數值方法 HW12

1.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy \qquad 0<x<\pi,\quad 0<y<\frac{\pi}{2}$$

$u(0,y) = \cos y$

$u(\pi,y) = -\cos y$

$u(x,0) = \cos x$

$u(x,\frac{\pi}{2}) = 0$

$h = k = 0.1\pi$

$h = \frac{\pi-0}{n_x+1} \Rightarrow n_x = 9$

$k = \frac{\frac{\pi}{2}-0}{n_y+1} \Rightarrow n_y = 4$

$\alpha = 1$

$$\frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{i,j} \qquad \alpha = \frac{h^2}{k^2}$$

$$(u_{i+1,j} - 2u_{ij} + u_{i-1,j}) + \alpha(u_{i,j+1} - 2u_{ij} + u_{i,j-1}) = h^2 f_{ij}$$

$$\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} = f_{i,j}$$

$u_{ij} \rightarrow U_{i+(j-1)n_1}$

$$\begin{bmatrix} -2(1+\alpha) & 1 & \cdots & \overset{n+1}{\alpha} \\ 1 & -2(1+\alpha) & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{12} \\ u_{22} \\ \vdots \end{bmatrix}$$

$$-4u_{ij} = h^2 f_{i,j} - (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})$$

$$u_{ij} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - h^2 f_{i,j})$$

C++ 算

u_{ij}

2. (a)

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{4K}\frac{\partial T}{\partial t}$$

$\frac{1}{2} \le r \le 1$, $0 \le t$

$T(1,t) = 100 + 40t$, $0 \le t \le 10$

$T(r,0) = 200(r-0.5)$, $0.5 \le r \le 1$

$\Delta t = 0.5$ $\quad K = 0.1$

$\Delta r = 0.1$

$$\frac{\partial T}{\partial r} + 3T = 0 \quad \text{at } r = \frac{1}{2}$$

$$\downarrow$$

$$\frac{T_{n+1} - T_n}{\Delta r} + 3T_n = 0$$

$$T_{n+1} = -3T_n \Delta r + T_n$$

$$\frac{T(x,t+1) - T(x,t)}{\Delta t} = \frac{\partial T}{\partial t}$$

$$\frac{T(x+1,t) - 2T(x,t) + T(x-1,t)}{\Delta r^2} = \frac{\partial^2 T}{\partial r^2}$$

$$\frac{1}{r_i} \cdot \frac{T(x+1,t) - T(x-1,t)}{2\Delta r} = \frac{1}{r_i}\frac{\partial T}{\partial r}$$

$$\frac{T(x,t+1) - T(x,t)}{\Delta t} = \frac{4K}{1}\left[\frac{T(x+1,t) - 2T(x,t) + T(x-1,t)}{\Delta r^2} + \frac{1}{r_i}\frac{T(x+1,t) - T(x,t)}{\Delta r}\right]$$

$$T(x,t+1) = T(x,t) + \underbrace{\frac{4K\Delta t}{\Delta r^2}}_{20}\left[\cdots\cdots + \frac{\Delta r}{r_i}\cdots\cdots\right]$$

$\alpha > \frac{1}{2}$ 數值不穩定 → C++ 解

$$\begin{array}{c} T_1 \\ \begin{bmatrix} 0 \\ \\ \\ \\ \\ 120 \end{bmatrix} \end{array} = \begin{bmatrix} 1+2\alpha & \alpha(1+\frac{\Delta r}{2r_i}) & & \\ \alpha(1-\frac{\Delta r}{2r_i}) & 1+2\alpha & \alpha(1+\frac{\Delta r}{2r_i}) & \\ & & \ddots & \\ & & & \ddots \end{bmatrix} \begin{array}{c} T_6 \\ \begin{bmatrix} 0 \\ 20 \\ 40 \\ 60 \\ 80 \\ 100 \end{bmatrix} \end{array}$$

