Lecture 7 9 Sep 2016

Theorems that guarantee convergence

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Theorem: Every monotone, bounded sequence converges.

uses completeness of $\mathbb{R}$!

More precisely:

(a) If $(a_n)_{n\in\mathbb{N}}$ is increasing (i.e. $a_1 \le a_2 \le a_3 \le \dots$)

and bounded from above,

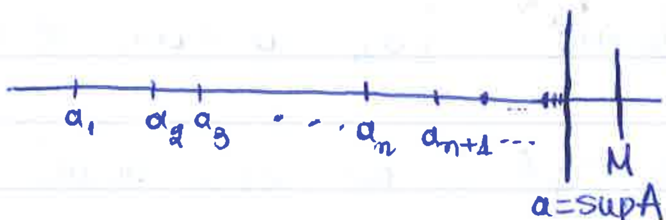
then $a_n \longrightarrow a$, for some $a \in \mathbb{R}$.

(b) If $(a_n)_{n\in\mathbb{N}}$ is decreasing (i.e. $a_1 \geq a_2 \geq a_3 \geq \dots$)

and bounded brom below,

then $a_n \longrightarrow a$, for some $a \in \mathbb{R}$.

Proof: (a)



(Idea: the limit of $(a_n)_{n\in\mathbb{N}}$ will be the $\sup\{a_n : n\in\mathbb{N}\}$.)

Let $A := \{a_n : n \in \mathbb{N}\}$. We have:

$A \neq \emptyset$ (as $a_1 \in A$, for instance) and A bounded from above

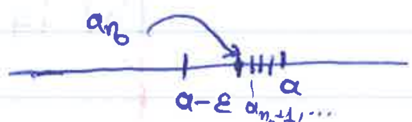
($(a_n)_{n\in\mathbb{N}}$ is bounded, so $\exists M>0$ s.t. $a_n \le M$, $\forall n\in\mathbb{N}$. This M is an upper bound of A).

Since $\mathbb{R}$ is complete, A has a least upper bound in $\mathbb{R}$, $\sup A$.

Let $a=\sup A$; we will show that $a_n \longrightarrow a$:

Let $\varepsilon>0$. $a-\varepsilon<a$, so $a\text{-}\varepsilon$ is not an upper bound

the least upper bound of A



of A, so $\exists n_0\in\mathbb{N}$ s.t. $a_{n_0} > a-\varepsilon$.

Now, since $(a_n)_{n\in\mathbb{N}}$ is increasing, we have $a_n \ge a_{n_0}$ $\forall n\ge n_0$, so $\boxed{a_n > a-\varepsilon \ \forall n\ge n_0}$ $(*_1)$.

And clearly $\boxed{a_n \le a < a+\varepsilon \ \forall n\ge n_0}$ $(*_2)$,

as $a=\sup A$ is an upper bound of A.

By $(*_1)$, $(*_2)$, we have that

$$a-\varepsilon < a_n < a+\varepsilon, \ \forall n\ge n_0.$$

Since ε was arbitrary, we have that $a_n \longrightarrow a$.

(b) Exercise.

→ **ex:** The sequence $a_n = \boxed{\left(1+\frac{1}{n}\right)^n}$, $\forall n \in \mathbb{N}$, converges.

Proof:

- $(a_n)_{n\in\mathbb{N}}$ is increasing (in fact, we can show it is strictly increasing):

We want to check if $\boxed{\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}, \forall n \in \mathbb{N}}$.

$$(*) \Leftrightarrow \left(\frac{n+1}{n}\right)^n < \left(\frac{n+2}{n+1}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^n \cdot \frac{n+2}{n+1}$$

$$\Leftrightarrow \underbrace{\frac{n+1}{n+2}}_{=\,1-\frac{1}{n+2}} < \Bigg(\underbrace{\frac{n\cdot(n+2)}{(n+1)^2}}_{=\,1-\frac{1}{(n+1)^2}}\Bigg)^n \Leftrightarrow 1-\frac{1}{n+2} < \left(1-\frac{1}{(n+1)^2}\right)^n.$$

By Bernoulli's inequality for $a = -\frac{1}{(n+1)^2} > -1$, we have:

$\left(1-\frac{1}{(n+1)^2}\right)^n \geq 1-\frac{n}{(n+1)^2}$, so it suffices to check that $1-\frac{n}{(n+1)^2} > 1-\frac{1}{n+2}$, which is true $\forall n\in\mathbb{N}$ (check it).

