Math 104, Weekly Assignment 1 : solutions.

1 (i) $a = 0 + a = \big((-c)+c\big) + a = (-c) + (c+a) =$
$= (-c) + (c+b) =$
$= \big((-c)+c\big) + b = 0 + b = b.$

(ii) $a = 1\cdot a \underset{c\neq 0}{=} \left(c^{-1}\cdot c\right)\cdot a = c^{-1}\cdot (ca) = c^{-1}\cdot (c b) =$
$= (c^{-1}\cdot c)\cdot b = 1\cdot b = b.$

(iii) $(-a) + a = 0$ (as $-a$ is the additive inverse of a)
and $(-a) + \big(-(-a)\big) = 0$ (as $-(-a)$ is the additive inverse of $-a$).

So, $(-a) + a = (-a) + \big(-(-a)\big) \xrightarrow{(i)} a = -(-a)$.
(Or: you can say: $(-a)+a = 0 \Rightarrow a$ is the additive inverse of $-a$).

(iv) $(a^{-1})\cdot a = 1$ (as a^{-1} is the multiplicative inverse of a)
and $(a^{-1})\cdot (a^{-1})^{-1} = 1$ (as $(a^{-1})^{-1}$ " " " " of a^{-1}).

So, $(a^{-1})\cdot a = (a^{-1})\cdot (a^{-1})^{-1} \xrightarrow{(ii)} a = (a^{-1})^{-1}$.
(Or: you can say: $(a^{-1})\cdot a = 1 \rightarrow a$ is the mult. inverse of a^{-1}).

(v) $(-a)\cdot b + ab = \big((-a)+a\big)\cdot b = 0\cdot b \overset{2(v)}{=} 0$,
and $-(a\cdot b) + ab = 0$ $\Big\}\xrightarrow{(i)}$

$\longrightarrow (-a)\cdot b = -(ab)$. Similarly for $a\cdot(-b) = -(ab)$.

(or: $(-a)b + ab = 0 \Rightarrow (-a)\cdot b$ is the additive inverse of ab).

(vi) $(-a)(-b) \overset{(v)}{=\!=} -\big((-a)\cdot b\big) \overset{(v)}{=\!=} -\big(-(a\cdot b)\big) \overset{(iii)}{=\!=} a b.$

(vii) $(-1)\cdot(-1) \overset{(vi)}{=\!=} 1\cdot 1 = 1.$

2 (i) Suppose that 0, $0'$ are both additive identities.

Then: $0+0' = 0'$ (as 0 is an add. identity)

and $0+0' = 0$ (as $0'$ " " " ")

So, $0=0'$.

(ii) Suppose that $1, 1'$ are both multiplicative identities.

Then: $1\cdot 1' = 1'$ (as 1 is a mult. identity)

and $1\cdot 1' = 1$ (as $1'$ " " " "),

so $1'=1$.

(iii) Suppose that a', a'' are both additive inverses of a.

Then, $\left.\begin{array}{l} a+a'=0 \\ \text{and } a+a''=0 \end{array}\right\} \Rightarrow a+a' = a+a'' \overset{1(i)}{\Longrightarrow} a'=a''.$

(iv) Suppose that a', a'' are both multiplicative inverses of $a \neq 0$.

Then: $\left.\begin{array}{r} a \cdot a' = 1 \\ \text{and } a \cdot a'' = 1 \end{array}\right\} \Longrightarrow \quad a \cdot a' = a \cdot a'' \xrightarrow[\text{as } a \neq 0]{1(ii),} a' = a''.$

(v) $0 = 0 + 0$, so $0 \cdot a = (0+0) \cdot a = 0 \cdot a + 0 \cdot a$.

So we have: $0 \cdot a + 0 \cdot a = 0 \cdot a + 0 \xrightarrow{1(i)} 0 \cdot a = 0$.

