Electric Fields Tutorial

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1 Question 1

1.1 Point 1

Finding the electric field at point 1:

$$\begin{aligned}
\vec{E}_1 &= \vec{E}_{-q} + \vec{E}_{+2q} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{a^2 + 2a^2}^2} \frac{-\hat{\imath} - 2\hat{\jmath}}{\sqrt{2^2 + 1^2}} + \frac{1}{4\pi\varepsilon_0} \frac{2q}{\sqrt{a^2 + 2a^2}^2} \frac{-\hat{\imath} + 2\hat{\jmath}}{\sqrt{2^2 + 1^2}} \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{q(-\hat{\imath} - 2\hat{\jmath})}{5a^2 \cdot \sqrt{5}} + \frac{q(-2\hat{\imath} + 4\hat{\jmath})}{5a^2 \cdot \sqrt{5}} \right) \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{q(-\hat{\imath} - 2\hat{\jmath} - 2\hat{\imath} + 4\hat{\jmath})}{5\sqrt{5}a^2} \right) \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{q(-3\hat{\imath} + 2\hat{\jmath})}{5\sqrt{5}a^2} \right) \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{5\sqrt{5}a^2} \right) (-3\hat{\imath} + 2\hat{\jmath})
\end{aligned}$$

1.2 Point 2

Finding the electric field at point 2:

$$\begin{aligned}
\vec{E}_2 &= \vec{E}_{-q} + \vec{E}_{+2q} \\
&= \frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \hat{\imath} + \frac{1}{4\pi\varepsilon_0} \frac{2q}{(3a)^2} \cdot -\hat{\imath} \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{a^2} - \frac{2q}{9a^2} \right) \hat{\imath} \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{9q}{9a^2} - \frac{2q}{9a^2} \right) \hat{\imath} \\
&= \frac{1}{4\pi\varepsilon_0} \left(\frac{7q}{9a^2} \right) \hat{\imath}
\end{aligned}$$

1.3 Neutral point

The neutral point must be somewhere along the x-axis in between the 2 charges.

Let x be the distance of the neutral point from charge $-q$:

$$\begin{aligned}
\vec{E} &= \vec{E}_{-q} + \vec{E}_{+2q} \\
0 &= \frac{1}{4\pi\varepsilon_0}\frac{q}{x^2} \cdot -\hat{i} + \frac{2q}{(2a-x)^2}\hat{i} \\
0 &= -\frac{1}{x^2} + \frac{2}{(2a-x)^2} \\
\frac{1}{x^2} &= \frac{2}{(2a-x)^2} \\
2a - x^2 &= 2x^2 \\
3x^2 &= 2a \\
x^2 &= \frac{2}{3}a \\
x &= \pm\sqrt{\frac{2}{3}a} \\
x &= \sqrt{\frac{2}{3}a} \quad (\because x > 0)
\end{aligned}$$

The neutral point will be $\sqrt{\frac{2}{3}a}$ away from the charge $-q$.

2 Question 2

When the charges on the spheres are in equilibrium, the electric potential of both will be equal:

$$
\begin{aligned}
V_1 &= V_2 \\
\frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} &= \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} \\
\frac{q_1}{r_1} &= \frac{q_2}{r_2} \\
\frac{r_2}{r_1} &= \frac{q_2}{q_1} \\
q_2 &= \frac{q_1 r_2}{r_1}
\end{aligned}
$$

Finding the ratio of E_1 to E_2:

$$
\begin{aligned}
\frac{E_1}{E_2} &= \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1^2} \div \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2^2} \\
&= \frac{q_1}{r_1^2}\cdot\frac{r_2^2}{q_2} \\
&= \frac{q_1}{r_1^2}\cdot\frac{r_2^2}{q_1\frac{r_2}{r_1}} \\
&= \frac{q_1}{r_1^2}\cdot\frac{r_2^2 r_1}{q_1 r_2} \\
&= \frac{r_2}{r_1}
\end{aligned}
$$

3 Question 3

An expression for electric potential:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

