Local Second-Order Limit Dynamics of the Alternating Direction Method of Multipliers for Semidefinite Programming

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Abstract

The alternating direction method of multipliers (ADMM) is widely used for solving large-scale semidefinite programs (SDPs), yet on degenerate instances it often enters prolonged slow-convergence regions where the Karush–Kuhn–Tucker (KKT) residuals nearly stall. To explain and predict the fine-grained dynamical behavior inside these regions, we develop a local second-order *limit dynamics* framework for ADMM near an *arbitrary* KKT point—not necessarily the eventual limit point of the iterates. Assuming the existence of a strictly complementary primal–dual solution pair, we derive a second-order local expansion of the ADMM dynamics by leveraging a refined and simplified variational characterization of the (parabolic) second-order directional derivative of the PSD projection operator. This expansion reveals a closed convex cone of directions along which the local first-order update vanishes, and it induces a second-order *limit map* that governs the persistent drift after transient effects are filtered out. We characterize fundamental properties of this mapping, including its kernel, range, and continuity. A primal–dual decoupling further yields a clean scaling law for the effect of the penalty parameter in ADMM. We connect these properties to second-order dynamical features of ADMM, including fixed points, almost-invariant sets, and microscopic phases. Three empirical phenomena in slow-convergence regions are then explained or predicted: (i) angles between consecutive iterate differences are small yet nonzero, except for sparse spikes; (ii) primal and dual infeasibilities are insensitive to penalty-parameter updates; and (iii) iterates can be transiently trapped in a low-dimensional subspace for an extended period. Extensive numerical experiments on the `Mittelmann` dataset corroborate our theoretical predictions.

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1 Introduction

Consider the following pair of primal–dual semidefinite programs (SDPs) in standard form:

$$\begin{array}{llll} \text{Primal:} & \text{minimize} & \langle C, X\rangle & \qquad \text{Dual:} & \text{maximize} & b^\mathsf{T} y \\ & \text{subject to} & \mathcal{A}X = b & & \text{subject to} & \mathcal{A}^* y + S = C \\ & & X \in \mathbb{S}^n_+ & & & S \in \mathbb{S}^n_+, \end{array} \tag{1}$$

with primal variable $X \in \mathbb{S}^n$ and dual variables $S \in \mathbb{S}^n$, $y \in \mathbb{R}^m$. $\mathbb{S}^n$ is the set of real symmetric $n \times n$ matrices and $\mathbb{S}^n_+$ is the set of positive semidefinite (PSD) matrices in $\mathbb{S}^n$. The linear operator $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is defined as $\mathcal{A}X := (\langle A_1, X\rangle, \cdots, \langle A_m, X\rangle)$. $\mathcal{A}^* y := \sum_{i=1}^m y_i A_i$ is its adjoint operator. The coefficients $C, A_1, \ldots, A_m$ are symmetric $n \times n$ matrices, and $b \in \mathbb{R}^m$. It is assumed that $\{A_i\}_{i=1}^m$ are linearly independent so that $\mathcal{A}\mathcal{A}^*$ is an invertible operator.

As the need to solve large-scale SDPs continues to grow—*e.g.,* those stemming from the moment and sums-of-squares (SOS) relaxations in polynomial optimization [23–25,34,48,50]—first-order methods (FOMs) have attracted increasing interest due to their low per-iteration cost and their ability to exploit problem structures such as sparsity. Among these methods, the Alternating Direction Method of Multipliers (ADMM) has become a particularly popular choice, supported by a wide range of implementations, applications, and algorithmic variants [16, 37, 49, 51, 53].

ADMM for SDP. Starting from $(X^{(0)}, y^{(0)}, S^{(0)})$, the classical three-step ADMM iteration for the SDP (1) reads [49]:

$$y^{(k+1)} = (\mathcal{A}\mathcal{A}^*)^{-1} \left(\sigma^{-1} b - \mathcal{A} \left(\sigma^{-1} X^{(k)} + S^{(k)} - C \right) \right), \tag{2a}$$

$$S^{(k+1)} = \Pi_{\mathbb{S}^n_+} \left(C - \mathcal{A}^* y^{(k+1)} - \sigma^{-1} X^{(k)} \right), \tag{2b}$$

$$X^{(k+1)} = X^{(k)} + \sigma \left(S^{(k+1)} + \mathcal{A}^* y^{(k+1)} - C \right), \tag{2c}$$

where $\Pi_{\mathbb{S}^n_+}(\cdot)$ denotes the orthogonal projection onto the PSD cone $\mathbb{S}^n_+$ and $\sigma > 0$ is the penalty parameter. Under mild conditions, $(X^{(k)}, y^{(k)}, S^{(k)})$ is convergent to $(\bar{X}, \bar{y}, \bar{S})$, one of the optimal solution pairs satisfying the Karush–Kuhn–Tucker (KKT) conditions [49, Theorem 2]:

$$\mathcal{A}\bar{X} = b,\ \mathcal{A}^*\bar{y} + \bar{S} = C,\ \langle \bar{X}, \bar{S} \rangle = 0,\ \bar{X} \in \mathbb{S}^n_+,\ \bar{S} \in \mathbb{S}^n_+. \tag{3}$$

The ADMM iteration applied to the dual SDP is equivalent to the Douglas–Rachford splitting (DRS) method applied to the primal SDP [29]:

$$Z^{(k+1)} = Z^{(k)} - \mathcal{P}(\Pi_{\mathbb{S}^n_+}(Z^{(k)}) - \widetilde{X}) + \mathcal{P}^\perp(\Pi_{\mathbb{S}^n_+}(-Z^{(k)}) - \sigma C), \tag{4}$$

where $\mathcal{P} := \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}\mathcal{A}$. $\mathcal{P}^\perp := \mathrm{Id} - \mathcal{P}$ (Id denotes the identity mapping). $\widetilde{X}$ is any constant matrix satisfying $\mathcal{A}\widetilde{X} = b$. We can recover the primal and dual variables from (4) as: $X^{(k)} := \Pi_{\mathbb{S}^n_+}(Z^{(k)})$ and $S^{(k)} := \frac{1}{\sigma}\Pi_{\mathbb{S}^n_+}(-Z^{(k)})$. Thus, each primal–dual optimal solution pair $(\bar{X}, \bar{S})$ corresponds to one optimal auxiliary variable $\bar{Z} := \bar{X} - \sigma\bar{S}$. We shall also call (4) as the *one-step* ADMM for solving the SDP. Define the primal optimal set $\mathcal{X}_\star$, the dual optimal set $\mathcal{S}_\star$, and the *difference* optimal set $\mathcal{Z}_\star$ as follows:

$$\text{Primal: } \mathcal{X}_\star := \left\{ X \;\middle|\; \mathcal{P}(X - \widetilde{X}) = 0,\ X \in \mathbb{S}^n_+,\ \langle X, S_\star \rangle = 0 \right\}, \tag{5a}$$

$$\text{Dual: } \mathcal{S}_\star := \left\{ S \;\middle|\; \mathcal{P}^\perp(S - C) = 0,\ S \in \mathbb{S}^n_+,\ \langle S, X_\star \rangle = 0 \right\}, \tag{5b}$$

$$\text{Difference: } \mathcal{Z}_\star := \mathcal{X}_\star - \sigma\mathcal{S}_\star. \tag{5c}$$

Here $(X_\star, S_\star)$ can be any primal–dual optimal solution pair in $\mathcal{X}_\star \times \mathcal{S}_\star$.

One-dimensional criteria. Both the three-step and the one-step ADMM for solving the SDP are high-dimensional dynamical systems. In practice, however, we often observe them through one-dimensional quantities. For the three-step ADMM (2) in particular, the primal infeasibility, dual infeasibility, and relative gap—collectively, the KKT residuals—are defined as

$$r_p^{(k)} := \frac{\|\mathcal{A}X^{(k)} - b\|}{1 + \|b\|}, \qquad r_d^{(k)} := \frac{\|\mathcal{A}^* y^{(k)} + S^{(k)} - C\|_{\mathsf{F}}}{1 + \|C\|_{\mathsf{F}}}, \qquad r_g^{(k)} := \frac{\left|\langle C, X^{(k)}\rangle - b^\mathsf{T} y^{(k)}\right|}{1 + \left|\langle C, X^{(k)}\rangle\right| + \left|b^\mathsf{T} y^{(k)}\right|}, \tag{6}$$

with $r_{\max}^{(k)} := \max\{r_p^{(k)}, r_d^{(k)}, r_g^{(k)}\}$ the maximum KKT residual. Since $X^{(k)} \succeq 0$ and $S^{(k)} \succeq 0$ at all iterations, we omit the PSD-violation terms from (6).

For the one-step ADMM (4), we denote $\Delta Z^{(k)} := Z^{(k+1)} - Z^{(k)}$, which is tightly related to $r_{\max}^{(k)}$ [22]. We write $\|\Delta Z^{(k)}\|_{\mathsf{F}}$ for its Frobenius norm. Similarly, we define $\Delta X^{(k)}$ and $\Delta S^{(k)}$ with their Frobenius norms $\|\Delta X^{(k)}\|_{\mathsf{F}}$ and $\|\Delta S^{(k)}\|_{\mathsf{F}}$. The angle between two consecutive $\Delta Z^{(k)}$ is denoted by $\angle(\Delta Z^{(k+1)}, \Delta Z^{(k)})$. We will frequently use these one-dimensional criteria in the subsequent analysis. [Hank: It is probably better to formally define the angle here with arccos?]

1.1 ADMM for SDP: Empirical Slow-Convergence Patterns

Despite its growing popularity and wide adoption, ADMM often suffers from slow-convergence issues when solving SDPs [18, 24, 53]: after several thousand iterations, progress often slows down dramatically and may nearly stall. This empirical observation *almost* aligns with existing theory. In general, ADMM for SDPs is widely understood to have sublinear convergence. Under additional regularity at the limiting KKT point—such as two-sided constraint nondegeneracy [8, 17] and strict complementarity [22]—one can establish local linear convergence. For general degenerate SDPs, however, metric subregularity of the KKT operator at the limiting point, which is required for local linear convergence of primal–dual splitting methods, may fail to hold [10, Example 1]. Consequently, slow-convergence regions are generally unavoidable for ADMM on degenerate SDPs, and characterizing these regions is of both practical and theoretical importance.

Empirical patterns in slow-convergence regions. A major motivation for this paper comes from the empirical observation that these slow-convergence regions exhibit remarkably consistent patterns. While a comprehensive numerical study is provided in §10, here we focus on four representative SDPs from the `Mittelmann` dataset[1]: `cnhil10`, `foot`, `neu1g`, and `texture`. These small- to medium-scale instances are among those for which ADMM struggles to reach high accuracy (*e.g.,* $r_{\max} \leq 10^{-10}$) within 10^6 iterations [22].

Experiment I. We first run three-step ADMM for about 10^6 iterations. In the first 20000 iterations, σ is updated using the classical heuristic that balances the primal and dual infeasibilities [49]; afterward, we fix σ. Figure 1 reports the trajectories of $\angle(\Delta Z^{(k+1)}, \Delta Z^{(k)})$, $\|\Delta Z^{(k)}\|_{\mathsf{F}}$, and $r_{\max}^{(k)}$. We observe the *first* noticeable pattern:

> During the prolonged period where $\|\Delta Z^{(k)}\|_{\mathsf{F}}$ and $r_{\max}^{(k)}$ nearly stall, $\angle(\Delta Z^{(k+1)}, \Delta Z^{(k)})$ tends to be small yet nonzero (typically around 10^{-3} to 10^{-5}), except for a few "sparse spikes".

This is unusual because even the smallest decision-variable dimension among these four SDPs exceeds 5000. In such high dimensions, two randomly generated vectors are typically nearly orthogonal, not nearly parallel.

Experiment II. We next perform a more delicate experiment. Taking $(X^{(40000)}, y^{(40000)}, S^{(40000)})$ as a new initialization, we gradually increase σ by a factor of 10 over the next 5000 iterations, mimicking the effect of σ updating in practice. Figure 2 shows the trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ as functions of σ. We observe the *second* noticeable pattern:

[1] https://plato.asu.edu/ftp/sparse_sdp.html

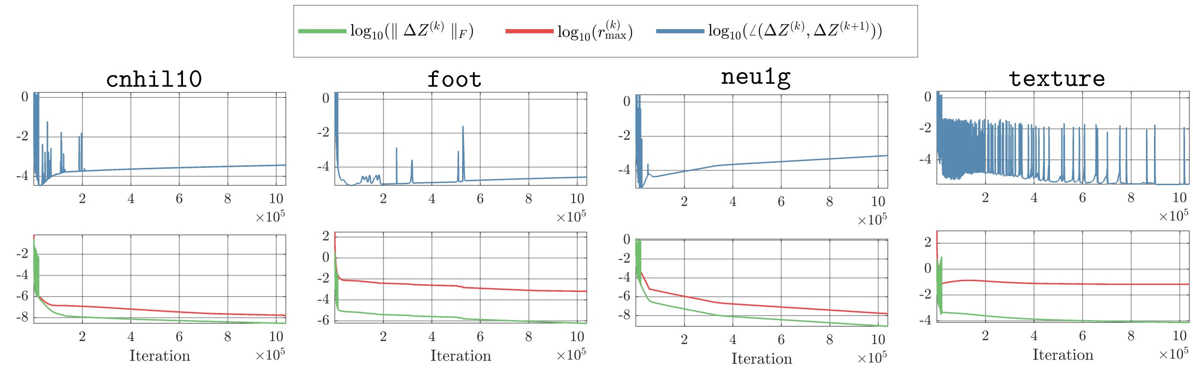


Figure 1: Trajectories of $r_{\max}^{(k)}$, $\|\Delta Z^{(k)}\|_{\mathsf{F}}$, and $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ in Experiment I. [Hank: Why the figures do not seem to span the entire width of the page?]

> As σ changes, $r_p^{(k)}$ and $r_d^{(k)}$ remain almost unchanged. Meanwhile, $\log_{10}(\|\Delta X^{(k)}\|_{\mathsf{F}})$ (resp. $\log_{10}(\|\Delta S^{(k)}\|_{\mathsf{F}})$) increases (resp. decreases) approximately linearly with $\log_{10}(\sigma)$, with slope close to $+1$ (resp. -1).

This observation conflicts with the common wisdom behind updating σ in practice, which aims to balance the primal and dual infeasibilities [49]. The apparent insensitivity of $r_p^{(k)}$ and $r_d^{(k)}$ to σ therefore poses a significant challenge for designing effective σ-update rules.

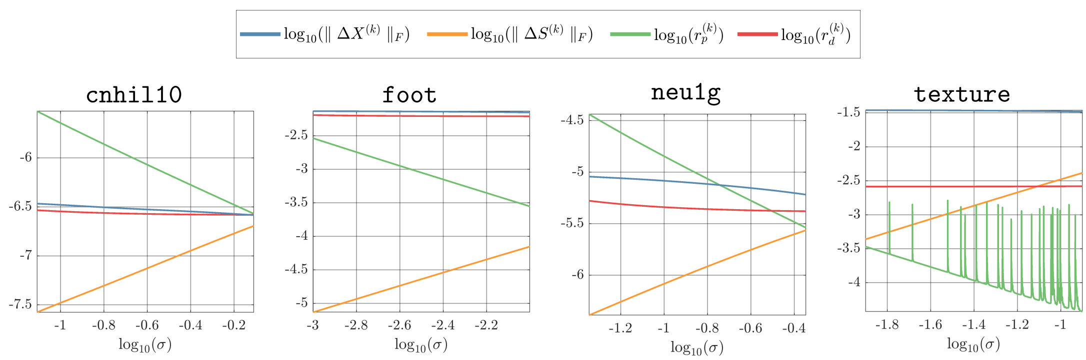


Figure 2: Trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ w.r.t σ in Experiment II. [Hank: Again, figures do not seem to span the entire width of the page. Also, four colors in the same plot is difficult to distinguish. Maybe you can use dashed lines to plot the two curves that respond at rates ± 1?]

In this paper, we aim to understand the mechanisms underlying ADMM's slow-convergence regions, with two goals: (i) to explain the two empirical patterns above; (ii) to predict additional qualitative behaviors in the slow-convergence regions and to shed light on algorithmic design for ADMM on degenerate SDPs.

1.2 Contributions

Assuming the *existence* of a strictly complementary primal–dual solution pair, we view ADMM for degenerate SDPs as a structured nonlinear dynamical system and study its *limiting* behavior in a neighborhood of an arbitrary $\bar{Z} \in \mathcal{Z}_\star$. Focusing on the cone of directions along which the local first-order update vanishes, we construct a local second-order *limit map* $\phi(\bar{Z}; \cdot)$ as a vector field. The induced local second-order *limit*

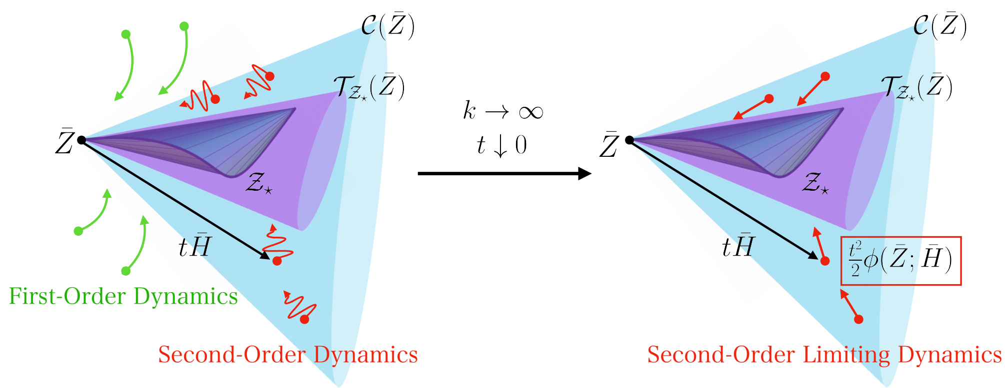


Figure 3: Illustration of the local second-order limit dynamics of ADMM for SDPs. The spectrahedron represents the optimal solution set $\mathcal{Z}_\star$. The blue cone depicts $\mathcal{C}(\bar{Z})$, the cone of directions along which ADMM's local first-order update vanishes. The purple cone depicts $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the tangent cone to $\mathcal{Z}_\star$ attached at $\bar{Z}$. In the left panel, the green points and flow indicate the transient local first-order dynamics, which vanishes as $k \to \infty$ and converges to $\mathcal{C}(\bar{Z})$. The red points and wavy trajectories illustrate the transient local second-order dynamics. For each point of the form $\bar{Z} + t\bar{H}$ with a stalled first-order direction $\bar{H} \in \mathcal{C}(\bar{Z})$, the second-order iterate difference converges to $\frac{t^2}{2}\phi(\bar{Z};\bar{H})$ (red arrows in the right panel), capturing ADMM's limiting behavior up to second order.

dynamics is

$$Z^{(k+1)} = Z^{(k)} + \frac{1}{2}\phi(\bar{Z}; Z^{(k)} - \bar{Z}) + o(\|Z^{(k)} - \bar{Z}\|_{\mathsf{F}}^2).$$

As a local surrogate for the nonlinear one-step dynamics (4) near $\bar{Z}$, this model captures ADMM's local limiting behavior while filtering out transient effects. We then analyze the fundamental properties of $\phi(\bar{Z};\cdot)$ (*e.g.,* kernel, range, continuity, and primal–dual partition) and connect them to qualitative features of the limit dynamics (*e.g.,* fixed points, almost-invariant sets, phase transitions, and the role of σ). A notable aspect of our framework is that it does *not* require $\bar{Z}$ to be the limiting point that ADMM eventually converges to; this shifts the analysis from a *pointwise, asymptotic* paradigm to a *region-wise, transient* one. This perspective is fundamentally different from the existing second-order analyses for (nonlinear) SDPs [15, 39, 43]. Concretely, our contributions are as follows.

A refined and simplified formula for the second-order directional derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$. A central technical ingredient in building our second-order analysis is the (parabolic) second-order directional derivative of the PSD projection $\Pi_{\mathbb{S}^n_+}(\cdot)$. Our derivation builds on [52, Theorem 4.1] and [32, Propositions 3.1–3.2], with two key refinements: (i) we correct several minor typos in both references, which yields a cleaner and more streamlined expression; (ii) we expose a *self-similar* structure between the first- and (parabolic) second-order directional derivatives of $\Pi_{\mathbb{S}^n_+}(\cdot)$. This self-similarity is repeatedly exploited in our second-order analysis. We expect the refined variational characterization to be useful beyond the present setting.

A local second-order limiting model for ADMM near any $\bar{Z} \in \mathcal{Z}_\star$. Starting from any $\bar{Z} \in \mathcal{Z}_\star$, we expand the one-step ADMM dynamics (4) around it up to second order, using the first- and (parabolic) second-order directional derivatives of $\Pi_{\mathbb{S}^n_+}(\cdot)$. For the operator governing the local first-order dynamics, we prove its firm nonexpansiveness and give a detailed characterization of its nonempty fixed-point set $\mathcal{C}(\bar{Z})$. For any stalled first-order direction $\bar{H} \in \mathcal{C}(\bar{Z})$, we show that the operator associated with the local second-order dynamics is also firmly nonexpansive but, in general, does *not* admit fixed points. Instead, we prove the existence of the limit of the iterate difference for the second-order dynamics and denote it by $\phi(\bar{Z};\bar{H})$.

By varying $\bar{H}$ over $\mathcal{C}(\bar{Z})$, we obtain the local second-order *limit map* $\phi(\bar{Z};\cdot) : \mathcal{C}(\bar{Z}) \mapsto \mathbb{S}^n$, which becomes the central object of the paper, and we define the induced limit dynamics accordingly. See Figure 3 for an illustration. We study four core properties of the limit map $\phi(\bar{Z};\cdot)$:

- **Kernel of $\phi(\bar{Z};\cdot)$.** We prove that $\ker(\phi(\bar{Z};\cdot))$ coincides with $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the tangent cone to $\mathcal{Z}_\star$ at $\bar{Z}$. This ties ADMM's local dynamics to Sturm's square-root error bound under strict complementarity [42]. From the limit-dynamics viewpoint, if ADMM is initialized with $Z^{(0)}$ satisfying $Z^{(0)} - \bar{Z} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, then $\Delta Z^{(k)}$ transiently tracks $\frac{1}{2}\phi(\bar{Z}; Z^{(k)} - \bar{Z})$. This mechanism explains the "small yet nonzero" behavior of $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ observed in Experiment I.
- **Range of $\phi(\bar{Z};\cdot)$.** We clarify the relationship between $\mathrm{ran}(\phi(\bar{Z};\cdot))$ and $\mathrm{aff}(\mathcal{C}(\bar{Z}))$: (i) in general, $\mathrm{ran}(\phi(\bar{Z};\cdot)) \nsubseteq \mathrm{aff}(\mathcal{C}(\bar{Z}))$; (ii) if, in addition, either the primal or the dual optimal solution is unique, then $\mathrm{ran}(\phi(\bar{Z};\cdot)) \subseteq \mathrm{aff}(\mathcal{C}(\bar{Z}))$. Interpreted through the limit dynamics, these results illuminate how $\mathcal{C}(\bar{Z})$ can act as a local almost-invariant structure and why second-order updates may remain confined to a low-dimensional subspace for a long time.
- **Continuity of $\phi(\bar{Z};\cdot)$.** We first construct explicit points of discontinuity of $\phi(\bar{Z};\cdot)$ on $\mathcal{C}(\bar{Z})$, and then establish an almost-sure type continuity statement for $\phi(\bar{Z};\cdot)$ with respect to the Lebesgue measure on $\mathrm{aff}(\mathcal{C}(\bar{Z}))$. In terms of limit dynamics, the "sparse" discontinuities of $\phi(\bar{Z};\cdot)$ provide a concrete explanation for the "sparse spikes" in $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ observed in Experiment I, and enable accurate predictions of these microscopic phase transitions.
- **Effect of σ on $\phi(\bar{Z};\cdot)$.** We show that, under the local second-order limit dynamics model, the second-order limits of both $r_p^{(k)}$ and $r_d^{(k)}$ are insensitive to σ. The proof relies on uncovering a primal–dual decoupling structure hidden in $\phi(\bar{Z};\bar{H})$. This result directly explains the response curves in Experiment II. We also discuss the implications for designing σ-update strategies in second-order-dominant regimes.

Numerical verification. We validate our theory on three (small-scale) degenerate SDP examples, where first- and second-order quantities can be computed explicitly. We further conduct experiments on the `Mittelmann` dataset. Across a substantial subset of hard instances, we observe empirical patterns that are explained by our local second-order limit dynamics, supporting the generality of the proposed framework.

1.3 Notation

Given a finite-dimensional Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and a convex set $C \subset \mathcal{H}$, we write $\mathrm{ri}(C)$ for the relative interior of C, $\mathrm{aff}(C)$ for its affine hull, and $\mathrm{cl}(C)$ for its closure. If, in addition, C is closed and convex, we denote by $\mathcal{T}_C(x)$ the tangent cone to C at any $x \in C$. For a convex cone $\mathcal{K} \subset \mathcal{H}$, we write $\mathcal{K}^\circ$ for its polar cone. Given a mapping $\mathcal{T} : \mathcal{U} \mapsto \mathcal{H}$ (with $\mathcal{U} \subset \mathcal{H}$), we define $\ker(\mathcal{T}) := \{x \in \mathcal{U} \mid \mathcal{T}(x) = 0\}$, $\mathrm{ran}(\mathcal{T}) := \{\mathcal{T}(x) \mid x \in \mathcal{U}\}$, and $\mathrm{Fix}(\mathcal{T}) := \{x \in \mathcal{U} \mid \mathcal{T}(x) = x\}$. We denote by $\Pi_C(x)$ the orthogonal projection of $x \in \mathcal{H}$ onto C. We define $\mathbb{B}_r(x) := \{y \in \mathcal{H} \mid \|y - x\| \le r\}$, where $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$.

For the space $\mathbb{S}^n$, the inner product is $\langle A, B\rangle = \mathrm{tr}\left(A^\mathsf{T} B\right) = \mathrm{tr}\left(AB\right)$ for all $A, B \in \mathbb{S}^n$. For any $H \in \mathbb{S}^n$, we denote by $\mathcal{O}^n(H)$ the set of orthonormal matrices that diagonalize H, and we write $\|H\|_\mathsf{F}$ for its Frobenius norm. We denote by $\mathbb{S}^n_-$ the set of negative semidefinite (NSD) matrices in $\mathbb{S}^n$. When the matrix size n is clear from the context, we abbreviate $\Pi_{\mathbb{S}^n_+}(H)$ (resp. $\Pi_{\mathbb{S}^n_-}(H)$) as $\Pi_+(H)$ (resp. $\Pi_-(H)$). For symmetric matrices, we display only the upper-triangular part for simplicity; the symmetric entries are indicated by "$\sim$". We denote by I_k the $k \times k$ identity matrix. For any $v \in \mathbb{R}^n$, we write $\|v\|$ for its Euclidean norm.

1.4 Outline

After a brief overview of related work in §2, we present in §3 a simplified formula for the (parabolic) second-order directional derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$. Building on this result, §4 develops a detailed second-order analysis around an arbitrary $\bar{Z} \in \mathcal{Z}_\star$, which naturally leads to the definition of the local second-order limit map and its induced dynamics—the core concepts of this paper. We then investigate four fundamental properties of

$\phi(\bar{Z};\cdot)$ in parallel: (i) its kernel in §5; (ii) its range in §6; (iii) its continuity in §7; and (iv) the effect of σ in §8. The connection between each property of $\phi(\bar{Z};\cdot)$ and ADMM's dynamical behavior is discussed at the end of the corresponding section. In §9, we present three SDP examples, which serve as sanity checks and illustrations of the theory and also contribute to our proofs. In §10, we report numerical experiments on the `Mittelmann` dataset. Finally, §11 concludes the paper.

2 Related Work

Our second-order analysis targets ADMM for degenerate SDPs and draws on tools from convex analysis, matrix analysis, monotone operator theory, and dynamical systems.

First-order proximal methods for SDP. ADMM can be viewed as a representative primal–dual proximal method arising from monotone operator theory [38]. Beyond the classical ADMM approach for SDPs [49], symmetric Gauss–Seidel (sGS)-ADMM [9] has attracted increasing attention as an efficient scheme for solving general SDPs to medium accuracy. Other proximal methods, such as the primal–dual hybrid gradient (PDHG) method [20], have likewise been investigated for SDP-type formulations. More recently, these proximal frameworks have been integrated with low-rank factorization schemes to better exploit problem structure and scalability [18,47]. On the theoretical side, sufficient conditions guaranteeing fast local linear convergence have been established, including two-sided constraint nondegeneracy [17] and strict complementarity [22] at the limiting KKT point. Numerical evidence supporting fast local convergence under such conditions can be found in [22, 47].

Variational properties of $\Pi_{\mathbb{S}^n_+}(\cdot)$. The PSD cone projection operator [19] can be viewed as a spectral function generated by the `ReLU` map $\max\{x, 0\}$. For spectral functions that are twice (continuously) Fréchet differentiable, explicit formulas for the first- and second-order Fréchet derivatives are provided in [26,28]. The broader class of second-order directional differentiability is systematically treated in [52], and [32] further leverages these results to derive the (parabolic) second-order directional derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$. Additional variational properties, including strong semismoothness, are studied in [44], with extensions to more general spectral operators discussed in [12]. Recent work also explores approximating the PSD cone projection via composite polynomial filtering motivated by homomorphic encryption considerations [21].

Second-order analysis for (nonlinear) SDP. Second-order variational analysis for (nonlinear) SDPs provides a systematic language for curvature, constraint qualifications, and stability. Foundational developments include first-order optimality and sensitivity frameworks [39] and second-order sufficient conditions together with constraint nondegeneracy-type regularity [43]. More recent studies introduce weaker second-order conditions [15] and extend such analyses to stratum-restricted settings [3]. These second-order conditions also play a role in characterizing critical points arising from reformulations such as the squared-variable approach versus the original nonlinear SDP [14]. In contrast, our work adopts a different viewpoint, emphasizing transient dynamical behavior in ADMM for SDPs rather than asymptotic optimality conditions at a single limiting point.

Optimization algorithms as dynamical systems. Many optimization algorithms—including primal–dual splitting methods for conic programming—can be naturally viewed as highly structured iterative maps [4, 38]. Compared with complexity analyses, however, dynamical features such as phases and (almost-)invariant sets remain relatively under-explored. Within the existing literature, partial smoothness and active-set identification [27] provide a powerful mechanism for explaining pronounced phase transitions from slow convergence to faster local regimes [30, 31]. For first-order methods in linear programming, dedicated geometric tools have also been developed to explain phase-transition phenomena even without partial smoothness assumptions [33]. In this paper, we investigate multiple dynamical features of ADMM for degenerate SDPs through the lens of "limit dynamics". This perspective is motivated by dynamical systems theory, where understanding limiting behaviors (*e.g.,* limit cycles [35] and center manifolds [7]) is a standard approach to analyzing complicated trajectories.

3 Simplified Second-Order Directional Derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$

Let $f:\mathbb{R}\mapsto\mathbb{R}$ be a (parabolically) second-order directionally differentiable scalar function. Its first- and (parabolic) second-order directional derivatives are defined by

$$f'(z;h) := \lim_{t\downarrow 0}\frac{f(z+th)-f(x)}{t}, \tag{7a}$$

$$f''(z;h,w) := \lim_{t\downarrow 0}\frac{f(z+th+\frac{t^2}{2}w^2)-f(z)-tf'(z;h)}{\frac{1}{2}t^2}. \tag{7b}$$

Next, let $F:\mathbb{S}^n\mapsto\mathbb{S}^n$ be a (parabolically) second-order directionally differentiable spectral function generated by f. Namely, for any $X\in\mathbb{S}^n$ with $Q\in\mathcal{O}^n(X)$,

$$F(X) = F(Q\mathrm{diag}\left(\{\lambda_i\}_{i=1}^n\right)Q^\mathsf{T}) = Q\mathrm{diag}\left(\{f(\lambda_i)\}_{i=1}^n\right)Q^\mathsf{T}.$$

The first- and (parabolic) second-order directional derivatives of F are defined as

$$F'(Z;H) := \lim_{t\downarrow 0}\frac{F(Z+tH)-F(Z)}{t}, \tag{8a}$$

$$F''(Z;H,W) := \lim_{t\downarrow 0}\frac{F(Z+tH+\frac{1}{2}t^2W)-F(Z)-tF'(Z;H)}{\frac{1}{2}t^2}. \tag{8b}$$

In particular, the PSD cone projection operator $\Pi_{\mathbb{S}^n_+}(\cdot)$ is a spectral function generated by $f(x)=\max\{x,0\}$. Moreover, $\Pi_{\mathbb{S}^n_+}(\cdot)$ is second-order directionally differentiable; see [52].

Nested eigen-structure description. We adopt notations from [52], with some adjustments tailored for the PSD cone projection:

1. *First-level description.* Start with a diagonal matrix Z. Denote its distinct positive eigenvalues (if any) by $\{\mu_a\}_{a\in\mathcal{I}_+}$ and its distinct negative eigenvalues (if any) by $\{\mu_b\}_{b\in\mathcal{I}_-}$. If Z has a zero eigenvalue, denote it by $\mu_0=0$. Define $\mathcal{I}:=\mathcal{I}_+\cup\mathcal{I}_-\cup\{0\}$. For each sub-block $k\in\mathcal{I}$, let the corresponding index set be α_k. That is,

$$Z_{\alpha_k\alpha_l} = \begin{cases}\mu_k\cdot I_{|\alpha_k|}, & k=l\in\mathcal{I},\\ 0, & k\neq l,\quad k,l\in\mathcal{I}.\end{cases}$$

   Here, for any matrix A, $A_{\alpha_k\alpha_l}$ denotes the sub-block of A with row indices α_k and column indices α_l. We also write A_{α_k} for the *columns* of A indexed by α_k. Finally, define $\alpha_+ := \cup_{a\in\mathcal{I}_+}\alpha_a$ and $\alpha_- := \cup_{b\in\mathcal{I}_-}\alpha_b$.

2. *Second-level description.* Let $Z\in\mathbb{S}^n$ be given by the first-level description, and let $H\in\mathbb{S}^n$ be another symmetric matrix. For any $k\in\mathcal{I}$, extract the corresponding sub-block of H, denoted by $H_{\alpha_k\alpha_k}$. Denote its distinct positive eigenvalues (if any) by $\{\eta_{k,i}\}_{i\in\mathcal{I}_{k,+}}$ and its distinct negative eigenvalues (if any) by $\{\eta_{k,j}\}_{j\in\mathcal{I}_{k,j}}$. If it has a zero eigenvalue, denote it by $\eta_{k,0}=0$. Define $\mathcal{I}_k:=\mathcal{I}_{k,+}\cup\mathcal{I}_{k,+}\cup\{0\}$. For $H_{\alpha_k\alpha_k}$, let each eigen-block $i\in\mathcal{I}_k$ be indexed by a set $\beta_{k,i}$. Equivalently, there exists $Q^k\in\mathcal{O}^{|\alpha_k|}(H_{\alpha_k\alpha_k})$ such that

$$(Q^k_{\beta_{k,i}})^\mathsf{T}H_{\alpha_k\alpha_k}Q^k_{\beta_{k,j}} = \begin{cases}\eta_{k,i}\cdot I_{|\beta_{k,i}|}, & i=j\in\mathcal{I}_k,\\ 0, & i\neq j,\quad i,j\in\mathcal{I}_k.\end{cases}$$

   Finally, define $\beta_{k,+} := \cup_{i\in\mathcal{I}_{k,+}}\beta_{k,i}$ and $\beta_{k,-} := \cup_{j\in\mathcal{I}_{k,-}}\beta_{k,j}$.

3. *Third-level description.* Let $Z \in \mathbb{S}^n$ be given by the first-level description, and let $H \in \mathbb{S}^n$ be given by the second-level description. Let $W \in \mathbb{S}^n$. For any $k \in \mathcal{I}$, define

$$V_k(H, W) := W_{\alpha_k \alpha_k} + \sum_{l \in \mathcal{I} \setminus \{k\}} \frac{2}{\mu_k - \mu_l} \cdot H_{\alpha_k \alpha_l} H_{\alpha_l \alpha_k}. \tag{9}$$

   We may abbreviate $V_k(H, W)$ as V_k if there is no ambiguity. Abbreviate $(Q^k_{\beta_{k,i}})^\mathsf{T} V_k Q^k_{\beta_{k,j}}$ as $\hat{V}_k^{i,j}$. For any $k \in \mathcal{I}, i \in \mathcal{I}_k$, we denote $\hat{V}_k^{i,i}$'s distinct positive eigenvalues (if any) by $\{\zeta_{k,i,i'}\}_{i' \in \mathcal{I}_{k,i,+}}$ and its distinct negative eigenvalues (if any) by $\{\zeta_{k,i,j'}\}_{j' \in \mathcal{I}_{k,i,-}}$. If $\hat{V}_k^{i,i}$ has a zero eigenvalue, denote it by $\zeta_{k,i,0} = 0$. Define $\mathcal{I}_{k,i} := \mathcal{I}_{k,i,+} \cup \mathcal{I}_{k,i,-} \cup \{0\}$. For $\hat{V}_k^{i,i}$, let each eigen-block $i' \in \mathcal{I}_{k,i}$ be indexed by a set $\gamma_{k,i,i'}$. Equivalently, there exists $\hat{Q}^{k,i} \in \mathcal{O}^{|\beta_{k,i}|}(\hat{V}_k^{i,i})$ such that

$$(\hat{Q}^{k,i}_{\gamma_{k,i,i'}})^\mathsf{T} \hat{V}_k^{i,i} \hat{Q}^{k,i}_{\gamma_{k,i,j'}} = \begin{cases} \zeta_{k,i,i'} \cdot I_{|\gamma_{k,i,i'}|}, & i' = j' \in \mathcal{I}_{k,i}, \\ 0, & i' \neq j', \ i', j' \in \mathcal{I}_{k,i}. \end{cases}$$

   Finally, define $\gamma_{k,i,+} := \cup_{i' \in \mathcal{I}_{k,i,+}} \gamma_{k,i,i'}$ and $\gamma_{k,i,-} := \cup_{j' \in \mathcal{I}_{k,i,-}} \gamma_{k,i,j'}$.

Now suppose we are given a triplet (Z, H, W) from the above three-level description. For visualization, we partition the $n \times n$ matrix into 3×3 sub-blocks, based on Z's positive-zero-negative eigenvalue structures. The partition is represented by *dashed* lines. For instance,

$$H = \left[\begin{array}{c:c:c} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0} & H_{\alpha_+\alpha_-} \\ \hdashline \sim & H_{\alpha_0\alpha_0} & H_{\alpha_0\alpha_-} \\ \hdashline \sim & \sim & H_{\alpha_-\alpha_-} \end{array}\right] = \left[\begin{array}{c:c:c} \{H_{\alpha_a\alpha_b}\}_{\substack{a \in \mathcal{I}_+ \\ b \in \mathcal{I}_+}} & \{H_{\alpha_a\alpha_0}\}_{a \in \mathcal{I}_+} & \{H_{\alpha_a\alpha_b}\}_{\substack{a \in \mathcal{I}_+ \\ b \in \mathcal{I}_-}} \\ \hdashline \sim & H_{\alpha_0\alpha_0} & \{H_{\alpha_0\alpha_b}\}_{b \in \mathcal{I}_-} \\ \hdashline \sim & \sim & \{H_{\alpha_a\alpha_b}\}_{\substack{a \in \mathcal{I}_- \\ b \in \mathcal{I}_-}} \end{array}\right].$$

Similarly, for the $\alpha_0\alpha_0$ block, we further partition it into 3×3 sub-blocks following $H_{\alpha_0\alpha_0}$'s positive-zero-negative eigenvalue structures. But since $H_{\alpha_0\alpha_0}$ is no longer diagonal, a basis change is necessary. For instance,

$$W_{\alpha_0\alpha_0} = Q^0 \left[\begin{array}{c:c:c} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}} & \widehat{W}_{\beta_{0,+}\beta_{0,-}} \\ \hdashline \sim & \widehat{W}_{\beta_{0,0}\beta_{0,0}} & \widehat{W}_{\beta_{0,0}\beta_{0,-}} \\ \hdashline \sim & \sim & \widehat{W}_{\beta_{0,-}\beta_{0,-}} \end{array}\right] (Q^0)^\mathsf{T} = Q^0 \left[\begin{array}{c:c:c} \left\{\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i \in \mathcal{I}_{0,+} \\ b \in \mathcal{I}_{0,+}}} & \left\{\widehat{W}_{\beta_{0,i}\beta_{0,0}}\right\}_{i \in \mathcal{I}_{0,+}} & \left\{\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i \in \mathcal{I}_{0,+} \\ b \in \mathcal{I}_{0,-}}} \\ \hdashline \sim & \widehat{W}_{\beta_{0,0}\beta_{0,0}} & \left\{\widehat{W}_{\beta_{0,0}\beta_{0,j}}\right\}_{j \in \mathcal{I}_{0,-}} \\ \hdashline \sim & \sim & \left\{\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i \in \mathcal{I}_{0,-} \\ b \in \mathcal{I}_{0,-}}} \end{array}\right] (Q^0)^\mathsf{T}.$$

where $\widehat{W} := (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0$.

First-order directional derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$. We provide the following classical result from [44, Theorem 4.7]:

Theorem 1 ($\Pi'_{\mathbb{S}^n_+}(Z; H)$)**.** *Let $Z \in \mathbb{S}^n$ be given by the first-level description. For any $H \in \mathbb{S}^n$ given by the second-level description,*

$$\Pi'_+(Z; H) = \left[\begin{array}{c:c:c} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0} & \left\{\frac{\mu_a}{\mu_a - \mu_b} H_{\alpha_a\alpha_b}\right\}_{\substack{a \in \mathcal{I}_+ \\ b \in \mathcal{I}_-}} \\ \hdashline \sim & \Pi_+(H_{\alpha_0\alpha_0}) & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right]. \tag{10}$$

For a non-diagonal $Z \in \mathbb{S}^n$: Pick $Q \in \mathcal{O}^n(Z)$. Denote $\widetilde{Z} := Q^\mathsf{T} Z Q$ diagonal and $\widetilde{H} := Q^\mathsf{T} H Q$:

$$\Pi'_+(Z; H) = Q \Pi'_+(\widetilde{Z}; \widetilde{H}) Q^\mathsf{T}. \tag{11}$$

(Parabolic) second-order directional derivative of $\Pi_{\mathbb{S}^n_+}(\cdot)$. Our result builds on [52, Theorem 4.1] and [32, Propositions 3.1–3.2], with two key refinements: (i) we correct several minor typos in both [52] and [32], which in turn yields a simplified formula; (ii) we reveal a *self-similar* structure between the $\alpha_0\alpha_0$ block of $\Pi''_+(Z;H,W)$ and $\Pi'_+(Z;H)$, which serves as a key ingredient in the subsequent second-order analysis.

