Math Module 5A Cheat Sheet

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Contents

1 Definitions

1.1 Vectors in $\mathbb{R}^n$

Let n be a positive integer.

We define $\mathbb{R}^n$ to be the set of all ordered n-tuples of elements from $\mathbb{R}$, i.e:

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}\}$$

An element $\boldsymbol{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ is called a **vector**. We usually use boldface symbols to denote vectors.

A vector is ordered, i.e:

$$(1, 2, 5) \neq (1, 5, 2)$$

Repetitions also matter:

$$(1, 1, 2) \neq (1, 2)$$

1.1.1 Example

$$\boldsymbol{x} = (-1, \pi, \sin 2, \frac{1}{2}, 1000) \text{ is a vector in } \mathbb{R}^5$$

1.2 Geometric interpretation of $\mathbb{R}^2$ or $\mathbb{R}^3$

- A vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ has length and direction, illustrated with an arrow.
- We use $\overrightarrow{P_1P_2}$ for the vector from P_1 to P_2.
- With $P_1 = (x_1, y_1, z_1), P_2 = (x_2, y_2, z_2)$, we have:

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

In particular, with $O = (0, 0, 0), P = (x, y, z)$, we get:

$$\overrightarrow{OP} = (x, y, z)$$

1.3 Scalars

Scalars are basically real numbers. For example, 1.2 and π are scalars.

1.4 Zero vector

The zero vector in $\mathbb{R}^n$ is the vector:

$$\mathbf{0} = (0, 0, \ldots, 0)$$

1.5 Negative vectors

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, its **additive inverse** (or **negative**) is denoted by $-\boldsymbol{x}$ and is given by:

$$-\boldsymbol{x} = (-1)\boldsymbol{x} = (-x_1, -x_2, \ldots, -x_n)$$

1.6 Sum of vectors

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$. The sum is defined as follows:

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

1.7 Difference of vectors

The difference $\boldsymbol{x} - \boldsymbol{y}$ of $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as:

$$\boldsymbol{x} - \boldsymbol{y} = \boldsymbol{x} + (-\boldsymbol{y})$$

1.8 Product of a vector and a scalar

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ be vectors in $\mathbb{R}^n$ and let c be a scalar. The product is defined as follows:

$$c\boldsymbol{x} = (cx_1, cx_2, \ldots, cx_n)$$

1.9 Dot product of vectors (Scalar product or Euclidean inner product)

For:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$$

Their scalar product (or Euclidean inner product, or dot product) $\boldsymbol{x} \cdot \boldsymbol{y}$, is defined as:

$$\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

Note that $\boldsymbol{x} \cdot \boldsymbol{y}$ is a scalar, not a vector.

1.9.1 Example:

With $\boldsymbol{x} = (1, 2, 3, 4), \boldsymbol{y} = (-2, 3, -1, 1)$, we have:

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= 1 \cdot (-2) + 2 \cdot 3 + 3 \cdot (-1) + 4 \cdot 1 \\ &= 5 \end{aligned}$$

1.10 Norm of a vector

The norm $||\boldsymbol{x}||$ of a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is defined by:

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$$

1.10.1 Example

The normal of the vector $\boldsymbol{x} = (1, 0, 3, -2)$ in $\mathbb{R}^4$ is:

$$\begin{aligned} ||\boldsymbol{x}|| &= \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} \\ &= \sqrt{1^2 + 0^2 + 3^2 + (-2)^2} \\ &= \sqrt{14} \end{aligned}$$

1.11 Geometric view of the dot product

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let θ be the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, where $\theta \in [0, \pi]$. Then:

$$\boldsymbol{x} \cdot \boldsymbol{y} = ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \cos \theta$$

Since $\theta \in [0, \pi]$, it follows that:

$$\theta = \arccos \left(\frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||} \right)$$

1.12 Cauchy-Schwarz inequality

Note that for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ or $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$, we have:

$$\begin{aligned} |\boldsymbol{x} \cdot \boldsymbol{y}| &= ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \cdot |\cos\theta| \\ &\leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}|| \quad (\because |\cos\theta| \leq 1 \text{ when } 0 \leq \theta \leq \pi) \end{aligned}$$

Hence, let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$. Then:

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}|| \cdot ||\boldsymbol{y}||$$

1.13 Triangle inequality

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$, then:

$$||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$$

1.14 Orthogonality

Two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^n$ are said to be orthogonal if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$.