Finding the work done, which is the potential energy, using potential:

$$W = \int_0^Q \frac{1}{4\pi\varepsilon_0}\frac{q}{R}\,dq$$

$$W = \frac{1}{4\pi\varepsilon_0 R}\int_0^Q q\,dq$$

$$W = \frac{1}{4\pi\varepsilon_0 R}\left[\frac{q^2}{2}\right]_0^Q$$

$$W = \frac{1}{4\pi\varepsilon_0 R}\left[\frac{Q^2}{2} - \frac{0}{2}\right]$$

$$W = \frac{1}{4\pi\varepsilon_0 R}\frac{Q^2}{2}$$

$$W = \frac{Q^2}{8\pi\varepsilon_0 R}$$

4 Question 4

4.1 (a)

Let ϕ be the angle between r and $\sqrt{r^2 + h^2}$.

Using the principle of linear super position for continuous charge distribution:

$$E = \int \frac{1}{4\pi\varepsilon_0 \sqrt{h^2 + r^2}^2} \, dq \cdot \sin\phi$$

$$E = \int \frac{1}{4\pi\varepsilon_0 (h^2 + r^2)} \, dq \cdot \sin\phi$$

$$E = \int \frac{1}{4\pi\varepsilon_0 (h^2 + r^2)} \sigma \, dA \cdot \sin\phi \quad (\because dq = \sigma \, dA)$$

$$E = \int \frac{1}{4\pi\varepsilon_0 (h^2 + r^2)} \sigma \cdot \delta r \cdot r \, d\theta \cdot \sin\phi \quad (\because dA = \delta r \cdot r \, d\theta)$$

$$E = \frac{\sigma r \delta r}{4\pi\varepsilon_0 (h^2 + r^2)} \int 1 \, d\theta \cdot \sin\phi$$

$$E = \frac{\sigma r \delta r}{4\pi\varepsilon_0 (h^2 + r^2)} \int_0^{2\pi} 1 \, d\theta \cdot \sin\phi$$

$$E = \frac{\sigma r \delta r}{4\pi\varepsilon_0 (h^2 + r^2)} [\theta]_0^{2\pi} \cdot \sin\phi$$

$$E = \frac{\sigma r \delta r}{4\pi\varepsilon_0 (h^2 + r^2)} [2\pi - 0] \cdot \sin\phi$$

$$E = \frac{2\pi \sigma r \delta r}{4\pi\varepsilon_0 (h^2 + r^2)} \cdot \sin\phi$$

$$E = \frac{\sigma r \delta r}{2\varepsilon_0 (h^2 + r^2)} \cdot \sin\phi$$

$$E = \frac{\sigma r \delta r}{2\varepsilon_0 (h^2 + r^2)} \cdot \frac{h}{\sqrt{h^2 + r^2}}$$

$$E = \frac{\sigma r \delta r h}{2\varepsilon_0 (h^2 + r^2)^{\frac{3}{2}}}$$

The electric force would be:

$$F = qE$$

$$F = q\frac{\sigma r \delta r h}{2\varepsilon_0 (h^2 + r^2)^{\frac{3}{2}}}$$

$$F = \frac{q\sigma r \delta r h}{2\varepsilon_0 (h^2 + r^2)^{\frac{3}{2}}} \quad \textbf{(Shown)}$$

4.2 (b)

The electric field of a large flat circular sheet would be the sum of all circular rings that eventually build up to a circular sheet.

Let δr be infinitesimally small, i.e. $\delta r = dr$. Using the electric field in part (a):

$$E = \int \frac{\sigma r h}{2\varepsilon_0 (h^2 + r^2)^{\frac{3}{2}}}\, dr$$

Let θ be the angle between h and $\sqrt{h^2 + r^2}$.

