MATH 104 - WEEKLY ASSIGNMENT 3

DUE 16 SEPTEMBER 2016, BY 16:00

(1) (i) Let $b_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}}$, for all $n \in \mathbb{N}$. What is the limit of $(b_n)_{n\in\mathbb{N}}$?
(Hint: Be very careful here: each of the terms above converges to 0, but that doesn't immediately imply that their sum will converge to 0: this is because the number of terms *also* depends on n. For instance, think of $1 = \frac{1}{n} + \ldots + \frac{1}{n}$; $\frac{1}{n} \to 0$, but $1 \nrightarrow 0$. Use that

$$\frac{1}{\sqrt{n^2+1}} \leq \frac{1}{\sqrt{n^2+k}} \leq \frac{1}{\sqrt{n^2+n}}, \text{ for all } k = 1, \ldots, n.)$$

(ii) Let $c_n = \frac{1+2^2+3^3+\ldots+n^n}{n^n}$. Show that $c_n \to 1$.
(Hint: Use the sandwich lemma, but don't allow such a loss as for the previous question: this time exploit that the power in the denominator is much larger than the powers of the terms in the numerator.)

(iii) Let $a_1, \ldots, a_k > 0$. Show that $\sqrt[n]{a_1^n + \ldots + a_k^n} \to \max\{a_1, \ldots, a_k\}$ as $n \to +\infty$.

(2) The aim of this exercise is to show (part of) the algebra of limits regarding convergence to $+\infty$:
(i) Show that, if $a_n \to +\infty$, and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\frac{1}{a_n} \to 0$.
(ii) Show that, if $a_n \to 0$ and $a_n > 0$ for all $n \in \mathbb{N}$, then $\frac{1}{a_n} \to +\infty$. (Note that $a_n > 0$ for all $n \in \mathbb{N}$ is required; look for instance at $\frac{(-1)^n}{n}$ for all $n \in \mathbb{N}$.)
(iii) Show that, if $a_n \to +\infty$ and $(b_n)_{n\in\mathbb{N}}$ is bounded from below, then $a_n + b_n \to +\infty$.
(iv) Show that, if $a_n \to +\infty$, then $a_n^2 \to +\infty$.
(v) Show that, if $a_n \to +\infty$, and $a_n > 0$ for all $n \in \mathbb{N}$, then $a_n^{\frac{1}{k}} \to +\infty$ as $n \to +\infty$, for any fixed $k \in \mathbb{N}$.
(vi) Show that, if $a_n > b_n$ for all $n \in \mathbb{N}$, and $b_n \to +\infty$, then $a_n \to +\infty$.
(vii) Show that, if $a_n \to +\infty$ and $\lambda > 0$, then $\lambda a_n \to +\infty$.

(Hint: These follow easily from the definition of limits, but keep in mind also if you can use any of these to show another of these. For instance, you may want to use (ii) a lot.)

(3) Provide a proof of the root test. (Start the same way as for the ratio test).

(4) Are the following true or false? Justify your answers.
(i) If $a_n > 0$ for all $n \in \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}}$ is not bounded from above, then $a_n \to +\infty$.
(ii) $a_n \to +\infty \Leftrightarrow$ for all $M > 0$, there exist infinitely many terms of $(a_n)_{n\in\mathbb{N}}$ larger than M. (Justify each direction here.)

(5) Find the limit of $(a_n)_{n\in\mathbb{N}}$ (as $n \to +\infty$), when:

(i) $a_n = \frac{n^5+4n^3-3}{2n^8+5n}$, for all $n \in \mathbb{N}$.

(ii) $a_n = \frac{n^2+\sqrt{n^{\frac{5}{2}}+1}}{n^4+3}$, for all $n \in \mathbb{N}$.

(iii) $a_n = \frac{2^n+n}{3^n-n}$, for all $n \in \mathbb{N}$.

(iv) $a_n = \sqrt[n]{10n^{91}}$, for all $n \in \mathbb{N}$.

(v) $a_n = \sqrt[n]{n^7 - 8n^3 + 11n^2 - 6}$, for all $n \in \mathbb{N}$, $n \geq 2$.

(vi) $a_n = \frac{n^n}{n!}$, for all $n \in \mathbb{N}$.

(vii) $a_n = \frac{n^k}{n!}$, for all $n \in \mathbb{N}$, where $k \in \mathbb{N}$ is fixed (imagine it like 2 or 3).

(viii) $a_n = \left(1 + \frac{1}{n}\right)^{n^2}$, for all $n \in \mathbb{N}$.

(ix) $a_n = \left(\frac{1+2^n}{n^2}\right)^n$, for all $n \in \mathbb{N}$.

(x) $a_n = \sqrt{n^2+2} - \sqrt{n^2+1}$, for all $n \in \mathbb{N}$. (Hint: the limit will be 0. You see that this is an indeterminate form (the limit of each of the terms in the difference is $+\infty$, and $+\infty - (+\infty)$ doesn't make any sense. Multiply and divide with an appropriate quantity (depending on n), which will eliminate the problem.)

(Hint: For each of these, it is up to you to choose the technique you think fits best: extracting the largest power, or the ratio or root tests, or the sandwich lemma, the definition of limit, any of the other questions in the assignment, etc.)

(6) Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{1 + \sqrt{a_n}}$, for all $n \in \mathbb{N}$. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ converges. You are not asked to find its limit; however, can you find an equation that the limit satisfies, which will give the limit once solved?