$$\text{Let } n_0 := \overbrace{\max\{n_1, n_2\}}^{\text{above this index, both the } a_n\text{'s and the } b_n\text{'s are in } (\ell-\varepsilon, \ell+\varepsilon)}. \text{ Then, } \forall n \geq n_0:$$

$$\ell-\varepsilon < \underbrace{\boxed{a_n \leq b_n \leq c_n}}_{\text{by assumption}} < \ell+\varepsilon$$

So, we have shown that:

$$\exists\, n_0 \in \mathbb{N} \text{ s.t. } \quad \ell-\varepsilon < b_n < \ell+\varepsilon,$$
$$\text{i.e. } |b_n - \ell| < \varepsilon.$$

Since ε was arbitrary, $b_n \longrightarrow \ell$. ■

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→ <u>Prop.:</u> Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$.
Suppose that: (i) $a_n \longrightarrow 0$
and (ii) $(b_n)_{n\in\mathbb{N}}$ bounded.

Then, $a_n \cdot b_n \longrightarrow 0$.

<u>Proof:</u> <u>Let $\varepsilon > 0$.</u> I am looking for $n_0 \in \mathbb{N}$ s.t.: $\forall n \geq n_0$, $|a_n b_n| < \varepsilon$.

$(b_n)_{n\in\mathbb{N}}$ is bounded $\Rightarrow$ $\exists\, M > 0$ s.t.: $\forall n \in \mathbb{N}$, $|b_n| < M$.

Since $a_n \longrightarrow 0$, there exists $n_0 \left(= n_0\left(\frac{\varepsilon}{M}\right)\right) \in \mathbb{N}$, s.t.:

$\forall n \geq n_0$, $|a_n| < \frac{\varepsilon}{M}$. (I apply the definition of limit for the positive number $\frac{\varepsilon}{M}$, i.e. the neighbourhood $(0-\frac{\varepsilon}{M}, 0+\frac{\varepsilon}{M})$ of 0.)

$a-\frac{\varepsilon}{M}$ $\quad a \quad$ $a+\frac{\varepsilon}{M}$

Then, for all $n \geq n_0$ we have:

$|a_n b_n| = |a_n| \cdot |b_n| < \frac{\varepsilon}{\cancel{M}} \cdot \cancel{M} = \varepsilon$, i.e.:

$\forall n \geq n_0, \ |a_n b_n| < \varepsilon.$

Since ε was arbitrary, the proof is complete. ■

Prop.: $\boxed{\text{If } a_n \to a \text{ and } b_n \to b, \text{ then } a_n + b_n \to a+b.}$

Proof: Let $\varepsilon > 0$. I know that:

Idea: If for large n a_n is close to a, and for large n b_n is close to b, then for large n: $a_n + b_n$ is close to $a+b$.

$a_n \to a$, so $\exists n_1 \in \mathbb{N}: \forall n \geq n_1, \ |a_n - a| < \frac{\varepsilon}{2}$,

$a - \frac{\varepsilon}{2} \quad a \quad a + \frac{\varepsilon}{2}$

and

$b_n \to b$, so $\exists n_2 \in \mathbb{N}: \forall n \geq n_2, \ |b_n - b| < \frac{\varepsilon}{2}$.

$b - \frac{\varepsilon}{2} \quad b \quad b + \frac{\varepsilon}{2}$

I define $n_0 := \max\{n_1, n_2\}$; then,

$\forall n \geq n_0$, we simultaneously have $|a_n - a| < \frac{\varepsilon}{2}$ and $|b_n - b| < \frac{\varepsilon}{2}$,

and thus $|(a_n + b_n) - (a+b)| = |(a_n - a) + (b_n - b)| \leq$ triangle inequality

$\leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

So, $\forall n \geq n_0: |(a_n + b_n) - (a+b)| < \varepsilon.$

Since ε was arbitrary, we have $a_n + b_n \to a+b$. ■

→ **Prop:** If $a_n \to a$ and $b_n \to b$, then $a_n \cdot b_n \longrightarrow a \cdot b$.

Proof: This is the first time we won't use the ε-definition of the limit, but simply the algebra of limits we've so far seen. If you want an ε-proof, use this idea together with the first 3 lines of this proof.

Idea: if for large n a_n is close to a, and for large n b_n is close to b, then for large n $a_n b_n$ is close to $a \cdot b$.