- $(a_n)_{n\in\mathbb{N}}$ is bounded from above:

To show this we introduce the sequence

$$b_n = \boxed{\left(1+\frac{1}{n}\right)^{n+1}}, \quad \forall n\in\mathbb{N}.$$

$(b_n)_{n\in\mathbb{N}}$ is decreasing (exercise; use Bernoulli's inequality again),

and clearly $a_n < b_n \quad \forall n\in\mathbb{N}$,

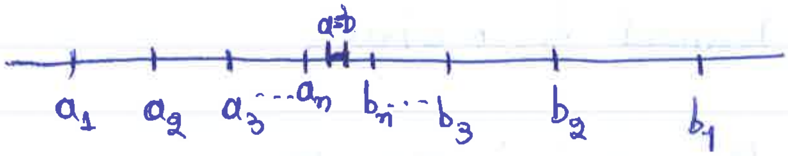
$\hookrightarrow < b_1$, as $(b_n)_{n\in\mathbb{N}}$ is decreasing

so $a_n < b_1 \quad \forall n\in\mathbb{N}$, which means that $b_1 = (1+1)^2 = 4$ is an upper bound for $(a_n)_{n\in\mathbb{N}}$.

So: since $(a_n)_{n\in\mathbb{N}}$ is increasing and bounded from above, it converges.

⚠ Similarly, $(b_n)_{n\in\mathbb{N}}$ is decreasing and bounded below by a_1 $(a_1 < a_n < b_n, \forall n\in\mathbb{N})$, so $(b_n)_{n\in\mathbb{N}}$ converges too.

In fact: let $a := \lim\limits_{n\to+\infty} a_n$ and $b := \lim\limits_{n\to+\infty} b_n$. Then, $\boxed{a=b}$:



$$b_n = \underbrace{a_n}_{\substack{\downarrow n\to+\infty \\ a}} \cdot \underbrace{\left(1+\frac{1}{n}\right)}_{\substack{\downarrow n\to+\infty \\ 1}} \longrightarrow a, \quad \text{so} \quad (b=) \lim_{n\to+\infty} b_n = a. \qquad \blacksquare$$

→ <u>Def:</u> $\boxed{e} := \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n \quad \left(= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1}\right).$

⚠ Using that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \quad \forall n\in\mathbb{N},$$

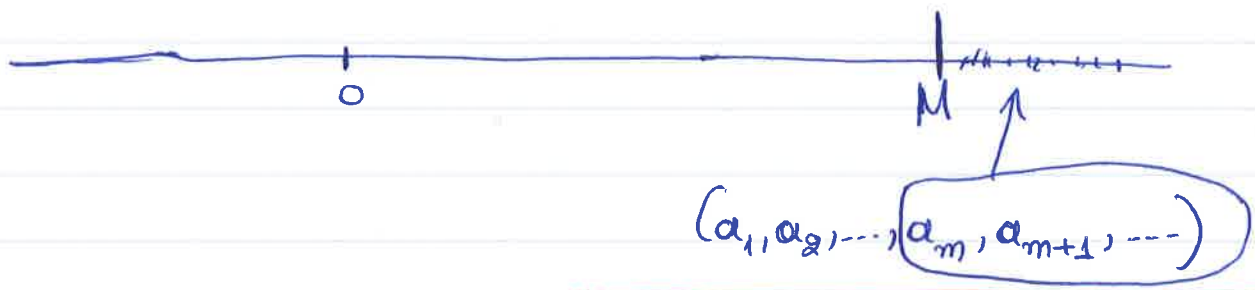
you can find very good approximations of e (as good as you wish). In particular: $e = 2.718\ldots$

→ **$+\infty$, $-\infty$ as limits:** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

I want a definition of $a_n \longrightarrow +\infty$ that will allow me to say that : "a_n is large for large n".

In particular I want the following to hold:

" No matter how large a number someone gives me, I can find whole final part of $(a_n)_{n\in\mathbb{N}}$ above that number ".



