$$\lim_{n\to+\infty} a_{k_n} = \lim_{n\to+\infty} b_n = \lim_{n\to+\infty} c_n = x.$$

So, $(a_{k_n})_{n\in\mathbb{N}}$ is convergent. ∎

⚠ Notice that both the proofs of the Bolzano–Weierstrass that we provided rely on the total order in $\mathbb{R}$. When we generalise the theorem to compact metric spaces, we will not have that advantage any more. So, we'll have to find a better way to exploit the generalisation of the nested intervals theorem that we mentioned earlier.

Lecture 8: 14 Sep 2016. 1

→ **Cauchy sequences:** This is another notion we will generalise to all metric spaces later.

The main observation that leads to the notion of a Cauchy sequence is that, if a sequence converges, then all its terms are close to the limit for large so, in particular, these terms should be close to each other. But does the converse hold ??

Def: A sequence $(a_n)_{n\in\mathbb{N}}$ is a $\boxed{\text{Cauchy sequence}}$ if: $\forall \varepsilon>0$, there exists $n_0\in\mathbb{N}$ s.t.: $\forall n\geq n_0$, $|a_n-a_m|<\varepsilon$.

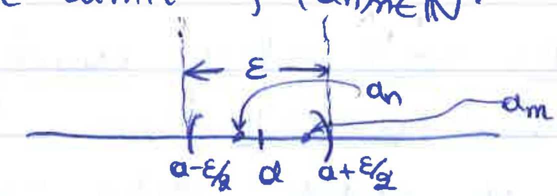
for instance, when $n\geq n_0$, we don't just have $|a_{n+1}-a_n|<\varepsilon$, but also $|a_{2n}-a_n|<\varepsilon$, $|a_{n^{1000}}-a_n|<\varepsilon$, etc.

⚠ We should think each $\varepsilon>0$ in the definition above as the "level of closeness" that we want the terms of $(a_n)_{n\in\mathbb{N}}$ to be achieving eventually (from some index onwards). There is no neighbourhood of any point involved.

Prop.: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. If $(a_n)_{n\in\mathbb{N}}$ converges, then $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof: Let a be the limit of $(a_n)_{n\in\mathbb{N}}$.

Let $\varepsilon>0$.



Since $a_n\to a$, there exists some $n_0\in\mathbb{N}$ s.t.:

$$\forall n\geq n_0,\ |a_n-a|<\frac{\varepsilon}{2}.$$

So, $\forall n,m\geq n_0$: $|a_n-a_m| = |(a_n-a)+(a-a_m)| \leq$ (triangle inequality)

$$\leq |a_n-a|+|a_m-a| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon>0$ was arbitrary, $(a_n)_{n\in\mathbb{N}}$ is Cauchy.

$\blacksquare$

⚠ In fact, we will later see that, in any metric space, convergent sequences are Cauchy.

However, the converse is not true in general; i.e., in a general metric space, Cauchy sequences don't necessarily converge

(i.e., the terms of a sequence being eventually as close as we want to each other doesn't mean that they are also all close to a fixed element of the metric space)

A metric space where Cauchy sequences converge is called complete. We will now see that $\mathbb{R}$ is a complete metric space (don't confuse this with the notion of a complete ordered field!)

→ **Thm:** $\boxed{\text{In } \mathbb{R}, \text{ a sequence converges} \iff \text{it is Cauchy.}}$

Proof: Due to the previous Proposition, we just need to show that every Cauchy sequence in $\mathbb{R}$ converges.

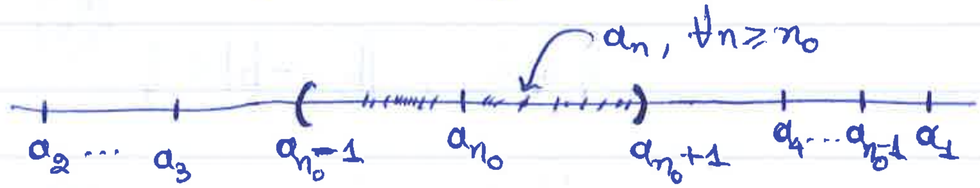
Let $(a_n)_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathbb{R}$. We finish the proof in 2 steps:

Step 1: Since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, there exists some $n_0 \in \mathbb{N}$

↓ Show that $(a_n)_{n\in\mathbb{N}}$ is bounded.
(works in every metric space)

s.t.: $\forall n, m \geq n_0$, $|a_n - a_m| < 1$ (we apply the definition of a Cauchy sequence for $\varepsilon = 1$.)

