Lecture 2 26 Aug 2016.

(vi) By the definition of an ordered field, exactly one of the following holds:

(*) $\boxed{1>0 \quad \text{or} \quad 1=0 \quad \text{or} \quad -1>0.}$

- Suppose that $1=0$. This is a contradiction, as it violates the definition of a field.

- Suppose that $-1>0$.
  Then, $(-1)\cdot(-1)>0$ (by definition of an ordered field).

  But $(-1)\cdot(-1)=1$ (by properties of a field).
  So, $1>0$. At the same time, $-1>0$, so two of the conditions (*) holds.
  So, we have a contradiction.

Therefore, $1>0$. ■

→ Problem: How to define an extension $\mathbb{R}$ of $\mathbb{Q}$, s.t.

(i) $\mathbb{R}$ is in a 1-1 correspondence with the number line, and

(ii) The operations $+$ and $\cdot$ that we know on $\mathbb{Q}$, as well as the order $<$ on $\mathbb{Q}$, are extended on $\mathbb{R}$, s.t. $(\mathbb{R}, \tilde{+}, \tilde{\cdot}, \tilde{<})$

the extensions of $+, \cdot$ — the extension of $<$

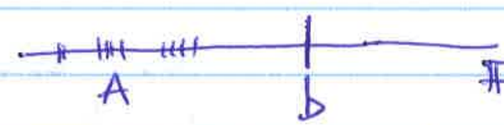
is an ordered field.

To do this, we need to understand what properties $\mathbb{Q}$ is <u>missing</u>, that prevent it from covering the whole number line.

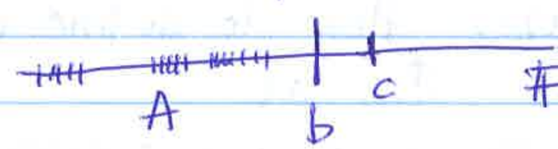
→ <u>Def.:</u> Let $(\mathbb{F}, +, \cdot)$ be an ordered field, and $A \subseteq \mathbb{F}$.

→ We say that A is <u>bounded from above</u> if there exists $b \in \underline{\mathbb{F}}$ s.t.

$$\boxed{a \leq b, \ \forall a \in A}$$



<u>Obs:</u> • Suppose that A is bounded from above, with $b \in \mathbb{F}$ an upper bound of A. If $c \in \mathbb{F}$ and $b \leq c$, then c is also an upper bound of A.



I.e.: A can have many upper bounds;

A doesn't have to have an upper bound; and if it does, that upper bound doesn't have to be in A - it just belongs to the ambient field, $\mathbb{F}$. If we look for upper bounds of A inside larger ordered fields that contain $\mathbb{F}$, then we will probably have more options for upper bounds.

ex: In $(\mathbb{Q}, +, \cdot, <)$:

- $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\{x \in \mathbb{Q} : x > 0\}$, $\{2^n : n \in \mathbb{N}\}$ are not bounded from above (in $\mathbb{Q}$).

- $\{1\}$ is bounded from above (by any $q \in \mathbb{Q}$ with $q \geq 1$).

- $\{x \in \mathbb{Q} : x < 0\}$ is bounded from above (by any $q \in \mathbb{Q}$ with $q \geq 0$).

→ Suppose the non-empty $A \subseteq \mathbb{F}$ is bounded from above. Suppose that $b \in \mathbb{F}$ is an upper bound of A. We say that b is a least upper bound of A if

$b \leq c$, for all c upper bounds of A (in $\mathbb{F}$)

Obs: • Note that b doesn't have to be in A

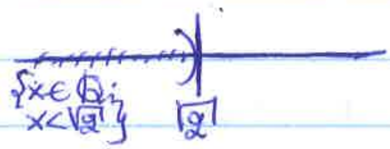
in order to be a least upper bound of A.

- A cannot have more than one least upper bounds. I.e., if A has a least upper bound (in $\mathbb{F}$), then that least upper bound is unique (exercise)
- A bounded from above $A \subseteq \mathbb{F}$ doesn't necessarily have a least upper bound (in $\mathbb{F}$). So, existence of least upper bounds is a special property; it is known as completeness:

Def: Let $(\mathbb{F}, +, \cdot, <)$ be an ordered field. We say that $(\mathbb{F}, +, \cdot, <)$ is complete if every (non-empty) subset of $\mathbb{F}$ that is bounded from above has a least upper bound (in $\mathbb{F}$).