Theorem 2 ($\Pi''_{\mathbb{S}^n_+}(Z;H,W)$)**.** *Let the triplet (Z,H,W) given by the three-level description. Then,*

$$\Pi''_+(Z;H,W) = \tag{12}$$
$$\left[\begin{array}{c:c:c} \left\{ \begin{array}{c} W_{\alpha_a\alpha_b} \\ +2\sum\limits_{c\in\mathcal{I}_-} \frac{-\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \end{array} \right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_+}} & \left\{ \begin{array}{c} W_{\alpha_a\alpha_0} \\ +2\sum\limits_{c\in\mathcal{I}_-} \frac{1}{\mu_a-\mu_c} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_0} \\ -2\frac{1}{\mu_a} H_{\alpha_a\alpha_0}\Pi_+(-H_{\alpha_0\alpha_0}) \end{array} \right\}_{a\in\mathcal{I}_+} & \left\{ \begin{array}{c} \frac{\mu_a}{\mu_a-\mu_b} W_{\alpha_a\alpha_b} \\ +2\sum\limits_{c\in\mathcal{I}_+} \frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \\ +2\frac{1}{\mu_a-\mu_b} H_{\alpha_a\alpha_0}H_{\alpha_0\alpha_b} \\ +2\sum\limits_{c\in\mathcal{I}_-} \frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \end{array} \right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \begin{array}{c} 2\sum\limits_{c\in\mathcal{I}_+} \frac{1}{\mu_c} H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0} \\ +\Pi'_+(H_{\alpha_0\alpha_0};V_0(H,W)) \end{array} & \left\{ \begin{array}{c} 2\sum\limits_{c\in\mathcal{I}_+} \frac{1}{\mu_c-\mu_b} H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_b} \\ +2\frac{1}{-\mu_b}\Pi_+(H_{\alpha_0\alpha_0})H_{\alpha_0\alpha_b} \end{array} \right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & \left\{ 2\sum\limits_{c\in\mathcal{I}_+} \frac{\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \right\}_{\substack{a\in\mathcal{I}_-\\ b\in\mathcal{I}_-}} \end{array}\right].$$

where $V_0(H,W)$ is defined in (9). $\Pi'_+(H_{\alpha_0\alpha_0};V_0(H,W))$ in the $\alpha_0\alpha_0$ block is calculated by (11), since $H_{\alpha_0\alpha_0}$ is not diagonal. For a non-diagonal $Z\in\mathbb{S}^n$: Pick $Q\in\mathcal{O}^n(Z)$. Denote $\widetilde{Z} := Q^\mathsf{T}ZQ$ diagonal, $\widetilde{H} := Q^\mathsf{T}HQ$, $\widetilde{W} := Q^\mathsf{T}WQ$:

$$\Pi''_+(Z;H,W) = Q\Pi''_+(\widetilde{Z};\widetilde{H},\widetilde{W})Q^\mathsf{T}. \tag{13}$$

For readability, we postpone the proof and discussion of Theorem 2 to Appendix A. One may have already noticed that $\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_0\alpha_0}$ exhibits a strong structural resemblance to $\Pi'_{\mathbb{S}^n_+}(Z;H)$. This is not a coincidence; rather, it stems from the *self-similarity* between the first- and (parabolic) second-order directional derivatives of $f(x)=\max\{x,0\}$:

$$f'(h;w) = f''(0;h,w) = \begin{cases} w, & h>0 \\ \max\{w,0\}, & h=0 \\ 0, & h<0 \end{cases}.$$

First- and (parabolic) second-order directional derivatives of $\Pi_{\mathbb{S}^n_-}(\cdot)$. We also derive $\Pi'_{\mathbb{S}^n_-}(Z;H)$ and $\Pi''_{\mathbb{S}^n_-}(Z;H,W)$ for further use.

Theorem 3 ($\Pi''_{\mathbb{S}^n_-}(Z;H)$)**.** *Let $Z\in\mathbb{S}^n$ be given by the first-level description. For any $H\in\mathbb{S}^n$ given by the second-level description,*

$$\Pi'_-(Z;H) = \left[\begin{array}{c:c:c} 0 & 0 & \left\{\frac{-\mu_b}{\mu_a-\mu_b}H_{\alpha_a\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \Pi_-(H_{\alpha_0\alpha_0}) & H_{\alpha_0\alpha_-} \\ \hdashline \sim & \sim & H_{\alpha_-\alpha_-} \end{array}\right]. \tag{14}$$

For a non-diagonal $Z\in\mathbb{S}^n$: Pick $Q\in\mathcal{O}^n(Z)$. Denote $\widetilde{Z} := Q^\mathsf{T}ZQ$ diagonal and $\widetilde{H} := Q^\mathsf{T}HQ$:

$$\Pi'_-(Z;H) = Q\Pi'_-(\widetilde{Z};\widetilde{H})Q^\mathsf{T}. \tag{15}$$

Proof. Since $\Pi_+(Z)+\Pi_-(Z)=Z$, we get

$$\Pi'_+(Z;H)+\Pi'_-(Z;H) = H.$$

Then, (14) is derived from (10) and simple calculation. ∎

Theorem 4 ($\Pi''_{\mathbb{S}^n_-}(Z;H,W)$)**.** *Let the triplet (Z,H,W) given by the three-level description. Then,*

$$
\Pi''_-(Z;H,W) = \tag{16}
$$
$$
\left[\begin{array}{c:c:c}
\left\{2\sum_{c\in\mathcal{I}_-}\frac{\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_+}} & \left\{\begin{array}{c}2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c-\mu_a}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_0}\\ +2\frac{1}{-\mu_a}H_{\alpha_a\alpha_0}\Pi_-(H_{\alpha_0\alpha_0})\end{array}\right\}_{a\in\mathcal{I}_+} & \left\{\begin{array}{c}\frac{-\mu_b}{\mu_a-\mu_b}W_{\alpha_a\alpha_b}\\ +2\sum_{c\in\mathcal{I}_+}\frac{\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\\ +2\frac{1}{\mu_b-\mu_a}H_{\alpha_a\alpha_0}H_{\alpha_0\alpha_b}\\ +2\sum_{c\in\mathcal{I}_-}\frac{-\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\end{array}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline
\sim & \begin{array}{c}2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0}\\ +\Pi'_-(H_{\alpha_0\alpha_0};V_0(H,W))\end{array} & \left\{\begin{array}{c}W_{\alpha_a\alpha_b}\\ +2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_b-\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_b}\\ -2\frac{1}{\mu_b}\Pi_-(-H_{\alpha_0\alpha_0})H_{\alpha_0\alpha_b}\end{array}\right\}_{b\in\mathcal{I}_-} \\ \hdashline
\sim & \sim & \left\{\begin{array}{c}W_{\alpha_a\alpha_b}\\ +2\sum_{c\in\mathcal{I}_+}\frac{-\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\end{array}\right\}_{\substack{a\in\mathcal{I}_-\\ b\in\mathcal{I}_-}}
\end{array}\right].
$$

where $V_0(H,W)$ is defined in (9). $\Pi'_-(H_{\alpha_0\alpha_0};V_0(H,W))$ in the $\alpha_0\alpha_0$ block is calculated by (15), since $H_{\alpha_0\alpha_0}$ is not diagonal. For a non-diagonal $Z\in\mathbb{S}^n$: Pick $Q\in\mathcal{O}^n(Z)$. Denote $\widetilde{Z}:=Q^\mathsf{T}ZQ$ diagonal, $\widetilde{H}:=Q^\mathsf{T}HQ$, $\widetilde{W}:=Q^\mathsf{T}WQ$:

$$
\Pi''_-(Z;H,W) = Q\Pi''_-(\widetilde{Z};\widetilde{H},\widetilde{W})Q^\mathsf{T}. \tag{17}
$$

Proof. Since $\Pi_+(Z)+\Pi_-(Z)=Z$, we get $\Pi'_+(Z;H)+\Pi'_-(Z;H)=H$ and

$$
\Pi''_+(Z;H,W)+\Pi''_-(Z;H,W)=W.
$$

Then, for $\Pi''_-(Z;H,W)$'s $\alpha_0\alpha_0$ block:

$$
\begin{aligned}
&\Pi''_-(Z;H,W)_{\alpha_0\alpha_0}\\
=&W_{\alpha_0\alpha_0}-2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0}-\Pi'_+(H_{\alpha_0\alpha_0};V_0(H,W))\\
=&W_{\alpha_0\alpha_0}-2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0}+\Pi'_-(H_{\alpha_0\alpha_0};V_0(H,W))-V_0(H,W)\\
=&W_{\alpha_0\alpha_0}-2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0}+\Pi'_-(H_{\alpha_0\alpha_0};V_0(H,W))-W_{\alpha_0\alpha_0}+2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}+2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}\\
=&2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}H_{\alpha_0\alpha_c}H_{\alpha_c\alpha_0}+\Pi'_-(H_{\alpha_0\alpha_0};V_0(H,W)),
\end{aligned}
$$

where we use (9) and the fact that $\Pi'_+(H_{\alpha_0\alpha_0},V_0)+\Pi'_-(H_{\alpha_0\alpha_0},V_0)=V_0$. The other blocks in $\Pi''_-(Z;H,W)$ can be derived from (12) and simple calculation. $\square$

4 Local Second-Order Limit Dynamics

As shown in [22], ADMM for SDPs converges locally at a linear rate in a neighborhood of a nonsingular KKT point Z_{sc}. In this section, we study ADMM's finer dynamical behavior near an arbitrary, possibly singular KKT point $\bar{Z}$. In this regime, the local dynamics can be effectively described by a *second-order limiting mapping* $\phi(\bar{Z};\cdot)$.

In §4.1, we state the standing assumptions used throughout the paper. In §4.2, we expand the one-step ADMM update (4) up to second order around $\bar{Z}$, leveraging the expression for $\Pi''_+(Z;H,W)$ in Theorem 2. In §4.3, we examine the geometry of $\mathcal{C}(\bar{Z})$, a closed convex cone along which the first-order updates vanish, and discuss its relationship with $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the tangent cone to the set of KKT points at $\bar{Z}$. In §4.4, we show that, under the local second-order expansion model, for every first-order direction $H \in \mathcal{C}(\bar{Z})$, the limitation of second-order drifting, denoted as $\phi(\bar{Z}; H)$, exists and need not vanish. This nonzero second-order effect motivates the central object of the paper: the *second-order limiting mapping* $\phi(\bar{Z}; \cdot) : \mathcal{C}(\bar{Z}) \mapsto \mathbb{S}^n$, viewed as a vector field. Specifically, at the points Z with $Z - \bar{Z} \in \mathcal{C}(\bar{Z})$ and $\|Z - \bar{Z}\|_{\mathsf{F}} \to 0$, we associate the second-order displacement $\frac{1}{2}\phi(\bar{Z}; Z - \bar{Z})$. The definition of the corresponding *second-order limiting dynamics* then follows immediately. As a local surrogate for the nonlinear dynamics (4) near $\bar{Z}$, it accurately captures the limiting behavior of (4). Finally, in §4.5, we simplify $\phi(\bar{Z}; \cdot)$ by exposing its primal–dual decoupling structure, which serves as a foundation for the subsequent characterization of $\phi(\bar{Z}; \cdot)$.

4.1 Assumptions

Assumption 1. *Two assumptions throughout the paper:*

1. *The linear operator $\mathcal{A} : \mathbb{S}^n \mapsto \mathbb{R}^m$ is surjective.*
2. *There exists a KKT point satisfying strict complementarity,* i.e., $\exists (X_{\mathsf{sc}}, y_{\mathsf{sc}}, S_{\mathsf{sc}})$ *satisfying* (3)*, s.t.* $\operatorname{rank}(X_{\mathsf{sc}}) + \operatorname{rank}(S_{\mathsf{sc}}) = n$*. Equivalently,* $Z_{\mathsf{sc}} = X_{\mathsf{sc}} - \sigma S_{\mathsf{sc}}$ *is nonsingular.*

Note that Assumption 1 requires neither a Slater condition nor constraint nondegeneracy. Many real-world SDPs (including degenerate instances) admit a strictly complementary primal–dual solution pair [22]. Moreover, for small- to medium-scale SDPs, interior-point methods (IPMs) can often be used to obtain a strictly complementary solution pair, provided that one exists. On the other hand, Assumption 1 rules out pathological SDPs whose optimal set has singularity degree greater than 1 [40, 42, 46]; for such problem classes, even IPMs can be challenging to apply effectively [40].

4.1.1 A 4×4 matrix block partition

By Assumption 1, there exists a strictly complementary optimal solution pair. Hence, without loss of generality, we can partition $\mathbb{S}^n$ into 2×2 blocks indexed by $(\alpha_{\mathsf{P}}, \alpha_{\mathsf{D}}) \times (\alpha_{\mathsf{P}}, \alpha_{\mathsf{D}})$, where $\alpha_{\mathsf{P}} \cup \alpha_{\mathsf{D}} = \{1, \ldots, n\}$. Then, up to a change of basis,

$$X = \left[\begin{array}{c|c} X_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} & 0 \\ \hline \sim & 0 \end{array}\right], \quad S = \left[\begin{array}{c|c} 0 & 0 \\ \hline \sim & S_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \end{array}\right], \quad \forall (X, S) \in \mathcal{X}_\star \times \mathcal{S}_\star, \tag{18}$$

with $[X_{\mathsf{sc}}]_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} \succ 0$ and $[S_{\mathsf{sc}}]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \prec 0$. Notice that the block partition based on strict complementarity is represented by *solid* lines, which distinguishes it from *dashed* lines partitioning the positive-zero-negative eigenvalue structures.

Now fix an optimal solution pair $(\bar{X}, \bar{S}) \in \mathcal{X}_\star \times \mathcal{S}_\star$. Without loss of generality, we further assume that both $\bar{X}$ and $\bar{S}$ are diagonal. Indeed, if this is not the case, then by the 2×2 block structure in (18), there exists an orthonormal matrix Q of the form

$$Q := \left[\begin{array}{c|c} Q_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} & 0 \\ \hline \sim & Q_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \end{array}\right],$$

which simultaneously diagonalizes $\bar{X}$ and $\bar{S}$:

$$Q^\mathsf{T} \bar{X} Q = \left[\begin{array}{c|c} \left[\begin{array}{c:c} \Lambda_{\mathsf{P}} & 0 \\ \hdashline \sim & 0 \end{array}\right] & 0 \\ \hline \sim & 0 \end{array}\right], \quad Q^\mathsf{T} \bar{S} Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline \sim & \left[\begin{array}{c:c} 0 & 0 \\ \hdashline \sim & \Lambda_{\mathsf{D}} \end{array}\right] \end{array}\right].$$

Moreover, for any $(X, S) \in \mathcal{X}_\star \times \mathcal{S}_\star$, the same 2×2 block partition (18) is preserved under this congruence transformation, since

$$Q^\mathsf{T} X Q = \left[\begin{array}{c|c} Q^\mathsf{T}_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} X_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} Q_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} & 0 \\ \hline \sim & 0 \end{array}\right], \quad Q^\mathsf{T} S Q = \left[\begin{array}{c|c} 0 & 0 \\ \hline \sim & Q^\mathsf{T}_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} S_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} Q_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \end{array}\right].$$

Accordingly, apply the following congruent transformation to the SDP data $\mathcal{A}$ and C:

$$\{A_i\}_{i=1}^m \leftarrow \left\{Q^\mathsf{T} A_i Q\right\}_{i=1}^m, \quad C \leftarrow Q^\mathsf{T} C Q.$$

Under the transformed data $(\mathcal{A}, b, C)$, the optimal sets $(\mathcal{X}_\star, \mathcal{S}_\star)$ still satisfy the 2×2 block partition (18), while the chosen pair $(\bar{X}, \bar{S})$ becomes diagonal. For $\bar{Z} := \bar{X} - \sigma\bar{S}$, we further assume that it satisfies the first-level description in §3. Under this assumption, the 2×2 partition in (18) refines the 3×3 block partition in §3 into a 4×4 one. Define $\alpha_0^\mathsf{P} := \alpha_\mathsf{P} \backslash \alpha_+$ and $\alpha_0^\mathsf{D} := \alpha_\mathsf{D} \backslash \alpha_-$. Then, for any $H \in \mathbb{S}^n$,

$$H = \left[\begin{array}{cc|cc} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^\mathsf{P}} & H_{\alpha_+\alpha_0^\mathsf{D}} & H_{\alpha_+\alpha_-} \\ \sim & H_{\alpha_0^\mathsf{P}\alpha_0^\mathsf{P}} & H_{\alpha_0^\mathsf{P}\alpha_0^\mathsf{D}} & H_{\alpha_0^\mathsf{P}\alpha_-} \\ \hline \sim & \sim & H_{\alpha_0^\mathsf{D}\alpha_0^\mathsf{D}} & H_{\alpha_0^\mathsf{D}\alpha_-} \\ \sim & \sim & \sim & H_{\alpha_-\alpha_-} \end{array}\right].$$

If we further assume that H satisfies the second-level description in §3, then for any $W \in \mathbb{S}^n$ the $\alpha_0\alpha_0$ block of W can be expressed — after a change of basis, since $H_{\alpha_0\alpha_0}$ is not diagonal — as

$$W_{\alpha_0\alpha_0} = Q^0 \left[\begin{array}{cc|cc} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^\mathsf{P}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^\mathsf{D}} & \widehat{W}_{\beta_{0,+}\beta_{0,-}} \\ \sim & \widehat{W}_{\beta_{0,0}^\mathsf{P}\beta_{0,0}^\mathsf{P}} & \widehat{W}_{\beta_{0,0}^\mathsf{P}\beta_{0,0}^\mathsf{D}} & \widehat{W}_{\beta_{0,0}^\mathsf{P}\beta_{0,-}} \\ \hline \sim & \sim & \widehat{W}_{\beta_{0,0}^\mathsf{D}\beta_{0,0}^\mathsf{D}} & \widehat{W}_{\beta_{0,0}^\mathsf{D}\beta_{0,-}} \\ \sim & \sim & \sim & \widehat{W}_{\beta_{0,-}\beta_{0,-}} \end{array}\right] (Q^0)^\mathsf{T},$$

where $(\beta_{0,\mathsf{P}}, \beta_{0,\mathsf{D}})$ is the primal-dual block partition for $H_{\alpha_0\alpha_0}$ with $\beta_{0,\mathsf{P}} \cup \beta_{0,\mathsf{D}} = \alpha_0$. $\widehat{W} := (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0$, $\beta_{0,0}^\mathsf{P} := \beta_{0,\mathsf{P}} \backslash \beta_{0,+}$, and $\beta_{0,0}^\mathsf{D} := \beta_{0,\mathsf{D}} \backslash \beta_{0,-}$.

4.2 Second-Order Local Expansion

Let $\bar{Z} \in \mathcal{Z}_\star$ be diagonal and satisfy the first-level description in §3. Suppose that $Z^{(k)}$ admits the following expansion in a neighborhood of $\bar{Z}$:

$$Z^{(k)} = \bar{Z} + tH^{(k)} + \frac{t^2}{2} W^{(k)} + o(t^2), \tag{19}$$

where $t \downarrow 0$ is a fixed hyper-parameter. Unlike [22], this local expansion does not require $\bar{Z}$ to be the eventual limit point of $Z^{(k)}$. Consequently, our local framework is more flexible, shifting from a *pointwise, asymptotic* perspective to a *region-wise, transient* one.

Recall the one-step ADMM update (4) and rewrite it in finite-difference form as $Z^{(k+1)} - Z^{(k)} = \delta(Z^{(k)})$, where the residual mapping $\delta(\cdot) : \mathbb{S}^n \mapsto \mathbb{S}^n$ is defined by

$$\delta(Z) := -\mathcal{P}(\Pi_+(Z) - \widetilde{X}) + \mathcal{P}^\perp(\Pi_+(-Z) - \sigma C) = -\mathcal{P}(\Pi_+(Z) - \widetilde{X}) - \mathcal{P}^\perp(\Pi_-(Z) + \sigma C). \tag{20}$$

Clearly, $\delta(\bar{Z}) = 0$. Since both $\Pi_+(\cdot)$ and $\Pi_-(\cdot)$ are (parabolically) second-order directionally differentiable around $\bar{Z}$, the mapping $\delta(\cdot)$ is also (parabolically) second-order directionally differentiable at $\bar{Z}$, with

$$\delta'(\bar{Z}; H) = -\mathcal{P}\Pi'_+(\bar{Z}; H) - \mathcal{P}^\perp\Pi'_-(\bar{Z}; H), \tag{21a}$$

$$\delta''(\bar{Z}; H, W) = -\mathcal{P}\Pi'_+(\bar{Z}; H, W) - \mathcal{P}^\perp\Pi''_-(\bar{Z}; H, W). \tag{21b}$$

Expanding $Z^{(k+1)}$ up to second order then yields

$$
\begin{aligned}
Z^{(k+1)} &= Z^{(k)} + \delta(\bar{Z}) + t\,\delta'(\bar{Z}; H^{(k)}) + \frac{t^2}{2}\,\delta''(\bar{Z}; H^{(k)}, W^{(k)}) + o(t^2) \\
&= \bar{Z} + t\,\underbrace{\left\{H^{(k)} - \mathcal{P}\Pi'_+(\bar{Z}; H^{(k)}) - \mathcal{P}^\perp\Pi'_-(\bar{Z}; H^{(k)})\right\}}_{=:H^{(k+1)}} \\
&\qquad + \frac{t^2}{2}\,\underbrace{\left\{W^{(k)} - \mathcal{P}\Pi'_+(\bar{Z}; H^{(k)}, W^{(k)}) - \mathcal{P}^\perp\Pi''_-(\bar{Z}; H^{(k)}, W^{(k)})\right\}}_{=:W^{(k+1)}} + o(t^2).
\end{aligned} \tag{22}
$$

Definition 1 (Local first- and second-order dynamics). *Define the local first-order dynamics as*

$$
H^{(k+1)} = (\mathrm{Id} + \delta'(\bar{Z};\cdot))(H^{(k)}) = H^{(k)} - \mathcal{P}\Pi'_+(\bar{Z}; H^{(k)}) - \mathcal{P}^\perp\Pi'_-(\bar{Z}; H^{(k)}), \tag{23}
$$

and the local second-order dynamics as

$$
W^{(k+1)} = (\mathrm{Id} + \delta''(\bar{Z}; H^{(k)},\cdot))(W^{(k)}) = W^{(k)} - \mathcal{P}\Pi''_+(\bar{Z}; H^{(k)}, W^{(k)}) - \mathcal{P}^\perp\Pi''_-(\bar{Z}; H^{(k)}, W^{(k)}). \tag{24}
$$

A notable feature of the second-order dynamics is that $W^{(k+1)}$ depends on both $H^{(k)}$ and $W^{(k)}$.

4.3 $\mathcal{C}(\bar{Z})$: the Cone where First-Order Updates Vanish

In this section, we analyze the local first-order dynamics (23). We shall first see that $H^{(k+1)} = (\mathrm{Id} + \delta'(\bar{Z};\cdot))(H^{(k)})$ will converge to one of $\mathrm{Id} + \delta'(\bar{Z};\cdot)$'s fixed points. Recall that an operator $\mathcal{T} : \mathbb{S}^n \mapsto \mathbb{S}^n$ is firmly nonexpansive on $(\mathbb{S}^n, \langle\cdot,\cdot\rangle)$, if

$$
\|\mathcal{T}(H) - \mathcal{T}(G)\|_\mathsf{F}^2 + \|(\mathrm{Id} - \mathcal{T})(H) - (\mathrm{Id} - \mathcal{T})(G)\|_\mathsf{F}^2 \le \|H - G\|_\mathsf{F}^2, \quad \forall H, G \in \mathbb{S}^n.
$$

Lemma 1 (Convergent first-order dyanmics). *Under Assumption 1,* $\mathrm{Id} + \delta'(\bar{Z};\cdot)$ *is firmly nonexpansive on* $(\mathbb{S}^n, \langle\cdot,\cdot\rangle)$*. Moreover, for any* $H^{(0)}$*,* $H^{(k+1)} = (\mathrm{Id} + \delta'(\bar{Z};\cdot))(H^{(k)})$ *converges to a fixed point of* $\mathrm{Id} + \delta'(\bar{Z};\cdot)$.

Proof. For ease of notation, for any $H \in \mathbb{S}^n$, we denote the mappings $(\mathrm{Id} + \delta'(\bar{Z};\cdot))(H)$ as $\mathcal{T}(H)$, $\Pi'_+(\bar{Z}; H)$ as $\Omega(H)$, and $\Pi'_-(\bar{Z}; H)$ as $\Omega^\perp(H)$.

(i) We first prove the firmly nonexpansiveness of $\mathcal{T}$. We have $\forall H, G \in \mathbb{S}^n$,

$$
\begin{aligned}
&\|\mathcal{T}(H) - \mathcal{T}(G)\|_\mathsf{F}^2 + \|(\mathrm{Id} - \mathcal{T})(H) - (\mathrm{Id} - \mathcal{T})(G)\|_\mathsf{F}^2 \\
=&\|\mathcal{P}^\perp[\Omega(H) - \Omega(G)]\|_\mathsf{F}^2 + \|\mathcal{P}[\Omega^\perp(H) - \Omega^\perp(G)]\|_\mathsf{F}^2 + \|\mathcal{P}[\Omega(H) - \Omega(G)]\|_\mathsf{F}^2 + \|\mathcal{P}^\perp[\Omega^\perp(H) - \Omega^\perp(G)]\|_\mathsf{F}^2 \\
=&\|\Omega(H) - \Omega(G)\|_\mathsf{F}^2 + \|\Omega^\perp(H) - \Omega^\perp(G)\|_\mathsf{F}^2 \\
=&\|H - G\|_\mathsf{F}^2 - 2\left\langle \Omega(H) - \Omega(G), \Omega^\perp(H) - \Omega^\perp(G)\right\rangle
\end{aligned}
$$

All we need to show is $\left\langle \Omega(H) - \Omega(G), \Omega^\perp(H) - \Omega^\perp(G)\right\rangle \ge 0$. Denote $U := \Omega(H) - \Omega(G)$ and $V := \Omega^\perp(H) - \Omega^\perp(G)$. Only consider the upper triangular parts of the symmetric matrix:

- If (1) $a \in \mathcal{I}_+, b \in \mathcal{I}_+$; or (2) $a \in \mathcal{I}_+, b = 0$; or (3) $a = 0, b \in \mathcal{I}_-$; or (4) $a \in \mathcal{I}_-, b \in \mathcal{I}_-$:

$$
\langle U_{\alpha_a\alpha_b}, V_{\alpha_a\alpha_b}\rangle = 0
$$

- (5) $a = 0, b = 0$:

$$
\begin{aligned}
&\langle U_{\alpha_0\alpha_0}, V_{\alpha_0\alpha_0}\rangle = \langle \Pi_+(H_{\alpha_0\alpha_0}) - \Pi_+(G_{\alpha_0\alpha_0}), \Pi_-(H_{\alpha_0\alpha_0}) - \Pi_-(G_{\alpha_0\alpha_0})\rangle \\
=&\langle \Pi_+(H_{\alpha_0\alpha_0}), -\Pi_-(G_{\alpha_0\alpha_0})\rangle + \langle \Pi_+(G_{\alpha_0\alpha_0}), -\Pi_-(H_{\alpha_0\alpha_0})\rangle \ge 0
\end{aligned}
$$

- (6) (5) $a \in \mathcal{I}_+, b \in \mathcal{I}_-$:

$$\langle U_{\alpha_a\alpha_b}, V_{\alpha_a\alpha_b}\rangle = \left\langle \frac{\mu_a}{\mu_a - \mu_b}(H-G)_{\alpha_a\alpha_b}, \frac{-\mu_b}{\mu_a - \mu_b}(H-G)_{\alpha_a\alpha_b} \right\rangle \geq 0$$

(ii) We second show $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Since $\delta'(\bar{Z};0) = 0$, we have $0 \in \mathrm{Fix}(\mathcal{T})$. Therefore, by [4, Example 5.18], $H^{(k+1)} = \mathcal{T}(H^{(k)})$ converges to one of the points in $\mathrm{Fix}(\mathcal{T})$. □

Denote $\mathcal{C}(\bar{Z})$ as $\mathrm{Fix}(\mathrm{Id} + \delta'(\bar{Z};\cdot))$ (or equivalently, $\ker(\delta'(\bar{Z};\cdot))$):

$$\mathcal{C}(\bar{Z}) := \left\{ H \in \mathbb{S}^n \;\middle|\; (\mathrm{Id} + \delta'(\bar{Z};\cdot))(H) = H \right\} = \left\{ H \in \mathbb{S}^n \;\middle|\; \delta'(\bar{Z};H) = 0 \right\} \tag{25}$$

We need an important lemma before starting to characterize $\mathcal{C}(\bar{Z})$'s structures:

Lemma 2. *For $G \in \mathbb{S}^n$, under Assumption 1:*

1. *If $\|\mathcal{P}G\|_{\mathsf{F}} \leq \epsilon$ and $\langle G, \bar{S}\rangle = 0$, then*

$$|\langle G, S\rangle| \leq \|S - \bar{S}\|_{\mathsf{F}} \cdot \epsilon, \quad \forall S \in \mathcal{S}_\star$$

2. *If $\|\mathcal{P}^\perp G\|_{\mathsf{F}} \leq \epsilon$ and $\langle G, \bar{X}\rangle = 0$, then*

$$|\langle G, X\rangle| \leq \|X - \bar{X}\|_{\mathsf{F}} \cdot \epsilon, \quad \forall X \in \mathcal{X}_\star$$

Proof. (1) Since $\bar{S}, S \in \mathcal{S}_\star$, we have $\mathcal{P}^\perp \bar{S} = \mathcal{P}^\perp S = \mathcal{P}^\perp C$. Thus,

$$\begin{aligned}
&|\langle G, S\rangle| = \left|\langle \mathcal{P}G, \mathcal{P}S\rangle + \langle \mathcal{P}^\perp G, \mathcal{P}^\perp S\rangle\right| = \left|\langle \mathcal{P}G, \mathcal{P}S\rangle + \langle \mathcal{P}^\perp G, \mathcal{P}^\perp \bar{S}\rangle\right| \\
=&\left|\langle \mathcal{P}G, \mathcal{P}\bar{S}\rangle + \langle \mathcal{P}^\perp G, \mathcal{P}^\perp \bar{S}\rangle + \langle \mathcal{P}G, \mathcal{P}(\bar{S}-S)\rangle\right| = \left|\langle G, \bar{S}\rangle + \langle \mathcal{P}G, \mathcal{P}(\bar{S}-S)\rangle\right| \\
=&\left|\langle \mathcal{P}G, \mathcal{P}(\bar{S}-S)\rangle\right| \\
\leq&\|\mathcal{P}G\|_{\mathsf{F}}\|\mathcal{P}(\bar{S}-S)\|_{\mathsf{F}} \leq \epsilon\|\mathcal{P}(\bar{S}-S)\|_{\mathsf{F}}
\end{aligned}$$

On the other hand, $\bar{S} - S = \mathcal{P}^\perp(\bar{S}-S) + \mathcal{P}(\bar{S}-S) = \mathcal{P}(\bar{S}-S)$.

(2) By primal-dual symmetry. □

Proposition 1 (Structure of $\mathcal{C}(\bar{Z})$)**.** *Under Assumption 1:*

1. *$\mathcal{C}(\bar{Z})$ is a nonempty, closed and convex cone.*
2. *$\mathcal{C}(\bar{Z}) = \mathcal{C}_{\mathsf{P}}(\bar{Z}) + \mathcal{C}_{\mathsf{D}}(\bar{Z})$, where*

$$\mathcal{C}_{\mathsf{P}}(\bar{Z}) := \left\{ H = \left[\begin{array}{c:c|c:c} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^{\mathsf{P}}} & H_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \hdashline \sim & H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}H = 0, \\ H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0 \end{array} \right\}, \tag{26a}$$

$$\mathcal{C}_{\mathsf{D}}(\bar{Z}) := \left\{ H = \left[\begin{array}{c:c|c:c} 0 & 0 & 0 & 0 \\ \hdashline \sim & 0 & 0 & H_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hline \sim & \sim & H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & H_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hdashline \sim & \sim & \sim & H_{\alpha_-\alpha_-} \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}^\perp H = 0, \\ H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0 \end{array} \right\}. \tag{26b}$$

Proof. (1) is directly from Lemma 1 and [4, Proposition 4.23].

(2) For ease of notation, denote $\Pi'_+(\bar{Z};\cdot)$ (resp. $\Pi'_-(\bar{Z};\cdot)$) as $\Omega(\cdot)$ (resp. $\Omega^\perp(\cdot)$). Then from (25), $H \in \mathcal{C}(\bar{Z})$ if and only if $\mathcal{P}\Omega(H) = 0$ and $\mathcal{P}^\perp\Omega^\perp(H) = 0$.

(i) We first prove $H_{\alpha_+\alpha_-} = 0, \forall H \in \mathcal{C}(\bar{Z})$. Notice that

$$\left\langle \Omega(H), \Omega^{\perp}(H) \right\rangle = \left\langle \mathcal{P}\Omega(H), \mathcal{P}\Omega^{\perp}(H) \right\rangle + \left\langle \mathcal{P}^{\perp}\Omega(H), \mathcal{P}^{\perp}\Omega^{\perp}(H) \right\rangle = 0.$$

On the other hand,

$$\left\langle \Omega(H), \Omega^{\perp}(H) \right\rangle = 2 \sum_{a \in \mathcal{I}_+, b \in \mathcal{I}_-} \frac{\mu_a \cdot (-\mu_b)}{\mu_a - \mu_b} \|H_{\alpha_a\alpha_b}\|_{\mathsf{F}}^2.$$

Thus, $H_{\alpha_a\alpha_b} = 0, \forall a \in \mathcal{I}_+, b \in \mathcal{I}_-$, *i.e.,* $H_{\alpha_+\alpha_-} = 0$.

(ii) We second prove $H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{D}}} = 0, \forall H \in \mathcal{C}(\bar{Z})$. By Theorem 1 and $\bar{S}_{\alpha_-\alpha_-} = 0$, we get $\left\langle \Omega(H), \bar{S} \right\rangle = 0$. Together with $\mathcal{P}\Omega(H) = 0$ and $S_{\mathsf{sc}} \in \mathcal{S}_\star$, we have $\langle \Omega(H), S_{\mathsf{sc}} \rangle = 0$ from Lemma 2. On the other hand,

$$\Omega(H) = \begin{bmatrix} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^{\mathsf{P}}} & H_{\alpha_+\alpha_0^{\mathsf{D}}} & H_{\alpha_+\alpha_-} \\ \sim & [\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & [\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} & 0 \\ \sim & \sim & [\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} & 0 \\ \sim & \sim & \sim & 0 \end{bmatrix}, \quad S_{\mathsf{sc}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & 0 \\ \sim & \sim & [S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & [S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & [S_{\mathsf{sc}}]_{\alpha_-\alpha_-} \end{bmatrix},$$

we have $\langle \Omega(H), S_{\mathsf{sc}} \rangle = \left\langle [\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}, [S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \right\rangle = 0$. Since $[S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \succ 0$ and $[\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} \succeq 0$, we get $[\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} = 0$. Symmetrically, $[\Pi_-(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} = 0$. Thus,

$$H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{D}}} = [H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} = [\Pi_+(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} + [\Pi_-(H_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} = 0.$$

(iii) From (i) and (ii), $\forall H \in \mathcal{C}(\bar{Z})$, it should be of the following form:

$$H = \underbrace{\begin{bmatrix} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^{\mathsf{P}}} & H_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \sim & H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{bmatrix}}_{=:U} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & H_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \sim & \sim & H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & H_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \sim & \sim & \sim & H_{\alpha_-\alpha_-} \end{bmatrix}}_{=:V}, \quad \text{with } U_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0, V_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0$$

Moreover,

$$\mathcal{P}\Omega(H) = \mathcal{P}U = 0, \quad \mathcal{P}^{\perp}\Omega^{\perp}(H) = \mathcal{P}^{\perp}V = 0.$$

Thus, $U \in \mathcal{C}_{\mathsf{P}}(\bar{Z}), V \in \mathcal{C}_{\mathsf{D}}(\bar{Z})$. This proves the "$\subseteq$" part. For the "$\supseteq$" part: take any $U \in \mathcal{C}_{\mathsf{P}}(\bar{Z}), V \in \mathcal{C}_{\mathsf{D}}(\bar{Z})$. Then,

$$\mathcal{P}\Omega(U+V) = \mathcal{P}U = 0, \quad \mathcal{P}^{\perp}\Omega^{\perp}(U+V) = \mathcal{P}^{\perp}V = 0,$$

which closes the proof. □

4.3.1 Relationships between $\mathcal{C}(\bar{Z})$ and $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$

The cone $\mathcal{C}(\bar{Z})$ consists of directions H along which $\delta(\bar{Z} + tH)$ (the backward error [42], i.e., the KKT residual) vanishes to first order, whereas $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ consists of directions H along which $\operatorname{dist}(\bar{Z} + tH, \mathcal{Z}_\star)$ (the forward error, *i.e.,* the distance to the optimal set) vanishes to first order. As we show below, these two cones are closely related.

Proposition 2 (Structure of $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$)**.** *Under Assumption 1,*

1. $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \mathcal{T}_{\mathcal{X}_\star}(\bar{X}) - \mathcal{T}_{\mathcal{S}_\star}(\bar{S})$, *where*

$$\mathcal{T}_{\mathcal{X}_\star}(\bar{X}) = \left\{ H = \left[\begin{array}{c|c} \begin{bmatrix} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^{\mathsf{P}}} \\ \sim & H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \end{bmatrix} & 0 \\ \hline \sim & 0 \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}H = 0, \\ H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0 \end{array} \right\}, \tag{27a}$$

$$\mathcal{T}_{\mathcal{S}_\star}(\bar{S}) = \left\{ H = \left[\begin{array}{c|c} 0 & 0 \\ \hline \sim & \begin{bmatrix} H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & H_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & H_{\alpha_-\alpha_-} \end{bmatrix} \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}^\perp H = 0, \\ H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0 \end{array} \right\}. \tag{27b}$$

2. $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \mathcal{C}(\bar{Z}) \cap \{H \in \mathbb{S}^n \mid H_{\alpha_+\alpha_0^{\mathsf{D}}} = 0, H_{\alpha_0^{\mathsf{P}}\alpha_-} = 0\}$.

Proof. (1) We first calculate $\mathcal{T}_{\mathcal{X}_\star}(\bar{X})$. Regularize $\mathcal{X}_\star$ to its affine hull: picking $(X_{\mathsf{sc}}, S_{\mathsf{sc}})$ as a maximal-rank primal-dual optimal solution pair,

$$\begin{aligned} \mathcal{X}_\star &= \left\{ X \;\middle|\; X \in \mathbb{S}^n_+ \cap \left\{ X \;\middle|\; \mathcal{P}X = \mathcal{P}\widetilde{X}, \langle X, S_{\mathsf{sc}} \rangle = 0 \right\} \right\} \\ &= \left\{ X = E_{\alpha_{\mathsf{P}}} U E_{\alpha_{\mathsf{P}}}^{\mathsf{T}} \;\middle|\; U \in \mathbb{S}_+^{|\alpha_{\mathsf{P}}|} \cap \left\{ U \;\middle|\; \mathcal{P}(E_{\alpha_{\mathsf{P}}} U E_{\alpha_{\mathsf{P}}}^{\mathsf{T}}) = \mathcal{P}\widetilde{X} \right\} \right\}, \end{aligned}$$

where $E_{\alpha_{\mathsf{P}}}$ denotes the first $|\alpha_{\mathsf{P}}|$ columns of I_n. Since $E_{\alpha_{\mathsf{P}}}^{\mathsf{T}} X_{\mathsf{sc}} E_{\alpha_{\mathsf{P}}} \in \mathrm{ri}(\mathbb{S}_+^{|\alpha_{\mathsf{P}}|} \cap \{U \mid \mathcal{P}(E_{\alpha_{\mathsf{P}}} U E_{\alpha_{\mathsf{P}}}^{\mathsf{T}}) = \mathcal{P}\widetilde{X}\})$, by [36, Theorem 6.42],

$$\begin{aligned} \mathcal{T}_{\mathcal{X}_\star}(\bar{X}) &= \left\{ H \;\middle|\; H_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} \in \mathcal{T}_{\mathbb{S}_+^{|\alpha_{\mathsf{P}}|}}(\bar{X}_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}}),\ \mathcal{P}H = 0,\ H_{\alpha_{\mathsf{P}}\alpha_{\mathsf{D}}} = H_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} = 0 \right\} \\ &= \left\{ H \;\middle|\; H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0,\ \mathcal{P}H = 0,\ H_{\alpha_{\mathsf{P}}\alpha_{\mathsf{D}}} = H_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} = 0 \right\}. \end{aligned}$$

Symmetrically, $\mathcal{T}_{\mathcal{S}_\star}(\bar{S})$ is of the form in (27). We notice that $\mathcal{T}_{\mathcal{X}_\star}(\bar{X}) \cap \mathcal{T}_{\mathcal{S}_\star}(\bar{S}) = \{0\}$. Thus, via [5, Collary 4.8 (v)] and [36, Exercise 6.44],

$$\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \mathrm{cl}(\mathcal{T}_{\mathcal{X}_\star}(\bar{X}) - \mathcal{T}_{\mathcal{S}_\star}(\bar{S})) = \mathcal{T}_{\mathcal{X}_\star}(\bar{X}) - \mathcal{T}_{\mathcal{S}_\star}(\bar{S})$$

.

