**Def.**: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and $a\in\mathbb{R}$. We say that $(a_n)_{n\in\mathbb{N}}$ converges to $a$, and that $a$ is the limit of $(a_n)_{n\in\mathbb{N}}$, and we write: "$a_n \longrightarrow a$ (as $n\to+\infty$)", if:

optional

$$\boxed{\forall \varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in\mathbb{N} :\ \forall n\geq n_0,\ |a_n-a|<\varepsilon}$$

⚠ Note that this definition can be rephrased as:

$(a_n)_{n\in\mathbb{N}}$ converges to $a$ if:

> for any neighbourhood $(a-\varepsilon, a+\varepsilon)$ of $a$, there exists a final part of $(a_n)_{n\in\mathbb{N}}$ contained in $(a-\varepsilon, a+\varepsilon)$.

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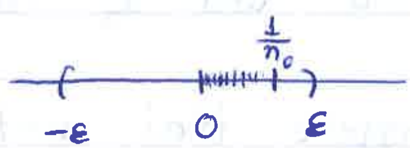
**ex:** $\boxed{a_n=\frac{1}{n} \longrightarrow 0 \text{ as } n\to+\infty.}$

**Proof:** Let $\varepsilon>0$.




I am looking for some $n_0\in\mathbb{N}$ (depending on the $\varepsilon$), such that:

for all $n\geq n_0$, $-\varepsilon<\frac{1}{n}<\varepsilon$.

In fact, I know that $\frac{1}{n}>0$ $\forall n\in\mathbb{N}$ (as $n>0$), so

I just want $n_0 \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, $\forall n \geq n_0$.

By the Archimedean property of the reals, we know that $\exists n_0 \in \mathbb{N} : \frac{1}{n_0} < \varepsilon$.

Then, $\forall n \geq n_0$, we have $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$.

So, indeed $\exists n_0 \in \mathbb{N}$ s.t.: $\forall n \geq n_0$, $\frac{1}{n} < \varepsilon$.

Since $\varepsilon$ was arbitrary, the proof is complete. ■

→ Obs.: The smallest $n_0$ satisfying $\frac{1}{n_0} < \varepsilon \Leftrightarrow n_0 > \frac{1}{\varepsilon}$

is $\lfloor \frac{1}{\varepsilon} \rfloor + 1$.

→ Prop. (Uniqueness of limits): Let $(a_n)_{n \in \mathbb{N}}$ a sequence.

If $a_n \longrightarrow a$ and $a_n \to b$, then $a = b$.

Idea: Since a,b are far apart, I can find 2 neighbourhoods that are disjoint. Since $a_n \to a$, I can find some final part of $(a_n)$ in the neigh. of $a$. Since $a_n \to b$, I can find some final part of $(a_n)$ in the neigh. of $b$. The smallest of the 2 final parts will be simultaneously in both neighbourhoods, contradiction.

Proof: Suppose that $a < b$.


Pick $\varepsilon > 0$ s.t. $a + \varepsilon < b - \varepsilon$ (this is $\Leftrightarrow 2\varepsilon < b - a \Leftrightarrow$
$\Leftrightarrow \varepsilon < \frac{b-a}{2}$; so, $\varepsilon = \frac{b-a}{3}$ will do)

Then, the neighbourhoods $(a-\varepsilon, a+\varepsilon)$ and $(b-\varepsilon, b+\varepsilon)$ are <u>disjoint</u>.

Since $a_n \longrightarrow a$, there exists $n_1 \in \mathbb{N}$ s.t.: $\forall n \geq n_1$, $a_n \in (a-\varepsilon, a+\varepsilon)$.

Since $a_n \longrightarrow b$, there exists $n_2 \in \mathbb{N}$ s.t.: $\forall n \geq n_2$, $a_n \in (b-\varepsilon, b+\varepsilon)$.

