Task 1

True ① $W_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 2y\}$

設 $y=0$, $z=0$ $\because x=2y$ $\therefore x=2\cdot 0=0$

得 $(0,0,0) \in W_1$, W_1含零向量

令 $u=(2y_1, y_1, z_1) \in W_1$, $v=(2y_2, y_2, z_2) \in W_1$

則 $u+v = (2(y_1+y_2), (y_1+y_2), (z_1+z_2))$

$u+v \in W_1$

W_1對加法封閉 (W_1 is closed under addition)

令 $u=(2y_3, y_3, z_3)$, $c \in \mathbb{R}$

$cu = (2cy_3, cy_3, cz_3)$, $x = 2cy_3 = 2\times(cy_3)$

W_1對純量乘法封閉 (W_1 is closed under scalar multiplication)

W_1滿足子空間(subspaces)的定義. W_1 is subspaces of $\mathbb{R}^3$

True ② $W_2 = \{(x, y, z) \in \mathbb{R}^3 : \cancel{x}\ y=0\}$

$(0,0,0) \in W_2$

令 $u=(x_1, 0, z_1) \in W_2$, $v=(x_2, 0, z_2) \in W_2$

$u+v = (x_1+x_2, 0, z_1+z_2) \in W_2$

令 $u=(x_3, 0, z_3)$, $c \in \mathbb{R}$

$cu = (cx_3, 0, cz_3) \in W_2$

W_2 is subspaces of $\mathbb{R}^3$

False ③ $W_3 = \{(x, y, z) \in \mathbb{R}^3 : x = 2y \text{ and } z = 2\}$

$(0, 0, 0) \notin W_3$

W_3 is not subspaces of $\mathbb{R}^3$

False ④ $W_3 = \{(x, y, z) \in \mathbb{R}^3 : x = y^2\}$

$(0, 0, 0) \in W_3$

令 $u = (x_1, y_1, z_1) = (y_1^2, y_1, z_1) \in W_3$

$v = (x_2, y_2, z_2) = (y_2^2, y_2, z_2) \in W_3$

得 $u + v = ((y_1^2 + y_2^2), (y_1 + y_2), (z_1 + z_2))$

$\because (y_1^2 + y_2^2) \neq (y_1 + y_2)^2$

$\therefore W_3$ is not closed under addition

W_3 is not subspaces of $\mathbb{R}^3$

Task2 $A = M_{n\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & a_{33} \end{bmatrix}$ 令 $A^T = B = M_{n\times n} = \begin{bmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & b_{33} \end{bmatrix}$

得 $a_{ij} = b_{ji}$

$$A + A^T = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots \\ a_{21}+a_{12} & a_{22}+a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$(A + A^T)^T = \begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & \cdots \\ a_{12}+a_{21} & a_{22}+a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$\because A + A^T = (A + A^T)^T \quad \therefore A + A^T$ is symmetric

Task 3

True ①

$-2x^2+3 = a(x^2+3x) + b(2x^2+4x-1)$

$(-2,0,3) = a(1,3,0) + b(2,4,-1)$

$(-2,0,3) = (a+2b, 3a+4b, -b)$

$\begin{cases} a+2b=-2 \\ a3+4b=0 \\ -b=3 \end{cases}$ $\begin{cases} a+2b=-2 \\ -2b=6 \\ -b=3 \end{cases}$ $\begin{cases} a=\cancel{-4}\ -8 \\ b=3 \end{cases}$

$-2x^3+3 = -8(x^3+3x) + 3(2x^2+4x-1)$

True ②

$x^2+2x-3 = a(-3x^2+2x+1) + b(2x^2-x-1)$

$(1,2,-3) = a(-3,2,1) + b(2,-1,-1)$

$= (-3a+2b, 2a-b, a-b)$

$\begin{cases} -3a+2b=1 \\ 2a-b=2 \\ a-b=-3 \end{cases}$ $\begin{cases} a-b=-3 \\ 2a-b=2 \\ -b=-8 \end{cases}$ $\begin{cases} a-b=-3 \\ b=8 \end{cases}$

$a=5,\ b=8$

$x^2+2x-3 = 5(-3x^2+2x+1) + 8(2x^2-x-1)$

False ③

$3x^2+4x+1 = a(x^2-2x+1) + b(-2x^2-x+1)$

$(3,4,1) = a(1,-2,1) + b(-2,-1,+1)$

$= (a-2b, -2a-b, a+b)$

$\begin{cases} a-2b=3 \\ -2a-b=4 \\ a+b=1 \end{cases}$ $\begin{cases} a+b=1 \\ b=6 \\ -3b=2 \end{cases}$ $\begin{cases} a+b=1 \\ b=6 \end{cases}$

Can't be expressed as a linear combination of these polynomials

Task 4

True ① $(2,-1,1) = a(1,0,2) + b(-1,1,1)$

$$\begin{cases} 2 = a - b \\ -1 = b \\ 1 = 2a + b \end{cases} \quad \begin{cases} a-b = 2 \\ b = -1 \\ 3b = -3 \end{cases} \quad \begin{cases} a = 1 \\ b = -1 \end{cases}$$

$(2,-1,1) = (1,0,2) + (-1)(-1,1,1)$

$(2,-1,1)$ is in the span of S

False ② $(-1,2,1) = a(1,0,2) + b(-1,1,1)$

$= (a-b, b, 2a+b)$

$$\begin{cases} a-b = -1 \\ b = 2 \\ 2a+b = 1 \end{cases} \quad \begin{cases} a-b = -1 \\ b = 2 \\ 3b = 3 \end{cases} \quad \begin{cases} a-b = -1 \\ b = 2 \\ b = 1 \end{cases}$$

$(-1,2,1)$ is not in the span of S

False ③ $(-1,1,1,2) = a(1,0,1,-1) + b(0,1,1,1)$

$= (a, b, a+b, b-a)$

$$\begin{cases} a = -1 \\ b = 1 \\ a+b = 1 \\ b-a = 2 \end{cases} \quad \begin{cases} a = -1 \\ b = 1 \\ b = 2 \\ b = 1 \end{cases}$$

$(-1,1,1,2)$ is not in the span of S

Task 5

令 $M_a \in S = \{M_1, M_2, M_3\}$

$M_a = aM_1 + bM_2 + cM_3 : a, b, c \in \mathbb{R}$

$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}^T$

$M_a = M_a^T$

$S = \{M_1, M_2, M_3\}$ is all symmetric 2×2 matrices