(2) Denote $\{H \in \mathbb{S}^n \mid H_{\alpha_+\alpha_0^{\mathsf{D}}} = 0, H_{\alpha_0^{\mathsf{P}}\alpha_-} = 0\}$ as $\mathcal{M}$. Suppose $H \in \mathcal{C}(\bar{Z}) \cap \mathcal{M}$. Then, from Proposition 1 (2), $H = U + V$, with $\mathcal{P}U = 0$ and $\mathcal{P}^\perp V = 0$, with

$$U = \left[\begin{array}{cc|cc} U_{\alpha_+\alpha_+} & U_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \sim & U_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0 & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{array}\right], \quad V = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & 0 \\ \hline \sim & \sim & V_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0 & V_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & V_{\alpha_-\alpha_-} \end{array}\right].$$

Thus, $U \in \mathcal{T}_{\mathcal{X}_\star}(\bar{X}), V \in -\mathcal{T}_{\mathcal{S}_\star}(\bar{S})$. This proves the "$\subseteq$" part. For the "$\supseteq$" part, take any $U \in \mathcal{T}_{\mathcal{X}_\star}(\bar{X}), V \in -\mathcal{T}_{\mathcal{S}_\star}(\bar{S})$. Then, $U \in \mathcal{C}_{\mathsf{P}}(\bar{Z}) \cap \mathcal{M}$ and $V \in \mathcal{C}_{\mathsf{D}}(\bar{Z}) \cap \mathcal{M}$. Thus, $U + V \in (\mathcal{C}_{\mathsf{P}}(\bar{Z}) + \mathcal{C}_{\mathsf{D}}(\bar{Z})) \cap \mathcal{M} = \mathcal{C}(\bar{Z}) \cap \mathcal{M}$. ☐

There are several special scenarios when $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$.

Corollary 1 (Special cases where $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$)**.** *Under Assumption 1, $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ under either of the two conditions:*

1. *If $\bar{X}$ satisfies primal constraint nondegeneracy and $\bar{S}$ satisfies dual constraint nondegeneracy;*

2. *If $(\bar{X}, \bar{S})$ is a strict complementary solution pair.*

Proof. (1) Under the two-side nondegenerate conditions, [22, Theorem 5] has already proven $\mathrm{Fix}(\mathrm{Id} + \delta'(\bar{Z}; \cdot)) = \{0\}$. Thus, $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \{0\}$.

(2) When $(\bar{X}, \bar{S})$ is of maximal rank, $H_{\alpha_{\mathsf{P}}\alpha_{\mathsf{D}}} = H_{\alpha_+\alpha_-} = 0$ for any $H \in \mathcal{C}(\bar{Z})$. Thus, $H_{\alpha_+\alpha_0^{\mathsf{D}}} = 0, H_{\alpha_0^{\mathsf{P}}\alpha_-} = 0$ naturally holds. By Proposition 2 (2), $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$. ☐

4.4 Second-Order Limit Map $\phi(\bar{Z};\cdot)$

In this section, we will develop the core concepts in the paper: local second-order *limiting* mapping and its induced local second-order *limiting* dynamics.

By Lemma 1, the local first-order dynamics (23) eventually vanishes and drives the iterates toward $\mathcal{C}(\bar{Z})$ for any initialization $H^{(0)} \in \mathbb{S}^n$. In contrast, the *true* one-step ADMM dynamics (4) need not vanish. This motivates us to investigate the local second-order dynamics (24) in the regime where the first-order dynamics has stalled. Fix an arbitrary $\bar{H} \in \mathcal{C}(\bar{Z})$. Then, by (23) and Lemma 1, we have $H^{(k)} \equiv \bar{H}$ for all $k \in \mathbb{N}$ if $H^{(0)}$ is set to $\bar{H}$. Consequently, (24) reduces to

$$W^{(k+1)} = (\mathrm{Id} + \delta''(\bar{Z};\bar{H},\cdot))(W^{(k)}) = W^{(k)} - \mathcal{P}\Pi''_+(\bar{Z};\bar{H},W^{(k)}) - \mathcal{P}^{\perp}\Pi''_-(\bar{Z};\bar{H},W^{(k)}). \tag{28}$$

As we will see later, a fundamental difference between (28) and (23) is that the sequence $\{W^{(k)}\}$ in (28) need not converge.

4.4.1 $\delta''(\bar{Z};\bar{H},W)$'s simplification under $\bar{H} \in \mathcal{C}(\bar{Z})$

Since $\bar{H} \in \mathcal{C}(\bar{Z})$, we have $\bar{H}_{\alpha_+\alpha_-} = 0$ from Proposition 1. Therefore, $\Pi''_+(\bar{Z};\bar{H},W)$ in (12) and $\Pi''_-(\bar{Z};\bar{H},W)$ in (16) can be simplified as:

$$\Pi''_+(\bar{Z};\bar{H},W) = \left[\begin{array}{c:c:c} W_{\alpha_+\alpha_+} & \left\{\begin{matrix} W_{\alpha_a\alpha_b} \\ -2\frac{1}{\mu_a}\overline{H}_{\alpha_a\alpha_0}\Pi_+(-\overline{H}_{\alpha_0\alpha_0}) \end{matrix}\right\}_{a\in\mathcal{I}_+} & \left\{\begin{matrix} \frac{\mu_a}{\mu_a-\mu_b}W_{\alpha_a\alpha_b} \\ +2\frac{1}{\mu_a-\mu_b}\overline{H}_{\alpha_a\alpha_0}\overline{H}_{\alpha_0\alpha_b} \end{matrix}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \begin{matrix} 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\overline{H}_{\alpha_0\alpha_c}\overline{H}_{\alpha_c\alpha_0} \\ +\Pi'_+(\overline{H}_{\alpha_0\alpha_0};V_0(\overline{H},W)) \end{matrix} & \left\{2\frac{1}{-\mu_b}\Pi_+(\overline{H}_{\alpha_0\alpha_0})\overline{H}_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right], \tag{29a}$$

$$\Pi''_-(\bar{Z};\bar{H},W) = \left[\begin{array}{c:c:c} 0 & \left\{2\frac{1}{-\mu_a}\overline{H}_{\alpha_a\alpha_0}\Pi_-(\overline{H}_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{\begin{matrix} \frac{-\mu_b}{\mu_a-\mu_b}W_{\alpha_a\alpha_b} \\ +2\frac{1}{\mu_b-\mu_a}\overline{H}_{\alpha_a\alpha_0}\overline{H}_{\alpha_0\alpha_b} \end{matrix}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \begin{matrix} 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\overline{H}_{\alpha_0\alpha_c}\overline{H}_{\alpha_c\alpha_0} \\ +\Pi'_-(\overline{H}_{\alpha_0\alpha_0};V_0(\overline{H},W)) \end{matrix} & \left\{\begin{matrix} W_{\alpha_a\alpha_b} \\ -2\frac{1}{\mu_b}\Pi_-(-\overline{H}_{\alpha_0\alpha_0})\overline{H}_{\alpha_0\alpha_b} \end{matrix}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & W_{\alpha_-\alpha_-} \end{array}\right]. \tag{29b}$$

We notice that $V_0(\bar{H},W)$ is linear in W. Therefore, define $\widetilde{W}$ as:

$$\widetilde{W}_{\alpha_a\alpha_b} := \begin{cases} V_0 = W_{\alpha_0\alpha_0} - 2\sum_{c\in\mathcal{I}_+\cup\mathcal{I}_-}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0}, & a=0, b=0 \\ W_{\alpha_a\alpha_b}, & \text{Otherwise} \end{cases}. \tag{30}$$

Define

$$\Upsilon(\bar{Z};\bar{H}) := \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_+\cup\mathcal{I}_-}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right]. \tag{31}$$

Then, $W - \widetilde{W} \equiv \Upsilon(\bar{H})$. Further more, define

$$\Theta(\bar{Z};\bar{H},\widetilde{W}) := \left[\begin{array}{c:c:c} \widetilde{W}_{\alpha_+\alpha_+} & \widetilde{W}_{\alpha_+\alpha_0} & \left\{\frac{\mu_a}{\mu_a-\mu_b}\widetilde{W}_{\alpha_a\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}_{\alpha_0\alpha_0}) & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right], \tag{32a}$$

$$\Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}) := \left[\begin{array}{c:c:c} 0 & 0 & \left\{\frac{-\mu_b}{\mu_a-\mu_b}\widetilde{W}_{\alpha_a\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \Pi'_-(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}_{\alpha_0\alpha_0}) & \widetilde{W}_{\alpha_0\alpha_-} \\ \hdashline \sim & \sim & \widetilde{W}_{\alpha_-\alpha_-} \end{array}\right], \tag{32b}$$

and

$$
\mathcal{E}(\bar{Z};\bar{H}) := \left[\begin{array}{c:c:c} 0 & \left\{-2\frac{1}{\mu_a}\bar{H}_{\alpha_a\alpha_0}\Pi_+(-\bar{H}_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{2\frac{1}{\mu_a-\mu_b}\bar{H}_{\alpha_a\alpha_0}\bar{H}_{\alpha_0\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0} & \left\{2\frac{1}{-\mu_b}\Pi_+(\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right], \tag{33a}
$$

$$
\mathcal{E}^{\perp}(\bar{Z};\bar{H}) := \left[\begin{array}{c:c:c} 0 & \left\{2\frac{1}{-\mu_a}\bar{H}_{\alpha_a\alpha_0}\Pi_-(\bar{H}_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{2\frac{1}{\mu_b-\mu_a}\bar{H}_{\alpha_a\alpha_0}\bar{H}_{\alpha_0\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0} & \left\{-2\frac{1}{\mu_b}\Pi_-(-\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right]. \tag{33b}
$$

For all $\bar{H}\in\mathcal{C}(\bar{Z})$ and $\widetilde{W}\in\mathbb{S}^n$, the following relationships holds:

$$
\Pi_+''(\bar{Z};\bar{H},W) = \Theta(\bar{Z};\bar{H},\widetilde{W}) + \mathcal{E}(\bar{Z};\bar{H}) \tag{34a}
$$

$$
\Pi_-''(\bar{Z};\bar{H},W) = \Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}) + \mathcal{E}^{\perp}(\bar{Z};\bar{H}) \tag{34b}
$$

$$
\Theta(\bar{Z};\bar{H},\widetilde{W}) + \Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}) = \widetilde{W} \tag{34c}
$$

$$
\mathcal{E}(\bar{Z};\bar{H}) + \mathcal{E}^{\perp}(\bar{Z};\bar{H}) = \Upsilon(\bar{Z};\bar{H}) \tag{34d}
$$

Now we are ready to simplify $\delta''(\bar{Z};\bar{H},W)$:

$$
\begin{aligned}
\delta''(\bar{Z};\bar{H},W) &= -\mathcal{P}\Pi_+''(\bar{Z};\bar{H},W) - \mathcal{P}^{\perp}\Pi_-''(\bar{Z};\bar{H},W) \\
&= -\mathcal{P}\Theta(\bar{Z};\bar{H},\widetilde{W}) - \mathcal{P}^{\perp}\Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}) - \mathcal{P}\mathcal{E}(\bar{Z};\bar{H}) - \mathcal{P}^{\perp}\mathcal{E}^{\perp}(\bar{Z};\bar{H}),
\end{aligned} \tag{35}
$$

where $\Theta(\bar{Z};\bar{H},\widetilde{W}),\Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W})$ are defined in (32) and $\mathcal{E}(\bar{Z};\bar{H}),\mathcal{E}^{\perp}(\bar{Z};\bar{H})$ are defined in (33).

4.4.2 $W^{(k+1)} - W^{(k)}$ is convergent

From (35) and (31),

$$
\widetilde{W}^{(k+1)} - \widetilde{W}^{(k)} = W^{(k+1)} - W^{(k)} = -\mathcal{P}\Theta(\bar{Z};\bar{H},\widetilde{W}^{(k)}) - \mathcal{P}^{\perp}\Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}^{(k)}) - \mathcal{P}\mathcal{E}(\bar{Z};\bar{H}) - \mathcal{P}^{\perp}\mathcal{E}^{\perp}(\bar{Z};\bar{H}).
$$

From (34), $\widetilde{W}^{(k)} = \Theta(\bar{Z};\bar{H},\widetilde{W}^{(k)}) + \Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}^{(k)})$. Thus,

$$
\widetilde{W}^{(k+1)} = \left\{\mathcal{P}^{\perp}\Theta(\bar{Z};\bar{H},\widetilde{W}^{(k)}) + \mathcal{P}\Theta^{\perp}(\bar{Z};\bar{H},\widetilde{W}^{(k)})\right\} + \underbrace{\left\{-\mathcal{P}\mathcal{E}(\bar{Z};\bar{H}) - \mathcal{P}^{\perp}\mathcal{E}^{\perp}(\bar{Z};\bar{H})\right\}}_{=:\Psi(\bar{Z};\bar{H})} \tag{36}
$$

From now on, suppose $\bar{H}$ follows the second-level description in §3.

Lemma 3 ($\mathcal{P}^{\perp}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}\Theta^{\perp}(\bar{Z};\bar{H},\cdot)$'s firmly nonexpansiveness)**.** *$\mathcal{P}^{\perp}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}\Theta^{\perp}(\bar{Z};\bar{H},\cdot)$ in (36) is firmly nonexpansive on $(\mathbb{S}^n,\langle\cdot,\cdot\rangle)$.*

Proof. The proof procedure is similar to the one in Lemma 1. For ease of notation, we abbreviate $\mathcal{P}^{\perp}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}\Theta^{\perp}(\bar{Z};\bar{H},\cdot)$ as $\mathcal{T}(\cdot)$, $\Theta(\bar{Z};\bar{H},\cdot)$ as $\mathcal{F}(\cdot)$, and $\Theta^{\perp}(\bar{Z};\bar{H},\cdot)$ as $\mathcal{F}^{\perp}(\cdot)$:

$$
\begin{aligned}
&\|\mathcal{T}(U)-\mathcal{T}(V)\|_{\mathsf{F}}^2 + \|(\mathrm{Id}-\mathcal{T})(U)-(\mathrm{Id}-\mathcal{T})(V)\|_{\mathsf{F}}^2 \\
=&\|\mathcal{P}^{\perp}[\mathcal{F}(U)-\mathcal{F}(V)]\|_{\mathsf{F}}^2 + \|\mathcal{P}[\mathcal{F}^{\perp}(U)-\mathcal{F}^{\perp}(V)]\|_{\mathsf{F}}^2 + \|\mathcal{P}[\mathcal{F}(U)-\mathcal{F}(V)]\|_{\mathsf{F}}^2 + \|\mathcal{P}^{\perp}[\mathcal{F}^{\perp}(U)-\mathcal{F}^{\perp}(V)]\|_{\mathsf{F}}^2 \\
=&\|\mathcal{F}(U)-\mathcal{F}(V)\|_{\mathsf{F}}^2 + \|\mathcal{F}^{\perp}(U)-\mathcal{F}^{\perp}(V)\|_{\mathsf{F}}^2 \\
=&\|U-V\|_{\mathsf{F}}^2 - 2\left\langle\mathcal{F}(U)-\mathcal{F}(V),\mathcal{F}^{\perp}(U)-\mathcal{F}^{\perp}(V)\right\rangle
\end{aligned}
$$

Thus, all we need to show is $\langle \mathcal{F}(U)-\mathcal{F}(V), \mathcal{F}^\perp(U)-\mathcal{F}^\perp(V)\rangle \geq 0$. Since

$$
\begin{aligned}
&\left\langle \mathcal{F}(U)-\mathcal{F}(V), \mathcal{F}^\perp(U)-\mathcal{F}^\perp(V)\right\rangle \\
=&\underbrace{2\sum_{a\in\mathcal{I}_+, b\in\mathcal{I}_-}\frac{\mu_a}{\mu_a-\mu_b}\cdot\frac{-\mu_b}{\mu_a-\mu_b}\|U_{\alpha_a\alpha_b}-V_{\alpha_a\alpha_b}\|_{\mathsf{F}}^2}_{\geq 0} \\
&+\underbrace{\left\langle \Pi_+'(\bar{H}_{\alpha_0\alpha_0};U_{\alpha_0\alpha_0})-\Pi_+'(\bar{H}_{\alpha_0\alpha_0};V_{\alpha_0\alpha_0}), \Pi_-'(\bar{H}_{\alpha_0\alpha_0};U_{\alpha_0\alpha_0})-\Pi_-'(\bar{H}_{\alpha_0\alpha_0};V_{\alpha_0\alpha_0})\right\rangle}_{=:\mathrm{LHS}}.
\end{aligned}
$$

It boils down to prove LHS ≥ 0. Abbreviate $\widehat{U}$ as $(Q^0)^\mathsf{T} U_{\alpha_0\alpha_0} Q^0$ and $\widehat{V}$ as $(Q^0)^\mathsf{T} V_{\alpha_0\alpha_0} Q^0$:

$$
\Pi_+'(\bar{H}_{\alpha_0\alpha_0};U_{\alpha_0\alpha_0}) = Q^0 \left[\begin{array}{c:c:c}
\widehat{U}_{\beta_{0,+}\beta_{0,+}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}} & \left\{\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\widehat{U}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline
\sim & \Pi_+(\widehat{U}_{\beta_{0,0}\beta_{0,0}}) & 0 \\ \hdashline
\sim & \sim & 0
\end{array}\right](Q^0)^\mathsf{T},
$$

$$
\Pi_-'(\bar{H}_{\alpha_0\alpha_0};V_{\alpha_0\alpha_0}) = Q^0 \left[\begin{array}{c:c:c}
0 & 0 & \left\{\frac{-\eta_{0,j}}{\eta_{0,i}-\eta_{0,j}}\widehat{V}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline
\sim & \Pi_-(\widehat{V}_{\beta_{0,0}\beta_{0,0}}) & \widehat{V}_{\beta_{0,0}\beta_{0,-}} \\ \hdashline
\sim & \sim & \widehat{V}_{\beta_{0,-}\beta_{0,-}}
\end{array}\right](Q^0)^\mathsf{T}.
$$

Thus,

$$
\begin{aligned}
\mathrm{LHS} =& \underbrace{2\sum_{i\in\mathcal{I}_{0,+}, j\in\mathcal{I}_{0,-}}\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\cdot\frac{-\eta_{0,j}}{\eta_{0,i}-\eta_{0,j}}\|\widehat{U}_{\beta_{0,i}\beta_{0,j}}-\widehat{V}_{\beta_{0,i}\beta_{0,j}}\|_{\mathsf{F}}^2}_{\geq 0} \\
&+\left\langle \Pi_+(\widehat{U}_{\beta_{0,0}\beta_{0,0}})-\Pi_+(\widehat{V}_{\beta_{0,0}\beta_{0,0}}), \Pi_-(\widehat{U}_{\beta_{0,0}\beta_{0,0}})-\Pi_-(\widehat{V}_{\beta_{0,0}\beta_{0,0}})\right\rangle \\
\geq& -\left\langle \Pi_+(\widehat{U}_{\beta_{0,0}\beta_{0,0}}), \Pi_-(\widehat{V}_{\beta_{0,0}\beta_{0,0}})\right\rangle - \left\langle \Pi_+(\widehat{V}_{\beta_{0,0}\beta_{0,0}}), \Pi_-(\widehat{U}_{\beta_{0,0}\beta_{0,0}})\right\rangle \geq 0,
\end{aligned}
$$

which closes the proof. □

One may have already noticed that the operator $\mathcal{P}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\cdot)$ in (36) closely resembles $\mathrm{Id}+\delta'(\bar{Z};\cdot)$ in Lemma 1. For instance, $\mathcal{P}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\cdot)$ is also positively homogeneous, and hence $0\in\mathrm{Fix}(\mathcal{P}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\cdot))$. The essential difference between the local first- and second-order dynamics, however, lies in the presence of the "constant term" $\Psi(\bar{Z};\bar{H})$.

Theorem 5 (Convergent $\widetilde{W}^{(k+1)}-\widetilde{W}^{(k)}$)**.** *For the dynamical system* (36)*,*

$$
\widetilde{W}^{(k+1)}-\widetilde{W}^{(k)} \to \phi(\bar{Z};\bar{H}) := \Psi(\bar{Z};\bar{H})-\Pi_{\mathcal{K}(\bar{Z};\bar{H})}(\Psi(\bar{Z};\bar{H})) = \Pi_{\mathcal{K}^\circ(\bar{Z};\bar{H})}(\Psi(\bar{Z};\bar{H})), \quad \text{as } k\to\infty \tag{38}
$$

where the closed convex cone $\mathcal{K}(\bar{Z};\bar{H})$ is defined as

$$
\mathcal{K}(\bar{Z};\bar{H}) := \mathrm{cl}\left(\left\{\mathcal{P}\Theta(\bar{Z};\bar{H},W)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},W) \;\middle|\; W\in\mathbb{S}^n\right\}\right) \tag{39}
$$

and its polar cone:

$$
\mathcal{K}^\circ(\bar{Z};\bar{H}) := \left\{Y\in\mathbb{S}^n \;\middle|\; \left\langle \mathcal{P}\Theta(\bar{Z};\bar{H},W)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},W), Y\right\rangle \leq 0,\ \forall W\in\mathbb{S}^n\right\} \tag{40}
$$

Proof. This is a standard result from Monotone operator theory. For ease of notation, abbreviate the operator $\mathcal{P}\Theta(\bar{Z};\bar{H},\cdot)+\mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\cdot)$ as $\mathcal{T}(\cdot)$, $\Psi(\bar{Z};\bar{H})$ as Ψ, and $\mathcal{K}(\bar{Z};\bar{H})$ as $\mathcal{K}$. Since $\mathcal{T}$ is firmly nonexpansive and positive homogeneous, $\mathcal{K} := \mathrm{cl}(\mathrm{ran}(\mathrm{Id} - \mathcal{T}))$ is a nonempty, closed, and convex cone. Denote $\mathcal{S}(\cdot) := \mathcal{T}(\cdot) + \Psi$ From [2, Corollary 2.3],

$$\widetilde{W}^{(k+1)} - \widetilde{W}^{(k)} \to -\Pi_{\mathrm{cl}(\mathrm{ran}(\mathrm{Id}-\mathcal{S}))}(0) \quad \text{as } k \to \infty$$

Further simplification:

$$\Pi_{\mathrm{cl}(\mathrm{ran}(\mathrm{Id}-\mathcal{S}))}(0) = \Pi_{\mathrm{cl}(\mathrm{ran}(\mathrm{Id}-\mathcal{T}))}(\Psi) - \Psi = \Pi_{\mathcal{K}}(\Psi) - \Psi$$

From polar cone's definition, $\Psi = \Pi_{\mathcal{K}}(\Psi) + \Pi_{\mathcal{K}^\circ}(\Psi)$. The closure in $\mathcal{K}$ does not affect $\mathcal{K}^\circ$, since for an arbitrary convex cone $\mathcal{C}$, $(\mathrm{cl}(\mathcal{C}))^\circ = \mathcal{C}^\circ$. ◻

4.4.3 Local second-order limit dynamics

By Lemma 3, the increment $W^{(k+1)} - W^{(k)}$ eventually converges to a constant second-order "drift" $\phi(\bar{Z}; \bar{H})$, and this limit is independent of the initialization $W^{(0)}$. From the viewpoint of time-scale separation, this convergence manifests as a second-order effect (scaled by $\frac{t^2}{2}$ with $t \downarrow 0$), whereas the evolution of $H^{(k)}$ is a first-order effect (scaled by t with $t \downarrow 0$). It is therefore reasonable to assume that, by the time $W^{(k+1)} - W^{(k)}$ has converged, $Z^{(k)}$ remains unchanged to first order. Consequently, by (22), the *limiting* dynamics after $W^{(k+1)} - W^{(k)} \to \phi(\bar{Z}; \bar{H})$ is

$$Z^{(k+1)} = Z^{(k)} + \frac{t^2}{2}\,\phi(\bar{Z}; \bar{H}) + o(t^2).$$

The update above produces a ray in $\mathbb{S}^n$ with a constant second-order "drift" as $t \downarrow 0$. Moreover, $Z^{(k)} - \bar{Z} = t\bar{H} + o(t)$ for any *fixed* k. However, one must account for the cumulative effect of this drift over many iterations: when $k \sim O(\frac{1}{t})$, the accumulated second-order displacement can become non-negligible compared to the first-order term $t\bar{H}$. In this regime, the original separation of first- and second-order dynamics in Definition 1 and (36) may no longer be accurate, and the effective direction $\bar{H}$ may need to be re-identified because t is fixed and positive. We address this issue by replacing the constant term $t\bar{H}$ with the "feedback" term $Z^{(k)} - \bar{Z}$. To this end, we use the following elementary scaling property of $\phi(\bar{Z}; \cdot)$.

Proposition 3 (Positive 2-homogeneity of $\phi(\bar{Z}; \cdot)$)**.** *For any $t > 0$ and any $\bar{H} \in \mathcal{C}(\bar{Z})$, it holds that $\phi(\bar{Z}; t\bar{H}) = t^2\,\phi(\bar{Z}; \bar{H})$.*

Proof. By (33), we have $\mathcal{E}(\bar{Z}; t\bar{H}) = t^2\mathcal{E}(\bar{Z}; \bar{H})$ and $\mathcal{E}^\perp(\bar{Z}; t\bar{H}) = t^2\mathcal{E}^\perp(\bar{Z}; \bar{H})$. Next, we show that $\mathcal{K}(\bar{Z}; \bar{H}) = \mathcal{K}(\bar{Z}; t\bar{H})$ for all $t > 0$. Indeed, for any $W \in \mathbb{S}^n$,

$$\Pi'_+(\bar{H}_{\alpha_0\alpha_0}; W_{\alpha_0\alpha_0}) = \Pi'_+(t\,\bar{H}_{\alpha_0\alpha_0}; W_{\alpha_0\alpha_0}),$$

since $\Pi_+(\cdot)$ is positively homogeneous. Hence, by (32), $\Theta(\bar{Z}; \bar{H}, W) = \Theta(\bar{Z}; t\bar{H}, W)$ for all $W \in \mathbb{S}^n$. By symmetry, $\Theta^\perp(\bar{Z}; \bar{H}, W) = \Theta^\perp(\bar{Z}; t\bar{H}, W)$ for all $W \in \mathbb{S}^n$. It then follows from (39) that $\mathcal{K}(\bar{Z}; \bar{H}) = \mathcal{K}(\bar{Z}; t\bar{H})$. Finally, using (38),

$$\phi(\bar{Z}; t\bar{H}) = \Psi(\bar{Z}; t\bar{H}) - \Pi_{\mathcal{K}(\bar{Z}; t\bar{H})}(\Psi(\bar{Z}; t\bar{H})) = t^2\Psi(\bar{Z}; \bar{H}) - \Pi_{\mathcal{K}(\bar{Z}; \bar{H})}(t^2\Psi(\bar{Z}; \bar{H})) = t^2\,\phi(\bar{Z}; \bar{H}).$$

◻

With Proposition 3, we have $\frac{t^2}{2}\phi(\bar{Z}; \bar{H}) = \frac{1}{2}\phi(\bar{Z}; t\bar{H})$ for all $t > 0$, which motivates the following definition.

Definition 2 (Local second-order limiting mapping and dynamics)**.** *At a point $\bar{Z} \in \mathcal{Z}_\star$, the local second-order limiting mapping $\phi(\bar{Z}; \cdot) : \mathcal{C}(\bar{Z}) \mapsto \mathbb{S}^n$ is defined by*

$$\phi(\bar{Z}; \cdot) := \Psi(\bar{Z}; \cdot) - \Pi_{\mathcal{K}(\bar{Z}; \cdot)}(\Psi(\bar{Z}; \cdot)) = \Pi_{\mathcal{K}^\circ(\bar{Z}; \cdot)}(\Psi(\bar{Z}; \cdot)). \tag{41}$$

Here $\Psi(\bar{Z}; \cdot)$ is defined in (36), $\mathcal{K}(\bar{Z}; \cdot)$ in (39), and $\mathcal{K}^\circ(\bar{Z}; \cdot)$ in (40). The local second-order limiting dynamics is defined as

$$Z^{(k+1)} = Z^{(k)} + \frac{1}{2}\phi(\bar{Z}; Z^{(k)} - \bar{Z}) + o(\|Z^{(k)} - \bar{Z}\|_{\mathsf{F}}^2). \tag{42}$$

From the perspective of a vector field, the limiting dynamics (42) assigns to each point $Z - \bar{Z} \in \mathcal{C}(\bar{Z})$ a displacement vector $\frac{1}{2}\phi(\bar{Z}; Z - \bar{Z})$, which satisfies $\frac{1}{2}\phi(\bar{Z}; Z - \bar{Z}) \sim \mathcal{O}(\|Z - \bar{Z}\|_{\mathsf{F}}^2)$ up to higher-order terms. Therefore, understanding the mapping $\phi(\bar{Z}; \cdot)$ in (41) becomes key to understanding the associated limiting dynamics (42). In the subsequent section, we will see that fundamental properties of $\phi(\bar{Z}; \cdot)$ (e.g., kernel, range, continuity, and primal–dual partition) are tightly linked to dynamical features of (42) (e.g., fixed points, almost-invariant sets, phases, and the effect of σ), which in turn explain and predict the limiting behavior of the one-step ADMM update (4) around $\bar{Z}$. Before proceeding, however, we first exploit several structural properties of $\phi(\bar{Z}; \cdot)$.

4.5 Polar Description and Primal-Dual Decoupling

4.5.1 Simplification of $\mathcal{K}^\circ(\bar{Z}; \bar{H})$

We first simplify the structure of $Q^0 \in \mathcal{O}^{|\alpha_0|}(\bar{H}_{\alpha_0\alpha_0})$, where $\bar{H} \in \mathcal{C}(\bar{Z})$.

Lemma 4 (Block-diagonal structure of Q^0)**.** *Under Assumption 1, fix any $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Suppose $\bar{Z}$ follows the first-level description in §3 and $\bar{H}$ the second-level description. Then, $Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} \equiv 0$ for all $Q^0 \in \mathcal{O}^{|\alpha_0|}(\bar{H}_{\alpha_0\alpha_0})$.*

Proof. From Proposition 1 (2), we get

$$\bar{H} = \left[\begin{array}{cc|cc} \bar{H}_{\alpha_+\alpha_+} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{P}}} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \sim & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hline \sim & \sim & \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & \bar{H}_{\alpha_-\alpha_-} \end{array}\right]$$

for any $\bar{H} \in \mathcal{C}(\bar{Z})$. Thus, $[\bar{H}_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}} = 0$, which closes the proof. $\square$

It turns out $\mathcal{K}^\circ(\bar{Z}; \bar{H})$ has the following nice structures:

Proposition 4 (Structure of $\mathcal{K}^\circ(\bar{Z}; \bar{H})$)**.** *Under Assumption 1, fix any $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Suppose $\bar{Z}$ follows the first-level description in §3 and $\bar{H}$ the second-level description. Then, $\mathcal{K}^\circ(\bar{Z}; \bar{H}) = \mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H}) + \mathcal{K}^\circ_{\mathsf{D}}(\bar{Z}; \bar{H})$, where*

$$\mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H}) := \left\{ W = \left[\begin{array}{c:c:c} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & Q^0 \left[\begin{array}{c:c|c:c} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & 0 \\ \hdashline \sim & \widehat{W}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right] (Q^0)^\mathsf{T} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}W = 0, \\ \widehat{W} = (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0, \\ \widehat{W}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} \succeq 0 \end{array} \right\}, \tag{43a}$$

$$\mathcal{K}^\circ_{\mathsf{D}}(\bar{Z}; \bar{H}) := \left\{ W = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & Q^0 \left[\begin{array}{c:c|c:c} 0 & 0 & 0 & 0 \\ \hdashline \sim & 0 & 0 & \widehat{W}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,-}} \\ \hline \sim & \sim & \widehat{W}_{\beta_{0,0}^{\mathsf{D}}\beta_{0,0}^{\mathsf{D}}} & \widehat{W}_{\beta_{0,0}^{\mathsf{D}}\beta_{0,-}} \\ \hdashline \sim & \sim & \sim & \widehat{W}_{\beta_{0,-}\beta_{0,-}} \end{array}\right] (Q^0)^\mathsf{T} & W_{\alpha_0\alpha_-} \\ \hdashline \sim & \sim & W_{\alpha_-\alpha_-} \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}^\perp W = 0, \\ \widehat{W} = (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0, \\ \widehat{W}_{\beta_{0,0}^{\mathsf{D}}\beta_{0,0}^{\mathsf{D}}} \preceq 0 \end{array} \right\}. \tag{43b}$$

Proof. "$\subseteq$": From (40), $Y \in \mathcal{K}^\circ(\bar{Z};\bar{H})$ if and only if

$$
\begin{aligned}
&\big\langle Y, \mathcal{P}\Theta(\bar{H};W) + \mathcal{P}^\perp\Theta^\perp(\bar{H};W)\big\rangle \le 0,\ \forall W \in \mathbb{S}^n \\
\iff &\big\langle \mathcal{P}Y, \mathcal{P}\Theta(\bar{H};W)\big\rangle + \big\langle \mathcal{P}^\perp Y, \mathcal{P}^\perp\Theta^\perp(\bar{H};W)\big\rangle \le 0,\ \forall W \in \mathbb{S}^n \\
\iff &Y = U + V,\ \mathcal{P}U = 0,\ \mathcal{P}^\perp V = 0,\ \big\langle V, \mathcal{P}\Theta(\bar{H};W)\big\rangle + \big\langle U, \mathcal{P}^\perp\Theta^\perp(\bar{H};W)\big\rangle \le 0,\ \forall W \in \mathbb{S}^n \\
\iff &Y = U + V,\ \mathcal{P}U = 0,\ \mathcal{P}^\perp V = 0,\ \big\langle V, \Theta(\bar{H};W)\big\rangle + \big\langle U, \Theta^\perp(\bar{H};W)\big\rangle \le 0,\ \forall W \in \mathbb{S}^n
\end{aligned} \tag{44}
$$

(i) Set $W_{\alpha_+\alpha_-} = 0, W_{\alpha_0\alpha_0} = 0, W_{\alpha_0\alpha_-} = 0, W_{\alpha_-\alpha_-} = 0$. Then, from (32),

$$
\Theta(\bar{H};W) = \left[\begin{array}{c:c:c} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & 0 & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right], \quad \Theta^\perp(\bar{H};W) = 0.
$$

Since $W_{\alpha_+\alpha_+}$ and $W_{\alpha_+\alpha_0}$ can be chosen arbitrarily, we have $V_{\alpha_+\alpha_+} = 0, V_{\alpha_+\alpha_0} = 0$. Symmetrically, $U_{\alpha_-\alpha_-} = 0, U_{\alpha_0\alpha_-} = 0$.

(ii) Set everything except $W_{\alpha_+\alpha_-}$ to be zero:

$$
\big\langle V, \Theta(\bar{H};W)\big\rangle + \big\langle U, \Theta^\perp(\bar{H};W)\big\rangle = 2\sum_{a\in\mathcal{I}_+}\sum_{b\in\mathcal{I}_-}\left\langle \frac{\mu_a}{\mu_a-\mu_b}V_{\alpha_a\alpha_b} + \frac{-\mu_b}{\mu_a-\mu_b}U_{\alpha_a\alpha_b}, W_{\alpha_a\alpha_b}\right\rangle
$$

Since $W_{\alpha_+\alpha_-}$ can be arbitrarily picked, we have

$$
\frac{\mu_a}{\mu_a-\mu_b}V_{\alpha_a\alpha_b} + \frac{-\mu_b}{\mu_a-\mu_b}U_{\alpha_a\alpha_b} = 0, \quad \forall a \in \mathcal{I}_+, b \in \mathcal{I}_- \tag{45}
$$

(iii) Now we zoom in to the $\alpha_0\alpha_0$ block. Set everything except $W_{\alpha_0\alpha_0}$ to be zero. Then from (32),

$$
\big\langle V, \Theta(\bar{H};W)\big\rangle + \big\langle U, \Theta^\perp(\bar{H};W)\big\rangle = \big\langle V_{\alpha_0\alpha_0}, \Pi'_+(\bar{H}_{\alpha_0\alpha_0}; W_{\alpha_0\alpha_0})\big\rangle + \big\langle U_{\alpha_0\alpha_0}, \Pi'_-(\bar{H}_{\alpha_0\alpha_0}; W_{\alpha_0\alpha_0})\big\rangle
$$

Further denote $\widehat{U} = (Q^0)^\mathsf{T} U_{\alpha_0\alpha_0} Q^0, \widehat{V} = (Q^0)^\mathsf{T} V_{\alpha_0\alpha_0} Q^0, \widehat{W} = (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0$:

$$
\begin{aligned}
&\big\langle V, \Theta(\bar{H};W)\big\rangle + \big\langle U, \Theta^\perp(\bar{H};W)\big\rangle = \\
&\left\langle \widehat{V}, \left[\begin{array}{c:c:c} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}} & \left\{\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline \sim & \Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}}) & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right]\right\rangle + \left\langle \widehat{U}, \left[\begin{array}{c:c:c} 0 & 0 & \left\{\frac{-\eta_{0,j}}{\eta_{0,i}-\eta_{0,j}}\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline \sim & \Pi_-(\widehat{W}_{\beta_{0,0}\beta_{0,0}}) & \widehat{W}_{\beta_{0,0}\beta_{0,-}} \\ \hdashline \sim & \sim & \widehat{W}_{\beta_{0,-}\beta_{0,-}} \end{array}\right]\right\rangle
\end{aligned}
$$

(a) Similar to (i), for $\widehat{W}$, set everything to 0 except $\widehat{W}_{\beta_{0,+}\beta_{0,+}}$ and $\widehat{W}_{\beta_{0,+}\beta_{0,0}}$. We get $\widehat{V}_{\beta_{0,+}\beta_{0,+}} = 0$ and $\widehat{V}_{\beta_{0,+}\beta_{0,0}} = 0$. Symmetrically, we get $\widehat{U}_{\beta_{0,-}\beta_{0,-}} = 0$ and $\widehat{U}_{\beta_{0,0}\beta_{0,0}} = 0$.

(b) Similar to (ii), for $\widehat{W}$, set everything to 0 except $\widehat{W}_{\beta_{0,+}\beta_{0,-}}$. We get

$$
\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\widehat{V}_{\beta_{0,i}\beta_{0,j}} + \frac{-\eta_{0,j}}{\eta_{0,i}-\eta_{0,j}}\widehat{U}_{\beta_{0,i}\beta_{0,j}} = 0, \quad \forall i \in \mathcal{I}_{0,+}, j \in \mathcal{I}_{0,-} \tag{46}
$$

(c) For $\widehat{W}$, set everything to 0 except $\widehat{W}_{\beta_{0,0}\beta_{0,0}}$. We get

$$
\left\langle \widehat{V}_{\beta_{0,0}\beta_{0,0}}, \Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})\right\rangle + \left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \Pi_-(\widehat{W}_{\beta_{0,0}\beta_{0,0}})\right\rangle \le 0,\ \forall \widehat{W}_{\beta_{0,0}\beta_{0,0}}
$$

Transversing $\widehat{W}_{\beta_{0,0}\beta_{0,0}}$ through $\mathbb{S}_+^{|\beta_{0,0}|}$, we get $\widehat{V}_{\beta_{0,0}\beta_{0,0}} \preceq 0$. Symmetrically, we get $\widehat{U}_{\beta_{0,0}\beta_{0,0}} \succeq 0$.

(iv) Since $\mathcal{P}U = 0, \mathcal{P}^{\perp}V = 0$, we have $\langle U, V\rangle = 0$. On the other hand, combining (i) - (iii):

$$\begin{aligned}\langle U, V\rangle &= 2\sum_{a\in\mathcal{I}_+}\sum_{b\in\mathcal{I}_-}\langle U_{\alpha_a\alpha_b}, V_{\alpha_a\alpha_b}\rangle + 2\sum_{i\in\mathcal{I}_{0,+}}\sum_{j\in\mathcal{I}_{0,-}}\left\langle \widehat{U}_{\beta_{0,i}\beta_{0,j}}, \widehat{V}_{\beta_{0,i}\beta_{0,j}}\right\rangle + \left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \widehat{V}_{\beta_{0,0}\beta_{0,0}}\right\rangle \\ =&2\sum_{a\in\mathcal{I}_+}\sum_{b\in\mathcal{I}_-}\frac{\mu_b}{\mu_a}\|U_{\alpha_a\alpha_b}\|_{\mathsf{F}}^2 + 2\sum_{i\in\mathcal{I}_{0,+}}\sum_{j\in\mathcal{I}_{0,-}}\frac{\eta_{0,j}}{\eta_{0,i}}\|\widehat{U}_{\beta_{0,i}\beta_{0,j}}\|_{\mathsf{F}}^2 + \left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \widehat{V}_{\beta_{0,0}\beta_{0,0}}\right\rangle = 0\end{aligned}$$

where the last equality comes from (45) and (46). Since $\mu_a > 0, \mu_b < 0, \eta_{0,i} > 0, \eta_{0,j} < 0$ and $\widehat{U}_{\beta_{0,0}\beta_{0,0}} \succeq 0, \widehat{V}_{\beta_{0,0}\beta_{0,0}} \preceq 0$, we get

$$U_{\alpha_+\alpha_-} = 0,\ V_{\alpha_+\alpha_-} = 0,\ \widehat{U}_{\beta_{0,+}\beta_{0,-}} = 0,\ \widehat{V}_{\beta_{0,+}\beta_{0,-}} = 0,\ \left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \widehat{V}_{\beta_{0,0}\beta_{0,0}}\right\rangle = 0$$

(v) Combining (i) - (iv), we know for any $Y = U + V \in \mathcal{K}(\bar{Z}; \bar{H})$, U and V have the following structure:

$$U = \left[\begin{array}{c:c:c} U_{\alpha_+\alpha_+} & U_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & Q^0\left[\begin{array}{c:c:c} \widehat{U}_{\beta_{0,+}\beta_{0,+}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}} & 0 \\ \hdashline \sim & \widehat{U}_{\beta_{0,0}\beta_{0,0}} \succeq 0 & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right](Q^0)^{\mathsf{T}} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right],\ V = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & Q^0\left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & \widehat{V}_{\beta_{0,0}\beta_{0,0}} \preceq 0 & \widehat{V}_{\beta_{0,0}\beta_{0,-}} \\ \hdashline \sim & \sim & \widehat{V}_{\beta_{0,-}\beta_{0,-}} \end{array}\right](Q^0)^{\mathsf{T}} & V_{\alpha_0\alpha_-} \\ \hdashline \sim & \sim & V_{\alpha_-\alpha_-} \end{array}\right]$$

Thus, $\langle U, \bar{S}\rangle = 0$. Since $\mathcal{P}U = 0$, we get $\langle U, S_{\mathsf{sc}}\rangle = 0$ from Lemma 2. Additional with Lemma 4, we get

$$\langle U, S_{\mathsf{sc}}\rangle = \left\langle Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\left[\begin{array}{c:c} \widehat{U}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} & 0 \\ \hdashline \sim & 0 \end{array}\right](Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}})^{\mathsf{T}}, [S_{\mathsf{sc}}]_{\alpha^{\mathsf{D}}_0\alpha^{\mathsf{D}}_0}\right\rangle = 0$$

Since $[S_{\mathsf{sc}}]_{\alpha^{\mathsf{D}}_0\alpha^{\mathsf{D}}_0} \succ 0$ and $\widehat{U}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} \succeq 0$, we get $\widehat{U}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} = 0$. Symmetrically, we get $\widehat{V}_{\beta^{\mathsf{P}}_{0,0}\beta^{\mathsf{P}}_{0,0}} = 0$. Since $\widehat{U}_{\beta_{0,0}\beta_{0,0}} \succeq 0$ and $\widehat{V}_{\beta_{0,0}\beta_{0,0}} \preceq 0$, fine-grained structure can be defined:

$$\widehat{U}_{\beta_{0,0}\beta_{0,0}} = \left[\begin{array}{c|c} \widehat{U}_{\beta^{\mathsf{P}}_{0,0}\beta^{\mathsf{P}}_{0,0}} \succeq 0 & 0 \\ \hline \sim & 0 \end{array}\right],\quad \widehat{V}_{\beta_{0,0}\beta_{0,0}} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \sim & \widehat{U}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} \preceq 0 \end{array}\right]$$

Notice that under this complementary structure, $\left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \widehat{V}_{\beta_{0,0}\beta_{0,0}}\right\rangle = 0$ automatically holds. This finishes the "$\subseteq$" part.

