Stat 134 lec 21

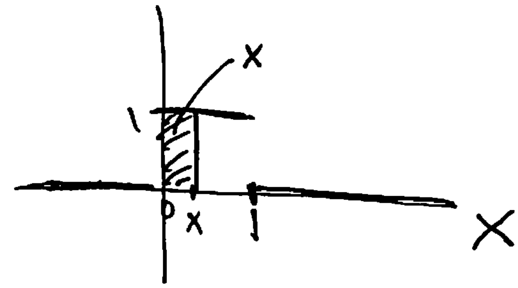
Sec 4.1

Let X be a RV. The commulative distribution function (cdf) of X is $F(x) = P(X \le x)$

ex $X \sim \text{Unif}(0,1)$

$F(x) = P(X \le x)$

$$F(x) = \begin{cases} 1 & \text{if } x \ge 1 \\ x & \text{if } 0 \le x \le 1 \\ 0 & x \le 0 \end{cases}$$



Relationship between cdf and the density function of X

1) The CDF is calculated from the density function by integrating!

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

2) The density function is calculated from the cdf by differentiating!

$$f(x) = F'(x)$$

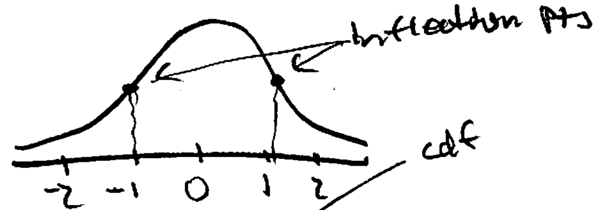
Consequence, a density function and cdf are equivalent descriptions of the distribution of a ~~continuous~~ RV.

cdf gives a way to talk about a distribution without worrying about whether it is discrete or not.

Standard Normal Density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$

Z:



cdf

$$P(a < Z < b) = \Phi(b) - \Phi(a)$$

were $\Phi(x) = \int_{-\infty}^{x} \phi(z)\,dz$ cummulative distribution function

$$= P(Z < x)$$

Facts

will prove in sec 5.3

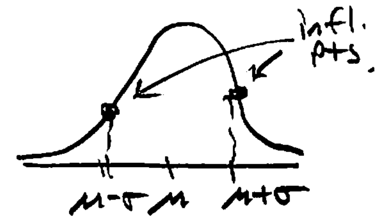
1. $\int_{-\infty}^{\infty} \phi(z)\,dz = 1$
2. $E(Z) = 0$ — need to check $E(|Z|) < \infty$.
3. $SD(Z) = 1$

Normal (μ, σ^2)

let $Z = \frac{X - \mu}{\sigma} \Rightarrow X = \mu + \sigma Z$ linear change of variable.

Fact (sec 4.4)

$$\boxed{f(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$



Central Limit theorem (CLT)

$X_1, \cdots X_n$ iid, mean μ, SD σ, $S_n = X_1 + \cdots + X_n$
Then for large n, the dist of S_n is approx $(n\mu, (\sqrt{n}\sigma)^2)$.
If X_i are normal themselves, S_n is exactly normal.
(similar for average $\frac{S_n}{n}$).

Sec 4.2 Exponential and Gamma Distribution

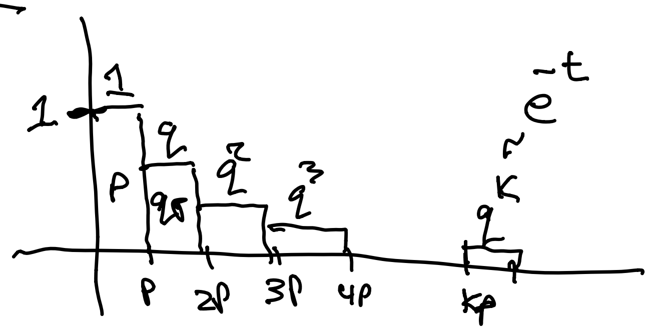
Exponential dist

Let $G \sim \text{Geom}(p)$ — # trials until 1^{st} success; outcomes $1, 2, 3, \ldots$

How can we make this a continuous RV so it is the time until the 1^{st} success?

let p be very small then pG has values $p(1,2,3,\ldots) = (p, 2p, 3p, \ldots)$

Picture



$P(pG = p) = P(G=1) = p$

$P(pG = 2p) = P(G=2) = qp$

The height of bar at time $t = kp$ is $q^k = (1-p)^k$

recall $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\Rightarrow \boxed{e^x \approx 1 + x}$ for small x

or $\boxed{e^{-x} \approx 1 - x}$

$\Rightarrow 1 - p \approx e^{-p}$ for small p

$\Rightarrow (1-p)^k = (e^{-p})^k = e^{-kp} = \boxed{e^{-t}}$

As $p \to 0$ we get $f(t) = e^{-t}$ which is density of exponential $(\lambda = 1)$

