Stat 134 lec 28

Sec 5.1, 5.2

X, Y indep

$$P(X \in dx, Y \in dy) = P(X \in dx) P(Y \in dy)$$

$$f(x,y)\,dx\,dy \qquad f(x)\,dx \quad f(y)\,dy$$

$$\Rightarrow \boxed{f(x,y) = f(x)f(y)}$$

ex X, Y iid std normal

$$f(x,y) = f(x)f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \boxed{\frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}}$$

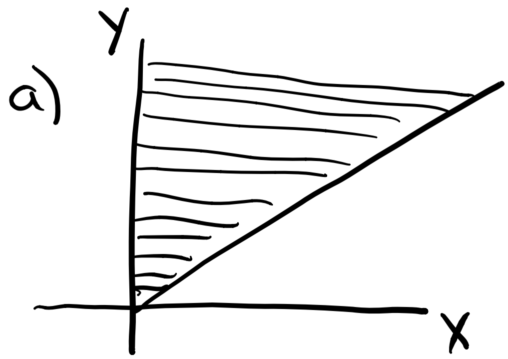
$x^2+y^2=R^2$ so $f(x,y)$ is constant on circles radius R.

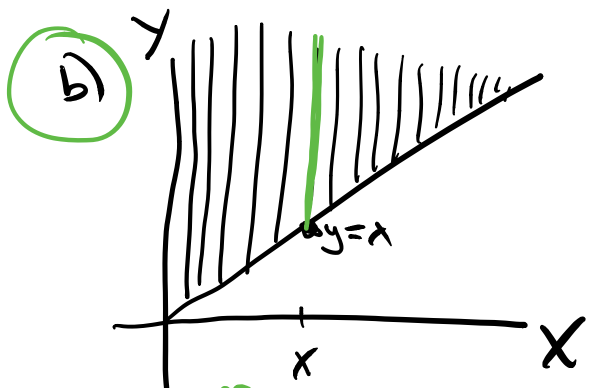
Density is shaped like a liberty bell.

Stat 134

Friday March 23 2018

1. Let $X \sim expon(\lambda)$ and $Y \sim expon(\mu)$ be independent lifetimes of two bulbs. To find $P(Y > X)$ which picture best illustrates the preferred order of integration?





Soln: Option b is better since $\int_{y=x}^{\infty} \lambda e^{-\mu y} dy = e^{-\mu y}$ is easy.

$$f(x,y) = \lambda e^{-\lambda x} \mu e^{-\mu y}$$

$$P(Y>x) = \lambda \cancel{\mu} \int_{x=0}^{\infty} e^{-\lambda x} \underbrace{\int_{y=x}^{\infty} e^{-\mu y} dy}_{= \frac{e^{-\mu x}}{\cancel{\mu}}} dx$$

$$= \lambda \int_{x=0}^{\infty} e^{-(\lambda+\mu)x} dx = \boxed{\frac{\lambda}{\lambda+\mu}}$$

Note $P(X>Y) = \frac{\mu}{\mu+\lambda}$ since $P(X>Y) + P(X<Y) = 1$

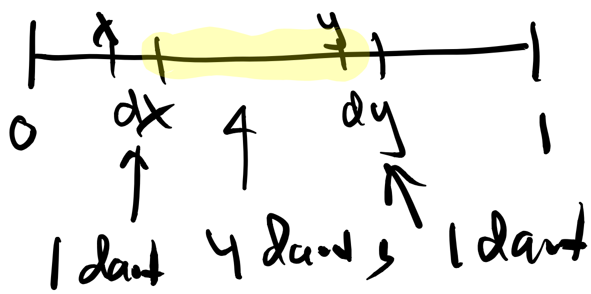
2. Is $f(x, y) = 30(y-x)^4$ on $0 < x < y < 1$ a joint density function?

a yes

b no

c not enough info to decide

$$\boxed{\begin{aligned} X &= U_{(1)} \\ Y &= U_{(6)} \end{aligned}}$$



$$6dx\binom{5}{4}(y-x)^4 dy$$

$$= 6\cdot 5\,(y-x)^4\,dx\,dy$$

<u>Additional practice</u>:

what joint density, $f(x,y)$, has variable part $x^2(y-x)^3(1-y)^7$?

