Stat 134: Final Review Sheet
Answers

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The proofs/calculations of most exercises here are omitted. Again, refer to your notes if unsure; all of the theoretical results have been discussed in lecture notes or in the textbook.

1 Relationships Between Distributions

1. cX, where $X \sim$ Exp (λ), $c > 0$:

   Exp $(\frac{\lambda}{c})$

2. $X_1 + X_2 + X_3$, where the X_i's are i.i.d. Gamma $(1, \mu)$. What about $\frac{X_1}{X_1+X_2+X_3}$?

   $X_1 + X_2 + X_3 \sim$ Gamma $(3, \mu)$; $\frac{X_1}{X_1+X_2+X_3} \sim$ Beta (1,2)

3. $X^2 + Y^2$, where X, Y are independent standard Normal. What is $\sqrt{X^2 + Y^2}$?

   Exp $(\frac{1}{2})$; standard Rayleigh

4. $2X + 3Y$, where X, Y independent Normal (μ, σ^2). What if X, Y are bivariate normal with correlation $\rho = 0.6$?

   $2X + 3Y \sim \mathcal{N}(5\mu, 13\sigma^2)$ if $\rho = 0$; $2X + 3Y \sim \mathcal{N}(5\mu, (3.8^2 + 1.92^2)\sigma^2)$ if $\rho = 0.6$

5. Consider each of the following common discrete distributions: Poisson, Binomial, Geometric, and Hypergeometric. For which of these is the sum of two independent RVs a known distribution? Under what conditions?

   Excluding the degenerate cases (e.g., $p = 0$ or $p = 1$), this holds for the first 3 distributions. For Binomial, the p values must be the same, in which case the n's are added; for the Geometric the result is Negative Binomial provided the p values are the same.

2 Symmetry

1. Under some conditions, we can quickly recognize the expectation of a random variable X to be zero. What are they?

   The distribution/density of X must be symmetric about the origin, and $\mathbb{E}(|X|) < \infty$; i.e. the expectation must be defined.

2. Find the probability that the last ace in a standard, well-shuffled deck is at position 47 or greater.

   Using symmetry between the front and back of the deck, this answer is

$$\frac{\binom{4}{1}\binom{48}{6}}{\binom{52}{6}}$$

3. You and I each roll a fair n sided die. Without using a summation, find the probability your roll is strictly greater than mine.

   $P(X < Y) = P(X > Y)$, and $P(X < Y) + P(X > Y) + P(X = Y) = 1$. Therefore,

   $$P(X > Y) = \frac{1 - \frac{1}{n}}{2}$$

4. Let X, Y be independent $\mathcal{N}(0, \sigma^2)$ random variables. Without using Φ, find $P(X + 2Y > 0, X > 0)$. What is the reason behind this answer?

   Using the rotational symmetry of the joint distribution of (X, Y),

   $$P(X + 2Y > 0, X > 0) = \frac{\frac{\pi}{4} + \arctan(\frac{1}{2})}{2\pi}$$

5. Suppose we have 5 independent Uniform $[0, 1]$ variables. Prove that $U_{(2)} - U_{(1)}$ is equal in distribution to $U_{(1)}$. Recognize this as a named distribution, and state the parameters.

   Both variables should be Beta (1,5).

3 To Infinity, and Beyond

1. Let $X \sim$ Geom (p) on $\{1, 2, \ldots\}$. It is not easy to directly find $\mathbb{E}(X)$ using the formula $\sum_{x=1}^{\infty} xP(X = x)$. We have shown two alternate methods for finding this expectation; what are they?

   The two methods are (i) the tail sum formula for expectations of discrete RVs and (ii) conditioning on the result of the first toss.

2. Two players simultaneously toss coins which land heads with probabilities p_1 and p_2 respectively. They continue until exactly one player's coin lands heads; that player is the winner. Show that the probability Player 1 wins is $\frac{p_1 q_2}{p_1 q_2 + q_1 p_2}$.

   Hint: write this out as a sequence of tosses. For the first player to win on the n^{th} toss, what has to happen?

3. Find $\mathbb{E}(X(X - 1))$ for $X \sim$ Pois (μ).

   Using the function rule, we observe that this value is μ^2.

4. Let Y have density $f_Y(y) = \frac{\lambda}{2} e^{-\lambda |y|}$, for $y \in \mathbb{R}$. With little computation, find $\mathbb{E}(|Y|)$ and $\mathbb{E}(Y)$.

   If we look at the graph of the density of Y, or through the change-of-variable formula, we observe that $|Y| \sim$ Exp (λ). Thus $\mathbb{E}(|Y|) = \frac{1}{\lambda}$, and $\mathbb{E}(Y) = 0$ by symmetry.

4 Could You Rephrase That?

1. Suppose there are $3n$ people in a room, divided into groups of 3. What is the chance that there is at least one group where two members share the same birthday?

   $$1 - \left(\frac{364 \cdot 363}{365^2}\right)^n$$

2. Let $X \sim$ Gamma (r, λ), where r is an integer. Use what we know about the Poisson process to obtain the CDF of X.

$$
\begin{aligned}
P(X < t) &= P(N_t \geq r) \\
&= 1 - P(N_t < r) \\
&= 1 - \sum_{k=0}^{r-1} e^{\lambda t} \frac{(\lambda t)^k}{k!}
\end{aligned}
$$

3. Let X be an arbitrary random variable with an invertible CDF F_X. What is the random variable formed by $F_X(X)$?

   Let $Z = F_X(X)$. Then,

$$
\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(F_X(X) \leq z) \\
&= P(F_X^{-1}(F_X(X)) \leq F_X^{-1}(z)) \\
&= P(X \leq F_X^{-1}(z)) \\
&= F_X(F_X^{-1}(z)) \\
&= z, \quad z \in [0,1]
\end{aligned}
$$

   We conclude that $Z \sim$ Unif (0,1).

5 Some Useful Results

1. Let $X \sim$ Pois (μ), $Y \sim$ Pois (λ), $X \perp Y$. What is the distribution of $X + Y$? (Hint: use the binomial theorem, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.) $X + Y \sim$ Pois $(\mu + \lambda)$

2. Use Markov's Inequality to derive Chebyshev's Inequality. See pg. 191 – 192 in Pitman's *Probability.*

3. Let $X \sim$ Exp (λ_X), $Y \sim$ Exp (λ_Y). What is $P(X < Y)$? $\frac{\lambda_X}{\lambda_X + \lambda_Y}$

4. Find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$, where the X_i's are independent Exp (λ) variables.

   Let M denote the minimum. Then $M \sim$ Exp $(n\lambda)$. Note this result generalizes to the case where the rates are all different; you simply add the rate parameters.

5. Suppose cars and trucks arrive at a bridge according to independent Poisson processes with rates λ_c and λ_r per minute respectively. Given that n vehicles arrive in t minutes, what is the distribution of $N_{c,t}$, the number of cars to arrive by time t?

   The easiest way to proceed here is using the conditional probability rule. We find that $N_{c,t} | N_t = n \sim$ Binom $(n, \frac{\lambda_c}{\lambda_c + \lambda_r})$.