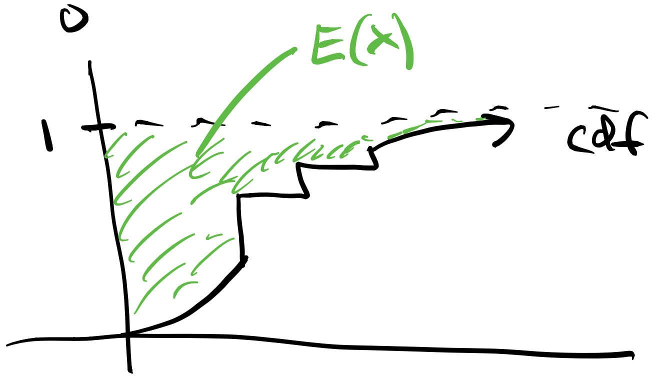
Stat 134 lec 26

Expectation of cdf

if $X > 0$

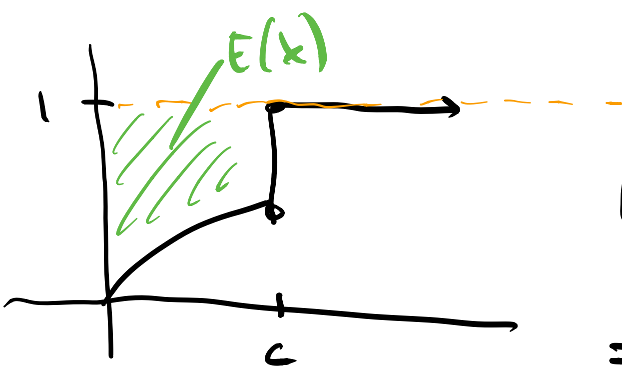
$$E(X) = \int_0^\infty (1 - F(x))\,dx$$

Picture



(see 4.5.9 how to generalize for any X)

ex $X = \min(T, c)$ $T \sim \text{expon}(\lambda)$, $c > 0$



Find $E(X)$

$$E(X) = \int_0^c 1 - (1 - e^{-\lambda x})\,dx$$

$$= \int_0^c e^{-\lambda x}\,dx$$

$$= -\frac{1}{\lambda} e^{-\lambda x} \Big|_0^c$$

$$= \boxed{\frac{1}{\lambda}(1 - e^{-\lambda c})}$$

$\underline{\underline{\text{or}}}$ $\quad X = \min(T, c)$, $T \sim \text{expon}(\lambda)$

Calculation of $E(X)$ using density (brace yourself)

$$E(X) = E(\min(T,c)) = \int_0^\infty \min(T,c) f_T(t)\,dt$$

function of T

$$= \int_0^\infty \min(T,c)\lambda e^{-\lambda t}\,dt$$

$$= \int_0^c \min(T,c)\lambda e^{-\lambda t}\,dt + \int_c^\infty \min(T,c)\lambda e^{-\lambda t}\,dt$$

$$= \int_0^c t\lambda e^{-\lambda t}\,dt + \int_c^\infty c\lambda e^{-\lambda t}\,dt$$

Integration by parts

$$= \lambda\left[-t\frac{1}{\lambda}e^{-\lambda t} - \frac{1}{\lambda^2}e^{-\lambda t}\right]_0^c + c\,e^{-c\lambda}$$

survival function

$$= \left(-t-\frac{1}{t}\right)e^{-\lambda t}\Big|_0^c + c e^{-c\lambda}$$

$$= \left(-c-\frac{1}{\lambda}\right)e^{-c\lambda} + \frac{1}{\lambda} + c e^{-c\lambda}$$

$$= \boxed{\frac{1}{\lambda}\left(1-e^{-c\lambda}\right)}$$

Stat 134

Friday March 16 2018

1. X is a RV with cdf

$$F(x) = \begin{cases} 1/4 & 0 \le x < 1 \\ 3/4 & 1 \le x \cancel{<} 2 \\ 1 & x \ge 2 \end{cases}$$

$P(X=0)=\frac{1}{4}$
$P(X=1)=1/2$
$P(X=2)=1/4$
$X \sim Bin(2,\frac{1}{2})$
$E(X)=2\cdot 1/2=1$

