Heterogenous Firms and the Dynamics of Investment

Matias Bayas-Erazo   Fergal Hanks

Abstract

We show that standard heterogeneous firm models of investment feature investment responses that decrease steadily after a shock as positive relationship between the magnitude of the initial response to a shock and the speed of convergence of capital. This is inconsistent with the hump shaped responses found in the data. To resolve this tension tension we add stochastic time to build to the standard model. This feature generates hump shaped responses and enables the model to match micro moments of investment as well as the response dynamics.

1 INTRODUCTION

Investment models struggle to replicate both the lumpiness of investment at the microeconomic level and the sluggishness at the macroeconomic level. The classical adjustment cost formulation (Hayashi, 1982) and its variations, typically favored by macroeconomists to match aggregate data, is starkly at odds with the infrequent and large investments observed at the microeconomic level. Models of lumpy investment were designed to address these limitations and can explain key moments of the investment distribution (Caballero and Engel, 1999; Khan and Thomas, 2008). The extreme responsiveness of investment in these models, however, implies that the heterogeneity and state-dependence that distinguishes them is irrelevant for general equilibrium dynamics (House, 2014). Since an economy with lumpy investment behaves like a neoclassical economy with frictionless capital adjustment, the class of investment models consistent with the micro data appears inconsistent with the macro data. In an attempt to match micro lumpiness without sacrificing macro sluggishness, state-of-the art investment models rely on a combination of fixed and convex capital adjustment costs. Convex capital adjustment costs dampen the responsiveness mentioned earlier, helping bring the models in line with existing evidence and breaking the general equilibrium irrelevance result (Winberry, 2021; Koby and Wolf, 2020).

This paper focuses on the dynamics of aggregate investment in in heterogenous firm models which is both important for the dynamics in general equilibrium but also useful for discriminating between different models of firms. It starts by showing that standard models of lumpy adjustment feature investment responses that peak on impact and then steadily decrease. Additionally these responses feature a strong positive relationship between the size of the initial response and the speed of convergence back to the steady state. These two features are inconsistent with the empirical evidence on investment dynamics such as Curtis et al., 2021. In order to resolve this inconsistency we add stochastic time to build to a model of lumpy capital adjustment. This leads to inertial dynamics in investment consistent with the empirical evidence on investment dynamics while still matching microeconomic moments of investment.

To analyse the dynamics of investment we use a general feature of firm investment models that the homogeneous of degree one in aggregate productivity and idiosyncratic capital. This assumption ensures that the importance of adjustment costs will not change with growth in aggregate productivity. In the absence of other idiosyncratic state variables this implies an equivalence between permanent shocks to aggregate productivity and capital destruction shocks[1]. Furthermore the response to a capital destruction shock captures the convergence dynamics of the model as it moves the distribution away from steady state but does not effect the policy functions. Thus the investment response to an permanent aggregate productivity shock will decrease over time as the distribution of capital converges to the new steady state distribution. Additionally these responses feature a strong positive relationship between the size of the initial response and the speed of convergence back to the steady state. As the size of the initial response

[1]This result has been used in papers such as Baley and Blanco (2021)

decreases, it takes longer for capital to converge back to the steady state level as the investment level at each distance from the new steady state level of capital will be lower. Thus, in models that produce large responses to shocks[2], such as models with only fixed costs that match micro moments, capital converges back to the steady state level quickly. On the other hand, models with smaller responses to shocks, such as models with convex costs, capital takes a long time to converge back to the steady state level.

These features are inconsistent with the empirical evidence on investment dynamics. Estimates from Curtis et al. (2021) show that response of investment does not steadily decrease in response to a permanent shock. Thus when calibrating to the initial response as estimated by Zwick and Mahon (2017) this implies a slow convergence of capital back to the steady state level. They dynamics are also inconsistent which Christiano et al. (2005) argued were needed to match the general equilibrium dynamics of investment estimated from VARs.

We then resolve this inconsistency by adding stochastic time to build to a model of lumpy capital adjustment. Firms can conduct immediate investment and their capital stock increases the next period but this investment is subject to convex costs. Additionally firms can choose to start investment projects which do not have convex costs of adjustment but only realise next period with probability κ. Otherwise the project is ongoing and the firm must wait till the next period for the next chance the project will realise. This feature leads to inertia in investment in response to a shock as when the shock hits many firms will have ongoing investment projects. Since the size of these projects was fixed prior to the shock their realisation will not increase the level of investment. Instead investment increases as both the number of firms starting new projects and the share of projects started post the shock increases. Thus leading to a hump shaped response of investment to a shock.

We then numerically confirm our earlier analysis by looking at the investment response as a function of the time to build parameter as well as the standard fixed and convex costs assumed in the literature. As the fixed and convex cost parameters increase[3] the initial response of investment decreases and the time taken for capital to converge back to the steady state level increases. When allowing for investment projects, the initial response of investment response decreases when the probability of project realisation decreases. In contrast to fixed and convex costs that lost initial response is concentrated in the periods after the initial period and not spread out over all periods and so does not lead to such drawn out dynamics. Thus for lower values of the time to build parameter the investment response is hump shaped consistent with empirical evidence.

These numerical results qualitatively confirm the analytical results but we also consider the quantitative error. The analytical results predict a path of investment based solely on the initial response to a productivity shock. Across a range of parameter values for the fixed and convex costs we find that this predicted

[2]Koby and Wolf (2020) show that there are strong linkages between the responses to different shocks, justifying the use of the productivity response as a general measure of the model's responsiveness

[3]While in previous work convex costs are usually used to match empirical impact responses Fang (2023) shows that fixed costs models can match these moments if not targeting aspects of the investment distribution

path very closely matches the actual path of investment.

We then calibrate the model to two sets of moments, cross-sectional distributional moments and response moments. The cross-sectional moments such as the proportion of firms with investment rates above 20% referred to as spikes are standard in the literature and ensure the model is consistent with micro level behaviour of firms. Our inclusion of response moments build on the argument of Koby and Wolf (2020) that the initial response of investment to a shock is a key target for models of firm investment for general equilibrium. We however extend this to include the dynamics over 10 years as estimated by Curtis et al. (2021).

We calibrate two versions of the model. The first allows for fixed and convex costs in adjustment of capital This model is unable to match the empirical response to a permanent change in capital costs as estimated by Curtis et al. (2021) due to the lack of inertia. While matching the initial response of investment after the first few periods the model response is far below the empirical response. This implies a long tail of investment with only 85% of the response occurring in the first 10 years. In contrast the model with time to build is able to much better match the empirical response of investment due to the inertia. For our calibration the 2nd and 3rd year investment responses are larger than the first year response. This implies less drawn out dynamics with 95% of the response occurring in the first 10 years. This model also matches the micro moments such as spike rate.