(b)

$$\frac{T_{(x,t)} - T_{(x,t-1)}}{\Delta t} = \frac{4k}{\Delta r^2}\left[T_{(x+1,t)} - 2T_{(x,t)} + T_{(x-1,t)} + \frac{\Delta r}{2r_i} T_{(x+1,t)} - T_{(x-1,t)}\right]$$

$$T_{(x,t)} - \alpha\left[T_{(x+1,t)} - 2T_{(x,t)} + T_{(x-1)t} + \frac{\Delta r}{2r_i}\left(T_{(x+1,t)} - T_{(x-1,t)}\right)\right]$$

$$\left(-\alpha\left(1+\frac{\Delta r}{2r_i}\right)\right)T_{(x+1)t} + (1+2\alpha)\,T_{(x,t)} + \left(-\alpha\left(1-\frac{\Delta r}{2r_i}\right)\right)T_{(x-1,t)} = T_{(x,t-1)}$$

$$\begin{bmatrix} 1+2\alpha & -\alpha\left(1+\frac{\Delta r}{2r_i}\right) & & \\ & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{array}{c} t=1 \\ \begin{bmatrix} T_{(0.5,1)} \\ T_{(0.6,1)} \\ T_{(0.7,1)} \\ T_{(0.8,1)} \\ T_{(0.9,1)} \\ T_{(1,1)} \end{bmatrix} \end{array} = \begin{array}{c} t=0 \\ \begin{bmatrix} 0 \\ 20 \\ 40 \\ 60 \\ 80 \\ 100 \end{bmatrix} \end{array}$$

$$\frac{\partial T}{\partial r} + 3T = 0$$

$$\frac{T_n - T_{n-1}}{\Delta r} + 3T_n = 0$$

$$\frac{1}{\Delta r}T_n - \frac{1}{\Delta r}T_{n-1} + 3T_n = 0$$

$$\left(\frac{1}{\Delta r} + 3\right)T_n = \frac{1}{\Delta r}T_{n-1}$$

$$= T_{n-1}/(3\Delta r+1)$$

$$T(1,1) = 100 + 20 = 120$$

$$T(0.5,1) = 0/(3\Delta r+1)$$

解 T

c++算

(c)

$$\frac{T_{(x,t+1)}-T_{(x,t)}}{\Delta t} = 4K\left[\frac{\left(\frac{T_{(x+1,t)}-2T_{(x,t)}+T_{(x-1,t)}}{\Delta r^2}+\frac{1}{r_i}\frac{T_{(x+1,t)}-T_{(x-1,t)}}{2\Delta r}\right)+\left(\frac{T_{(x+1,t+1)}-2T_{(x,t+1)}+T_{(x-1,t+1)}}{\Delta r^2}+\frac{1}{r_i}\frac{T_{(x+1,t+1)}-T_{(x-1,t+1)}}{2\Delta r}\right)}{2}\right]$$

$$T_{(x,t+1)}-T_{(x,t)} = \underset{\substack{\parallel \\ \frac{2K\Delta t}{\Delta r^2}=10}}{\alpha}\left[\qquad\qquad\right]$$

$$-\alpha\left(1+\frac{\Delta r}{2r_i}\right)T_{(x+1,t+1)}+(1+2\alpha)T_{(x,t+1)}-\alpha\left(1-\frac{\Delta r}{2r_i}\right)T_{(x-1,t+1)} = \alpha\left(1+\frac{\Delta r}{2r_i}\right)T_{(x+1,t)}+(1+2\alpha)T_{(x,t)}+\alpha\left(1-\frac{\Delta r}{2r_i}\right)T_{(x-1,t)}$$

$$\begin{bmatrix} 1+2\alpha & -\alpha\left(1+\frac{\Delta r}{2r_i}\right) & & \\ -\alpha\left(1-\frac{\Delta r}{2r_i}\right) & 1+2\alpha & -\alpha\left(1+\frac{\Delta r}{2r_i}\right) & \\ & & \ddots & \\ & & & \end{bmatrix}\begin{array}{c} T_1 \\ \begin{bmatrix} 0 \\ \vdots \\ 120 \end{bmatrix}\end{array} = \begin{bmatrix} 1+2\alpha & \alpha\left(1+\frac{\Delta r}{2r_i}\right) & & \\ \alpha\left(1-\frac{\Delta r}{2r_i}\right) & 1+2\alpha & \alpha\left(1+\frac{\Delta r}{2r_i}\right) & \\ & & \ddots & \\ & & & \end{bmatrix}\begin{array}{c} T_0 \\ \begin{bmatrix} 0 \\ 20 \\ 40 \\ 60 \\ 80 \\ 100 \end{bmatrix}\end{array}$$

↑
解

$$= \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

重復

↓ 以此解

3. $\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$ $\qquad \frac{1}{2} \le r \le 1 \qquad 0 \le t \le \frac{\pi}{3}$

