Math 104 - Weekly Assignment 2, Solutions.

(1) Let $A := \{x \in \mathbb{R}\setminus\mathbb{Q} : a < x < b\}$.

We have shown that $A \neq \emptyset$. So, A finite or infinite.

Suppose that A is finite, then $A = \{x_1, \dots, x_N\}$,

for some $N \in \mathbb{N}$, with $x_1 < x_2 < \dots < x_N$.

We know that $\exists\, x_{N+1}$ irrational with $x_N < x_{N+1} < b$,

so $x_{N+1} \in A$ and x_{N+1} is larger than the largest element of A, contradiction. So, A is infinite.

(2) Let's show: $\big| |a| - |b| \big| \leq |a+b|$.

This is equivalent to $-|a+b| \leq |a| - |b| \leq |a+b|$

$$\Leftrightarrow \left\{\begin{array}{l} |a|-|b| \leq |a+b| \\ \text{and } |a|-|b| \geq -|a+b| \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} |a| \leq |a+b| + |b| \\ \text{and } |b| \leq |a+b| + |a|. \end{array}\right.$$

Now, $a = a+b + (-b) \Longrightarrow |a| \underset{\substack{\text{triangle}\\ \text{ineq.}}}{\leq} |a+b| + |-b| = |a+b| + |b|$,

and $b = a+b + (-a) \Longrightarrow |b| \leq |a+b| + |-a| = |a+b| + |a|$. ∎

Let's show: $\big||a|-|b|\big| \leq |a-b|$:

One can work as above, or just apply what we just proved, for the real numbers $a, -b$:

$$\big||a|-\underbrace{|-b|}_{=|b|}\big| \leq |\underbrace{a+(-b)}_{=a-b}| \text{ , i.e. } \big||a|-|b|\big| \leq |a-b| .$$

3 Since $p(\lambda)$ is a sum of squares, we have
$p(\lambda) \geq 0, \ \forall \lambda \in \mathbb{R}$.

Also, $p(\lambda) = a_1^2 + 2\lambda a_1 b_1 + \lambda^2 b_1^2 + \dots + a_n^2 + 2\lambda a_n b_n + \lambda^2 b_n^2 =$

$$= (a_1^2 + \dots + a_n^2) + \lambda \cdot (2a_1b_1 + \dots + 2a_nb_n) + \lambda^2 \cdot (b_1^2 + \dots + b_n^2) =$$

$$= \lambda^2 \cdot (b_1^2 + \dots + b_n^2) + \lambda \cdot \big(2 \cdot (a_1b_1 + \dots + a_nb_n)\big) + (a_1^2 + \dots + a_n^2).$$

This is a degree 2 polynomial in λ, with discriminant $\Delta = [2(a_1b_1 + \dots + a_nb_n)]^2 -$
$- 4 \cdot (b_1^2 + \dots + b_n^2) \cdot (a_1^2 + \dots + a_n^2)$.

And, since $p(\lambda) \geq 0 \ \forall \lambda \in \mathbb{R}$, we have $\Delta \leq 0$, so

$$\underbrace{\left[2\left(a_1 b_1 + \dots + a_n b_n\right)\right]^2}_{\substack{\shortparallel \\ 4\cdot\left(a_1 b_1 + \dots + a_n b_n\right)^2}} \leq 4\left(b_1^2 + \dots + b_n^2\right)\cdot\left(a_1^2 + \dots + a_n^2\right) \Leftrightarrow$$

$$\Leftrightarrow \left(a_1 b_1 + \dots + a_n b_n\right)^2 \leq \left(b_1^2 + \dots + b_n^2\right)\left(a_1^2 + \dots + a_n^2\right).$$

If you prefer proof by induction:

Obvious for $n = 1, 2$.

Suppose you know it for $n = m$, you want it for $n = m+1$:

$$\left|a_1 b_1 + \dots + a_m b_m + a_{m+1} b_{m+1}\right| \underset{\substack{\text{triangle} \\ \text{ineq.}}}{\leq} \underbrace{\left|a_1 b_1 + \dots + a_m b_m\right|}_{\substack{\text{C-S} \\ \text{for this}}} + \left|a_{m+1} b_{m+1}\right|$$

$$\leq \underbrace{\sqrt{a_1^2 + \dots + a_m^2}} \cdot \underbrace{\sqrt{b_1^2 + \dots + b_m^2}} + \underbrace{\left|a_{m+1}\right|} \cdot \underbrace{\left|b_{m+1}\right|}$$

$$\underset{\substack{\text{Cauchy-Schwarz} \\ \text{for } n=2}}{\leq} \left(\sqrt{a_1^2 + \dots + a_m^2}^{\,2} + \left|a_{m+1}\right|^2\right)^{1/2} \cdot \left(\sqrt{b_1^2 + \dots + b_m^2}^{\,2} + \left|b_{m+1}\right|^2\right)^{1/2}$$

$$= \left(a_1^2 + \dots + a_{m+1}^2\right)^{1/2} \cdot \left(b_1^2 + \dots + b_{m+1}^2\right)^{1/2}.$$