Def: We say that $a_n \longrightarrow \underline{+\infty}$ as $n \longrightarrow +\infty$ if: $\forall M>0, \exists n_0 = n_0(M) \in \mathbb{N}$ s.t.: $a_n > M, \forall n \geq n_0$

This is equivalent to saying:

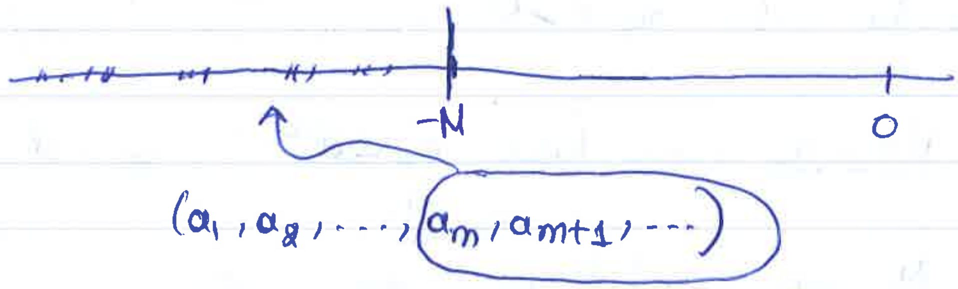
$a_n \longrightarrow \underline{+\infty}$ as $n \longrightarrow +\infty$ if $\forall M>0$, there exists a whole final part of $(a_n)_{n\in\mathbb{N}}$ in $(M, +\infty)$.

Similarly: We say that $a_n \longrightarrow \underline{-\infty}$ as $n \longrightarrow +\infty$ if: $\forall M>0, \exists n_0 = n_0(M) \in \mathbb{N}$ s.t.: $a_n < -M, \forall n \geq n_0$.

This is equivalent to saying:

$a_n \longrightarrow +\infty$ as $n \longrightarrow +\infty$ if

$\forall M>0$, there exists some final part of $(a_n)_{n\in\mathbb{N}}$ in $(-\infty,-M)$.



$(a, a^2, a^3, \dots)$

Prop: Let $a>0$. We define $a_n := a^n$, $\forall n\in\mathbb{N}$. Then:

- If $0<a<1$, then $a_n = a^n \longrightarrow 0$.
- If $a=1$, then $a_n = 1 \longrightarrow 1$.
- If $a>1$, then $a_n = a^n \longrightarrow +\infty$.

Proof: - Let $\boxed{a>1}$. Let $M>0$. We will show that there exists some $n_0\in\mathbb{N}$ s.t.: $\forall n\geq n_0$, $a_n > M$. Indeed:

Since $a>1$, there exists some $\vartheta>0$ s.t. $a=1+\vartheta$ (actually, $\vartheta = a-1$).

(you can just use ② vii and vi from the 3rd weekly assignment, instead of what follows)

Then, $a_n = a^n = (1+\vartheta)^n \geq 1 + n\vartheta > n\vartheta, \; \forall n \in \mathbb{N}.$

by Bernoulli's inequality, which we can apply because $\vartheta > 0$ (and thus > -1)

By the Archimedean property of the reals, there exists some $n_0 \in \mathbb{N}$ s.t. $n_0 \cdot \vartheta > M$ (i.e., we can make $n\vartheta$ as large as we want).

Then, $\underline{\forall n \geq n_0}$, we have $a_n > n\vartheta \geq n_0\vartheta > M$,

so $\underline{a_n > M}$.

Since M was arbitrary, $a_n \longrightarrow +\infty$.

- Let $0 < a < 1$. Then, $\frac{1}{a} > 1 \implies \underbrace{\left(\frac{1}{a}\right)^n}_{= \frac{1}{a^n}} \longrightarrow +\infty \implies a^n \to 0.$

use that if $c_n \to +\infty$ then $\frac{1}{c_n} \to 0$ (exercise).

(A second proof: Since $0 < a < 1$, there exists some $\vartheta > 0$ s.t. $a = \frac{1}{1+\vartheta}$. Then: $\forall n \in \mathbb{N}$,

$0 < a_n = a^n = \frac{1}{(1+\vartheta)^n} \leq \frac{1}{1+n\vartheta} < \frac{1}{n\vartheta} = \frac{1}{\vartheta} \cdot \frac{1}{n}.$)

Bernoulli's inequality

So, $0 < a_n < \frac{1}{\theta} \cdot \frac{1}{n}$, $\forall n \in \mathbb{N}$.

