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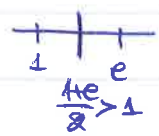
4 Show that $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.

__1st way:__ $\left(1+\frac{1}{n}\right)^{n^2} = \Bigg[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{\downarrow \; e,\text{ so } > \frac{1+e}{2} \, (>1) \text{ for large } n}\Bigg]^{n}$.



So, $\left(1+\frac{1}{n}\right)^{n^2} \underset{\text{for large } n}{>} \left(\frac{1+e}{2}\right)^{n} \longrightarrow +\infty$ as $n \to +\infty$ (since $\frac{1+e}{2} > 1$)

So, $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.

__2nd way:__ $\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \left(1+\frac{1}{n}\right)^{n} \longrightarrow e > 1$,

so, by the root test, $\left(1+\frac{1}{n}\right)^{n^2} \longrightarrow +\infty$.

__Lecture 8:__ 12 Sep 2016. 1

→ __Def:__ Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

A sequence $(b_n)_{n\in\mathbb{N}}$ is called a subsequence of $(a_n)_{n\in\mathbb{N}}$

if there exist $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots$ (indices of the original sequence) in $\mathbb{N}$,

s.t. $b_n = a_{k_n}$, $\forall n \in \mathbb{N}$.

In other words, the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of the sequence $(a_n)_{n\in\mathbb{N}}$

$$\left(\text{which was a map } a: \begin{array}{ccccccccc} 1 & & 2 & \cdots & \boxed{k_1} & \cdots & \boxed{k_2} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ a(1) & & a(2) & & a(k_1) & & a(k_2) & \end{array}\right)$$

is another map from $\mathbb{N}$ to $\mathbb{R}$, that only keeps the information of where a sends $k_1, k_2, k_3, \dots$,

and which we see as the map

$$(a_{k_n})_{n\in\mathbb{N}}: \begin{array}{cccc} 1 & 2 & 3 & \cdots \\ \downarrow & \downarrow & \downarrow & \\ a(k_1) & a(k_2) & a(k_3) & \end{array}$$

That is, $(a_n)_{n\in\mathbb{N}}$ is $(a_1, a_2, \dots, a_{k_1}, \dots, a_{k_2}, \dots, a_{k_3}, \dots)$

and $(a_{k_n})_{n\in\mathbb{N}}$ is $(a_{k_1}, a_{k_2}, a_{k_3}, \dots)$.

⚠ I am NOT allowed to jump from the index k_2 back to k_1: the terms will appear in the subsequence in the same order as in the sequence (by definition)

strictly increasing

the indices we keep for the subsequence

ex. • Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and $k_n = 2n$. What is $(a_{k_n})_{n\in\mathbb{N}}$?

It is the sequence $(a_{2n})_{n\in\mathbb{N}} = (a_2, a_4, a_6, \dots)$.

strictly increasing

- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and $k_n = 2n-1$. What is $(a_{k_n})_{n\in\mathbb{N}}$?

  It is the sequence $(a_{2n-1})_{n\in\mathbb{N}} = (a_1, a_3, a_5, \dots)$

strictly increasing.

- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and $k_n = n^2$. What is $(a_{k_n})_{n\in\mathbb{N}}$?

  It is the sequence $(a_{n^2})_{n\in\mathbb{N}} = (a_1, a_4, a_9, a_{16}, \dots)$.

- Every final part of $(a_n)_{n\in\mathbb{N}}$ is a subsequence of $(a_n)_{n\in\mathbb{N}}$:

  Let $(a_m, a_{m+1}, \dots)$ be a final part of $(a_n)_{n\in\mathbb{N}}$.
  Then, $(a_m, a_{m+1}, \dots) = (a_{m+n-1})_{n\in\mathbb{N}}$,

  and $m+1-1 < m+2-1 < m+3-1 < \dots$,

  so $(a_{m+n-1})_{n\in\mathbb{N}}$ is indeed a subsequence of $(a_n)_{n\in\mathbb{N}}$.

⚠ We require $k_1 \lneq k_2 \lneq k_3 \lneq \dots$ in the definition of a subsequence so that the same index isn't repeated.

→ From the above, we see that, to create a subsequence of $(a_n)_{n\in\mathbb{N}}$:

- We pick some $k_1 \in \mathbb{N}$; a_{k_1} will be the first term of the subsequence.