"$\supseteq$": Now we shall prove that for any $U \in \mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H})$ and $V \in \mathcal{K}^\circ_{\mathsf{D}}(\bar{Z}; \bar{H})$, (44) holds. To see this, $\forall W \in \mathbb{S}^n$:

$$\left\langle V, \Theta(\bar{H}; W)\right\rangle + \left\langle U, \Theta^{\perp}(\bar{H}; W)\right\rangle = \left\langle \widehat{V}_{\beta_{0,0}\beta_{0,0}}, \Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})\right\rangle + \left\langle \widehat{U}_{\beta_{0,0}\beta_{0,0}}, \Pi_-(\widehat{W}_{\beta_{0,0}\beta_{0,0}})\right\rangle \leq 0$$

This finishes the "$\supseteq$" part. $\square$

Corollary 2 (Relationship between $\mathcal{C}(\bar{Z})$ and $\mathcal{K}^\circ(\bar{Z}; \bar{H})$)**.** *Under Assumption 1, for any $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$, $\mathcal{C}_{\mathsf{P}}(\bar{Z}) \subseteq \mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H})$, $\mathcal{C}_{\mathsf{D}}(\bar{Z}) \subseteq \mathcal{K}^\circ_{\mathsf{D}}(\bar{Z}; \bar{H})$.*

Proof. Take any $H \in \mathcal{C}_{\mathsf{P}}(\bar{Z})$. From Proposition 1 (2) and Lemma 4,

$$H = \left[\begin{array}{c:c:c} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & Q^0\left[\begin{array}{c|c} (Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}})^{\mathsf{T}} H_{\alpha^{\mathsf{P}}_0\alpha^{\mathsf{P}}_0} Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0 \\ \hline \sim & 0 \end{array}\right](Q^0)^{\mathsf{T}} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right],$$

with $H_{\alpha^{\mathsf{P}}_0\alpha^{\mathsf{P}}_0} \succeq 0$ and $\mathcal{P}H = 0$. Thus, $H \in \mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H})$ from Proposition 4. The relationship between $\mathcal{C}_{\mathsf{D}}(\bar{Z})$ and $\mathcal{K}^\circ_{\mathsf{D}}(\bar{Z}; \bar{H})$ can be proven symmetrically. $\square$

4.5.2 Second-order limitation of $X^{(k+1)} - X^{(k)}$ and $S^{(k+1)} - S^{(k)}$

Under the local first- and second-order dynamics in Definition 1, if we initialize with $H^{(0)} = \bar{H} \in \mathcal{C}(\bar{Z})$, then $H^{(k)} \equiv \bar{H}$ and $W^{(k+1)} - W^{(k)} \to \phi(\bar{Z}; \bar{H})$ by Theorem 5. Within this local model, it is natural to ask whether the second-order limits of the primal and dual increments also exist. We give an affirmative answer.

For the primal variable $X^{(k)} = \Pi_+(Z^{(k)})$, we have

$$
\begin{aligned}
&X^{(k+1)} - X^{(k)} = \Pi_+(Z^{(k+1)}) - \Pi_+(Z^{(k)}) \\
&= t(\Pi'_+(\bar{Z}; H^{(k+1)}) - \Pi'_+(\bar{Z}; H^{(k)})) + \frac{t^2}{2}(\Pi''_+(\bar{Z}; H^{(k+1)}, W^{(k+1)}) - \Pi''_+(\bar{Z}; H^{(k)}, W^{(k)})) + o(t^2),
\end{aligned}
$$

where $Z^{(k)}$ is of the form (19). In the present regime, the first-order updates have stalled, i.e., $H^{(k+1)} = H^{(k)} = \bar{H}$. Hence, it suffices to analyze the limit of

$$
\Pi''_+(\bar{Z}; \bar{H}, W^{(k+1)}) - \Pi''_+(\bar{Z}; \bar{H}, W^{(k)}) \quad \text{as } k \to \infty.
$$

Similarly, for the dual variable $S^{(k)} = -\frac{1}{\sigma}\Pi_-(Z^{(k)})$, it suffices to study the limit of

$$
-\frac{1}{\sigma}(\Pi''_-(\bar{Z}; \bar{H}, W^{(k+1)}) - \Pi''_-(\bar{Z}; \bar{H}, W^{(k)})).
$$

To proceed, we first establish the following auxiliary lemma.

Lemma 5 (Convergent difference of $\Pi_+(\cdot)$)**.** *For a symmetric matrix sequence $\{X_k\}_{k=0}^{\infty}$, we have*

$$
\lim_{k\to\infty} (X_{k+1} - X_k) = \Delta \Longrightarrow \lim_{k\to\infty} (\Pi_+(X_{k+1}) - \Pi_+(X_k)) = \Pi_+(\Delta).
$$

Proof. Let $Y_k = \frac{X_k}{k}$ for all $k \geq 1$, and denote $\Delta_k := X_{k+1} - X_k$. Then

$$
\lim_{k\to\infty} Y_k = \lim_{k\to\infty} \left(\frac{X_0}{k} + \frac{1}{k}\sum_{i=0}^{k-1} \Delta_i \right) = \Delta.
$$

Since $\Pi_+(\cdot)$ is positively homogeneous,

$$
\Pi_+(X_{k+1}) - \Pi_+(X_k) = (k+1)\Pi_+(Y_{k+1}) - k\Pi_+(Y_k) = \Pi_+(Y_{k+1}) + k(\Pi_+(Y_{k+1}) - \Pi_+(Y_k)).
$$

The first term satisfies $\Pi_+(Y_{k+1}) \to \Pi_+(\Delta)$ by continuity of $\Pi_+(\cdot)$. For the second term, note that

$$
Y_{k+1} - Y_k = \frac{1}{k+1}(\Delta_k - Y_k), \qquad \Delta_k - Y_k \to 0,
$$

hence $Y_{k+1} - Y_k = o(\frac{1}{k})$ as $k \to \infty$. Since $\Pi_+(\cdot)$ is 1-Lipschitz,

$$
\|\Pi_+(Y_{k+1}) - \Pi_+(Y_k)\|_{\mathsf{F}} \leq \|Y_{k+1} - Y_k\|_{\mathsf{F}} = o\left(\frac{1}{k}\right). \tag{47}
$$

Therefore, $k(\Pi_+(Y_{k+1}) - \Pi_+(Y_k)) \to 0$, and the claim follows. □

Theorem 6 ($\phi_{\mathsf{P}}(\bar{Z}; \bar{H})$ and $\phi_{\mathsf{D}}(\bar{Z}; \bar{H})$)**.** *Under the local first- and second-order dynamics in Definition 1, with initialization $H^{(0)} = \bar{H} \in \mathcal{C}(\bar{Z})$, the local second-order limit of $X^{(k+1)} - X^{(k)}$ is*

$$
\phi_{\mathsf{P}}(\bar{Z}; \bar{H}) := \lim_{k\to\infty} \left\{ \Pi''_+(\bar{Z}; \bar{H}, W^{(k+1)}) - \Pi''_+(\bar{Z}; \bar{H}, W^{(k)}) \right\} = \Theta(\bar{Z}; \bar{H}, \phi(\bar{Z}; \bar{H})), \tag{48}
$$

and the local second-order limit of $S^{(k+1)} - S^{(k)}$ is

$$
\phi_{\mathsf{D}}(\bar{Z}; \bar{H}) := -\frac{1}{\sigma}\lim_{k\to\infty} \left\{ \Pi''_-(\bar{Z}; \bar{H}, W^{(k+1)}) - \Pi''_-(\bar{Z}; \bar{H}, W^{(k)}) \right\} = -\frac{1}{\sigma}\Theta^{\perp}(\bar{Z}; \bar{H}, \phi(\bar{Z}; \bar{H})), \tag{49}
$$

where $\Theta(\bar{Z}; \bar{H}, \cdot)$ and $\Theta^{\perp}(\bar{Z}; \bar{H}, \cdot)$ are defined in (32). Moreover, $\phi(\bar{Z}; \bar{H}) = \phi_{\mathsf{P}}(\bar{Z}; \bar{H}) - \sigma\phi_{\mathsf{D}}(\bar{Z}; \bar{H})$.

Proof. (i) For the primal part, by (34),

$$\Pi''_+(\bar{Z};\bar{H},W^{(k+1)}) - \Pi''_+(\bar{Z};\bar{H},W^{(k)}) = \Theta(\bar{Z};\bar{H},\widetilde{W}^{(k+1)}) - \Theta(\bar{Z};\bar{H},\widetilde{W}^{(k)})$$
$$= \left[\begin{array}{c:c:c} \widetilde{W}^{(k+1)}_{\alpha_+\alpha_+} - \widetilde{W}^{(k)}_{\alpha_+\alpha_+} & \widetilde{W}^{(k+1)}_{\alpha_+\alpha_0} - \widetilde{W}^{(k)}_{\alpha_+\alpha_0} & \left\{\frac{\mu_a}{\mu_a-\mu_b}(\widetilde{W}^{(k+1)}_{\alpha_a\alpha_b} - \widetilde{W}^{(k)}_{\alpha_a\alpha_b})\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k+1)}_{\alpha_0\alpha_0}) - \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k)}_{\alpha_0\alpha_0}) & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right].$$

Since $\widetilde{W}^{(k+1)} - \widetilde{W}^{(k)} \to \phi(\bar{Z};\bar{H})$ as $k\to\infty$ by Theorem 5, we have

$$\widetilde{W}^{(k+1)}_{\alpha_a\alpha_b} - \widetilde{W}^{(k)}_{\alpha_a\alpha_b} \to \phi(\bar{Z};\bar{H})_{\alpha_a\alpha_b}, \quad \forall a\in\mathcal{I},\ \forall b\in\mathcal{I}.$$

It remains to handle the term $\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k+1)}_{\alpha_0\alpha_0}) - \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k)}_{\alpha_0\alpha_0})$. For any $W\in\mathbb{S}^n$,

$$\Pi'_+(\bar{H}_{\alpha_0\alpha_0};W_{\alpha_0\alpha_0}) = Q^0\left[\begin{array}{c:c:c} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}} & \left\{\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}(\widehat{W}_{\beta_{0,i}\beta_{0,j}})\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline \sim & \Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}}) & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right](Q^0)^\mathsf{T}$$

where $\widehat{W} = (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0$. The only nonlinear component is the PSD projector. By Lemma 5,

$$\Pi_+(\widehat{W}^{(k+1)}_{\beta_{0,0}\beta_{0,0}}) - \Pi_+(\widehat{W}^{(k)}_{\beta_{0,0}\beta_{0,0}}) \to \Pi_+(\widehat{\phi}_{\beta_{0,0}\beta_{0,0}}) \quad \text{as } k\to\infty,$$

where $\widehat{\phi} := (Q^0)^\mathsf{T}\phi(\bar{Z};\bar{H})_{\alpha_0\alpha_0}Q^0$. Therefore,

$$\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k+1)}_{\alpha_0\alpha_0}) - \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^{(k)}_{\alpha_0\alpha_0}) \to \Pi'_+(\bar{H}_{\alpha_0\alpha_0};\phi(\bar{Z};\bar{H})_{\alpha_0\alpha_0}) \quad \text{as } k\to\infty,$$

which implies that

$$\Pi''_+(\bar{Z};\bar{H},W^{(k+1)}) - \Pi''_+(\bar{Z};\bar{H},W^{(k)}) \to \Theta(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})).$$

(ii) The dual part follows by symmetry: one can similarly show that

$$\Pi''_-(\bar{Z};\bar{H},W^{(k+1)}) - \Pi''_-(\bar{Z};\bar{H},W^{(k)}) \to \Theta^\perp(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})) \quad \text{as } k\to\infty.$$

The final identity $\phi(\bar{Z};\bar{H}) = \phi_\mathsf{P}(\bar{Z};\bar{H}) - \sigma\phi_\mathsf{D}(\bar{Z};\bar{H})$ follows from $\phi(\bar{Z};\bar{H}) = \Theta(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})) + \Theta^\perp(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H}))$ in (34). ☐

4.5.3 Primal-dual decoupling of $\phi(\bar{Z};\bar{H})$

Theorem 6 connects $\phi_\mathsf{P}(\bar{Z};\bar{H})$ (resp. $\phi_\mathsf{D}(\bar{Z};\bar{H})$) with the limiting behavior of $X^{(k+1)} - X^{(k)}$ (resp. $S^{(k+1)} - S^{(k)}$). The next theorem further reveals a deeper connection between $\phi_\mathsf{P}(\bar{Z};\bar{H})$ (resp. $\phi_\mathsf{D}(\bar{Z};\bar{H})$) and $\mathcal{K}^\circ_\mathsf{P}(\bar{Z};\bar{H})$ (resp. $\mathcal{K}^\circ_\mathsf{D}(\bar{Z};\bar{H})$).

Theorem 7 (Primal-dual decoupling of $\phi(\bar{Z};\bar{H})$)**.** *Let $\phi_\mathsf{P}(\bar{Z};\bar{H})$ be defined in (48) and $\phi_\mathsf{D}(\bar{Z};\bar{H})$ in (49). Let $\mathcal{K}^\circ_\mathsf{P}(\bar{Z};\bar{H})$ and $\mathcal{K}^\circ_\mathsf{D}(\bar{Z};\bar{H})$ be defined in (43). Then, under Assumption 1,*

$$\phi_\mathsf{P}(\bar{Z};\bar{H}) = \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_\mathsf{P}(\bar{Z};\bar{H})} \|W + \mathcal{E}^\perp(\bar{Z};\bar{H})\|_\mathsf{F}^2, \tag{50a}$$

$$\phi_\mathsf{D}(\bar{Z};\bar{H}) = -\frac{1}{\sigma}\operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_\mathsf{D}(\bar{Z};\bar{H})} \|W + \mathcal{E}(\bar{Z};\bar{H})\|_\mathsf{F}^2. \tag{50b}$$

where $\mathcal{E}(\bar{Z};\bar{H})$ and $\mathcal{E}^\perp(\bar{Z};\bar{H})$ are defined in (33).

Proof. Since $\mathcal{K}^\circ(\bar{Z};\bar{H})$ is closed and convex, $\inf_{W\in\mathcal{K}^\circ(\bar{Z};\bar{H})}\|W-\Psi(\bar{Z};\bar{H})\|_{\mathsf{F}}^2$'s optimal solution can be attained and is unique, where $\Psi(\bar{Z};\bar{H})$ is defined in (36). Additional with Proposition 4, we get

$$
\begin{aligned}
&\phi(\bar{Z};\bar{H}) = \Pi_{\mathcal{K}^\circ(\bar{Z};\bar{H})}\Psi(\bar{Z};\bar{H}) = \mathop{\arg\min}_{W\in\mathcal{K}^\circ(\bar{Z};\bar{H})}\|W-\Psi(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\mathop{\arg\min}_{U\in\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H}),V\in\mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H})}\|U+V-\Psi(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\mathop{\arg\min}_{U\in\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H}),V\in\mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H})}\|U+V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2.
\end{aligned}
$$

Since $U\in\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H})$, we get $\mathcal{P}U=0$. Symmetrically, $\mathcal{P}^\perp V=0$. Therefore,

$$
\begin{aligned}
&\|U+V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\|\mathcal{P}^\perp U+\mathcal{P}V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2=\|\mathcal{P}^\perp U+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2+\|\mathcal{P}V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\|U+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2+\|V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\|U+\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2+\|V+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2-\|\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2-\|\mathcal{P}^\perp\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2,
\end{aligned}
$$

where in the last equality, we use $\left\langle U,\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\right\rangle=0$ and

$$
\begin{aligned}
&\|U+\mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
=&\left\langle U+\mathcal{E}^\perp(\bar{Z};\bar{H}),U+\mathcal{E}^\perp(\bar{Z};\bar{H})\right\rangle+\left\langle\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H}),\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\right\rangle-2\left\langle U+\mathcal{E}^\perp(\bar{Z};\bar{H}),\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\right\rangle\\
=&\|U+\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2-\|\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2.
\end{aligned}
$$

Symmetrically, $\left\langle V,\mathcal{P}^\perp\mathcal{E}(\bar{Z};\bar{H})\right\rangle=0$ and $\|V+\mathcal{P}\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2=\|V+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2-\|\mathcal{P}^\perp\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2$. Notice that $-\|\mathcal{P}\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2-\|\mathcal{P}^\perp\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2$ is an constant and does not affect the arg min. After observing that $U\in\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H})$ and $V\in\mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H})$ are totally decoupled in the objective, we get

$$
\phi(\bar{Z};\bar{H})=\underbrace{\mathop{\arg\min}_{U\in\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H})}\|U+\mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2}_{=:\bar{U}}+\underbrace{\mathop{\arg\min}_{V\in\mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H})}\|V+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2}_{=:\bar{V}},
$$

where $\bar{U}$ (resp. $\bar{V}$) is attainable and unique since $\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H})$ (resp. $\mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H})$) is closed and convex. Now we shall prove that $\bar{U}=\phi_{\mathsf{P}}(\bar{Z};\bar{H})$ and $\bar{V}=-\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H})$. From Proposition 4,

$$
\phi(\bar{Z};\bar{H})=\bar{U}+\bar{V}=\left[\begin{array}{c:c:c}
\bar{U}_{\alpha_+\alpha_+} & \bar{U}_{\alpha_+\alpha_0} & 0\\ \hdashline
\sim & Q^0\left[\begin{array}{c:c|c:c}
\widehat{U}_{\beta_{0,+}\beta_{0,+}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & 0\\ \hdashline
\sim & \widehat{U}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}}\succeq 0 & 0 & \widehat{V}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,-}}\\ \hline
\sim & \sim & \widehat{V}_{\beta_{0,0}^{\mathsf{D}}\beta_{0,0}^{\mathsf{D}}}\preceq 0 & \widehat{V}_{\beta_{0,0}^{\mathsf{D}}\beta_{0,-}}\\ \hdashline
\sim & \sim & \sim & \widehat{V}_{\beta_{0,-}\beta_{0,-}}
\end{array}\right](Q^0)^{\mathsf{T}} & \bar{V}_{\alpha_0\alpha_-}\\ \hdashline
\sim & \sim & \bar{V}_{\alpha_-\alpha_-}
\end{array}\right],
$$

where $\widehat{U}=(Q^0)^{\mathsf{T}}\bar{U}_{\alpha_0\alpha_0}Q^0$ and $\widehat{V}=(Q^0)^{\mathsf{T}}\bar{V}_{\alpha_0\alpha_0}Q^0$. Then, from Theorem 2,

$$
\begin{aligned}
&\Pi_+'(\bar{H}_{\alpha_0\alpha_0};\phi(\bar{Z};\bar{H})_{\alpha_0\alpha_0})=\Pi_+'(\bar{H}_{\alpha_0\alpha_0};\bar{U}_{\alpha_0\alpha_0}+\bar{V}_{\alpha_0\alpha_0})=Q^0\Pi_+'(\widehat{H};\widehat{U}+\widehat{V})(Q^0)^{\mathsf{T}}\\
=&Q^0\left[\begin{array}{c:c|c:c}
\widehat{U}_{\beta_{0,+}\beta_{0,+}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & 0\\ \hdashline
\sim & \widehat{U}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} & 0 & 0\\ \hline
\sim & \sim & 0 & 0\\ \hdashline
\sim & \sim & \sim & 0
\end{array}\right](Q^0)^{\mathsf{T}},
\end{aligned}
$$

where $\widehat{H} := (Q^0)^\mathsf{T} \bar{H}_{\alpha_0\alpha_0} Q^0$ is diagonal. Thus, from (32),

$$\Theta(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})) = \left[\begin{array}{c:c:c} \bar{U}_{\alpha_+\alpha_+} & \bar{U}_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & Q^0 \left[\begin{array}{c:c|c:c} \widehat{U}_{\beta_{0,+}\beta_{0,+}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat{U}_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & 0 \\ \hdashline \sim & \widehat{U}_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} \succeq 0 & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right] (Q^0)^\mathsf{T} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right] = \bar{U},$$

where in the last equality, we use $\bar{U} \in \mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z};\bar{H})$ and Proposition 4 again. Symmetrically, $\bar{V} = \Theta^\perp(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H}))$. We close the proof by recalling Theorem 6: $\Theta(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})) = \phi_{\mathsf{P}}(\bar{Z};\bar{H})$ and $\Theta^\perp(\bar{Z};\bar{H},\phi(\bar{Z};\bar{H})) = -\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H})$. $\square$

5 Kernel of $\phi(\bar{Z};\cdot)$

The first property of $\phi(\bar{Z};\cdot)$ that we study is its kernel $\ker(\phi(\bar{Z};\cdot))$, i.e., $\{\bar{H} \in \mathcal{C}(\bar{Z}) \mid \phi(\bar{Z};\bar{H}) = 0\}$. The set $\ker(\phi(\bar{Z};\cdot))$ directly characterizes when the local second-order limiting dynamics (42) is *effective*, in the sense that the higher-order term $o(\|Z^{(k)} - \bar{Z}\|_{\mathsf{F}}^2)$ in (42) can be neglected. When $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\ker(\phi(\bar{Z};\cdot))$, we have $\phi(\bar{Z};\bar{H}) \neq 0$, and the second-order term dominates the evolution in (42). Conversely, if $\bar{H} \in \ker(\phi(\bar{Z};\cdot))$ (i.e., the second-order term vanishes) yet the true one-step ADMM iteration (4) does not converge, then higher-order terms must be taken into account. It turns out that $\ker(\phi(\bar{Z};\cdot))$ admits a clean characterization:

Proposition 5 (Kernel of $\phi(\bar{Z};\cdot)$)**.** *Under Assumption 1, for any $\bar{Z} \in \mathcal{Z}_\star$, $\ker(\phi(\bar{Z};\cdot)) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$.*

The proof of Proposition 5 is divided into two parts. First, we show that $\phi(\bar{Z};\bar{H}) \neq 0$ for any $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ (Lemma 6 in §5.1). Second, we show that $\phi(\bar{Z};\bar{H}) = 0$ for any $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ (Lemma 7 in §5.2).

In §5.3, we discuss one application of $\ker(\phi(\bar{Z};\cdot))$. In slow-convergence regions of the one-step ADMM iteration (4), a typical pattern is that $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ tends to be very small yet is generally nonzero. We explain this phenomenon using our local second-order limiting dynamics model (42), with the initialization $Z^{(0)}$ chosen in $\bar{Z} + (\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}))$.

5.1 Proof of "$\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \Longrightarrow \phi(\bar{Z};\bar{H}) \neq 0$"

The motivation for this part comes from Sturm's square-root error bound under the existence of a strictly complementary primal–dual pair [42]. Since $\bar{H} \in \ker(\delta'_{\bar{Z}})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the forward error $\mathrm{dist}(\bar{Z} + t\bar{H}, \mathcal{Z}_\star)$ is of order t. Consequently, under Assumption 1, the backward error $\delta(\bar{Z} + t\bar{H})$ must exhibit a nonzero response at order t^2.

Lemma 6 ($\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \Longrightarrow \phi(\bar{Z};\bar{H}) \neq 0$)**.** *Under Assumption 1, pick any $\bar{H} \in \mathcal{C}(\bar{Z})$. If $\phi(\bar{Z};\bar{H}) = 0$, then $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$.*

Proof. We aim to show that if $\phi(\bar{Z};\bar{H}) = 0$, then $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} = 0$ and $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} = 0$. From Theorem 5, if $\phi(\bar{Z};\bar{H}) = \Psi(\bar{Z};\bar{H}) - \Pi_{\mathcal{K}(\bar{Z};\bar{H})}(\Psi(\bar{Z};\bar{H})) = 0$, we have $\Psi(\bar{Z};\bar{H}) \in \mathcal{K}(\bar{Z};\bar{H})$. By (39), there exists a convergent sequence $\{\Psi^i\}_{i=1}^\infty \to \Psi(\bar{Z};\bar{H})$, such that for each Ψ^i, there exists $\widetilde{W}^i \in \mathbb{S}^n$ with:

$$\mathcal{P}\Theta(\bar{Z};\bar{H},\widetilde{W}^i) + \mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\widetilde{W}^i) = \Psi^i.$$

By $\{\Psi^i\}_{i=1}^\infty$'s definition, $\forall \epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$, such that $\forall i \geq N_\epsilon$, $\|\Psi(\bar{Z};\bar{H}) - \Psi^i\|_{\mathsf{F}} \leq \epsilon$. Substituting $\Psi(\bar{Z};\bar{H})$'s formula from (36):

$$\begin{aligned} & \|\mathcal{P}\Theta(\bar{Z};\bar{H},\widetilde{W}^i) + \mathcal{P}^\perp\Theta^\perp(\bar{Z};\bar{H},\widetilde{W}^i) - (-\mathcal{P}\mathcal{E}(\bar{Z};\bar{H}) - \mathcal{P}^\perp\mathcal{E}^\perp(\bar{Z};\bar{H}))\|_{\mathsf{F}} \leq \epsilon \\ \Longrightarrow & \begin{cases} \|\mathcal{P}\left\{\Theta(\bar{Z};\bar{H},\widetilde{W}^i) + \mathcal{E}(\bar{Z};\bar{H})\right\}\|_{\mathsf{F}} \leq \epsilon \\ \|\mathcal{P}^\perp\left\{\Theta^\perp(\bar{Z};\bar{H},\widetilde{W}^i) + \mathcal{E}^\perp(\bar{Z};\bar{H})\right\}\|_{\mathsf{F}} \leq \epsilon \end{cases} \end{aligned}$$

We first focus on the primal part: $\|\mathcal{P}\left\{\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})\right\}\|_{\mathsf{F}}\le\epsilon$. Do expansion for $\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})$ from (32) and (33):

$$\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})=\left[\begin{array}{c:c:c} \widetilde{W}^i_{\alpha_+\alpha_+} & \left\{\begin{array}{c}\widetilde{W}^i_{\alpha_a\alpha_0}\\ -2\frac{1}{\mu_a}\bar{H}_{\alpha_a\alpha_0}\Pi_+(-\bar{H}_{\alpha_0\alpha_0})\end{array}\right\}_{a\in\mathcal{I}_+} & \left\{\begin{array}{c}\frac{\mu_a}{\mu_a-\mu_b}\widetilde{W}^i_{\alpha_a\alpha_b}\\ +2\frac{1}{\mu_a-\mu_b}\bar{H}_{\alpha_a\alpha_0}\bar{H}_{\alpha_0\alpha_b}\end{array}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & \begin{array}{c}\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^i_{\alpha_0\alpha_0})\\ +2\sum\limits_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0}\end{array} & \left\{2\frac{1}{-\mu_b}\Pi_+(\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right].$$

Now our goal is to show

$$[\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}}=\left[\begin{array}{c:c}[\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & 0\\ \hdashline \sim & 0\end{array}\right]\text{ with }[\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}}\succeq 0. \tag{51}$$

(a) For $2\frac{1}{-\mu_b}\Pi_+(\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}$, $\forall b\in\mathcal{I}_-$, we notice from Proposition 1 (2) that

$$\Pi_+(\bar{H}_{\alpha_0\alpha_0})=\left[\begin{array}{c|c}[\Pi_+(\bar{H}_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline \sim & 0\end{array}\right].$$

Thus,

$$\Pi_+(\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}=\left[\begin{array}{c|c}[\Pi_+(\bar{H}_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline \sim & 0\end{array}\right]\left[\begin{array}{c}\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_b}\\ \hline \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_b}\end{array}\right]=\left[\begin{array}{c}[\Pi_+(\bar{H}_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}}\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_b}\\ \hline 0\end{array}\right],$$

which directly implies $[\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})]_{\alpha_0^{\mathsf{D}}\alpha_-}=0$.
(b) For $[\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H})]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}}=[\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^i_{\alpha_0\alpha_0})+2\sum\limits_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}$:
since $2\sum\limits_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0}\succeq 0$, all we need to prove is $[\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^i_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\succeq 0$. From (11), we have

$$\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^i_{\alpha_0\alpha_0})=Q^0\left\{\Pi'_+\left((Q^0)^{\mathsf{T}}\bar{H}_{\alpha_0\alpha_0}Q^0,(Q^0)^{\mathsf{T}}\widetilde{W}^i_{\alpha_0\alpha_0}Q^0\right)\right\}(Q^0)^{\mathsf{T}}=$$

$$Q^0\left[\begin{array}{c:c|c:c} \widehat{W}_{\beta_{0,+}\beta_{0,+}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat{W}_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & \left\{\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\widehat{W}_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}} \\ \hdashline \sim & [\Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{P}}\gamma_{0,0,\mathsf{P}}} & [\Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{P}}\gamma_{0,0,\mathsf{D}}} & 0 \\ \hline \sim & \sim & [\Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{D}}\gamma_{0,0,\mathsf{D}}} & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right](Q^0)^{\mathsf{T}},$$

where we abbreviate $(Q^0)^{\mathsf{T}}\widetilde{W}^i_{\alpha_0\alpha_0}Q^0$ as $\widehat{W}$. $\gamma_{0,0,\mathsf{P}}$ and $\gamma_{0,0,\mathsf{D}}$ divide $\Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})$'s indices by the primal and dual part. Moreover, from Lemma 4, $Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}}\in\mathcal{O}^{|\beta_{0,\mathsf{P}}|}(\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}})$ and $Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\in\mathcal{O}^{|\beta_{0,\mathsf{D}}|}(\bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}})$. Therefore,

$$[\Pi'_+(\bar{H}_{\alpha_0\alpha_0};\widetilde{W}^i_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}=Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\left[\begin{array}{c:c}[\Pi_+(\widehat{W}_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{D}}\gamma_{0,0,\mathsf{D}}} & 0\\ \hdashline \sim & 0\end{array}\right](Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}})^{\mathsf{T}}\succeq 0$$

Combining (a) and (b), we prove (51).

On the other hand, we notice that $\left\langle\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H}),\bar{S}\right\rangle=0$ regardless $\widetilde{W}^i$'s choice. From Lemma 2,

$$\left|\left\langle\Theta(\bar{Z};\bar{H},\widetilde{W}^i)+\mathcal{E}(\bar{Z};\bar{H}),S_{\mathsf{sc}}\right\rangle\right|\le\epsilon\|S_{\mathsf{sc}}-\bar{S}\|_{\mathsf{F}}.$$

Since $[S_{\mathsf{sc}}]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \succ 0$, $[\Theta(\bar{Z}; \bar{H}, \widetilde{W}^i) + \mathcal{E}(\bar{Z}; \bar{H})]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \succeq 0$, and

$$\left\langle \Theta(\bar{Z}; \bar{H}, \widetilde{W}^i) + \mathcal{E}(\bar{Z}; \bar{H}), S_{\mathsf{sc}} \right\rangle = \left\langle [\Theta(\bar{Z}; \bar{H}, \widetilde{W}^i) + \mathcal{E}(\bar{Z}; \bar{H})]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}}, [S_{\mathsf{sc}}]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \right\rangle,$$

we have

$$\|[\Theta(\bar{Z}; \bar{H}, \widetilde{W}^i) + \mathcal{E}(\bar{Z}; \bar{H})]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}}\|_{\mathsf{F}} \leq \frac{\|S_{\mathsf{sc}} - \bar{S}\|_{\mathsf{F}}}{\lambda_{\min}([S_{\mathsf{sc}}]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}})} \cdot \epsilon.$$

Together with $[\Pi'_+(\bar{H}_{\alpha_0\alpha_0}; \widetilde{W}^i_{\alpha_0\alpha_0})]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} \succeq 0$, $[2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} \succeq 0$, and

$$[\Theta(\bar{Z}; \bar{H}, \widetilde{W}^i) + \mathcal{E}(\bar{Z}; \bar{H})]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} = [\Pi'_+(\bar{H}_{\alpha_0\alpha_0}; \widetilde{W}^i_{\alpha_0\alpha_0}) + 2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}},$$

we get

$$\|[2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\|_{\mathsf{F}} \leq \frac{\|S_{\mathsf{sc}} - \bar{S}\|_{\mathsf{F}}}{\lambda_{\min}([S_{\mathsf{sc}}]_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}})} \cdot \epsilon.$$

Observing that the above error bound does not contain $\widetilde{W}^i$ and ϵ could be picked arbitrarily small, we get

$$\|[2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\|_{\mathsf{F}} = 0.$$

Due to positive semi-definiteness of $\{\bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}\}_{c \in \mathcal{I}_+}$'s, we get $\|[\bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\|_{\mathsf{F}} = 0$, for all $c \in \mathcal{I}_+$. Thus,

$$\|\bar{H}_{\alpha_c\alpha_0^{\mathsf{D}}}\|_{\mathsf{F}}^2 = \operatorname{tr}\left(\bar{H}^{\mathsf{T}}_{\alpha_c\alpha_0^{\mathsf{D}}} \bar{H}_{\alpha_c\alpha_0^{\mathsf{D}}}\right) \leq \sqrt{|\alpha_0^{\mathsf{D}}|} \cdot \|\bar{H}^{\mathsf{T}}_{\alpha_c\alpha_0^{\mathsf{D}}} \bar{H}_{\alpha_c\alpha_0^{\mathsf{D}}}\|_{\mathsf{F}} = \sqrt{|\alpha_0^{\mathsf{D}}|} \cdot \|[\bar{H}_{\alpha_0\alpha_c} \bar{H}_{\alpha_c\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\|_{\mathsf{F}} = 0, \ \forall c \in \mathcal{I}_+.$$

Therefore, $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} = 0$. By primal-dual symmetry, $\bar{H}_{\alpha_-\alpha_0^{\mathsf{P}}} = 0$. Finally, by Proposition 2 (2), we get $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$. $\square$

5.2 Proof of "$\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \Longrightarrow \phi(\bar{Z}; \bar{H}) = 0$"

Intuitively, if $\bar{H} \in \operatorname{ri}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}))$, then $\phi(\bar{Z}; \bar{H}) = 0$. The following stronger result shows that $\phi(\bar{Z}; \bar{H})$ will vanish even when $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \backslash \operatorname{ri}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}))$.

Lemma 7 ($\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \Longrightarrow \phi(\bar{Z}; \bar{H}) = 0$)**.** *Under Assumption 1, if $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, then $\phi(\bar{Z}; \bar{H}) = 0$.*

Proof. Proof by construction. Since $(X_{\mathsf{sc}}, S_{\mathsf{sc}})$ is a strict complementary pair, $[X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succ 0$ and $[S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \succ 0$. Define two constants

$$\kappa_{\mathsf{P}} = \frac{\lambda_{\max}(2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_c} \bar{H}_{\alpha_c\alpha_0^{\mathsf{P}}})}{\lambda_{\min}([X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}})}, \quad \kappa_{\mathsf{D}} = \frac{\lambda_{\max}(-2 \sum_{c \in \mathcal{I}_-} \frac{1}{\mu_c} \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_c} \bar{H}_{\alpha_c\alpha_0^{\mathsf{D}}})}{\lambda_{\min}([\sigma S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}})}. \tag{52}$$

Construct W as follows:

$$\left[\begin{array}{c:c|c:c} \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}} - \bar{X}]_{\alpha_+\alpha_+} & \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}}]_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hdashline \sim & \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} - 2 \sum_{c \in \mathcal{I}_+} \frac{1}{\mu_c} \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_c} \bar{H}_{\alpha_c\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & -\kappa_{\mathsf{D}} \cdot [\sigma S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} - 2 \sum_{c \in \mathcal{I}_-} \frac{1}{\mu_c} \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_c} \bar{H}_{\alpha_c\alpha_0^{\mathsf{D}}} & -\kappa_{\mathsf{D}} \cdot [\sigma S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \hdashline \sim & \sim & \sim & -\kappa_{\mathsf{D}} \cdot [\sigma S_{\mathsf{sc}} - \sigma \bar{S}]_{\alpha_-\alpha_-} \end{array}\right].$$

We shall prove $\Psi(\bar Z;\bar H)=\mathcal{P}\Theta(\bar Z;\bar H,W)+\mathcal{P}^\perp\Theta^\perp(\bar Z;\bar H,W)$, which implies $\Psi(\bar Z;\bar H)\in\mathcal{K}(\bar Z;\bar H)$.

