Stat 134 lec 32

Sec 5.4

Last Time:

Convolution formula for sum $\quad f_{X+Y}(z) = \int_{x=-\infty}^{x=\infty} f_X(x) f_Y(z-x)\,dx$

Gamma $\quad r>0, \lambda>0$

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, \quad x>0$$

where $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x}, \quad r>0$

Beta $\quad r>0, s>0$

$$f_X(x) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1}(1-x)^{s-1}\,dx$$

still need to show that this is a density!

ex $\left.\begin{array}{l} X \sim \text{gamma}(r,\lambda) \\ Y \sim \text{gamma}(s,\lambda) \end{array}\right\}$ indep, $r,s>0$

$Z = X+Y$

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x)\,dx, \quad z>0$$

$$= \int_0^z \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} \cdot \frac{\lambda^s}{\Gamma(s)} (z-x)^{s-1} e^{-\lambda(z-x)}\,dx$$

$$= \int_0^z \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} x^{r-1} (z-x)^{s-1} e^{-\lambda z} \, dx$$

$$= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\lambda z} \int_0^z x^{r-1} (z-x)^{s-1} \, dx$$

let $u = \frac{x}{z}$ to change limit of integration to 1

$x = uz$, $dx = z\,du$

$$= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} e^{-\lambda z} \int_0^1 (uz)^{r-1} (z-uz)^{s-1} z \, du$$

$= z^{r+s-1} u^{r-1} (1-u)^{s-1}$

$$\Rightarrow f_Z(z) = \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \left(\int_0^1 u^{r-1} (1-u)^{s-1} \, du \right) \cdot z^{r+s-1} \cdot e^{-\lambda z}$$

constant part (no z)

variable part (z)

we see variable part is from gamma$(r+s, \lambda)$

so, $f_Z(z)$ must be gamma$(r+s, \lambda)$.

so constant part must be $\dfrac{\lambda^{r+s}}{\Gamma(r+s)}$

<u>i.e</u> $\dfrac{\cancel{\lambda^{r+s}}}{\Gamma(r)\Gamma(s)} \displaystyle\int_0^1 u^{r-1}(1-u)^{s-1}\,du = \dfrac{\cancel{\lambda^{r+s}}}{\Gamma(r+s)}$

<u>i.e</u> $\displaystyle\int_0^1 \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} u^{r-1}(1-u)^{s-1}du = \boxed{1}$

So we shown 2 things.

① $Z = X+Y \sim \text{Gamma}(r+s, \lambda)$, $r,s>0$

② $\dfrac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} u^{r-1}(1-u)^{s-1}$ is a density called beta (r,s) $r,s>0$.

Distribution of ratios

let X, Y indep positive RV

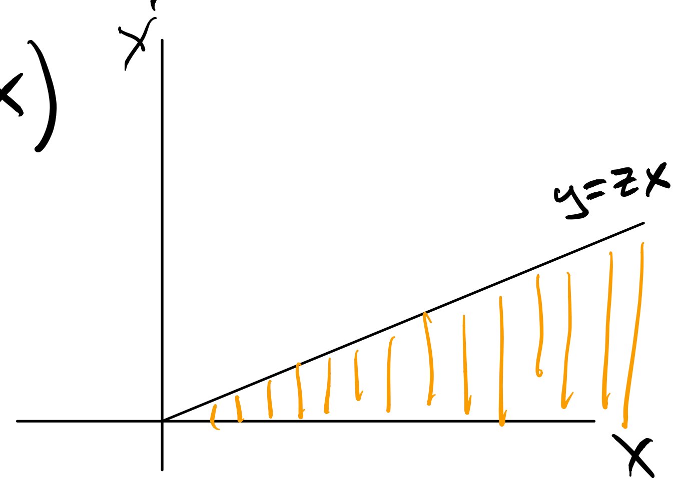
let $Z = \frac{Y}{X}$

find $f_Z(z)$

cdf method:

$$F_Z(z) = P(Z < z) = P\left(\frac{Y}{X} < z\right)$$

$$= P(Y < zX)$$



$$= \int_{x=0}^{x=\infty} \int_{y=0}^{y=zX} f_X(x) f_Y(y)\, dy\, dx$$

$$= \int_0^{\infty} f_X(x) \underbrace{\int_{y=0}^{y=zx} f_Y(y)\, dy}_{= F_Y(zx)}\, dx$$

$$= \int_0^\infty f_X(x)\, F_Y(zx)\, dx \,,$$

$$\frac{d}{dz} F_Y(\overset{y(z)}{zx}) = \underbrace{F_Y'(y(z))}_{= f_Y(zx)} \cdot y'(z) \quad x$$

$$\frac{d}{dz} F(z) = \int_0^\infty f_X(x) \frac{d}{dz} F_Y(zx)\, dx$$

$$= \boxed{\int_0^\infty f_X(x)\, f_Y(zx)\, x\, dx}$$

Convolution formula for ratios,

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1. Let X,Y be i.i.d. expon(1) and $Z = Y/X$. The integrand in the convolution formula

$$f(z) = \int_0^\infty f_X(x) f_Y(zx) x dx$$

is:

$X \sim \text{Gamma}(r, \lambda)$

$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}$

a gamma(1,z+1)

b gamma(2,z+1)

c gamma(2,z)

d none of the above

$X, Y \sim \exp(1)$

$f_X(z) = \int_0^\infty e^{-x} e^{-zx} x\,dx$

$= \int_0^\infty x e^{-(z+1)x}$

gamma (2, z+1)

using the fact that the integrand is gamma (2, z+1) we can solve the integral with a trick

$= \frac{\Gamma(2)}{(z+1)^2} \underbrace{\int_0^\infty \frac{(z+1)^2}{\Gamma(2)} x^{2-1} e^{-(z+1)x}}_{=1} = \frac{\Gamma(2)^{1}}{(z+1)^2}$

$= \frac{1}{(z+1)^2}$