- We pick some $k_2 > k_1$ in $\mathbb{N}$; a_{k_2} will be the second term of the subsequence.

- We pick some $k_3 > k_2$ in $\mathbb{N}$: a_{k_3} will be the third term of the subsequence

⋮

→ Observation: If $k_1 < k_2 < \dots < k_n < k_{n+1} < \dots$ in $\mathbb{N}$,

then $k_n \geq n$, $\forall n \in \mathbb{N}$.

i.e., the n-th term of the subsequence comes after the n-th term of the original sequence (or they are the same).

Proof: - $k_1 \geq 1$, since $k_1 \in \mathbb{N}$.

- Suppose that $k_m \geq m$, for some $m \in \mathbb{N}$. Then,

$k_{m+1} > k_m$ (in $\mathbb{N}$), i.e. $k_{m+1} \geq k_m + 1 \geq m+1$. ∎

→ <u>Observation:</u> Every sequence has infinitely many subsequences.

<u>Proof:</u> A sequence has infinitely many final parts, each of which is a subsequence of the sequence. ∎

→ <u>Prop:</u> Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, with $a_n \to a$ ($a \in \mathbb{R}$ or $a = +\infty$ or $a = -\infty$).

Then, $a_{k_n} \xrightarrow[n\to+\infty]{} a$, for any subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$.

<u>I.e.:</u> If a sequence has a limit, then <u>all</u> its subsequences have <u>the same limit</u>, that of the original sequence.

<u>Proof:</u> - Suppose that $a \in \mathbb{R}$. Let $(a_{k_n})_{n\in\mathbb{N}}$ be a subsequence of $(a_n)_{n\in\mathbb{N}}$.

<u>Let $\varepsilon > 0$.</u> We want to show that, for some $n_0 \in \mathbb{N}$: $|a_{k_n} - a| < \varepsilon$, $\forall n \geq n_0$.

Since $a_n \to a$, there exists some $n_0 \in \mathbb{N}$: $|a_n - a| < \varepsilon$, $\forall n \geq n_0$.

Now, by the observation earlier, we have that:
$\forall n \in \mathbb{N}$, $k_n \geq n$; in particular, $k_n \geq n \geq n_0$, $\forall n \geq n_0$.

So, $\forall n \geq n_0$: $|a_{k_n} - a| < \varepsilon$. Since ε was arbitrary,

we have that $a_{k_n} \to a$ as $n \to +\infty$.

- Work similarly for $a = +\infty$ and $a = -\infty$. ■

→ Application: The sequence $a_n = (-1)^n$, $n \in \mathbb{N}$, doesn't converge, and also $a_n \not\to +\infty$, $a_n \not\to -\infty$.

Proof: Suppose $a_n \to a$, for some $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. Then, for any subsequence $(a_{k_n})_{n \in \mathbb{N}}$ we also

have $a_{k_n} \xrightarrow[n \to +\infty]{} a$. But:

$a_{2n} = (-1)^{2n} = 1 \quad \forall n \in \mathbb{N}$, so $a_{2n} \to 1$,

while $a_{2n-1} = (-1)^{2n-1} = -1 \quad \forall n \in \mathbb{N}$, so $a_{2n-1} \to -1$,

and $1 \neq -1$, a contradiction.

Bolzano - Weierstrass theorem :

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: For the proof we need the Proposition that follows, that states that : every sequence in $\mathbb{R}$ has a monotone subsequence.

Once we know this, the proof of the Bolzano-Weierstrass theorem follows as such:

- Let $(a_n)_{n\in\mathbb{N}}$ be a bounded real sequence.

  (that is, $\exists M>0$ s.t. $|a_n|<M$, $\forall n\in\mathbb{N}$).

- By the Proposition that follows, $(a_n)_{n\in\mathbb{N}}$ has a monotone subsequence $(a_{k_n})_{n\in\mathbb{N}}$

  (note that boundedness of $(a_n)_{n\in\mathbb{N}}$ is not required for this).

- $(a_{k_n})_{n\in\mathbb{N}}$ is bounded (as $|a_n|<M\ \forall n\in\mathbb{N}$
  $\Rightarrow |a_{k_n}|<M\ \forall n\in\mathbb{N}$).

