→ Def: $\forall A \subseteq \mathbb{R}$, let $\underbrace{\sup A}$:= the least upper bound of A in $\mathbb{R}$. (→ the supremum of A)

⇝ We have shown that $\nexists\, q \in \mathbb{Q}$ with $q^2 = 2$.
However:

→ Prop: There exists a unique $x \in \mathbb{R}$, $x > 0$ with $x^2 = 2$.

Proof: Exercise. ■

We denote by $\sqrt{2}$ this unique positive real.
Moreover, the following holds:

→ Prop: For any $r > 0$ in $\mathbb{R}$ and any $n \in \mathbb{N}$, there exists a unique $x \in \mathbb{R}$, $x > 0$ with $x^n = r$.

Proof: See Theorem 1.21 in p. 10 of Rudin's book. ■

We denote by $r^{\frac{1}{n}}$ this unique positive real.

We will now see how the extra property of completeness gives $\mathbb{R}$ the amazing properties that make it so useful.

Some basic consequences of completeness of $(\mathbb{R}, +, \cdot, <)$

1. The real numbers have the Archimedean property:

The Archimedean property can be expressed in the following 3 ways:

→ Prop. 1: $\mathbb{N}$ is not bounded from above in $\mathbb{R}$.

Proof: Suppose that $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\mathbb{N}$ has a least upper bound $\alpha \in \mathbb{R}$.

Then:

for all $n \in \mathbb{N}$, $n \le \alpha$ (α an upper bound),

so, for all $n \in \mathbb{N}$, $\underbrace{n+1}_{\in \mathbb{N}} \le \alpha$,

i.e., for all $n \in \mathbb{N}$, $n \le \alpha - 1$.

So, $\alpha - 1$ is an upper bound of A in $\mathbb{R}$. However, $\alpha - 1 < \alpha$, the least upper bound of A. This is a contradiction. So,

$\mathbb{N}$ is not bounded from above.

⚠ Prop. 1 tells us that we can find as large natural numbers as we wish. The next one gives us another way to quantify this information.

→ <u>Prop. 2:</u> Let $a, \varepsilon \in \mathbb{R}$, with $\varepsilon > 0$. Then, there exists $n \in \mathbb{N}$ with $n \cdot \varepsilon > a$

(⚠ We tend to always think of ε as <u>very small</u>. This proposition tells us that, no matter how small ε is, we can always make it as large as we want by multiplying it with an appropriately large natural number. (<u>Note</u>: Prop. 1 is Prop. 2 for $\varepsilon = 1$).)

<u>Proof:</u> Consider the element $\frac{a}{\varepsilon} \in \mathbb{R}$. We know that $\mathbb{N}$ is not bounded from above in $\mathbb{R}$, so $\frac{a}{\varepsilon}$ is not an upper bound of $\mathbb{N}$.

Thus, there exists some $n \in \mathbb{N}$ with $n > \frac{a}{\varepsilon}$

$$\overset{\varepsilon > 0}{\Downarrow}$$

$$n \cdot \varepsilon > a. \quad \blacksquare$$

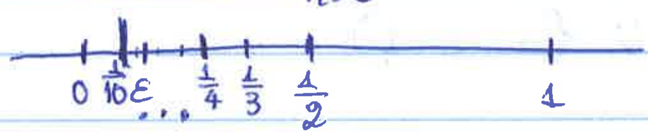
→ <u>Prop. 3:</u> Let $\varepsilon > 0$. Then, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

(⚠ This tells us that, no matter how small ε is, we can always divide 1 in so many equal line segments that each will be smaller than ε.)

<u>Proof:</u> Consider the element $\frac{1}{\varepsilon} \in \mathbb{R}$.

Since $\mathbb{N}$ is not bounded from above, there exists $n \in \mathbb{N}$ such that

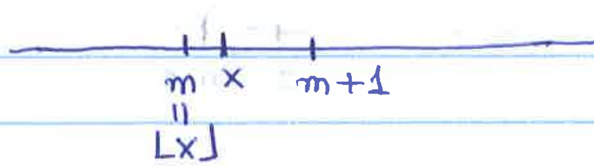
$$n > \frac{1}{\varepsilon} \xrightarrow[\varepsilon > 0,\ n > 0]{} \frac{1}{n} < \varepsilon.$$



$\blacksquare$

② Existence of integer part of every real:

→ <u>Prop.:</u> Let $x \in \mathbb{R}$. There exists a unique integer $m \in \mathbb{Z}$, such that $m \leq x < m+1$.



We say that this m is <u>the integer part of x</u>, and we denote it by $\lfloor x \rfloor$.

NOT required

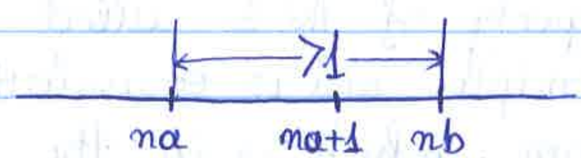
"Proof": The above may seem obvious (in fact, in the proof of 1.20(b) in p. 9 of Rudin's book this fact seems to be derived from the fact that we can find $m_1, m_2 \in \mathbb{N}$ s.t. $-m_1 < x < m_2$. However, one also needs to use that <u>every subset of $\mathbb{N}$ has a minimal element</u> (eventually, $m+1$ will be the minimal element of $\{k \in \mathbb{Z} : k > x\}$, practically a copy of $\mathbb{N}$ which will imply that $m \leq x$). This property of $\mathbb{N}$ is called the well-ordering principle, and is equivalent to the <u>induction axiom</u>, which is in the axiomatic definition of the natural numbers. You don't need to know these for the exam, but you should investigate further if you are curious.