(i) Primal part. Since $\bar H\in\mathcal{T}_{\mathcal{Z}_\star}(\bar Z)$, we have $\bar H_{\alpha_0^{\mathsf{D}}\alpha_a}=0,\forall a\in\mathcal{I}_+$ and $\bar H_{\alpha_0^{\mathsf{P}}\alpha_b}=0,\forall b\in\mathcal{I}_-$. Additional with

$$\bar H_{\alpha_0\alpha_0}=\left[\begin{array}{c|c}[\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline \sim & [\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\end{array}\right],\quad \text{where } [\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}}\succeq 0,\ [\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\preceq 0,$$

we have

$$-2\frac{1}{\mu_a}\bar H_{\alpha_a\alpha_0}\Pi_+(-\bar H_{\alpha_0\alpha_0})=-2\frac{1}{\mu_a}\left[\begin{array}{c|c}\bar H_{\alpha_a\alpha_0^{\mathsf{P}}} & 0\end{array}\right]\left[\begin{array}{c|c}0 & 0\\ \hline \sim & -[\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\end{array}\right]=0,\quad \forall a\in\mathcal{I}_+,$$

$$2\frac{1}{-\mu_b}\bar H_{\alpha_b\alpha_0}\Pi_+(\bar H_{\alpha_0\alpha_0})=2\frac{1}{-\mu_b}\left[\begin{array}{c|c}0 & \bar H_{\alpha_b\alpha_0^{\mathsf{D}}}\end{array}\right]\left[\begin{array}{c|c}[\bar H_{\alpha_0\alpha_0}]_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline \sim & 0\end{array}\right]=0,\quad \forall b\in\mathcal{I}_-.$$

Thus,

$$\mathcal{E}(\bar Z;\bar H)=\left[\begin{array}{c:c:c}0 & 0 & 0\\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0\alpha_c}\bar H_{\alpha_c\alpha_0} & 0\\ \hdashline \sim & \sim & 0\end{array}\right]=\left[\begin{array}{c:c|c:c}0 & 0 & 0 & 0\\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{P}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{P}}} & 0 & 0\\ \hline \sim & \sim & 0 & 0\\ \hdashline \sim & \sim & \sim & 0\end{array}\right].$$

Now we calculate $\Theta(\bar Z;\bar H,W)$. The most complex part is $\Pi_+'(\bar H_{\alpha_0\alpha_0};W_{\alpha_0\alpha_0})$. With κ_{P} set as in (52), $\lambda_{\min}(\kappa_{\mathsf{P}}\cdot[X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}})=\lambda_{\max}(2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{P}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{P}}})$. Thus, $\kappa_{\mathsf{P}}\cdot[X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}}-2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{P}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{P}}}\succeq 0$. Symmetrically, $\kappa_{\mathsf{D}}\cdot[\sigma S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}}+2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{D}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{D}}}\succeq 0$. Define

$$\begin{aligned}\widehat W&:=(Q^0)^{\mathsf{T}}W_{\alpha_0\alpha_0}Q^0\\ &=(Q^0)^{\mathsf{T}}\left[\begin{array}{c|c}\kappa_{\mathsf{P}}\cdot[X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}}-2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{P}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{P}}} & 0\\ \hline \sim & -\kappa_{\mathsf{D}}\cdot[\sigma S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}}-2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\bar H_{\alpha_0^{\mathsf{D}}\alpha_c}\bar H_{\alpha_c\alpha_0^{\mathsf{D}}}\end{array}\right]Q^0,\end{aligned}$$

with the block-diagonal Q^0 defined in Lemma 4. Thus, $\widehat W_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}}\succeq 0$, $\widehat W_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\preceq 0$, and $\widehat W_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{D}}}=0$. Consequently,

$$\Pi_+(\widehat W_{\beta_{0,0}\beta_{0,0}})=\left[\begin{array}{c|c}\widehat W_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} & 0\\ \hline \sim & 0\end{array}\right],$$

because $\widehat W_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}}$ (resp. $\widehat W_{\beta_{0,0}^{\mathsf{D}}\beta_{0,0}^{\mathsf{D}}}$) is a principle submatrix of $\widehat W_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}}$ (resp. $\widehat W_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}$). Now,

$$\begin{aligned}&\Pi_+'(\bar H_{\alpha_0\alpha_0};W_{\alpha_0\alpha_0})=\\ &Q^0\left[\begin{array}{c:c|c:c}\widehat W_{\beta_{0,+}\beta_{0,+}} & \widehat W_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & \widehat W_{\beta_{0,+}\beta_{0,0}^{\mathsf{D}}} & \left\{\frac{\eta_{0,i}}{\eta_{0,i}-\eta_{0,j}}\widehat W_{\beta_{0,i}\beta_{0,j}}\right\}_{\substack{i\in\mathcal{I}_{0,+}\\ j\in\mathcal{I}_{0,-}}}\\ \hdashline \sim & [\Pi_+(\widehat W_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{P}}\gamma_{0,0,\mathsf{P}}} & [\Pi_+(\widehat W_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{P}}\gamma_{0,0,\mathsf{D}}} & 0\\ \hline \sim & \sim & [\Pi_+(\widehat W_{\beta_{0,0}\beta_{0,0}})]_{\gamma_{0,0,\mathsf{D}}\gamma_{0,0,\mathsf{D}}} & 0\\ \hdashline \sim & \sim & \sim & 0\end{array}\right](Q^0)^{\mathsf{T}}\\ &=\left[\begin{array}{c|c}Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline 0 & Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\end{array}\right]\left[\begin{array}{c:c|c:c}\widehat W_{\beta_{0,+}\beta_{0,+}} & \widehat W_{\beta_{0,+}\beta_{0,0}^{\mathsf{P}}} & 0 & 0\\ \hdashline \sim & \widehat W_{\beta_{0,0}^{\mathsf{P}}\beta_{0,0}^{\mathsf{P}}} & 0 & 0\\ \hline \sim & \sim & 0 & 0\\ \hdashline \sim & \sim & \sim & 0\end{array}\right]\left[\begin{array}{c|c}Q^0_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0\\ \hline 0 & Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}\end{array}\right]^{\mathsf{T}}=\left[\begin{array}{c|c}W_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0\\ \hline \sim & 0\end{array}\right].\end{aligned}$$

Therefore, from (32),

$$\Theta(\bar{Z};\bar{H},W) = \left[\begin{array}{cc|cc} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \sim & W_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{array}\right] = \left[\begin{array}{c|c} W_{\alpha_{\mathsf{P}}\alpha_{\mathsf{P}}} & 0 \\ \sim & 0 \end{array}\right].$$

Combining $\mathcal{E}(\bar{Z};\bar{H})$ and $\Theta(\bar{Z};\bar{H},W)$:

$$\Theta(\bar{Z};\bar{H},W) + \mathcal{E}(\bar{Z};\bar{H}) = \left[\begin{array}{cc|cc} \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}} - \bar{X}]_{\alpha_+\alpha_+} & \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}}]_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \sim & \kappa_{\mathsf{P}} \cdot [X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{array}\right] = \kappa_{\mathsf{P}} \cdot (X_{\mathsf{sc}} - \bar{X}).$$

By X_{sc} and $\bar{X}$'s definition, $\mathcal{P}X_{\mathsf{sc}} = \mathcal{P}\bar{X} = \mathcal{P}\tilde{X}$. Thus,

$$\mathcal{P}(\Theta(\bar{Z};\bar{H},W) + \mathcal{E}(\bar{Z};\bar{H})) = \kappa_{\mathsf{P}} \cdot \mathcal{P}(X_{\mathsf{sc}} - \bar{X}) = 0.$$

(ii) Dual part. The same as primal part's procedure, we can show that

$$\mathcal{E}^{\perp}(\bar{Z};\bar{H}) = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & 0 \\ \hline \sim & \sim & 2\sum_{c\in\mathcal{I}_-} \frac{1}{\mu_c} \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_c} \bar{H}_{\alpha_c\alpha_0^{\mathsf{P}}} & 0 \\ \sim & \sim & \sim & 0 \end{array}\right] \text{ and } \Theta^{\perp}(\bar{Z};\bar{H},W) = \left[\begin{array}{c|c} 0 & 0 \\ \sim & W_{\alpha_{\mathsf{D}}\alpha_{\mathsf{D}}} \end{array}\right].$$

With the fact that $\mathcal{P}^{\perp}S_{\mathsf{sc}} = \mathcal{P}^{\perp}\bar{S} = \mathcal{P}^{\perp}C$:

$$\mathcal{P}^{\perp}(\Theta^{\perp}(\bar{Z};\bar{H},W) + \mathcal{E}^{\perp}(\bar{Z};\bar{H})) = -\kappa_{\mathsf{D}} \cdot \mathcal{P}^{\perp}(\sigma S_{\mathsf{sc}} - \sigma\bar{S}) = 0.$$

(iii) Combining the primal and dual part:

$$\Psi(\bar{Z};\bar{H}) = -\mathcal{P}\mathcal{E}(\bar{Z};\bar{H}) - \mathcal{P}^{\perp}\mathcal{E}^{\perp}(\bar{Z};\bar{H}) = \mathcal{P}\Theta(\bar{Z};\bar{H},W) + \mathcal{P}^{\perp}\Theta^{\perp}(\bar{Z};\bar{H},W),$$

which directly implies $\Psi(\bar{Z};\bar{H}) \in \mathcal{K}(\bar{Z};\bar{H})$ and $\phi(\bar{Z};\bar{H}) = 0$. □

5.3 Discussion: Small yet Non-Zero $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$

From Proposition 5, as long as $\bar{Z} \in \mathcal{Z}_\star$ and $Z^{(k)} - \bar{Z} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, $\phi(\bar{Z}; Z^{(k)} - \bar{Z})$ is guaranteed to be non-zero. In this case, the higher-order term in the second-order local limiting dynamics (42) can be (transiently) omitted, and $\Delta Z^{(k)} \approx \frac{1}{2}\phi(\bar{Z}; Z^{(k)} - \bar{Z}) \sim o(\|Z^{(k)} - \bar{Z}\|_{\mathsf{F}})$. The approximation becomes more and more accurate as $Z - \bar{Z} \to 0$. In this case,

$$\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)}) \approx \angle(\phi(\bar{Z}; Z^{(k)} - \bar{Z}), \phi(\bar{Z}; Z^{(k)} - \bar{Z} + \Delta Z^{(k)})).$$

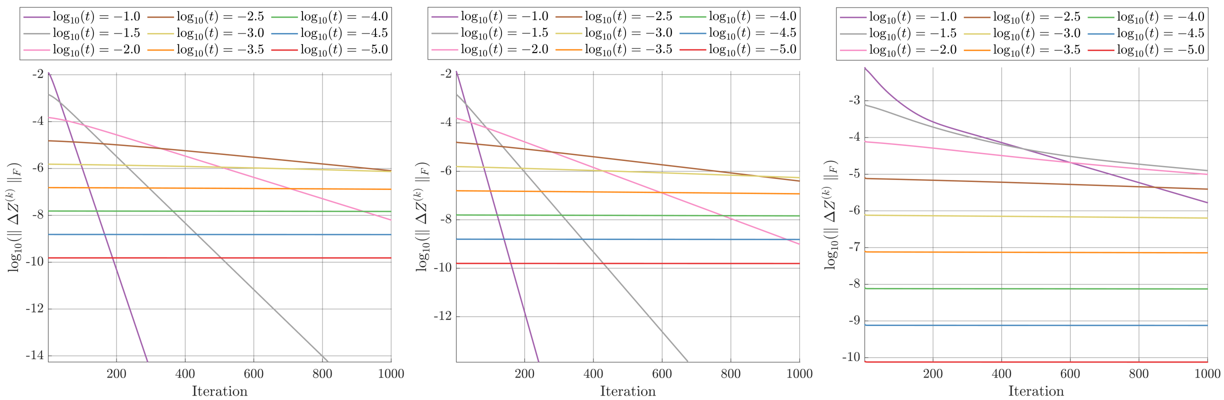
Therefore, as long as $\phi(\bar{Z};\cdot)$ can exhibit certain type of continuity at $Z^{(k)} - \bar{Z}$ (and the "almost-sure" type continuity will be established in §7.2), one could expect $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)}) \to 0$ as $Z^{(k)} - \bar{Z} \to 0$. The "small yet non-zero" effect may be due to the presence of higher-order terms. We empirically verify our analysis with three toy examples defined in §9. Across all examples, we fix σ to 1 and the tolerance for $r_{\max}$ is set to 10^{-14}. The maximum ADMM iteration number is 1000. $\bar{H} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ is picked as:

1. For (SDP-I), $a = 1$ and $b = 1$ in (57). The corresponding $\phi(\bar{Z};\bar{H})$ is defined in (58).
2. For (SDP-II), $\bar{H}_{12} = 1, \bar{H}_{22} = 1, \bar{H}_{23} = 1$ in (59). The corresponding $\phi(\bar{Z};\bar{H})$ is defined in (60).
3. For (SDP-III), $h = 1$ and $\epsilon = 0$ in (61). The corresponding $\phi(\bar{Z};\bar{H})$ is defined in (62).

We check four quantities:

$$\|\Delta Z^{(k)}\|_{\mathsf{F}}, \quad \angle(\Delta Z^{(k)}, \Delta Z^{(k+1)}), \quad \frac{\|0.5\phi(\bar{Z}; Z^{(0)} - \bar{Z}) - \Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}}, \quad \frac{\|0.5\phi(\bar{Z}; Z^{(k)} - \bar{Z}) - \Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}}.$$

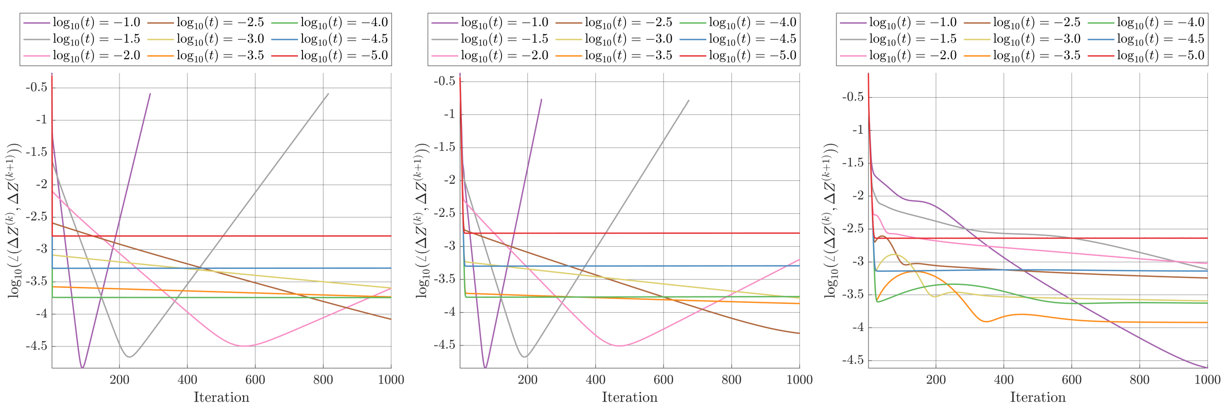
Discussion on $\|\Delta Z^{(k)}\|_{\mathsf{F}}$. The results are shown in Figure 4. When t is relatively large (e.g., $\log_{10}(t) > -2$), ADMM still exhibits an observable linear convergence rate. However, as $t \downarrow 0$, this rate decays to 0. On the other hand, when t is sufficiently small (e.g., $\log_{10}(t) < -3.5$), the second-order term starts to dominate the dynamics, and $\Delta Z^{(k)}$ transiently converges to $\frac{t^2}{2}\phi(\bar{Z}; \bar{H})$. This quadratic relationship is evident in Figure 4: as $\log_{10}(t)$ decreases by 1, the transiently convergent $\log_{10}(\|\Delta Z^{(k)}\|_{\mathsf{F}})$ decreases by approximately 2 across all three toy examples.



(SDP-I) (SDP-II) (SDP-III)

Figure 4: $\log_{10}(\|\Delta Z^{(k)}\|_{\mathsf{F}})$ in three toy examples. In each example, the initialization is chosen as $Z^{(0)} = \bar{Z} + t\bar{H}$, where $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, and we sweep t from 10^{-1} to 10^{-5}. σ is fixed to 1.

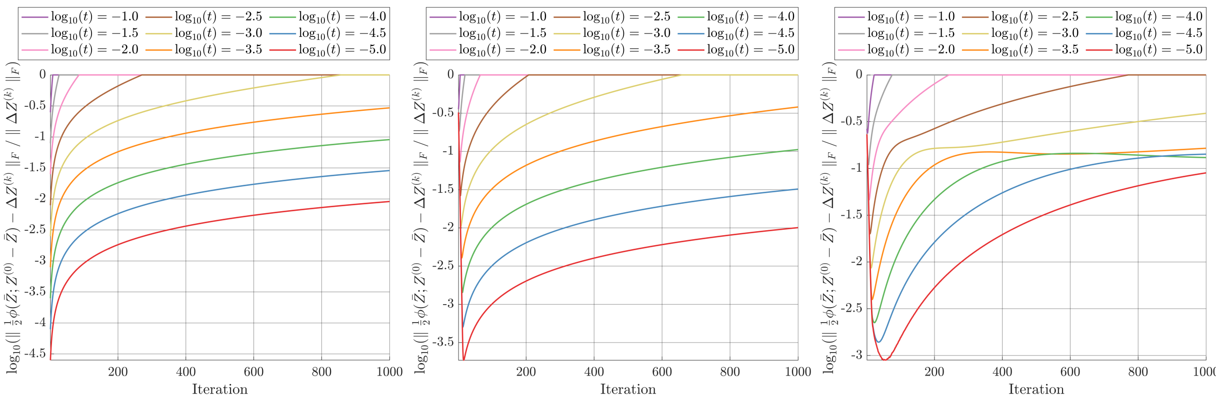
Discussion on $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$. The results are shown in Figure 5. As $t \downarrow 0$, the transiently convergent $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ tends to become smaller. One noticeable feature is that the convergent angle does not appear to decrease monotonically: in all examples, as t decreases from 10^{-3} to 10^{-4}, the transiently convergent angle actually increases. This behavior may be caused by numerical issues when computing angles between two extremely small vectors in double precision.



(SDP-I) (SDP-II) (SDP-III)

Figure 5: $\log_{10}(\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)}))$ in three toy examples. In each example, the initialization is chosen as $Z^{(0)} = \bar{Z} + t\bar{H}$, where $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, and we sweep t from 10^{-1} to 10^{-5}. σ is fixed to 1.

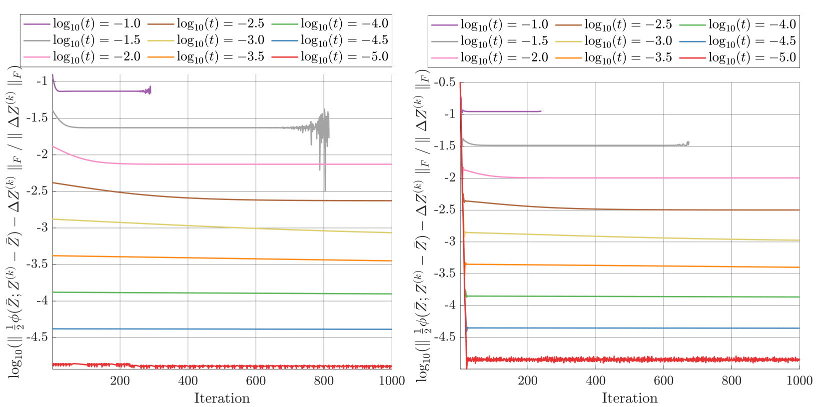
Discussion on $\frac{\|0.5\phi(\bar{Z};Z^{(0)}-\bar{Z})-\Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}}$. The results are shown in Figure 6. As $t \downarrow 0$, $\Delta Z^{(k)}$ first transiently converges to $0.5\,\phi(\bar{Z};Z^{(0)}-\bar{Z}) = \frac{t^2}{2}\phi(\bar{Z};\bar{H})$, and then gradually deviates from it. This deviation is caused by the change of $Z^{(k)}$ discussed in §4.4.



(SDP-I) (SDP-II) (SDP-III)

Figure 6: $\log_{10}(\frac{\|0.5\,\phi(\bar{Z};Z^{(0)}-\bar{Z})-\Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}})$ in the three toy examples. For visualization, we upper-clamp the values at 1. In each example, the initialization is chosen as $Z^{(0)} = \bar{Z} + t\bar{H}$, where $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, and we sweep t from 10^{-1} to 10^{-5}. σ is fixed to 1.

Discussion on $\frac{\|0.5\phi(\bar{Z};Z^{(k)}-\bar{Z})-\Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}}$. The results are shown in Figure 7. Since the complete description of $\mathcal{C}(\bar{Z})$ and the corresponding $\phi(\bar{Z};\cdot)$ is hard to obtain in (SDP-III), we report only the results for (SDP-I) and (SDP-II). Unlike Figure 6, the second-order limiting predictor $0.5\,\phi(\bar{Z};Z^{(k)}-\bar{Z})$ stably tracks $\Delta Z^{(k)}$ in both examples. Interestingly, as $\log_{10}(t)$ decreases by 1, the log of relative tracking error also decreases by 1.



(SDP-I) (SDP-II)

Figure 7: $\log_{10}(\frac{\|0.5\,\phi(\bar{Z};Z^{(k)}-\bar{Z})-\Delta Z^{(k)}\|_{\mathsf{F}}}{\|\Delta Z^{(k)}\|_{\mathsf{F}}})$ in the first two toy examples. In each example, the initialization is chosen as $Z^{(0)} = \bar{Z} + t\bar{H}$, where $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, and we sweep t from 10^{-1} to 10^{-5}. σ is fixed to 1.

6 Range of $\phi(\bar{Z};\cdot)$

The second property of $\phi(\bar{Z};\cdot)$ that we study is its range, i.e., $\operatorname{ran}(\phi(\bar{Z};\cdot))$. For the one-step ADMM iteration (4), $\phi(\bar{Z};\cdot)$ can be interpreted as a second-order local "steady-state response" of $\Delta Z^{(k)}$ from any initialization $Z^{(0)} = Z$ satisfying $Z - \bar{Z} \in \mathcal{C}(\bar{Z})$ and $Z \to \bar{Z}$, after filtering out all transient directions. It is therefore natural to expect that $\operatorname{ran}(\phi(\bar{Z};\cdot))$ lies in a subset whose dimension is much lower than that of the ambient space $\mathbb{S}^n$. We are particularly interested in how $\operatorname{ran}(\phi(\bar{Z};\cdot))$ relates to $\mathcal{C}(\bar{Z})$ (up to an affine hull). For example, if one could establish that $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \mathcal{C}(\bar{Z})$ under suitable conditions, then it would follow immediately that $\bar{Z} + \mathcal{C}(\bar{Z})$ is an invariant set for the local second-order limiting dynamics (42) when the higher-order term $o(\|Z^{(k)} - \bar{Z}\|_{\mathsf{F}}^2)$ is neglected.

In §6.1, we present a negative result: under Assumption 1 alone, one cannot even guarantee $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$. In §6.2, however, we show that the inclusion $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$ does hold once uniqueness of either the primal or the dual optimal solution is imposed. Finally, in §6.3, we discuss almost-invariant sets around $\bar{Z}$, leveraging the structure of $\operatorname{ran}(\phi(\bar{Z};\cdot))$.

6.1 General Case: $\operatorname{ran}(\phi(\bar{Z};\cdot)) \nsubseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$

From (SDP-I) and (SDP-II), one may conjecture that $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \mathcal{C}(\bar{Z})$, or at least $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$. However, this is not true in general.

Proposition 6 **($\operatorname{ran}(\phi(\bar{Z};\cdot)) \nsubseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$ in general cases).** *There exists an SDP data triplet $(\mathcal{A}, b, C)$ satisfying Assumption 1, equipped with $\bar{Z} \in \mathcal{Z}_\star$, such that $\operatorname{ran}(\phi(\bar{Z};\cdot)) \nsubseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$.*

Proof. From Proposition 1 (2), $H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{D}}} = 0$ for all $H \in \mathcal{C}(\bar{Z})$. Thus, as long as we can construct an SDP satisfying Assumption 1, such that there exists $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$ with $\phi(\bar{Z};\bar{H})_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{D}}} \neq 0$, the claim holds. Please see (SDP-III) for a concrete construction. Specifically, consider $\bar{H} = \bar{H}(h, 0)$ in (61) with $h > \sqrt{2}$. Then, from (63):

$$
\phi(\bar{Z};\bar{H}) = \left[\begin{array}{c:cccc:c}
-\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} & 0 & 0 & -\frac{h}{3} & 0 \\
\hdashline
\sim & \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 & -\frac{2}{3\sigma} \\
 & \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} & 0 \\
 & \sim & \sim & 0 & -\frac{h}{3} & 0 \\
 & \sim & \sim & \sim & -\frac{h^2}{6} & 0 \\
\hdashline
\sim & & \sim & & & \frac{h^2}{6}
\end{array}\right].
$$

Clearly, $\phi(\bar{Z};\bar{H})_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{D}}} \neq 0$. $\square$

6.2 Under One-Side Uniqueness: $\operatorname{ran}(\phi(\bar{Z};\cdot)) \subseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$

Although in general $\operatorname{ran}(\phi(\bar{Z};\cdot)) \nsubseteq \operatorname{aff}(\mathcal{C}(\bar{Z}))$, we will show that this inclusion does hold once additional conditions are imposed. Before proceeding, we first state a few elementary lemmas on relative interiors and affine hulls.

Lemma 8. *Given a finite-dimensional Hilbert space $\mathcal{H}$. Assume M is an affine set and $S \subseteq M$. If there exists $\bar{x} \in M$ and $\epsilon > 0$, such that $(\bar{x} + \epsilon\mathbb{B}) \cap M \subseteq S$, then $\operatorname{aff}(S) = M$.*

Proof. Clearly, $\operatorname{aff}(S) \subseteq M$.

For the other direction, denote M as $\bar{x} + L$, where $L = M - M$ is the linear subspace parallel to M. Then, by assumption, $((\bar{x} + \epsilon\mathbb{B}) \cap M) - \bar{x} = \epsilon\mathbb{B} \cap L \subseteq S - \bar{x}$. But $\operatorname{aff}(\epsilon\mathbb{B} \cap L) = L$: picking any $v \in L$ and $t \in \mathbb{R}$ large enough, we get $v/t \in \epsilon\mathbb{B}$. Thus, $v/t \in \epsilon\mathbb{B} \cap L$ and $v = t \cdot v/t \in \operatorname{span}(\epsilon\mathbb{B} \cap L)$. Since $0 \in \epsilon\mathbb{B} \cap L$, we get $\operatorname{span}(\epsilon\mathbb{B} \cap L) = \operatorname{aff}(\epsilon\mathbb{B} \cap L)$. By the minimality of the affine hull, $L = \operatorname{aff}(\epsilon\mathbb{B} \cap L) \subseteq \operatorname{aff}(S - \bar{x})$. The proof is closed by showing $\operatorname{aff}(S) = \bar{x} + \operatorname{span}(S - \bar{x}) \supseteq \bar{x} + L = M$. $\square$

Lemma 9. *Given a finite-dimensional Hilbert space $\mathcal{H}$ and two convex sets $C_1, C_2 \subset \mathcal{H}$. If $\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) \neq \emptyset$, then $\mathrm{aff}(C_1) \cap \mathrm{aff}(C_2) = \mathrm{aff}(C_1 \cap C_2)$.*

Proof. For ease of notation, set $M_1 := \mathrm{aff}(C_1), M_2 = \mathrm{aff}(C_2), M = M_1 \cap M_2$. Take any $\bar{x} \in \mathrm{ri}(C_1) \cap \mathrm{ri}(C_2)$. By definition, $\exists \epsilon_1, \epsilon_2 > 0$, s.t. $(\bar{x} + \epsilon_1 \mathbb{B}) \cap M_1 \subseteq C_1$ and $(\bar{x} + \epsilon_2 \mathbb{B}) \cap M_2 \subseteq C_2$. Set $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, we get

$$(\bar{x} + \epsilon \mathbb{B}) \cap M = ((\bar{x} + \epsilon \mathbb{B}) \cap M_1) \cap ((\bar{x} + \epsilon \mathbb{B}) \cap M_2) \subseteq C_1 \cap C_2$$

Thus, by $(C_1 \cap C_2) \subseteq M$ and Lemma 8, we get $\mathrm{aff}(C_1 \cap C_2) = M$. □

Lemma 10. *Given a finite-dimensional Hilbert space $\mathcal{H}$ and two sets $C_1, C_2 \subset \mathcal{H}$. $\mathrm{aff}(C_1 + C_2) = \mathrm{aff}(C_1) + \mathrm{aff}(C_2)$.*

Proof. The "$\subseteq$" part. Since $C_1 \subseteq \mathrm{aff}(C_1), C_2 \subseteq \mathrm{aff}(C_2)$, we have $C_1 + C_2 = \mathrm{aff}(C_1) + \mathrm{aff}(C_2)$. Thus, $\mathrm{aff}(C_1 + C_2) \subseteq \mathrm{aff}(C_1) + \mathrm{aff}(C_2)$.

The "$\supseteq$" part. Take any $u \in \mathrm{aff}(C_1)$ and $v \in \mathrm{aff}(C_2)$. By affine hull's definition, there exist $x_i \in C_1$ and $\sum_i \alpha_i = 1$, s.t. $u = \sum_i \alpha_i x_i$. Similarly, there exist $y_j \in C_2$ and $\sum_j \beta_j = 1$, s.t. $v = \sum_j \beta_j y_j$. Thus,

$$u + v = \sum_i \alpha_i x_i + \sum_j \beta_j y_j = \sum_{i,j} (\alpha_i \beta_j)(x_i + y_j)$$

Observing that $\sum_{i,j} \alpha_i \beta_j = 1$ and $x_i + y_j \in C_1 + C_2$, we get $u + v \in \mathrm{aff}(C_1 + C_2)$. □

Proposition 7 ($\mathrm{ran}(\phi(\bar{Z}; \cdot)) \subseteq \mathrm{aff}(\mathcal{C}(\bar{Z}))$ under one-side uniqueness)**.** *Under Assumption 1, if either the primal or the dual optimal solution is unique, then $\mathrm{ran}(\phi_{\mathsf{P}}(\bar{Z}; \cdot)) \subseteq \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))$ and $\mathrm{ran}(\phi_{\mathsf{D}}(\bar{Z}; \cdot)) \subseteq \mathrm{aff}(\mathcal{C}_{\mathsf{D}}(\bar{Z}))$. Consequently, $\mathrm{ran}(\phi(\bar{Z}; \cdot)) \subseteq \mathrm{aff}(\mathcal{C}(\bar{Z}))$.*

Proof. We only prove for the case when the primal solution is unique. The dual solution unique case can be proven symmetrically. Since the primal optimal solution is unique, $X_{\mathsf{sc}} = \bar{X}$. Thus, picking any $\bar{H} \in \mathcal{C}(\bar{Z})$, we have $\bar{H}_{\alpha_0^{\mathsf{P}} \alpha_0^{\mathsf{P}}} = 0, \bar{H}_{\alpha_0^{\mathsf{P}} \alpha_0^{\mathsf{D}}} = 0, \bar{H}_{\alpha_0^{\mathsf{P}} \alpha_-} = 0$. Thus, Q^0 is degraded to $Q^0_{\beta_{0,\mathsf{D}} \beta_{0,\mathsf{D}}}$. The cones in (26) are degraded to:

$$\mathcal{C}_{\mathsf{P}}(\bar{Z}) = \left\{ H = \left[\begin{array}{c:c:c} H_{\alpha_+ \alpha_+} & H_{\alpha_+ \alpha_0^{\mathsf{P}}} & 0 \\ \hdashline \sim & 0 & 0 \\ \hdashline \sim & \sim & 0 \end{array} \right] \;\middle|\; \mathcal{P} H = 0 \right\},$$

$$\mathcal{C}_{\mathsf{D}}(\bar{Z}) = \left\{ H = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & H_{\alpha_0^{\mathsf{D}} \alpha_0^{\mathsf{D}}} & H_{\alpha_0^{\mathsf{D}} \alpha_-} \\ \hdashline \sim & \sim & H_{\alpha_- \alpha_-} \end{array} \right] \;\middle|\; \begin{array}{l} \mathcal{P}^\perp H = 0, \\ H_{\alpha_0^{\mathsf{P}} \alpha_0^{\mathsf{P}}} \preceq 0 \end{array} \right\} = \underbrace{\left\{ H \;\middle|\; \mathcal{P}^\perp H = 0 \right\}}_{=:\mathcal{C}_1} \cap \underbrace{\left\{ H = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & H_{\alpha_0^{\mathsf{D}} \alpha_0^{\mathsf{D}}} \preceq 0 & H_{\alpha_0^{\mathsf{D}} \alpha_-} \\ \hdashline \sim & \sim & H_{\alpha_- \alpha_-} \end{array} \right] \right\}}_{=:\mathcal{C}_2}.$$

Since $\mathcal{C}_{\mathsf{P}}(\bar{Z})$ is already affine, $\mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z})) = \mathcal{C}_{\mathsf{P}}(\bar{Z})$. For $\mathcal{C}_{\mathsf{D}}(\bar{Z}) = \mathcal{C}_1 \cap \mathcal{C}_2$, we shall prove that

$$-S_{\mathsf{sc}} + \bar{S} \in \mathrm{ri}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2).$$

To see this: since $\mathcal{P}^\perp S_{\mathsf{sc}} = \mathcal{P}^\perp \bar{S} = \mathcal{P}^\perp C$, then $-S_{\mathsf{sc}} + \bar{S} \in \mathcal{C}_1 = \mathrm{ri}(\mathcal{C}_1)$; since $[-S_{\mathsf{sc}} + \bar{S}]_{\alpha_0^{\mathsf{D}} \alpha_0^{\mathsf{D}}} = -[S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}} \alpha_0^{\mathsf{D}}} \prec 0$, then $-S_{\mathsf{sc}} + \bar{S} \in \mathrm{ri}(\mathcal{C}_2)$. Therefore, invoking Lemma 9:

$$\mathrm{aff}(\mathcal{C}_{\mathsf{D}}(\bar{Z})) = \mathrm{aff}(\mathcal{C}_1) \cap \mathrm{aff}(\mathcal{C}_2) = \left\{ H \;\middle|\; \mathcal{P}^\perp H = 0 \right\} \cap \left\{ H = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline \sim & H_{\alpha_0^{\mathsf{D}} \alpha_0^{\mathsf{D}}} & H_{\alpha_0^{\mathsf{D}} \alpha_-} \\ \hdashline \sim & \sim & H_{\alpha_- \alpha_-} \end{array} \right] \right\}.$$

On the other hand, the cones in (43) are degraded to:

$$\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z};\bar{H}) = \left\{ W = \begin{bmatrix} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 0 \end{bmatrix} \;\middle|\; \mathcal{P}W = 0 \right\},$$

$$\mathcal{K}_{\mathsf{D}}^{\circ}(\bar{Z};\bar{H}) = \left\{ W = \begin{bmatrix} 0 & 0 & 0 \\ \sim & Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}} \begin{bmatrix} \widehat{W}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} & \widehat{W}_{\beta^{\mathsf{D}}_{0,0}\beta_{0,-}} \\ \sim & \widehat{W}_{\beta_{0,-}\beta_{0,-}} \end{bmatrix} (Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}})^{\mathsf{T}} & W_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & W_{\alpha_-\alpha_-} \end{bmatrix} \;\middle|\; \begin{array}{c} \mathcal{P}^{\perp}W = 0, \\ \widehat{W} = (Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}})^{\mathsf{T}} W_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} Q^0_{\beta_{0,\mathsf{D}}\beta_{0,\mathsf{D}}}, \\ \widehat{W}_{\beta^{\mathsf{D}}_{0,0}\beta^{\mathsf{D}}_{0,0}} \preceq 0 \end{array} \right\}.$$

Thus,

$$\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z};\bar{H}) = \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z})), \quad \mathcal{K}_{\mathsf{D}}^{\circ}(\bar{Z};\bar{H}) \subseteq \mathrm{aff}(\mathcal{C}_{\mathsf{D}}(\bar{Z})).$$

By Theorem 7, $\phi_{\mathsf{P}}(\bar{Z};\bar{H}) \in \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))$ and $-\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) \in \mathrm{aff}(\mathcal{C}_{\mathsf{D}}(\bar{Z}))$. Thus,

$$\phi(\bar{Z};\bar{H}) = \phi_{\mathsf{P}}(\bar{Z};\bar{H}) - \sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) \in \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z})) + \mathrm{aff}(\mathcal{C}_{\mathsf{D}}(\bar{Z})) = \mathrm{aff}(\mathcal{C}(\bar{Z})).$$

by Lemma 10. □

It remains unclear to us under what conditions the stronger inclusion $\mathrm{ran}(\phi(\bar{Z};\cdot)) \subseteq \mathcal{C}(\bar{Z})$ holds.

Remark 1. *In [13], one-side uniqueness of optimal solution in addition with Assumption 1 is called the* simplicity *condition.*

6.3 Discussion: Connections to Almost Invariant Sets

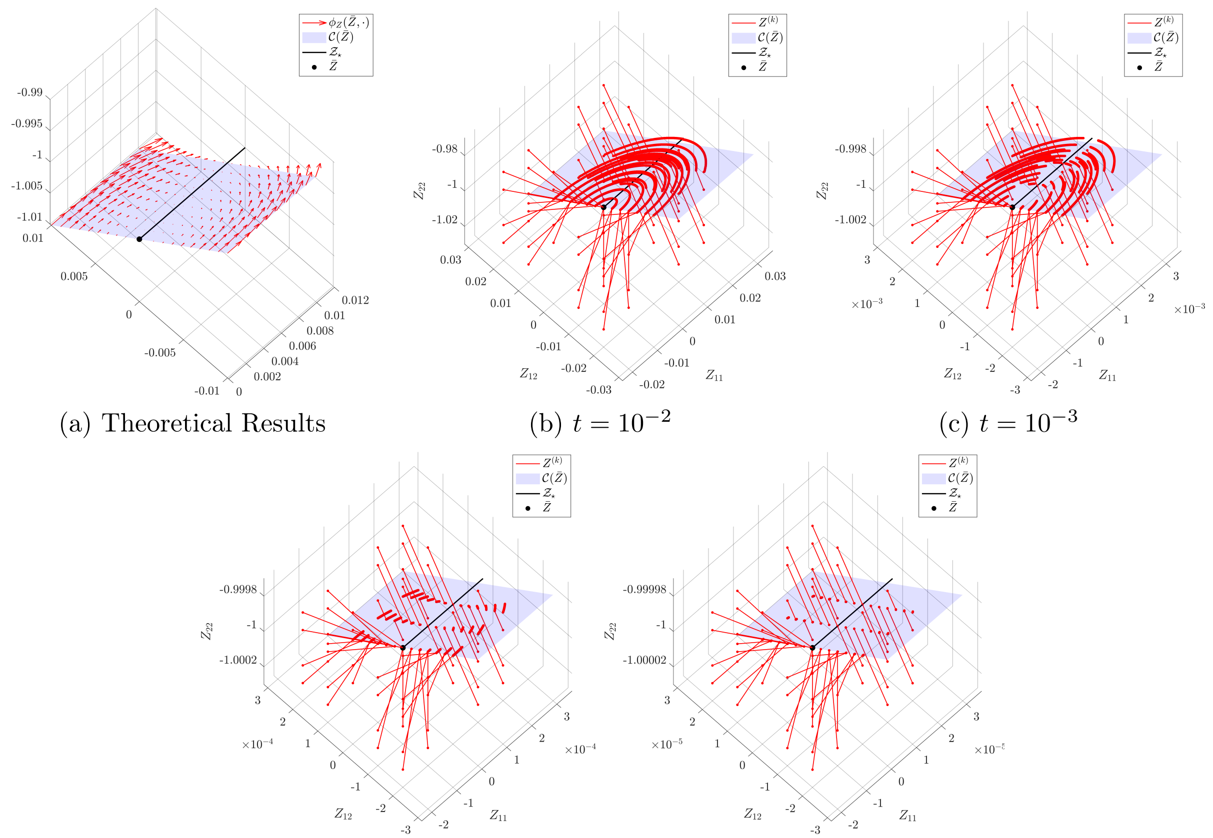
Proposition 7 indicates that, under the local second-order limiting dynamics (42), $\Delta Z^{(k)}$ lies in $\mathrm{aff}(\mathcal{C}(\bar{Z}))$ whenever $Z^{(k)} \in \mathcal{C}(\bar{Z})$ (up to higher-order terms), provided the additional one-sided uniqueness condition holds. Indeed, in both (SDP-I) and (SDP-II), the stronger inclusion $\mathrm{ran}(\phi(\bar{Z};\cdot)) \subseteq \mathcal{C}(\bar{Z})$ holds. One can readily verify that dual uniqueness holds in both examples.

On the other hand, in the nonlinear dynamics literature there is the notion of an *almost invariant set* [11]. Informally, an almost invariant set is a region of the state space that trajectories tend to remain in for a long time, with only a small probability (or small "leakage") of leaving over a prescribed time horizon. This raises the question of whether $\mathcal{C}(\bar{Z}) \cap \mathbb{B}_r(\bar{Z})$, for some fixed small $r > 0$, can serve as a *local* almost invariant set for the one-step ADMM dynamics (4). This question is difficult to answer in general, because two competing forces must be balanced: (i) the local first-order dynamics (23) tends to drive $Z^{(k)}$ toward $\mathcal{C}(\bar{Z})$ (Lemma 1); (ii) the local second-order dynamics (24) may drive $Z^{(k)}$ outside $\mathcal{C}(\bar{Z})$, as suggested by Proposition 6.

A simple visualization. We illustrate the two-level effects using (SDP-I). The results are shown in Figure 8. Figure 8(a) depicts the vector field induced by $\phi(\bar{Z};\cdot)$. Figures 8(b)–(e) show one-step ADMM trajectories initialized at $Z^{(0)} = \bar{Z} + tH$ for different choices of t and H. Across all experiments, we fix $\sigma = 1$ and set the maximum number of iterations to 1000. We make three empirical observations: (i) Starting from any initialization, $Z^{(k)}$ collapses to $\mathcal{C}(\bar{Z})$ in a single ADMM step, regardless of the choice of t. (ii) As $t \to 0$, the decrease in $Z^{(k)}$ is much faster than $Z^{(k)} - \bar{Z}$, which remains of order $O(t)$. (iii) The trajectories of $Z^{(k)}$ closely resemble the theoretical vector field in (a), regardless of the choice of t. Taken together, these observations suggest that $\mathcal{C}(\bar{Z})$ in (SDP-I) is very likely to be an almost invariant set.

7 Continuity of $\phi(\bar{Z};\cdot)$

The third property of $\phi(\bar{Z};\cdot)$ that we study is its continuity on $\mathcal{C}(\bar{Z})$. Perhaps surprisingly, although the residual mapping of the one-step ADMM update (20) is continuous on the entire ambient space $\mathbb{S}^n$, the



(a) Theoretical Results (b) $t = 10^{-2}$ (c) $t = 10^{-3}$

(d) $t = 10^{-4}$ (e) $t = 10^{-5}$

Figure 8: (a) The theoretical vector field $\phi(\bar{Z}; H)$ in (SDP-I), where $H \in \mathcal{C}(\bar{Z})$. (b)–(e) In (SDP-I), trajectories of $Z^{(k)}$ from different initializations $Z^{(0)}$ with varying t and first-order perturbations H. We sweep t from 10^{-2} to 10^{-5}. For each fixed t, (H_{11}, H_{12}, H_{22}) is sampled from $\{-2, -1, 1, 2\}^3$, yielding 64 initial points in total.

induced second-order limiting mapping can be discontinuous. Indeed, as defined in (43), the cone-valued mapping $\mathcal{K}^\circ(\bar{Z}; \cdot)$ may lose inner semicontinuity at a point $\bar{H}$ satisfying $\det(\bar{H}_{\alpha_0\alpha_0}) = 0$, which provides a potential source of discontinuity for $\phi(\bar{Z}; \cdot)$. We construct an explicit example in Proposition 8 (§7.1). On the positive side, we show that the set of discontinuity points of $\phi(\bar{Z}; \cdot)$ has Lebesgue measure zero on $\operatorname{aff}(\mathcal{C}(\bar{Z}))$ (cf. Proposition 9 in §7.2). Moreover, except for the trivial case $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the set $\mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$—where $\phi(\bar{Z}; \cdot)$ is nonzero (cf. Proposition 5)—has infinite Lebesgue measure (cf. Proposition 10 in §7.2). Together, these results establish an "almost-sure" type continuity of $\phi(\bar{Z}; \cdot)$ on $\mathcal{C}(\bar{Z})$.

In §7.3, we discuss a subtle phenomenon in slow-convergence regions. For most iterations, the angle $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ tends to be small and varies smoothly, as described in §5.3. Occasionally, however, $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ can spike to a large value (often close to $\frac{\pi}{2}$) before quickly returning to a small value. We use the almost-sure continuity of $\phi(\bar{Z}; \cdot)$ to explain these "sparse spikes" in the slow-convergence regime. For small-scale SDP instances, our surrogate limiting model (42) can even accurately predict such microscopic phase transitions.

7.1 Existence of Discontinuity

Proposition 8 (Discontinuity in $\phi(\bar{Z};\cdot)$)**.** *There exists an SDP data triplet $(\mathcal{A}, b, C)$ satisfying Assumption 1 with $\bar{Z} \in \mathcal{Z}_\star$, $\{H^i\}_{i=1}^{\infty} \in \ker(\delta'_{\bar{Z}})$, and $\bar{H} \in \ker(\delta'_{\bar{Z}})$, s.t.*

$$\lim_{i\to\infty} H^i = \bar{H}, \text{ yet } \lim_{i\to\infty} \phi(\bar{Z}, H^i) \neq \phi(\bar{Z}; \bar{H}).$$

Proof. Please see (SDP-III) for a constructive example. Concretely, under the SDP data provided by (SDP-III), we choose a real sequence $\epsilon_i \to 0$ as $i \to \infty$. For $\bar{H}(h,\epsilon)$ in (61), define $H^i := \bar{H}(h,\epsilon_i)$, $\bar{H} := \bar{H}(h,0)$. As long as $\epsilon \geq 0$, $\{H^i\}_{i=1}^{\infty}$ and $\bar{H}$ all belong to $\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$. On the other hand, from (62) and (63), as long as $h > \sqrt{2}$, we have

$$\lim_{i\to\infty} \phi(\bar{Z}; H^i) = \lim_{i\to\infty} \left[\begin{array}{c:c:c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} \quad 0 \quad -\frac{\epsilon_i}{3} \quad -\frac{h}{3} & 0 \\ \hdashline \sim & \left[\begin{array}{c:c|c:c} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \hdashline \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} \\ \hline \sim & \sim & \frac{h^2-2}{9} & -\frac{h}{3} \\ \hdashline \sim & \sim & \sim & -\frac{2h^2-1}{9} \end{array}\right] & \begin{array}{c} -\frac{2}{3\sigma} \\ 0 \\ \frac{2\epsilon_i}{3\sigma} \\ 0 \end{array} \\ \hdashline \sim & \sim & \frac{h^2+1}{9} \end{array}\right] = \left[\begin{array}{c:c:c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} \quad 0 \quad 0 \quad -\frac{h}{3} & 0 \\ \hdashline \sim & \left[\begin{array}{c:c|c:c} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \hdashline \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} \\ \hline \sim & \sim & \frac{h^2-2}{9} & -\frac{h}{3} \\ \hdashline \sim & \sim & \sim & -\frac{2h^2-1}{9} \end{array}\right] & \begin{array}{c} -\frac{2}{3\sigma} \\ 0 \\ 0 \\ 0 \end{array} \\ \hdashline \sim & \sim & \frac{h^2+1}{9} \end{array}\right],$$

and

$$\phi(\bar{Z}; \bar{H}) = \left[\begin{array}{c:c:c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} \quad 0 \quad 0 \quad -\frac{h}{3} & 0 \\ \hdashline \sim & \left[\begin{array}{c:c|c:c} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \hdashline \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} \\ \hline \sim & \sim & 0 & -\frac{h}{3} \\ \hdashline \sim & \sim & \sim & -\frac{h^2}{6} \end{array}\right] & \begin{array}{c} -\frac{2}{3\sigma} \\ 0 \\ 0 \\ 0 \end{array} \\ \hdashline \sim & \sim & \frac{h^2}{6} \end{array}\right].$$

Clearly, $\lim_{i\to\infty} \phi(\bar{Z}, H^i) \neq \phi(\bar{Z}; \bar{H})$. $\square$

7.2 Almost-Sure Continuity

In (SDP-III), the point $\bar{H}$ at which $\phi(\bar{Z};\cdot)$ loses continuity corresponds to $\bar{H}_{\alpha_0\alpha_0}$ being rank deficient, i.e., $\det(\bar{H}_{\alpha_0\alpha_0}) = 0$. In the next lemma, we show that $\phi(\bar{Z};\cdot)$ is continuous at every $\bar{H}$ whose $\bar{H}_{\alpha_0\alpha_0}$ is nonsingular.

