Stat 134 MGF homework problems

Problem 1 (properties of the moment generating function). We defined the moment generating function (m.g.f.) of a random variable X as $\psi_X(t) = E(e^{Xt})$ which is a function of t (note: I sometimes use $\psi_X(t)$ instead of $M_X(t)$ as the name of a moment generating function). More explicitly, it has one of the two forms:

$$\psi_X(t) = E(e^{Xt}) = \sum_{\text{all } x} e^{xt} P(X = x) \qquad \text{or} \qquad \psi_X(t) = E(e^{Xt}) = \int_{-\infty}^{\infty} e^{xt} f_X(x) \, dx,$$

if X is a *discrete* or *continuous* random variable, respectively. (In case you are asking yourself about the convergence of the summation or the integral, one can prove, using the theory of Laplace transforms, that they converge at least in an interval of $\mathbb{R}$ that contains the point $t = 0$; we take this interval as the domain of definition of the moment generating function $\psi_X(t)$).

(a) Show that, for any r.v. X, it is the case that $\psi_X(0) = 1$. (So the m.g.f. is always defined at $t = 0$.)

(b) Show that if $Y = aX + b$, where a and b are constants, then $\psi_Y(t) = e^{bt}\psi_X(at)$.

(c) Show that if the random variables X and Y are *independent* and $Z = X + Y$, then $\psi_Z(t) = \psi_X(t)\psi_Y(t)$. *Hint:* you have to use the fact that if two random variables X and Y are independent and f and g are two functions, then the two new random variables $U = f(X)$ and $V = g(Y)$ are also independent.

Problem 2 Another property of the moment generating function is that if two random variables have the same m.g.f., then they have the same p.d.f. (we say that the mapping p.d.f. $\mapsto$ m.g.f. is one-to-one). In class we showed that if we have $X \sim \text{Gamma}(\alpha, \lambda)$, then $\psi_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$. Use these facts and part **(c)** of the previous problem to show that if $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ are independent random variables and we define $Z = X + Y$, then $Z \sim \text{Gamma}(\alpha + \beta, \lambda)$.

Problem 3 (m.g.f. of Binomial random variables). We proved that the m.g.f. $\psi_X(t)$ "generates" the moments of the random variable X by *differentiation*, and computation at $t = 0$ (rather than by integration or summation, which is typically harder). For example,

$$\psi_X'(0) = E(X), \quad \psi_X''(0) = E(X^2), \quad \ldots, \quad \psi_X^{(n)}(0) = E(X^n),$$

where the superscript $^{(n)}$ indicates the nth derivative.

(a) Assume that $X \sim \text{Binomial}(n, p)$, i.e. with p.f. $f_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, for $k = 0, 1, \ldots, n$.

Find the function $\psi_X(t)$, in terms of the parameters n and p.

Hint: You will need to use the binomial theorem, which states that $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$.

(b) Find $E(X)$ and $\text{Var}(X)$ using the method described above.

Note that this is *much* simpler than computing, for example, $E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$.

Problem 4 (m.g.f. of Normal random variables).

(a) Consider a *standard* Normal random variable $Z \sim \mathcal{N}(0,1)$, and compute its m.g.f. ψ_Z. *Hint:* You need to "complete the square" at the exponent: $\frac{z^2}{2} - tz = \frac{1}{2}(z^2 - 2tz + t^2) - \frac{t^2}{2} = \frac{1}{2}(z-t)^2 - \frac{t^2}{2}$, and then perform a change of variable in the integral.

(b) Now use part (b) of Problem 1 to compute the m.g.f. of a *generic* Normal r.v. $X \sim N(\mu, \sigma^2)$.

(c) Finally, use part (c) of Problem 1 to show that if $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent and we define $Z = X + Y$, then $Z \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Remark: it should be clear by now that using the moment generating functions to prove the above fact is a much simpler method than computing the convolution between two Normal p.d.f.'s!

Problem 5 Suppose that X is a random variable for which the m.g.f. is as follows:

$$\psi_X(t) = \frac{1}{6}\left(4 + e^t + e^{-2t}\right).$$

Find the probability distribution of X. *Hint:* It is a simple discrete distribution for example, pmf $P(X = 0) = .3$, $P(X = 1) = .7$.