$$\tan\theta = \frac{r}{h}$$

$$r = h\tan\theta$$

$$\frac{dr}{d\theta} = h\sec^2\theta$$

$$dr = h\sec^2\theta\, d\theta$$

Using the above substitutions, the electric field would be:

$$\begin{aligned}
E &= \int \frac{\sigma h \tan\theta h}{2\varepsilon_0 (h^2 + (h\tan\theta)^2)^{\frac{3}{2}}}\, h\sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{h^3 \tan\theta}{(h^2 + h^2\tan^2\theta)^{\frac{3}{2}}}\, \sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{h^3 \tan\theta}{(h^2(1 + \tan^2\theta))^{\frac{3}{2}}}\, \sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{h^3 \tan\theta}{(h^2(1 + \tan^2\theta))^{\frac{3}{2}}}\, \sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{h^3 \tan\theta}{h^3(1 + \tan^2\theta)^{\frac{3}{2}}}\, \sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{\tan\theta}{(1 + \tan^2\theta)^{\frac{3}{2}}}\, \sec^2\theta \, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \frac{\sec^2\theta \tan\theta}{(1 + \tan^2\theta)^{\frac{3}{2}}}\, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int \sec^2\theta \tan\theta (1 + \tan^2\theta)^{-\frac{3}{2}}\, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot 2\sec^2\theta \tan\theta (1 + \tan^2\theta)^{-\frac{3}{2}}\, d\theta \\
&= \frac{\sigma}{2\varepsilon_0} \left[\frac{1}{2} \cdot \frac{1}{-\frac{1}{2}} (1 + \tan^2\theta)^{-\frac{1}{2}} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\sigma}{2\varepsilon_0} \left[\frac{1}{2} \cdot -2(\sec^2\theta)^{-\frac{1}{2}} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\sigma}{2\varepsilon_0} \left[-|\sec\theta|^{-1} \right]_0^{\frac{\pi}{2}} \\
&= \frac{\sigma}{2\varepsilon_0} \left[-|\cos\theta| \right]_0^{\frac{\pi}{2}} \\
&= \frac{\sigma}{2\varepsilon_0} \left[-\left|\cos\frac{\pi}{2}\right| - (-|\cos 0|) \right] \\
&= \frac{\sigma}{2\varepsilon_0} [0 - (-1)] \\
&= \frac{\sigma}{2\varepsilon_0} [1] \\
&= \frac{\sigma}{2\varepsilon_0} \ \textbf{(Shown)}
\end{aligned}$$

5 Question 5

Using the definition of Gauss' Law:

$$\Phi = \frac{Q_{encl}}{\varepsilon_0}$$

Electric flux through S1:

$$\begin{aligned}\Phi_{S1} &= \frac{-2Q + Q}{\varepsilon_0} \\ &= \frac{-Q}{\varepsilon_0}\end{aligned}$$

Electric flux through S2:

$$\begin{aligned}\Phi_{S2} &= \frac{Q - Q}{\varepsilon_0} \\ &= 0\end{aligned}$$

Electric flux through S3:

$$\begin{aligned}\Phi_{S3} &= \frac{-2Q + Q - Q}{\varepsilon_0} \\ &= \frac{-3Q}{\varepsilon_0}\end{aligned}$$

Electric flux through S4:

$$\begin{aligned}\Phi_{S4} &= \frac{0}{\varepsilon_0} \\ &= 0\end{aligned}$$

6 Question 6

6.1 (a)

Using Gauss' Law:

$$\begin{aligned}\oint \vec{E} \cdot d\vec{A} &= \frac{Q_{encl}}{\varepsilon_0} \\ -15000 \cdot 4\pi(8 \cdot 10^{-2})^2 &= \frac{Q_{encl}}{8.85 \cdot 10^{-12}} \quad (\because E \text{ is pointing inwards}) \\ Q_{encl} &= -1.067638847 \cdot 10^{-9} \\ Q_{encl} &\approx -1.07 \cdot 10^{-9}\end{aligned}$$

6.2 (b)

Using Gauss' Law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

$$15000 \cdot 4\pi(8 \cdot 10^{-2})^2 = \frac{Q_{encl}}{8.85 \cdot 10^{-12}} \quad (\because E \text{ is pointing outwards})$$

$$Q_{encl} = 1.067638847 \cdot 10^{-9}$$

$$Q_{encl} \approx 1.07 \cdot 10^{-9}$$

6.3 (c)