4

Let $x_1, \ldots, x_n > 0 \longrightarrow \frac{1}{x_1}, \ldots, \frac{1}{x_n} > 0$, so

we can apply the geometric-arithmetic mean inequality for $\frac{1}{x_1}, \ldots, \frac{1}{x_n}$:

$$\left(\frac{1}{x_1} \cdot \ldots \cdot \frac{1}{x_n}\right)^{\frac{1}{n}} \leq \frac{\frac{1}{x_1} + \ldots + \frac{1}{x_n}}{n} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}} \leq \frac{1}{\left(\frac{1}{x_1} \cdot \ldots \cdot \frac{1}{x_n}\right)^{1/n}} = \frac{1}{\frac{1}{x_1^{1/n}} \cdot \ldots \cdot \frac{1}{x_n^{1/n}}} =$$

$$= \frac{1}{\frac{1}{x_1^{1/n} \cdot \ldots \cdot x_n^{1/n}}} = x_1^{1/n} \cdot \ldots \cdot x_n^{1/n}.$$

5 Let $(a_n)_{n\in\mathbb{N}}$ be bounded. This means that

$\exists\, b, c \in \mathbb{R}$, with $b \leq a_n \leq c$, $\forall n \in \mathbb{N}$.

Let $A := \max\{|b|, |c|\}$. Then,

$|a_n| \leq A$, $\forall n \in \mathbb{N}$: Indeed, $|b| \leq A \Longleftrightarrow -A \leq b \leq A$,
$|c| \leq A \Longleftrightarrow -A \leq c \leq A$,

so $-A \leq b \leq a_n \leq c \leq A$, $\forall n \in \mathbb{N} \longrightarrow -A \leq a_n \leq A$, i.e.

$|a_n| \leq A$, $\forall n \in \mathbb{N}$.

If $A > 0$, then we let $M := A$.

If $A = 0$ ($\Leftrightarrow b = c = 0$), then $a_n = 0 \ \forall n \in \mathbb{N}$, so $M = 1$ will do.

6 (i) Let $(a_m, a_{m+1}, \ldots)$ be a final part of $(a_n)_{n\in\mathbb{N}}$.

Let $\varepsilon > 0$. Since $a_n \to a$, there exists some

$n_1 \in \mathbb{N}$ s.t. $\forall n \geq n_1$, $a_n \in (a-\varepsilon, a+\varepsilon)$.

Let $n_0 = \max\{n_1, m\}$; then, $a_n \in (a-\varepsilon, a+\varepsilon)$, $\forall n \geq n_0$,

i.e. all the terms of the sequence $(a_{n_0}, a_{n_0+1}, \ldots)$ are in $(a-\varepsilon, a+\varepsilon)$.

But $(a_{n_0}, a_{n_0+1}, \ldots)$ is a final part of $(a_m, a_{m+1}, \ldots)$ (since $n_0 \geq m$).

So, we have shown that there exists a final part of $(a_m, a_{m+1}, \ldots)$ inside $(a-\varepsilon, a+\varepsilon)$. Since ε was arbitrary, $(a_m, a_{m+1}, \ldots)$ converges to a.

(ii) Let $\varepsilon > 0$. Since $(a_m, a_{m+1}, \ldots)$ converges to a, there exists some final part $(a_{n_0}, a_{n_0+1}, \ldots)$ of $(a_m, a_{m+1}, \ldots)$ (i.e. some $n_0 \in \mathbb{N}$), s.t. all the terms of $(a_{n_0}, a_{n_0+1}, \ldots)$ are in $(a-\varepsilon, a+\varepsilon)$

(i.e, $a_n \in (a-\varepsilon, a+\varepsilon), \forall n \geq n_0$)

Since ε was arbitrary, $a_n \xrightarrow[n \to +\infty]{} a$.