$dr = 0.1 \qquad d\theta = \frac{\pi}{15}$

$T(r,0) = 0 \qquad T(r, \frac{\pi}{5}) = 0 \qquad T(1/2, \theta) = 50 \qquad T(1, \theta) = 100$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta r^2} + \frac{1}{r_i} \cdot \frac{T_{i+1,j} - T_{i-1,j}}{2\Delta r} + \frac{1}{r_i^2} \cdot \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta\theta^2} = 0$$

$$T_{i+1,j} - 2T_{i,j} + T_{i-1,j} + \frac{\Delta r}{2r_i}(T_{i+1,j} - T_{i-1,j}) + \frac{\Delta r^2}{\Delta\theta^2}\frac{1}{r_i^2}(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}) = 0$$

$$\underbrace{\left(1+\frac{\Delta r}{2r_i}\right)}_{a}(T_{i+1,j}) + \underbrace{\left(1-\frac{\Delta r}{2r_i}\right)}_{b}(T_{i-1,j}) \quad \underbrace{\left(-2-2\frac{1}{r_i^2}\frac{\Delta r^2}{\Delta\theta^2}\right)}_{c}T_{i,j} + \underbrace{\frac{\Delta r^2}{\Delta\theta^2}\frac{1}{r_i^2}}_{d}(T_{i,j+1} + T_{i,j-1}) = 0$$

$$T_{ij} = \frac{\cdots\cdots\cdots}{\left(2+2\frac{1}{r_i^2}\frac{\Delta r^2}{\Delta\theta^2}\right)}$$

$$\begin{bmatrix} c & a & 0 & 0 & d & \cdots & & & \\ b & c & a & 0 & 0 & d & & & \\ 0 & b & c & a & 0 & 0 & d & & \\ & & b & c & 0 & 0 & 0 & d & \\ & & & 0 & c & a & & & \\ & & & & & & & & \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \\ u_{12} \\ u_{22} \\ u \end{bmatrix} = \begin{bmatrix} -bu_{01} - du_{10} \\ -du_{20} \\ -du_{30} \\ -au_{51} - du_{10} \\ \vdots \\ \vdots \end{bmatrix}$$

C++ 算

T_{ij}

4.
$$\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} \qquad 0 \le x \le 1 \qquad 0 \le t \qquad \Delta x = \Delta t = 0.1$$

$P(0,t)=1$, $P(1,t)=2$, $P(x,0)=\cos(2\pi x)$, $\frac{\partial P}{\partial t}(x,0) = 2\pi \sin(2\pi x)$

$$P_{(x,t+1)} - 2P_{(x,t)} + P_{(x,t-1)} = \frac{\Delta t^2}{\Delta x^2}\left(P_{(x+1,t)} - 2P_{(x,t)} + P_{(x-1,t)}\right)$$

$$P_{(x,t+1)} = 2P_{(x,t)} - P_{(x,t-1)} + \frac{\Delta t^2}{\Delta x^2}\left(P_{(x+1,t)} - 2P_{(x,t)} + P_{(x-1,t)}\right)$$

$$\frac{P_{n+1} - P_n}{\Delta t} = 2\pi \sin(2\pi x)$$

$$P_{n+1} = P_n + (\Delta t)\, 2\pi \sin(2\pi x)$$

$$P_1 = P_0 + \Delta t \cdot \frac{\partial P}{\partial t} + \frac{\Delta t^2}{2}\left(\frac{\partial^2 P}{\partial t^2}\right)$$

$$\Rightarrow \frac{\partial^2 P}{\partial^2 x}$$

$$\frac{P_{(x+1,t)} - 2P_{(x,t)} + P_{(x-1,t)}}{\Delta x^2}$$

$P(x,0) =$

x	
0.0	1
0.1	
0.2	
0.3	
0.4	$\cos(2\pi x)$
0.5	
0.6	
0.7	
0.8	
0.9	
1.0	2

$\rightarrow P(x,1)$

$$P_1 = P_0 + \Delta t\, 2\pi \sin(2\pi x) + \frac{P_{(x+1,0)} - 2P_{(x,0)} + P_{(x-1,0)}}{(\Delta x)^2}$$

$$P_n = 2P_{n-1} - P_{n-2} + \frac{\Delta t^2}{\Delta x^2}\left(P_{(x+1,n)} - 2P_{(x,n)} + P_{(x-1,n)}\right)$$

$\downarrow$ C++