- for $\underline{x \leq -1}$: $(x^n)_{n\in\mathbb{N}}$ doesn't converge, so $(s_n)_{n\in\mathbb{N}}$ doesn't converge either.

Thus, $\sum_{k=0}^{+\infty} x^k$ diverges (and the infinite sum $1+x+x^2+\dots$ is not defined).

- To sum up:

$$\boxed{\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \text{ for } |x|<1, \text{ and } \sum_{k=0}^{+\infty} x^k \text{ diverges for } |x|\geq 1. \left(\text{In particular, } \sum_{k=0}^{+\infty} x^k = +\infty \text{ for } x\geq 1\right)}$$

Lecture 10: 16 Sep 2016. 1

→ **Telescopic series:** The series $\sum_{k=1}^{+\infty} a_k$ is called <u>telescopic</u> if there exists a sequence $(b_n)_{n\in\mathbb{N}}$, s.t.:

$$a_k = b_{k+1} - b_k \quad , \forall k\in\mathbb{N}.$$

In that case:

$$\begin{aligned} S_n = a_1 + a_2 + \dots + a_n = \ & b_2 - b_1 \\ + \ & b_3 - \cancel{b_2} \\ + \ & \dots \\ + \ & b_n - \cancel{b_{n-1}} \\ + \ & b_{n+1} - \cancel{b_n} \end{aligned} = b_{n+1} - b_1$$

$(s_n)_{n\in\mathbb{N}}$ converges $\Longleftrightarrow$ $(b_n)_{n\in\mathbb{N}}$ converges.

And : $\sum_{k=1}^{+\infty} a_k = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} (b_{n+1} - b_1) = \lim_{n\to+\infty} b_n - b_1.$

<u>ex:</u> $\underline{\sum_{k=1}^{+\infty} \frac{1}{k(k+1)}}$: $a_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$, $\forall k\in\mathbb{N}$.

So; $a_k = b_{k+1} - b_k \quad \forall k\in\mathbb{N}$,

where $b_k = -\frac{1}{k} \quad \forall k\in\mathbb{N}$.

Thus : $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} = \lim_{k\to+\infty} b_k - b_1 = 0-(-1)=1.$

→ <u>Prop:</u> Let $(a_k)_{k\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences, and let $\lambda,\mu\in\mathbb{R}$. We consider the sequence $(\lambda a_k + \mu b_k)_{k\in\mathbb{N}}$.

If $\sum_{k=1}^{+\infty} a_k$, $\sum_{k=1}^{+\infty} b_k$ converge,

then $\sum_{k=1}^{+\infty} (\lambda a_k + \mu b_k)$ converges as well,

and $\sum_{k=1}^{+\infty} (\lambda a_k + \mu b_k) = \lambda\cdot\sum_{k=1}^{+\infty} a_k + \mu\cdot\sum_{k=1}^{+\infty} b_k.$

<u>Proof:</u> Since $\sum_{k=1}^{+\infty} a_k$, $\sum_{k=1}^{+\infty} b_k$ converge,

we have (by definition of series convergence) that

$$s_n := a_1+a_2+\dots+a_n \xrightarrow[n\to+\infty]{} a$$

and $$t_n := b_1+b_2+\dots+b_n \xrightarrow[n\to+\infty]{} b$$,

for some $a, b \in \mathbb{R}$. Note that

$$\sum_{k=1}^{+\infty} a_n = \lim_{n\to+\infty} s_n = a, \text{ and } \sum_{k=1}^{+\infty} b_n = \lim_{n\to+\infty} t_n = b.$$

We now consider the n-th partial sum of $\sum_{k=1}^{+\infty}(\lambda a_k+\mu b_k)$:

it equals $u_n = (\lambda a_1+\mu b_1)+(\lambda a_2+\mu b_2)+\dots+(\lambda a_n+\mu b_n)$

$$= \lambda\cdot(a_1+a_2+\dots+a_n)+\mu\cdot(b_1+b_2+\dots+b_n)=$$

$$= \lambda\cdot s_n+\mu\cdot t_n \xrightarrow[n\to+\infty]{} \lambda a+\mu b$$

So, $\sum_{k=1}^{+\infty}(\lambda a_k+\mu b_k)$ converges, and

$$\sum_{k=1}^{+\infty}(\lambda a_k+\mu b_k) = \lim_{n\to+\infty} u_n = \lambda a+\mu b = \lambda\cdot\sum_{k=1} a_k+\mu\cdot\sum_{k=1} b_k.$$

$\blacksquare$

→ **Prop:** For any $m \in \mathbb{N}$, (fixed!)

$$\sum_{k=1}^{+\infty} a_k \text{ converges} \iff \sum_{k=m}^{+\infty} a_k \text{ converges}$$

⚠ This tells us that convergence of a series doesn't depend on the first terms of the series.