This paper contributes to the long literature on lumpiness in firm investment starting with Doms and Dunne (1998), Caballero et al. (1995) and Caballero and Engel (1999). Particularly related is recent work by Winberry (2021) and Koby and Wolf (2020) who argued that models with solely fixed costs of adjustment such as the seminal paper of Khan and Thomas (2008) could not match the impulse response of investment on impact and resolved this by adding convex costs. In this paper we consider the full dynamics of investment and we show that convex costs are unable to match these. and instead time to build is needed. The paper of Baley and Blanco (2021) is also strongly related as they focus on the full dynamics in the form of the cumulative impulse response which allows them to derive analytical results. In contrast we take a quantitative approach and match the period by period dynamics.

This paper also obviously relates to the literature on time to build starting with the seminal work of Kydland and Prescott (1982). Papers such as Ramey et al. (2020) and Leeper et al. (2010) have argued that time to build is important to understand the dynamics caused by government investment. Casares (2006), Edge (2007) and Lucca (2007) all argued that time to build in a representative firm framework was important for macro dynamics when investment is heterogenous and complementary. We find that time to build with heterogenous firms can generate the hump shaped response even without aggregate complementary.

Finally there is are two empirical literatures that this paper builds upon. The first is the studies of Zwick and Mahon (2017) and Curtis et al. (2021) who estimate the response of investment to a change in tax depreciation rules in the US. We argue these estimates are a key target for heterogenous firm models to be able to match similar to the argument in Auclert et al. (2024) and Auclert et al. (2020) that the dynamic spending response is a key target for heterogenous household models. Related is macro estimates of the investment response such as Christiano et al. (2005) in which the investment response is hump shaped. The second is the empirical literature on time to build, such as Fernandes and Rigato (2023) which illustrates that time to build is a relevant feature of the world.

2 Dynamics of Investment

In models of heterogenous households, recent work such as Auclert et al. (2024) has emphasised the importance of understanding the dynamic response of heterogenous agent blocks for their impact on aggregate dynamics. In this section, we develop a framework in which to compare the dynamic response of investment in different models. A key feature of this framework is that the cumulative net investment response will be fixed so that the relative dynamics can be compared similar to how the iMPCs in Auclert et al. (2024) sum to one in net present value terms. We will then show that in standard models of capital investment these investment dynamics are inconsistent with empirical evidence.

Start by considering a model of a firm investing in capital $k_{i,t}$ subject to adjustment costs in capital facing idiosyncratic and aggregate productivity shocks. The value function of a firm is then given by

$$V(Z, e_{i,t}, k_{i,t}) = \max_{k'} F(Z_t, e_{i,t}, k_{i,t}) + AC(Z_t, k_{i,t}, k') \tag{1}$$

$$+ \beta \mathbb{E}_{e'|e_{i,t}} \left[V(Z_t, e', k') \right] \tag{2}$$

Where $F(Z_t, e_{i,t}, k_{i,t})$ is the production function of the firm, $AC(Z_t, k_{i,t}, k')$ is the adjustment cost function and β is the discount factor.

We then need to make two key assumptions. The first is that the economy is in the aggregate non stochastic steady state. This implies the firm does not have to form expectations over aggregate productivity. We will therefore analyse the response of firms to unanticipated MIT shocks to aggregate productivity. This relies on approximate certainty equivalence as laid out in Boppart et al. (2018) for the impulse response to an MIT shock to resemble the impulse response to a shock in the full stochastic model.

The second key assumption is that $F(Z_t, e_{i,t}, k_{i,t}) + AC(Z_t, k_{i,t}, k')$ is homogeneous of degree one in in k and Z. While this assumption is somewhat restrictive it is usually satisfied by models in the literature. It ensures that the importance of the adjustment costs does not depend on the level of productivity and thus that there exists a balanced growth path. Writing $F(Z_t, e_{i,t}, k_{i,t})$ to be homogeneous of degree one in k and Z is generally achievable by a redefinition of Z. For example while $F(Z_t, e, k) = Zek^{\alpha}$ is not homogeneous of degree one. By defining $\tilde{Z}^{1-\alpha} = Z$ means $F(\tilde{Z}, e, k) = \tilde{Z}^{1-\alpha} e k^{\alpha}$ is homogeneous of degree one in k and $\tilde{Z}$. For $AC(Z_t, k_{i,t}, k')$ this assumption is more restrictive but for adjustment costs such as the cost of buying capital or convex adjustment costs this condition is satisfied by these costs being of degree 1 in k. For adjustment costs which aren't homogeneous of degree 1 in k such as fixed costs which are independent of k, scaling the fixed cost with Z can satisfies condition 2 and ensures that the fixed cost remains relevant in the case of growth as was done by Khan and Thomas (2008).

With these assumptions the firm's value function can be written as

$$V(Z, e_{i,t}, k_{i,t}) = ZV(e_{i,t}, \frac{k_{i,t}}{Z}) \tag{3}$$

This implies that the policy functions of the firm in $(e, \frac{k}{Z})$ space are invariant to changes in Z. So in response to a permanent unanticipated MIT shock to Z of 1% the long run average change in capital will be 1%.

An important implication of this result is in the $\frac{k}{Z}$ space two shocks will be equivalent, a positive productivity shock and a uniform proportional capital destruction shock. These shocks reduce $\frac{k_{i,t}}{Z}$ by the same fraction for all firms shifting the distribution to the left in $\frac{k}{Z}$ space. The capital destruction shock will have no impact on policy functions however the aggregate productivity shock will. But as mentioned about the policy functions of won't change in the $\frac{k}{Z}$ space. Thus the response of investment divided by Z will be the same. This result and implication is not in and of itself novel with Baley and Blanco (2021) showing that in continuous time it is possible to write the value function in terms of the log of the capital to productivity ratio. Where productivity includes both idiosyncratic and aggregate productivity.

It is possible to draw out further novel implications via another observation. A capital destruction shock moves the distribution of capital away from the steady state distribution this will approximate the convergence behaviour of the model. To formalise this consider the level of net investment for a given distribution $f(e, k)$.

$$\int\int \mathbb{E}[\Delta k(e,k)]f(e,k)dkde- \tag{4}$$

Where $\mathbb{E}[\Delta k(e,k)]$ reflects the expected difference in k for a firm with productivity e and capital k taking the expectation over iid shocks that affect the firm's investment decision such as the random fixed costs of investment that are common in the literature. Then approximate $f(e,k)$ with $\tilde{f}(e,k)$ where $\tilde{f}(e, \lambda k) = f_{ss}(e,k)$ and λ is defined as

$$\lambda = \frac{\int\int kf(e,k)dkde}{\int\int kf_{ss}(e,k)dkde} \tag{5}$$

This implies that

$$\int\int kf(e,k)dkde = \int\int k\tilde{f}(e,k)dkde \tag{6}$$

So the net investment level can be approximation as

$$\int\int \mathbb{E}[\Delta k(e,k)]\tilde{f}(e,k)dkde \tag{7}$$

$$\int\int \mathbb{E}[\Delta k(e,k)]f_{ss}(e, \frac{k}{\lambda})v \tag{8}$$

Then in the term do a substitution of $x = \frac{k}{\lambda}$.