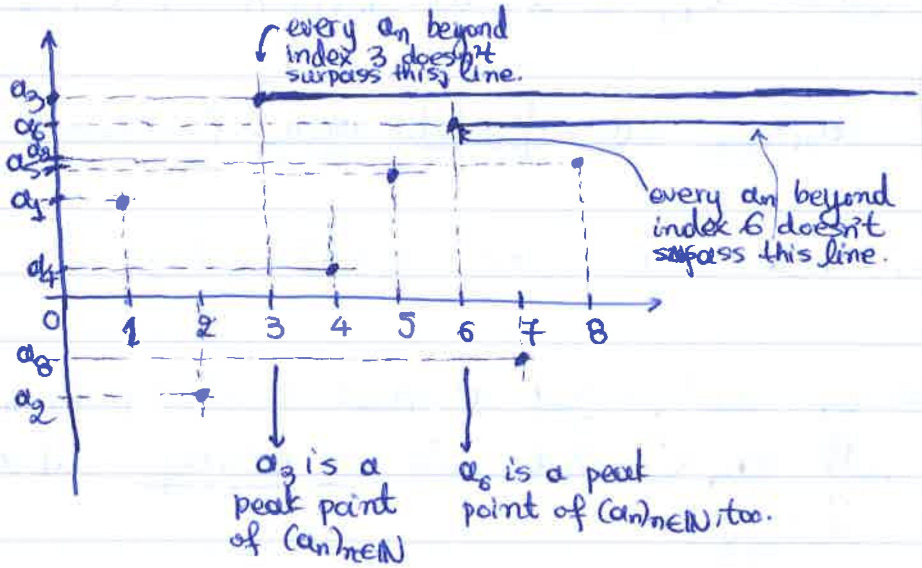
So, $(a_{k_n})_{n\in\mathbb{N}}$ is monotone and bounded $\Rightarrow (a_{k_n})_{n\in\mathbb{N}}$ converges

So, the proof will be complete once we prove the following Proposition.

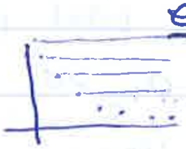
→ Prop: $\boxed{\text{Every sequence in } \mathbb{R} \text{ has a monotone subsequence.}}$

Proof: First, we need the following definition:

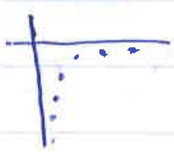
→ Def: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We say that a term a_m of $(a_n)_{n\in\mathbb{N}}$ is a peak point of $(a_n)_{n\in\mathbb{N}}$ if: $a_m \geq a_n, \forall n \geq m$.



ex:



- The sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = \frac{1}{n}$ $\forall n\in\mathbb{N}$ is decreasing, so all its terms are peak points.



- The sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n = -\frac{1}{n}$ $\forall n\in\mathbb{N}$ is increasing, so it has no peak points.

Let us now go back to our proof:

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

Case 1: $(a_n)_{n\in\mathbb{N}}$ has infinitely many peak points.

I.e., $\exists\, k_1 < k_2 < \dots < k_n < k_{n+1} < \dots \in \mathbb{N}$ s.t.: a_{k_n} is a peak point of $(a_n)_{n\in\mathbb{N}}$, for all $n\in\mathbb{N}$.

Then, $(a_{k_n})_{n\in\mathbb{N}}$ is decreasing: $\forall n\in\mathbb{N}$, $a_{k_n} \geq a_{k_{n+1}}$, because a_{k_n} is a peak point of $(a_n)_{n\in\mathbb{N}}$.

Case 2: $(a_n)_{n\in\mathbb{N}}$ has finitely many (or none) peak points.

Then, there exists some $n_0\in\mathbb{N}$ s.t.:

$\forall n \geq n_0$, a_n is not a peak point of $(a_n)_{n\in\mathbb{N}}$.

We will now construct an increasing subsequence of $(a_n)_{n\in\mathbb{N}}$:

We set $k_1 = n_0$. Since a_{k_1} is not a peak point of $(a_n)_{n\in\mathbb{N}}$, there exists $k_2 > k_1$ s.t. $a_{k_2} > a_{k_1}$. Since a_{k_2} is not a peak point of $(a_n)_{n\in\mathbb{N}}$, there exists $k_3 > k_2$ s.t. $a_{k_3} > a_{k_2}$, and so on.

