Stat 134 lec 24

Sec 4.4 Change of variable

X cont RV w/ density f_X

we write $P(X \in dx) = f_X(x)dx$ where $dx > 0$ is a pos length.

$dx > 0$, so you could write

$$P(X \in dx) = f_X(x)|dx|$$

← abs value

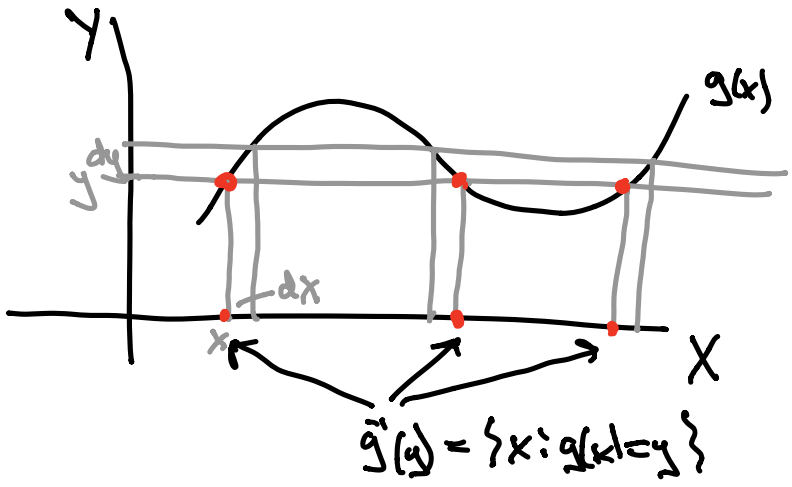
let $Y = g(X)$, g strictly decreasing

$$P(X \in dx) = P(Y \in dy)$$

$$\Rightarrow f_X(x)|dx| = f_Y(y)|dy| \Rightarrow f_X(x) = f_Y(y)\left|\frac{dy}{dx}\right|$$

$$\Rightarrow \boxed{f_Y(y) = \frac{f_X(x)}{\left|\frac{dy}{dx}\right|} \text{ where } x = \vec{g}(y)}$$

many to one g:



$\vec{g}(y) = \{x : g(x) = y\}$

addition rule

$$P(Y \in dy) \overset{\text{addition rule}}{=} \sum_{x \in g^{-1}(y)} P(X \in dx) = \sum_{x \in g^{-1}(y)} f_X(x)|dx|$$

$$\| \quad f_Y(y)|dy|$$

$$\Rightarrow \boxed{f_Y(y) = \sum_{x \in g^{-1}(y)} f_X(x)\left|\frac{dx}{dy}\right| = \sum_{x \in g^{-1}(y)} f_X(x) \cdot \frac{1}{\left|\frac{dy}{dx}\right|}}$$

$$\left|\frac{dy}{dx}\right| = |g'(x)|$$

where $x \in g^{-1}(y)$,

ex Let Z be std normal

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \qquad -\infty < z < \infty$$

Find the density of $Y = \frac{1}{Z^2}$

Steps

1) range of Y — $0 < y < \infty$

2) $\frac{dy}{dz} = -2\frac{1}{z^3}$

3) z in terms of y — $z = \pm\sqrt{\frac{1}{y}}$

4) $f_Y(y) = \sum_{z \in \pm\sqrt{\frac{1}{y}}} \frac{1}{\left|\frac{dy}{dz}\right|} \cdot f_Z(z)$

$\left|\frac{dy}{dz}\right| = \left|\frac{z^3}{2}\right|$

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\pm\sqrt{\frac{1}{y}}\right)^2}$

