MATH 104 - WEEKLY ASSIGNMENT 1

DUE 5 SEPTEMBER 2016, BY 16:00

(1) Let $(\mathbb{F}, +, \cdot)$ be a field. Show the following:
   (i) For $a, b, c \in \mathbb{F}$, $a + c = b + c \Rightarrow a = b$.
   (ii) For $a, b, c \in \mathbb{F}$ and $c \neq 0$, $ac = bc \Rightarrow a = b$.
   (iii) For every $a \in \mathbb{F}$, $-(-a) = a$.
   (iv) For every $a \in \mathbb{F}$ with $a \neq 0$, $(a^{-1})^{-1} = a$.
   (v) For every $a, b \in \mathbb{F}$, $(-a)b = a(-b) = -(ab)$.
   (vi) For every $a, b \in \mathbb{F}$, $(-a)(-b) = ab$.
   (vii) $(-1)(-1) = 1$.

(2) Let $(\mathbb{F}, +, \cdot)$ be a field. Show the following:
   (i) 0 (the additive identity) is unique.
   (ii) 1 (the multiplicative identity) is unique.
   (iii) For every $a \in \mathbb{F}$, the additive inverse of a is unique.
   (iv) For every $a \in \mathbb{F}$, $a \neq 0$, the multiplicative inverse of a is unique.
   (v) $0 \cdot a = 0$, for all $a \in \mathbb{F}$.
   (vi) For $a, b \in \mathbb{F}$, $ab = 0 \Rightarrow a = 0$ or $b = 0$.

(3) Let $(\mathbb{F}, +, \cdot, <)$ be an ordered field. Show the following:
   (i) If $a, b, c \in \mathbb{F}$ and $a > b$, then $a + c > b + c$.
   (ii) If $a, b, c \in \mathbb{F}$, $a > b$ and $c > 0$, then $ac > bc$.
   (iii) If $a, b, c \in \mathbb{F}$, $a > b$ and $c < 0$, then $ac < bc$
   (iv) If $a, b, c, d \in \mathbb{F}$, $a > b$ and $c > d$, then $a + c > b + d$.
   (v) If $a \in \mathbb{F}$ and $a > 0$, then $-a < 0$.
   (vi) If $a \in \mathbb{F}$ and $a \neq 0$, then $a^2 > 0$.
   (vii) If $a, b \in \mathbb{F}$ and $a, b > 0$, then $a < b \Leftrightarrow a^2 < b^2$.
   (viii) If $a, b \in \mathbb{F}$ and $a, b \geq 0$, then $a = b \Leftrightarrow a^2 = b^2$.
   (ix) If $a, b \in \mathbb{F}$ and $a, b > 0$, then $a < b \Leftrightarrow b^{-1} < a^{-1}$.

(4) Show that $\mathbb{Z}_2$ is not an ordered field.

(5) Show that there exists a unique $x > 0$ in $\mathbb{R}$ such that $x^2 = 2$.
(Hint: Follow the proof we did to show that $\nexists q \in \mathbb{Q}$ with $x^2 = 2$, but this time for $\mathbb{R}$.)

(6) Let $(\mathbb{F}, +, \cdot)$ be an ordered field, and $\emptyset \neq A \subset \mathbb{F}$ bounded from above. Show that, if A has a least upper bound, then it is unique.

(7) For $(0,1) \subset \mathbb{R}$, show that $\sup(0,1) = 1$.

(8) Are the following RIGHT or WRONG in $\mathbb{R}$? Justify your answer.
  (i) Let $\emptyset \neq A \subset \mathbb{R}$, bounded from above. Then, $\forall x \in A$, $x \leq \sup A$.
  (ii) Let $\emptyset \neq A \subset \mathbb{R}$, bounded from above, and let $x \in \mathbb{R}$. Then:
  $$x \text{ upper bound of } A \Leftrightarrow \sup A \leq x.$$
  (iii) Let $\emptyset \neq A \subset \mathbb{R}$, bounded from above. Then, $\sup A \in A$.

(9) (i) Let A, B non-empty subests of $\mathbb{R}$, bounded from above. Show that, if $A \subset B$, then $\sup A \leq \sup B$.
  (ii) Let $A \subset \mathbb{R}$. We say that m is a maximal element of A if m is an upper bound of A *such that* $m \in A$. Suppose that A has a maximal element $\max A$. Show that $\max A = \sup A$. Does $(0,1) \subset \mathbb{R}$ have a maximal element?

(10) Let $A \subset \mathbb{R}$. We say that A is *bounded from below* if there exists $b \in \mathbb{R}$ such that $b \leq a$, for all $a \in A$. If such a b exists, it is called *a lower bound of* A. Moreover, we say that b is *a greatest lower bound* of A (we also call it *infimum of* A and denote it by $\inf A$) if
  (i) b is a lower bound of A, and
  (ii) $b \geq c$, for all c lower bounds of A.
  Show that every $\emptyset \neq A \subset \mathbb{R}$ that is bounded from below has a greatest lower bound.