$$\int \int \mathbb{E}[\Delta k(e, \lambda x)] f_{ss}(e, x) dx de \tag{9}$$

Then the first order response can be calculated by taking the derivative with respect to λ around $\lambda = 1$. We are interested in $-\frac{\partial I}{\partial \lambda}$ as we are considering a capital destruction shock.

$$-\frac{\partial I}{\partial \lambda} = \int \int \mathbb{E}[\Delta k_2(e, \lambda x) x] f_{ss}(e, x) dx de \tag{10}$$

Where k'_2 is the derivative of the capital policy function with respect to its second argument. Then since the derivative is being evaluated at $\lambda = 1$, this simplifies to

$$-\frac{\partial I}{\partial \lambda} = \int \int \mathbb{E}[\Delta k_2(e, k) k] f_{ss}(e, k) dk de \tag{11}$$

$$= \int \int \mathbb{E}[\frac{\partial \Delta k(e, k)}{\partial \log(k)}] f_{ss}(e, k) dk de \tag{12}$$

Where $\frac{\partial \Delta k(e,k)}{\partial \log(k)}$ is the semi elasticity of the change in capital with respect to capital. So the level of net investment can be approximate as

$$I \approx -\frac{\partial I}{\partial \lambda}(\lambda - 1) \tag{13}$$

$$= \frac{\partial I}{\partial \lambda}(1 - \lambda) \tag{14}$$

This implies that the convergence behaviour to first order is approximately the distance of aggregate capital from the steady state level multiplied by the response of investment to a capital destruction shock[4]. In a later section when calculating the convergence dynamics numerically we will discuss the accuracy of this approximation.

Thus net investment will response to a permanent shock in a steadily decreasing manner. Since initially the distribution of capital is far from the steady state but over time it will converge to the distribution. Additionally, this implies a relationship between the impact response of investment and the speed of convergence. The larger the initial investment response the faster the rate of convergence.

These features are inconsistent with the empirical evidence on investment dynamics. Curtis et al. (2021) estimate the response of investment to changes in depreciation timing in the US. We compare this empirically estimated response to the response of investment in a standard model of investment in Figure 1.

They do not find a steadily decreasing response of investment to a permanent shock. Instead investment is elevated for a number of years before quickly returning to zero. In comparison a standard model of

[4]This approximation will be accurate when the mean capital level is a good predictor of investment, i.e. the method of Krusell and Smith (1998) works well

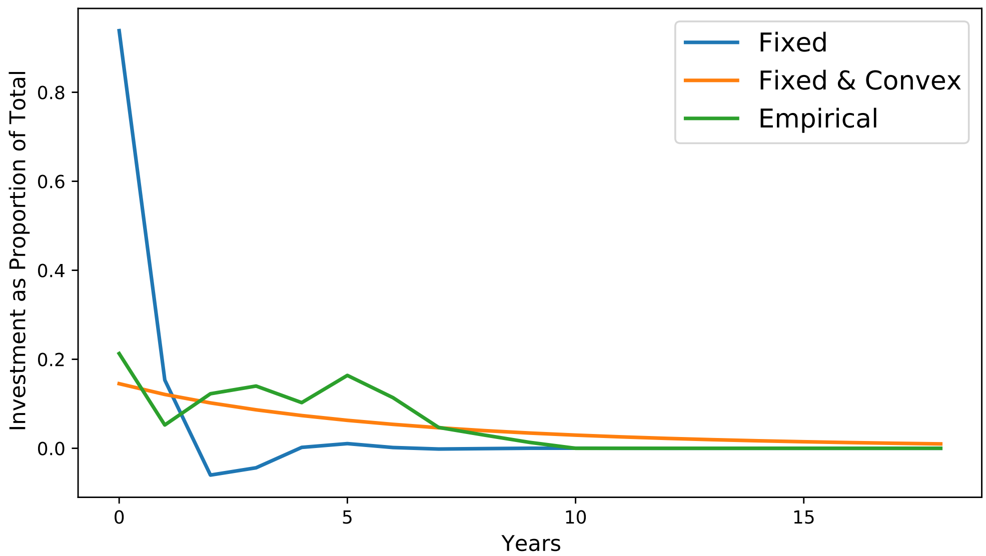

Figure 1: Comparison of Investment Dynamics in Standard Models and Empirical

investment featuring only fixed costs such as Khan and Thomas (2008) generates a much larger response on impact and the investment response quickly returns to 0. On the other hand a model that adds convex costs such as Winberry (2021) or Koby and Wolf (2020) is able to much more closely match the response on impact. However the investment response declines at a steady rate in contrast to the empirical evidence. This leads to investment not returning to zero even after 20 years, much more drawn out than the empirical evidence.

There is also related evidence at the macro level which these dynamics are inconsistent with. Christiano et al. (2005) argued that to match the dynamics of investment estimated in VARs, investment needed respond in a hump shaped manner. It was for this reason they assumed adjustment costs on investment.

3 Model

We consider a discrete-time economy populated by a continuum of firms that maximize the expected net present value of profits through optimal investment decisions. The key innovation in our framework is the incorporation of time-to-build frictions in the capital accumulation process, which creates delays between investment decisions and capital formation. Each firm j has an idiosyncratic productivity level $e_{j,t}$ that follows an AR(1) process in logs with normal innovations:

$$\log(e_{j,t+1}) = \rho_e \log(e_{j,t}) + \varepsilon_{j,t+1}, \quad \varepsilon_{j,t+1} \sim \mathcal{N}(0, \sigma_e^2) \tag{15}$$

Firms produce output using a decreasing returns to scale production function:

$$F(Z_t, e_{j,t}, k_{j,t}) = Z_t e_{j,t} k_{j,t}^{\alpha} \tag{16}$$

where Z_t represents aggregate productivity, $k_{j,t}$ is the firm's capital stock, and $0 < \alpha < 1$ is the returns-to-scale parameter. The firm's capital stock depreciates at rate δ each period. Adjustments to the capital stock can happen in two ways:

The first way is immediate investment, the firm with capital k can choose a new level of capital after production and depreciation subject to a linear cost $\gamma(k' - (1-\delta)k)$ and quadratic cost $\phi\frac{(k'-(1-\delta)k)^2}{2}k$. The convex cost term allows our model to nest recent specifications like those in Winberry (2021) and Koby and Wolf (2020), which have shown that convex costs help match the impact response of investment to shocks. The γ term is to allow us to capture the effects of the tax depreciation rate changes studied by Zwick and Mahon (2017) and Curtis et al. (2021). These tax depreciation changed moved when investment could be deducted against taxes earlier in time thus reducing the net present cost of investment. We will use a decrease in γ to mimic this policy change in the model and thus calibrate the responses of investment in the model to the estimated empirical responses.

The second way is an investment project: the firm can initiate an investment project of size i subject to time to build frictions , which will increase its capital stock by i once the project completes. The firm must pay a linear cost γi upon completion of the project, this is again to mimic tax depreciation rate changes. The time to build friction is stochastic so each period an investment project has a probability κ of completing and a probability $1-\kappa$ of continuing.