Proof: We have that, $\forall n > m$:

$$\underbrace{a_1 + a_2 + \dots + a_n}_{\substack{= \\ s_n,}} = (a_1 + \dots + a_{m-1}) + \underbrace{(a_m + a_{m+1} + \dots + a_n)}_{\substack{= \\ \text{the sum of the first } n-(m-1) \\ \text{terms of } \sum_{k=m}^{+\infty} a_k \\ = \\ t_{n-(m-1)},}}$$

s_n, the n-th partial sum of $\sum_{k=1}^{+\infty} a_k$

where t_k is the k-th partial sum of $\sum_{k=1}^{+\infty} b_k$.

I.e.: $s_n = \underbrace{(a_1 + a_2 + \dots + a_{m-1})}_{\text{a constant}} + t_{n-\underbrace{(m-1)}_{\text{fixed}}}, \forall n \in \mathbb{N}$.

Thus: $(s_n)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $\underbrace{(t_{n-(m-1)})_{n\in\mathbb{N}}}_{\text{a final part of } (t_n)_{n\in\mathbb{N}}}$ converges.

a final part of $(t_n)_{n\in\mathbb{N}}$, so converges $\Leftrightarrow$ $(t_n)_{n\in\mathbb{N}}$ converges (in fact, they have the same limit).

So:

$(s_n)_{n\in\mathbb{N}}$ converges

$\Longleftrightarrow$

$(t_n)_{n\in\mathbb{N}}$ converges,

i.e. $\sum_{k=1}^{+\infty} a_k$ converges $\Longleftrightarrow$ $\sum_{k=m}^{+\infty} a_k$ converges.

⚠ Notice, in particular, that

" $\lim_{n\to+\infty} t_n$

$$\lim_{n\to+\infty} s_n = (a_1+\dots+a_{m-1}) + \lim_{n\to+\infty} t_{n-(m-1)},$$

so

$$\boxed{\sum_{k=1}^{+\infty} a_k = (a_1+\dots+a_{m-1}) + \sum_{k=m}^{+\infty} a_k, \quad \forall m\in\mathbb{N},}$$

when $\sum_{k=1}^{+\infty} a_k$ is a convergent series.

■

Corollary: If the sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are <u>eventually equal</u> (i.e., if $\exists n_0\in\mathbb{N}$ s.t. $a_n=b_n\ \forall n\geq n_0$),

then $\sum_{k=1}^{+\infty} a_k$ converges $\Leftrightarrow$ $\sum_{k=1}^{+\infty} b_k$ converges.

Proof: $\sum_{k=1}^{+\infty} a_k$ converges $\Leftrightarrow$ $\sum_{k=n_0}^{+\infty} \underbrace{a_k}_{=b_k}$ converges

previous Proposition

$\Leftrightarrow$ $\sum_{k=n_0}^{+\infty} b_k$ converges

$\Leftrightarrow$ $\sum_{k=1}^{+\infty} b_k$ converges. ∎

previous Proposition

Prop: Let $(a_k)_{k\in\mathbb{N}}$ be a sequence.

If $\sum_{k=1}^{+\infty} a_k$ converges, then: $a_k \xrightarrow[k\to+\infty]{} 0$.

Proof: Let $s_n := a_1+a_2+\dots+a_n$, the n-th partial sum of $\sum_{k=1}^{+\infty} a_k$.