Lemma 11 ($\phi(\bar{Z};\cdot)$ is continuous at nonsingular $\bar{H}_{\alpha_0\alpha_0}$)**.** *Under Assumption 1, suppose further that $\bar{Z} \in \mathcal{Z}_\star$ is singular, i.e., $|\alpha_0| > 0$. Then $\phi(\bar{Z};\cdot)$ is continuous at every $\bar{H} \in \mathcal{C}(\bar{Z})$ such that $\bar{H}_{\alpha_0\alpha_0}$ is nonsingular.*

Proof. We prove the continuity of $\phi_{\mathsf{P}}(\bar{Z};\cdot)$ and $\phi_{\mathsf{D}}(\bar{Z};\cdot)$ separately, and begin with the primal part.

Since $\bar{H}_{\alpha_0\alpha_0}$ is nonsingular, we have $|\beta_{0,0}| = 0$. Hence, by (43), the cone $\mathcal{K}^\circ_{\mathsf{P}}(\bar{Z};\bar{H})$ reduces to

$$\mathcal{K}^\circ_{\mathsf{P}}(\bar{Z};\bar{H}) = \left\{ W = \left[\begin{array}{c:c:c} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0} & 0 \\ \hdashline \sim & Q^0 \left[\begin{array}{c|c} \widehat{W}_{\beta_{0,\mathsf{P}}\beta_{0,\mathsf{P}}} & 0 \\ \hline \sim & 0 \end{array}\right] (Q^0)^\mathsf{T} & 0 \\ \hdashline \sim & \sim & 0 \end{array}\right] \;\middle|\; \begin{array}{c} \mathcal{P}W = 0, \\ \widehat{W} = (Q^0)^\mathsf{T} W_{\alpha_0\alpha_0} Q^0 \end{array} \right\}$$

$$= \underbrace{\left\{ W = \left[\begin{array}{c:c|c:c} W_{\alpha_+\alpha_+} & W_{\alpha_+\alpha_0^\mathsf{P}} & W_{\alpha_+\alpha_0^\mathsf{D}} & 0 \\ \hdashline \sim & W_{\alpha_0^\mathsf{P}\alpha_0^\mathsf{P}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right] \right\}}_{=:\mathcal{M}_1} \cap \underbrace{\{W \mid \mathcal{P}W = 0\}}_{=:\mathcal{M}_2},$$

where the last equality uses Lemma 4. By Proposition 1 (2), define

$$\mathcal{C}_1 := \left\{ H = \left[\begin{array}{cc|cc} H_{\alpha_+\alpha_+} & H_{\alpha_+\alpha_0^{\mathsf{P}}} & H_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \hdashline \sim & H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \hdashline \sim & \sim & \sim & 0 \end{array}\right] \;\middle|\; H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0 \right\}, \quad \mathcal{C}_2 = \mathcal{M}_2,$$

Then, $\mathcal{C}_{\mathsf{P}}(\bar{Z}) = \mathcal{C}_1 \cap \mathcal{C}_2$, $\mathrm{aff}(\mathcal{C}_1) = \mathcal{M}_1$, and $\mathrm{aff}(\mathcal{C}_2) = \mathcal{M}_2$. With the observation that $X_{\mathsf{sc}} - \bar{X} \in \mathrm{ri}(\mathcal{C}_1) \cap \mathrm{ri}(\mathcal{C}_2)$, we have $\mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H}) = \mathrm{aff}(\mathcal{C}_1) \cap \mathrm{aff}(\mathcal{C}_2) = \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))$ by Lemma 9. In particular, the above description of $\mathcal{K}^\circ_{\mathsf{P}}(\bar{Z}; \bar{H})$ at nonsingular $\bar{H}_{\alpha_0\alpha_0}$ is independent of $\bar{H}$.

Next, by Weyl's theorem, for any fixed such $\bar{H}$, there exists $\epsilon > 0$ such that for all $H \in \mathbb{B}_\epsilon(\bar{H})$, we have $\det(H_{\alpha_0\alpha_0}) \neq 0$. Therefore, for all $H \in \mathbb{B}_\epsilon(\bar{H}) \cap \mathcal{C}(\bar{Z})$,

$$\phi_{\mathsf{P}}(\bar{Z}; H) = \underset{W \in \mathcal{K}^\circ_{\mathsf{P}}(\bar{Z};H)}{\arg\min} \|W + \mathcal{E}^\perp(\bar{Z}; H)\|_{\mathsf{F}}^2 = \underset{W \in \mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))}{\arg\min} \|W + \mathcal{E}^\perp(\bar{Z}; H)\|_{\mathsf{F}}^2 = \Pi_{\mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))}(-\mathcal{E}^\perp(\bar{Z}; H)),$$

by Theorem 4.5. Since $\Pi_{\mathbb{S}^n_+}(\cdot)$ (resp. $\Pi_{\mathbb{S}^n_-}(\cdot)$) is continuous on $\mathbb{S}^n$, it follows from (33) that $-\mathcal{E}^\perp(\bar{Z}; \cdot)$ is continuous on $\mathbb{B}_\epsilon(\bar{H}) \cap \mathcal{C}(\bar{Z})$. Moreover, the projection mapping $\Pi_{\mathrm{aff}(\mathcal{C}_{\mathsf{P}}(\bar{Z}))}(\cdot)$ is continuous on $\mathbb{B}_\epsilon(\bar{H}) \cap \mathcal{C}(\bar{Z})$. Therefore, $\phi_{\mathsf{P}}(\bar{Z}; \cdot)$ is continuous on $\mathbb{B}_\epsilon(\bar{H}) \cap \mathcal{C}(\bar{Z})$, and in particular continuous at $\bar{H}$.

The continuity for $\phi_{\mathsf{D}}(\bar{Z}; \cdot)$ at such an $\bar{H}$ can be deduced symmetrically. Thus, $\phi(\bar{Z}; \cdot) = \phi_{\mathsf{P}}(\bar{Z}; \cdot) - \sigma\phi_{\mathsf{D}}(\bar{Z}; \cdot)$ is continuous at $\bar{H}$ with $\det(\bar{H}_{\alpha_0\alpha_0}) \neq 0$. ☐

From now on, abbreviate $\mathrm{aff}(\mathcal{C}(\bar{Z}))$ as $\mathcal{L}$. Suppose the dimension of $\mathcal{L}$ is d. Let ρ_d be the standard Lebesgue measure on $\mathbb{R}^d$. Fix $\mathcal{F}$ as any linear isomorphism from $\mathbb{R}^d$ to $\mathcal{L}$. Then, the d-dimension Lebesgue measure on $\mathcal{L}$ is defined by

$$\rho_{\mathcal{L}}(A) = \rho_d(\mathcal{F}^{-1}(A)), \quad \forall A \subset \mathcal{L} \text{ Borel.} \tag{53}$$

Please note that the choice of $\mathcal{F}$ will only affect $\rho_{\mathcal{L}}$ by a positive constant. We first show that the set of points making $\phi(\bar{Z}; \cdot)$ discontinuous is of measure zero in terms of $\rho_{\mathcal{L}}$.

Proposition 9 (Discontinuity is of measure zero)**.** *Under Assumption 1, fix any $\bar{Z} \in \mathcal{Z}_\star$. Suppose $\rho_{\mathcal{L}}$ is defined in (53). Then:*

$$\rho_{\mathcal{L}}\left(\left\{\bar{H} \in \mathcal{C}(\bar{Z}) \;\middle|\; \phi(\bar{Z}; \bar{H}) \textit{ is discontinuous at } \bar{H}\right\}\right) = 0.$$

Proof. If $\bar{Z}$ is nonsingular, then from Corollary 1 (2), $\mathcal{C}(\bar{Z}) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$. We get $\phi(\bar{Z}; \bar{H}) \equiv 0$ for all $\bar{H} \in \mathcal{C}(\bar{Z})$ from Proposition 5. Thus, the claim trivially holds. Now let us consider the case when $\bar{Z}$ is singular. Invoking Lemma 11, discontinuity only occurs when $\bar{H}_{\alpha_0\alpha_0}$ is singular. Denote the polynomial $p : \mathbb{S}^n \mapsto \mathbb{R}$ as $p(H) = \det(H_{\alpha_0\alpha_0})$:

$$\begin{aligned} &\left\{\bar{H} \in \mathcal{C}(\bar{Z}) \;\middle|\; \phi(\bar{Z}; \bar{H}) \text{ is discontinuous at } \bar{H}\right\} \\ \subseteq &\left\{\bar{H} \in \mathcal{C}(\bar{Z}) \;\middle|\; \det(\bar{H}_{\alpha_0\alpha_0}) = 0\right\} \subseteq \left\{\bar{H} \in \mathcal{L} \;\middle|\; p(\bar{H}) = 0\right\} =: \mathcal{D}. \end{aligned}$$

All we need to prove is $\rho_{\mathcal{L}}(\mathcal{D}) = 0$.

(i) We first prove that there exists $\widetilde{H} \in \mathcal{C}(\bar{Z})$, s.t. $p(\widetilde{H}) \neq 0$. Set $\widetilde{H}$ as

$$\widetilde{H} = (X_{\mathsf{sc}} - \bar{X}) - (S_{\mathsf{sc}} - \bar{S}) = \left[\begin{array}{cc|cc} [X_{\mathsf{sc}} - \bar{X}]_{\alpha_+\alpha_+} & [X_{\mathsf{sc}}]_{\alpha_+\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hdashline \sim & [X_{\mathsf{sc}}]_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succ 0 & 0 & 0 \\ \hline \sim & \sim & -[S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \prec 0 & -[S_{\mathsf{sc}}]_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \hdashline \sim & \sim & \sim & -[S_{\mathsf{sc}} - \bar{S}]_{\alpha_-\alpha_-} \end{array}\right].$$

It is easy to verify that

$$\mathcal{P}\Pi'_+(\bar{Z};\widetilde{H}) + \mathcal{P}^\perp\Pi'_-(\bar{Z};\widetilde{H}) = \mathcal{P}(X_{\mathsf{sc}} - \bar{X}) - \mathcal{P}^\perp(S_{\mathsf{sc}} - \bar{S}) = 0,$$

and $\det(\widetilde{H}_{\alpha_0\alpha_0}) \neq 0$.

(ii) Consider the restriction $q = p_{|\mathcal{L}}: \mathcal{L} \mapsto \mathbb{R}$. Under the identification $\mathcal{L} \simeq \mathbb{R}^d$ via $\mathcal{F}$, $\widetilde{q} = q \circ \mathcal{F}: \mathbb{R}^d \mapsto \mathbb{R}$ is a polynomial on $\mathbb{R}^d$. Since $\widetilde{q}(\mathcal{F}^{-1}(\widetilde{H})) = q(\widetilde{H}) \neq 0$ with $\widetilde{H} \in \mathcal{L}$, $\widetilde{q}$ is a nonzero polynomial on $\mathbb{R}^d$. From [6], the set $\widetilde{\mathcal{D}} := \mathcal{F}^{-1}(\mathcal{D}) = \{x \in \mathbb{R}^d \mid \widetilde{q}(x) = 0\}$ is of Lebesgue measure zero, *i.e.*, $\rho_d(\widetilde{\mathcal{D}}) = 0$. From (53),

$$\rho_{\mathcal{L}}(\mathcal{D}) = \rho_d(\mathcal{F}^{-1}(\mathcal{D})) = \rho_d(\widetilde{\mathcal{D}}) = 0.$$

□

Proposition 9 shows that the set of discontinuity points has measure zero. However, this does not rule out the possibility that $\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$—the set on which $\phi(\bar{Z};\bar{H})$ does not vanish—also has measure zero. The following proposition dispels this concern:

Proposition 10 (Measure of $\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$)**.** *Under Assumption 1, let $\rho_{\mathcal{L}}$ be defined in (53). Then either of the two cases holds: (i) $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \mathcal{C}(\bar{Z})$; (ii) $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) \subsetneq \mathcal{C}(\bar{Z})$ and $\rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) = \infty$.*

Proof. Case (i) is the trivial case, where $\phi(\bar{Z};\bar{H}) \equiv 0$ for all $\bar{H} \in \mathcal{C}(\bar{Z})$.

For case (ii), there exists $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, s.t. at least one of the following two conditions holds: $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} \neq 0$ and $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} \neq 0$. Otherwise, from Proposition 2 (2), $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$. Thus, from Proposition 2 (1), $\bar{H} \notin \operatorname{span}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}))$. This gives us $\operatorname{span}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) \subsetneq \operatorname{span}(\mathcal{C}(\bar{Z}))$, which implies $\dim(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) < \dim(\mathcal{C}(\bar{Z}))$. Since $\mathcal{L} = \operatorname{aff}(\mathcal{C}(\bar{Z}))$, $\rho_{\mathcal{L}}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) = 0$. Thus, by countable additivity,

$$\rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})) = \rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) + \rho_{\mathcal{L}}(\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) = \rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})).$$

On the other hand, since $\mathcal{C}(\bar{Z})$ is a nonempty closed convex cone of dimension $d \geq 1$, $\operatorname{ri}(\mathcal{C}(\bar{Z}))$ at least contains a ray $\mathcal{R} := \{t\bar{H} \mid t > 0\}$ with a nonzero $\bar{H} \in \operatorname{ri}(\mathcal{C}(\bar{Z}))$. By definition, there exists $r > 0$, s.t. $A := \mathbb{B}_r(\bar{H}) \cap \mathcal{L} \subset \mathcal{C}$ and $\rho_{\mathcal{L}}(A) > 0$. Define $tA := \{tH \mid H \in A\}$ for any $t > 0$, we get

$$\rho_{\mathcal{L}}(tA) = t^d\rho_{\mathcal{L}}(A) \to \infty, \quad \text{as } t \to \infty.$$

Finally, since $tA \subset \mathcal{C}(\bar{Z})$ for all $t > 0$, we conclude that $\rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})) = \infty$, and hence $\rho_{\mathcal{L}}(\mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})) = \infty$. □

The Proposition 10, together with Proposition 9, describes the "almost-sure" type continuity of $\phi(\bar{Z};\cdot)$ over $\mathcal{C}(\bar{Z})$.

7.3 Discussion: "Spikes" in $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$

The almost-sure type discontinuity of $\phi(\bar{Z};\cdot)$ provides a natural explanation for the microscopic phase transitions observed inside ADMM's slow-convergence regions: (i) When the iterates $Z^{(k)}$ pass through regions where $\phi(\bar{Z};Z^{(k)} - \bar{Z})$ varies continuously, the angle $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ remains small and evolves smoothly, as discussed in §5.3; (ii) when $Z^{(k)}$ hits a discontinuity point of $\phi(\bar{Z};\cdot)$, the approximation $\Delta Z^{(k)} \approx \frac{1}{2}\phi(\bar{Z};Z^{(k)} - \bar{Z})$ abruptly switches to a different displacement vector and quickly stabilizes again. Since $\Delta Z^{(k)}$ is closely related to the KKT residuals in ADMM [22, Lemma 4], we also expect an observable jump in $r_{\max}^{(k)}$.

We use (SDP-III) to illustrate the validity and accuracy of this explanation. By (62) and (63), if $h > \sqrt{2}$, then $\bar{H}(h,0)$ defined in (61) is a discontinuity point of $\phi(\bar{Z};\cdot)$, whereas if $h \leq \sqrt{2}$, $\bar{H}(h,0)$ is a continuity point. Now consider the initialization $Z^{(0)} := \bar{Z} + t\,\bar{H}(h,\epsilon)$. When $t \to 0$ and ϵ is set to a small positive value, we expect $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ to exhibit a spike when $h > \sqrt{2}$. Moreover, ϵ should affect the spike's arrival time: the smaller ϵ is, the earlier the spike occurs.

The results are shown in Figure 9. When $\epsilon = 10^{-2}$ is too large, no spike is observed even when $h = 1.6$. When $\epsilon = 10^{-3}$, larger values of h lead to earlier spike times. In addition, when $h \leq 1.40$, no spike is observed within the first 1000 iterations. Whenever $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ spikes, there is also a clearly observable jump in $r_{\max}^{(k+1)} - r_{\max}^{(k)}$. By comparison, the jump in $\|\Delta Z^{(k+1)}\|_{\mathsf{F}} - \|\Delta Z^{(k)}\|_{\mathsf{F}}$ is less pronounced. The behavior for $\epsilon = 10^{-4}$ is similar to that for $\epsilon = 10^{-3}$. When $\epsilon = 10^{-5}$, the spike occurs so early that it becomes indistinguishable from the initial transient phase before $Z^{(k)}$ has converged to $\mathcal{C}(\bar{Z})$.

8 σ's Effect on $\phi(\bar{Z};\cdot)$

The fourth property of $\phi(\bar{Z};\cdot)$ that we study concerns its dependence on σ, the tunable penalty parameter in (2). This issue is both theoretically and computationally important, since the choice of σ can significantly affect ADMM's convergence behavior. In general, the dependence of the one-step residual $\delta(\cdot)$ in (20) on σ is highly intricate. Under our local second-order limiting dynamics model, however, this relationship becomes much simpler.

Specifically, in §8.1, we show that when σ is updated to σ' (and $Z = \bar{Z} + t\bar{H} + o(t)$ is updated to Z'), both the primal and dual iterates remain unchanged to first order as long as $\bar{H} \in \mathcal{C}(\bar{Z})$. Moreover, the updated point Z' continues to lie in $\bar{Z}' + \mathcal{C}(\bar{Z}')$ up to first order, so a corresponding first-order direction $\bar{H}' \in \mathcal{C}(\bar{Z}')$ is well defined. At the second-order level, we obtain a clean scaling law: $\phi_{\mathsf{P}}(\bar{Z};\bar{H})$ in (48) is updated to $\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H})$, and $\phi_{\mathsf{D}}(\bar{Z};\bar{H})$ in (49) is updated to $\phi_{\mathsf{D}}(\bar{Z}';\bar{H}') = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H})$ (cf. §8.2). An immediate corollary is that, under the second-order limiting model, both the primal and dual infeasibilities are invariant to σ tuning. Finally, in §8.3, we discuss practical strategies for updating σ in the second-order-dominant regime.

8.1 First-Order Effect

Suppose we change σ to σ'. Then, for $Z = \bar{Z} + t\bar{H} + \frac{t^2}{2}W + o(t^2)$ with $X = \Pi_+(Z)$ and $S = -\frac{1}{\sigma}\Pi_-(Z)$, it is updated to:

$$Z' := \Pi_+(Z) + \frac{\sigma'}{\sigma}\Pi_-(Z) = \bar{Z}' + t\bar{H}' + \frac{t^2}{2}W' + o(t^2). \tag{54}$$

For the KKT point $\bar{Z} := \bar{X} - \sigma\bar{S}$, it is updated to $\bar{Z}' := \bar{X} - \sigma'\bar{S}$. Its corresponding eigenvalues are updated as follows:

$$\mu_k' = \begin{cases} \frac{\sigma'}{\sigma}\mu_k, & k \in \mathcal{I}_- \\ \mu_k, & \text{otherwise} \end{cases}$$

The corresponding optimal set is changed to $\mathcal{Z}_\star' := \{X - \sigma' S \mid X \in \mathcal{X}_\star, S \in \mathcal{S}_\star\}$. From (54),

$$\bar{H}' = \Pi_+'(\bar{Z};\bar{H}) + \frac{\sigma'}{\sigma}\Pi_-'(\bar{Z};\bar{H}). \tag{55}$$

For an arbitrary $\bar{H} \in \mathbb{S}^n$, it does not in general that $\Pi_+'(\bar{Z}';\bar{H}') = \Pi_+'(\bar{Z};\bar{H})$ and $\Pi_-'(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\Pi_-'(\bar{Z};\bar{H})$. However, as we will see, if $\bar{H} \in \mathcal{C}(\bar{Z})$, these equalities do hold.

Lemma 12 (New partition for $\bar{H}'$)**.** *Under Assumption 1, if $\bar{H} \in \mathcal{C}(\bar{Z})$, then $\Pi_+'(\bar{Z}';\bar{H}') = \Pi_+'(\bar{Z};\bar{H})$ and $\Pi_-'(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\Pi_-'(\bar{Z};\bar{H})$.*

Proof. From Proposition 1 (2), we have

$$\bar{H} = \left[\begin{array}{cc|cc} \bar{H}_{\alpha_+\alpha_+} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{P}}} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \sim & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hline \sim & \sim & \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & \bar{H}_{\alpha_-\alpha_-} \end{array}\right], \text{ with } \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0,\ \bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0.$$

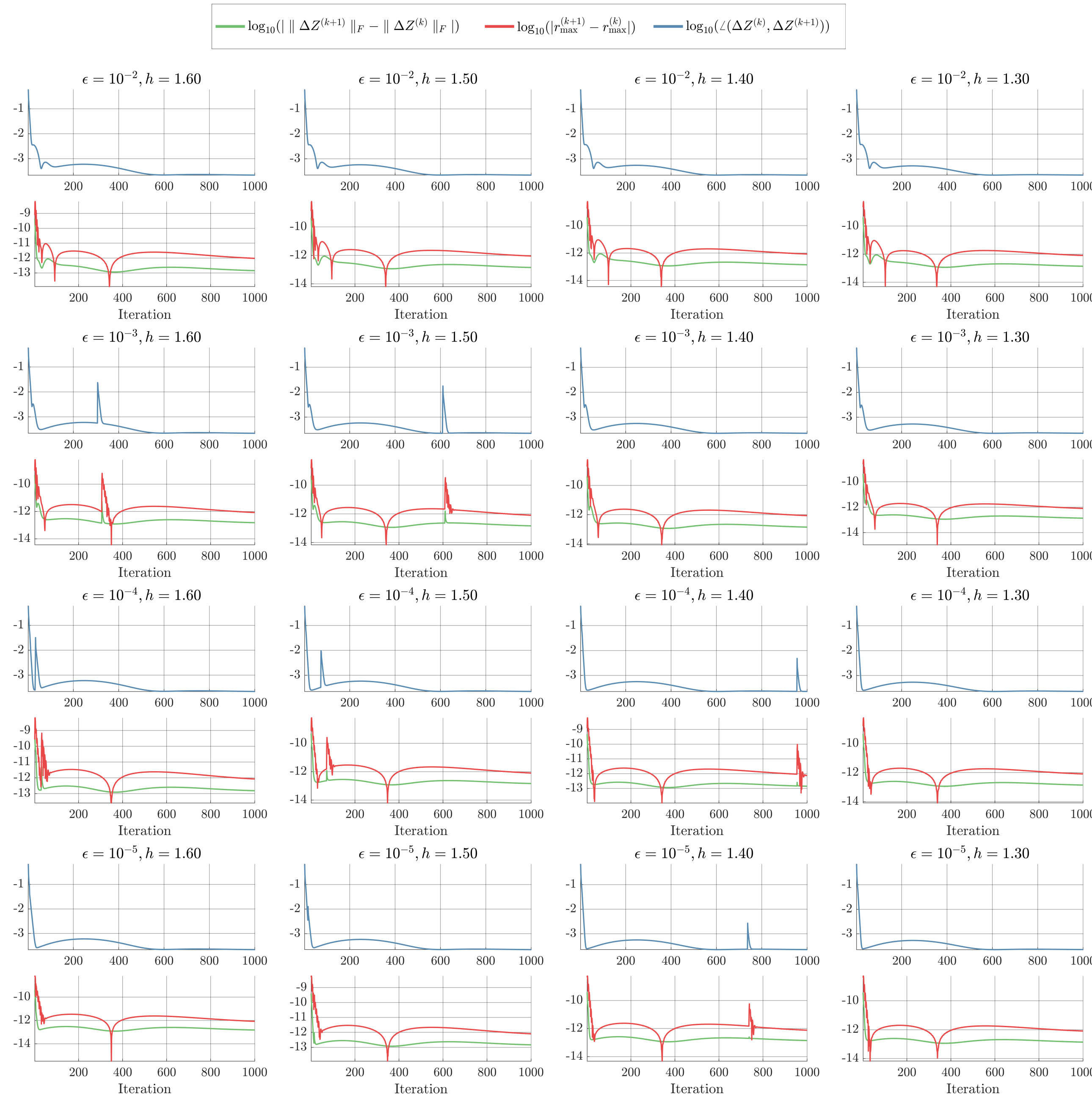


Figure 9: Trajectories of $\log_{10}(\Delta Z^{(k)}, \Delta Z^{(k+1)})$, $\log_{10}(|\|\Delta Z^{(k+1)}\|_{\mathsf{F}} - \|\Delta Z^{(k)}\|_{\mathsf{F}}|)$, and $\log_{10}(|r_{\max}^{(k+1)} - r_{\max}^{(k)}|)$ in (SDP-III) with different $Z^{(0)} := \bar{Z} + t\bar{H}(h, \epsilon)$. t is fixed as 10^{-4} and σ is fixed as 1. The maximum iteration number of ADMM is 1000. We sweep (h, ϵ) from $\{1.6, 1.5, 1.4, 1.3\} \times \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$, 16 points in total.

Thus,

$$\bar{H}' = \Pi'_+(\bar{Z};\bar{H}) + \frac{\sigma'}{\sigma}\Pi'_-(\bar{Z};\bar{H}) = \left[\begin{array}{cc|cc} \bar{H}_{\alpha_+\alpha_+} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{P}}} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \sim & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hline \sim & \sim & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_-\alpha_-} \end{array}\right], \tag{56}$$

and

$$\Pi'_+(\bar{Z}';\bar{H}') = \left[\begin{array}{cc|cc} \bar{H}_{\alpha_+\alpha_+} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{P}}} & \bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} & 0 \\ \sim & \bar{H}_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} & 0 & 0 \\ \hline \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{array}\right] = \Pi'_+(\bar{Z};\bar{H}),$$

$$\Pi'_-(\bar{Z}';\bar{H}') = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} \\ \hline \sim & \sim & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_0^{\mathsf{D}}\alpha_-} \\ \sim & \sim & \sim & \frac{\sigma'}{\sigma}\bar{H}_{\alpha_-\alpha_-} \end{array}\right] = \frac{\sigma'}{\sigma}\Pi'_-(\bar{Z};\bar{H}).$$

$\square$

Since $X' = \Pi_+(Z')$ and $S' = -\frac{1}{\sigma'}\Pi_-(Z')$, from Lemma 12:

$$\begin{aligned} X' &= \Pi_+(\bar{Z}') + t\Pi'_+(\bar{Z}';\bar{H}') + o(t) = \bar{X} + t\Pi'_+(\bar{Z};\bar{H}) + o(t) = X + o(t), \\ S' &= -\frac{1}{\sigma'}\Pi_-(\bar{Z}') - t\cdot\frac{1}{\sigma'}\Pi'_-(\bar{Z}';\bar{H}') + o(t) = \bar{S} - t\cdot\frac{1}{\sigma}\Pi'_-(\bar{Z};\bar{H}) + o(t) = S + o(t). \end{aligned}$$

Therefore, both primal and dual iterates remains unchanged up to first-order. The following results shows that if $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, then only updating σ cannot escape from the second-order-dominant region.

Lemma 13 ($\bar{H}'$ still in $\mathcal{C}(\bar{Z}')\backslash\mathcal{T}_{\mathcal{Z}'_\star}(\bar{Z}')$)**.** *Under Assumption 1, if $\bar{H} \in \mathcal{C}(\bar{Z})\backslash\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, then $\bar{H}' \in \mathcal{C}(\bar{Z}')\backslash\mathcal{T}_{\mathcal{Z}'_\star}(\bar{Z}')$.*

Proof. (i) We first show that $\bar{H}' \in \mathcal{C}(\bar{Z}')$. From Lemma 12, $\Pi'_+(\bar{Z}';\bar{H}') = \Pi'_+(\bar{Z};\bar{H})$ and $\Pi'_-(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\Pi'_-(\bar{Z};\bar{H})$. Thus,

$$\begin{aligned} \mathcal{P}\Pi'_+(\bar{Z}';\bar{H}') &= \mathcal{P}\Pi'_+(\bar{Z};\bar{H}) = 0 \\ \mathcal{P}^\perp\Pi'_-(\bar{Z}';\bar{H}') &= \mathcal{P}^\perp\frac{\sigma'}{\sigma}\Pi'_-(\bar{Z};\bar{H}) = 0 \end{aligned}$$

Thus, $\delta'(\bar{Z}';\bar{H}') = 0$.

(ii) We second show that $\bar{H}' \neq \mathcal{T}_{\mathcal{Z}'_\star}(\bar{Z}')$. Proof by contradiction. Suppose $\bar{H}' \in \mathcal{T}_{\mathcal{Z}'_\star}(\bar{Z}')$. From Proposition 2 (1), $\bar{H}'_{\alpha_{\mathsf{P}}\alpha_{\mathsf{D}}} = 0$. Thus, $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}} = 0$ and $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} = 0$ from (56). Combining Proposition 2 (2), $\bar{H} \in \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, which results in a contradiction. $\square$

8.2 Second-Order Effect

We show the following clean result:

Proposition 11 (σ's second-order effect)**.** *Under Assumption 1, suppose $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. When σ is updated to σ':*

$$\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H}), \quad \phi_{\mathsf{D}}(\bar{Z}';\bar{H}') = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H}).$$

Proof. (i) We first show that

$$\mathcal{K}^\circ_{\mathsf{P}}(\bar Z';\bar H') = \mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H), \quad \mathcal{K}^\circ_{\mathsf{D}}(\bar Z';\bar H') = \mathcal{K}^\circ_{\mathsf{D}}(\bar Z;\bar H).$$

To see this: from Lemma 12,

$$\bar H'_{\alpha_0\alpha_0} = \left[\begin{array}{c|c} \bar H_{\alpha_0^{\mathsf{P}}\alpha_0^{\mathsf{P}}} \succeq 0 & 0 \\ \hline \sim & \frac{\sigma'}{\sigma}\bar H_{\alpha_0^{\mathsf{D}}\alpha_0^{\mathsf{D}}} \preceq 0 \end{array}\right].$$

Thus,

$$(Q^0)' = Q^0 \text{ and } \begin{cases} \eta'_{0,i} = \eta_{0,i}, & i \in \beta_{0,+} \\ \eta'_{0,j} = \frac{\sigma'}{\sigma}\eta_{0,j}, & j \in \beta_{0,-} \\ \eta'_{0,k} = 0, & k \in \beta_{0,0} \end{cases}$$

Since $\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)$ and $\mathcal{K}^\circ_{\mathsf{D}}(\bar Z;\bar H)$ only depends on Q^0 and the partition $(\beta_{0,+},\beta_{0,0},\beta_{0,-})$, we finished the proof.

(ii) For the primal part: due to $\bar H'_{\alpha_0\alpha_0}$'s structure, $\Pi_+(\bar H'_{\alpha_0\alpha_0}) = \Pi_+(\bar H_{\alpha_0\alpha_0})$ and $\Pi_-(\bar H'_{\alpha_0\alpha_0}) = \frac{\sigma'}{\sigma}\Pi_-(\bar H_{\alpha_0\alpha_0})$. From (33),

$$\mathcal{E}^\perp(\bar Z';\bar H') = \left[\begin{array}{c:c:c} 0 & \left\{2\frac{1}{-\mu'_a}\bar H'_{\alpha_a\alpha_0}\Pi_-(\bar H'_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{2\frac{1}{\mu'_a-\mu'_b}\bar H'_{\alpha_a\alpha_0}\bar H'_{\alpha_0\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu'_c}\bar H'_{\alpha_0\alpha_c}\bar H'_{\alpha_c\alpha_0} & \left\{-2\frac{1}{\mu'_b}\Pi_-(-\bar H'_{\alpha_0\alpha_0})\bar H'_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right]$$

$$= \left[\begin{array}{c:c:c} 0 & \left\{2\frac{\sigma'}{\sigma}\frac{1}{-\mu_a}\bar H_{\alpha_a\alpha_0}\Pi_-(\bar H_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{2\frac{\frac{\sigma'}{\sigma}}{\mu_a-\frac{\sigma'}{\sigma}\mu_b}\bar H_{\alpha_a\alpha_0}\bar H_{\alpha_0\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & 2\frac{\sigma'}{\sigma}\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\bar H_{\alpha_0\alpha_c}\bar H_{\alpha_c\alpha_0} & \left\{-2\frac{1}{\mu_b}\Pi_-(-\bar H_{\alpha_0\alpha_0})\bar H_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right].$$

Therefore, by Proposition 4:

$$\phi_{\mathsf{P}}(\bar Z';\bar H') = \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z';\bar H')}\|W+\mathcal{E}^\perp(\bar Z';\bar H')\|_{\mathsf{F}}^2 = \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)}\|W+\mathcal{E}^\perp(\bar Z';\bar H')\|_{\mathsf{F}}^2$$

$$= \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)} 2\sum_{a\in\mathcal{I}_+}\|W_{\alpha_a\alpha_0} + \frac{\sigma'}{\sigma}\cdot 2\frac{1}{-\mu_a}\bar H_{\alpha_a\alpha_0}\Pi_-(\bar H_{\alpha_0\alpha_0})\|_{\mathsf{F}}^2 + \|\overline{W}_{\alpha_0\alpha_0} + \frac{\sigma'}{\sigma}\cdot 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_c}\bar H_{\alpha_0\alpha_c}\bar H_{\alpha_c\alpha_0}\|_{\mathsf{F}}^2$$

$$= \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)} 2\sum_{a\in\mathcal{I}_+}\|W_{\alpha_a\alpha_0} + \frac{\sigma'}{\sigma}\cdot[\mathcal{E}^\perp(\bar Z;\bar H)]_{\alpha_a\alpha_0}\|_{\mathsf{F}}^2 + \|\overline{W}_{\alpha_0\alpha_0} + \frac{\sigma'}{\sigma}\cdot[\mathcal{E}(\bar Z;\bar H)]_{\alpha_0\alpha_0}\|_{\mathsf{F}}^2$$

$$= \operatorname*{arg\,min}_{W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)}\|W + \frac{\sigma'}{\sigma}\cdot\mathcal{E}^\perp(\bar Z;\bar H)\|_{\mathsf{F}}^2 = \Pi_{\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)}(-\frac{\sigma'}{\sigma}\cdot\mathcal{E}^\perp(\bar Z;\bar H)) = \frac{\sigma'}{\sigma}\cdot\phi_{\mathsf{P}}(\bar Z;\bar H).$$

One may notice that the key observation here is $W_{\alpha_+\alpha_-}\equiv 0, \forall W\in\mathcal{K}^\circ_{\mathsf{P}}(\bar Z;\bar H)$. The last equality comes from the fact that for a closed convex cone $\mathcal{C}\subset\mathbb{S}^n$, $\Pi_{\mathcal{C}}(\alpha x) = \alpha\Pi_{\mathcal{C}}(x)$ for all $\alpha > 0$.

(iii) For the dual part, similar to the primal part:

$$\mathcal{E}(\bar Z;\bar H) = \left[\begin{array}{c:c:c} 0 & \left\{-2\frac{\sigma'}{\sigma}\frac{1}{\mu_a}\bar H_{\alpha_a\alpha_0}\Pi_+(-\bar H_{\alpha_0\alpha_0})\right\}_{a\in\mathcal{I}_+} & \left\{-2\frac{\frac{\sigma'}{\sigma}}{\mu_a-\frac{\sigma'}{\sigma}\mu_b}\bar H_{\alpha_a\alpha_0}\bar H_{\alpha_0\alpha_b}\right\}_{\substack{a\in\mathcal{I}_+\\ b\in\mathcal{I}_-}} \\ \hdashline \sim & 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar H_{\alpha_0\alpha_c}\bar H_{\alpha_c\alpha_0} & \left\{2\frac{1}{-\mu_b}\Pi_+(\bar H_{\alpha_0\alpha_0})\bar H_{\alpha_0\alpha_b}\right\}_{b\in\mathcal{I}_-} \\ \hdashline \sim & \sim & 0 \end{array}\right].$$

Thus, by Proposition 4:

$$
\begin{aligned}
\phi_{\mathsf{D}}(\bar{Z}';\bar{H}') &= -\frac{1}{\sigma'} \operatorname*{arg\,min}_{W\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z}';\bar{H}')} \|W+\mathcal{E}(\bar{Z}';\bar{H}')\|_{\mathsf{F}}^2 = -\frac{1}{\sigma'} \operatorname*{arg\,min}_{W\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})} \|W+\mathcal{E}(\bar{Z}';\bar{H}')\|_{\mathsf{F}}^2 \\
&= -\frac{1}{\sigma'} \operatorname*{arg\,min}_{W\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})} 2\sum_{b\in\mathcal{I}_-} \|W_{\alpha_0\alpha_b} + 2\frac{1}{-\mu_b}\Pi_+(\bar{H}_{\alpha_0\alpha_0})\bar{H}_{\alpha_0\alpha_b}\|_{\mathsf{F}}^2 + \|W_{\alpha_0\alpha_0} + 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}\bar{H}_{\alpha_0\alpha_c}\bar{H}_{\alpha_c\alpha_0}\|_{\mathsf{F}}^2 \\
&= -\frac{1}{\sigma'} \operatorname*{arg\,min}_{W\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})} \|W+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 = \frac{\sigma}{\sigma'}\cdot -\frac{1}{\sigma} \operatorname*{arg\,min}_{W\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})} \|W+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H}).
\end{aligned}
$$

Again, the key observation is $W_{\alpha_+\alpha_-} \equiv 0, \forall W \in \mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})$. □

An immediate corollary from Proposition 11 is that, the limiting behaviors of primal/dual infeasibility is *irrelevant* to σ in the second-order-dominant regions.

Corollary 3. *Under Assumption 1, let $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Under the first- and second-order local dynamics model in Definition 1, when σ is updated to σ', the limits of both $r_p^{(k)}$ and $r_d^{(k)}$ in (6) remain unchanged up to second-order.*

Proof. From [49, Corollary 1],

$$
\mathcal{A}X^{(k)} - b = \sigma\mathcal{A}(S^{(k+1)} - S^{(k)}), \quad \mathcal{A}^* y^{(k)} + S^{(k)} - C = \frac{1}{\sigma}(X^{(k+1)} - X^{(k)}).
$$

Thus,

$$
r_p^{(k)} = \sigma\frac{\|\mathcal{A}(S^{(k+1)} - S^{(k)})\|}{1+\|b\|}, \quad r_d^{(k)} = \frac{1}{\sigma}\frac{\|X^{(k+1)} - X^{(k)}\|_{\mathsf{F}}}{1+\|C\|_{\mathsf{F}}}.
$$

From Theorem 6, the local second-order limit of $X^{(k+1)} - X^{(k)}$ (resp. $S^{(k+1)} - S^{(k)}$) is $\phi_{\mathsf{P}}(\bar{Z};\bar{H})$ (resp. $\phi_{\mathsf{D}}(\bar{Z};\bar{H})$). Thus,

$$
\lim_{k\to\infty} r_p^{(k)} \propto \sigma\|\mathcal{A}\phi_{\mathsf{D}}(\bar{Z};\bar{H})\|, \quad \lim_{k\to\infty} r_d^{(k)} \propto \frac{1}{\sigma}\|\phi_{\mathsf{P}}(\bar{Z};\bar{H})\|_{\mathsf{F}}.
$$

On the other hand, from Proposition 11,

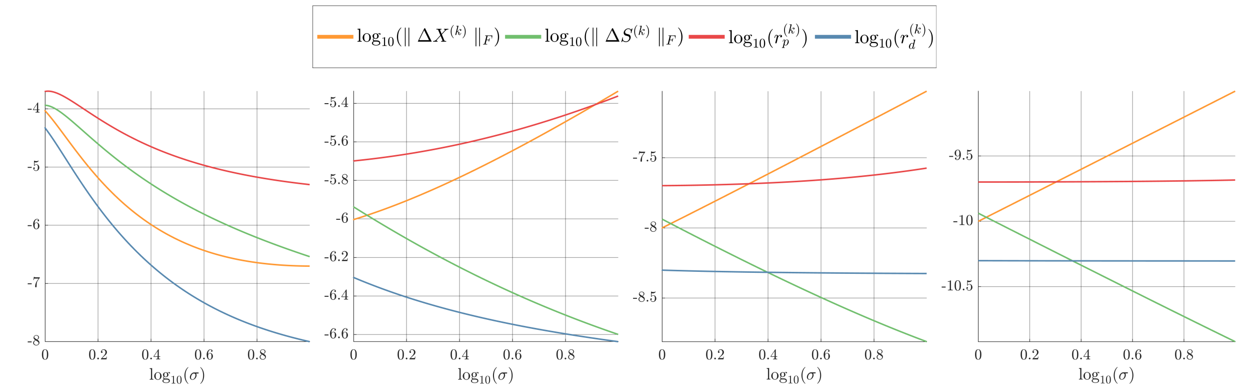
$$
\begin{aligned}
\sigma'\phi_{\mathsf{D}}(\bar{Z}';\bar{H}') &= \sigma'\frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}), \\
\frac{1}{\sigma'}\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') &= \frac{1}{\sigma'}\frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H}) = \frac{1}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H}).
\end{aligned}
$$

which closes the proof. □

8.3 Discussion: σ's Updating Rules

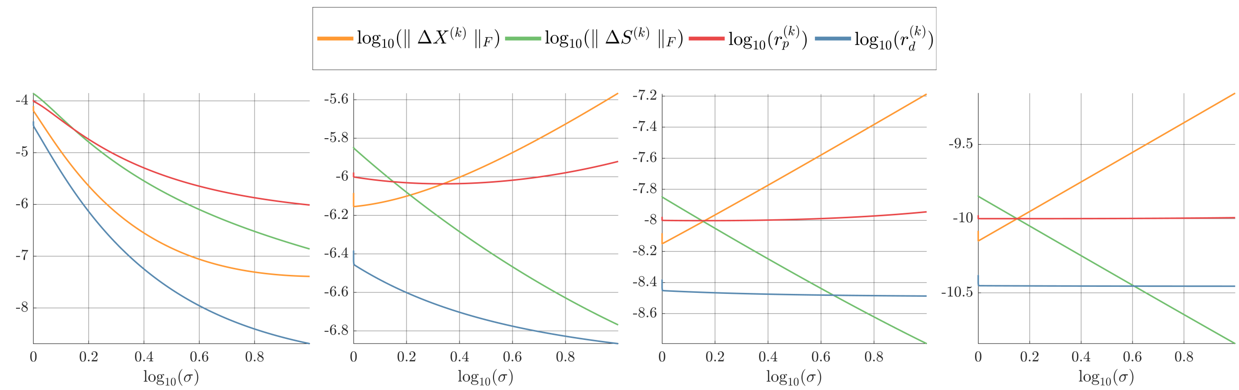
Traditional σ-updating heuristics typically aim to balance the primal and dual infeasibilities, under the implicit assumption that $\Delta X^{(k)} := X^{(k+1)} - X^{(k)}$ and $\Delta S^{(k)} := S^{(k+1)} - S^{(k)}$ are nearly independent of σ [24, 49]. However, Proposition 11 and Corollary 3 indicate that such heuristics become ineffective in second-order-dominant regions, since $r_p^{(k)}$ and $r_d^{(k)}$ are (locally) insensitive to σ.