In particular, $|a_n - a_{n_0}| < 1 \ \forall n \geq n_0$,
i.e. $a_n \in (a_{n_0} - 1, a_{n_0} + 1)$, $\forall n \geq n_0$



So: $a_n \in \left(\min\{a_1, a_2, \ldots, a_{n_0-1}, a_{n_0} - 1\}, \max\{a_1, a_2, \ldots, a_{n_0-1}, a_{n_0}+1\}\right)$, $\forall n \in \mathbb{N}$.

So, $(a_n)_{n\in\mathbb{N}}$ is bounded.

Step 2: Since $(a_n)_{n\in\mathbb{N}}$ is bounded, it has a convergent subsequence (by the Bolzano-Weierstrass theorem)

⚠ Not true in every metric space!

It thus suffices to show the following:

Holds in every metric space.

If $(b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and it has a subsequence $(b_{k_n})_{n\in\mathbb{N}}$ with $b_{k_n} \xrightarrow[n\to+\infty]{} b \in \mathbb{R}$, then $b_n \xrightarrow[n\to+\infty]{} b$.

Proof: Let $\varepsilon>0$.

Since $b_{k_n} \xrightarrow[n\to+\infty]{} b$, there exists some $n_1\in\mathbb{N}$, s.t.:

$$\boxed{\forall n\geq n_1,\ |b_{k_n}-b|<\frac{\varepsilon}{2}} \quad (*_1)$$

And:

Since $(b_n)_{n\in\mathbb{N}}$ is Cauchy, there exists $n_2\in\mathbb{N}$, s.t.:

$$\boxed{\forall n,m\geq n_2,\ |b_n-b_m|<\frac{\varepsilon}{2}} \quad (*_2)$$

So, for all $n\geq n_0 := \max\{n_1,n_2\}$,

$$|b_n - b| = |(b_n - b_{k_n}) + (b_{k_n} - b)| \leq \underbrace{|b_n - b_{k_n}|}_{<\frac{\varepsilon}{2}} + \underbrace{|b_{k_n} - b|}_{<\frac{\varepsilon}{2}}$$

$< \frac{\varepsilon}{2}$, because $k_n \geq n \ \forall n \in \mathbb{N}$, so $k_n, n \geq n_0 \geq n_2$ when $n \geq n_0$.

$< \frac{\varepsilon}{2}$, because $k_n \geq n \ \forall n \in \mathbb{N}$, so $k_n \geq n_0 \geq n_1$ $\forall n \geq n_0$. $\blacksquare$

By Step 2, $(a_n)_{n\in\mathbb{N}}$ has a subsequence $(a_{k_n})_{n\in\mathbb{N}}$, with $a_{k_n} \longrightarrow a$, for some $a \in \mathbb{R}$. By the above, $a_n \longrightarrow a$, so $(a_n)_{n\in\mathbb{N}}$ converges. $\blacksquare$

A common mistake:

It holds that, if $(a_n)_{n\in\mathbb{N}}$ is Cauchy, then $a_{n+1} - a_n \xrightarrow[n\to+\infty]{} 0$. (Exercise!)

However, the converse is not true:

find one!

$$\boxed{\text{there exists } (a_n)_{n\in\mathbb{N}} \text{ with } a_{n+1} - a_n \longrightarrow 0, \text{ but with } (a_n)_{n\in\mathbb{N}} \text{ not Cauchy!}}$$

Thus: for $(a_n)_{n\in\mathbb{N}}$ to be Cauchy, we need all terms from some index onwards to be close to each other; $\boxed{\text{not just consecutive terms!!!}}$

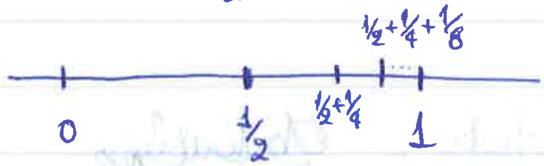
Series in $\mathbb{R}$

→ **Problem:** How do we add infinitely many real numbers together? I.e.:
Given a sequence $(a_n)_{n\in\mathbb{N}}$, what do we mean by $a_1+a_2+a_3+\dots$?