$\downarrow_{n \to +\infty}$ 0 $\qquad \downarrow_{n \to +\infty}$ 0

By the sandwich lemma, $a_n \longrightarrow 0$. ■

→ **The ratio test:** Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, with $a_n \neq 0$ $\forall n \in \mathbb{N}$.

Consider $b_n = \dfrac{|a_{n+1}|}{|a_n|}$, $\forall n \in \mathbb{N}$.

- If $\lim\limits_{n\to+\infty} b_n = p < 1$, then $a_n \longrightarrow 0$.
- If $\lim\limits_{n\to+\infty} b_n = p > 1$, (maybe $+\infty$) then $|a_n| \longrightarrow +\infty$.
- If $\lim\limits_{n\to+\infty} b_n = 1$, then the test is inconclusive.

Proof: - Suppose that $(0 \leq) \lim\limits_{n\to+\infty} b_n = p < 1$.

a final part of $(b_n)_{n\in\mathbb{N}}$ in here.

Pick some number ϑ between p and 1; say, $\vartheta = \frac{1+p}{2}$, and indeed for this ϑ we have $\rho < \vartheta < 1$.

Since $b_n \longrightarrow \rho < \vartheta$, there exists some $n_0 \in \mathbb{N}$ s.t.:

$\forall n \geq n_0, \quad b_n < \vartheta$

(apply the definition of limit for $\varepsilon = \vartheta - \rho$, i.e. the neighbourhood $(\rho - (\vartheta-\rho), \underbrace{\rho + (\vartheta - \rho)}_{= \vartheta})$ of ρ)

i.e., $\forall n \geq n_0$, we have: $\dfrac{|a_{n+1}|}{|a_n|} < \vartheta \Leftrightarrow |a_{n+1}| < \vartheta \cdot |a_n|$.

So:
$$\left.\begin{cases} |\cancel{a_{n_0+1}}| < \vartheta \cdot |a_{n_0}| \\ |\cancel{a_{n_0+2}}| < \vartheta \cdot |\cancel{a_{n_0+1}}| \\ \vdots \\ |a_{n_0+k}| < \vartheta \cdot |\cancel{a_{n_0+k-1}}| \end{cases}\right\} , \forall k \in \mathbb{N} \Rightarrow$$

$\Rightarrow |a_{n_0+k}| < \vartheta^k \cdot |a_{n_0}|, \ \forall k \in \mathbb{N}$.

So: $\underset{\substack{\downarrow k \to +\infty \\ 0}}{0} < |a_{n_0+k}| < \underset{\substack{\downarrow k \to +\infty \\ 0}}{\vartheta^k \cdot |a_{n_0}|}$, so, by the sandwich lemma,

$a_{n_0+k} \xrightarrow[k \to +\infty]{} 0$.

Notice that the sequence $(a_{n_0+k})_{k\in\mathbb{N}} = (a_{n_0+1}, a_{n_0+2}, \dots)$ is a final part of $(a_n)_{n\in\mathbb{N}}$, so $(a_n)_{n\in\mathbb{N}}$ has the same limit as $(a_{n_0+k})_{k\in\mathbb{N}}$, i.e. $a_n \longrightarrow 0$ as $n \longrightarrow +\infty$.

- Suppose that $\rho > 1$. Whether $\rho \in \mathbb{R}$ or $\rho = +\infty$, since $b_n \longrightarrow \rho$ there exists some $\underline{\vartheta > 1}$ s.t. : $\forall n \geq n_0$, $b_n > \vartheta$

$\Longleftrightarrow$ $\forall n \geq n_0$, $|a_{n+1}| > \vartheta \cdot |a_n|$.

Work as before to show that $|a_n| \longrightarrow +\infty$ ∎

→ $\boxed{\text{The root test :}}$ Let $(a_n)_{n\in\mathbb{N}}$ be a sequence

Consider $b_n = \sqrt[n]{|a_n|}$, $\forall n \in \mathbb{N}$.

- If $b_n \longrightarrow \rho < 1$, then $a_n \longrightarrow 0$.
- If $b_n \longrightarrow \rho > 1$, then $|a_n| \to +\infty$
  maybe $+\infty$.
- If $b_n \longrightarrow 1$, then the test is inconclusive.