We thus inductively find $k_1 < k_2 < \ldots < k_n < k_{n+1} < \ldots \in \mathbb{N}$,

$$\text{s.t. } a_{k_1} < a_{k_2} < \ldots < a_{k_n} < a_{k_{n+1}} < \ldots$$

The subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ is (strictly) increasing. ■

⚠ We will later prove that every sequence has a convergent subsequence, <u>in a compact metric space</u>. The above is a corollary of this more general result because a bounded sequence in $\mathbb{R}$ is always contained in some closed interval, which is a compact metric space.

Let us now see a second proof of the Bolzano-Weierstrass theorem. It will follow from the following theorem on nested intervals, which generalises to any metric space for nested compact sets (Cantor's intersection theorem).

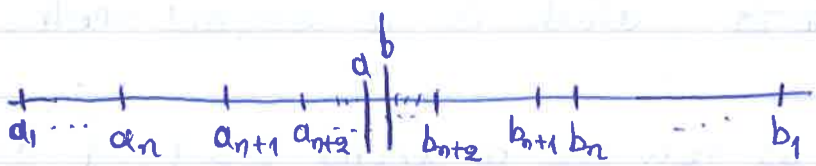
Nested intervals theorem:

Let $[a_1,b_1] \supseteq [a_2,b_2] \supseteq \dots \supseteq [a_n,b_n] \supseteq [a_{n+1},b_{n+1}] \supseteq \dots$, a sequence of nested intervals. Then:

- $\bigcap_{n=1}^{+\infty} [a_n,b_n] \neq \emptyset$ (In fact, $\bigcap_{n=1}^{+\infty} [a_n,b_n] = [a,b]$, where $a = \lim_{n\to+\infty} a_n = \sup\{a_n : n\in\mathbb{N}\}$ and $b = \lim_{n\to+\infty} b_n = \inf\{b_n : n\in\mathbb{N}\}$)
- If, in particular, $b_n - a_n \longrightarrow 0$, then

$$\bigcap_{n=1}^{+\infty} [a_n,b_n] = \{x\}, \text{ for some } x\in\mathbb{R}.$$
$$\left(x = \lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n\right)$$

Proof:



Since $[a_1,b_1] \supseteq [a_2,b_2] \supseteq \dots \supseteq [a_n,b_n] \supseteq [a_{n+1},b_{n+1}] \supseteq \dots$,

we have that $(a_n)_{n\in\mathbb{N}}$ is increasing
and $(b_n)_{n\in\mathbb{N}}$ is decreasing,

and $(a_n)_{n\in\mathbb{N}}$ bounded from above (as $a_n \leq b_1 \ \forall n\in\mathbb{N}$),

and $(b_n)_{n\in\mathbb{N}}$ bounded from below (as $b_n \geq a_1 \ \forall n\in\mathbb{N}$).

So: $a_n \to a$ for $a = \sup\{a_n : n \in \mathbb{N}\}$,
and $b_n \to b$ for $b = \inf\{b_n : n \in \mathbb{N}\}$.

Since $a_n \leq b_n$ $\forall n \in \mathbb{N}$, we have $a \leq b$ (exercise 10(i) in Weekly Assignment 2).

We will now show that $\bigcap_{n=1}^{+\infty} [a_n, b_n] = [a, b]$:

- Let $x \in \bigcap_{n=1}^{+\infty} [a_n, b_n]$. Then, $x \in [a_n, b_n]$, $\forall n \in \mathbb{N}$; i.e.:

$$\underset{\downarrow n \to +\infty}{a_n} \leq \underset{\downarrow n \to +\infty}{x} \leq \underset{\downarrow n \to +\infty}{b_n}, \quad \forall n \in \mathbb{N}. \text{ Thus, } a \leq x \leq b$$
$$a \qquad x \qquad b$$

(again by 10(i) in Weekly Assignment 2).

So, $x \in [a, b]$. Therefore: $\boxed{\bigcap_{n=1}^{+\infty} [a_n, b_n] \subseteq [a, b]}$ $(*_1)$

- Let $x \in [a, b]$. Then, $a_n \leq a \leq x \leq b \leq b_n$ $\forall n \in \mathbb{N}$,
so $a_n \leq x \leq b_n$, $\forall n \in \mathbb{N}$,
i.e. $x \in [a_n, b_n]$, $\forall n \in \mathbb{N}$.