③ Denseness of $\mathbb{Q}$ in $\mathbb{R}$:

→ <u>Prop.</u> For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ with $a < q < b$.

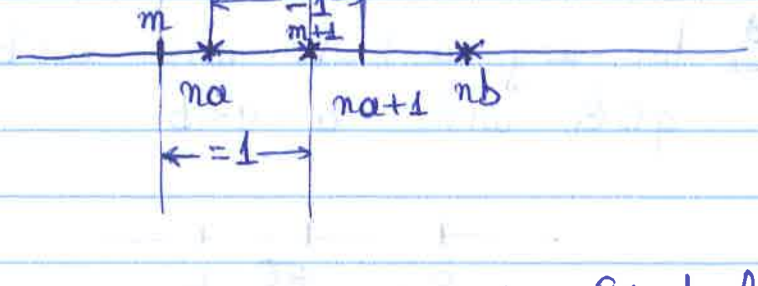
<u>Proof:</u> (<u>Idea:</u> If two real numbers differ by more than 1, then there should exist an integer between them, which is of course rational! Since we don't know if a and b differ by more than 1, we'll multiply their difference with an $n \in \mathbb{N}$ large enough to make the difference larger than 1, and see what happens...)

$b-a>0$. So, by the Archimedean property of the reals, there exists $n \in \mathbb{N}$ such that $n(b-a)>1$, i.e. $nb-na>1$.



So, $$na < na+1 < nb$$
(check both inequalities formally!)

Let $m = \lfloor na \rfloor$; by the definition of integer part, we have that $m \leq na < m+1$. So:



$m \leq na < m+1 \leq na+1 < nb$ (check formally).

What we will use from this is that

$$na < m+1 < nb$$

$$\Big\Downarrow n>0$$

$$a < \underbrace{\frac{m+1}{n}}_{\in \mathbb{Q}} < b \qquad \blacksquare$$

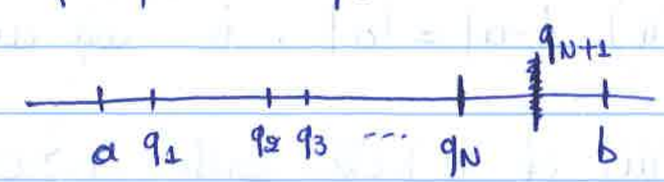
→ <u>Corollary:</u> for any $a, b \in \mathbb{R}$ with $a<b$, there exist <u>infinitely many</u> rationals q with $a<q<b$.

<u>Proof:</u> We know that there exists at least one $q \in \mathbb{Q}$ such that $a<q<b$. So, the set $\{q \in \mathbb{Q} : a<q<b\}$ is non-empty.

Suppose that $\{q \in \mathbb{Q} : a<q<b\}$ is finite; let

$$\{q_1, q_2, \dots, q_N\} = \{q \in \mathbb{Q} : a<q<b\},$$

with $q_1 < q_2 < \dots < q_N$.



Since $q_N, b \in \mathbb{R}$ with $q_N < b$, it follows by the last proposition that $\exists\, q_{N+1} \in \mathbb{Q}$ with $q_N < q_{N+1} < b$.

So, $q_{N+1} \in \mathbb{Q}$ is larger than the largest rational between a and b, contradiction.

So, $\{q \in \mathbb{Q} : a < q < b\}$ is not finite. Since it is non-empty, it has to be infinite.

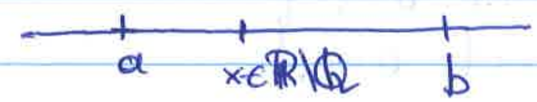
4 Denseness of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$:

We know that $\mathbb{Q} \subsetneq \mathbb{R}$: indeed, we have seen that $\nexists\, q \in \mathbb{Q}$ with $q^2 = 2$, while $\exists\, x \in \mathbb{R}$ with $x^2 = 2$.

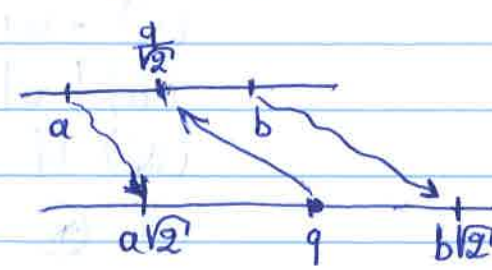
→ Def: We define the set of irrational numbers to be $\mathbb{R} \setminus \mathbb{Q}$.

→ Prop: For any $a, b \in \mathbb{R}$ with $a < b$, there exists

$$x \in \mathbb{R} \setminus \mathbb{Q} \text{ with } a < x < b.$$



Proof: Since $a < b$ and $\sqrt{2} > 0$, we have
$$a\sqrt{2} < b\sqrt{2}$$



By denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q \in \mathbb{Q}$, $q \neq 0$,

s.t. $a\sqrt{2} < q < b\sqrt{2}$.
(it is the Corollary earlier, rather than the Proposition, that ensures that we can find such q that is non-zero).

Since $\sqrt{2} > 0$, we have $a < \frac{q}{\sqrt{2}} < b$.

And $\frac{q}{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$ (indeed, if $\frac{q}{\sqrt{2}} = q' \in \mathbb{Q}$, then $q \neq 0 \Rightarrow q' \neq 0$, so
$\sqrt{2} = \frac{q}{q'} \in \mathbb{Q}$, a contradiction) $\blacksquare$

Corollary: For any $a, b \in \mathbb{R}$ with $a < b$, there exist infinitely many irrationals x with $a < x < b$.

Proof: Exercise.