"$\sum_{k=1}^{+\infty} a_k$ converges" means that $s_n \xrightarrow[n\to+\infty]{} s$, for some $s \in \mathbb{R}$. Now: $a_{n+1} = \underset{\substack{\downarrow n\to+\infty \\ s}}{s_{n+1}} - \underset{\substack{\downarrow n\to+\infty \\ s}}{s_n} \xrightarrow[n\to+\infty]{} 0.$

So, $a_n \xrightarrow[n\to+\infty]{} 0$ (as $(a_{n+1})_{n\in\mathbb{N}}$ is just a final part of $(a_n)_{n\in\mathbb{N}}$). ■

The idea for the above is that, to go from one partial sum to the next, we just add ~~th~~ one more term of the sequence. So, since, for large n, the s_n's all cluster around some point (as $(s_n)_{n\in\mathbb{N}}$ converges), we cannot possibly be adding a lot to go from s_n to s_{n+1}!

⚠ The above Proposition is very important! It provides the simplest, most basic way to test if a series converges. It is formulated in the following way:

test

$\boxed{\text{Preliminary test:}}$ If $a_k \underset{k\to+\infty}{\not\longrightarrow} 0$, then $\sum_{k=1}^{+\infty} a_k$ diverges

if $\sum_{k=1}^{+\infty} a_k$ converged, then $a_k \to 0$, contradiction.

Note that this is equivalent to the Proposition above: it just uses the simple fact that

$A \Rightarrow B \iff (\text{not } B) \Rightarrow (\text{not } A)$ (easy by contradiction, for instance)

Examples:

- $\sum_{k=1}^{+\infty} (-1)^k$: $(-1)^k \underset{k\to+\infty}{\not\longrightarrow} 0$, so $\sum_{k=1}^{+\infty} (-1)^k$ diverges.

- $\sum_{k=1}^{+\infty} x^k$: When $|x|<1$, then $|x^k| = |x|^k \xrightarrow[k\to+\infty]{} 0$,

  so $x^k \longrightarrow 0$.

  This tells me nothing! Notice that

  the preliminary test doesn't imply convergence if the sequence goes to 0!

  It just implies divergence if the sequence doesn't go to 0.

  So, even though I know that the geometric series above converges when $|x|<1$, this doesn't follow from the preliminary test.

  But: When $|x| \geq 1$, then $|x^k| \xrightarrow[k\to+\infty]{} +\infty \neq 0$,

  so $x^k \underset{k\to+\infty}{\not\longrightarrow} 0$, so $\sum_{k=1}^{+\infty} x^k$ diverges.

- $\sum_{k=1}^{+\infty} \frac{1}{k}$: $\frac{1}{k} \xrightarrow[k\to+\infty]{} 0$, so the preliminary test tells me nothing! In fact, we will later see that this series diverges.

Thus: $\boxed{\text{When } a_k \xrightarrow[k\to+\infty]{} 0, \text{ ANYTHING CAN HAPPEN.}}$

I am allowed to write this: $\forall n \in \mathbb{N}$, $\sum_{k=n+1}^{+\infty} a_k$ converges, as it just misses some initial terms of $\sum_{k=1}^{+\infty} a_k$.

Prop.: $\boxed{\text{If } \sum_{k=1}^{+\infty} a_k \text{ converges, then: } \sum_{k=n+1}^{+\infty} a_k \xrightarrow[n\to+\infty]{} 0.}$

Proof: **Idea:** Since $\sum_{k=1}^{+\infty} a_k$ converges, we have that

$$a_1+a_2+\dots+a_n \xrightarrow[n\to+\infty]{} \sum_{k=1}^{+\infty} a_k \in \mathbb{R};$$

so, for large n, $\underbrace{\sum_{k=1}^{+\infty} a_k}$ is practically $\underbrace{a_1+a_2+\dots+a_n}$;

So, their difference, which is $\sum_{k=n+1}^{+\infty} a_k$, must be very small:

$$\underbrace{a_1+a_2+a_3+\dots = (a_1+a_2+\dots+a_n)}_{\text{pretty much equal}} \Longrightarrow \underbrace{(a_{n+1}+a_{n+2}+\dots)}_{\text{pretty much } 0.}$$

We have shown that, if $\sum_{k=1}^{+\infty} a_k$ converges, then

$$\sum_{k=1}^{+\infty} a_k = (a_1+a_2+\dots+a_n) + \sum_{k=n+1}^{+\infty} a_k, \quad \forall n \in \mathbb{N}.$$

($\sum_{k=1}^{+\infty} a_k \in \mathbb{R}$, $\sum_{k=n+1}^{+\infty} a_k \in \mathbb{R}$)

