MATH 104 - WEEKLY ASSIGNMENT 2

DUE 9 SEPTEMBER 2016, BY 16:00

(1) Show that, for any $a, b \in \mathbb{R}$ with $a < b$, there exist infinitely many irrational numbers x with $a < x < b$.

(2) Show that, for any $a, b \in \mathbb{R}$, it holds that

$$\big||a| - |b|\big| \leq |a + b|$$

and

$$\big||a| - |b|\big| \leq |a - b|.$$

(Hint: These are both corollaries of the triangle inequality.)

(3) Show the Cauchy–Schwarz inequality: For any $a_1, \dots, a_n \in \mathbb{R}$, $b_1, \dots, b_n \in \mathbb{R}$,

$$(a_1 b_1 + \ldots + a_n b_n)^2 \leq (a_1^2 + \ldots + a_n^2) \cdot (b_1^2 + \ldots + b_n^2).$$

(Hint: Consider the polynomial

$$p(\lambda) = (a_1 + \lambda b_1)^2 + \ldots + (a_n + \lambda b_n)^2, \ \lambda \in \mathbb{R}.$$

What is the degree of p? What is the sign of $p(\lambda)$ for $\lambda \in \mathbb{R}$? What does this imply for the discriminant of p?

(4) Use the geometric-arithmetic mean inequality to show the harmonic-geometric mean inequality. I.e., use that

$$(x_1 \cdots x_n)^{1/n} \leq \frac{x_1 + \ldots + x_n}{n}, \text{ for all } x_1, \ldots, x_n > 0$$

to prove that

$$\frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}} \leq (x_1 \cdots x_n)^{1/n}, \text{ for all } x_1, \ldots, x_n > 0.$$

(5) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence. Show that $(a_n)_{n \in \mathbb{N}}$ is bounded if and only if there exists some $M > 0$ with $|a_n| \leq M$, for all $n \in \mathbb{N}$.

(6) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence. (The aim of this exercise is to show that the convergence behaviour of a sequence doesn't depend on the first terms of the sequence.)

(i) Suppose that $(a_n)_{n \in \mathbb{N}} \to a$, for some $a \in \mathbb{R}$. Show that any final part $(a_m, a_{m+1}, \ldots)$ of $(a_n)_{n \in \mathbb{N}}$ also converges to a.

(ii) Suppose that some final part $(a_m, a_{m+1}, \ldots)$ of $(a_n)_{n \in \mathbb{N}}$ converges to some $a \in \mathbb{R}$. Show that also $a_n \to a$.

(iii) Deduce that:

(a) $a_n \to a$ if and only if $a_{n+3} \to a$.
(b) If $a_n \leq b_n \leq c_n$ for all $n \geq n_0$ in $\mathbb{N}$, for some $n_0 \in \mathbb{N}$, and if $a_n \to l$, $c_n \to l$, then also $b_n \to l$. (This means that we can use the sandwich lemma even if b_n is not between a_n and c_n for finitely many n).

(7) Show that, for all $x \in \mathbb{R}$, there exists a sequence of irrational numbers converging to x.

(8) Show that $\nexists a \in \mathbb{R}$ with $(-1)^n \to a$ as $n \to +\infty$ (i.e, the sequence $(-1)^n$, for $n \in \mathbb{N}$, doesn't converge).

(Hint: Let $a \in \mathbb{R}$. How would we show that $a_n = (-1)^n$ doesn't converge to a? We would see what the negation of $a_n \to a$ is. The definition is: for any neighbourhood of a, I can find a final part of $(a_n)_{n\in\mathbb{N}}$ in the neighbourhood. So, the negation is: there exists a neighbourhood of a that doesn't contain any final part of $(a_n)_{n\in\mathbb{N}}$. That's what we need to show for our sequence. Or, if you prefer it this way: we should show that $\exists \epsilon > 0$ such that, for all $n_0 \in \mathbb{N}$, there exists some $n \geq n_0$ in $\mathbb{N}$, with $|a_n - a| > \epsilon$. This the negation of the more formal definition of convergence.)

(9) True of False? Explain your answer:
(i) Every convergent sequence of irrational numbers converges to an irrational number.
(ii) Every bounded sequence converges.
(iii) If $a_n \to 0$ and $(b_n)_{n\in\mathbb{N}}$ is bounded, then $(a_n b_n)_{n\in\mathbb{N}}$ is bounded.
(iv) If $a_n \to a$ and $a > 0$, then $a_n > 0$ for large n (i.e., there exists some $n_0 \in \mathbb{N}$ such that: for all $n \geq n_0$, $a_n > 0$).
(v) If $a_n \to a$ and $a \geq 0$, then $a_n > 0$ for large n.

(10) (i) Let $a_n \to a$ and $b_n \to b$. Show that, if $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
(Hint: For contradiction, suppose that $a > b$. Find two disjoint neighbourhoods of a and b, and find final parts of the corresponding sequences in there. What goes wrong?)
(ii) Let $a_n \to a$ and $b_n \to b$ with $a_n \lneq b_n$, for all $n \in \mathbb{N}$. Is it necessarily true that $a \lneq b$?