So, $x \in \bigcap_{n=1}^{+\infty} [a_n, b_n]$. Therefore, $\boxed{[a, b] \subseteq \bigcap_{n=1}^{+\infty} [a_n, b_n]}$ $(*_2)$

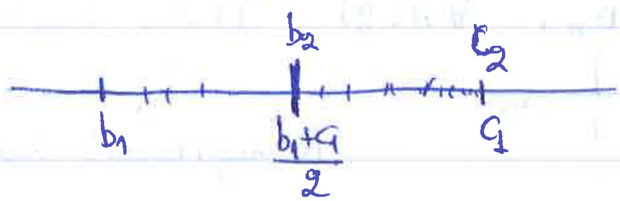
By $(*_1)$ and $(*_2)$, $\bigcap_{n=1}^{+\infty} [a_n, b_n] = [a, b] \Rightarrow \bigcap_{n=1}^{+\infty} [a_n, b_n] \neq \emptyset$.

In particular, if $b_n - a_n \to 0$, then $\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n$, i.e. $a=b$, and $[a,b]=\{a\}$. So, if $b_n - a_n \to 0$,

then $\bigcap_{n=1}^{+\infty} [a_n, b_n]$ is a singleton. ∎

→ <u>Second proof of Bolzano-Weierstrass theorem :</u>

Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. We will show that it has a convergent subsequence:



Since $(a_n)_{n\in\mathbb{N}}$ is bounded, there exist $b_1 < c_1 \in \mathbb{R}$ s.t.:

$$b_1 \leq a_n \leq c_1, \quad \forall n \in \mathbb{N}.$$

We split $[b_1, c_1]$ in two intervals of equal length,

$[b_1, \frac{b_1+c_1}{2}]$ and $[\frac{b_1+c_1}{2}, c_1]$. At least one of these

two intervals contains infinitely many terms of $(a_n)_{n\in\mathbb{N}}$. We pick one such interval, and denote it by $[b_2, c_2]$.

Inductively, we find $[b_1, c_1] \supseteq [b_2, c_2] \supseteq \dots \supseteq [b_n, c_n] \supseteq \dots$, s.t.:

① $[b_n, c_n]$ contains infinitely terms of $(a_n)_{n\in\mathbb{N}}$, $\forall n \in \mathbb{N}$

② $[b_n, c_n]$ has length $\dfrac{c_1 - b_1}{2^{n-1}}$, $\forall n \in \mathbb{N}$.

By the nested intervals theorem, $\bigcap_{n=1}^{+\infty} [b_n, c_n] \neq \emptyset$. In particular, since

$$c_n - b_n = \frac{c_1 - b_1}{2^{n-1}} \xrightarrow[n\to+\infty]{} 0,$$ we have that

$$\bigcap_{n=1}^{+\infty} [b_n, c_n] = \{x\}, \text{ where } x = \lim_{n\to+\infty} b_n = \lim_{n\to+\infty} c_n.$$

We will now define a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ with $a_{k_n} \xrightarrow[n\to+\infty]{} x$:

- Let $k_1 = 1$; clearly, $b_1 \leq a_{k_1} \leq c_1$.

- In $[b_2, c_2]$, there are infinitely many terms of $(a_n)_{n\in\mathbb{N}}$; so, in particular $\exists\, a_{k_2} \in [b_2, c_2]$, with $k_2 > k_1$.

- In $[b_3, c_3]$, there are infinitely many terms of $(a_n)_{n\in\mathbb{N}}$; so, in particular $\exists\, a_{k_3} \in [b_3, c_3]$, with $k_3 > k_2$,

and so on. Eventually:

$b_n \leq a_{k_n} \leq c_n$ $\forall n \in \mathbb{N}$. By the sandwich lemma,

$$\lim_{n\to+\infty} a_{k_n} = \lim_{n\to+\infty} b_n = \lim_{n\to+\infty} c_n = x.$$

So, $(a_{k_n})_{n\in\mathbb{N}}$ is convergent. ∎

⚠ Notice that both the proofs of the Bolzano–Weierstrass that we provided rely on the total order in $\mathbb{R}$. When we generalise the theorem to compact metric spaces, we will not have that advantage any more. So, we'll have to find a better way to exploit the generalisation of the nested intervals theorem that we mentioned earlier.