1) Definition of a periodic function:

$$f(x+p) = f(x),\ x, p \in \mathbb{R}$$

Let $P(n)$ be the statement:

$$f(x+np) = f(x)$$

Base case:

when $n=1$,

$$f(x+p) = f(x)$$

Assuming $P(k)$ is correct for $k \in \mathbb{Z}^+, k>1$,

$$\therefore f(x+kp) = f(x)$$

$$P(k+1) = f(x+(k+1)p)$$

$$= f(x+kp+p)$$

$$= f(x+p) \because f(x+kp) = f(x)$$

$$= f(x)$$

2) $f(x+p) = f(x) \quad - (1)$

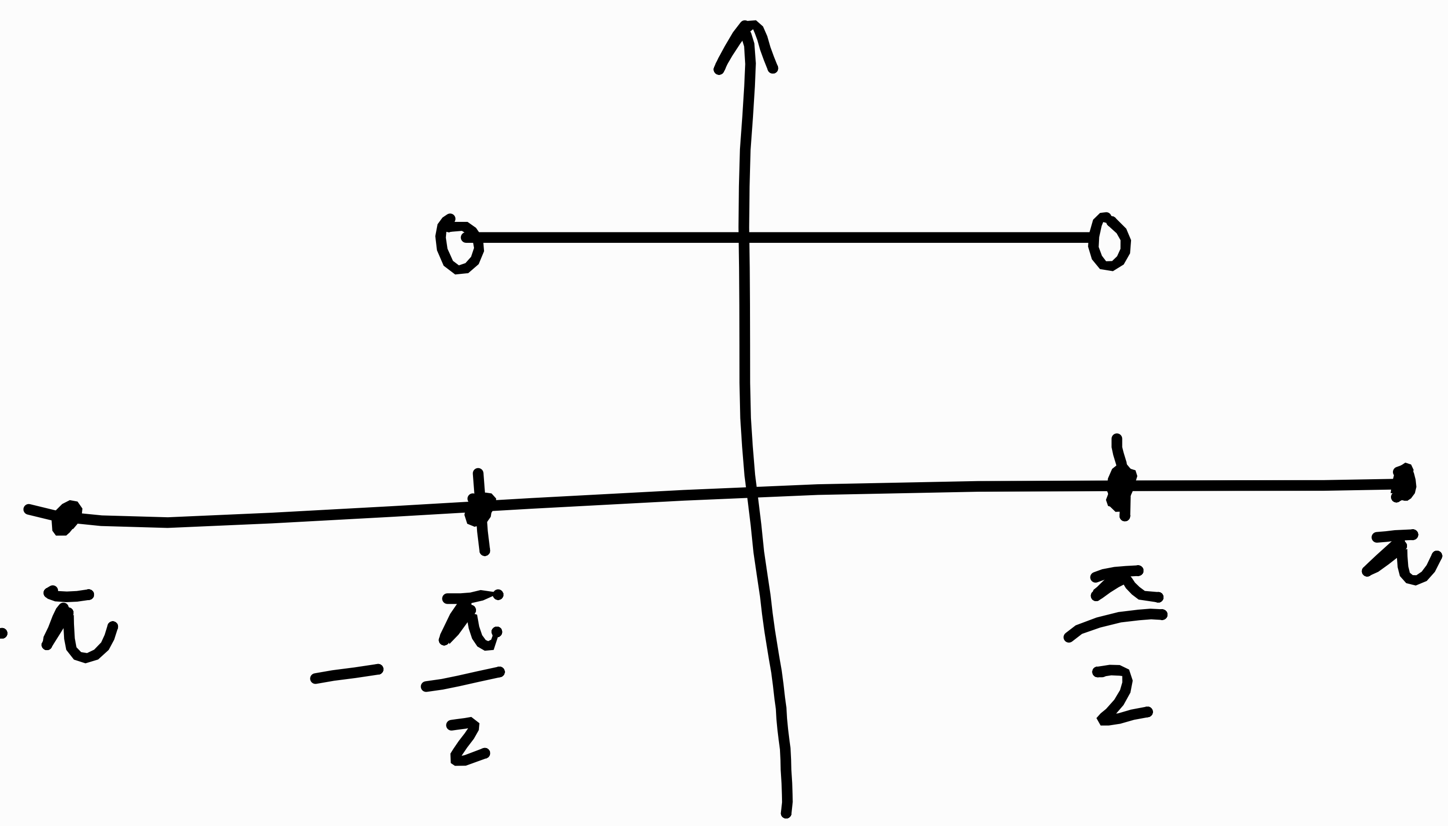
Sub $x = ax$ into (1)