In order to engage in either of these types of investment the firm must pay a fixed cost drawn from a distribution. We will assume the firm will make independent draws of the fixed cost of each investment type and may choose to engage in neither, one or both types of investment in a period. There are two motivations for these fixed costs, the first is to generate (s,S) type investment behaviour where firms will put off investment until the fixed cost draw is low enough or the marginal product of capital is high enough to justify investment. This behaviour generates the lumpiness of investment observed in the data. The second motivation is computational tractability, the random fixed cost smooths out the adjustment maximisation decision ensuring the value function is differentiable. To maximise the computational tractability we will assume that the fixed costs are drawn from separate exponential distributions governed by parameters $\lambda^{immediate}$ and λ^{ttb}.[5]. Allowing for separate fixed costs numerically smooths out the decision to make an investment project or immediate investment ensuring the value function is differentiable.

3.1 Timing and Value Functions

The timing within each period proceeds as follows:

[5]The exponential distribution has three features that make it computationally tractable. First in comparison to the commonly used uniform distribution it does not have a finite support which introduces a kink into the value function when the value gain from adjustment goes over the upper bound of the support. Secondly in comparison to other infinite support distributions like the log normal its probability density function is monotone decreasing. Non monotonicity of the pdf leads to non monotonicity of the derivative of the value function making first order conditions no longer sufficient for global optimality. Finally in comparison to related distributions such as the Gamma or Weibull distributions it is governed by a single parameter

1. Firms produce using their current capital stock then capital depreciates

2. Firms draw a fixed cost for immediate investment and then decide on their immediate investment and pay the associated costs

3. Firms without ongoing investment projects draw a fixed cost for initiating one and then decide whether to initiate new ones and the size of the project

4. Investment projects complete with probability κ

We will define the value function and policies starting from the end of the period and working backwards. The last event is the completion of projects so the value function of a firm with an ongoing project of size i prior to the realisation of the project completion is:

$$V^{\text{pre ttb}}(e_{j,t}, k, i) = \kappa\beta\mathbb{E}[V(e', k+i, 0) - \gamma i] + (1-\kappa)\beta\mathbb{E}[V(e', k, i)] \tag{17}$$

Where $V(e_{j,t}, k, i)$ is the value function of a firm with productivity e, capital k and investment project i at the beginning of the period. The expectation is taken over the idiosyncratic productivity process.

Given $V^{\text{pre ttb}}(e_{j,t}, k, i)$ the optimal project size $i^*(e, k)$ is the given as

$$\arg\max_i V^{\text{pre ttb}}(e, k, i) \tag{18}$$

Thus the value function prior to the realisation of the draw of the investment project fixed cost is

$$V^{\text{pre project decision}}(e_{j,t}, k, i) = \mathbb{E}[\max\{V^{\text{pre ttb}}(e_{j,t}, k, i^*(e_{j,t}, k)) - \xi^{\text{ttb}}_{j,t}, V^{\text{pre ttb}}(e_{j,t}, k, i)\}] \tag{19}$$

This setup leads to a threshold rule for the adjustment cost the firm is willing to pay to start a project $\hat{\xi}^{ttb}(e, k, i) = V^{\text{pre ttb}}(e, k, i^*(e, k)) - V^{\text{pre ttb}}(e, k, i)$. Thus the above can be written

$$\begin{aligned} V^{\text{pre project decision}}(e_{j,t}, k, i) =& P(\xi^{\text{ttb}}_{j,t} < \hat{\xi}^{\text{ttb}}(e_{j,t}, k, i))(V^{\text{pre ttb}}(e_{j,t}, k, i^*(e_{j,t}, k)) - \mathbb{E}[\xi^{\text{ttb}}_{j,t} | \xi^{\text{ttb}}_{j,t} < \hat{\xi}^{\text{ttb}}(e_{j,t}, k, i)]) \\ &+ (1 - P(\xi^{\text{ttb}}_{j,t} \leq \hat{\xi}^{\text{ttb}}(e_{j,t}, k)))V^{\text{pre ttb}}(e_{j,t}, k, i) \end{aligned} \tag{20}$$

The restriction that firms with ongoing projects cannot start new projects can be imposed by setting $\hat{\xi}^{ttb}(e_{j,t}, k, i) = 0$ for $i > 0$.

The next decision firms make is the immediate investment decision. The optimal capital adjustment policy $k'^*(e, k, i)$ will be given by

$$\arg\max_{k'} V^{\text{pre project decision}}(e, k', i) - \gamma(k' - (1-\delta)k) - \phi\frac{(k' - (1-\delta)k)^2}{2}k \tag{21}$$

Then with a separate fixed cost draw for immediate investment $\xi_{j,t}^{\text{cap}}$ the firm will again optimally adopt a threshold rule $\hat{\xi}^{cap}(e,k,i)$ the value after production but before the fixed cost draw is:

$$
\begin{aligned}
V^{\text{pre capital decision}}(e_{j,t},k,i) =& P(\xi_{j,t}^{\text{cap}} < \hat{\xi}^{\text{cap}}(e_{j,t},k,i))(V^{\text{pre project decision}}(e_{j,t},k'^{*}(e_{j,t},k,i),i) - \mathbb{E}[\xi_{j,t}^{\text{cap}}|\xi_{j,t}^{\text{cap}} < \hat{\xi}^{\text{cap}}(e_{j,t},k,i)]) \\
&+ (1 - P(\xi_{j,t}^{\text{cap}} \leq \hat{\xi}^{\text{cap}}(e_{j,t},k)))V^{\text{pre project decision}}(e_{j,t},(1-\delta)k,i)
\end{aligned} \tag{22}
$$

Note that this step accounts for the depreciation of the capital stock. Now we can finally define the value function of a firm at the beginning of the period as:

$$
V(e_{j,t},k,i) = Ze_{j,t}k_{j,t}^{\alpha} + \mathbb{E}[V^{\text{pre capital decision}}(e_{j,t},k,i)] \tag{23}
$$

4 NUMERICAL ILLUSTRATION OF THE DYNAMICS OF INVESTMENT

Having described the model we now illustrate the results relating to the shape of investment dynamics as well as the relationship between the initial response and the convergence speed numerically. To do this we vary one of the parameters of the model at a time and compare how the dynamics of investment change in response to each parameter. The three parameters we vary are $\lambda^{immediate}$ of the fixed cost distribution, the convex investment cost parameter ϕ and the time to build parameter κ. In order to highlight the differential responses of models with and without time to build features we keep $\kappa = 0$ when varying $\lambda^{immediate}$ and ϕ thus illustrating the responses generated by standard adjustment cost models. When not the parameter being varied we set $\phi = 0.2$ and $\lambda^{immediate} = 2.934$. The choice of $\phi > 0$ is necessary to incentivise firms to start investment projects when time to build is present. If $\phi = 0$ then firms can make even large investments via immediate investment therefore bypassing time to build frictions making them irrelevant. Finally we set $\lambda^{ttb} = 2.934$ for consistency with $\lambda^{immediate}$.

The results can be seen in 2. These three panels show the dynamics of net investment in response to a permanent productivity shock as the three parameters are varied. The cumulative percentage response of net investment will be constant across the parameter space due to the scaling properties of these models. So on the y-axis is the share of the cumulative net investment response that occurs in each period.

