Task 1

linear independent ①

若 $a\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} + b\begin{bmatrix}0 & -1\\ 3 & 1\end{bmatrix} + c\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$: $a,b,c \in \mathbb{R}$

則為 dependent $(a,b,c \neq 0)$

$$\begin{cases} a+c=0 \\ 2a-b-c=0 \\ 3b+2c=0 \\ b-c=0 \end{cases} \quad \begin{cases} a+c=0 \\ 2a-2c=0 \\ 5c=0 \end{cases} \quad \begin{matrix} a=0 \\ b=0 \\ c=0 \end{matrix}$$

因 $a=b=c=0$，故該 set 為 independent

linear dependent ②

$a\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 2\\ 0 & 4\end{bmatrix} + c\begin{bmatrix}-2 & 3\\ 2 & 6\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$$\begin{cases} a-2c=0 \\ 2b+3c=0 \\ -a+2c=0 \\ 4b+6c=0 \end{cases} \quad \begin{cases} a-2c=0 \\ 2b+3c=0 \end{cases} \quad \begin{cases} 3a=6c \\ 6c=-4b \end{cases}$$

$3a=-4b=6c$，不僅有 $a=b=c=0$ 之唯一解

linear ③ dependent

$$\begin{cases} a - c + 2d = 0 \\ -b + 2c + d = 0 \\ -2a + b + c - 4d = 0 \\ a + b + 4d = 0 \end{cases} \quad \begin{cases} a - c + 2d = 0 \\ -b + 2c + d = 0 \\ b - c = 0 \\ -a + b + 2c = 0 \end{cases} \quad \begin{cases} a - c + 2d = 0 \\ c + d = 0 \\ b = c \\ -a + 3c = 0 \end{cases}$$

$$\begin{cases} 2c + 2d = 0 \\ c = -d \\ b = c \\ a = 3c \end{cases} \quad \begin{cases} a = 3b \\ b = c \\ c = -d \end{cases}$$ 非唯一解

linear ④ independent

$$\begin{cases} a - b + 2d = 0 \\ -b + 2c + d = 0 \\ -2a + b + c + 2d = 0 \\ a + b - 2d = 0 \end{cases} \quad \begin{cases} a - b + 2d = 0 \\ -b + 2c + d = 0 \\ -3a + 2b + c = 0 \\ 2a = 0 \end{cases} \quad \begin{cases} -b + 2d = 0 \\ -b + 2c + d = 0 \\ 2b + c = 0 \\ a = 0 \end{cases}$$

$$\begin{cases} b = +2d \\ 2c - d = 0 \\ 4d + c = 0 \\ a = 0 \end{cases} \quad \begin{cases} b = 2d \\ 2c = d \\ 9c = 0 \\ a = 0 \end{cases} \quad \begin{cases} b = 0 \\ d = 0 \\ c = 0 \\ a = 0 \end{cases}$$ $a = b = c = d = 0$

Task 2

$S = \{u, v, (0,0,0)\} : u, v \in \mathbb{R}^3$

S in linear dependent

令 $au + bv + c(0,0,0) = (0,0,0) : a, b, c \in \mathbb{R}$

令 $a = b = 0$, $c \neq 0$, 等式成立

且 a, b, c 不全為0, 得 S 為 linear dependent

Task 3

False ①

$(1,0)$ and $(0,1)$ are vector in basis of $\mathbb{R}^2$

but $\{(-1,0), (0,-1)\}$ is also a basis of $\mathbb{R}^2$

True ②

False ③

A basis of V should stisty:

1. It contain n vectors, where n is the dimenstion of V
2. These vectors are linear independent

However, the question does not state that the vectors in S are linear independent.

For example: $V = \mathbb{R}^2$, $S = \{(1,1), (2,2)\}$

$a(1,1) + b(2,2) = (0,0)$, $2a = -b$, S is linear dependent

Task 4

is basis ① If $\{1-x^2, 2+5x+x^2, -4x+3x^2\}$ is indenpendent, it is basis of $P_2(\mathbb{R})$

$$a(-1,0,1)+b(1,5,2)+c(3,-4,0)=(0,0,0)$$

$$\begin{cases} -a+b+3c=0 \\ 5b-4c=0 \\ a+2b+0=0 \end{cases} \quad \begin{cases} -a+b+3c=0 \\ 5b=4c \\ a=-2b \end{cases} \quad \begin{cases} 3b+3c=0 \\ 5b=4c \\ a=-2b \end{cases}$$

$$\begin{cases} b=-c \\ 5b=4c \\ a=2b \end{cases} \quad \begin{cases} b=0 \\ c=0 \\ a=0 \end{cases} \quad a=b=c=0$$

$\{1-x^2, 2+5x+x^2, -4x+3x^2\}$ is basis of $P_2(\mathbb{R})$

not basis ② $S=\{2-4x+x^2, 3x-x^2, 6-x^2\}$

$$a(1,-4,2)+b(-1,3,0)+c(-1,0,6)=(0,0,0)$$

$$\begin{cases} a-b-c=0 \\ -4a+3b=0 \\ 2a+6c=0 \end{cases} \quad \begin{cases} a-b-c=0 \\ 4a=3b \\ a=-3c \end{cases} \quad \begin{cases} -4c=b \\ 4a=3b \\ a=-3c \end{cases} \quad 4a=3b=-12c$$

S is linear dependent, Not a basis of $P_2(\mathbb{R})$

③ $S = \{1+2x-x^2, 1+2x^2, 2+x+x^2\}$

is basis

$a(-1,2,1) + b(2,0,1) + c(1,1,2) = (0,0,0)$

$\begin{cases} -a+2b+c=0 \\ 2a+c=0 \\ a+b+2c=0 \end{cases}$ $\begin{cases} -a+2b+c=0 \\ 2a+c=0 \\ 3a+3c=0 \end{cases}$ $\begin{cases} b=0 \\ a=0 \\ c=0 \end{cases}$

$a=b=c=0$, S is linear independent

S is a basis of $P_2(\mathbb{R})$

Task 5

Dimensions $= 4$

$a_1 - a_3 - a_4 = 0$

$a_1 = a_3 + a_4$

$W = (a_3+a_4, a_2, a_3, a_4, a_5)$

$= (a_3+a_4)(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0)$
$+ a_4(0,0,0,1,0) + a_5(0,0,0,0,1)$

$= a_2(0,1,0,0,0) + a_3(1,0,1,0,0) + a_4(1,0,0,1,0)$
$+ a_5(0,0,0,0,1)$

Get basis $S = \{(0,1,0,0,0), (1,0,1,0,0), (1,0,0,1,0), (0,0,0,0,1)\}$

S have 4 Vector, then dimensions of W is 4