For example:

- When $a_n=\frac{1}{2^n}$, $\forall n\in\mathbb{N}$, what is

  $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots$ equal to? (What is it even defined as?)



  It is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots$, so one would think that we start from 0, then add $\frac12$, then $\frac14$, then $\frac18$, etc.
  Since at every step we add half the distance of where we are from 1, it may not come as a surprise that the infinite sum will eventually be shown to equal 1:
  $1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^n}$ gets closer and closer to 1 for n large.

- When $a_n=(-1)^n$, $\forall n\in\mathbb{N}$, what is

  $a_1+a_2+a_3+a_4+a_5+a_6+\dots$
  $=(-1)+1+(-1)+1+(-1)+1+\dots$ equal to?

We have: $a_1 = -1$,

$a_1 + a_2 = 0$,

$a_1 + a_2 + a_3 = -1$,

$a_1 + a_2 + a_3 + a_4 = 0$, etc.

So, as we add more terms, it doesn't look like $a_1 + a_2 + \ldots + a_n$ "stabilises" around some fixed value for n large. Thus, it may not come as a surprise that $(-1) + 1 + (-1) + 1 + (-1) + 1 + \ldots$ is not defined.

From these examples, it looks like the infinite sum $a_1 + a_2 + a_3 + \ldots$ is determined by the behaviour of

$a_1 + a_2 + \ldots + a_n$ for n large. Indeed, this is how we <u>define</u> the infinite sum $a_1 + a_2 + a_3 + \ldots$:

→ <u>Def:</u> Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

- We call the symbol $\sum_{k=1}^{+\infty} a_k$ "the series of a_k".

- Let $s_1 := a_1$,

  $s_2 := a_1 + a_2$,

  $s_3 := a_1 + a_2 + a_3$,

  $\vdots$

  $s_n := a_1 + a_2 + a_3 + \ldots + a_n$, $\forall n \in \mathbb{N}$.

  We call s_n the <u>n-th partial sum</u> of the series $\sum_{k=1}^{+\infty} a_k$.

- If the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums of $\sum_{k=1}^{+\infty} a_k$ <u>converges</u> to some $\underline{s\in\mathbb{R}}$, then we $\boxed{\underline{\text{define}} \quad \sum_{k=1}^{+\infty} a_k \text{ to be } s.}$

  this was just a symbol up to now!

  We say that the series $\sum_{k=1}^{+\infty} a_k$ <u>converges to s</u>, and we write $\underline{\sum_{k=1}^{+\infty} a_k = s}$.

  In other words: When $(s_n)_{n\in\mathbb{N}}$ <u>converges</u>, then $\sum_{k=1}^{+\infty} a_k := \lim_{n\to+\infty} s_n$, i.e. $s_n \xrightarrow[n\to+\infty]{} \sum_{k=1}^{+\infty} a_n$.

- If the sequence $(s_n)_{n\in\mathbb{N}}$ of partial sums of $\sum_{k=1}^{+\infty} a_k$ <u>is not convergent</u>, then we say that the series $\sum_{k=1}^{+\infty} a_k$ <u>diverges.</u> In particular:

  - If $s_n \longrightarrow +\infty$, then we $\boxed{\underline{\text{define}} \ \sum_{k=1}^{+\infty} a_k \text{ to be } +\infty.}$

We say that the series $\sum_{k=1}^{+\infty} a_k$ diverges to $+\infty$,

and we write $\sum_{k=1}^{+\infty} a_k = +\infty$.

In other words: When $s_n \longrightarrow +\infty$,

then $\sum_{k=1}^{+\infty} a_k := \lim_{n\to+\infty} s_n = +\infty$,

i.e. $s_n \xrightarrow[n\to+\infty]{} \sum_{k=1}^{+\infty} a_k$.

- If $s_n \longrightarrow -\infty$, then we define $\sum_{k=1}^{+\infty} a_k$ to be $-\infty$.

We say that the series $\sum_{k=1}^{+\infty} a_k$ diverges to $-\infty$,

and we write $\sum_{k=1}^{+\infty} a_k = -\infty$.

