the following improper integrals:

If $f(x)$ is unbounded on $[a, b)$ but integrable on $[a, c]$ for $c < b$, put:

$$\int_a^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(t)\,dt$$

And also:

$$\int_a^{+\infty} f(x)\,dx = \lim_{R \to +\infty} \int_a^R f(t)\,dt$$

$$\int_{-\infty}^b f(x)\,dx = \lim_{R \to -\infty} \int_R^b f(t)\,dt$$

1.4.1 Example

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Here, f is unbounded on $(0,1]$ (not even defined at $x = 0$), but for $c \in (0,1]$, the integral below exists:

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx$$

Hence, $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral, that can be evaluated as:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{\sqrt{x}}\,dx$$

This is only true if the limit exists.

2 Theorems and lemmas

2.1 Continuous functions are integrable

If a function is continuous on $[a,b]$, then it is integrable on $[a,b]$.

2.2 Linearity

If f and g are both integrable on $[a,b]$ and $c,d \in \mathbb{R}$, then $cf + dg$ is also integrable on $[a,b]$ and:

$$\int_a^b [cf(x) + dg(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

2.3 Additivity

For f integrable on an interval containing the points a,b,c, we have:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

2.4 The value of the integral follows the output of the function

If f and g are both integrable on $[a, b]$ and if:

$$f(x) \leq g(x), \quad \text{for all } x \in [a, b]$$

Then:

$$\int_a^b f(x)\,dx \leq \int_b^a g(x)\,dx$$

2.5 Triangle inequality for integrals

Note that the triangle inequality $|x+y| \leq |x| + |y|$ generalises to sums with more terms, i.e.

$$\left| \sum_{i=i}^{n} x_i \right| \leq \sum_{i=i}^{n} |x_i|$$

Using the definition of integrals and the properties of limits, and given that f and $|f|$ are integrable on $[a, b]$, it also follows that:

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

2.6 Continuity

Given an integrable function $f : [a, b] \to \mathbb{R}$ and let

$$F(x) = \int_a^x f(t)\,dt$$

Then $F \in C([a, b])$. This is to show that for every $x_0 \in [a, b]$, $\lim_{x \to x_0} F(x) = F(x_0)$

2.7 The integral mean value theorem

Suppose $f \in C([a, b])$. Then there exists a point $c \in (a, b)$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

2.8 The fundamental theorem of calculus

Suppose that $f \in C([a,b])$ and let $F : [a,b] \to \mathbb{R}$ be defined by:

$$F(x) = \int_a^x f(t)\,dt$$

Then $F'(x) = f(x)$ for any $x \in (a,b)$.

2.9 Newton-Leibniz' Formula

If f is continuous and $F' = f$ on $[a,b]$, then:

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) \\ &= [F(x)]_a^b \\ &= F(x)|_a^b\end{aligned}$$

3 Variable of integration

The name for the variable of integration is like a summation index. It is **arbitrary**. However, please **avoid** writing:

$$\int_a^x f(x)\,dx$$

4 What kind of functions are integrable?

The definition requires that for f to be integrable on $[a,b]$,
the limit $\lim_{\Delta x \to 0} \sum_{i=1}^n f(x_i^*)\Delta x_i$ must exist and be independent of how the partition points a_i and sample points x_i^* are chosen.

A previous theorem stated that **continuous** functions on $[a,b]$ are integrable.

Also, if $f : [a,b] \to \mathbb{R}$ is **bounded** and is continuous on $[a,b]$ except at finitely many points, f is still integrable. Moreover, changing the value of $f(x)$ at only finitely many points, does not affect the value of the integral $\int_a^b f(x)\,dx$.

5 Non-integrable functions

5.1 Example

Let $f : [0, 1] \to \mathbb{R}$ be given by:

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \notin \mathbb{Q} \end{cases}$$

Is $f(x)$ integrable on $[0, 1]$?

Let $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$ be a partition of $[0, 1]$. In each subinterval $[a_{i-1}, a_i]$, we can pick a point $x_i^* \in \mathbb{Q}$ and a point $t_i^* \notin \mathbb{Q}$.