Take $n_0 = \max\{n_1, n_2\}$. Then, $a_{n_0} \in (a-\varepsilon, a+\varepsilon) \cap (b-\varepsilon, b+\varepsilon)$,

a <u>contradiction</u>, as $(a-\varepsilon, a+\varepsilon) \cap (b-\varepsilon, b+\varepsilon) = \emptyset$. Similarly, we get a contradiction when we assume $b<a$. So, $b=a$. ■

→ <u>Notation</u>: When $a_n \longrightarrow a \in \mathbb{R}$, we denote this <u>unique</u> limit of $(a_n)_{n\in\mathbb{N}}$ by

$$\lim_{n\to+\infty} a_n \quad \text{or} \quad \lim a_n .$$

→ <u>Observation 1</u>: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and $(a_m, a_{m+1}, a_{m+2}, \dots) = (a_{m+n-1})_{n\in\mathbb{N}}$ a final part of $(a_n)_{n\in\mathbb{N}}$.

The proof is left as an exercise.

Then: $a_n \longrightarrow a \iff a_{m+n-1} \longrightarrow a$ as $n \longrightarrow +\infty$.

→ **Observation 2:** If $a_n \to a$, $b_n \to b$ and

$(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ are eventually equal (that is, they differ for at most finitely many indices),

then $a = b$.

**Proof:** Since $(a_n)$, $(b_n)$ are eventually equal, there exists some $m \in \mathbb{N}$ s.t.: $a_n = b_n$, $\forall n \geq m$.

$$\text{So: } \underbrace{(a_m, a_{m+1}, \dots)}_{\substack{\text{a final} \\ \text{part of } (a_n), \\ \text{so } \to a.}} = \underbrace{(b_m, b_{m+1}, \dots)}_{\substack{\text{a final} \\ \text{part of } (b_n), \\ \text{so } \to b}}.$$

By uniqueness of limits, $a = b$.

→ **Def:** The sequence $(a_n)_{n\in\mathbb{N}}$ is:

- bounded from above if $\exists\, b \in \mathbb{R}$ s.t. $a_n \leq b$ $\forall n \in \mathbb{N}$.


- bounded from below if $\exists\, c \in \mathbb{R}$ s.t. $a_n \geq c$, $\forall n \in \mathbb{N}$.


- bounded if $\exists\, b, c \in \mathbb{R}$ s.t. $c \leq a_n \leq b$, $\forall n \in \mathbb{N}$.

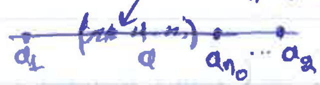
→ **Observation:** $(a_n)_{n\in\mathbb{N}}$ is bounded $\Longleftrightarrow$ $\exists\, M>0$ s.t. $|a_n|\le M, \forall n\in\mathbb{N}$.

**Proof:** Exercise.

→ **Prop:** Every convergent sequence is bounded.

**Proof:** ( **Idea:** Let $a_n \to a$. Pick some neighbourhood of $a$.

the $a_n$'s for large $n$ (say, $n>n_0$).




for large $n$, the $a_n$'s cluster in that neighbourhood.

The $a_n$'s outside the neighbourhood are only finitely many. So, the $a_n$'s cannot go infinitely far from $a$. )

Let $(a_n)_{n\in\mathbb{N}}$ a convergent sequence, and $a$ its limit.

Let $\varepsilon = 1$ $(>0)$. Since $a_n \to a$, there exists $n_0\in\mathbb{N}$ s.t.

$$|a_n - a| < 1 \quad , \ \forall n > n_0 ;$$

i.e., $\quad a-1 < a_n < a+1, \ \forall n > n_0.$

So, $\forall n\in\mathbb{N}$:

$$\min\{a_1, a_2, \dots, a_{n_0}, a-1\} \in \mathbb{R}$$
$$\le$$
$$a_n$$
$$\le$$
$$\max\{a_1, a_2, \dots, a_{n_0}, a-1\} \in \mathbb{R}$$

So, $(a_n)_{n\in\mathbb{N}}$ bounded. ∎

## Algebra of limits:

**Prop:** $\boxed{\underbrace{a_n \longrightarrow a}_{A} \Longleftrightarrow \underbrace{a_n - a \longrightarrow 0}_{B} \Longleftrightarrow \underbrace{|a_n - a| \longrightarrow 0}_{C}.}$