- In $\mathbb{R}^2$ or $\mathbb{R}^3$, two non-zero vectors are perpendicular if and only if they are orthogonal. It is useful to think about orthogonal vectors in higher dimensions in the same way, even if it makes no geometrical sense.
- Since $\mathbf{0} \cdot \boldsymbol{x} = 0$, the zero vector is orthogonal to all vectors.

1.14.1 Example

The vectors $\boldsymbol{x} = (1, 2, 1, 2)$ and $\boldsymbol{y} = (3, 3, -3, -3)$ are orthogonal vectors in $\mathbb{R}^4$, since:

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= 1 \cdot 3 + 2 \cdot 3 + 1 \cdot (-3) + 2 \cdot (-3) \\ &= 0 \end{aligned}$$

1.15 Pythagoras' theorem in $\mathbb{R}^n$

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal vectors in $\mathbb{R}^n$, then:

$$||\boldsymbol{x} + \boldsymbol{y}||^2 = ||\boldsymbol{x}||^2 + ||\boldsymbol{y}||^2$$

1.16 Projections in $\mathbb{R}^n$

Suppose $\boldsymbol{x}, \boldsymbol{a} \in \mathbb{R}^n, \boldsymbol{a} \neq 0$.

There exist **unique** $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$ such that:

$$\boldsymbol{x} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \boldsymbol{x}_1 = k\boldsymbol{a}, l \in \mathbb{R}, \quad \text{and } \boldsymbol{x}_2 \cdot \boldsymbol{a} = 0$$

This unique representation is given by:

$$\boldsymbol{x}_1 = \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{||\boldsymbol{a}||^2}\boldsymbol{a}, \quad \boldsymbol{x}_2 = \boldsymbol{x} - \frac{\boldsymbol{x} \cdot \boldsymbol{a}}{||\boldsymbol{a}||^2}\boldsymbol{a}$$

$\boldsymbol{x}_1$ is called the **orthogonal projection** of $\boldsymbol{x}$ onto $\boldsymbol{a}$ and is denoted $proj_a\boldsymbol{x}$.

1.17 Cross product of vectors

Consider $\boldsymbol{x} = (x_1, x_2, x_3), \boldsymbol{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$.

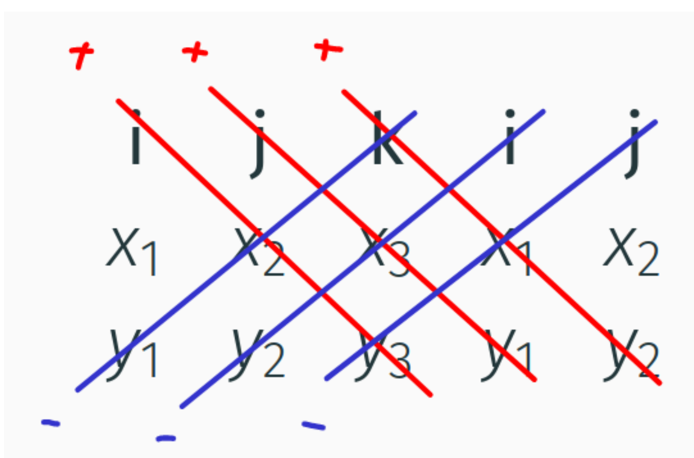
The **cross product** $\boldsymbol{x} \times \boldsymbol{y}$ of $\boldsymbol{x}$ and $\boldsymbol{y}$ is:

$$\boldsymbol{x} \times \boldsymbol{y} = (x_2y_3 - x_3y_2, -x_1y_3 + x_3y_1, x_1y_2 - x_2y_1) \in \mathbb{R}^3$$

To remember the formula above, let:

$$\boldsymbol{i} = (1, 0, 0), \quad \boldsymbol{j} = (0, 1, 0), \quad \boldsymbol{k} = (0, 0, 1)$$

And use this scheme:



$$\begin{aligned} &x_2y_3\boldsymbol{i} + x_3y_1\boldsymbol{j} + x_1y_2\boldsymbol{k} - x_2y_1\boldsymbol{k} - x_3y_2\boldsymbol{i} - x_1y_3\boldsymbol{j} \\ &= (x_2y_3 - x_3y_2)\boldsymbol{i} + (x_3y_1 - x_1y_3)\boldsymbol{j} + (x_1y_2 - x_2y_1)\boldsymbol{k} \end{aligned}$$

1.18 Planes

With a **normal vector** to a plane in $\mathbb{R}^3$, we mean a non-zero vector that is perpendicular to the plane.