Using Gauss' Law:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\varepsilon_0}$$

$$15000 \cdot 4\pi(17 \cdot 10^{-2})^2 = \frac{Q_{encl}}{8.85 \cdot 10^{-12}} \quad (\because E \text{ is pointing outwards})$$

$$Q_{encl} = 4.82105667 \cdot 10^{-9}$$

$$Q_{encl} \approx 4.82 \cdot 10^{-9}$$

7 Question 7

7.1 (a)

$$\tau = \vec{F} \times \vec{d}$$

$$\tau = q\vec{E} \times \vec{d}$$

$$\tau = \vec{E} \times q\vec{d}$$

$$\tau = \vec{E} \times \vec{p}$$

$$\tau = \vec{p} \times \vec{E} \textbf{ (Shown)}$$

7.2 (b)

$$\begin{aligned}
\Delta U &= -\int \vec{F} \cdot d\vec{l} \\
&= -\int_{W_i}^{W_f} dW \\
&= -\int_{\theta_i}^{\theta_f} \tau \, d\theta \\
&= -\int_{\theta_i}^{\theta_f} pE \sin\theta \, d\theta \\
&= -[pE\cos\theta]_{\theta_i}^{\theta_f} \\
&= -[pE\cos\theta_f - pE\cos\theta_i] \\
&= -pE(\cos\theta_f - \cos\theta_i) \ \textbf{(Shown)}
\end{aligned}$$

8 Question 8

8.1 (a)

The dipole moment in the y direction will cancel out, hence, the net dipole moment is:

$$p_{net} = p_1 \cos\theta + p_2 \cos\theta$$

$$p_{net} = \cos\theta(p_1 + p_2)$$

Using the definition of dipole moment:

$$p = qd$$

$$p_{net} = \cos\theta(qd + qd)$$

$$p_{net} = 2qd\cos\theta$$

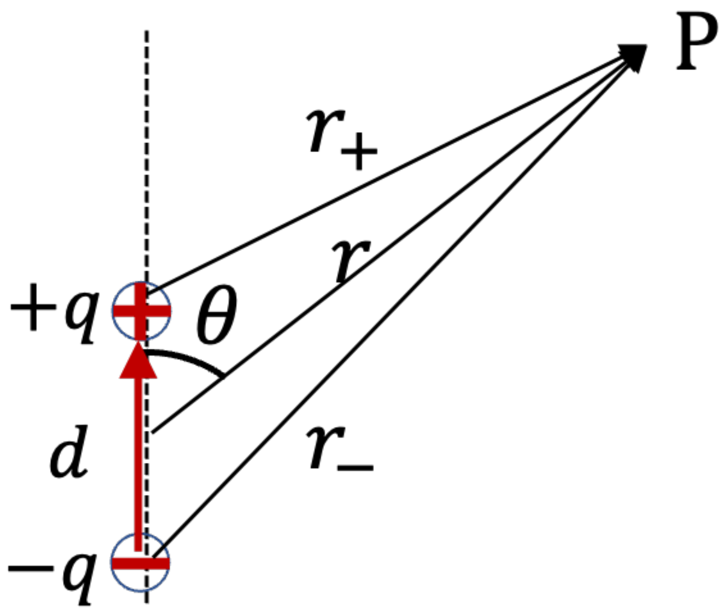
$$6.1 \cdot 10^{-30} = (q \cdot 0.96 \cdot 10^{-10} + q \cdot 0.96 \cdot 10^{-10}) \cos 52^\circ$$

$$6.1 \cdot 10^{-30} = 2q \cdot 0.96 \cdot 10^{-10} \cdot \cos 52^\circ$$

$$q = 5.160438749 \cdot 10^{-20}$$

$$q \approx 5.16 \cdot 10^{-20}$$

8.2 (b)



The actual potential when adding up the two charges is:

$$V = \frac{q}{4\pi\varepsilon_0 r_+} - \frac{q}{4\pi\varepsilon_0 r_-}$$

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{1}$$

8.2.1 Finding an approximation for $\frac{1}{r_+}$

Using the law of cosines:

$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos\theta$$

$$r_+^2 = r^2 + \left(\frac{d}{2}\right)^2 - rd\cos\theta$$

$$r_+^2 = r^2 + \frac{d^2}{4} - rd\cos\theta$$

$$r_+^2 = r^2\left(1 + \frac{d^2}{4r^2} - \frac{d}{r}\cos\theta\right)$$

$$r_+^2 = r^2\left(1 + \frac{1}{4}\left(\frac{d}{r}\right)^2 - \frac{d}{r}\cos\theta\right)$$

$$r_+^2 = r^2\left(1 - \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)$$

$$r_+ = r\left(1 - \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)^{\frac{1}{2}}$$

$$\frac{1}{r_+} = \frac{1}{r}\left(1 - \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)^{-\frac{1}{2}}$$

$$\frac{1}{r_+} = \frac{1}{r}\left(1 + \frac{1}{2}\frac{d}{r}\cos\theta - \frac{1}{8}\left(\frac{d}{r}\right)^2 + \frac{3}{8}\left(\frac{d}{r}\right)^2\cos^2\theta + \cdots\right)$$

$$\frac{1}{r_+} = \frac{1}{r} + \frac{1}{2}\frac{d}{r^2}\cos\theta - \frac{1}{8}\frac{d^2}{r^3} + \frac{3}{8}\frac{d^2}{r^3}\cos^2\theta + \cdots$$

When $\frac{d}{r} << 1$, we can ignore the powers of $\frac{d}{r}$ that are greater than 2:

$$\frac{1}{r_+} \approx \frac{1}{r} + \frac{1}{2}\frac{d}{r^2}\cos\theta \tag{2}$$

8.2.2 Finding an approximation for $\frac{1}{r_-}$

Using the law of cosines:

$$r_-^2 = r^2 + \left(\frac{d}{2}\right)^2 - 2r\frac{d}{2}\cos(\pi - \theta)$$

$$r_-^2 = r^2 + \frac{d^2}{4} - rd\cos(\pi - \theta)$$

$$r_-^2 = r^2 + \frac{d^2}{4} + rd\cos\theta$$

$$r_-^2 = r^2 + rd\cos\theta + \frac{d^2}{4}$$

$$r_-^2 = r^2\left(1 + \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$r_-^2 = r^2\left(1 + \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)$$

$$r_- = r\left(1 + \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)^{\frac{1}{2}}$$

$$\frac{1}{r_-} = \frac{1}{r}\left(1 + \frac{d}{r}\cos\theta + \frac{1}{4}\left(\frac{d}{r}\right)^2\right)^{-\frac{1}{2}}$$

$$\frac{1}{r_-} = \frac{1}{r}\left(1 - \frac{1}{2}\frac{d}{r}\cos\theta - \frac{1}{8}\left(\frac{d}{r}\right)^2 - \frac{3}{8}\left(\frac{d}{r}\right)^2\cos^2\theta + \cdots\right)$$

$$\frac{1}{r_-} = \frac{1}{r} - \frac{1}{2}\frac{d}{r^2}\cos\theta - \frac{1}{8}\frac{d^2}{r^3} - \frac{3}{8}\frac{d^2}{r^3}\cos^2\theta + \cdots$$

When $\frac{d}{r} << 1$, we can ignore the powers of $\frac{d}{r}$ that are greater than 2:

$$\frac{1}{r_-} \approx \frac{1}{r} - \frac{1}{2}\frac{d}{r^2}\cos\theta \tag{3}$$

8.2.3 Substituting the approximations back into the original equation

Substituting (2) and (3) into (1):

$$V \approx \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{1}{2}\frac{d}{r^2}\cos\theta-\left(\frac{1}{r}-\frac{1}{2}\frac{d}{r^2}\cos\theta\right)\right)$$

$$V \approx \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r}+\frac{1}{2}\frac{d}{r^2}\cos\theta-\frac{1}{r}+\frac{1}{2}\frac{d}{r^2}\cos\theta\right)$$

$$V \approx \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{2}\frac{d}{r^2}\cos\theta+\frac{1}{2}\frac{d}{r^2}\cos\theta\right)$$

$$V \approx \frac{q}{4\pi\varepsilon_0}\frac{d}{r^2}\cos\theta$$

$$V \approx \frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

Since $p = qd$:

$$V \approx \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\ \textbf{(Shown)}$$