2 equal terms in the sum

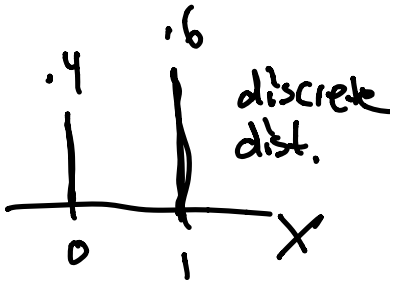
$$= 2 \cdot \left| \left(\frac{1}{y}\right)^{3/2} \right| \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{1}{y}\right)}$$

$$= \boxed{2 \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{y^{3/2}} \cdot e^{-1/2y}, \quad y > 0}$$

sec 4.5 cdf

defn X RV

$$F_X(x) = P(X \le x)$$



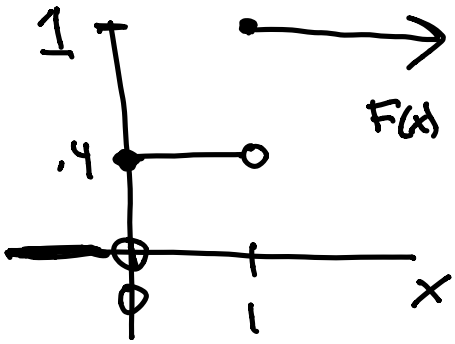
ex $X \sim$ Bernoulli $(p = .6)$

$F(-1) = 0$

$F(0) = .4$

$F(1/2) = .4$

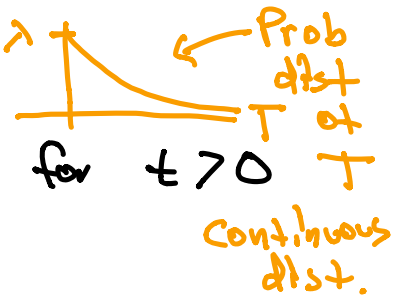
$F(1) = 1$



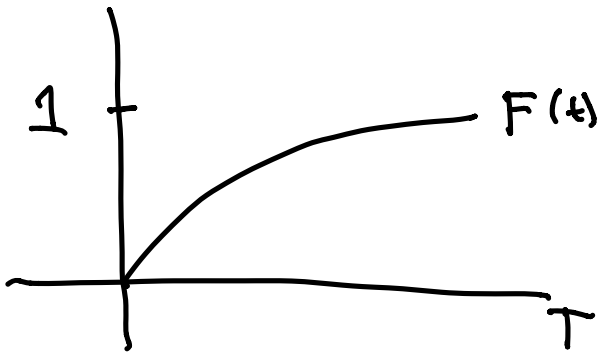
discrete cdf,

Ex $T \sim \text{expon}\left(\frac{1}{2}\right)$

$F(t) = 1 - e^{-\frac{1}{2}t}$ for $t > 0$



continuous dist.



continuous cdf

Ex $X =$ "T killed at c", $c > 0$

$T \sim \text{expon}(\lambda)$

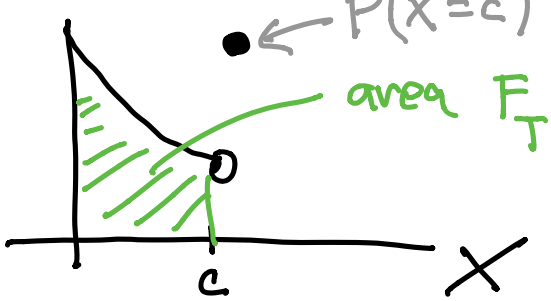
This defined using indicators:

$$X = T \cdot I(T < c) + c \cdot I(T \geq c)$$

i.e. when $T < c$, $X = T$

otherwise, $X = c$

Picture

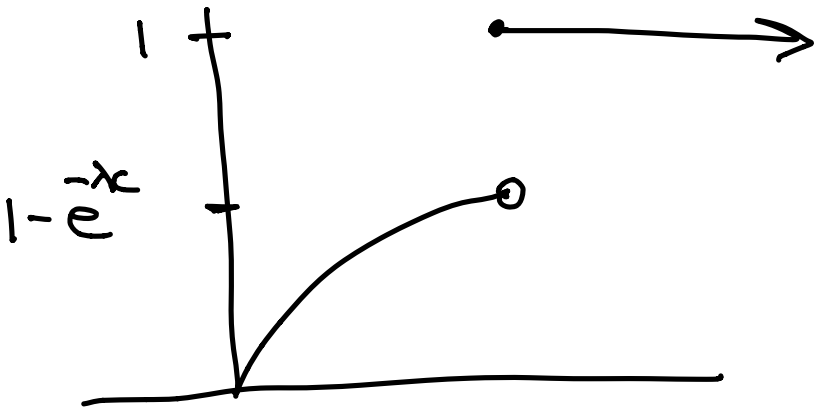


"mixed" distribution.

Notice that this distribution is both continuous and discrete. (called a "mixed" distribution.)

Here $P(X=c) = e^{-\lambda c}$ so that the total probability is 1.

Draw the cdf:



This is a mixed cdf.

Properties of cdf

a) $F_X(x)$ is non decreasing function

b) $\lim_{x \to -\infty} F(x) = 0$

c) $\lim_{x \to \infty} F(x) = 1$

d) $P(a < X \leq b) = F_X(b) - F_X(a)$

e) A prob distribution is completely determined by its cdf.

f) $X_1, \ldots, X_n$ iid RV w/ cdf F

(i) $F_{\max(X_1,\ldots,X_n)} = (F(x))^n$

$F_{\min(X_1,\ldots,X_n)} = 1 - (1 - F(x))^n$

ex $X_i \overset{\text{indep}}{\sim} \text{expon}(\lambda_i)$

$F_{X_i}(x) = 1 - e^{-\lambda_i x}$ — $(0, \infty)$

let $M = \min(X_1, \ldots, X_n)$

for $t > 0$, $P(M > t) = P(X_1 > t, X_2 > t, \ldots, X_n > t)$

$$= P(X_1 > t) \cdot P(X_2 > t) \cdots$$

$$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t} \cdots e^{-\lambda_n t}$$

$$= e^{-(\lambda_1 + \cdots + \lambda_n)t}$$

$$\Rightarrow F_M(t) = 1 - e^{-(\lambda_1 + \cdots + \lambda_n)t}$$

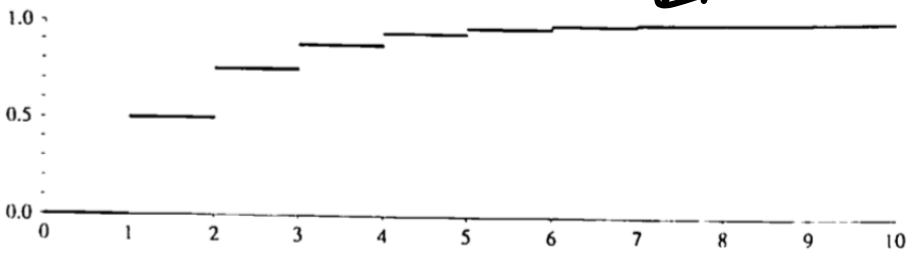
$$\Rightarrow M \sim \text{expon}(\lambda_1 + \cdots + \lambda_n)$$

Stat 134

Monday March 14 2018

there are many little steps here.

1. Below is the cdf of which distribution?



a binomial(10,1/2)

b geometric(1/2) on $\{1, 2, 3, ...\}$

c uniform(0,10)

d none of the above

Not (a) since $P(X=0) = \left(\frac{1}{2}\right)^{10}$ and
$P(X=1) = 10 \cdot \left(\frac{1}{2}\right)^{10}$ so
$F(1) = 11 \cdot \left(\frac{1}{2}\right)^{10} \neq .5$

Not (c) since cdf of unif is a continuous cdf not a discrete cdf.

(b) is good since $P(X=0) = 0$, $P(X=1) = 1/2$
$P(X=2) = 1/4 \Rightarrow F(2) = .75$ ✓