Answ: throw down 14 darts on $(0,1)$

let, $X = U_{(3)}$

$Y = U_{(7)}$

$$f(x,y) = \binom{14}{2}x^2\binom{12}{1}\cdot 1\cdot\binom{11}{3}(y-x)^3\cdot\binom{8}{1}\cdot 1\cdot\binom{7}{7}(1-y)^7$$

ex (5.2.9a)

$X = \min(S,T)$
$Y = \max(S,T)$ for S,T iid expon(λ)

Find the joint density of X and Y.
Are X, Y indep?

$$P(X \in dx, Y \in dy) = f(x,y)\,dx\,dy$$

trick

$$P(X \in dx, Y \in dy)$$

Event $X \in dX$, $Y \in dY$ is equivalent to $S \in dS$, $T \in dT$ with $X=S$, $Y=T$ or $S \in dS$, $T \in dT$ with $X=T$, $Y=S$. Apply addition rule.

$$= 2P(S \in dS, T \in dT)$$

$S+T = X+Y$
$ds\,dt = dx\,dy$

$$= 2\lambda e^{-\lambda s} ds\, \lambda e^{-\lambda t} dt = 2\lambda^2 e^{-\lambda(x+y)} dx\,dy$$

$$\Rightarrow f(x,y) = \boxed{2\lambda^2 e^{-\lambda(x+y)}}$$

$X = \min(S,T)$

$$P(X > x) = P(S > x, T > x)$$
$$= P(S > x)^2 = (e^{-\lambda x})^2$$
$$= e^{-2\lambda x}$$

$\Rightarrow F(x) = 1 - e^{-2\lambda x}$

$$\boxed{f(x) = 2\lambda e^{-2\lambda x}}$$

Find $f_y(y)$

$F(y) = P(Y < y) = P(S < y, T < y)$

$= P(S < y)^2 = (1 - e^{-\lambda y})^2$

$$\Rightarrow \boxed{f(y) = 2\lambda(1 - e^{-\lambda y})(e^{-\lambda y})}$$

$\Rightarrow x, y$ not indep.

ex

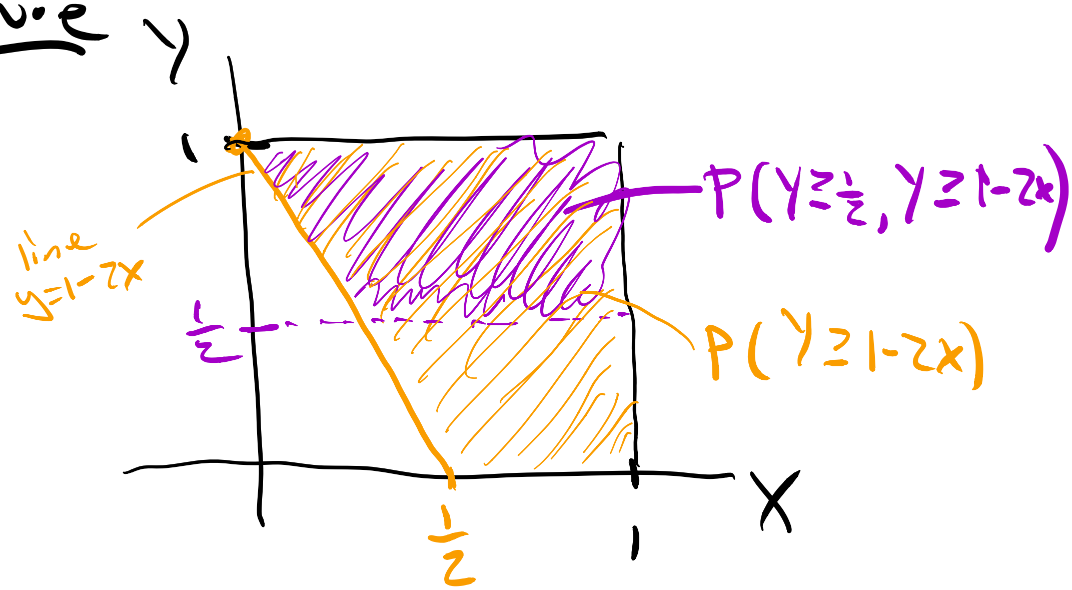
If $x, y \overset{iid}{\sim} U(0,1)$

then $f(x,y) = \underset{1}{\underset{\shortparallel}{f(x)}}\,\underset{1}{\underset{\shortparallel}{f(y)}} = 1$ for $0 < x, y < 1$

Find $P\left(Y \geq \frac{1}{2} \,\middle|\, Y \geq 1 - 2x\right)$

$$= \frac{P\left(Y \geq \frac{1}{2}, Y \geq 1 - 2x\right)}{P(Y \geq 1 - 2x)}$$

Picture



Note: $P(Y \geq 1-2x) = \iint_{Y \geq 1-2x} 1 \, dx\,dy$ = area with $Y \geq 1-2x$

$$\frac{P(Y \geq \frac{1}{2}, Y \geq 1-2x)}{P(Y \geq 1-2x)} = \frac{\frac{1}{2} - \frac{1}{2}\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{1 - \frac{1}{2}(1)\left(\frac{1}{2}\right)} = \frac{7/16}{3/4}$$

$$= 7/12$$

Have a great Spring Break!