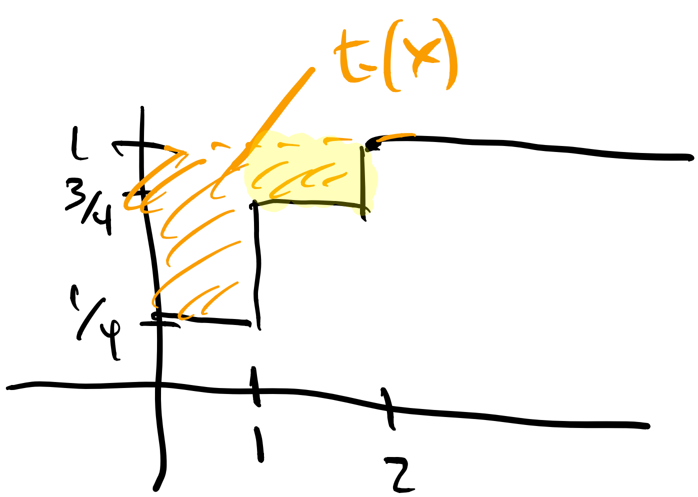
$E(X)$ equals

a $\frac{1}{2}$

b 1

c $\frac{3}{2}$

d none of the above



$\frac{3}{4}+\frac{1}{4}=1$

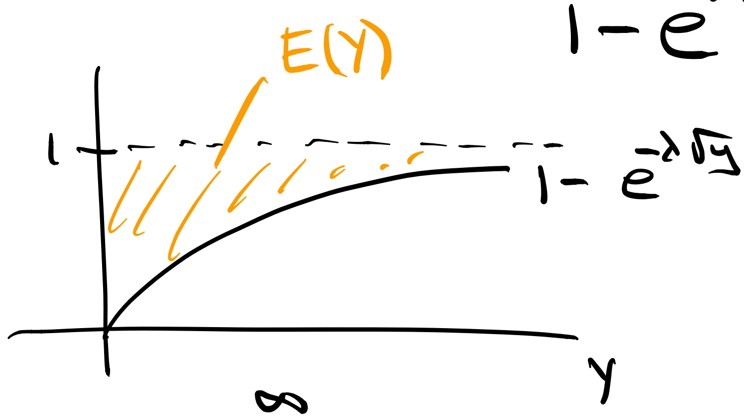
ex $T \sim \text{expon}(\lambda)$

Find $Var(T) = E(T^2) - E(T)^2$

idea use cdf to find $E(T^2)$

$Y = T^2$ —— $(0, \infty)$

$$F_Y(y) = P(T^2 \le y) = P(T \le \sqrt{y}) = 1 - e^{-\lambda\sqrt{y}}$$



$$E(T^2) = \int_0^\infty e^{-\lambda\sqrt{y}}\, dy$$

subst method

$t = \sqrt{y}$

$dt = \frac{1}{2\sqrt{y}}\, dy \Rightarrow dy = 2\sqrt{y}\, dt = 2t\, dt$

$$= \int_0^\infty e^{-\lambda t} \cdot 2t\, dt = 2\int_0^\infty t e^{-\lambda t}\, dt$$

we will now use:

Principle of ignoring constants

(see 4.R.8)

If $f(x)$ is a density

and $f(x) = c\,h(x)$ for constant c

then $\int_{-\infty}^{\infty} h(x)\,dx = \frac{1}{c}$

to use this in our example:

recall $X \sim \text{Gamma}(r, \lambda)$

$$f_X(x) = \underbrace{\frac{\lambda^r}{\Gamma(r)}}_{= c} \underbrace{x^{r-1} e^{-\lambda x}}_{\text{variable part } h(x)}$$

$$\int_0^{\infty} x^{r-1} e^{-\lambda x} = \frac{\Gamma(r)}{\lambda^r}$$

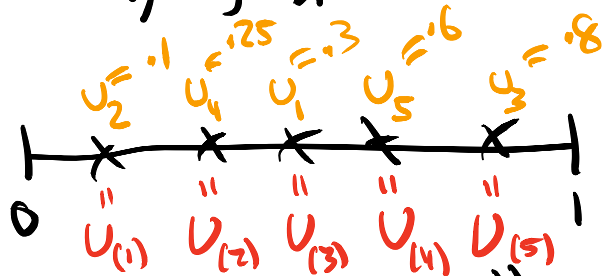
For us $r = 2$
$\lambda = \lambda$

$$E(T^2) = 2\int_0^{\infty} t e^{-\lambda t}\,dt = 2\frac{\Gamma(2)}{\lambda^2} = \frac{2 \cdot 1}{\lambda^2}$$

$$\text{var}(T) = E(T^2) - E(T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Sec 4.6 Order statistics of Unif (0,1)

let $U_1, \dots, U_n \overset{iid}{\sim} \text{Unif}(0,1)$



let $U_{(k)}$ = called the k^{th} order statistic

= k^{th} from the bottom (smallest)

of $\{U_1, U_2, \dots, U_n\}$

We will assume no ties.