Staring with the first panel in which ϕ the convex cost parameter varies, we confirm the analytical results, that the investment response is decreasing after the shock. As well as there being a strong positive relationship between the size of the response and the speed of convergence back to zero net investment. Of particular note is that even moderate values of ϕ can lead to slow convergence speeds. This is unsurprising as convex adjustment costs strongly incentivise firms to spread the adjustment out over time. However this is a cautionary note for models that rely on convex adjustment costs to generate an empirically plausible response on impact to a shock.

"In the second panel, we vary $\lambda^{immediate}$, the parameter of the fixed cost distribution. Note that the mean of the exponential distribution equals $\frac{1}{\lambda^{immediate}}$, so decreasing $\lambda^{immediate}$ raises the average fixed cost.

The results again confirm that as adjustment costs increase, the initial response decreases and the speed of convergence slows down. Importantly, the relationship between initial response and long-run dynamics appears quantitatively similar across different adjustment cost types. For example, cases with similar initial responses—such as $\lambda^{immediate} = 2.5$ with $\phi = 0.2$ and $\lambda^{immediate} = 0.5$ with $\phi = 0.6$—also exhibit very similar long-run dynamics, a pattern we will return to at the end of this section.

Finally we come to the third panel where κ the time to build parameter is varied. Strikingly for the values of κ from 0.6 downwards the initial investment response is not the largest in contrast to all values considered in the panels to the left. Indeed for $\kappa = 0.2$ it is only 4 years later that the level of net investment is lower than the initial response. This illustrates that models with stochastic time to build can generate more inertial investment responses qualitatively in line with estimated empirical responses. We will return to the quantitative fit in the next section when we calibrate the model to the empirical data.

The cause of this humped shaped behaviour is inertia in the distribution of ongoing investment projects. When the shock hits some firms will already have ongoing projects whose size was determined before the realisation of the shock. This mutes the initial response as the realisation of these projects does not increase investment relative to the pre-shock level. Instead as the composition of ongoing investment projects shifts towards projects started after the shock the investment response increases. Thus generating a hump shaped response to the shock.

In order to generate inertia in the distribution of ongoing investment projects a low value of κ is required. This can be seen by calculating the degree of inertia in the steady state distribution of projects. Fix a period t, in period $t + 1$ a mass of projects will be started equal to the mass of investment projects that realised at the end of period t which is κ times the total steady state mass, due to the distribution being constant in steady state. So before the realisation of time to build in $t + 1$, κ of the projects will have been started in $t + 1$ and $1 - \kappa$ will have been started in t or earlier. Then in period $t + 2$ the mass of projects that will be started again will make up a κ share of investment projects while those started in and before t will be $(1 - \kappa)^2$. So the share projects started in or before t will decay exponentially leading to little inertia for large values of κ. For example if $\kappa = 0.8$ the share of projects started in or before t will be $1, 0.2, 0.04, 0.008$ while for $\kappa = 0.2$ the share will be $1, 0.8, 0.64, 0.512$.

With hump shaped responses, the strong link between the size of the initial response and the speed of convergence brakes down. Consider a comparison between $\kappa = 0.4$ and $\phi = 0.6$. The initial response is smaller for $\kappa = 0.4$ is below 0.15 while the case of $\phi = 0.6$ is above 0.15. Despite this the long term dynamics of $\kappa = 0.4$ are less prolonged,net investment visually reaches 0 after 12 years. While the case of $\phi = 0.6$ is still visually above 0 after 20 years. The cause of this difference is due to the hump-shaped response When $\kappa = 0.4$ the level of net investment in years 2-4 all exceed the net investment in year 1. Consequently, the cumulative response in the first 4 years is much larger than when $\kappa = 0$ and $\phi = 0.6$ despite the smaller initial response. Therefore there is less adjustment in later years leading net investment to return to 0 faster. With more adjustment concentrated in these early years, less adjustment remains for later periods, allowing

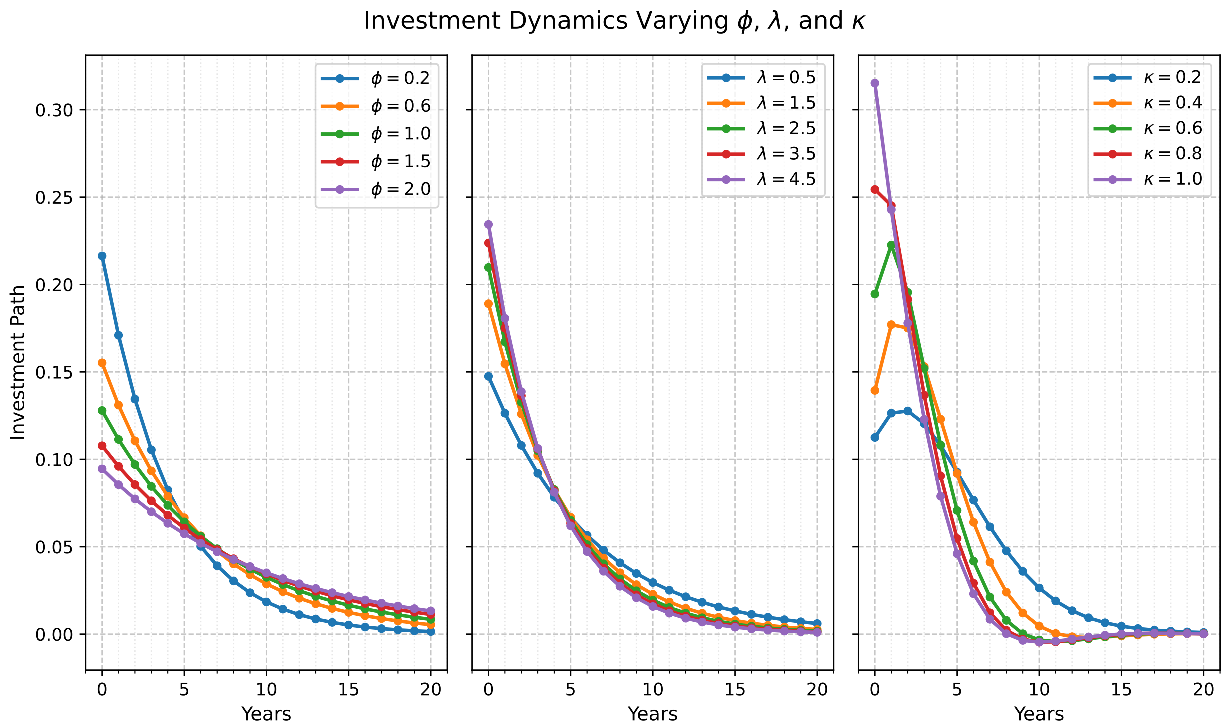

Figure 2: The effect of varying $\lambda^{immediate}$, ϕ and κ on the dynamics of investment in response to a permanent positive productivity shock.

net investment to return to steady state more rapidly.