(vi) Suppose that $a, b \neq 0$. Then, the multiplicative inverses of a and b exist, and:

$$a \cdot b = 0 \rightarrow \underbrace{(a^{-1}) \cdot (ab)}_{((\underbrace{(a^{-1}) \cdot a}_{1}) \cdot b = 1 \cdot b = b} = \underbrace{(a^{-1}) \cdot 0}_{0, \text{ by (v)}} \Longrightarrow b = 0, \text{ a contradiction.}$$

So, at least one of a, b is 0.

3 (i) $(a+c) + (\underbrace{-(b+c)}_{(-b)+(-c)}) = a + \cancel{c} + (-b) + (\cancel{-c}) = \underbrace{a + (-b)}_{\text{since } a>b} > 0$,

(by definition of $a>b$)

so $a + c > b + c$.

(ii) $ac - bc = ac + (-(b \cdot c)) \underset{①(v)}{=\!=} ac + (-b)\cdot c = (a + (-b))\cdot c.$

And : $c > 0$
and $\underbrace{a + (-b)}_{\substack{\parallel \\ a-b}} > 0$ $\Bigg\}$ $\xrightarrow[\text{of ordered field}]{\text{by definition}}$

$(a + (-b))\cdot c > 0.$

(iii) $bc + (-(ac)) \overset{①(v)}{=\!=} bc + (-a)\cdot c = (b + (-a))\cdot c =$
$\underset{①(vi)}{=\!=} [-(b + (-a))]\cdot(-c)$

And: $-c > 0$ (this is what $c < 0$ means),
and $-(b + (-a)) = -b + a = a - b > 0$ $\Bigg\} \Rightarrow$

$\xrightarrow[\text{of ordered field}]{\text{by definition}}$ $[-(b+(-a))]\cdot(-c) > 0.$

So, $bc + (-(ac)) > 0$,
so $bc > ac$ (that is what $bc > ac$ means by definition),

so $ac < bc$.

(iv) $a+c+\left(-(b+d)\right) = a+c+(-b)+(-d) =$

$$= \underbrace{\left(a+(-b)\right)}_{\substack{>0, \\ \text{as } a>b \\ \text{is defined} \\ \text{to mean } a+(-b)>0}} + \underbrace{\left(c+(-d)\right)}_{>0} > 0,$$ by definition of an ordered field.

So, $a+c > b+d$

(v) $a>0 \overset{(i)}{\Longrightarrow} \underbrace{a+(-a)}_{=0} > \underbrace{0+(-a)}_{=-a}$, i.e. $-a<0$.

(vi) If $a>0$, then $\underbrace{a\cdot a}_{=a^2} > 0$, by definition of an ordered field.

If $a<0$, then $-a>0$ $\left(\text{as for (v)}: a<0 \overset{(i)}{\Longrightarrow} \underbrace{(-a)+a}_{=0} < \underbrace{(-a)+0}_{=-a}\right)$,

so $\underbrace{(-a)\cdot(-a)}_{\substack{=\text{(1)(vi)} \\ a\cdot a \\ = \\ a^2}} > 0$ (by def. of an ordered field).

(vii) $b^2 + (-a^2) = (b+a)\cdot(b-a)$. And

$a, b > 0 \xrightarrow{\text{F ordered}} a+b>0$ $\Big\}\Rightarrow (b+a)(b-a) > 0.$

and $a<b$ means that $b-a>0$

So, $b^2+(-a^2) > 0$, so $b^2 > a^2$.

(viii) • $a=b \Rightarrow a^2 = a\cdot a = b\cdot b = b^2$.

• Suppose that $a^2=b^2 \Rightarrow b^2-a^2=0 \Rightarrow (b+a)(b-a)=0$.

- If $b+a\neq 0$, then it has a multiplicative inverse,
so $b-a = \left((b+a)^{-1}\cdot(b+a)\right)\cdot(b-a) =$

$$= (b+a)^{-1}\cdot\left((b+a)(b-a)\right) = (b+a)^{-1}\cdot 0 \underset{\text{by } ②(v)}{=} 0,$$

so $b=a$.