$$U_{(1)} = \min(U_1, \dots, U_n)$$

$$U_{(n)} = \max(U_1, \dots, U_n)$$

$$F_{U_{(n)}}(x) = P(U_{(n)} \le x) = P(U_1 \le x, U_2 \le x, \dots, U_n \le x)$$

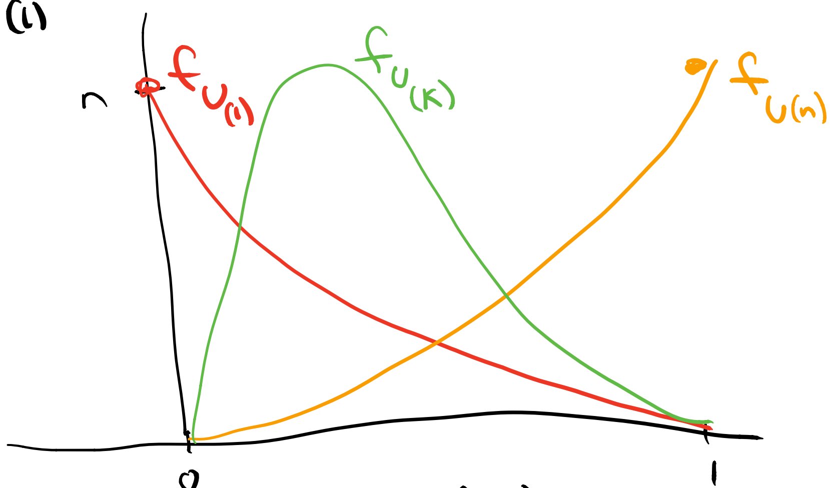
$$= (P(U_1 \le x))^n = (F_{U_1}(x))^n = x^n \quad 0 \le x \le 1$$

$$\boxed{f_{U_{(n)}}(x) = n x^{n-1}, \quad 0 \le x \le 1}$$

Similarly

$$F_{U_{(1)}}(x) = P(U_{(1)} \le x) = 1 - (1-x)^n, \quad 0 \le x \le 1$$

$$f_{U_{(1)}}(x) = n(1-x)^{n-1}, \quad 0 \le x \le 1$$



Order statistic of $U(0,1)$ provides a family of densities on the unit interval.

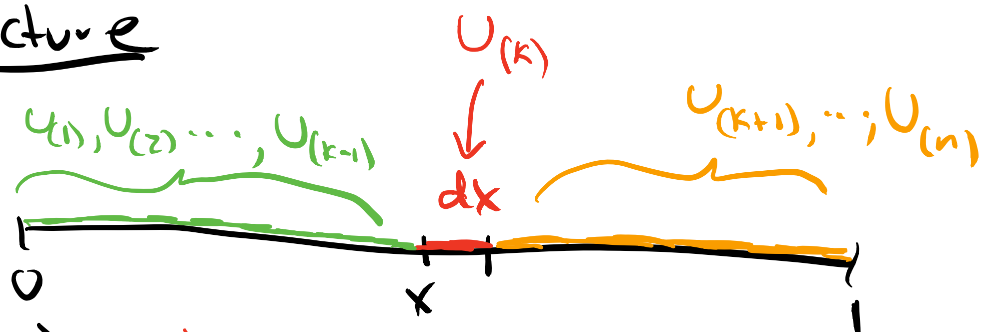
Find density of $U_{(k)}$:

Strategy:

write $P(U_{(k)} \in dx) = f(x)\,dx$

$U_{(k)} \in dx$ means that one of the $U_1, U_2, \ldots, U_n$ are in dx and $k-1$ of them are between 0 and x, $n-k$ of them are between x and 1

Picture



$$P(U_{(k)} \in dx) = \binom{n}{1} 1\,dx \binom{n-1}{k-1} x^{k-1} \binom{n-k}{n-k} (1-x)^{n-k}$$

$$= \underbrace{\frac{n(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{(n-k+1)-1}}_{f_{U_{(k)}}(x)} dx$$

$$\Rightarrow \boxed{f_{U_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{(n-k+1)-1}}$$