**Proof:** A $\Longleftrightarrow$ $\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0,\ \underbrace{|a_n - a|}_{\| \ \ \|} < \varepsilon.$

$$|(a_n - a) - 0| \qquad \big|\,|a_n - a| - 0\,\big|$$

Since B $\Longleftrightarrow$ $\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0,\ |(a_n - a) - 0| < \varepsilon.$

and C $\Longleftrightarrow$ $\forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N} : \forall n \geq n_0,\ \big|\,|a_n - a| - 0\,\big| < \varepsilon,$

we have that A $\Longleftrightarrow$ B $\Longleftrightarrow$ C (the same index $n_0$ happens to work for the same $\varepsilon$ in all three cases) ■

**Corollary:** $a_n \longrightarrow 0 \Longleftrightarrow |a_n| \longrightarrow 0.$

**Proof:** $\boxed{\text{If } a_n \longrightarrow a, \text{ then } |a_n| \longrightarrow |a|.}$

**Proof:** I want to show: $|a_n| \longrightarrow |a|$.
Let $\varepsilon > 0$. I am looking for $n_0 \in \mathbb{N}$ s.t.: $\forall n \geq n_0$,
$\big|\,|a_n| - |a|\,\big| < \varepsilon$.

I know that $a_n \to a$; so, for this $\varepsilon > 0$, $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$ s.t.: $\forall n \geq n_0$, $|a_n - a| < \varepsilon$.

And: $\big||a_n|-|a|\big| \leq |a_n-a| \ \forall n\in\mathbb{N}$ (by properties of absolute value).

So, $\forall n \geq n_0$ : $\big||a_n|-|a|\big| < \varepsilon$.
Since $\varepsilon$ was arbitrary, the proof is complete. ■

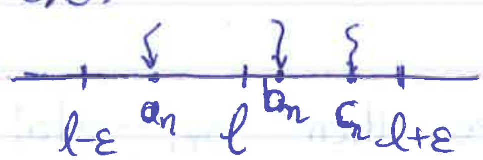
⚠ It is not true in general than $|a_n| \to |a| \implies a_n \to a$.
For example, for $a_n = (-1)^n \ \forall n\in\mathbb{N}$, we have
$|a_n| = 1 \longrightarrow 1$, but $(a_n)_{n\in\mathbb{N}}$ doesn't converge (exercise!).

→ **Prop.:** Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, $(c_n)_{n\in\mathbb{N}}$ be sequences.
Suppose that: (i) $\forall n\in\mathbb{N}$, $a_n \leq b_n \leq c_n$.
and (ii) $a_n \to \ell$ and $b_n \to \ell$.

Then: $b_n \longrightarrow \ell$.

Squeeze Theorem or Sandwich Lemma.

**Proof:** Let $\varepsilon > 0$.




Idea: If for large $n$ $a_n$ is close to $\ell$, and for large $n$ $a_n$ is close to $\ell$, and $b_n$ is squeezed between $a_n$ and $b_n$, then $b_n$ should also be close to $\ell$ for large $n$.

Since $a_n \to \ell$, there exists $n_1\ (=n_1(\varepsilon)) \in\mathbb{N}$, s.t. $\forall n \geq n_1$, $\ell-\varepsilon < a_n < \ell+\varepsilon$.

Since $c_n \longrightarrow \ell$, there exists $n_2\ (=n_2(\varepsilon)) \in\mathbb{N}$, s.t. $\forall n \geq n_2$, $\ell-\varepsilon < c_n < \ell+\varepsilon$.

Let $n_0 := \overbrace{\max\{n_1, n_2\}}^{\text{above this index, both the } a_n\text{'s and the } b_n\text{'s are in } (\ell-\varepsilon,\, \ell+\varepsilon)\text{.}}$. Then, $\forall n \geq n_0$:

$$\ell-\varepsilon < \underbrace{\boxed{a_n \leq b_n \leq c_n}}_{\text{by assumption.}} < \ell+\varepsilon$$

So, we have shown that:

$\exists\, n_0 \in \mathbb{N}$ s.t. $\ell-\varepsilon < b_n < \ell+\varepsilon$,
i.e. $|b_n - \ell| < \varepsilon$.

Since $\varepsilon$ was arbitrary, $b_n \longrightarrow \ell$. ∎