4.1 Robustness of Approximation

While the results in 2 are qualitatively in line with the analytical results, in this section we look at the quantitative accuracy of the approximation. The prediction of the analytical approximation is that the net investment response is a linear function of the ratio of current aggregate capital to aggregate capital in steady state γ.

$$I = \frac{\partial I}{\partial \gamma}(1 - \gamma) \tag{24}$$

Denoting the steady state level of capital as K^* and the level of capital in period t as K_t we can write the investment response as

$$I_t = \frac{\partial I}{\partial \gamma}(\frac{K^* - K_t}{K^*}) \tag{25}$$

It will then be useful to normalise both I_t and $\frac{\partial I}{\partial \gamma}$ by the steady state level of capital. We will use i_t to denote $\frac{I_t}{K^*}$ and ρ to denote $\frac{\frac{\partial I}{\partial \gamma}}{K^*}$. Then the investment response can be written as

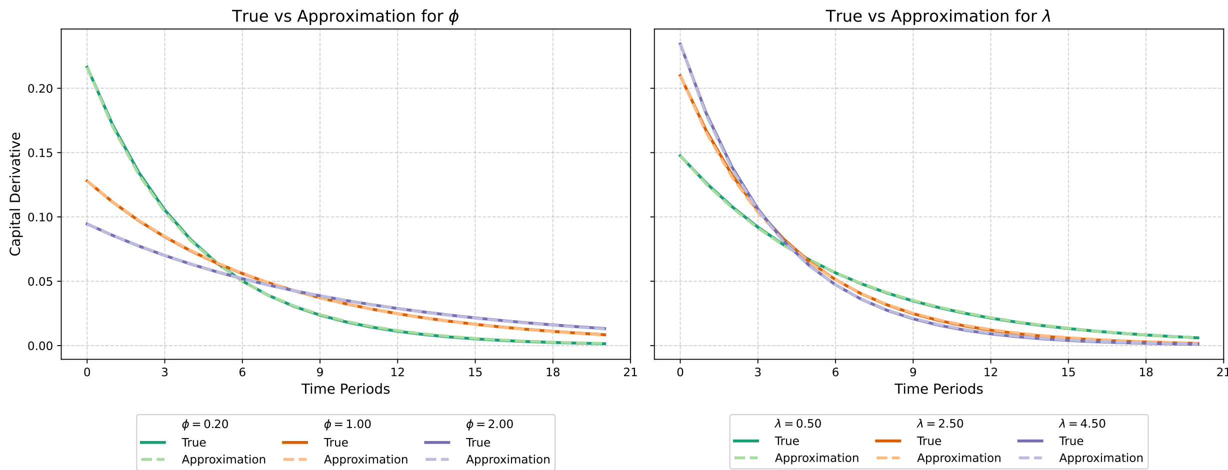

Figure 3: The accuracy of the approximation in predicting the dynamics of investment in response to a permanent positive productivity shock.

$$i_t = \rho(\frac{K^* - K_t}{K^*}) \tag{26}$$

This means that we need only ρ to calculate the entire path of investment responses predicted by the approximation. Fortunately this parameter can be calculated directly from the initial response $i_0 = \rho\frac{K^* - K_0}{K^*}$.

Then the next period's investment response is $i_1 = \rho\frac{K^* - K_1}{K^*}$ but this can be written as

$$i_1 = \rho\frac{K^* - K_1}{K^*} = \rho\frac{K* - (K_0 + K^* i_0)}{K^*} \tag{27}$$

$$= \rho(K* - K_0 - \rho(K* - K_0)) \tag{28}$$

$$= \rho(1 - \rho)(K* - K_0) \tag{29}$$

Similarly $i_2 = \rho(1-\rho)^2(K* - K_0)$ and so on. We then normalise by the total response $(K* - K_0)$ for both clarity and because as a percentage of the initial capital stock the size of the response does not depend on parameters. We then plot the actual responses against the predicted responses when varying μ and ϕ in 3.

The fit of the approximation is very good though note the initial investment response will always be hit exactly due to how the approximation is constructed. In all cases pictured the approximation lies visually on top of the true response. The maximum absolute error occurs for $\lambda^{immediate} = 2.5$ with $\phi = 0.2$ in year 4 and is 0.0016. In relative terms the error is only 1.2% of the true response. The relative error is higher late on in the response but this is driven by the low levels of the true level of net investment at the time. Given the performance of the approximation in this setting, applications to other related settings may be fruitful.

The good fit of the approximation further shows that fixed and convex costs have the same effect on the dynamics of investment. It doesn't matter what combination of $\lambda^{immediate}$ and ϕ are used as long as the initial response is the same. Then the dynamics produced by the model will be nearly identical as discussed earlier in relation to 4. Thus is it not possible to change the convex and fixed paramters to change the dynamics of the model while keeping the initial response fixed.

5 Calibration

We have shown that for some parameters models with time to build can generate inertia investment responses but are these parameters reasonable in the sense that they can also match cross-sectional moments of firm investment. To do this we calibrate two versions of the model one with time to build and one without, to a set of moments including estimated dynamic responses as well as moments of the micro distribution that have been targeted in the previous literature.

We start by setting parameters not directly related to investment frictions to values standard in the literature. With the model period chosen to be a year we set a discount factor of $\beta = 0.977$ and a depreciation rate of $\delta = 0.067$. We assume the decreasing returns to scale parameter to be equal to 0.67. For the parameters of the idiosyncratic productivity process we set the persistence $\rho_e = 0.890$ and the standard deviation $\sigma_e = 0.1$.

In the calibration we have two sets of moments, micro and response moments. We target 3 micro moments all in terms of the investment to capital ratio and we take all these estimates from Cooper and Haltiwanger (2006)[6]. The first moment is the average investment to capital ratio which is estimated to be 12%. The second moment is the standard deviation of the investment rate which is estimated to be 33.7%. This is the first moment that captures the micro "lumpiness" in the data as it requires that there is a large dispersion in investment rates implying some firms must make large investments. The third moment is the proportion of firms that have an investment to capital ratio greater than 20%, which the previous literature has referred to as a spike. This ensures that the standard deviation is not driven by a few firms with very large investment rates but instead by a sizable number of firms with large investment rates.

While the micro moments have been the focus of the literature, we also include moments relating to the aggregated dynamics of investment in response to a shock[7]. Koby and Wolf (2020) argued for targeting the initial response to a shock to limit the excessive responsiveness that previous calibrations exhibited. The shock moment they proposed targeting was the investment semi elasticity to a permanent shock to the cost of capital estimated to be 7.2 by Zwick and Mahon (2017) using changes to tax depreciation schedules. While the cumulative response to a permanent shock to productivity has a constant scale across all models

[6] Winberry (2021) relies on estimates from Zwick and Mahon (2017) due to the more recent sample and the more complete coverage of the economy. We prefer the Cooper and Haltiwanger (2006) estimates as they include structures, a large form of investment particularly likely to have time to build properties. The main difference between the two sets of estimates are the standard deviations with Zwick and Mahon (2017) finding a standard deviation of 16%

[7] These are the average partial equilibrium responses of firms to aggregate shocks not the general equilibrium response

the response to a cost of capital shock need not be the same. Thus this moment fixes the scale of the total dynamic response.