I notice that:

$$a_n b_n - ab = a_n \cdot b_n - a b_n + a b_n - ab =$$
$$= b_n \cdot (a_n - a) + a \cdot (b_n - b).$$

Let's look at the sequence $\left(b_n \cdot (a_n - a)\right)_{n \in \mathbb{N}}$. We have:

- $a_n - a \longrightarrow 0$, because $a_n \to a$.
- $(b_n)_{n \in \mathbb{N}}$ is bounded, because it is convergent.

So, $b_n \cdot (a_n - a) \longrightarrow 0$. $(*_1)$

Similarly for $a \cdot (b_n - b)$:

- $b_n - b \to 0$, because $b_n \longrightarrow b$.
- $(a)_{n \in \mathbb{N}}$ is bounded, because it is a constant sequence

So, $a \cdot (b_n - b) \longrightarrow 0$. $(*_2)$

By $(*_1)$ and $(*_2)$, $b_n(a_n - a) + a(b_n - b) \longrightarrow 0$,

i.e. $a_n b_n \longrightarrow a \cdot b$. ∎

Corollary: $\boxed{\text{If } a_n \to a \text{ as } n \to +\infty \text{ and } k \in \mathbb{N}, \text{ then } a_n^k \to a^k \text{ as } n \to +\infty}$

Proof: By the previous proposition:

$$\underbrace{a_n^2}_{= a_n \cdot a_n} \to \underbrace{a^2}_{= a \cdot a}, \quad \underbrace{a_n^2 \cdot a_n}_{= a_n^3} \to \underbrace{a^2 \cdot a}_{= a^3}, \text{ etc.}$$

ex: $\frac{1}{n^2} \to 0$, $\frac{1}{n^3} \to 0$, $\frac{1}{n^{10}} \to 0$.

Prop: $\boxed{\text{If } b_n \neq 0 \ \forall n \in \mathbb{N} \text{ and } b_n \to b, \text{ then } \frac{1}{b_n} \to \frac{1}{b}.}$

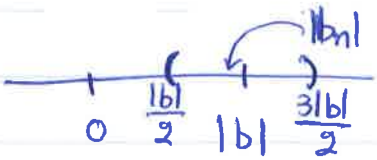
Proof: **Idea:** We notice that $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b - b_n|}{|b_n b|} =$

$= \frac{|b_n - b|}{|b_n| \cdot |b|}$. If I show that this quantity is small for large n, I am done. I notice that if the denominator is larger than some constant, then the fraction is at most $|b_n - b|$ (times a constant), which is small. So, I just need to bound the denominator from below.

Let $\varepsilon > 0$. I want to show that there exists some $n_0 \in \mathbb{N}$,

$$\text{s.t.: } \forall n \geq n_0, \underbrace{\left| \frac{1}{b_n} - \frac{1}{b} \right|}_{= \frac{|b_n - b|}{|b_n| \cdot |b|}} < \varepsilon$$

I know that $b_n \longrightarrow b$; so, $|b_n| \longrightarrow |b|$. By the



definition of limit for the neighbourhood $\left(\frac{|b|}{2}, \frac{3|b|}{2} \right)$ of $|b|$

(i.e., for $\varepsilon' = \frac{|b|}{2}$),

there exists some $n_1 \in \mathbb{N}$ s.t.: $\forall n \geq n_1$, $|b_n| \in \left(\frac{|b|}{2}, \frac{3|b|}{2} \right)$.

In particular: $|b_n| > \frac{|b|}{2}$, $\forall n \geq n_1$.

$(*_1)$

$$\boxed{\text{So, } \forall n \geq n_1: \left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b_n - b|}{|b_n| \cdot |b|} < \frac{|b_n - b|}{\frac{|b|}{2} \cdot |b|} = \frac{2}{|b|^2} \cdot |b_n - b|}$$

For the $\varepsilon > 0$ I originally picked, $\exists\, n_2 \in \mathbb{N}$ s.t.:

$$\boxed{\forall n \geq n_2, \quad |b_n - b| < \frac{\varepsilon \cdot |b|^2}{2}} \quad (*_2)$$

I now combine (*1) and (*2):

$$\left.\begin{aligned} &\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2}{|b|^2}\cdot|b_n - b|, \quad \forall n \geq n_1, \\ \text{and} \quad &|b_n - b| < \frac{\varepsilon\cdot|b|^2}{2}, \quad \forall n \geq n_2 \end{aligned}\right\} \Longrightarrow$$