(iii) (a) $(a_{n+3})_{n\in\mathbb{N}}$ is the sequence $(a_4, a_5, a_6, \ldots)$, which is a final part of $(a_n)_{n\in\mathbb{N}}$.

By (i) and (ii), $a_n \to a \Leftrightarrow a_{n+3} \to a$.

(b)

Consider the sequences

$$(a_{n+n_0-1})_{n\in\mathbb{N}}, \quad (b_{n+n_0-1})_{n\in\mathbb{N}}, \quad (c_{n+n_0-1})_{n\in\mathbb{N}}$$
$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$
$$(a_{n_0}, a_{n_0+1}, \ldots) \quad (b_{n_0}, b_{n_0+1}, \ldots) \quad (c_{n_0}, c_{n_0+1}, \ldots)$$

These are final parts of $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$, respectively. So, they all converge to ℓ (by (i)).

And: $a_n \leq b_n \leq c_n \quad \forall n \geq n_0$

$$\Longrightarrow \underbrace{a_{n+n_0-1}}_{\substack{\downarrow n\to+\infty \\ \ell}} \leq b_{n+n_0-1} \leq \underbrace{c_{n+n_0-1}}_{\substack{\downarrow n\to+\infty \\ \ell}} \quad \forall n \in \mathbb{N}.$$

By the sandwich lemma, $b_{n+n_0-1} \xrightarrow[n\to+\infty]{} \ell$.

By (ii), also $b_n \to \ell$ as $n\to+\infty$.

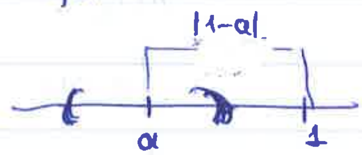
7 - If $x \in \mathbb{R}\backslash\mathbb{Q}$: the sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = x + \frac{1}{n}$ $\forall n\in\mathbb{N}$ is a sequence of irrational numbers converging to x:

$$\left.\begin{aligned} \frac{1}{n} &\xrightarrow[n\to+\infty]{} 0 \\ \text{and} \quad x &\xrightarrow[n\to+\infty]{} x \end{aligned}\right\} \longrightarrow \quad x + \frac{1}{n} \xrightarrow[n\to+\infty]{} x + 0 = x.$$

- If $x \in \mathbb{Q}$: the sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n = x + \frac{\sqrt{2}}{n}$ $\forall n\in\mathbb{N}$ is a sequence of <u>irrational</u> numbers converging to x:

$$x+\frac{\sqrt{2}}{n} = \underbrace{x}_{\downarrow \, x} + \underbrace{\sqrt{2}}_{\text{bounded}}\cdot \underbrace{\frac{1}{n}}_{\downarrow \, 0} \xrightarrow[n\to+\infty]{} x+\sqrt{2}\cdot 0 = x .$$

(8) Suppose that $(-1)^n \xrightarrow[n\to+\infty]{} a$, for some $a\in\mathbb{R}$



Suppose that $a \neq 1$.
Then, consider the neighbourhood $\left(a-\frac{|1-a|}{2}, a+\frac{|1-a|}{2}\right)$ of a

$$\left((a-\varepsilon, a+\varepsilon) \text{ for } \varepsilon = \frac{|1-a|}{2} > 0 \right).$$

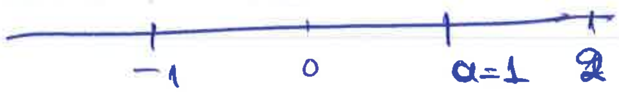
Then, $1 \notin \left(a-\frac{|1-a|}{2}, a+\frac{|1-a|}{2}\right)$,

(otherwise $|1-a| < \frac{|1-a|}{2}$),

<u>so there doesn't exist any final part of $((-1)^n)_{n\in\mathbb{N}}$ that fully lies in $\left(a-\frac{|1-a|}{2}, a+\frac{|1-a|}{2}\right)$</u>

(as all final parts contain 1 as a term).
This contradicts the definition of limit,
so $a=1$.

Now, consider a neighbourhood around $a(=1)$ that doesn't contain -1;



for instance, $(0,2)$.

There doesn't exist any final part of $((-1)^n)_{n\in\mathbb{N}}$ fully inside $(0,2)$ (as all final parts have -1 as a term),

so we have a contradiction to the limit being 1.

So, we ended up to:

if $(-1)^n \xrightarrow[n\to+\infty]{} a \in \mathbb{R}$, then $a=1$, which is eventually a contradiction.