The novelty of our calibration strategy is to additionally target the dynamics of investment in response to a shock. We take estimates from Curtis et al. (2021) who studied the long run responses to the same changes to tax depreciation schedules studied by Zwick and Mahon (2017). Due to the nature of their estimation these estimates are not in the form of an semi-elasticity. However the level of net investment returns very close to 0 within the 10 year time frame of their estimation. Therefore we make the assumption that net investment after 10 years is equal to 0. We can then normalise the investment path by the total cumulative net investment over 10 years and compare this to the investment path in the model divided by the total net investment in the model required to reach the new steady state. Thus the cumulative net investment in the empirical moments is equal to 1. If in the model net investment is positive beyond 10 years then this will reduce the cumulative sum of normalised net investment in the first 10 years. Thus excessively slow convergence will lead to divergence between the model moments and the data moments.

In the first calibration we set the time to build parameter $\kappa = 0$ and allow the fixed cost parameter $\lambda^{immediate}$ as well as the convex cost parameter ϕ to vary to match the data as closely as possible.

Table 1: No Time to Build Model: Results and Moment Comparison

Parameters			
$\lambda^{immediate}$		0.45	
ϕ		0.14	
Moment	**Description**	**Model**	**Data**
1	Year 1 Investment	0.158	0.140
2	Year 2 Investment	0.135	0.129
3	Year 3 Investment	0.114	0.121
4	Year 4 Investment	0.096	0.122
5	Year 5 Investment	0.081	0.125
6	Year 6 Investment	0.0677	0.118
7	Year 7 Investment	0.057	0.098
8	Year 8 Investment	0.047	0.070
9	Year 9 Investment	0.040	0.044
10	Year 10 Investment	0.033	0.030
11	Avg i_j/k_j	0.10093252	0.122
12	Std Dev i_j/k_j	0.3167519	0.337
12	Avg Spike	0.11145838	0.186
13	Inv Semi-Elasticity	7.431897	7.200

The results in 1 show the model with only fixed and convex costs is unable to jointly match the micro and the response moments. In order to best fit the initial few years of investment the initial response overshoots the empirical response as a share of cumulative investment. Then as the response steadily decreases it crosses the empirical response thus the error in moment matching in the first three years is relatively smooth However this error increases as the model responses continues to decline in years 4 to 7. After this the empirical response declines towards the model response. However after 10 years the empirical response remains above 0 due to the slow rate at which the response decays.

The fit to the remaining moments is fairly good. The average and standard deviation of i_j/k_j in the model are close to that of the data. The investment semi-elasticity also matches the data well. The only moment that is not matched well is the proportion of spikes which is much lower in the model than in the data.

Table 2: Time to Build Model: Results and Moment Comparison

Parameters			
κ	0.22		
λ^{ttb}	2.288		
$\lambda^{immediate}$	2.3		
ϕ	0.12		
Moment	**Description**	**Model**	**Data**
1	Year 1 Investment	0.131	0.140
2	Year 2 Investment	0.139	0.129
3	Year 3 Investment	0.135	0.121
4	Year 4 Investment	0.124	0.122
5	Year 5 Investment	0.108	0.125
6	Year 6 Investment	0.091	0.118
7	Year 7 Investment	0.074	0.098
8	Year 8 Investment	0.058	0.070
9	Year 9 Investment	0.044	0.044
10	Year 10 Investment	0.032	0.030
11	Avg I/K	0.099	0.122
12	Std Dev I/K	0.341	0.337
12	Avg Spike	0.130	0.186
13	Inv Semi-Elasticity	5.94	7.200

By comparison the model with time to build is able to match the data much more closely as can be seen in 2. This comes in a large part as when $kappa \neq 0$ the model is able to generate a hump shaped response to shocks. In the calibrated model's response the peak occurs in the second year and the investment rate in the third year is higher than the first year. Thus the response better matches the inertia seen in the empirical response leading to lower moment errors in years 4 to 7 of the response.

The fit on the micro moments does not decrease in a trade off for the fit on the response moments. The average investment ratio is very close to that produced by the model without time to build. While the standard deviation and proportion of spikes are closer to the data. The moment that sees the largest discrepancy is the investment semi elasticity which is lower than the data though not near the magnitudes that models with only fixed costs miss this moment. This is also consistent with the discrepancy with the proportion of the net investment response that occurs in the first year where in the data the response peaks.

5.1 Comparison of Investment Dynamics

Given the focus on the dynamics of investment we compare the investment dynamics of the two calibrated models with the empirical estimates from Curtis et al. (2021) in 4. This figure makes it clear how the hump shaped response from time to build frictions allows for a much better match of the investment dynamics. Due to the steadily decreasing nature of the investment response in the baseline model, it cannot match both the size of the initial response and the investment levels from years 4 to 8. However the time to build model generates much more investment in the medium run while still matching the initial response. So after 10 years in the time to build model the cumulative net investment response is 95% of the total net investment response. While in the baseline model this figure is only 85% despite this implicitly being targeted to be 100% in the calibration. This shows that even when targeting the dynamics of investment in a model without time to build the model generates large long term dynamics not supported by the data.

6 Conclusion

In this paper we show that the standard models of lumpy capital adjustment are not able to match the full dynamics of investment observed in the data. We resolved this issue by introducing time to build into the standard model. This feature generates hump shaped responses in investment to shocks and thus enables the model to match both the micro and macro moments of investment. Further research both more precisely estimating investment dynamics as well as understanding how these dynamics vary across depending on firm characteristics is needed to more fully understand the both macroeconomic dynamics as well as the impact of policies aimed at stimulating investment.