To empirically verify Proposition 11 and Corollary 3, we fix $\bar{Z}$ and $\bar{H}$ as in §5 for all three examples. Starting from $Z^{(0)} := \bar{Z} + t\bar{H}$ with different choices of t, we uniformly increase $\log_{10}(\sigma)$ from 0 to 1 over 1000 ADMM iterations. Since the change in σ is gradual and the iteration horizon is moderate, we may assume that $\Delta X^{(k)}$ (resp. $\Delta S^{(k)}$) steadily tracks its second-order limit as $t \downarrow 0$. The results for (SDP-I), (SDP-II), and (SDP-III) are shown in Figures 10, 11, and 12, respectively. When $t = 10^{-5}$, the dependence of $(\Delta X^{(k)}, \Delta S^{(k)}, r_p^{(k)}, r_d^{(k)})$ on σ is consistent across all three examples: (i) $\log_{10}(\|\Delta X^{(k)}\|_{\mathsf{F}})$ (resp. $\log_{10}(\|\Delta S^{(k)}\|_{\mathsf{F}})$) increases (resp. decreases) approximately linearly with $\log_{10}(\sigma)$, with slope close to $+1$ (resp. -1); (ii) $r_p^{(k)}$ and $r_d^{(k)}$ remain essentially unchanged as σ varies.



(a) $t = 10^{-2}$ (b) $t = 10^{-3}$ (c) $t = 10^{-4}$ (d) $t = 10^{-5}$

Figure 10: Trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ in (SDP-I). Fix $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Pick $t \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ such that $Z^{(0)} = \bar{Z} + t\bar{H}$. $\log_{10}(\sigma)$ is uniformly increased from 0 to 1 in 1000 iterations.

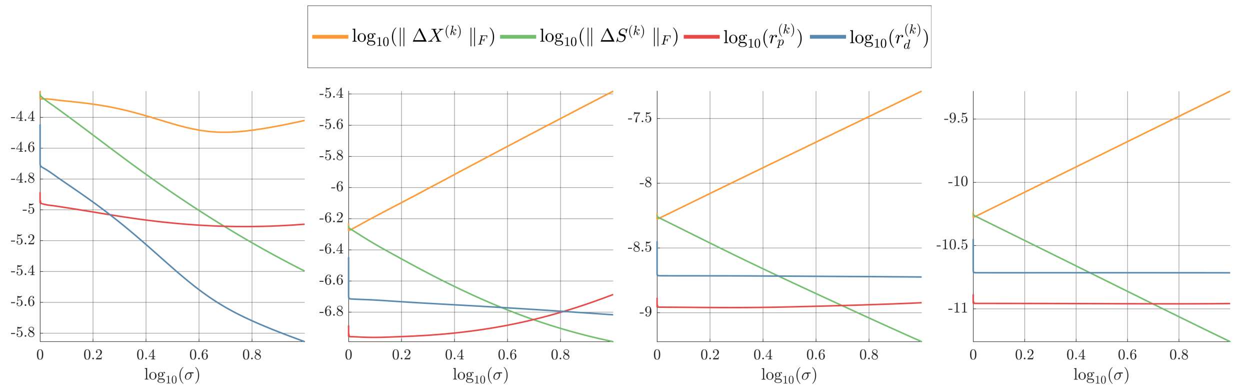


(a) $t = 10^{-2}$ (b) $t = 10^{-3}$ (c) $t = 10^{-4}$ (d) $t = 10^{-5}$

Figure 11: Trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ in (SDP-II). Fix $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Pick $t \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ such that $Z^{(0)} = \bar{Z} + t\bar{H}$. $\log_{10}(\sigma)$ is uniformly increased from 0 to 1 in 1000 iterations.

Discussion on the one-side uniqueness condition. It is difficult to design a "universally" good σ-updating strategy in the second-order-dominant regime. For instance, when both primal and dual constraint nondegeneracy fail, it is likely that $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}}$ in the $\mathcal{C}_{\mathsf{P}}(\bar{Z})$ part and $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-}$ in the $\mathcal{C}_{\mathsf{D}}(\bar{Z})$ part are simultaneously nonzero (e.g., (61) in (SDP-III)). In this case, enlarging σ amplifies $\Delta X^{(k)}$, which may help reduce $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}}$. On the other hand, it also suppresses $\Delta S^{(k)}$, which may worsen the $\mathcal{C}_{\mathsf{D}}(\bar{Z})$ component. The situation can be more favorable when one-sided uniqueness holds in either the primal or the dual optimal solution set. For example, when the primal solution is unique, Proposition 7's proof implies $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-} = 0$. In this case, we only need to eliminate $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}}$ in the $\mathcal{C}_{\mathsf{P}}(\bar{Z})$ part, and it may be beneficial to choose a large σ. Symmetrically, when the dual optimal solution is unique, we only need to eliminate $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-}$ in the $\mathcal{C}_{\mathsf{D}}(\bar{Z})$ part, and it may be beneficial to choose a small σ.

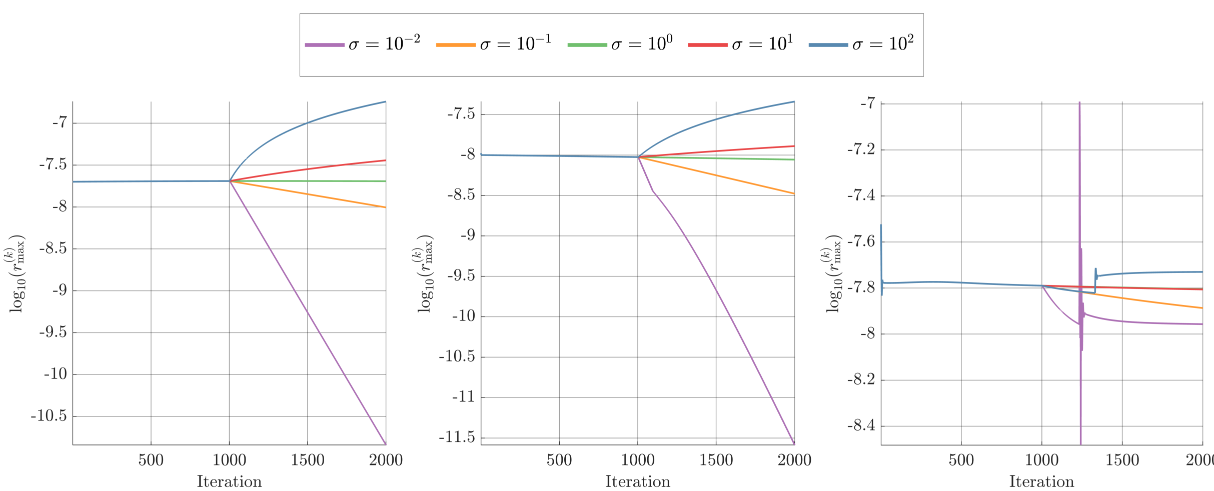
We empirically verify this analysis using three toy examples. The initial $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$ are the same as in §5. We fix $t = 10^{-4}$ and initialize $\sigma = 1$. After running 1000 ADMM iterations, we update σ to a value in $\{10^{-2}, 10^{-1}, 1, 10, 10^2\}$. We then record the change in the maximum KKT residual $r_{\max}^{(k)}$. The results are shown in Figure 13. In both (SDP-I) and (SDP-II), we observe a significant acceleration when



(a) $t = 10^{-2}$ (b) $t = 10^{-3}$ (c) $t = 10^{-4}$ (d) $t = 10^{-5}$

Figure 12: Trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ in (SDP-III). Fix $\bar{Z} \in \mathcal{Z}_\star$ and $\bar{H} \in \mathcal{C}(\bar{Z})$. Pick $t \in \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$ such that $Z^{(0)} = \bar{Z} + t\bar{H}$. $\log_{10}(\sigma)$ is uniformly increased from 0 to 1 in 1000 iterations.

σ is updated from 1 to 10^{-2}. This is consistent with our analysis, since the dual optimal solution is unique in both examples. For (SDP-III), changing σ does not help the iterates escape the slow-convergence region. This is unsurprising, since $\bar{H}(1,0)$ defined in (61) has nonzero entries in both $\bar{H}_{\alpha_+\alpha_0^{\mathsf{D}}}$ and $\bar{H}_{\alpha_0^{\mathsf{P}}\alpha_-}$.



(SDP-I) (SDP-II) (SDP-III)

Figure 13: Trajectories of $\log_{10}(r_{\max}^{(k)})$ in the three toy examples. For each example, we fix $\bar{Z} \in \mathcal{Z}_\star$, $\bar{H} \in \mathcal{C}(\bar{Z})$, and $t = 10^{-4}$, and initialize $Z^{(0)} := \bar{Z} + t\bar{H}$. The penalty parameter is initialized at $\sigma = 1$. After 1000 iterations, we update σ to a value in $\{10^{-2}, 10^{-1}, 1, 10, 10^2\}$ and run an additional 1000 ADMM iterations.

9 Examples

We present three SDP examples in this section. For each instance and the associated rank-deficient $\bar{Z} \in \mathcal{Z}_\star$, we compute the relevant first-order objects (*e.g.*, $\mathcal{C}(\bar{Z})$, $\mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$) and second-order objects (*e.g.*, $\mathcal{K}(\bar{Z};\bar{H})$, $\mathcal{K}^\circ(\bar{Z};\bar{H})$, $\Psi(\bar{Z};\bar{H})$, $\phi(\bar{Z};\bar{H})$). These calculations serve three purposes:

1. These examples provide a sanity check for the validity of our second-order analysis.

2. More importantly, the examples serve as constructive demonstrations of the key properties of $\phi(\bar{Z};\cdot)$. For instance, (SDP-III) simultaneously establishes Proposition 6 and Proposition 8.

3. When discussing the connection between properties of $\phi(\bar{Z};\cdot)$ and ADMM's empirical behavior, we use numerical results on these examples for illustration.

9.1 Example I

SDP data. We consider a 2×2 SDP from [10, Example 1]:

$$C = \begin{bmatrix} 0 & 0 \\ \sim & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ \sim & -1 \end{bmatrix}, \quad b = 0. \tag{SDP-I}$$

Its optimal sets are:

$$\mathcal{X}_\star = \left\{ \begin{bmatrix} X_{11} & 0 \\ \sim & 0 \end{bmatrix} \;\middle|\; X_{11} \geq 0 \right\},\ \mathcal{S}_\star = \left\{ \begin{bmatrix} 0 & 0 \\ \sim & 1 \end{bmatrix} \right\},\ \mathcal{Z}_\star = \left\{ \begin{bmatrix} Z_{11} & 0 \\ \sim & -\sigma \end{bmatrix} \;\middle|\; Z_{11} \geq 0 \right\}.$$

Clearly, the primal optimal set is unbounded. Moreover, except $Z_{11}=0$, all other optimal solutions satisfies strict complementarity. We are typically interested in the rank deficient optimal solution ($Z_{11}=0$).

First-order information. Denote $(x)_+$ (resp. $(x)_-$) as an abbreviation for $\Pi_{\mathbb{S}^1_+}(x)$ (resp. $\Pi_{\mathbb{S}^1_-}(x)$). Since

$$\mathcal{P}X = \frac{1}{3}(2X_{12} - X_{22}) \begin{bmatrix} 0 & 1 \\ \sim & -1 \end{bmatrix}, \quad \mathcal{P}^\perp X = \begin{bmatrix} X_{11} & \frac{1}{3}(X_{12}+X_{22}) \\ \sim & \frac{2}{3}(X_{12}+X_{22}) \end{bmatrix},$$

and

$$\Pi'_+(\bar{Z};H) = \begin{bmatrix} (H_{11})_+ & 0 \\ \sim & 0 \end{bmatrix}, \quad \Pi'_-(\bar{Z};H) = \begin{bmatrix} (H_{11})_- & H_{12} \\ \sim & H_{22} \end{bmatrix}.$$

we have

$$\mathcal{P}\Pi'_+(\bar{Z};H) = 0, \quad \mathcal{P}^\perp\Pi'_-(\bar{Z};H) = \begin{bmatrix} (H_{11})_- & \frac{1}{3}(H_{12}+H_{22}) \\ \sim & \frac{2}{3}(H_{12}+H_{22}) \end{bmatrix}.$$

Thus,

$$\mathcal{C}(\bar{Z}) = \left\{ \begin{bmatrix} a & b \\ \sim & -b \end{bmatrix} \;\middle|\; a \geq 0 \right\}, \quad \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \mathcal{C}(\bar{Z}) \cap \left\{ H \in \mathbb{S}^2 \mid H_{12} = 0 \right\} = \left\{ \begin{bmatrix} a & 0 \\ \sim & 0 \end{bmatrix} \;\middle|\; a \geq 0 \right\}. \tag{57}$$

we choose an arbitrary $\bar{H} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ with $b \neq 0$.

Second-order information. For this simple 2×2 SDP example, we use $\mathcal{K}(\bar{Z};\bar{H})$'s formula in (41) to calculate $\phi(\bar{Z};\bar{H})$. Through careful calculation:

$$\mathcal{E}(\bar{Z};\bar{H}) = \begin{bmatrix} 0 & 2\frac{ab}{\sigma} \\ \sim & 0 \end{bmatrix}, \quad \mathcal{E}^\perp(\bar{Z};\bar{H}) = \begin{bmatrix} -2\frac{b^2}{\sigma} & -2\frac{ab}{\sigma} \\ \sim & 0 \end{bmatrix}, \quad \Psi(\bar{Z};\bar{H}) = \begin{bmatrix} 2\frac{b^2}{\sigma} & -\frac{2}{3}\frac{ab}{\sigma} \\ \sim & \frac{8}{3}\frac{ab}{\sigma} \end{bmatrix}.$$

For $\mathcal{K}(\bar{Z};\bar{H})$, there are two cases:

- Case I: $a > 0$. In this case,

$$\Theta(\bar{Z};\bar{H},W) = \begin{bmatrix} W_{11} & 0 \\ \sim & 0 \end{bmatrix}, \quad \Theta^\perp(\bar{Z};\bar{H},W) = \begin{bmatrix} 0 & W_{12} \\ \sim & W_{22} \end{bmatrix}, \quad \mathcal{K}(\bar{Z};\bar{H}) = \left\{ \begin{bmatrix} 0 & \frac{1}{3}(W_{12}+W_{22}) \\ \sim & \frac{2}{3}(W_{12}+W_{22}) \end{bmatrix} \right\}.$$

  Thus,

$$\Pi_{\mathcal{K}(\bar{Z};\bar{H})}(\Psi(\bar{Z};\bar{H})) = \begin{bmatrix} 0 & \frac{2}{3}\frac{ab}{\sigma} \\ \sim & \frac{4}{3}\frac{ab}{\sigma} \end{bmatrix}, \quad \phi(\bar{Z};\bar{H}) = \begin{bmatrix} 2\frac{b^2}{\sigma} & -\frac{4}{3}\frac{ab}{\sigma} \\ \sim & \frac{4}{3}\frac{ab}{\sigma} \end{bmatrix}. \tag{58}$$

- Case II: $a = 0$. In this case,

$$\Theta(\bar{Z};\bar{H},W) = \begin{bmatrix} (W_{11})_+ & 0 \\ \sim & 0 \end{bmatrix},\ \Theta^{\perp}(\bar{Z};\bar{H},W) = \begin{bmatrix} (W_{11})_- & W_{12} \\ \sim & W_{22} \end{bmatrix},\ \mathcal{K}(\bar{Z};\bar{H}) = \left\{ \begin{bmatrix} (W_{11})_- & \frac{1}{3}(W_{12}+W_{22}) \\ \sim & \frac{2}{3}(W_{12}+W_{22}) \end{bmatrix} \right\}.$$

Thus,

$$\Pi_{\mathcal{K}(\bar{Z};\bar{H})}(\Psi(\bar{Z};\bar{H})) = \begin{bmatrix} 0 & 0 \\ \sim & 0 \end{bmatrix}, \quad \phi(\bar{Z};\bar{H}) = \begin{bmatrix} 2\frac{b^2}{\sigma} & 0 \\ \sim & 0 \end{bmatrix}.$$

Clearly, the two cases can be combined.

σ updating. Consider updating σ to σ'. From Lemma 12 and Lemma 13:

$$\bar{H} = \begin{bmatrix} a & b \\ \sim & -b \end{bmatrix} \Longrightarrow \bar{H}' = \begin{bmatrix} a' & b' \\ \sim & -b' \end{bmatrix} = \begin{bmatrix} a & \frac{\sigma'}{\sigma} b \\ \sim & -\frac{\sigma'}{\sigma} b \end{bmatrix}.$$

Since

$$\phi_{\mathsf{P}}(\bar{Z};\bar{H}) = \begin{bmatrix} 2\frac{b^2}{\sigma} & 0 \\ \sim & 0 \end{bmatrix}, \quad \phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \begin{bmatrix} 0 & \frac{4}{3}\frac{ab}{\sigma^2} \\ \sim & -\frac{4}{3}\frac{ab}{\sigma^2} \end{bmatrix},$$

we have

$$\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma} \begin{bmatrix} 2\frac{b^2}{\sigma} & 0 \\ \sim & 0 \end{bmatrix} = \frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H}), \quad \phi_{\mathsf{D}}(\bar{Z}';\bar{H}') = \frac{\sigma}{\sigma'} \begin{bmatrix} 0 & \frac{4}{3}\frac{ab}{\sigma^2} \\ \sim & -\frac{4}{3}\frac{ab}{\sigma^2} \end{bmatrix} = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H}).$$

9.2 Example II

SDP data. Consider the following SDP instance:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 1 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 & 0 \\ \sim & 1 & 0 \\ \sim & \sim & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & 1 \\ \sim & \sim & 0 \end{bmatrix},\ b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{SDP-II}$$

The primal-dual optimal set:

$$\mathcal{X}_\star = \left\{ \begin{bmatrix} a & u & 0 \\ \sim & 1-a & 0 \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; 0 \le a \le 1,\ u^2 \le a(1-a) \right\},\ \mathcal{S}_\star = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 1 \end{bmatrix} \right\},\ \mathcal{Z}_\star = \mathcal{X}_\star - \sigma\mathcal{S}_\star.$$

For (SDP-II), we can easily check that it satisfies two-side Slater conditions. Also, there exists an strict complementary solution pair. We are typically interested in the rank-deficient solutions, *i.e.,* $\operatorname{rank}(\bar{X}) = 1$ of the form:

$$\bar{X} = \bar{X}(a) = \begin{bmatrix} a & \pm\sqrt{a}\sqrt{1-a} & 0 \\ \sim & 1-a & 0 \\ \sim & \sim & 0 \end{bmatrix}, \quad a \in [0,1].$$

Our framework requires $\bar{X}$ to be diagonal. Therefore, some changes of basis are needed. Define the orthonormal matrix Q as:

$$Q = \begin{bmatrix} \sqrt{a} & \mp\sqrt{1-a} & 0 \\ \pm\sqrt{1-a} & \sqrt{a} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Under this basis:

$$C \leftarrow Q^\top C Q = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 1 \end{bmatrix},\ A_1 \leftarrow Q^\top A_1 Q = \begin{bmatrix} 1 & 0 & 0 \\ \sim & 1 & 0 \\ \sim & \sim & 1 \end{bmatrix},\ A_2 \leftarrow Q^\top A_2 Q = \begin{bmatrix} 0 & 0 & \pm\sqrt{1-a} \\ \sim & 0 & \sqrt{a} \\ \sim & \sim & 0 \end{bmatrix},$$

$$\bar{X} \leftarrow Q^\top \bar{X} Q = \begin{bmatrix} 1 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 0 \end{bmatrix},\ \bar{S} \leftarrow Q^\top \bar{S} Q = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & 1 \end{bmatrix}.$$

First-order information. For a fixed a,

$$\mathcal{P}X = \tfrac{1}{3}(X_{11}+X_{22}+X_{33}) \begin{bmatrix} 1 & 0 & 0 \\ \sim & 1 & 0 \\ \sim & \sim & 1 \end{bmatrix} + (\pm\sqrt{1-a}X_{13} + \sqrt{a}X_{23}) \begin{bmatrix} 0 & 0 & \pm\sqrt{1-a} \\ \sim & 0 & \sqrt{a} \\ \sim & \sim & 0 \end{bmatrix},$$

$$\Pi'_+(\bar{Z};H) = \begin{bmatrix} H_{11} & H_{12} & \frac{1}{1+\sigma}H_{13} \\ \sim & (H_{22})_+ & 0 \\ \sim & \sim & 0 \end{bmatrix},\quad \Pi'_-(\bar{Z};H) = \begin{bmatrix} 0 & 0 & \frac{\sigma}{1+\sigma}H_{13} \\ \sim & (H_{22})_- & H_{23} \\ \sim & \sim & H_{33} \end{bmatrix}.$$

Via careful calculation:

$$\mathcal{C}(\bar{Z}) = \begin{cases} \left\{ \begin{bmatrix} -H_{22} & H_{12} & 0 \\ \sim & H_{22} & 0 \\ \sim & \sim & 0 \end{bmatrix} \,\middle|\, H_{22} \geq 0 \right\}, & a \in [0,1) \\ \left\{ \begin{bmatrix} -H_{22} & H_{12} & 0 \\ \sim & H_{22} & H_{23} \\ \sim & \sim & 0 \end{bmatrix} \,\middle|\, H_{22} \geq 0 \right\}, & a = 1 \end{cases},\ \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}) = \left\{ \begin{bmatrix} -H_{22} & H_{12} & 0 \\ \sim & H_{22} & 0 \\ \sim & \sim & 0 \end{bmatrix} \,\middle|\, H_{22} \geq 0 \right\},\ \forall a \in [0,1]. \tag{59}$$

Therefore, to pick up $\bar{H} \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$, the only nontrivial case is $a = 1$ and $\bar{H}_{23} \neq 0$.

Second-order information. We adopt the polar description in Proposition 4 to calculate $\phi(\bar{Z};\bar{H})$. Via careful calculation:

$$\mathcal{E}(\bar{Z};\bar{H}) = \begin{bmatrix} 0 & 0 & \frac{2}{1+\sigma}\bar{H}_{12}\bar{H}_{23} \\ \sim & 2\bar{H}_{12}^2 & \frac{2}{\sigma}\bar{H}_{22}\bar{H}_{23} \\ \sim & \sim & 0 \end{bmatrix},\quad \mathcal{E}^\perp(\bar{Z};\bar{H}) = \begin{bmatrix} 0 & 0 & -\frac{2}{1+\sigma}\bar{H}_{12}\bar{H}_{23} \\ \sim & -\frac{2}{\sigma}\bar{H}_{23}^2 & -\frac{2}{\sigma}\bar{H}_{22}\bar{H}_{23} \\ \sim & \sim & 0 \end{bmatrix},$$

and

$$\mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H}) = \begin{cases} \left\{ W = \begin{bmatrix} W_{11} & W_{12} & 0 \\ \sim & W_{22} & 0 \\ \sim & \sim & 0 \end{bmatrix} \,\middle|\, W_{11}+W_{22} = 0 \right\}, & \bar{H}_{22} > 0 \\ \left\{ W = \begin{bmatrix} W_{11} & W_{12} & 0 \\ \sim & W_{22} & 0 \\ \sim & \sim & 0 \end{bmatrix} \,\middle|\, \begin{bmatrix} W_{11}+W_{22} = 0, \\ W_{22} \geq 0 \end{bmatrix} \right\}, & \bar{H}_{22} = 0 \end{cases},\quad \mathcal{K}_{\mathsf{D}}^\circ(\bar{Z};\bar{H}) = \left\{ W = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & W_{23} \\ \sim & \sim & 0 \end{bmatrix} \right\}.$$

(i) For the primal part, we need to consider two cases:
(a) $\bar{H}_{22} > 0$. In this case, from Theorem 7,

$$\phi_{\mathsf{P}}(\bar{Z};\bar{H}) = \operatorname*{arg\,min}_{W \in \mathcal{K}_{\mathsf{P}}^\circ(\bar{Z};\bar{H})} \|W + \mathcal{E}^\perp(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 = \operatorname*{arg\,min}_{W_{11}+W_{22}=0} (W_{22} - \tfrac{2}{\sigma}\bar{H}_{23}^2)^2 + 2W_{12}^2 + W_{11}^2 = \begin{bmatrix} -\frac{1}{\sigma}\bar{H}_{23}^2 & 0 & 0 \\ \sim & \frac{1}{\sigma}\bar{H}_{23}^2 & 0 \\ \sim & \sim & 0 \end{bmatrix}.$$

(b) $\bar{H}_{22} = 0$. Similar to case (a),

$$\phi_{\mathsf{P}}(\bar{Z};\bar{H}) = \underset{W\in\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z};\bar{H})}{\arg\min} \|W + \mathcal{E}^{\perp}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 = \underset{\substack{W_{11}+W_{22}=0,\\ W_{22}\geq 0}}{\arg\min} (W_{22} - \tfrac{2}{\sigma}\bar{H}_{23}^2)^2 + 2W_{12}^2 + W_{11}^2 = \begin{bmatrix} -\frac{1}{\sigma}\bar{H}_{23}^2 & 0 & 0 \\ \sim & \frac{1}{\sigma}\bar{H}_{23}^2 & 0 \\ \sim & \sim & 0 \end{bmatrix}.$$

Clearly, case (a) and (b) can be combined.

(ii) For the dual part, from Theorem 7:

$$-\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \underset{W\in\mathcal{K}_{\mathsf{D}}^{\circ}(\bar{Z};\bar{H})}{\arg\min} \|W + \mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 = \underset{W_{23}\in\mathbb{R}}{\arg\min}(W_{23} + \frac{2}{\sigma}\bar{H}_{22}\bar{H}_{23})^2 = \begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & -\frac{2}{\sigma}\bar{H}_{22}\bar{H}_{23} \\ \sim & \sim & 0 \end{bmatrix}.$$

(iii) Combining the primal and dual part:

$$\phi(\bar{Z};\bar{H}) = \phi_{\mathsf{P}}(\bar{Z};\bar{H}) - \sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \begin{bmatrix} -\frac{1}{\sigma}\bar{H}_{23}^2 & 0 & 0 \\ \sim & \frac{1}{\sigma}\bar{H}_{23}^2 & -\frac{2}{\sigma}\bar{H}_{22}\bar{H}_{23} \\ \sim & \sim & 0 \end{bmatrix}. \tag{60}$$

σ updating. When σ is updated to σ', $\bar{H}$ is updated to

$$\bar{H}' = \begin{bmatrix} -\bar{H}_{22} & \bar{H}_{12} & 0 \\ \sim & \bar{H}_{22} & \frac{\sigma'}{\sigma}\bar{H}_{23} \\ \sim & \sim & 0 \end{bmatrix}$$

from (55). Thus,

$$\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') = \begin{bmatrix} -\frac{1}{\sigma'}(\frac{\sigma'}{\sigma}\bar{H}_{23})^2 & 0 & 0 \\ \sim & \frac{1}{\sigma'}(\frac{\sigma'}{\sigma}\bar{H}_{23})^2 & 0 \\ \sim & \sim & 0 \end{bmatrix} = \frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H}),$$

$$\phi_{\mathsf{D}}(\bar{Z}';\bar{H}') = -\frac{1}{\sigma'}\begin{bmatrix} 0 & 0 & 0 \\ \sim & 0 & -\frac{2}{\sigma'}\bar{H}_{22}(\frac{\sigma'}{\sigma}\bar{H}_{23}) \\ \sim & \sim & 0 \end{bmatrix} = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H}).$$

9.3 Example III

SDP data. Consider a 6 by 6 SDP example. For ease of notation, define $E_{ij} \in \mathbb{S}^6$ $(1 \leq i, j \leq 6)$ as:

$$E_{ij}(m,n) := \begin{cases} 1, & m = i, n = j \\ 0, & \text{otherwise} \end{cases}$$

Moreover, $0_{m\times n}$ is an abbreviation of all-zero matrix of size $m \times n$ and I_m is an identity matrix of size $m \times m$. Define an orthonormal matrix Q as

$$Q := \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{bmatrix}.$$

The SDP data is

$$b = \begin{bmatrix} 6 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{15},\ C = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ \sim & I_3 \end{bmatrix},\ A_1 = I_6,\ A_2 = \begin{bmatrix} Q^{\mathsf{T}}\begin{bmatrix} 1 & 0 & 0 \\ \sim & -1 & 0 \\ \sim & \sim & 0 \end{bmatrix}Q & 0_{3\times 3} \\ \sim & 0_{3\times 3} \end{bmatrix},\ A_3 = \begin{bmatrix} Q^{\mathsf{T}}\begin{bmatrix} 1 & 0 & 0 \\ \sim & 0 & 0 \\ \sim & \sim & -1 \end{bmatrix}Q & 0_{3\times 3} \\ \sim & 0_{3\times 3} \end{bmatrix},$$

$$A_4 = E_{44} - E_{55},\ A_5 = E_{55} - E_{66},\ A_6 = E_{24} + E_{42},\ A_7 = E_{25} + E_{52},\ A_8 = E_{26} + E_{62},\ A_9 = E_{34} + E_{43},$$
$$A_{10} = E_{35} + E_{53},\ A_{11} = E_{36} + E_{63},\ A_{12} = E_{45} + E_{54},\ A_{13} = E_{46} + E_{64},\ E_{14} = E_{56} + E_{65},\ A_{14} = E_{16} + E_{61}. \tag{SDP-III}$$

One can verify that for (SDP-III), there exist strict complementary and rank-deficient solution pairs:

$$(X_{\mathsf{sc}}, S_{\mathsf{sc}}) = \left(\begin{bmatrix} 2I_3 & 0_{3\times 3} \\ \sim & 0_{3\times 3} \end{bmatrix}, \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ \sim & I_3 \end{bmatrix} \right), \quad (\bar{X}, \bar{S}) = (6E_{11},\ 3E_{66}).$$

Thus, we pick $\bar{Z} = 6E_{11} - 3\sigma E_{66}$.

First-order information. $H \in \mathcal{C}(\bar{Z})$ if and only if:

$$\mathcal{P} \left[\begin{array}{c:cc:cc:c} H_{11} & H_{12} & H_{13} & H_{14} & H_{15} & 0 \\ \hdashline \sim & \multicolumn{2}{c:}{\begin{bmatrix} H_{22} & H_{23} \\ \sim & H_{33} \end{bmatrix} \succeq 0} & \multicolumn{2}{c:}{0_{2\times 2}} & 0_{2\times 1} \\ \hline \sim & \multicolumn{2}{c:}{\sim} & \multicolumn{2}{c:}{0_{2\times 2}} & 0_{2\times 1} \\ \hdashline \sim & \multicolumn{2}{c:}{\sim} & \multicolumn{2}{c:}{\sim} & 0 \end{array} \right] = 0 \text{ and } \mathcal{P}^{\perp} \left[\begin{array}{c:c:cc:c} 0 & 0_{1\times 2} & \multicolumn{2}{c:}{0_{1\times 2}} & 0 \\ \hdashline \sim & 0_{2\times 2} & \multicolumn{2}{c:}{0_{2\times 2}} & \begin{matrix} H_{26} \\ H_{36} \end{matrix} \\ \hline \sim & \sim & \multicolumn{2}{c:}{\begin{bmatrix} H_{44} & H_{45} \\ \sim & H_{55} \end{bmatrix} \preceq 0} & \begin{matrix} H_{46} \\ H_{56} \end{matrix} \\ \hdashline \sim & \sim & \multicolumn{2}{c:}{\sim} & H_{66} \end{array} \right] = 0.$$

Via calculation, we find a family of $\bar{H}$'s belonging to $\mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$:

$$\bar{H} = \bar{H}(h, \epsilon) = \left[\begin{array}{c:cccc:c} -1 & 0 & -\frac{\sqrt{2}}{4} & 1 & h & 0 \\ \hdashline \sim & 1 & 0 & 0 & 0 & 1 \\ & \sim & 0 & 0 & 0 & 1 \\ & \sim & \sim & -\epsilon & 0 & 1 \\ & \sim & \sim & \sim & -1 & 1 \\ \hdashline \sim & & & \sim & & 1+\epsilon \end{array} \right] \in \mathcal{C}(\bar{Z}) \backslash \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z}), \quad \forall \epsilon \geq 0, h \in \mathbb{R}. \tag{61}$$

Second-order information. We adopt the polar description in Proposition 4 to calculate $\phi(\bar{Z}; \bar{H}(h, \epsilon))$.

(i) For the primal part, calculating $\mathcal{E}(\bar{Z}; \bar{H})$ from (33) as:

$$\mathcal{E}^{\perp}(\bar{Z}; \bar{H}) = \left[\begin{array}{c:cccc:c} 0 & 0 & 0 & \frac{\epsilon}{3} & \frac{h}{3} & -\frac{4h-\sqrt{2}+4}{6(\sigma+2)} \\ \hdashline \sim & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} \\ & \sim & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & 0 \\ & \sim & \sim & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} & \frac{2\epsilon}{3\sigma} \\ & \sim & \sim & \sim & -\frac{2}{3\sigma} & 0 \\ \hdashline \sim & & & \sim & & 0 \end{array} \right].$$

Calculate $\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z}; \bar{H})$ from (43):

$$\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z}; \bar{H}) = \left\{ U = \left[\begin{array}{c:cccc:c} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & 0 \\ \hdashline \sim & U_{22} & U_{23} & U_{24} & 0 & 0 \\ & \sim & U_{33} \geq 0 & 0 & 0 & 0 \\ & \sim & \sim & 0 & 0 & 0 \\ & \sim & \sim & \sim & 0 & 0 \\ \hdashline \sim & & & \sim & & 0 \end{array} \right] \middle| \ \mathcal{P} U = 0 \right\}$$

$$= \left\{ U = \left[\begin{array}{c:cccc:c} U_{11} & U_{12} & U_{13} & U_{14} & U_{15} & 0 \\ \hdashline \sim & U_{22} & U_{23} & U_{24} & 0 & 0 \\ & \sim & U_{33} & 0 & 0 & 0 \\ & \sim & \sim & 0 & 0 & 0 \\ & \sim & \sim & \sim & 0 & 0 \\ \hdashline \sim & & & \sim & & 0 \end{array} \right] \middle| \ \begin{array}{c} \widetilde{U} = Q \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ \sim & U_{22} & U_{23} \\ \sim & \sim & U_{33} \end{bmatrix} Q^{\mathsf{T}}, \\ \widetilde{U}_{11} = \widetilde{U}_{22} = \widetilde{U}_{33} = 0, \\ U_{33} \geq 0,\ U_{24} = 0,\ U_{14} \in \mathbb{R},\ U_{15} \in \mathbb{R} \end{array} \right\}.$$

Thus, from Theorem 7:

$$
\begin{aligned}
\phi_{\mathsf{P}}(\bar{Z};\bar{H}) &= \operatorname*{arg\,min}_{U\in\mathcal{K}_{\mathsf{P}}^{\circ}(\bar{Z};\bar{H})} \|U+\mathcal{E}^{\perp}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2 \\
&= \operatorname*{arg\,min}_{\substack{U_{33}\geq 0, U_{24}=0,\\ \widetilde{U}_{11}=\widetilde{U}_{22}=\widetilde{U}_{33}=0}} \|\widetilde{U}+Q\begin{bmatrix} 0 & 0 & 0 \\ \sim & -\frac{2}{3\sigma} & -\frac{2}{3\sigma} \\ \sim & \sim & -\frac{2}{3\sigma}\end{bmatrix}Q^{\mathsf{T}}\|_{\mathsf{F}}^2 + 2(U_{14}+\frac{\epsilon}{3})^2 + 2(U_{15}+\frac{h}{3})^2 + 2(U_{24}-\frac{2}{3\sigma})^2 \\
&= \left[\begin{array}{c|cccc|c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} & 0 & -\frac{\epsilon}{3} & -\frac{h}{3} & 0 \\ \hline \sim & \multicolumn{4}{c|}{\begin{bmatrix} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \sim & \frac{2}{9\sigma} & 0 & 0 \\ \sim & \sim & 0 & 0 \\ \sim & \sim & \sim & 0 \end{bmatrix}} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \\ \hline \sim & \multicolumn{4}{c|}{\sim} & 0 \end{array}\right].
\end{aligned}
$$

(ii) For the dual part: from (33),

$$
\mathcal{E}(\bar{Z};\bar{H}) = \left[\begin{array}{c|cccc|c} 0 & 0 & 0 & -\frac{\epsilon}{3} & -\frac{h}{3} & \frac{4h-\sqrt{2}+4}{6(\sigma+2)} \\ \hline \sim & \multicolumn{4}{c|}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sim & \frac{1}{24} & -\frac{\sqrt{2}}{12} & -\frac{\sqrt{2}h}{12} \\ \sim & \sim & \frac{1}{3} & \frac{h}{3} \\ \sim & \sim & \sim & \frac{h^2}{3} \end{bmatrix}} & \begin{matrix} \frac{2}{3\sigma} \\ 0 \\ -\frac{2\epsilon}{3\sigma} \\ 0 \end{matrix} \\ \hline \sim & \multicolumn{4}{c|}{\sim} & 0 \end{array}\right].
$$

However, $\mathcal{K}_{\mathsf{D}}^{\circ}(\bar{Z};\bar{H}(h,\epsilon))$ shows discontinuity at $\epsilon=0$.