$\Longrightarrow$ for $n \geq n_0 := \max\{n_1, n_2\}$, we simultaneously have that $\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{2}{|b|^2}|b_n - b|$ and $|b_n - b| < \frac{\varepsilon|b|^2}{2}$, and thus

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| < \frac{\cancel{2}}{\cancel{|b|^2}} \cdot \frac{\varepsilon \cdot \cancel{|b|^2}}{\cancel{2}} = \varepsilon, \quad \forall n \geq n_0.$$

Since ε was arbitrary, $\frac{1}{b_n} \longrightarrow \frac{1}{b}$. ■

Corollary: If $b_n \neq 0 \ \forall n \in \mathbb{N}$, $a_n \longrightarrow a$ and $b_n \longrightarrow b$, then $\frac{a_n}{b_n} \longrightarrow \frac{a}{b}$.

Proof:

$$\frac{a_n}{b_n} = \underbrace{\boxed{a_n}}_{\downarrow\, a} \cdot \underbrace{\boxed{\frac{1}{b_n}}}_{\downarrow\, \frac{1}{b}} \longrightarrow a \cdot \frac{1}{b} = \frac{a}{b}.$$ ■

this is just another notation for $a^{1/k}$, the unique positive real x with $x^k = a$ (see Lecture 3)

→ **Prop.:** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence with non-negative terms, and $k \geq 2$.

If $a_n \longrightarrow a$, then $\sqrt[k]{a_n} \longrightarrow \sqrt[k]{a}$ as $n \longrightarrow +\infty$.

Imagine k to be fixed here (such as $2, 3, 10 \ldots$).

which will be ≥ 0 since $a_n \geq 0 \ \forall n \in \mathbb{N}$

Proof: Idea: We want to show that $|\sqrt[k]{a_n} - \sqrt[k]{a}|$ is small for large n. How can I relate $|\sqrt[k]{a_n} - \sqrt[k]{a}|$ to the quantity $|a_n - a|$, which I know is small? I will use the identity

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + x^{k-3}y^2 + \ldots + xy^{k-2} + y^{k-1}),$$

which implies that

$$a_n - a = (\sqrt[k]{a_n} - \sqrt[k]{a}) \cdot \left(\sqrt[k]{a_n}^{\,k-1} + \sqrt[k]{a_n}^{\,k-2}\sqrt[k]{a} + \ldots + \sqrt[k]{a_n}\,\sqrt[k]{a}^{\,k-2} + \sqrt[k]{a}^{\,k-1} \right)$$

Case 1: $a = 0$. I.e., $a_n \longrightarrow 0$, and we want to show that $\sqrt[k]{a_n} \longrightarrow 0$. Let $\varepsilon > 0$.

Since $a_n \to 0$, by the definition of limit for $\varepsilon' = \varepsilon^k$,

there exists $n_0 \in \mathbb{N}$ s.t.: $\forall n \geq n_0$, $|a_n| < \varepsilon^k \Leftrightarrow \sqrt[k]{|a_n|} < \varepsilon \Leftrightarrow$

show this! (by contradiction)

$\Leftrightarrow \left|\sqrt[k]{a_n}\right| < \varepsilon$. Since ε was arbitrary, $\sqrt[k]{a_n} \to 0$.

Case 2: $a>0$.

We have:

$$a_n - a = \left(\sqrt[k]{a_n} - \sqrt[k]{a}\right)\cdot\left(\sqrt[k]{a_n}^{\,k-1} + \sqrt[k]{a_n}^{\,k-2}\sqrt[k]{a} + \dots + \sqrt[k]{a_n}\,\sqrt[k]{a}^{\,k-2} + \sqrt[k]{a}^{\,k-1}\right),$$

so

$$\left|\sqrt[k]{a_n} - \sqrt[k]{a}\right| = \frac{|a_n - a|}{\underbrace{\Big(\underbrace{\sqrt[k]{a_n}^{\,k-1} + \sqrt[k]{a_n}^{\,k-2}\sqrt[k]{a} + \dots + \sqrt[k]{a_n}\,\sqrt[k]{a}^{\,k-2}}_{\geq 0} + \underbrace{\sqrt[k]{a}^{\,k-1}}_{>0}\Big)}_{\geq \sqrt[k]{a}^{\,k-1}}},$$

thus

$$\underset{\substack{\downarrow n\to+\infty \\ 0}}{0} \leq \left|\sqrt[k]{a_n} - \sqrt[k]{a}\right| \leq \underset{\substack{\downarrow n\to+\infty \\ 0}}{\frac{|a_n - a|}{\sqrt[k]{a}^{\,k-1}}}, \quad \forall n \in \mathbb{N}.$$

$\sqrt[k]{a}^{\,k-1}$ → a constant.