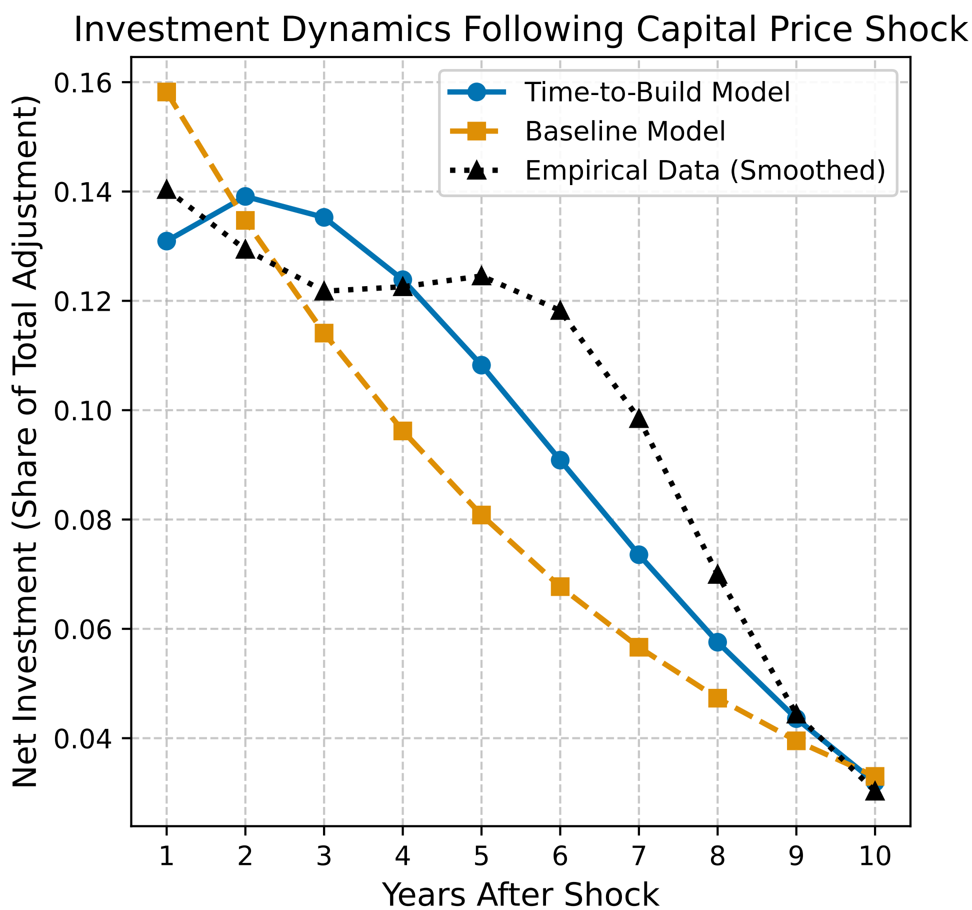

Figure 4: The empirical data comes from Curtis et al. (2021) which studies the response of investment to changes in tax depreciation schedules. Both the model responses are in response to a permanent decrease in the price of capital γ. This mimics the acceleration in tax depreciation schedules as both decrease the net present cost of capital. All responses are normalised by the total cumulative net investment response with the data response being assumed to be 0 after 10 years. The empirical response is also smoothed using a gaussian filter with the standard deviation of the kernel set to 1.5.

References

Auclert, A., Rognlie, M., and Straub, L. (2020). Micro jumps, macro humps: Monetary policy and business cycles in an estimated hank model. Technical report, National Bureau of Economic Research.

Auclert, A., Rognlie, M., and Straub, L. (2024). The intertemporal keynesian cross. *Journal of Political Economy*, 132(12):4068–4121.

Baley, I. and Blanco, A. (2021). Aggregate Dynamics in Lumpy Economies. *Econometrica*.

Boppart, T., Krusell, P., and Mitman, K. (2018). Exploiting mit shocks in heterogeneous-agent economies: the impulse response as a numerical derivative. *Journal of Economic Dynamics and Control*, 89:68–92.

Caballero, R. J., Engel, E., and Haltiwanger, J. C. (1995). Aggregate employment dynamics: Building from microeconomic evidence.

Caballero, R. J. and Engel, E. M. R. A. (1999). Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S, s) Approach. *Econometrica*, 67(4):783–826. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1111/1468-0262.00053.

Casares, M. (2006). Time-to-build, monetary shocks, and aggregate fluctuations. *Journal of Monetary Economics*, 53(6):1161–1176.

Christiano, L. J., Eichenbaum, M., and Evans, C. L. (2005). Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of political Economy*, 113(1):1–45.

Cooper, R. W. and Haltiwanger, J. C. (2006). On the nature of capital adjustment costs. *The Review of Economic Studies*, 73(3):611–633.

Curtis, M. E., Garrett, D. G., Ohrn, E., Roberts, K. A., and Suárez Serrato, J. C. (2021). Capital investment and labor demand: Evidence from 21st century tax policy.

Doms, M. and Dunne, T. (1998). Capital adjustment patterns in manufacturing plants. *Review of economic dynamics*, 1(2):409–429.

Edge, R. M. (2007). Time-to-build, time-to-plan, habit-persistence, and the liquidity effect. *Journal of Monetary Economics*, 54(6):1644–1669.

Fang, M. (2023). A note on nonconvex adjustment costs in lumpy investment models: Mean versus variance. *Macroeconomic Dynamics*, 27(4):1166–1177.

Fernandes, A. and Rigato, R. (2023). K wasn't built in a day: Investment with endogenous time to build. Technical report, Working Paper.

Hayashi, F. (1982). Tobin's Marginal q and Average q: A Neoclassical Interpretation. *Econometrica*, 50(1):213–224. Publisher: [Wiley, Econometric Society].

House, C. L. (2014). Fixed costs and long-lived investments. *Journal of Monetary Economics*, 68:86–100.

Khan, A. and Thomas, J. K. (2008). Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics. *Econometrica*.

Koby, Y. and Wolf, C. K. (2020). Aggregation in Heterogeneous-Firm Models: Theory and Measurement. *Working Paper*.

Krusell, P. and Smith, A. A. (1998). Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*.

Kydland, F. E. and Prescott, E. C. (1982). Time to build and aggregate fluctuations. *Econometrica: Journal of the Econometric Society*, pages 1345–1370.

Leeper, E. M., Walker, T. B., and Yang, S.-C. S. (2010). Government investment and fiscal stimulus. *Journal of monetary Economics*, 57(8):1000–1012.

Lucca, D. (2007). Resuscitating time-to-build. *Manuscript, Federal Reserve Board*.

Ramey, V. A. et al. (2020). *The macroeconomic consequences of infrastructure investment*, volume 10. National Bureau of Economic Research Cambridge, MA.

Winberry, T. (2021). Lumpy investment, business cycles, and stimulus policy. *American Economic Review*.

Zwick, E. and Mahon, J. (2017). Tax policy and heterogeneous investment behavior. *American Economic Review*, 107(1):217–248.

A Continuous time model

In the continuous time version of the model, the firm solves a combined stochastic control and impulse control problem. When there is no adjustment, capital evolves as

$$dk_t = -\delta k_t dt + \sigma dB_t.$$

With impulse Δk, the capital stock jumpts from k to $k' = k + \Delta k$ but the firm has to pay a cost $\mathcal{G}$ that can be given by

$$\mathcal{G}(\Delta k) = F + p \max\{\Delta k, 0\} + p(1-\gamma)\min\{\Delta k, 0\}$$

As in the discrete time model, we assume that the firm faces convex and symmetric adjustment costs in labour. It can hire workers at rate n_t but must pay a cost $\mathcal{C}$ that is allowed to depend on current employment at the firm. A fraction ρ of workers leave the firm each period, so the evolution of labour is governed by

$$dl_t = [n_t - \rho l_t]\, dt$$

The program of the firm can be written as

$$v(k_0, l_0) = \sup_{\{\Delta k_t, n_t\}_{t\geq 0}} \mathbb{E}_0 \int_0^\infty e^{-rt} \left[eF(k_t, l_t) - wl_t - AC(\Delta k_t, n_t, k_t, l_t)\right] dt$$

subject to the evolution of capital and labour above and with

$$AC(\Delta k_t, n_t) := \mathcal{G}(\Delta k_t, k_t) + \mathcal{C}(n_t, l_t)$$