Stat 134: Section 21

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April 16th, 2018

Problem 1

Let $N \sim$ Pois (μ). Suppose given $N = n$, random variables $X_1, X_2, \ldots, X_n$ are i.i.d. Unif $(0, 1)$. I.e., the number of X_i's depends on N.

a. Given $N = n$, what is the chance that all the X_i's are less than t?

b. What is the unconditional probability that all the X_i's are less than t?

c. Let $S_N = X_1 + \ldots + X_N$ denote the sum of the (random) number of X_i's. Find $\mathbb{E}(S_N)$.

Ex 6.2.6 in Pitman's Probability

Problem 2

Suppose you have a coin which lands heads with probability p. Let X denote the number of tosses required to observe both heads and tails.

Hint: condition on the result of the first toss. No summations necessary!

a. Find $\mathbb{E}(X)$.

b. Find $Var(X)$.

Problem 3: The Beta-Binomial

Let $S_n = \sum_{i=1}^{n} X_i$ be the number of successes in a sequence of Bernoulli (Π) trials, where $\Pi \sim$ Beta (r, s). That is, given $\Pi = p$, $S_n \sim$ Binomial (n, p). This arises as a natural model in Bayesian inference when we are uncertain about the true value of p.

a. Given $S_n = k$, show that the posterior distribution of Π is Beta $(r + k, s + n - k)$.

b. Use the fact that the total integral of the beta $(r + k, s + n - k)$ density is 1 to find a formula for the unconditional probability $P(S_n = k)$.

c. Find $\mathbb{E}(\Pi \mid S_n = k)$ and $Var(\Pi \mid S_n = k)$. (Note that these facts can be used to show as $n \to \infty$, $\Pi \to \frac{S_n}{n}$, the observed sample proportion of successes, regardless of the values of r, s.)

Ex 6.3.15 in Pitman's Probability

Prepared by Brian Thorsen