(a) When $\epsilon=0$: in this case,

$$
\begin{aligned}
\mathcal{K}_{\mathsf{D}}^{\circ}(\bar{Z};\bar{H}) &= \left\{ V = \left[\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \sim & \multicolumn{4}{c|}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & V_{35} \\ \sim & \sim & V_{44}\leq 0 & V_{45} \\ \sim & \sim & \sim & V_{55} \end{bmatrix}} & \begin{matrix} V_{26} \\ V_{36} \\ V_{46} \\ V_{56} \end{matrix} \\ \hline \sim & \multicolumn{4}{c|}{\sim} & V_{66} \end{array}\right] \;\middle|\; \mathcal{P}^{\perp}V = 0 \right\} \\
&= \left\{ V = \left[\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \sim & \multicolumn{4}{c|}{\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sim & 0 & 0 & V_{35} \\ \sim & \sim & V_{44} & V_{45} \\ \sim & \sim & \sim & V_{55} \end{bmatrix}} & \begin{matrix} V_{26} \\ V_{36} \\ V_{46} \\ V_{56} \end{matrix} \\ \hline \sim & \multicolumn{4}{c|}{\sim} & V_{66} \end{array}\right] \;\middle|\; \begin{array}{c} V_{44}\leq 0, \\ \begin{bmatrix} V_{44} \\ V_{55} \\ V_{66} \end{bmatrix} \in a\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ a,b\in\mathbb{R}, \\ V_{26},V_{35},V_{36},V_{45},V_{46},V_{56}\in\mathbb{R} \end{array} \right\}.
\end{aligned}
$$

Thus, from Theorem 7, we get

$$
\begin{aligned}
&-\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \underset{V\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})}{\arg\min}\ \|V+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
&= \underset{\substack{V_{44}\le 0,\\ V_{44}=a,\\ V_{55}=-a+b,\\ V_{66}=-b}}{\arg\min}\ \left\|\begin{bmatrix}V_{44}\\V_{55}\\V_{66}\end{bmatrix}+\begin{bmatrix}\frac{1}{3}\\ \frac{h^2}{3}\\ 0\end{bmatrix}\right\|_{\mathsf{F}}^2 + 2(V_{26}+\tfrac{2}{3\sigma})^2+2(V_{35}-\tfrac{\sqrt{2}h}{12})^2+2(V_{45}+\tfrac{h}{3})^2+2V_{36}^2+2V_{46}^2+2V_{56}^2\\
&= \left[\begin{array}{c:c:c}
0 & 0\quad 0\quad 0\quad 0 & 0\\ \hdashline
\sim & \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0\\
\sim & 0 & 0 & \frac{\sqrt{2}h}{12}\\ \hline
\sim & \sim & a^{\star} & -\frac{h}{3}\\
\sim & \sim & \sim & -a^{\star}+b^{\star}
\end{array}\right] & \begin{array}{c}-\frac{2}{3\sigma}\\0\\0\\0\end{array}\\ \hdashline
\sim & \sim & -b^{\star}
\end{array}\right].
\end{aligned}
$$

where $(a^{\star},b^{\star})$ is defined as:

$$
(a^{\star},b^{\star}) = \underset{a\le 0}{\arg\min}(a+\frac{1}{3})^2+(-a+b+\frac{h^2}{3})^2+b^2 = \begin{cases}(\frac{1}{9}(h^2-2),\ -\frac{1}{9}(h^2+1)), & |h|\le\sqrt{2}\\ (0,\ -\frac{1}{6}h^2), & |h|>\sqrt{2}\end{cases}
$$

(b) When $\epsilon>0$: in this case,

$$
\begin{aligned}
\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H}) &= \left\{ V = \left[\begin{array}{c:c:c}
0 & 0\quad 0\quad 0\quad 0 & 0\\ \hdashline
\sim & \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0\\
\sim & 0 & 0 & V_{35}\\ \hline
\sim & \sim & V_{44} & V_{45}\\
\sim & \sim & \sim & V_{55}
\end{array}\right] & \begin{array}{c}V_{26}\\V_{36}\\V_{46}\\V_{56}\end{array}\\ \hdashline
\sim & \sim & V_{66}
\end{array}\right] \ \middle|\ \mathcal{P}^{\perp}V = 0 \right\}\\
&= \left\{ V = \left[\begin{array}{c:c:c}
0 & 0\quad 0\quad 0\quad 0 & 0\\ \hdashline
\sim & \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0\\
\sim & 0 & 0 & V_{35}\\ \hline
\sim & \sim & V_{44} & V_{45}\\
\sim & \sim & \sim & V_{55}
\end{array}\right] & \begin{array}{c}V_{26}\\V_{36}\\V_{46}\\V_{56}\end{array}\\ \hdashline
\sim & \sim & V_{66}
\end{array}\right] \ \middle|\ \begin{array}{c}\begin{bmatrix}V_{44}\\V_{55}\\V_{66}\end{bmatrix}\in a\begin{bmatrix}1\\-1\\0\end{bmatrix}+b\begin{bmatrix}0\\1\\-1\end{bmatrix},\ a,b\in\mathbb{R},\\ V_{26},V_{35},V_{36},V_{45},V_{46},V_{56}\in\mathbb{R}\end{array} \right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&-\sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \underset{V\in\mathcal{K}^{\circ}_{\mathsf{D}}(\bar{Z};\bar{H})}{\arg\min}\ \|V+\mathcal{E}(\bar{Z};\bar{H})\|_{\mathsf{F}}^2\\
&= \underset{\substack{V_{44}=a,\\ V_{55}=-a+b,\\ V_{66}=-b}}{\arg\min}\ \left\|\begin{bmatrix}V_{44}\\V_{55}\\V_{66}\end{bmatrix}+\begin{bmatrix}\frac{1}{3}\\ \frac{h^2}{3}\\ 0\end{bmatrix}\right\|_{\mathsf{F}}^2 + 2(V_{26}+\tfrac{2}{3\sigma})^2+2(V_{35}-\tfrac{\sqrt{2}h}{12})^2+2(V_{45}+\tfrac{h}{3})^2+2V_{36}^2+2(V_{46}-\tfrac{2\epsilon}{3\sigma})^2+2V_{56}^2\\
&= \left[\begin{array}{c:c:c}
0 & 0\quad 0\quad 0\quad 0 & 0\\ \hdashline
\sim & \left[\begin{array}{cc|cc}
0 & 0 & 0 & 0\\
\sim & 0 & 0 & \frac{\sqrt{2}h}{12}\\ \hline
\sim & \sim & \frac{1}{9}(h^2-2) & -\frac{h}{3}\\
\sim & \sim & \sim & -\frac{1}{9}(2h^2-1)
\end{array}\right] & \begin{array}{c}-\frac{2}{3\sigma}\\0\\ \frac{2\epsilon}{3\sigma}\\0\end{array}\\ \hdashline
\sim & \sim & \frac{1}{9}(h^2+1)
\end{array}\right].
\end{aligned}
$$

(iii) Combining (i) and (iii), we get:
(a) If case I: $|h| \le \sqrt{2}$; or case II: $|h| > \sqrt{2}$ and $\epsilon > 0$,

$$\phi(\bar{Z};\bar{H}) = \phi_{\mathsf{P}}(\bar{Z};\bar{H}) - \sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \left[\begin{array}{c:c:c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} \quad 0 \quad -\frac{\epsilon}{3} \quad -\frac{h}{3} & 0 \\ \hdashline \sim & \left[\begin{array}{c:c|c:c} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \hdashline \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} \\ \hline \sim & \sim & \frac{h^2-2}{9} & -\frac{h}{3} \\ \hdashline \sim & \sim & \sim & -\frac{2h^2-1}{9} \end{array}\right] & \begin{array}{c} -\frac{2}{3\sigma} \\ 0 \\ \frac{2\epsilon}{3\sigma} \\ 0 \end{array} \\ \hdashline \sim & \sim & \frac{h^2+1}{9} \end{array}\right]. \tag{62}$$

(b) If $|h| > \sqrt{2}$ and $\epsilon = 0$,

$$\phi(\bar{Z};\bar{H}) = \phi_{\mathsf{P}}(\bar{Z};\bar{H}) - \sigma\phi_{\mathsf{D}}(\bar{Z};\bar{H}) = \left[\begin{array}{c:c:c} -\frac{4}{9\sigma} & -\frac{2\sqrt{2}}{9\sigma} \quad 0 \quad 0 \quad -\frac{h}{3} & 0 \\ \hdashline \sim & \left[\begin{array}{c:c|c:c} \frac{2}{9\sigma} & \frac{4}{9\sigma} & 0 & 0 \\ \hdashline \sim & \frac{2}{9\sigma} & 0 & \frac{\sqrt{2}h}{12} \\ \hline \sim & \sim & 0 & -\frac{h}{3} \\ \hdashline \sim & \sim & \sim & -\frac{h^2}{6} \end{array}\right] & \begin{array}{c} -\frac{2}{3\sigma} \\ 0 \\ 0 \\ 0 \end{array} \\ \hdashline \sim & \sim & \frac{h^2}{6} \end{array}\right]. \tag{63}$$

σ updating. Following the exact same procedure in §9.1 and §9.2, we can get $\phi_{\mathsf{P}}(\bar{Z}';\bar{H}') = \frac{\sigma'}{\sigma}\phi_{\mathsf{P}}(\bar{Z};\bar{H})$ and $\phi_{\mathsf{D}}(\bar{Z}';\bar{H}') = \frac{\sigma}{\sigma'}\phi_{\mathsf{D}}(\bar{Z};\bar{H})$ as we update σ to σ' for all $\bar{H} = \bar{H}(h,\epsilon)$. We omit the details here.

10 Numerical Experiments

Experiment set-up. To further qualitatively evaluate our analysis framework, we run the three-step ADMM (2) on the `Mittelmann` dataset, a widely used benchmark for SDP solvers [1, 18, 41, 45]. For concreteness, we select the single-block instances with block size no greater than 3000, yielding 25 instances in total. All experiments were executed on the Harvard Faculty of Arts and Sciences Research Computing (FASRC) cluster. Jobs were submitted to the `seas_compute` partition, and each run requested 48 CPU cores and 64 GB of memory.[2]

Experiment I. After rescaling the SDP data and applying diagonal preconditioning to the constraint matrix $\mathcal{A}$, we run ADMM using a *fixed* σ-updating strategy that aims to balance the primal and dual infeasibilities [49] for 20000 iterations. After that, σ is kept unchanged. At the 40000th iteration, we record the current penalty parameter as σ_0 and set $(X^{(40000)}, y^{(40000)}, S^{(40000)})$ as (X_0, y_0, S_0) for subsequent use. For each instance, we set the maximum number of iterations to 10^6 and the maximum running time to 168 hours. We terminate once the maximum KKT residual satisfies $r_{\max} \le 10^{-10}$.

We report the trajectories of $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$, $\|\Delta Z^{(k)}\|_{\mathsf{F}}$, and $r_{\max}^{(k)}$. The results are shown in Figure 14. Based on whether ADMM solves an instance to $r_{\max}$ below 10^{-10}, we divide the 25 SDPs into two groups:

- "Easy" SDPs: `1et2048`, `1zc1024`, `cphil12`, `G48mb`, `G48mc`, `hamming8`, `hamming9`, `theta12`, `theta102`, `theta123`. (10 instances.)
- "Hard" SDPs: `1dc1024`, `1tc2048`, `cancer100`, `cnhil10`, `foot`, `G40mb`, `hand`, `neosfbr25`, `neosfbr30e8`, `neu1g`, `neu2g`, `neu3g`, `r12000`, `swissroll`, `texture`. (15 instances.)

Across the slow-convergence regions of all "hard" instances, we observe that $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ remains small yet nonzero for most iterations (typically around 10^{-3} to 10^{-5}), except for several sparse spikes. This behavior is consistent with our local second-order limiting dynamics model (42), as discussed in §5.3 and §7.3.

[2] https://docs.rc.fas.harvard.edu/kb/running-jobs/

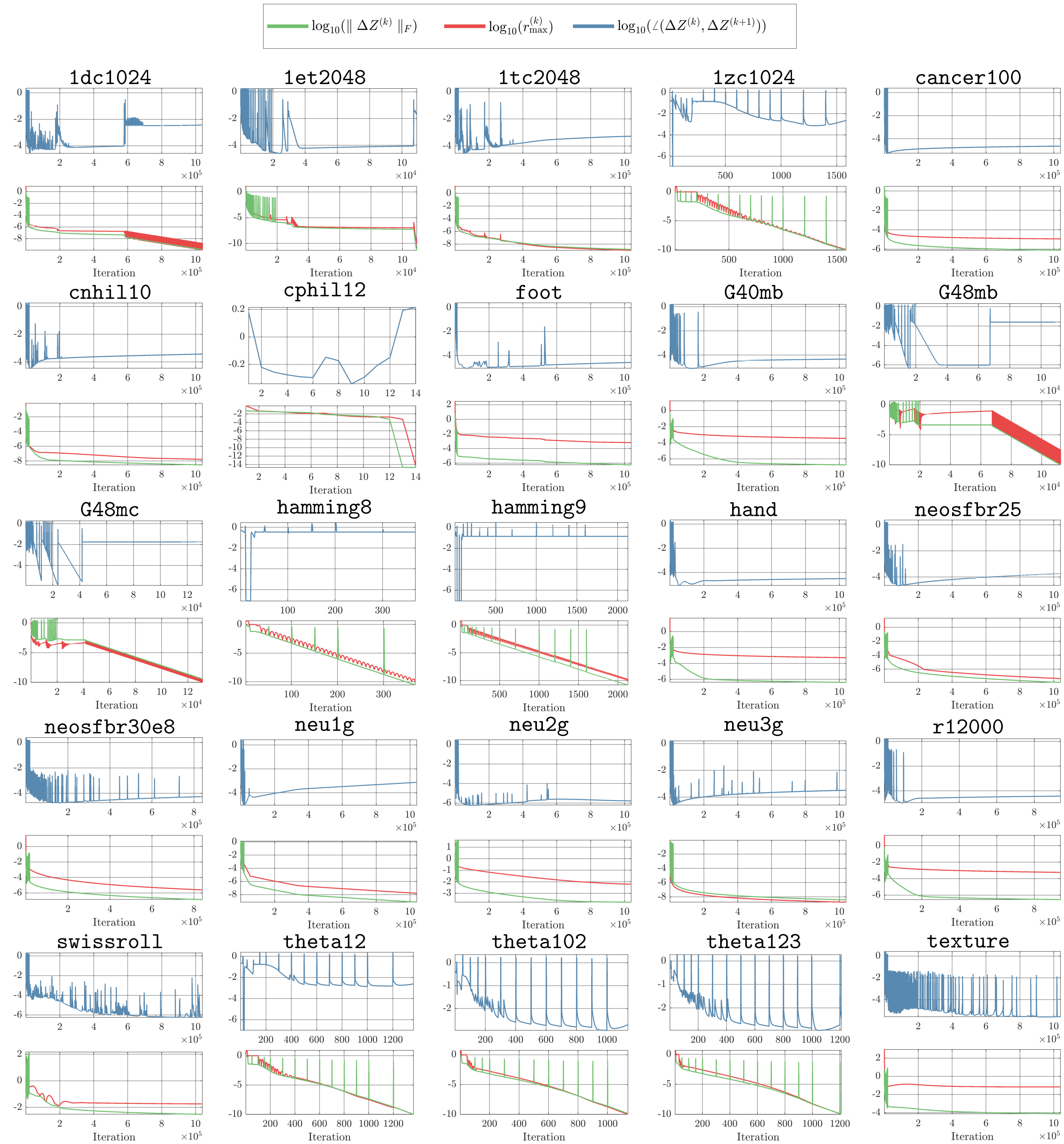


Figure 14: $r_{\max}^{(k)}$, $\|\Delta Z^{(k)}\|_{\mathsf{F}}$, and $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ for 25 `Mittelmann` SDP datasets (single-block, block size ≤ 3000). For *all* instances exhibiting slow convergence, $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ remains small for an extended period, except for a few sparse spikes. In contrast, for many instances that display an observable sharp linear convergence phase, $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ is large.

Experiment II. To further probe the slow-convergence regimes of the 15 "hard" instances, we restart from (X_0, y_0, S_0) and σ_0, and then uniformly increase $\log_{10}(\sigma)$ from $\log_{10}(\sigma_0)$ to $\log_{10}(10\sigma_0)$ over the next 5000 iterations. As in §8.3, we plot the resulting "response curves" of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$ as functions of σ. The results are shown in Figure 15. We further divide them into three groups:

- Group I: `cnhil10`, `foot`, `neu1g`, `neu3g`, `texture`. (5 instances.)
- Group II: `1dc1024`, `G40mb`, `hand`, `neosfbr25`, `r12000`, `swissroll`. (6 instances.)
- Group III: `1tc2048`, `cancer100`, `neosfbr30e8`, `neu2g`. (4 instances.)

The response curves of Group I are compatible with our limiting dynamics model. For these instances, $\log_{10}(\|\Delta X^{(k)}\|_{\mathsf{F}})$ (resp. $\log_{10}(\|\Delta S^{(k)}\|_{\mathsf{F}})$) increases (resp. decreases) approximately linearly with $\log_{10}(\sigma)$, with slope close to $+1$ (resp. -1). For these 5 SDPs, the slow-convergence regions are therefore likely driven by the second-order limiting dynamics (42).

For Group II, the response curves partially resemble those in Group I. When σ is small, the slope of $\log_{10}(\|\Delta X^{(k)}\|_{\mathsf{F}})$ (resp. $\log_{10}(\|\Delta S^{(k)}\|_{\mathsf{F}})$) is close to $+1$ (resp. -1). However, as σ increases, the curves distort, either smoothly or abruptly. For these 6 SDPs, we conjecture that updating σ helps the iterates escape the current second-order-dominant regions.

For Group III, the response curves deviate substantially from those predicted by the local second-order limiting dynamics model. The mechanisms underlying the slow-convergence behavior of these 4 instances therefore remain unclear to us.

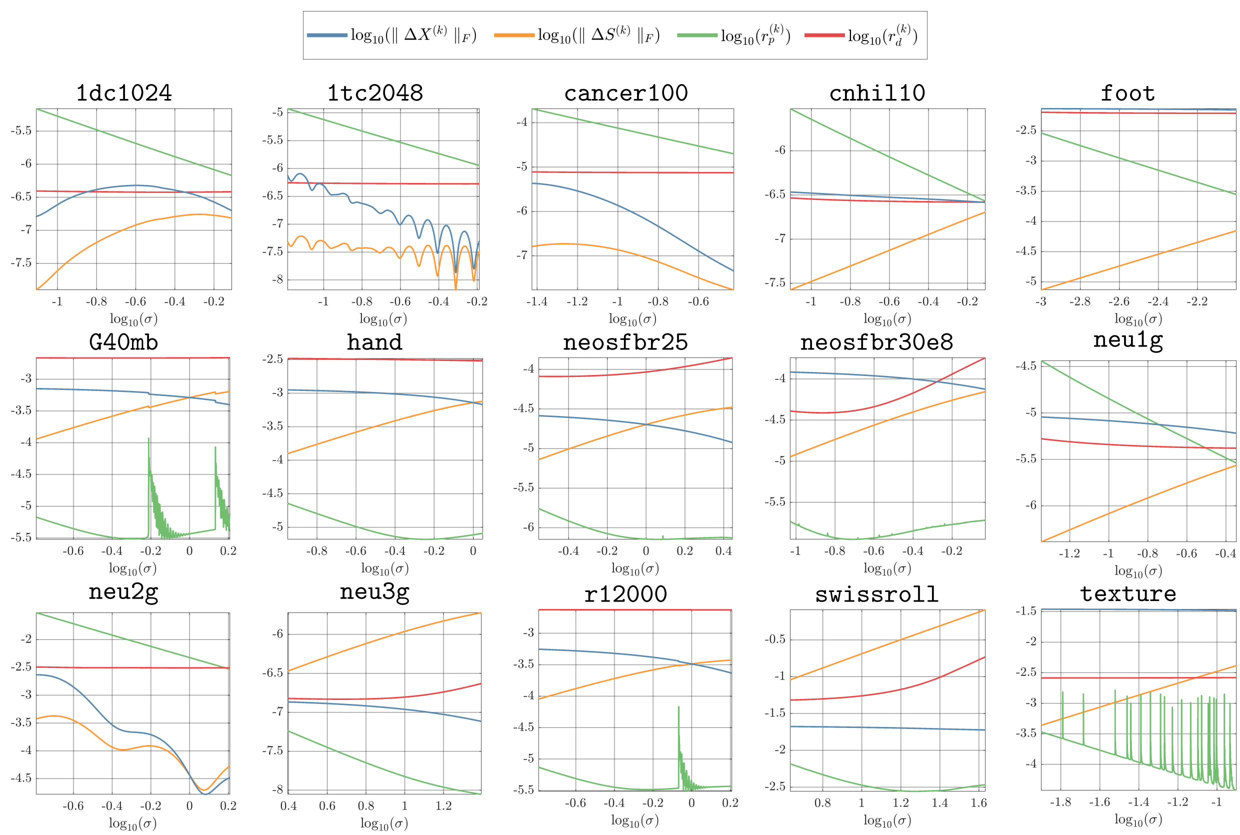


Figure 15: For the selected 15 "hard" SDP instances in the `Mittelmann` dataset, we run ADMM iterations with initial guess (X_0, y_0, S_0) and initial $\sigma = \sigma_0$. We uniformly increase $\log_{10}(\sigma)$ from $\log_{10}(\sigma_0)$ to $\log_{10}(10\sigma_0)$ over the next 5000 iterations. We report the trajectories of $\|\Delta X^{(k)}\|_{\mathsf{F}}$, $\|\Delta S^{(k)}\|_{\mathsf{F}}$, $r_p^{(k)}$, and $r_d^{(k)}$.

11 Conclusion

This paper developed a transient, region-wise perspective on the slow-convergence behavior observed in ADMM for degenerate SDPs. We refined and streamlined the (parabolic) second-order directional derivative formula for the PSD projection operator, and leveraged it to derive a detailed second-order expansion of the ADMM dynamics around an arbitrary KKT point $\bar{Z}$. This expansion isolates the cone $\mathcal{C}(\bar{Z})$ of first-order stalled directions and leads to the central object of the paper: the local second-order limit map $\phi(\bar{Z};\cdot) : \mathcal{C}(\bar{Z}) \mapsto \mathbb{S}^n$ and its induced limit dynamics. This limit dynamics serves as a local surrogate for the nonlinear ADMM update after transient effects have decayed.

We then analyzed four structural properties of $\phi(\bar{Z};\cdot)$: its kernel, range, continuity, and dependence on the penalty parameter σ. These results explain or predict three empirical slow-convergence patterns:

- Using the characterization $\ker(\phi(\bar{Z};\cdot)) = \mathcal{T}_{\mathcal{Z}_\star}(\bar{Z})$ together with the almost-sure type continuity of the limit map, we showed that $\angle(\Delta Z^{(k)}, \Delta Z^{(k+1)})$ tends to be small yet nonzero, except for sparse spikes.
- By relating $\mathrm{ran}(\phi(\bar{Z};\cdot))$ to $\mathrm{aff}(\mathcal{C}(\bar{Z}))$, we showed that $Z^{(k)}$ can be transiently trapped in a low-dimensional subspace for an extended period of time.
- Exploiting a primal–dual decoupling of $\phi(\bar{Z};\cdot)$, we showed that primal/dual infeasibilities are locally insensitive to σ in the second-order-dominant regimes, clarifying why classical balancing heuristics can become ineffective.

Extensive experiments on the `Mittelmann` dataset corroborate these theoretical predictions.

Future directions. Several directions remain open. On the singularity-degree side, it would be valuable to relax strict complementarity and understand how the resulting picture changes. On the dynamics side, a quantitative theory for the leakage from $\mathcal{C}(\bar{Z})$ would strengthen the connection to almost-invariant set phenomena. On the algorithmic side, the limit map suggests the possibility of principled σ-adaptation and acceleration mechanisms tailored to degenerate regimes. Finally, it would be interesting to extend the present approach beyond ADMM to other splitting schemes (*e.g.,* sGS-ADMM and PDHG) and to other conic programs where projection operators admit a comparable second-order structure.

Acknowledgments

We sincerely thank Henry Wolkowicz for valuable discussions.

Appendix A Proof of Theorem 2

Let $F : \mathbb{S}^n \mapsto \mathbb{S}^n$ be a (parabolically) second-order directional differentiable spectral function generated by a (parabolically) second-order directional differentiable scalar function $f : \mathbb{R} \mapsto \mathbb{R}$. For $x \neq y \neq z$, define first- and (parabolic) second-order divided difference of f as

$$\begin{aligned} f^{[1]}(x,y) &:= \frac{f(x)-f(y)}{x-y} = \frac{f(x)}{x-y} + \frac{f(y)}{y-x}, \\ f^{[2]}(x,y,z) &:= \frac{f^{[1]}(x,y)-f^{[1]}(x,z)}{y-z} = \frac{f(x)}{(x-y)(x-z)} + \frac{f(y)}{(y-x)(y-z)} + \frac{f(z)}{(z-x)(z-y)}. \end{aligned}$$

Now let (Z,H,W) is given by the three-level description in §3. For any $k \in \mathcal{I}$, denote $\Phi_k : \mathbb{S}^{|\alpha_k|} \mapsto \mathbb{S}^{|\alpha_k|}$ as the spectral function generated by the scalar function as $f'(\mu_k;\cdot)$. For any $k \in \mathcal{I}, i \in \mathcal{I}_k$, denote $\Psi_{k,i} : \mathbb{S}^{|\beta_{k,i}|} \mapsto \mathbb{S}^{|\beta_{k,i}|}$, such that $\Psi_{k,i}$ is generated by the scalar function is $f''(\mu_k;\eta_{k,i},\cdot)$. For any $a,b \in \mathcal{I}$, define

$$\Gamma_1(H,W)_{\alpha_a\alpha_b} := \begin{cases} f^{[1]}(\mu_a,\mu_b)W_{\alpha_a\alpha_b} + \sum\limits_{c\neq\{a,b\}} 2f^{[2]}(\mu_a,\mu_b,\mu_c)H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} - \frac{2(f(\mu_a)-f(\mu_b))}{(\mu_a-\mu_b)^2}(H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b} - H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}), & a \neq b \\ -\sum\limits_{c\in\mathcal{I}\setminus\{a\}} \frac{2(f(\mu_a)-f(\mu_c))}{(\mu_a-\mu_c)^2} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}, & a = b \end{cases}, \tag{64}$$

and

$$\Gamma_2(H,W)_{\alpha_a\alpha_b} = \begin{cases} Q^a[\Omega^a \circ (Q^a)^\top V_a Q^a]\,(Q^a)^\top, & a = b \\ \Phi_a(H_{\alpha_a\alpha_a})\frac{2H_{\alpha_a\alpha_b}}{\mu_a-\mu_b} + \frac{2H_{\alpha_a\alpha_b}}{\mu_b-\mu_a}\Phi_b(H_{\alpha_b\alpha_b}), & a \neq b \end{cases}, \tag{65}$$

and

$$\Gamma_3(H,W)_{\alpha_a\alpha_b} = \begin{cases} Q^a \mathrm{diag}\left(\left\{\Psi_{a,i}(\hat{V}_a^{i,i})\right\}_{i\in\mathcal{I}_a}\right)(Q^a)^\top, & a = b \\ 0, & a \neq b \end{cases}. \tag{66}$$

To prove Theorem 2, we first need the (parabolic) second-order directional derivative's formula for a general spectral function F from [52, Theorem 4.1]:

Theorem 8 ($F''(Z;H,W)$). *Let the triplet (Z,H,W) given by the three-level description in §3. Then, for any $a,b \in \mathcal{I}$,*

$$F''(Z;H,W)_{\alpha_a\alpha_b} = \Gamma_1(H,W)_{\alpha_a\alpha_b} + \Gamma_2(H,W)_{\alpha_a\alpha_b} + \Gamma_3(H,W)_{\alpha_a\alpha_b},$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ is defined in (64) to (66). [3]

Now we are already to prove Theorem 2. For PSD cone projection operator $\Pi_{\mathbb{S}^n_+}(\cdot)$, $f(\mu_k) = \max\{\mu_k, 0\}$. Thus,

$$f'(\mu_k;\eta_{k,i}) = \begin{cases} \eta_{k,i}, & \mu_k > 0 \\ \max\{\eta_{k,i},0\}, & \mu_k = 0 \\ 0, & \mu_k < 0 \end{cases},$$

and

$$f''(\mu_k;\eta_{k,i},\zeta_{k,i,i'}) = \begin{cases} \zeta_{k,i,i'}, & \mu_k > 0 \\ \begin{cases} \zeta_{k,i,i'}, & \eta_{k,i} > 0 \\ \max\{\zeta_{k,i,i'},0\}, & \eta_{k,i} = 0 \\ 0, & \eta_{k,i} = 0 \end{cases}, & \mu_k = 0 \\ 0, & \mu_k < 0 \end{cases}.$$

[3] In [52, Eq. (4.4) - (4.5)], the authors drops the multiplier 2, and it should be $-\frac{f(\mu_l)-f(\mu_k)}{(\mu_l-\mu_k)^2}$, instead of $\frac{f(\mu_l)-f(\mu_k)}{(\mu_l-\mu_k)^2}$. In [52, Theorem 4.1], $[F''(Z;H,W)]_{\alpha_a\alpha_a}$ drops the term $C(H,W)_{\alpha_a\alpha_a}$.

Case (1)(i): $a \in \mathcal{I}_+, b \in \mathcal{I}_+$ **and** $a \neq b$. For $\Gamma_1(H,W)_{\alpha_a\alpha_b}$, $f^{[1]}(\mu_a,\mu_b) = 1$, $f(\mu_a) - f(\mu_b) = \mu_a - \mu_b$, and

$$f^{[2]}(\mu_a,\mu_b,\mu_c) = \begin{cases} 0, & c \in \mathcal{I}_+ \cup \mathcal{I}_0 \\ \frac{1}{\mu_a-\mu_b}\left(\frac{\mu_a}{\mu_a-\mu_c} - \frac{\mu_b}{\mu_b-\mu_c}\right) = \frac{-\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)}, & c \in \mathcal{I}_- \end{cases}$$

Thus, [4]

$$\Gamma_1(H,W)_{\alpha_a\alpha_b} = W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_-} \frac{-\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} - \frac{2}{\mu_a-\mu_b}(H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b} - H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}).$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_b}$, we have $\Phi_a(H_{\alpha_a\alpha_a}) = H_{\alpha_a\alpha_a}$ since $f'(\mu_a;\eta_{a,i}) = \eta_{a,i}$. Symmetrically, $\Phi_b(H_{\alpha_b\alpha_b}) = H_{\alpha_b\alpha_b}$. Thus,

$$\Gamma_2(H,W)_{\alpha_a\alpha_b} = \frac{2}{\mu_a-\mu_b}H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b} - \frac{2}{\mu_a-\mu_b}H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}.$$

For $\Gamma_3(H,W)_{\alpha_a\alpha_b}$, it is 0. Thus,

$$\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_b} = W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_-}\frac{-\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}.$$

Case (1)(ii): $a \in \mathcal{I}_+, b \in \mathcal{I}_+$ **and** $a = b$. For $\Gamma_1(H,W)_{\alpha_a\alpha_a}$,

$$\begin{aligned}\Gamma_1(H,W)_{\alpha_a\alpha_a} &= -2\sum_{c\in\mathcal{I}\setminus\{a\}}\frac{f(\mu_a)-f(\mu_c)}{(\mu_a-\mu_c)^2}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} \\ &= -2\sum_{c\in\mathcal{I}_+\setminus\{a\}}\frac{1}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} - 2\sum_{c\in\mathcal{I}_0}\frac{1}{\mu_a}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} - 2\sum_{c\in\mathcal{I}_-}\frac{\mu_a}{(\mu_a-\mu_c)^2}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}.\end{aligned}$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_a}$, since $[f'(\mu_a,\cdot)]^{[1]}(\eta_{a,i},\eta_{a,j}) = 1$, we have

$$\Omega^a_{\beta_{a,i},\beta_{a,j}} = \begin{cases} E_{|\beta_{a,i}|\times|\beta_{a,j}|}, & i \neq j \\ 0, & i = j \end{cases}$$

Thus,

$$\begin{aligned}\Gamma_2(H,W)_{\alpha_a\alpha_a} =& Q^a\left[\Omega^a \circ \left((Q^a)^\mathsf{T} V_a(H,W)Q^a\right)\right](Q^a)^\mathsf{T} \\ =& V_a(H,W) - Q^a\mathrm{diag}\left(\left\{\hat{V}_a^{i,i}(H,W)\right\}_{i\in\mathcal{I}_a}\right)(Q^a)^\mathsf{T} \\ =& W_{\alpha_a\alpha_a} + \sum_{c\in\mathcal{I}\setminus\{a\}}\frac{2}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} - Q^a\mathrm{diag}\left(\left\{\hat{V}_a^{i,i}(H,W)\right\}_{i\in\mathcal{I}_a}\right)(Q^a)^\mathsf{T}.\end{aligned}$$

For $\Gamma_3(H,W)_{\alpha_a\alpha_a}$, since $f''(\mu_a;\eta_{a,i},\zeta_{a,i,i'}) = \zeta_{a,i,i'}$, $\Psi_{a,i}(\hat{V}_a^{i,i}) = \hat{V}_a^{i,i}$. Thus,

$$\Gamma_3(H,W)_{\alpha_a\alpha_a} = Q^a\mathrm{diag}\left(\left\{\hat{V}_a^{i,i}(H,W)\right\}_{i\in\mathcal{I}_a}\right)(Q^a)^\mathsf{T}.$$

[4] In [32, Eq. (10a)], the μ_j in the numerator should be $-\mu_j$.

Summing up all three terms:

$$\begin{aligned}
&\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_a} = \Gamma_1(H,W)_{\alpha_a\alpha_a} + \Gamma_2(H,W)_{\alpha_a\alpha_a} + \Gamma_3(H,W)_{\alpha_a\alpha_a} \\
=&W_{\alpha_a\alpha_a} + 2\sum_{c\in\mathcal{I}_-} \frac{1}{\mu_a-\mu_c} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} - 2\sum_{c\in\mathcal{I}_-}\frac{\mu_a}{(\mu_a-\mu_c)^2} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} \\
=&W_{\alpha_a\alpha_a} + 2\sum_{c\in\mathcal{I}_-} \frac{-\mu_c}{(\mu_a-\mu_c)^2} H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}.
\end{aligned}$$

Clearly, Case (1)(i) and Case (1)(ii)'s results can be merged.

Case (2): $a\in\mathcal{I}_+, b\in\mathcal{I}_0$. $\Gamma_1(H,W)_{\alpha_a\alpha_b}$ is the same as Case (1)(i): $a\in\mathcal{I}_+, b\in\mathcal{I}_+$ and $a\neq b$, except that $\mu_b=0$:

$$\Gamma_1(H,W)_{\alpha_a\alpha_b} = W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} - \frac{2}{\mu_a}(H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b} - H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}).$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_b}$, we have $f'(\mu_a;\eta_{a,i}) = \eta_{a,i}$ and $f'(\mu_b;\eta_{b,i}) = f'(0;\eta_{b,i}) = \max\{\eta_{b,i},0\}$. Therefore,

$$\Phi_a(H_{\alpha_a\alpha_a}) = H_{\alpha_a\alpha_a}, \quad \Phi_b(H_{\alpha_b\alpha_b}) = \Pi_+(H_{\alpha_b\alpha_b}).$$

Consequently,

$$\Gamma_2(H,W)_{\alpha_a\alpha_b} = H_{\alpha_a\alpha_a}\frac{2H_{\alpha_a\alpha_b}}{\mu_a} - \frac{2H_{\alpha_a\alpha_b}}{\mu_a}\Pi_+(H_{\alpha_b\alpha_b}).$$

For $\Gamma_3(H,W)_{\alpha_a\alpha_b}$, it is 0 since $a\neq b$. Thus,

$$\begin{aligned}
&\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_b} \\
=&W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} + \frac{2}{\mu_a}H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b} - \frac{2}{\mu_a}H_{\alpha_a\alpha_b}\Pi_+(H_{\alpha_b\alpha_b}) \\
=&W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_-}\frac{1}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} - 2\frac{1}{\mu_a}H_{\alpha_a\alpha_b}\Pi_+(-H_{\alpha_b\alpha_b}).
\end{aligned}$$

Case (3): $a\in\mathcal{I}_+, b\in\mathcal{I}_-$. In this case, $f^{[1]}(\mu_a,\mu_b) = \frac{\mu_a}{\mu_a-\mu_b}$, and

$$f^{[2]}(\mu_a,\mu_b,\mu_c) = \begin{cases} \frac{1-\frac{-\mu_c}{\mu_b-\mu_c}}{\mu_a-\mu_b} = \frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)}, & c\in\mathcal{I}_+\backslash\{a\} \\ \frac{1}{\mu_a-\mu_b}, & c\in\mathcal{I}_0 \\ \frac{\frac{\mu_a}{\mu_a-\mu_c}}{\mu_a-\mu_b} = \frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)}, & c\in\mathcal{I}_-\backslash\{b\} \end{cases}$$

Thus,

$$\begin{aligned}
\Gamma_1(H,W)_{\alpha_a\alpha_b} =& \frac{\mu_a}{\mu_a-\mu_b}W_{\alpha_a\alpha_b} + 2\sum_{c\in\mathcal{I}_+\backslash\{a\}}\frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \\
&+ 2\sum_{c\in\mathcal{I}_0}\frac{1}{\mu_a-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} + 2\sum_{c\in\mathcal{I}_-\backslash\{b\}}\frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} \\
&- 2\frac{\mu_a}{(\mu_a-\mu_b)^2}(H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b} - H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}).
\end{aligned}$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_b}$: since $\Phi_b(H_{\alpha_b\alpha_b})=0$, we have

$$\Gamma_2(H,W)_{\alpha_a\alpha_b} = 2\frac{1}{\mu_a-\mu_b}H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b}.$$

$\Gamma_3(H,W)_{\alpha_a\alpha_b}=0$ since $a\neq b$. Thus,

$$\begin{aligned}\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_b}=&\frac{\mu_a}{\mu_a-\mu_b}W_{\alpha_a\alpha_b}+2\sum_{c\in\mathcal{I}_+\setminus\{a\}}\frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\\&+2\sum_{c\in\mathcal{I}_0}\frac{1}{\mu_a-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}+2\sum_{c\in\mathcal{I}_-\setminus\{b\}}\frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\\&+2\frac{-\mu_b}{(\mu_a-\mu_b)^2}H_{\alpha_a\alpha_a}H_{\alpha_a\alpha_b}+2\frac{\mu_a}{(\mu_a-\mu_b)^2}H_{\alpha_a\alpha_b}H_{\alpha_b\alpha_b}\\=&\frac{\mu_a}{\mu_a-\mu_b}W_{\alpha_a\alpha_b}+2\sum_{c\in\mathcal{I}_+}\frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\\&+2\sum_{c\in\mathcal{I}_0}\frac{1}{\mu_a-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}+2\sum_{c\in\mathcal{I}_-}\frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}.\end{aligned}$$

Case (4): $a\in\mathcal{I}_0, b\in\mathcal{I}_0$. This case implies $a=b$. For $\Gamma_1(H,W)_{\alpha_a\alpha_a}$:

$$\Gamma_1(H,W)_{\alpha_a\alpha_a}=2\sum_{c\in\mathcal{I}\setminus\{a\}}\frac{f(\mu_c)}{\mu_c^2}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}=2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}.$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_a}$,

$$[f'(\mu_a;\cdot)]^{[1]}(\eta_{a,i},\eta_{a,j})=\frac{\max\{\eta_{a,i},0\}-\max\{\eta_{a,j},0\}}{\eta_{a,i}-\eta_{a,j}}.$$

Thus,

$$\Omega^a_{\beta_{a,i},\beta_{a,j}}=\begin{cases}\frac{\max\{\eta_{a,i},0\}-\max\{\eta_{a,j},0\}}{\eta_{a,i}-\eta_{a,j}}E_{|\beta_{a,i}|\times|\beta_{a,j}|}, & i\neq j\\ 0, & i=j\end{cases}$$

For $\Gamma_3(H,W)_{\alpha_a\alpha_a}$, since

$$f''(\mu_a;\eta_{a,i},\zeta_{a,i,i'})=\begin{cases}\zeta_{a,i,i'}, & \eta_{a,i}>0\\ \max\{\zeta_{a,i,i'},0\}, & \eta_{a,i}=0\\ 0, & \eta_{a,i}<0\end{cases}$$

We have

$$\Psi^{a,i}(\hat{V}_a^{i,i})=\begin{cases}\hat{V}_a^{i,i}, & i\in\mathcal{I}_{a,+}\\ \Pi_+(\hat{V}_a^{i,i}), & i\in\mathcal{I}_{a,0}\\ 0, & i\in\mathcal{I}_{a,-}\end{cases}$$

Now we simplify $\Gamma_2(H,W)_{\alpha_a\alpha_a}+\Gamma_3(H,W)_{\alpha_a\alpha_a}$. Notice that from (11),

$$\Pi'_+(H_{\alpha_a\alpha_a};V_a)=Q^a\Upsilon^a(Q^a)^\intercal,$$

where

$$\Upsilon^a_{\beta_{a,i}\beta_{a,j}}=\begin{cases}\frac{\max\{\eta_{a,i},0\}-\max\{\eta_{a,j},0\}}{\eta_{a,i}-\eta_{a,j}}\hat{V}_a^{i,j}=\Omega^a_{\beta_{a,i}\beta_{a,j}}\circ\hat{V}_a^{i,j}, & i\neq j\\ \Psi^{a,i}(\hat{V}_a^{i,i}), & i=j\end{cases}$$

Therefore,

$$\Gamma_2(H,W)_{\alpha_a\alpha_a} + \Gamma_3(H,W)_{\alpha_a\alpha_a} = \Pi_+'(H_{\alpha_a\alpha_a}; V_a).$$

Also,

$$\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_a} = 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} + \Pi_+'(H_{\alpha_a\alpha_a}; V_a(H,W)),$$

where

$$\begin{aligned}V_a(H,W) =& W_{\alpha_a\alpha_a} + 2\sum_{c\in\mathcal{I}\setminus\{a\}}\frac{1}{\mu_a-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}\\ =& W_{\alpha_a\alpha_a} - 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a} + 2\sum_{c\in\mathcal{I}_-}\frac{1}{-\mu_c}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}.\end{aligned}$$

Case (5): $a\in\mathcal{I}_0, b\in\mathcal{I}_-$. In this case, $\mu_a=0$, $f^{[1]}(\mu_a,\mu_b)=0$, and

$$f^{[2]}(\mu_a,\mu_b,\mu_c) = \begin{cases}\frac{-\mu_b}{(\mu_b-\mu_a)(\mu_b-\mu_c)} = \frac{1}{\mu_c-\mu_b}, & c\in\mathcal{I}_+\\ \frac{\mu_a}{(\mu_a-\mu_b)(\mu_a-\mu_c)} = 0, & c\in\mathcal{I}_-\setminus\{b\}\end{cases}$$

Thus,

$$\Gamma_1(H,W)_{\alpha_a\alpha_b} = 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}.$$

For $\Gamma_2(H,W)_{\alpha_a\alpha_b}$, $\Phi_a(H_{\alpha_a\alpha_a}) = \Pi_+(H_{\alpha_a\alpha_a})$ and $\Phi_b(H_{\alpha_b\alpha_b})=0$. Thus,

$$\Gamma_2(H,W)_{\alpha_a\alpha_b} = \frac{2}{-\mu_b}\Pi_+(H_{\alpha_a\alpha_a})H_{\alpha_a\alpha_b}.$$

$\Gamma_3(H,W)_{\alpha_a\alpha_b}=0$ since $a\neq b$. Thus,

$$\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_b} = 2\sum_{c\in\mathcal{I}_+}\frac{1}{\mu_c-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} + 2\frac{1}{-\mu_b}\Pi_+(H_{\alpha_a\alpha_a})H_{\alpha_a\alpha_b}.$$

Case (6)(i): $a\in\mathcal{I}_-, b\in\mathcal{I}_-$ **and** $a\neq b$. In this case, $\Gamma_2(H,W)_{\alpha_a\alpha_b} = \Gamma_3(H,W)_{\alpha_a\alpha_b}=0$, $f(\mu_a)=f(\mu_b)=0$. Thus,

$$\begin{aligned}&\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_b} = \Gamma_1(H,W)_{\alpha_a\alpha_b} = 2\sum_{c\in\mathcal{I}_+}f^{[2]}(\mu_a,\mu_b,\mu_c)H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}\\ =&2\sum_{c\in\mathcal{I}_+}\frac{\frac{\mu_c}{\mu_c-\mu_a}-\frac{\mu_c}{\mu_c-\mu_b}}{\mu_a-\mu_b}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b} = 2\sum_{c\in\mathcal{I}_+}\frac{\mu_c}{(\mu_c-\mu_a)(\mu_c-\mu_b)}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_b}.\end{aligned}$$

Case (6)(ii): $a\in\mathcal{I}_-, b\in\mathcal{I}_-$ **and** $a=b$. In this case, $\Gamma_2(H,W)_{\alpha_a\alpha_b} = \Gamma_3(H,W)_{\alpha_a\alpha_b}=0$, $\Omega^a=0$, and $\Psi_{a,i}(\hat{V}_a^{i,i})=0$. Thus,

$$\begin{aligned}&\Pi''_{\mathbb{S}^n_+}(Z;H,W)_{\alpha_a\alpha_a} = \Gamma_1(H,W)_{\alpha_a\alpha_a} = -2\sum_{c\in\mathcal{I}\setminus\{a\}}\frac{f(\mu_a)-f(\mu_c)}{(\mu_a-\mu_c)^2}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}\\ =&2\sum_{c\in\mathcal{I}_+}\frac{\mu_c}{(\mu_a-\mu_c)^2}H_{\alpha_a\alpha_c}H_{\alpha_c\alpha_a}.\end{aligned}$$

Clearly, Case (6)(i) and Case (6)(ii)'s results can be merged. This closes the proof of Theorem 2.

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