→ Prop: The ordered field $(\mathbb{Q}, +, \cdot, <)$ is not complete.

Proof: Idea: We have shown that $\sqrt{2} \notin \mathbb{Q}$; however, one feels that the set of rationals smaller than $\sqrt{2}$ should



have $\sqrt{2}$ as a least upper bound. <u>So</u>:
We will use essentially $\{x\in\mathbb{Q} : x<\sqrt{2}\}$ as an example of a non-empty subset of $\mathbb{Q}$ without a least upper bound <u>in $\mathbb{Q}$</u>.
<u>However</u>, we are not even allowed to write $\sqrt{2}$ yet; we haven't defined anything beyond $\mathbb{Q}$, and we know that $\nexists q\in\mathbb{Q}$ s.t. $q^2=2$... And, even if I could write $\sqrt{2}$, I have not defined any order relation involving $\sqrt{2}$ (as my order is so far only defined in $\mathbb{Q}$); so writing $x<\sqrt{2}$ doesn't make sense.

<u>So</u>, we will write $\{x\in\mathbb{Q} : x<\sqrt{2}\}$ as $\{x\in\mathbb{Q} : x\leq 0 \text{ or } x^2<2\}$; this set makes sense with respect to everything we have defined so far. In fact, we will take a smaller subset of it, that only contains positive elements (for technical reasons).

The set $A=\{x\in\mathbb{Q} : x>0 \text{ and } x^2<2\}$ is non-empty, bounded from above, and doesn't have a least upper bound (in $\mathbb{Q}$). Indeed:

- $A\neq\emptyset$: $1\in A$ ($1\in\mathbb{Q}$, $1>0$, $1^2<2$)
- A is bounded from above (in $\mathbb{Q}$): For instance,

⑥

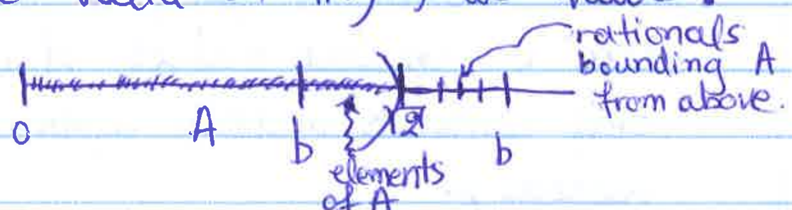
4 is an upper bound of A, because, $\forall x \in A$, $4^2 > 2 > x^2 \Longrightarrow 4^2 > x^2 \underset{4,x>0}{\Longrightarrow} 4 > x$.

(indeed, $x<y \Leftrightarrow x^2<y^2$ for $x>0, y>0$ in an ordered field; exercise).

Suppose that A has a least upper bound, say $b \in \mathbb{Q}$. Then, exactly one of the following holds (as $\mathbb{Q}$ is an ordered field):

(*) $$\boxed{b^2<2 \quad \text{or} \quad b^2=2 \quad \text{or} \quad b^2>2}$$

Idea: in the picture below, which we have explained we cannot officially use yet (but which we know will eventually be valid in $\mathbb{R}$), we have:



- If $b<\sqrt{2}$, then there would exist elements of A larger than b, contradiction, as b is the least upper bound of A.
- If $b>\sqrt{2}$, then there would exist rationals in $(\sqrt{2}, b)$, bounding A from above, thus smaller than the least

upper bound of A, contradiction.

- If $b=\sqrt{2} \Rightarrow \sqrt{2} \in \mathbb{Q}$, contradiction

So, A cannot have a least upper bound.

- Suppose that $\underline{b^2=2}$. Then there exists a rational that squares to 2; contradiction.

- Suppose that $\underline{b^2<2}$. We will find $\varepsilon>0$ s.t. $b+\varepsilon \in A$, in which case $b+\varepsilon>b$, where b is ~~the~~ least upper bound of A; contradiction.

<u>Details</u>: Indeed, we want $\varepsilon \in \mathbb{Q}$, $\varepsilon>0$, $(b+\varepsilon)^2<2$.

Now, $(b+\varepsilon)^2<2 \Leftrightarrow b^2+2b\varepsilon+\varepsilon^2<2 \Leftrightarrow$
$\Leftrightarrow 2b\varepsilon+\varepsilon^2<2-b^2$. (*)

So, if we look for $\varepsilon<1$ with the above properties, we will be able to use that $\varepsilon^2<\varepsilon$, which implies that $2b\varepsilon+\varepsilon^2<$ $<2b\varepsilon+\varepsilon=(2b+1)\cdot\varepsilon$. So, if we find

properties of ordered field.