<u>Proof:</u> As for the ratio test (exercise). $\blacksquare$

→ <u>Examples:</u>

<u>1</u> Find the limit of : $a_n = \dfrac{n^3+5n^2+2}{2n^3+9}$, $n \in \mathbb{N}$.

$$a_n = \frac{n^3+5n^2+2}{2n^3+9} = \frac{\cancel{n^3}\cdot\left(1+\frac{5}{n}+\frac{2}{n^3}\right)}{\cancel{n^3}\cdot\left(2+\frac{9}{n^3}\right)} = \frac{\overset{\to 1}{1}+\overset{\to 0}{\frac{5}{n}}+\overset{\to 0}{\frac{2}{n^3}}}{\underset{\downarrow\, 2}{2}+\underset{\downarrow\, 0}{\frac{9}{n^3}}} \xrightarrow[n\to+\infty]{} \frac{1}{2}.$$

extract the largest power in the numerator and the denominator; because a polynomial behaves like its monomial of largest power.

<u>2</u> Find the limit of : $b_n = \underbrace{\dfrac{1}{n^2}+\dfrac{1}{(n+1)^2}+\dots+\dfrac{1}{(2n)^2}}_{n+1 \text{ terms}}$, $n \in \mathbb{N}$,

$n+1$ terms; so, even though each converges to 0, we cannot deduce their sum converges to 0.

generally, when the number of terms <u>depends on n</u>, we think of the <u>sandwich lemma</u>.

Notice that: $\dfrac{1}{(2n)^2} \leq \dfrac{1}{(n+k)^2} \leq \dfrac{1}{n^2}$, $\forall k = 0, 1, \dots, n$.

So : $$\underbrace{\frac{1}{(2n)^2}+\dots+\frac{1}{(2n)^2}}_{n+1 \text{ terms}} \leq \underbrace{\frac{1}{n^2}+\frac{1}{(n+1)^2}+\dots+\frac{1}{(2n)^2}}_{b_n} \leq \underbrace{\frac{1}{n^2}+\dots+\frac{1}{n^2}}_{n+1 \text{ terms}},$$

i.e.
$$\frac{n+1}{(2n)^2} \leq b_n \leq \frac{n+1}{n^2}, \quad \forall n \in \mathbb{N}$$

$$\frac{n+1}{(2n)^2} = \frac{\cancel{n}(1+\frac{1}{n})}{n^{\cancel{2}}(4)} = \frac{1}{4n}\cdot\left(1+\frac{1}{n}\right) \xrightarrow[n\to+\infty]{} 0.$$

$$\frac{n+1}{n^2} = \frac{n(1+\frac{1}{n})}{n^2} = \frac{1}{n}\left(1+\frac{1}{n}\right) \xrightarrow[n\to+\infty]{} 0$$

So, $b_n \longrightarrow 0$ (by the sandwich lemma).

3| Find the limit of: $c_n = \dfrac{n}{2^n}$, $n \in \mathbb{N}$.

this is a ratio, so we hope that, if we try the ratio test, a lot will cancel out.

We have:
$$\frac{|c_{n+1}|}{|c_n|} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{(n+1)\cdot 2^n}{n\cdot 2^{n+1}} = \frac{1}{2}\cdot\frac{n+1}{n} =$$

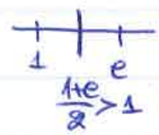
$$= \frac{1}{2}\cdot\frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}} = \frac{1}{2}\cdot\left(1+\frac{1}{n}\right) \longrightarrow \frac{1}{2} < 1.$$

So, $c_n \longrightarrow 0$.

4 Show that $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.

<u>1st way:</u> $\left(1+\frac{1}{n}\right)^{n^2} = \left[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{\downarrow}\right]^{n}$.

e, so $> \frac{1+e}{2}$ (>1) for large n.



So, $\left(1+\frac{1}{n}\right)^{n^2} \underset{\text{for large } n}{>} \left(\frac{1+e}{2}\right)^{n} \longrightarrow +\infty$ as $n \to +\infty$ (since $\frac{1+e}{2} > 1$)

So, $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.

<u>2nd way:</u> $\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^{n} \longrightarrow e > 1$,

so, by the root test, $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.