- If $b+a=0 \Rightarrow b=-a$.
So, if $b>0 \Rightarrow -a>0 \xrightarrow[③(v)]{} \underbrace{-(-a)}_{\overset{①(iii)}{=}\, a} < 0$, i.e. $a<0$, contradiction.

So, $b=0$, i.e. $b+0=0$.
And: $b+a=0$ $\Big\} \Rightarrow b+0=b+a \underset{1(i)}{\Longrightarrow} a=0$,

so $b=a(=0)$

(ix) - $a<b \Rightarrow b^{-1}<a^{-1}$ for $a,b>0$:

First, we show that, <u>for $a>0$, then $a^{-1}>0$</u>:

Indeed, if $a\neq 0$ then $a^{-1}\neq 0$ (because $a^{-1}=0$
$\Rightarrow a^{-1}\cdot a = 0\cdot a = 0$
$\Rightarrow 1=0$, contradiction)

And suppose that $a^{-1}<0$. Then:

Since $a>0$, $a\cdot a^{-1} \underset{\text{by } 3(iii)}{<} 0\cdot a^{-1}=0$,

i.e. $1<0$, a contradiction. So:

we don't have $a^{-1}<0$, or $a^{-1}=0$ $\Rightarrow a^{-1}>0$
($\mathbb{F}$ is ordered, so exactly one of these three has to hold).

So, we have that $a^{-1}, b^{-1} > 0$.

And : $a < b \xrightarrow{3(ii)} \underbrace{a\cdot(a^{-1})}_{\substack{\| \\ 1}} < b\cdot(a^{-1})$

$\xrightarrow{3(ii)}$ $b^{-1}\cdot 1 < \underbrace{b^{-1}\cdot\left(b\cdot(a^{-1})\right)}_{\substack{\| \\ ((b^{-1})\cdot b)\cdot a^{-1} = 1\cdot a^{-1} = a^{-1}}}$,

i.e. $b^{-1} < a^{-1}$.

- $b^{-1} < a^{-1} \xrightarrow[\text{by what we just showed}]{} \underbrace{(a^{-1})^{-1}}_{\substack{\| 1(iv) \\ a}} < \underbrace{(b^{-1})^{-1}}_{\substack{\| 1(iv) \\ b}}$, i.e. $a < b$.

(4) Suppose that $\mathbb{Z}_2$ is ordered. Then, $1 > 0$

$\underset{\mathbb{Z}_2 \text{ ordered}}{\longrightarrow} \underbrace{1+1}_{\substack{\| \\ 0}} > 0$, i.e. $0 > 0$, a contradiction.

So, $\mathbb{Z}_2$ is not ordered.

5 Let $A := \{x \in \mathbb{R}$ s.t. $x > 0$ and $x^2 < 2\}$.
- $A \neq \emptyset$ ($1 \in A$). - A bounded from above (say, by 2).

So: Since $\mathbb{R}$ is complete, A has a least upper bound $b \in \mathbb{R}$.

Since $\mathbb{R}$ is ordered, exactly one of the following holds:

$b^2 < 2$ or $b^2 = 2$ or $b^2 > 2$.

- Suppose that $b^2 < 2$. Then, find $\underset{\in \mathbb{R}}{\varepsilon > 0}$ s.t. $(b+\varepsilon)^2 < 2$; then, $b+\varepsilon \in A$, and $b+\varepsilon > \underset{\text{an upper bound of } A}{b}$, a contradiction.

- Suppose that $b^2 > 2$. Then, find $\underset{\in \mathbb{R}}{\varepsilon > 0}$ s.t. $(b-\varepsilon)^2 > 2$ and $b-\varepsilon > 0$. Then, $b-\varepsilon$ will be an upper bound of A, smaller than b, the least upper bound of A; contradiction.

<u>So:</u> $b^2 = 2$.

(6) Suppose that b, c are both least upper bounds of A. Then:

$b \le c$, as b is a least upper bound of A and c is an upper bound of A.

And: If $b < c$, then we have a contradiction, as the upper bound b of A cannot be smaller than the least upper bound c of A.

So, $b = c$.