Suppose the plane Π contains the point $P_0 = (x_0, y_0, z_0)$ and has normal vector $\boldsymbol{n} = (a, b, c)$. We see that a point $P = (x, y, z)$ lies in Π if and only if $\overrightarrow{P_0P}$ is orthogonal to $\boldsymbol{n}$.

1.18.1 Equations for a plane

$$P = (x, y, z) \text{ lies in the plane } \Pi$$

$$\Updownarrow$$

$$\overrightarrow{P_0P} \text{ is orthogonal to } \boldsymbol{n}$$

$$\Updownarrow$$

$$\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$$

$$\Updownarrow$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\Updownarrow$$

$$ax + by + cz + d = 0, \quad d = -ax_0 - by_0 - cz_0$$

1.19 Parametric equations for a straight line

Suppose P_0 is a point on the line l and suppose $\boldsymbol{v}$ is a vector parallel to l. Then a point P lies on l if and only if:

$$\overrightarrow{OP} = \overrightarrow{OP_0} + t\boldsymbol{v}, \quad \text{for some } t \in \mathbb{R}$$

$$\begin{gathered} P \text{ lies on } l \\ \Updownarrow \\ \overrightarrow{P_0P} = t\boldsymbol{v}, \quad \text{for some } t \in \mathbb{R} \\ \Updownarrow \\ \overrightarrow{OP} = \overrightarrow{OP_0} + t\boldsymbol{v}, \quad \text{for some } t \in \mathbb{R} \end{gathered}$$

If $P_0 = (x_0, y_0, z_0), \boldsymbol{v} = (a, b, c)$ and $P = (x, y, z)$, then:

$$\begin{gathered} \overrightarrow{OP} = \overrightarrow{OP_0} + t\boldsymbol{v}, \quad \text{for some } t \in \mathbb{R} \\ \Updownarrow \end{gathered}$$

$$\begin{aligned} x &= x_0 + at, \\ y &= x_0 + bt, \quad t \in \mathbb{R} \\ z &= z_0 + ct. \end{aligned}$$

2 Arithmetic rules for operations on vectors

For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n, k, m \in \mathbb{R}$, we have:

1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
2. $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{x} = \boldsymbol{x}$
3. $k(m\boldsymbol{x}) = (km)\boldsymbol{x}$
4. $(k + m)\boldsymbol{x} = k\boldsymbol{x} + m\boldsymbol{x}$
5. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$
6. $\boldsymbol{x} + (-\boldsymbol{x}) = \boldsymbol{0}$
7. $k(\boldsymbol{x} + \boldsymbol{y}) = k\boldsymbol{x} + k\boldsymbol{y}$
8. $1\boldsymbol{x} = \boldsymbol{x}$

3 Arithmetic rules for the dot product

For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n, k \in \mathbb{R}$, we have:

1. $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{y} \cdot \boldsymbol{x}$
2. $(\boldsymbol{x} + \boldsymbol{y}) \cdot \boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{z} + \boldsymbol{y} \cdot \boldsymbol{z}$
3. $(k\boldsymbol{x}) \cdot \boldsymbol{y} = k(\boldsymbol{x} \cdot \boldsymbol{y})$
4. $\boldsymbol{x} \cdot \boldsymbol{x} \geq 0$. Also, $\boldsymbol{x} \cdot \boldsymbol{x} = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$

4 Rules for the norm of a vector

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors in $\mathbb{R}^n$ and k a scalar. Then we have:

1. $||\boldsymbol{x}|| \geq 0$
2. $||\boldsymbol{x}|| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$
3. $||k\boldsymbol{x}|| = |k| \cdot ||\boldsymbol{x}||$
4. $||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$