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a)

$$y'' = -(x+1)y' + 2y + (1-x^2)e^{-x} \qquad y(0)=1 \quad y(1)=2$$

$$y_1'' = -(x-1)y_1' + 2y_1 + (1-x^2)e^{-x} \qquad y_1(0)=1 \quad y_1'(0)=0$$

$$y_2'' = -(x-1)y_2' + 2y_2 \qquad y_2(0)=0 \quad y_2'(0)=1$$

$$y(x) = y_1(x) + C\,y_2(x) \qquad C = \frac{2-y_1(1)}{y_2(1)} = \frac{y(1)-y_1(1)}{y_2(1)}$$

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① 先用 y_1

$$y_1'' = -(x-1)y_1' + 2y_1 + (1-x^2)e^{-x}$$

$$z_1 = y$$
$$z_2 = y'$$

$$z_2' = -(x-1)z_2 + 2z_1 + (1-x^2)e^{-x} \triangleq f_2(\underset{t}{x}, \underset{z_1}{y}, \underset{z_2}{dy})$$

$$z_1' = z_2 \triangleq f_1(dy)$$

$\Rightarrow$ C++

$h=0.1$

x from $0 \to 1$

K_{11} K_{21}
K_{12} K_{22}
K_{13} K_{23}
K_{14} K_{24}

$\Rightarrow y_1(1)=$

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② 再用 y_2

$$y_2'' = -(x-1)y_2' + 2y_2$$

$$z_1 = y$$
$$z_2 = y'$$

$$z_2' = -(x-1)z_2 + 2z_1 \triangleq f_2(x, y, dy)$$

$$z_1' = z_2 \triangleq f_1(dy)$$

$\Rightarrow$ C++

$\Rightarrow$

$y_2(1)=$

③ 求 $c = \dfrac{2 - y_1(1)}{y_2(1)}$ $\Rightarrow$ C++ $= 0.024157$

④ $y(x) = y_1(x) + c\, y_2(x)$

$\Downarrow$

C++

x	y
0	1.000000
0.1	1.016650
0.2	1.059293
0.3	1.124476
0.4	1.209121
0.5	1.310528
0.6	1.426377
0.7	1.554712
0.8	1.693917
0.9	1.842688
1.0	2.000000

$\rightarrow y(0) = 1$

$\hookrightarrow y(1) = 2$ 符合 B.C.

b)

$y_0 = 1$ $\quad x_0 = 0$ $\quad h=0.1$

$$y'' = -(x+1)y' + 2y + (1-x^2)e^{-x} \qquad \begin{aligned} y(0) &= 1 \\ y(1) &= 2 \end{aligned} \qquad x_{10} = 1$$

$y_{10} = 2$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad r(x) = (1-x^2)e^{-x}$$

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = -(x+1) \cdot \frac{y_{i+1} - y_{i-1}}{2h} + 2y_i + (1-x_i^2)e^{-x_i}$$

$$y_{i+1} - 2y_i + y_{i-1} + (x+1)\frac{h}{2}(y_{i+1} - y_{i-1}) - h^2 2y_i = r_i h^2$$

$$\left(1 - \frac{h}{2}(x+1)\right)y_{i-1} - (2 + h^2 2)y_i + \left(1 + \frac{h}{2}(x+1)\right)y_{i+1} = r_i h^2$$

$$\begin{bmatrix} -(2+h^2 2), & 1+\frac{h}{2}(x+1), & \cdots & \\ 1-\frac{h}{2}(x+1), & -(2+h^2 2), & 1+\frac{h}{2}(x+1) & \cdots \\ & \vdots & & \\ & \cdots & 1-\frac{h}{2}(x+1), & -(2+h^2 2) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_9 \end{bmatrix} = \begin{bmatrix} r_1 h^2 - \left(1-\frac{h}{2}(0+1)\right)\times 1 \\ \\ \vdots \\ \\ r_9 h^2 - \left(1+\frac{h}{2}(1+1)\right)\times 2 \end{bmatrix}$$

$$\begin{bmatrix} -2.02, & 1.055 & & \\ 0.94 & -2.02 & 1.06 & \\ & & \ddots & \\ & & 0.905 & -2.02 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_9 \end{bmatrix} = \begin{bmatrix} -0.93604 \\ \\ \vdots \\ -1.8092275 \end{bmatrix}$$

⇒ C++ 解

x	y
0	1.000000
0.1	1.016532
0.2	1.059102
0.3	1.124251
0.4	1.208890
0.5	1.310313
0.6	1.426194
0.7	1.554570
0.8	1.693822
0.9	1.842642
1.0	2.000000

$$y'' = -(x+1)y' + 2y + (1-x^2)e^{-x} \qquad \begin{aligned} y(0)&=1 \\ y(1)&=2 \end{aligned}$$

$$-y'' + 2y = (x+1)y' - (1-x^2)e^{-x}$$

$$y_1(x) = (1-x)\times 1 + (x)\times 2 = 1+x$$

$$y(x) = y_1(x) + y_2(x)$$

$$(y_1+y_2)'' = -(x+1)(y_1+y_2)' + 2(y_1+y_2) + (1-x^2)e^{-x}$$

$$y_2'' = -(x+1)y_2' + 2y_2 + R(x)$$

$$\begin{aligned} R(x) &= -(x+1)y_1' + 2y_1 + (1-x^2)e^{-x} \\ &= -(x+1) + (2+2x) + (1-x^2)e^{-x} \\ &= (x+1) + (1-x^2)e^{-x} \end{aligned}$$

Let: $y_2(x) = \sum_{i=1}^{n} c_i \phi_i(x) \qquad \phi_i = \sin(i x \pi)$

$$\sum_{i=1}^{n} c_i \phi_i''(x) = -(x+1)\sum c_i \phi_i' + 2\sum c_i \phi_i + R(x)$$

残差 $R_G(x) = \sum c_i(\phi_i'' + (x+1)\phi_i' - 2\phi_i) - R(x)$

$$\int_0^1 R_G(x)\cdot \phi_j(x)\,dx = 0 \qquad j = 1 \sim n$$

$$\sum_{i=1}^{n} c_i \int_0^1 (\phi_i'' + (x+1)\phi_i' - 2\phi_i)\,\phi_j(x)\,dx = \int_0^1 R(x)\phi_j(x)\,dx$$

$$\begin{bmatrix} \int_0^1 (\phi_1'' + (x+1)\phi_1' - 2\phi_1)\phi_1 \, dx ; & \int_0^1 (\phi_2'' + (x+1)\phi_2' - 2\phi_2)\phi_1 \, dx , & \int_0^1 (\phi_3'' + (x+1)\phi_3' - 2\phi_3)\phi_1 \, dx \\ \vdots \ \phi_2 \ \vdots , & \vdots \ \phi_2 \ \vdots , & \vdots \ \phi_2 \ \vdots \\ \vdots \ \phi_3 \ \vdots , & \vdots \ \phi_3 \ \vdots , & \vdots \ \phi_3 \ , \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \int_0^1 R(x)\phi_1(x) \, dx \\ \vdots \ \phi_2 \ \vdots \\ \vdots \ \phi_3 \ \vdots \end{bmatrix}$$

c++ 高斯

解 $c_1 \ c_2 \ c_3$

$$y(x) = 1 + x + \sum_{i=1}^{3} c_i \phi_i(x)$$

c++

x	y
0	1.000000
0.1	1.021459
0.2	1.058856
0.3	1.121064
0.4	1.207554
0.5	1.311938
0.6	1.428148
0.7	1.554387
0.8	1.692179
0.9	1.842053
1.0	2.000000