(7) • 1 is an upper bound of $(0,1)$ (as $a \le 1, \forall a \in (0,1)$).

• Suppose that there exists an upper bound c of $(0,1)$,

with $c < 1$. We have that

$\frac{1}{2} \le c$ (as $\frac{1}{2} \in (0,1)$), so $\frac{c+1}{2} \in (0,1)$.

And: $\underbrace{\frac{c+1}{2}}_{\text{an element of } (0,1)} > \underbrace{c}_{\text{an upper bound of } (0,1)}$, a contradiction.

8 (i) Correct: $\sup A$ is an upper bound of A.

(ii) Correct: If x is an upper bound of A, then

$\sup A \leq x$ (by definition of least upper bound).

And: if $x \geq \underbrace{\sup A}_{\text{which in turn is } \geq a \ \forall a \in A, \text{ as an upper bound of } A}$, then $x \geq a$, $\forall a \in A$, so x is an upper bound of A.

(iii) Wrong: see 7.

9 (i) It suffices to show that $\sup B$ is an upper bound of A (because then, by definition of the least upper bound $\sup A$ of A: $\sup A \leq \sup B$)

So: let $a \in A$. We will show that $a \leq \sup B$.

Indeed, $a \in A \subseteq B$, so $a \in B$, so $a \leq \sup B$,

as $\sup B$ is an upper bound of B.

So, $a \leq \sup B \ \forall a \in A \longrightarrow \sup B$ an upper bound of $A \Rightarrow$

$\Rightarrow \sup A \leq \sup B.$

(ii) $\max A$:

– is an upper bound of A (by its definition)

and – If b is an upper bound of A, then
$\max A \leq b$
(as $a \leq b$ $\forall a \in A$, since b is an upper bound of A, so in particular $\underbrace{\max A}_{\in A} \leq b$).

So, $\max A$ is the least upper bound of A.

$\sup (0,1) = 1 \notin A$, so $(0,1)$ doesn't have a maximal element.

10 Consider the set $-A = \{-a : a \in A\}$.

- $-A$ is bounded from above: Indeed, A is bounded from below; let b be a lower bound of A. Then, $b \leq a$ $\forall a \in A$
$\longrightarrow -b \geq -a, \ \forall a \in A,$
because $a \geq 0 \Rightarrow -a \leq 0$, $\forall a \in$ (ordered field).
i.e. $-b \geq \tilde{a}, \ \forall \tilde{a} \in -A \rightarrow$

$\Rightarrow -b$ is a lower bound of A.

- Since $-A \subseteq \mathbb{R}$ is bounded from above,

  $-A$ has a least upper bound $c \in \mathbb{R}$. I.e:

$\rightsquigarrow$ c is an upper bound of A,

and $\rightsquigarrow$ $c \leq d$, $\forall d$ upper bound of A.

We will show that $-c$ is the greatest lower bound of $-A$. To do that, we need to show the following two:

$\rightsquigarrow$ $-c$ is a lower bound of A: Let $a \in A$.

Then, $-a \in -A \underset{\substack{c \text{ upper} \\ \text{bound of } A}}{\Longrightarrow} -a \leq c \Rightarrow a \geq -c$.

So: $-c \leq a$, $\forall a \in A \Rightarrow -c$ a lower bound of A.

$\rightsquigarrow$ $d \leq -c$, $\forall d$ lower bound of A:

Let d be a lower bound of $A \Rightarrow$

$d \leq a$, $\forall a \in A \Longrightarrow -a \leq -d$, $\forall a \in A$, i.e.

$\tilde{a} \leq -d$, $\forall \tilde{a} \in -A \Longrightarrow$

$\Rightarrow -d$ an upper bound of A

$\Rightarrow \underbrace{c}_{\substack{\text{the least} \\ \text{upper bound of} \\ -A}} \leq -d \Rightarrow d \leq -c.$

So, $d \leq -c \ \ \forall d$ lower bound of A.

So, $-c$ is the greatest lower bound of A.
So, A has a greatest lower bound.