By the sandwich lemma, $\left|\sqrt[k]{a_n} - \sqrt[k]{a}\right| \to 0 \Leftrightarrow \sqrt[k]{a_n} \to \sqrt[k]{a}$.

We can use the ε-definition of limit instead.

$\blacksquare$

ex: $\frac{1}{\sqrt{n}} \to 0$, $\left(\frac{1}{n}\right)^{\frac{1}{10}} \to 0$.

Prop.: $\boxed{\text{If } a>0, \text{ then } \sqrt[n]{a} \longrightarrow 1 \text{ as } n \to +\infty.}$

Proof: We want: $\sqrt[n]{a} \longrightarrow 1 \iff \underbrace{\sqrt[n]{a}-1}_{= \vartheta_n} \longrightarrow 0.$

Case 1: $a \geq 1$.

In this case; $(*_1)$ $\boxed{\vartheta_n \geq 0, \ \forall n \in \mathbb{N}}$ (as $a \geq 1 \Rightarrow \sqrt[n]{a} \geq 1, \forall n \in \mathbb{N}$). show this (hint: contradiction otherwise).

So, we can apply Bernoulli's inequality for ϑ_n:

$$\underbrace{(1+\vartheta_n)^n}_{= \sqrt[n]{a}^{\,n} = a} \geq 1 + n \cdot \vartheta_n, \quad \forall n \in \mathbb{N},$$

⚠ Notice that it would have sufficed to just use that $(1+\vartheta_n)^n \geq n \cdot \vartheta_n$, which comes from Bernoulli's inequality (weaker).

i.e. $n\vartheta_n \leq a-1 \iff \boxed{\vartheta_n \leq \frac{a-1}{n}, \ \forall n \in \mathbb{N}}$ $(*_2)$

It is for this to hold that we required $a \geq 1$; Bernoulli's inequality holds for $a \geq -1$.

By $(*_1)$, $(*_2)$, we have $0 \leq \vartheta_n \leq \frac{a-1}{n}, \ \forall n \in \mathbb{N}$.

$$0 \xrightarrow{n \to +\infty} 0, \qquad \frac{a-1}{n} \xrightarrow{n \to +\infty} 0$$

By the sandwich lemma, $\vartheta_n \longrightarrow 0 \iff \sqrt[n]{a} \longrightarrow 1$.

Case 2: $a < 1$. In this case, $\frac{1}{a} > 1 \xRightarrow{\text{by Case 1}} \sqrt[n]{\frac{1}{a}} \longrightarrow 1 \Rightarrow \sqrt[n]{a} \longrightarrow 1$. ∎

$\left(\sqrt[n]{\frac{1}{a}} = \frac{1}{\sqrt[n]{a}}\right)$

- ex: $\left(\frac{1}{2}\right)^{1/n} \to 1$, $3^{1/n} \to 1$, $1000^{1/n} \to 1$.

→ $\boxed{\sqrt[n]{n} \longrightarrow 1 \quad \text{as} \quad n \to +\infty.}$

Proof: Let's try to work as for $\sqrt[n]{a}$:

We want: $\sqrt[n]{n} \to 1 \iff \underbrace{\sqrt[n]{n} - 1}_{= \vartheta_n} \to 0$.

We have (*) $\boxed{\vartheta_n \geq 0, \ \forall n \in \mathbb{N}}$ (as $n \geq 1 \Rightarrow \sqrt[n]{n} \geq 1$).

Thoughts...