In other words: When $s_n \longrightarrow -\infty$,

then $\sum_{k=1}^{+\infty} a_k := \lim_{n\to+\infty} s_n = -\infty$,

i.e. $s_n \xrightarrow[n\to+\infty]{} \sum_{k=1}^{+\infty} a_k$.

- If $\lim_{n\to+\infty} s_n$ doesn't exist (in $\mathbb{R}\cup\{+\infty,-\infty\}$), then $\sum_{k=1}^{+\infty} a_k$ diverges, and the infinite sum $a_1+a_2+\ldots$ is not defined.

To sum up:

- If $\lim_{n\to+\infty} s_n$ exists (in $\mathbb{R}\cup\{+\infty,-\infty\}$), then $\sum_{k=1}^{+\infty} a_k := \lim_{n\to+\infty} s_n$.
- If $\lim_{n\to+\infty} s_n$ doesn't exist, then $\sum_{k=1}^{+\infty} a_k$ is not defined.

⚠ Sometimes, a sequence may be given in the form $(a_n)_{n=0}^{+\infty}$, or $(a_n)_{n=4}^{+\infty}$, etc.

$$(a_n)_{n=0}^{+\infty} = (a_0, a_1, a_2, \dots) \qquad (a_n)_{n=4}^{+\infty} = (a_4, a_5, a_6, \dots)$$

No matter what, the n-th partial sum s_n of the series corresponding to the sequence is the sum of the first n terms of the sequence.

the number of terms in the sum

(for the sequences above, for instance, s_3 is $a_0+a_1+a_2$ and $a_4+a_5+a_6$, respectively).

→ So, to find $\sum_{k=1}^{+\infty} a_k$, we start adding up the terms one by one in the order that they appear (first we have a_1, then a_1+a_2, then $a_1+a_2+a_3$, etc), and see whether, as the sum gets longer and longer, it has a limit.

Let us see some important examples:

Geometric series:

The geometric series with ratio $x \in \mathbb{R}$ is:

$$\boxed{\sum_{k=0}^{+\infty} x^k}$$

(i.e., the series $\sum_{k=0}^{+\infty} a_k$, for $(a_n)_{n=0}^{+\infty} = (1, x, x^2, x^3, \ldots)$.)

For this series, $s_n = a_0 + a_1 + \ldots + a_{n-1} =$

$$= 1 + x + \ldots + x^{n-1} = \begin{cases} \dfrac{1-x^n}{1-x} \\ n \quad , \text{ if } x = 1 \end{cases}$$

So:

- For $\underline{x=1}$, $s_n = n \longrightarrow +\infty$, so $\sum_{k=0}^{+\infty} x^k = +\infty$.

- For $\underline{|x| < 1}$, $\underbrace{|x|^n}_{= |x^n|} \xrightarrow[n \to +\infty]{} 0$, i.e. $x^n \xrightarrow[n \to +\infty]{} 0$,

  so $s_n = \dfrac{1-x^n}{1-x} \xrightarrow[n \to +\infty]{} \dfrac{1-0}{1-x} = \dfrac{1}{1-x}$.

  Thus, for $|x| < 1$, $\sum_{k=0}^{+\infty} x^k = \dfrac{1}{1-x}$; the series converges.

- For $\underline{x > 1}$, $x^n \xrightarrow[n \to +\infty]{} +\infty$, so $s_n = \dfrac{1-x^n}{\underbrace{(1-x)}_{<0}} \xrightarrow[n \to +\infty]{} +\infty$.

  Thus, for $x > 1$, $\sum_{k=0}^{+\infty} x^k = +\infty$; the series diverges.

- for $\underline{x \leq -1}$: $(x^n)_{n\in\mathbb{N}}$ doesn't converge, so $(s_n)_{n\in\mathbb{N}}$ doesn't converge either.

Thus, $\sum_{k=0}^{+\infty} x^k$ diverges (and the infinite sum $1+x+x^2+\dots$ is not defined).

– To sum up :

$$\boxed{\begin{array}{l} \displaystyle\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x} \text{ for } |x|<1, \\ \text{and } \displaystyle\sum_{k=0}^{+\infty} x^k \text{ diverges for } |x|\geq 1. \left(\text{In particular, } \displaystyle\sum_{k=0}^{+\infty} x^k = +\infty \text{ for } x\geq 1\right) \end{array}}$$