$f(ax+p) = f(ax)$

$f\left(x + \frac{p}{a}\right) = f(x)$

$\therefore f(ax)$ is periodic with period $\frac{p}{a}$

3)



$$\text{F.S} = a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

$$= a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right) + b_n \sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right) \right\}$$

$$= a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos(nx) + b_n \sin(nx) \right\}$$

3) $a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$

$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{2\pi}\left[\cancel{\int_{-\pi}^{-\frac{\pi}{2}} 0\,dx} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dx + \cancel{\int_{\frac{\pi}{2}}^{\pi} 0\,dx}\right]$

$= \frac{1}{2\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$

$= \frac{1}{2}$

$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{n\pi x}{\pi}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$

$= \frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(nx)\,dx\right]$

$= \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$

3) $b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$

$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin(nx)\,dx$

$= \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{1}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) - \left(-\cos\left(-\frac{n\pi}{2}\right)\right)\right]$

$= \frac{1}{n\pi}\left[-\cos\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)\right]$

$= 0$

$\therefore\ F.S = \frac{1}{2} + \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx)\right\}$

$F.S @ x=\frac{\pi}{2} = \frac{1}{2} + \cancel{\sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right)\right\}}$

$= \frac{1}{2}$

$\frac{1}{2}\left(\lim_{x\to(\frac{\pi}{2})^-} f(x) + \lim_{x\to(\frac{\pi}{2})^+} f(x)\right) = \frac{1}{2}(0+1)$

$= \frac{1}{2}$ (verified)

4) $$a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)\,dx$$

$$= \frac{1}{n^3\pi}\left[\cancel{n^2x^2\sin(nx)} \cancel{- 2\sin(nx)} + 2nx\cos(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n^3\pi}\left[2n\pi\cos(n\pi) - \left(-2n\pi\cos(-n\pi)\right)\right]$$

$$= \frac{4}{n^2}\cos(n\pi)$$

4) $b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(nx)\, dx$$

$$= \frac{1}{n^3\pi}\left[\cancel{2\cos(nx)} - \cancel{n^2x^2\cos(nx)} + \cancel{2nx\sin(nx)}\right]_{-\pi}^{\pi}$$

$$= 0$$

$$\therefore\ F.S = \frac{1}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^2}\cos(nx)$$

5) $a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$

$= \frac{1}{2}\int_{-1}^{1} f(x)\,dx$

$= \frac{1}{2}\left[\int_{-1}^{0} -1\,dx + \int_{0}^{1} 1\,dx\right]$

$= \frac{1}{2}(-1+1)$

$= 0$

$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx$

$= \int_{-1}^{0} -\cos(n\pi x)\,dx + \int_{0}^{1}\cos(n\pi x)\,dx$

$= -\left[\frac{\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin(n\pi x)}{n\pi}\right]_{0}^{1}$

$= 0$

5) $$b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} -\sin(n\pi x)\,dx + \int_{0}^{1}\sin(n\pi x)\,dx$$

$$= \left[\frac{\cos(n\pi x)}{n\pi}\right]_{-1}^{0} - \left[\frac{\cos(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$= \frac{1}{n\pi}\left[1-\cos(n\pi) - \left(\cos(n\pi) - 1\right)\right]$$

$$= \frac{2}{n\pi}\left[1-\cos(n\pi)\right]$$

$$F.S = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos(n\pi)\right]\sin(n\pi x)$$