$\varepsilon \in \mathbb{Q}$ with $0<\varepsilon<1$ s.t. $(2b+1)\cdot\varepsilon < 2-b^2$,

we automatically have (*) as well.

Therefore, it suffices to find

$\varepsilon \in \mathbb{Q}$ s.t. $\varepsilon>0$, $\varepsilon<1$, and

$$(2b+1)\cdot\varepsilon < 2-b^2 \underset{\substack{2b+1>0 \\ \text{(check!)}}}{\Longleftrightarrow} \varepsilon < \frac{2-b^2}{2b+1}$$

Notice that $\varepsilon = \frac{1}{2}\cdot\min\left\{1, \frac{2-b^2}{2b+1}\right\}$

satisfies all these conditions; thus, for this ε, $b+\varepsilon \in A$, and $b+\varepsilon > b$, the least upper bound of A, a contradiction.

- Suppose $b^2>2$. We will find $\varepsilon>0$ (in $\mathbb{Q}$), s.t $b-\varepsilon$ is an upper bound of A (in $\mathbb{Q}$). In this case $b-\varepsilon$ is an upper bound smaller than the least upper bound, a contradiction.

Details: for $b-\varepsilon$ to be an upper bound of A for some $\varepsilon\in\mathbb{Q}$, it suffices to have $\underbrace{(b-\varepsilon)^2}_{b^2-2b\varepsilon+\varepsilon^2}>2$ and $b-\varepsilon>0$ (prove this!).

So, it suffices to have:

$\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$, $\varepsilon < b$ and $b^2 - 2b\varepsilon + \varepsilon^2 > 2$.

Notice that, if I find $\varepsilon \in \mathbb{Q}$ s.t. $\varepsilon > 0$, $\varepsilon < 0$ and $b^2 - 2b\varepsilon > 2$, then I automatically also have $b^2 - 2b\varepsilon + \varepsilon^2 > 2$ (as $\varepsilon^2 > 0$ in the ordered field $\mathbb{Q}$) so I am done.

So, it suffices to find $\varepsilon \in \mathbb{Q}$ s.t. $\varepsilon > 0$, $\varepsilon < b$

and $b^2 - 2b\varepsilon > 2 \Leftrightarrow 2b\varepsilon < b^2 - 2 \underset{\substack{2b>0 \\ \text{(check!)}}}{\Longleftrightarrow} \varepsilon < \dfrac{b^2-2}{2b}$.

Notice that $\varepsilon = \frac{1}{2} \cdot \min\left\{b, \dfrac{b^2-2}{2b}\right\}$ satisfies all

these conditions; thus, for this ε, $b-\varepsilon$ is an upper bound of A (in $\mathbb{Q}$). And $b - \varepsilon < b$, the least upper bound of A, a <u>contradiction</u>.

<u>Eventually</u>, we have shown that (*)

is false. This is a contradiction, so our initial assumption that A has a least upper bound is false. ∎

So, we have shown that $\mathbb{Q}$ is missing the completeness property! $\mathbb{R}$ will be the unique extension of $(\mathbb{Q}, +, \cdot, <)$ to an ordered field that is complete (this essentially covers the gaps on the number line).

→ <u>Theorem (the real numbers):</u>

(1) There exists an extension of $(\mathbb{Q}, +, \cdot, <)$ to a <u>complete ordered field</u> $(\mathbb{R}, +, \cdot, <)$.

I.e.
- $\mathbb{Q} \subseteq \mathbb{R}$.
- The operations $+$ and $\cdot$ on $\mathbb{R}$, when restricted on $\mathbb{Q}$, are the original operations $+$ and $\cdot$ on $\mathbb{Q}$.
- The order $<$ on $\mathbb{R}$, restricted on $\mathbb{Q}$, is the same as the order $<$ on $\mathbb{Q}$.
- Every $A \subseteq \mathbb{R}$, $A \neq \emptyset$ that is bounded from above has a least upper bound (in $\mathbb{R}$).

(2) There exists a <u>unique</u> complete ordered field (up to isomorphism).

→ <u>Corollary:</u> The extension of $(\mathbb{Q}, +, \cdot, <)$ to a complete ordered field is <u>unique.</u> We call this unique extension the field of real numbers.

We will not worry about the proof of the existence and uniqueness of the real field. If you are interested, you can find all the details in Spivak's book. It is actually not a hard proof, just a very long one.