Lecture 4: ① 31 Aug 2016.

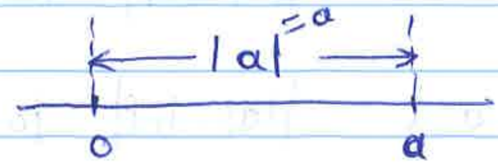
$\boxed{\text{Absolute value:}}$

→ Def: For any $a \in \mathbb{R}$, we define its absolute value as
$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}.$$

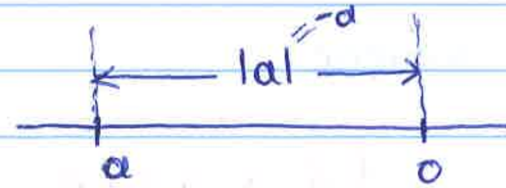
→ Obs.: (i) $|a| \geq 0$ $\forall a \in \mathbb{R}$, with equality only if $a = 0$.
(ii) $-|a| \leq a \leq |a|$, $\forall a \in \mathbb{R}$.
(iii) $|-a| = |a|$, $\forall a \in \mathbb{R}$.
(iv) If $a, b \in \mathbb{R}$ with $b \geq 0$, then
(*) $|a| \leq b \Leftrightarrow -b \leq a \leq b$ (exercise!).

All the above become obvious once one realises that $|a|$ is actually the <u>distance of a from 0</u>:

- $a>0$:



- $a<0$:



→ <u>Prop. (triangle inequality)</u>:

If $a, b \in \mathbb{R}$, then $|a+b| \leq |a| + |b|$

<u>Proof</u>: It suffices (by ⊛) to show that

$$\underbrace{-(|a|+|b|)}_{= -|a|-|b|} \leq a+b \leq |a|+|b| \qquad (\text{since } |a|+|b| \geq 0)$$

Indeed:
$$\left.\begin{array}{l} -|a| \leq a \leq |a| \\ \text{and } -|b| \leq b \leq |b| \end{array}\right\} \Rightarrow -|a|-|b| \leq a+b \leq |a|+|b|$$

by properties of ordered field $\mathbb{R}$

∎

Corollary: If $a, b \in \mathbb{R}$, then

$$\big||a|-|b|\big| \leq |a-b|$$

and $$\big||a|-|b|\big| \leq |a+b|.$$

Proof: Exercise. ■

Some useful equalities and inequalities:

1 Bernoulli's inequality: If $a \geq 1$, then $(1+a)^n \geq 1+na, \forall n \in \mathbb{N}$.

Proof: Exercise (by induction). ■

2 Binomial expansion: If $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

Note: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

then $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^k \cdot b^{n-k}$$

Proof: You don't have to know this proof. However, it would be good practice if you tried it. You can do it by induction, where you will need that

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$ (which can be proved directly)

Another way to see it is to use the fact that $\binom{n}{k}$ is the number of ways one can choose k elements out of n. Since

$$(a+b)^n = \underbrace{(a+b)\cdot \ldots \cdot(a+b)}_{n \text{ times}},$$

$(a+b)^n$ will be the sum of all possible terms created by picking a from k of the brackets and b from the rest $n-k$, for all $k=0,1,\ldots,n$. Each such term will equal $a^k b^{n-k}$; so, since there are $\binom{n}{k}$ ways to choose the k brackets a will be picked from, $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

Once you learn combinatorics, this will be immediate. ∎

③ Cauchy - Schwarz inequality:

If $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}$, then

$$(a_1b_1 + a_2b_2 + \ldots + a_nb_n)^2 \le (a_1^2+a_2^2+\ldots+a_n^2)\cdot(b_1^2+b_2^2+\ldots+b_n^2).$$

I.e., $\left|\sum_{k=1}^{n} a_k b_k\right| \le \sqrt{\sum_{k=1}^{n} a_k^2}\cdot\sqrt{\sum_{k=1}^{n} b_k^2}$.

Proof: Exercise. You can prove this by induction. Another, more imaginative way is to consider the polynomial

$$p(\lambda) = (a_1+\lambda b_1)^2 + (a_2+\lambda b_2)^2 + \dots + (a_n+\lambda b_n)^2, \ \lambda\in\mathbb{R}.$$

What do we know about the sign of $p(\lambda)$?
What is the discriminant of $p(\lambda)$?