With sample points x_i^*, we get:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} 1 \cdot \Delta x_i = 1 \to 1 \text{ as } \Delta x \to 0$$

On the other hand, with sample points t_i^*, we get:

$$\sum_{i=1}^{n} f(t_i^*)\Delta x_i = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0 \to 0 \text{ as } \Delta x \to 0$$

Since the limit of Riemann sums as $\Delta x \to 0$ is not independent of our choice of sample points, the function f is **not integrable**.

5.2 Unbounded functions are not integrable

If f is unbounded on $[a, b]$, then f is **not** integrable on $[a, b]$.

6 Average value of a function

For a finite set of numbers $a_1, a_2, \ldots, a_n$, their mean (average) value a_{avg} is:

$$a_{avg} = \frac{a_1 + a_2 + \ldots + a_n}{n}$$

The idea is that if we replaced all the different a_i with one fixed value, the average a_{avg}, we would still have the same sum, i.e.

$$a_{avg} + a_{avg} + \ldots + a_{avg} = na_{avg} = a_1 + a_2 + \ldots + a_n$$

$$\sum_{i=1}^{n} a_{avg} = \sum_{i=1}^{n} a_i$$

The average value f_{avg} of a function $f : [a, b] \to \mathbb{R}$ we choose such that if we replace $y = f(x)$ with the constant $f = f_{avg}$, we still get the same **integral**.

7 Applications to physics

7.1 Work

The amount of work W is the product of the force F and the distance s the object is moved:

$$W = F \cdot s$$

This assumes that the force is **constant** and acts in the direction of motion.

If the force is not constant, suppose $F = F(x)$.

Let's assume that $F(x)$ is continuous, and moves an object from $x = a$ to $x = b$. Divide $[a, b]$ into n subintervals, $[a_{i-1}, a_i]$ where:

$$a = a_0 < a_1 < a_2 < \ldots < a_n = b$$

Let $\Delta x_i = a_i - a_{i-1}$ and take $x_i^* \in [a_{i-1}, a_i]$. Since F is continuous, if Δx_i is small, we have:

$$F \approx F(x_i^*), \quad \text{for } x \in [a_{i-1}, a_i]$$

The work ΔW_i required to move the object along $\$[\mathrm{a}_{\mathrm{i - 1}}, \mathrm{a}_{\mathrm{i}}]$ is:

$$\Delta W_i \approx F(x_i^*) \Delta x_i$$

And the total work to move from a to b is:

$$W = \sum_{i=1}^{n} \Delta W_i \approx \sum_{i=1}^{n} F(x_i^*) \Delta x_i$$

Taking more but smaller subintervals, the approximation gets better, so:

$$W \int_a^b F(x)\, dx$$

7.2 Centre of mass

Consider a system of n masses m_i at positions x_i respectively ($i = 1, \ldots, n$).

It's centre of mass, is the point $\bar{x}$ about which the total moment is zero.

$$\sum_{i=1}^{n}(x_i - \bar{x})m_i = \sum_{i=1}^{n} x_i m_i - \bar{x}\sum_{i=1}^{n} m_i = 0$$

I.e.

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} = \frac{M_{x=0}}{m}$$

Where $M_{x=0}$ is the total moment about $x = 0$ and m is the total mass.

7.3 Continuous mass distribution

Consider a one-dimensional distribution of mass with continuously variable line density $\rho(x)$ along the interval $[a, b]$.

Consider an element of length dx at position x. It has mass $dm = \rho(x)\,dx$ and has a moment $x = x_0$ of:

$$dM_{x=x_0} = (x - x_0)\,dm = (x - x_0)\rho(x)\,dx$$

It's centre of mass, is the point $\bar{x}$ about which the total moment is zero, i.e.

$$\int_{x=a}^{b} dM_{x=\bar{x}} = \int_{a}^{b}(x - \bar{x})\rho(x)\,dx = \int_{a}^{b} x\rho(x)\,dx - \bar{x}\int_{a}^{b}\rho(x)\,dx = 0$$

Hence:

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx} = \frac{M_{x=0}}{m}$$

Where $M_{x=0}$a is the total moment about $x = 0$ and m is the total mass.