However, if we apply Bernoulli's inequality for ϑ_n, then we get $\underbrace{(1+\vartheta_n)^n}_{= \sqrt[n]{n}^{\,n} = n} \geq 1 + n \cdot \vartheta_n$, i.e.

$$\vartheta_n \leq \frac{n-1}{n} = \frac{n(1-\frac{1}{n})}{n} = 1 - \frac{1}{n} \xrightarrow[n \to +\infty]{} 1 \, \boxed{\neq 0}$$

So, Bernoulli's inequality doesn't provide an upper bound for $(\vartheta_n)_{n \in \mathbb{N}}$ good enough for the sandwich lemma to work; we need an even better upper bound for $(\vartheta_n)_{n \in \mathbb{N}}$, which will actually converge to 0. We get this from the binomial expansion of $(1+\vartheta_n)^n$ (which, notice, implies also Bernoulli's inequality for $\vartheta_n \geq 0$ (rather than ≥ -1)):

We have: $\forall n \in \mathbb{N},\ n \geq 2$:

$$\underbrace{(1+\vartheta_n)^n}_{=n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \vartheta_n^k$$

$$= \underbrace{1 + n\vartheta_n}_{\substack{\geq 0, \\ \text{as } \vartheta_n \geq 0\ \forall n \in \mathbb{N}}} + \underbrace{\binom{n}{2}}_{=\frac{n\cdot(n-1)}{2}} \cdot \vartheta_n^2 + \underbrace{\binom{n}{3}\cdot\vartheta_n^3 + \dots + \binom{n}{n-1}\vartheta_n^{n-1} + \binom{n}{n}\vartheta_n^n}_{\geq 0,\ \text{as } \vartheta_n \geq 0\ \forall n \in \mathbb{N}.}$$

So, $$\underbrace{(1+\vartheta_n)^n}_{= \sqrt[n]{n}^{\,n} = n} \geq \binom{n}{2}\vartheta_n^2 = \frac{n\cdot(n-1)}{2}\cdot\vartheta_n^2$$

$$\longrightarrow \frac{n\cdot(n-1)}{2}\cdot\vartheta_n^2 \leq n \Longleftrightarrow \vartheta_n^2 \leq \frac{2\cancel{n}}{\cancel{n}(n-1)} = \frac{2}{n-1}$$

$$\xleftrightarrow[\vartheta_n \geq 0\ \forall n \in \mathbb{N}]{} \boxed{\vartheta_n \leq \sqrt{\frac{2}{n-1}},\ \forall n \in \mathbb{N},\ n \geq 2} \quad (*_2).$$

By $(*_1)$ and $(*_2)$, we have

$$0 \leq \vartheta_n \leq \sqrt{\frac{2}{n-1}},\quad \forall n \in \mathbb{N},\ n \geq 2.$$

$$0 \xrightarrow{n\to+\infty} 0, \qquad \sqrt{\frac{2}{n-1}} \xrightarrow{n\to+\infty} 0$$

This is by the algebra of limits we have proved so far:
$\frac{1}{n} \longrightarrow 0 \Rightarrow \frac{1}{n-1} \longrightarrow 0 \Rightarrow$
$\Rightarrow \frac{2}{n-1} \longrightarrow 2\cdot 0 = 0 \longrightarrow \sqrt{\frac{2}{n-1}} \longrightarrow 0.$

By the sandwich lemma, $\vartheta_n \longrightarrow 0$. So, $\sqrt[n]{n} \longrightarrow 1$. ∎

⚠ Observe that, in the proof of $\sqrt[n]{n} \to 1$,

the problem that made things harder than for $\sqrt[n]{a} \to 1$

is that $(1+\vartheta_n)^n = n$, rather than a constant.

So, $1+n\vartheta_n$, which is also linear in n, cannot help us. We need to use something like $\underbrace{(1+\vartheta_n)^n}_{= n} \geq n^k \cdot \vartheta_n^k$, for some fixed $\boxed{k > 1}$ (or something like that),

so that n^k grows faster than n

to demonstrate how truly small ϑ_n is (indeed, notice that the above implies that
$$\vartheta_n^k \leq \frac{n}{n^k} = \frac{1}{n^{k-1}} \xrightarrow[n\to+\infty]{} 0 \quad, \text{ as } \underline{k>1})$$

That's why we chose $\binom{n}{2}\vartheta_n^2$ as the appropriate lower bound for $(1+\vartheta_n)^n = n$; $\binom{n}{2} = \frac{n\cdot(n-1)}{2}$,

which should "behave like" n^2 for n large; whatever that means.

Notice that, instead of $\binom{n}{2}\vartheta_n^2$, we could have used $\binom{n}{3}\vartheta_n^3$, or $\binom{n}{4}\vartheta_n^4$, ..., or $\binom{n}{k}\vartheta_n^k$, for <u>explicit</u> $k\in\mathbb{N}$ <u>independent of n</u>. (of course with $k<n$).