[Sidenote: Cauchy-Schwarz is an inequality that generalises in every inner product space. You will learn more about this in your mathematical future.]

4 Arithmetic - geometric - harmonic mean inequality:

If $x_1, x_2, \dots, x_n > 0$, then

$$\underbrace{\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\dots+\frac{1}{x_n}}}_{\text{harmonic mean}} \le \underbrace{\sqrt[n]{x_1\cdot x_2\cdot\ldots\cdot x_n}}_{\text{geometric mean}} \le \underbrace{\frac{x_1+x_2+\dots+x_n}{n}}_{\text{arithmetic mean}}$$

Equality holds only if $x_1 = x_2 = \dots = x_n$.

Proof: You don't need to know the geometric–arithmetic mean inequality proof. However, you can try it by induction (it is easy for $n = 2^k, k \in \mathbb{N}$, but trickier for all n). You can find where to read in the Further Reading chapter of Spivak's book (3rd edition).

The harmonic-geometric mean inequality is a straightforward application of the geometric-arithmetic mean inequality, and is left as an exercise.

$\blacksquare$

$\boxed{\text{Sequences of real numbers.}}$

→ Def: A sequence is a map $a : \mathbb{N} \to \mathbb{R}$.

$$
\begin{array}{ccccccc}
1 & 2 & 3 & & n & n+1 & \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \cdots \\
a(1) & a(2) & a(3) & & a(n) & a(n+1) &
\end{array}
$$

We denote each $a(n)$ by a_n, for simplicity.

We also denote the sequence a by:

$(a_n)_{n \in \mathbb{N}}$, $(a_n)_{n=1}^{\infty}$, (a_n), $(a_1, a_2, a_3, \ldots)$.

→ ex:

(i) Let $c \in \mathbb{R}$. The sequence $a_n = c$ $\forall n \in \mathbb{N}$ is

$$(\underset{a_1}{\overset{\shortparallel}{c}}, \underset{a_2}{\overset{\shortparallel}{c}}, \underset{a_3}{\overset{\shortparallel}{c}}, \dots, c, \dots)$$, a <u>constant</u> sequence.

(ii) $a_n = n$, $\forall n \in \mathbb{N}$: $(a_n)_{n \in \mathbb{N}} = (1, 2, 3, \dots, n, \dots)$.

(iii) $a_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$: $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots)$.

(iv) $a_n = n^2 - n + 1$, $\forall n \in \mathbb{N}$: $(a_n) = (1^2 - 1 + 1, 2^2 - 2 + 1,$

$3^2 - 3 + 1, \dots, n^2 - n + 1, \dots)$.

(v) $a_1 = 1$, $a_{n+1} = \sqrt{1 + a_n}$, $\forall n \in \mathbb{N}$ (this sequence is defined inductively).

Then, $a_2 = \sqrt{1 + a_1} = \sqrt{1+1} = \sqrt{2}$,

$a_3 = \sqrt{1 + a_2} = \sqrt{1 + \sqrt{2}}$,

$a_4 = \sqrt{1 + a_3} = \sqrt{1 + \sqrt{1 + \sqrt{2}}}$, etc.

→ <u>Definitions + Observations:</u>

① <u>The set of terms</u> of the sequence $(a_n)_{n \in \mathbb{N}}$

is the set $\{a_n : n \in \mathbb{N}\}$.

⚠ The set of terms of a sequence is <u>not</u> the same as the sequence! Indeed:

- First of all, a sequence is a map, not a set. More particularly, a sequence contains the information of where <u>each</u> $n \in \mathbb{N}$ is sent to, while that information is not preserved in the set of terms. <u>ex:</u> for $a_n = (-1)^n \ \forall n \in \mathbb{N}$,
$$(a_n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, 1, \dots),$$
$$\text{while } \{a_n : n \in \mathbb{N}\} = \{-1, 1\}.$$

- Two different sequences may have the same set of terms. <u>ex:</u> both
$$(-1, 1, -1, 1, -1, 1, \dots)$$
and $(1, -1, 1, -1, 1, -1, \dots)$
have the same set of terms.