So, $((-1)^n)_{n\in\mathbb{N}}$ doesn't converge in $\mathbb{R}$.

9 (i) False : $\frac{\sqrt{2}}{n}$ irrational $\forall n\in\mathbb{N}$,

but $\frac{\sqrt{2}}{n} \xrightarrow[n\to+\infty]{} 0$, a rational.

(ii) False : $((-1)^n)_{n\in\mathbb{N}}$ bounded, but it doesn't converge.

(iii) True: • $a_n \longrightarrow 0 \Longrightarrow (a_n)_{n\in\mathbb{N}}$ bounded, so

$\exists\, M_1>0$ s.t. $|a_n| \le M_1\ \forall n\in\mathbb{N}$.

• $(b_n)_{n\in\mathbb{N}}$ bounded $\Longrightarrow \exists\, M_2>0$ s.t.

$|b_n| \le M_2,\ \forall n\in\mathbb{N}$.

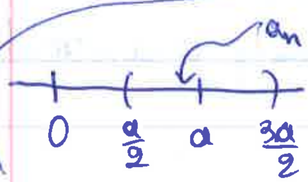
$\Longrightarrow$

$\Longrightarrow |a_n\cdot b_n| = |a_n|\cdot|b_n| \le M_1\cdot M_2\ \forall n\in\mathbb{N}$,

so $(a_n b_n)_{n\in\mathbb{N}}$ bounded.

(iv) True: Idea: $a>0$, so the neighbourhood

$$\left(\underbrace{a-\frac{a}{2}}_{>0},\ a+\frac{a}{2}\right) = \left(\frac{a}{2}, \frac{3a}{2}\right)$$



of a doesn't contain 0.

Since $a_n \longrightarrow a$, $\exists\, n_0\in\mathbb{N}$ s.t.

$\forall n \ge n_0$, $a_n \in \left(\frac{a}{2}, \frac{3a}{2}\right)$.

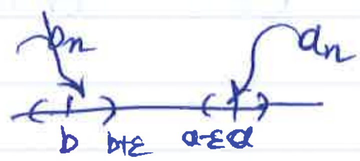
So, $a_n > \frac{a}{2} > 0$, $\forall n \ge n_0$.

(v) False: If $a>0$, this is true (iv).
But if $a=0$, it can be false: For instance,

$$\frac{(-1)^n}{n} = \underbrace{(-1)^n}_{\text{bounded}} \cdot \underbrace{\frac{1}{n}}_{\downarrow n\to+\infty \atop 0} \xrightarrow[n\to+\infty]{} 0,$$

but $\frac{(-1)^n}{n}$ isn't positive for all n large.

10 (i) Suppose that $b<a$.



Consider $\varepsilon = \frac{|b-a|}{3}$;

then,

$b+\varepsilon < a-\varepsilon$.

And: Since $a_n \to a$, $\exists n_1 \in \mathbb{N}: \forall n \geq n_1,\ a_n \in (a-\varepsilon, a+\varepsilon)$
$\Rightarrow a_n > a-\varepsilon,\ \forall n \geq n_1$.

Since $b_n \to b$, $\exists n_2 \in \mathbb{N}: \forall n \geq n_2,\ b_n \in (b-\varepsilon, b+\varepsilon)$
$\Rightarrow b_n < b-\varepsilon,\ \forall n \geq n_2$.

Pick $n_0 = \max\{n_1, n_2\}$; then,

$a_{n_0} > a-\varepsilon$ (since $n_0 \geq n_1$)

and $b_{n_0} < b+\varepsilon$ (since $n_0 \geq n_2$)

So, $b_{n_0} < b+\varepsilon < a-\varepsilon < a_{n_0}$

$\Rightarrow$ $b_{n_0} < a_{n_0}$, contradiction, as $a_n \leq b_n \; \forall n \in \mathbb{N}$.

So, $a \leq b$.

(ii) No : Consider $a_n = -\frac{1}{n}$, $\forall n \in \mathbb{N}$,

and $b_n = \frac{1}{n}$, $\forall n \in \mathbb{N}$.

Then : $a_n \leq b_n \; \forall n \in \mathbb{N}$,

but $a_n = -\frac{1}{n} = -1 \cdot \frac{1}{n} \xrightarrow[n \to +\infty]{} -1 \cdot 0 = 0$,

$b_n = \frac{1}{n} \xrightarrow[n \to +\infty]{} 0$,

so $a = b$ in this case.