Stat 134 lec 24

Sec 4.4 Change of variable

X cont RV w/ density f_X

we write $P(X \in dx) = f_X(x)dx$ where $dx > 0$ is a pos length.

$dx > 0$, so you could write

$$P(X \in dx) = f_X(x)|dx|$$ ← abs value

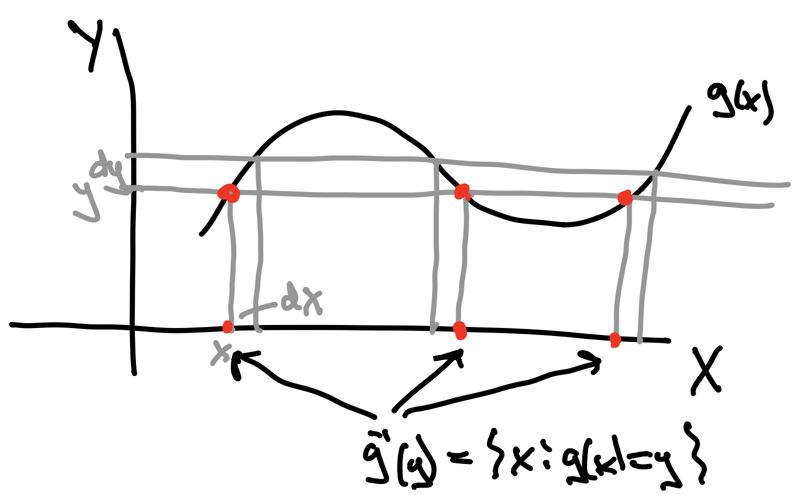
let $Y = g(X)$, g strictly decreasing

$$P(X \in dx) = P(Y \in dy)$$

$$\Rightarrow f_X(x)|dx| = f_Y(y)|dy| \Rightarrow f_X(x) = f_Y(y)\left|\frac{dy}{dx}\right|$$

$$\Rightarrow \boxed{f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \text{ where } x = g^{-1}(y)}$$

many to one g:



addition rule

$$P(Y \in dy) = \sum_{x \in g^{-1}(y)} P(X \in dx) = \sum_{x \in g^{-1}(y)} f_X(x)|dx|$$

$$\| \quad f_Y(y)|dy|$$

$$\Rightarrow \boxed{f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)\left|\frac{dx}{dy}\right| = \sum_{x \in g^{-1}(y)} f_X(x) \cdot \frac{1}{\left|\frac{dy}{dx}\right|} \quad \text{where } x \in g^{-1}(y).}$$

$$\left|\frac{dy}{dx}\right| = |g'(x)|$$

ex Let Z be std normal

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

Find the density of $Y = \frac{1}{Z^2}$

Steps

1) range of Y — $0 < y < \infty$

2) $\frac{dy}{dz} = -2\frac{1}{z^3}$

3) z in terms of y — $z = \pm\sqrt{\frac{1}{y}}$

4) $$f_Y(y) = \sum_{z \in \pm\sqrt{\frac{1}{y}}} \frac{1}{\left|\frac{dy}{dz}\right|} \cdot f_Z(z)$$

$$\frac{1}{\left|\frac{dy}{dz}\right|} = \left|\frac{z^3}{2}\right| \qquad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\pm\sqrt{\frac{1}{y}}\right)^2}$$

2 equal terms in the sum

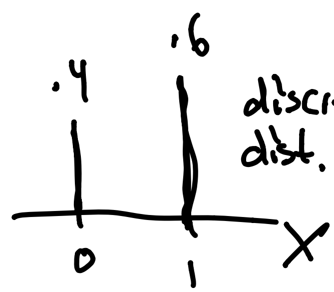
$$= 2 \cdot \left| \frac{\left(\frac{1}{y}\right)^{3/2}}{2} \right| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{y}\right)}$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \frac{1}{y^{3/2}} \cdot e^{-1/2y}, \quad y > 0}$$

sec 4.5 cdf

defn X RV

$$F_X(x) = P(X \le x)$$



discrete dist.

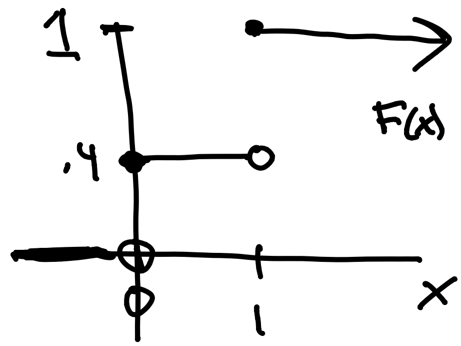
ex $X \sim \text{Bernoulli}(p = .6)$

$F(-1) = 0$

$F(0) = .4$

$F(1/2) = .4$

$F(1) = 1$



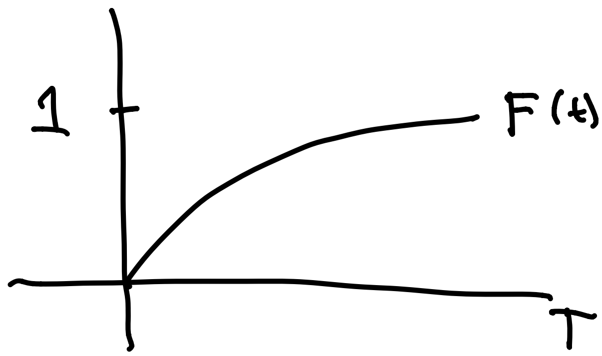
discrete cdf.