② If $(a_n)_{n=1}^{\infty}$ is a sequence and $m \in \mathbb{N}$, then the sequence $(a_m, a_{m+1}, a_{m+2}, \dots)$ <u>is called a final part</u> of $(a_n)_{n=1}^{\infty}$.

$$(a_1, a_2, a_3, \dots, a_{m-1}, \underbrace{\boxed{a_m, a_{m+1}, \dots}}_{\text{a final part of } (a_n)_{n\in\mathbb{N}}})$$

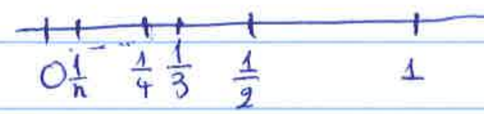
Note that $(a_m, a_{m+1}, \dots) = (a_{m+n-1})_{n\in\mathbb{N}}$.

ex: The sequence $a_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$ has final parts $\left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right)$ (the sequence itself),

$$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right), \left(\frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \ldots\right) \text{(among others)}.$$

Limit of a sequence:

ex: $a_n = \frac{1}{n}$, $n \in \mathbb{N}$.



I want to have a definition for the limit of a sequence that will allow me to say that

$$a_n = \frac{1}{n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow +\infty.$$

Observe that what really happens for $a_n = \frac{1}{n}$ is the following:

"No matter how close to 0 I look at, I can find $a_n = \frac{1}{n}$ for large n there."

→ "for large n" means "for all $n \in \mathbb{N}$ from some natural number onwards".

→ "No matter how close to 0 I look at" means

"no matter how small an interval I pick around 0",

or, more specifically,

"no matter how small a neighbourhood of 0 I pick",

where:

Def: For any $a \in \mathbb{R}$, we define a neighbourhood of a to be any interval of the form $(a-\varepsilon, a+\varepsilon)$, for $\varepsilon > 0$.

⚠ Side note: In topology, a neighbourhood of a point is defined as any open set containing the point. So, for instance, every open interval containing the point is a neighbourhood of the point (whether it is symmetric around the point or not). However, in the case of limits in $\mathbb{R}$ it suffices to consider only symmetric intervals around each point, so, for simplicity, we will call "neighbourhoods" only such symmetric intervals.

Thus, I require the following:

"For any neighbourhood of 0, I can find in the neighbourhood all $a_n = \frac{1}{n}$ from some n natural number onwards"

a whole final part of $(a_n)_{n\in\mathbb{N}}$!

I.e.:

"For any neighbourhood of 0, there exists some final part of $(a_n)_{n\in\mathbb{N}}$ contained in the neighbourhood".

$(a_{n_0}, a_{n_0+1}, a_{n_0+2}, \ldots)$, for some $n_0 \in \mathbb{N}$ that depends on the neighbourhood.

I.e.:

"For any interval of the form $(0-\varepsilon, 0+\varepsilon)$, where $\varepsilon > 0$,

there exists some $n_0 \in \mathbb{N}$, depending on ε, s.t.

$|a_n - 0| < \varepsilon$ $a_n \in (0-\varepsilon, 0+\varepsilon)$, $\forall n \geqslant n_0$".

I.e.: For all $\varepsilon > 0$, $\exists n_0 = n_0(\varepsilon) \in \mathbb{N}$, s.t.: $\left(|a_n - 0| < \varepsilon, \forall n \geqslant n_0 \right)$

This is exactly the definition of "$a_n \longrightarrow 0$ as $n \longrightarrow +\infty$".

In general:

Def.: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and $a\in\mathbb{R}$. We say that $(a_n)_{n\in\mathbb{N}}$ converges to a, and that a is the limit of $(a_n)_{n\in\mathbb{N}}$, and we write: "$a_n \longrightarrow a$ (as $n \to +\infty$)", if:

optional

$$\boxed{\forall \varepsilon>0,\ \exists n_0=n_0(\varepsilon)\in\mathbb{N}:\ \forall n\geq n_0,\ |a_n-a|<\varepsilon}$$

⚠ Note that this definition can be rephrased as:

$(a_n)_{n\in\mathbb{N}}$ converges to a if:

> for any neighbourhood $(a-\varepsilon, a+\varepsilon)$ of a, there exists a final part of $(a_n)_{n\in\mathbb{N}}$ contained in $(a-\varepsilon, a+\varepsilon)$.