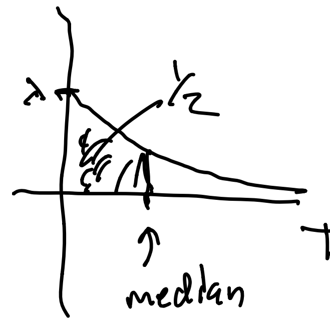
Here the rate of success (death) is p per unit time which is p so

$$\lambda = \frac{P(\text{success})}{\text{unit time}} = \frac{p}{p} = 1$$

Fix parameter $\lambda > 0$ (λ is instantaneous success rate)

T has exponential(λ) density

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else} \end{cases}$$



If $G \sim \text{Geom}(p)$, $pG \xrightarrow{\text{dist}}$ exponential(1).

T represents time until first success where λ is rate of success per unit time.

ex T = time till lightbulb burns out.

CDF and survival function

$$F(s) = P(T < s) = \int_0^s \lambda e^{-\lambda t}\,dt$$ — prob bulb dies before time s.

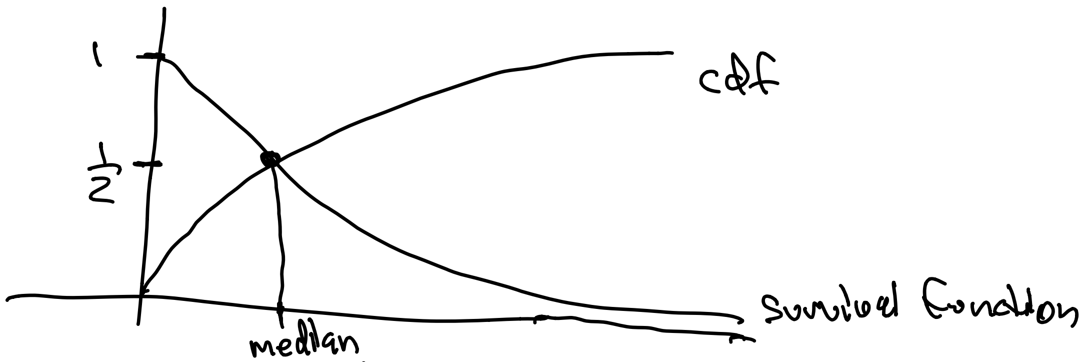
$$= \lambda \frac{e^{-\lambda t}}{-\lambda}\Big|_0^s$$

$$= -e^{-\lambda s} + 1 = \boxed{1 - e^{-\lambda s}}$$

$P(T > s) = P($"survive beyond s)

$$= 1 - F(s) = \boxed{e^{-\lambda s}}$$

Picture



$$e^{-\lambda s} = 1 - e^{-\lambda s} \Rightarrow 2e^{-\lambda s} = 1$$

$$\Rightarrow e^{-\lambda s} = 1/2$$

Next mean and variance.

$$E(X) = \lambda \int t e^{-\lambda t} dt$$

use tabular method for integration by parts

works when your function is a prod of two expressions, where one expression has some nth derivative equal to zero.

ex $x^4 e^{3x}$ ✓ $\sin(x) e^{3x}$

$\frac{d}{dt}$		$\int$
t	$+$	$e^{-\lambda t}$
1	$-$	$-\frac{e^{-\lambda t}}{\lambda}$
0		$\frac{e^{-\lambda t}}{\lambda^2}$

$$E(T) = \cancel{\lambda}\int_0^\infty t e^{-\lambda t}\,dt = \cancel{\lambda}\left[t\left(-\frac{e^{-\lambda t}}{\cancel{\lambda}}\right) - \frac{e^{-\lambda t}}{\lambda^{\cancel{2}}}\right]_0^\infty$$

$$= 0 - \left(0 - \frac{1}{\lambda}\right)$$

$$= \frac{1}{\lambda}$$

$$E(T^2) = \lambda\int_0^\infty t^2 e^{-\lambda t}\,dt$$

$\frac{d}{dt}$		$\int$
t^2	$+$	$e^{-\lambda t}$
$2t$	$-$	$-\frac{1}{\lambda}e^{-\lambda t}$
2	$+$	$\frac{1}{\lambda^2}e^{-\lambda t}$
0		$-\frac{1}{\lambda^3}e^{-\lambda t}$

$$E(T^2) = \lambda \left[t^2\left(-\frac{1}{\lambda}e^{-\lambda t}\right) - 2t\left(\frac{1}{\lambda^2}e^{-\lambda t}\right) + 2\left(-\frac{1}{\lambda^3}e^{-\lambda t}\right) \right]\Bigg|_0^\infty$$

$$= \frac{2}{\lambda^2}$$

$$\mathrm{Var}(T) = E(T^2) - E(T)^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$

$$SD(T) = \boxed{\frac{1}{\lambda}}$$