ex $T \sim \text{expon}\left(\frac{1}{2}\right)$

$$F(t) = 1 - e^{-\frac{1}{2}t} \quad \text{for } t > 0$$

Prob dist of T

continuous dist.



continuous cdf

ex

$T \sim \text{expon}(\lambda)$

$X =$ "T killed at c", $c > 0$

$X = \min(T, c)$

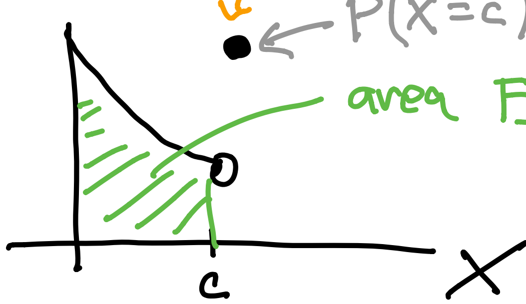
i.e. when $T \leq c$, $X = T$

otherwise, $X = c$

we have a jump here because the rest of the probability gets concentrated at c

$P(X=c) = e^{-\lambda c}$

<u>Picture</u>



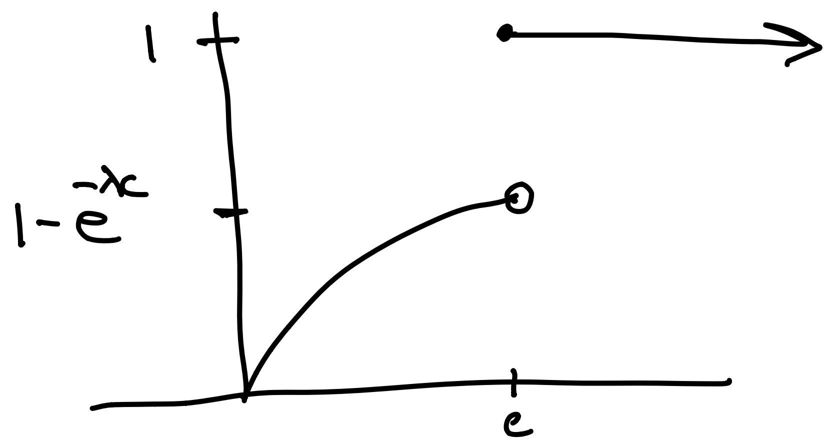
area $F_T(c) = 1 - e^{-\lambda c}$

"mixed" distribution.

Notice that this distribution is both continuous and discrete (called a "mixed" distribution.)

Here $P(X=c) = e^{-\lambda c}$ so that the total probability is 1.

Draw the cdf:



This is a mixed cdf.

Properties of cdf

a) $F_X(x)$ is non decreasing function

b) $\lim_{x \to -\infty} F(x) = 0$

c) $\lim_{x \to \infty} F(x) = 1$

d) $P\left(a < X \leq b\right) = F_X(b) - F_X(a)$

e) A prob distribution is completely determined by its cdf.

f) $X_1, \dots, X_n$ iid RV w/ cdf F

(i) $F_{\max(X_1,\dots,X_n)} = (F(x))^n$

$F_{\min(X_1,\dots,X_n)} = 1 - (1-F(x))^n$

ex $X_i \overset{\text{indep}}{\sim} \text{expon}(\lambda_i)$

$F_{X_i}(x) = 1 - e^{-\lambda_i x}$ $(0, \infty)$

let $M = \min(X_1, \dots, X_n)$

for $t > 0$, $P(M > t) = P(X_1 > t, X_2 > t, \dots, X_n > t)$

$$= P(X_1 > t) \cdot P(X_2 > t) \cdots$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \cdots + \lambda_n) t}$$

$$\Rightarrow F_M(t) = 1 - e^{-(\lambda_1 + \cdots + \lambda_n) t}$$

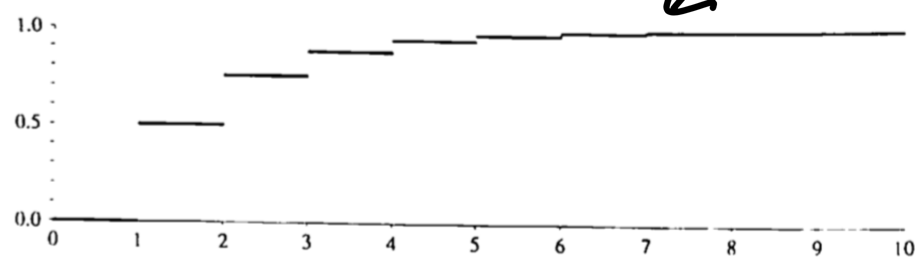
$$\Rightarrow M \sim \text{expon}(\lambda_1 + \cdots + \lambda_n)$$

Stat 134

Monday March 14 2018

there are many little steps here.

1. Below is the cdf of which distribution?



a binomial(10,1/2)

b geometric(1/2) on $\{1, 2, 3, ...\}$

c uniform(0,10)

d none of the above

Not (a) since $P(X=0)=(\frac{1}{2})^{10}$ and
$P(X=1)=10\cdot(\frac{1}{2})^{10}$ so
$F(1)=11\cdot(\frac{1}{2})^{10} \neq .5$

Not (c) since cdf of unif is a continuous cdf not a discrete cdf.

(b) is good since $P(X=0)=0$, $P(X=1)=\frac{1}{2}$
$P(X=2)=\frac{1}{4} \Rightarrow F(2)=.75$ ✓