6) $$a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$= \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}\left[\int_{-2}^{0} 0\,dx + \int_{0}^{2} 2\,dx\right]$$

$$= 1$$

$$a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\int_{0}^{2} 2\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= -\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right]_0^2$$

$$= \frac{-2\sin(n\pi)}{n\pi}$$

$$= 0$$

6) $$b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_{0}^{2} \sin\left(\frac{n\pi x}{2}\right)dx$$

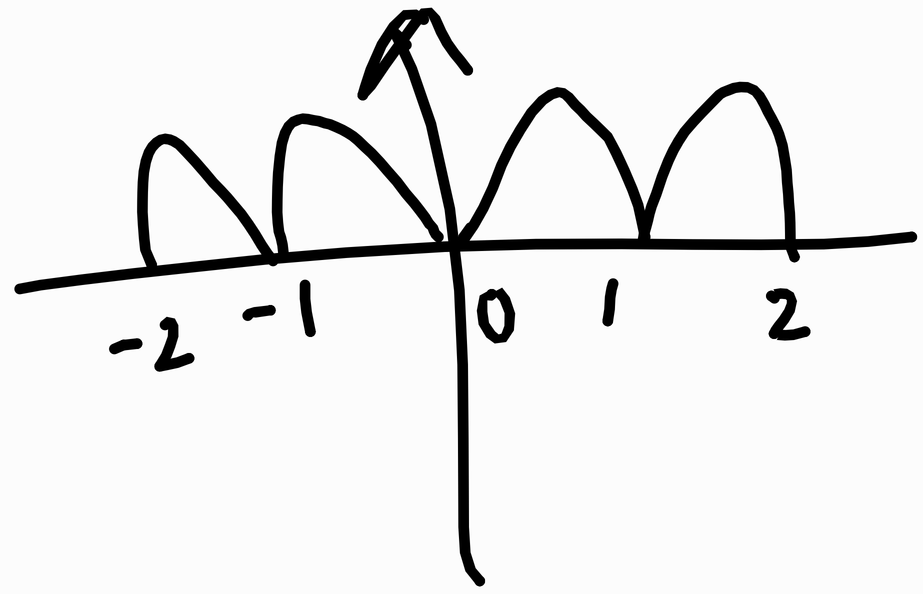
$$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$= \frac{2}{n\pi}\left[1-\cos(n\pi)\right]$$

F.S is $1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-\cos(n\pi)\right]\sin\left(\frac{n\pi x}{2}\right)$

7) $f(x) = \sin(\pi x)$



$$a_0 = \frac{1}{2L}\int_{\alpha}^{\alpha+2L} f(x)\,dx$$

$$= \int_0^1 \sin(\pi x)\,dx$$

$$= -\left[\frac{\cos(\pi x)}{\pi}\right]_0^1$$

$$= -\left(-\frac{1}{\pi} - \frac{1}{\pi}\right)$$

$$= \frac{2}{\pi}$$

7) $a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$= 2\int_0^1 \sin(\pi x)\cos(2n\pi x)\,dx$$

$$= -\frac{1}{\pi^2(1-4n^2)}\Big[(\pi+2n\pi)\overset{-1}{\underline{\cos((\pi-2n\pi)x)}} +$$

$$(\pi-2n\pi)\overset{-1}{\underline{\cos((\pi+2n\pi)x)}}\Big]_0^1$$

$$= -\frac{1}{\pi^2(1-4n^2)}\Big[-\pi\cancel{-2n\pi}-\pi\cancel{+2n\pi}-$$

$$(\pi\cancel{+2n\pi}+\pi\cancel{-2n\pi})\Big]$$

$$= \frac{4}{\pi(1-4n^2)}$$

$$= \frac{-4}{\pi(4n^2-1)}$$

7) $b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$$= 2\int_0^1 \sin(\pi x)\sin(2n\pi x)\,dx$$

$$= \frac{1}{\pi^2(1-4n^2)}\Big[(\pi+2n\pi)\overset{0}{\underline{\sin((\pi-2n\pi)x)}} + (2n\pi-\pi)\overset{0}{\underline{\sin((\pi+2n\pi)x)}}\Big]_0^1$$

$$= 0$$

$\therefore$ F.S is $\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos(2n\pi x)$