So:

$$\sum_{k=n+1}^{+\infty} a_k = \underbrace{\sum_{k=1}^{+\infty} a_k}_{\text{a constant}} - \underbrace{(a_1+a_2+\dots+a_n)}_{\downarrow n\to+\infty \ \sum_{k=1}^{+\infty} a_k}.$$

$$\xrightarrow[n\to+\infty]{} \underbrace{\sum_{k=1}^{+\infty} a_k}_{\in\mathbb{R}} - \underbrace{\sum_{k=1}^{+\infty} a_k}_{\in\mathbb{R}} = 0. \quad \blacksquare$$

The following test for convergence is very important; it is used to prove other basic tests. It says really that $(s_n)_{n\in\mathbb{N}}$ converges $\Longleftrightarrow$ $(s_n)_{n\in\mathbb{N}}$ Cauchy

test

Cauchy criterion: The series $\sum_{k=1}^{+\infty} a_k$ converges

$\Longleftrightarrow$ $\forall \varepsilon>0,\ \exists n_0\in\mathbb{N}$ s.t. : $\forall n>m\geq n_0$, $|a_{m+1}+a_{m+2}+\ldots+a_n|<\varepsilon$

Remember: this is just saying that $(s_n)_{n\in\mathbb{N}}$ is Cauchy!

Proof: $\sum_{k=1}^{+\infty} a_k$ converges $\Longleftrightarrow$ the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums converges.

$\Longleftrightarrow$ the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums is Cauchy.

a sequence in $\mathbb{R}$ converges $\Longleftrightarrow$ it is Cauchy

And : $(s_n)_{n\in\mathbb{N}}$ is Cauchy $\Longleftrightarrow$ $\forall\varepsilon>0,\ \exists n_0\in\mathbb{N}$ s.t. $\forall n,m\geq n_0$, $|s_n-s_m|<\varepsilon$.

definition of Cauchy sequence

Notice that this is equivalent to saying that:

$$\forall \varepsilon>0,\ \exists n_0\in\mathbb{N} \text{ s.t.}: \forall n>m\geq n_0,\ |s_n-s_m|<\varepsilon$$

(because $|s_n-s_m|=|s_m-s_n|$).

And: for $n>m\geq n_0$, $s_n-s_m=$

$$=(a_1+a_2+\dots+a_m+a_{m+1}+\dots+a_n)-(a_1+a_2+\dots+a_m)=$$

$$=a_{m+1}+a_{m+2}+\dots+a_m.$$

So, $\sum_{k=1}^{+\infty} a_k$ converges $\Longleftrightarrow$

$$\forall \varepsilon>0,\ \exists n_0\in\mathbb{N} \text{ s.t.}: \forall n>m\geq n_0,\ |a_{m+1}+a_{m+2}+\dots+a_n|<\varepsilon.$$

→ **Example:**

⇝ **Harmonic series:** This is the series $\boxed{\sum_{k=1}^{+\infty}\frac{1}{k}}$.

It diverges:

Suppose that it converges. Then, by the Cauchy criterion for $\varepsilon=\frac{1}{4}$, there should exist $n_0\in\mathbb{N}$ s.t.:

$\forall\ n > m \geq n_0, \left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| < \frac{1}{4}$

Notice that $\left| \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \right| =$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \underset{n \geq m+1}{\geq} \underbrace{\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n-m \text{ terms}} = \frac{n-m}{n}.$$

So, we should have that

$$\frac{n-m}{n} < \frac{1}{4}, \quad \forall n > m \geq n_0. \quad (*)$$

You can probably already see that this is impossible: if $m = n_0$ and $n \to +\infty$, then $\frac{n-m}{n} \xrightarrow[n \to +\infty]{} 1$, which is larger than $\frac{1}{4}$. One can also prove that $\boxed{(*) \text{ cannot}}$ hold this way: $\boxed{\text{Since } (*) \text{ should}}$ hold $\forall n > m \geq n_0$, it should hold in particular for $m = n_0$ and $n = 2n_0$; so,

$$\frac{1}{2} = \frac{2n_0 - n_0}{2n_0} < \frac{1}{4},$$ contradiction. So, $\sum_{k=1}^{+\infty} \frac{1}{k}$ diverges. ∎