Stat 134 lec 31

Gamma and Beta densities

For $r>0$, we define the gamma function

$$\Gamma(r) = \int_0^\infty t^{r-1} e^{-t}\, dt$$

(not a density its a number)

For r pos integer
$\Gamma(r) = (r-1)!$

This function allows us to generalize the gamma density for $r>0$ (not necessarily integers).

$$X \sim \text{gamma}(r, \lambda) \quad r>0,\ \lambda>0$$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$$

lets check this integrates to 1.

$$\int_0^\infty f_X(x) = \frac{1}{\Gamma(r)} \int_0^\infty \lambda^r x^{r-1} e^{-\lambda x}\, dx$$

set $t = \lambda x \Rightarrow x = \frac{t}{\lambda}$

$dx = \frac{dt}{\lambda}$

$$= \frac{1}{\Gamma(r)} \int_0^\infty \cancel{\lambda^r} \frac{t^{r-1}}{\cancel{\lambda^{r-1}}} \cdot e^{-t} \frac{dt}{\cancel{\lambda}} = \frac{1}{\Gamma(r)} \Gamma(r) = 1 \checkmark$$

Recall beta(r, s) has variable part

$x^{r-1}(1-x)^{s-1}$ $\quad 0<x<1$ w/ constant

$$\frac{(r+s-1)!}{(r-1)!\,(s-1)!}$$

these are gamma.

we can generalize beta(r,s) so $r, s > 0$ (not nec integers).

For $r>0$, $s>0$ (not nec integers) we define

$X \sim$ beta(r, s) to have density

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}, \quad 0<x<1$$

we need

$$\int_0^1 f_X(x)\,dx = 1 \quad \text{i.e.} \quad \int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$$

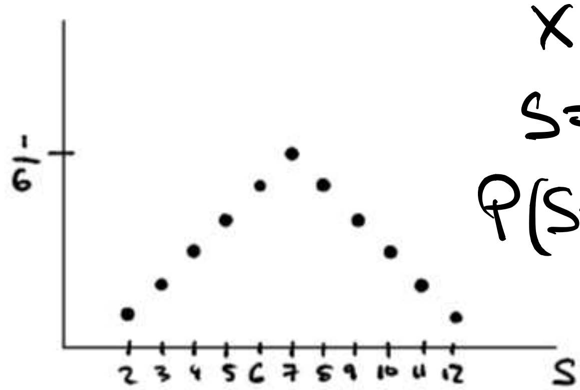
we will prove this by a powerful result in sec 5.4 that seems completely unrelated.

Sec 5.4 Sums of independent random variables.

Stat 134

Friday April 6 2018

1. Let S be the sum of the roll of two fair die.
   The distribution of S is given below:



$X, Y \sim \text{Unif}\{1,2,3,4,5,6\}$

$S = X + Y$

$$P(S=s) = \sum_{x=1}^{s-1} P(X=x, Y=s-x)$$

convolution formula

a true

b false

ex

$$P(S=3) = \sum_{x=1}^{2} P(X=x, Y=2-x)$$

$$= P(X=1, Y=2) + P(X=2, Y=1)$$

$$= \frac{1}{36} + \frac{1}{36} = \frac{2}{36}$$

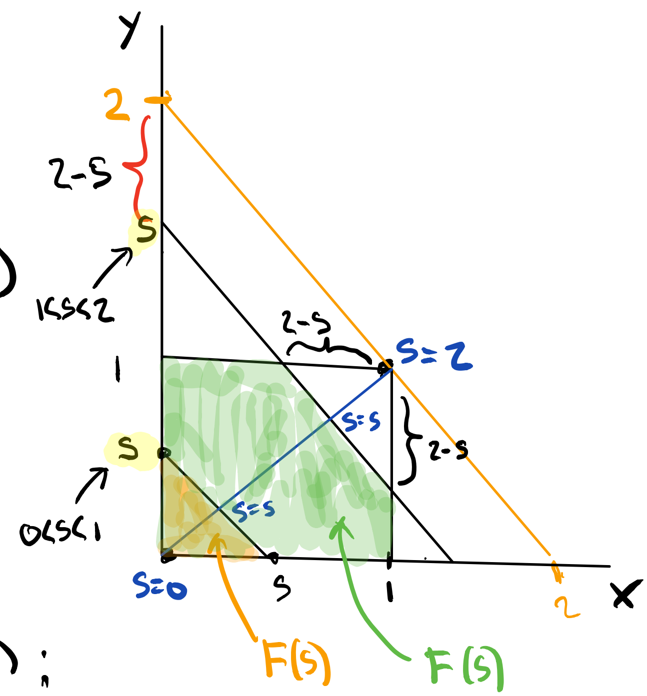
Continuous case:

$X, Y \overset{iid}{\sim} U(0,1)$

$S = X + Y$ — $(0,2)$

$s = x + y$

$y = s - x$



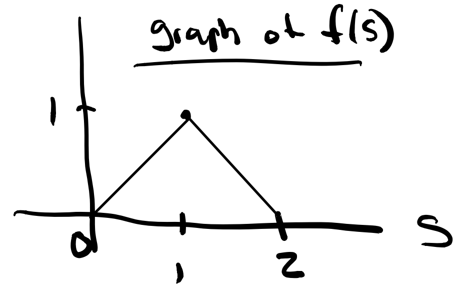
find $f_S(s)$:

cdf method:

$$F(s) = P(S \le s) = \begin{cases} \frac{1}{2}s^2 & 0<s<1 \\ 1 - \frac{1}{2}(2-s)^2 & 1<s<2 \end{cases}$$

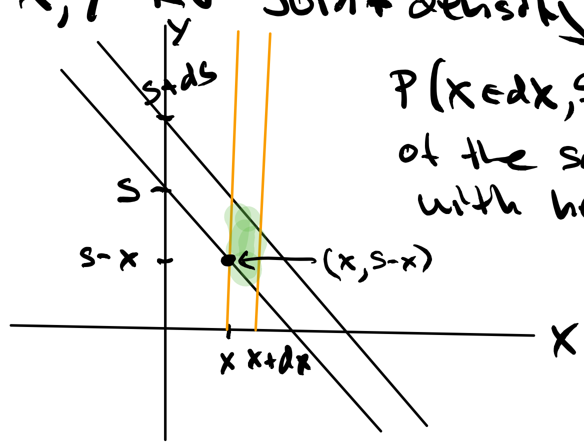
$$f(s) = \begin{cases} s & 0<s<1 \\ 2-s & 1<s<2 \end{cases}$$

graph of $f(s)$



X, Y RV Joint density $f(x,y)$, $S = X+Y$

$P(x \in dx, S \in ds)$ is the volume of the solid over the parallelogram with height $f(x, s-x)$



so $P(X \in dx, S \in ds) = f(x, s-x)\,dx\,ds$

to find the marginal density we integrate out x:

$$\boxed{f_S(s) = \int_{-\infty}^{\infty} f_{(X,Y)}(x, s-x)\,dx \underset{\text{if } X,Y \text{ indep}}{=} \int_{-\infty}^{\infty} f_X(x) f_Y(s-x)\,dx}$$

convolution formula

Ex $X, Y \overset{iid}{\sim} \text{expon}(\lambda)$ $S = X + Y$

$$f_S(s) = \int_0^s f_X(x)\, f_Y(s-x)\,dx$$

$$= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)}\,dx$$

$$= \int_0^s \lambda^2 e^{-\lambda s}\,dx = \lambda^2 e^{-\lambda s} x \Big|_0^s$$

$$= \boxed{\lambda^2 e^{-\lambda s} \cdot s}$$

variable part of gamma$(2, \lambda)$

$\Rightarrow S \sim \text{gamma}(2, \lambda)$.

Next time we will prove that $\int_0^1 x^{r-1}(1-x)^{s-1}\,dx = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$.