

NE 795 Assignment 4

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Question 1 Differential Fourier Analysis

1(a) Iterative Error Equations Using the definitions of iterative error,

$$\delta\psi^{(s+1/2)} = \psi^* - \psi^{(s+1/2)} \quad (1)$$

$$\delta F^{(s+1/2)} = F^* - F^{(s+1/2)} \quad (2)$$

$$\delta J^{(s+1)} = J^* - J^{(s+1)} \quad (3)$$

$$\delta\phi^{(s+1)} = \phi^* - \phi^{(s+1)}, \quad (4)$$

The equations for iterative error of the transport equation can be found by subtracting the equation for the $(s + 1/2)$ iterate from the equation for the true (converged) solution:

$$\mu \frac{d\psi^*}{dx} - \mu \frac{d\psi^{(s+1/2)}}{dx} + \sigma_t \psi^* - \sigma_t \psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \phi^* + q) - \frac{1}{2} (\sigma_s \phi^{(s)} + q) \quad (5)$$

$$\mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \delta\phi^{(s)}) \quad (6)$$

where

$$\delta\psi^{(s+1/2)}(0, \mu) = (\psi_{in}^+ - \psi_{in}^+) = 0 \quad \text{for } \mu > 0 \quad (7)$$

$$\delta\psi^{(s+1/2)}(X, \mu) = (\psi_{in}^- - \psi_{in}^-) = 0 \quad \text{for } \mu < 0 \quad (8)$$

and $*$ denotes the true (converged) solution. The equations for error in the closure terms are found in a similar way:

$$\delta F^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu \quad (9)$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(0, \mu) d\mu \quad (10a)$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(X, \mu) d\mu. \quad (10b)$$

The equations for the errors in the SM equations are found the same way:

$$\frac{d\delta J^{(s+1)}}{dx} + (\sigma_t - \sigma_s) \delta \phi^{(s+1)} = 0 \quad (11)$$

$$\frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} + \sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} \quad (12)$$

$$\delta J^{(s+1)}(0) = -\frac{1}{2} \delta \phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (13)$$

$$\delta J^{(s+1)}(X) = \frac{1}{2} \delta \phi^{(s+1)}(X) - \delta P_R^{(s+1/2)} \quad (14)$$

These equations resemble the original iterative equations, with source (and incoming flux boundary conditions) set to zero. By taking the derivative of Equation 12 with respect to x and plugging into equation Equation 11, these equations can be reduced to a single equation for $\delta \phi$ only:

$$\sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} \quad (15)$$

$$\delta J^{(s+1)} = \frac{1}{\sigma_t} \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3\sigma_t} \frac{d\delta \phi^{(s+1)}}{dx} \quad (16)$$

$$\frac{d\delta J^{(s+1)}}{dx} = \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2}, \quad (17)$$

evaluating Equation 11 using this equation for $(\delta J)_x$,

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{d\delta J^{(s+1)}}{dx} = 0 \quad (18)$$

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2} = 0. \quad (19)$$

The boundary conditions $J(0)$ and $J(X)$ can be used to formulate auxiliary conditions for the equation for ϕ . Evaluating Equation 16 using Equations 13 and 14 yields the auxiliary conditions:

$$\frac{1}{\sigma_t(0)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=0} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (20a)$$

$$\frac{1}{\sigma_t(X)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=X} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \quad (20b)$$

Then the full set of equations for iterative error for the SM iterative algorithm is:

$$\mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \delta\phi^{(s)}) \quad (21a)$$

$$\delta\psi^{(s+1/2)}(0, \mu) = 0 \quad \text{for } \mu > 0 \quad \text{and} \quad \delta\psi^{(s+1/2)}(X, \mu) = 0 \quad \text{for } \mu < 0 \quad (21b)$$

$$\delta F^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu \quad (21c)$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(0, \mu) d\mu \quad (21d)$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(X, \mu) d\mu \quad (21e)$$

$$(\sigma_t - \sigma_s) \delta\phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta\phi^{(s+1)}}{dx^2} = 0 \quad (21f)$$

$$\frac{1}{\sigma_t(0)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=0} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (21g)$$

$$\frac{1}{\sigma_t(X)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=X} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \quad (21h)$$

1(b) Fourier Analysis

Validity of Fourier analysis depends on the conditions:

- Infinite spatial domain
- Constant coefficients in all equations ($\sigma_t(x) = \sigma_t$ and $\sigma_s(x) = \sigma_s$).

If these conditions are true, the Fourier Ansatz is used to formulate an assumed form of the solutions to the system defined in 21:

$$\delta\psi^{(s)} = \int_{-\infty}^{\infty} f^{(s)}(\lambda, \mu) e^{i\lambda\sigma_t x} d\lambda \quad (22a)$$

$$\delta F^{(s)} = \int_{-\infty}^{\infty} g^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22b)$$

$$\delta\phi^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22c)$$

where λ is the Fourier mode wavenumber. Substituting the Fourier Ansatz into the system 21:

$$\int_{-\infty}^{\infty} \left[\sigma_t(i\mu\lambda + 1) f^{(s+1/2)}(\lambda, \mu) e^{i\lambda\sigma_t x} = \frac{\sigma_s}{2} h^{(s)}(\lambda) e^{i\lambda\sigma_t x} \right] d\lambda \quad (23a)$$

$$\int_{-\infty}^{\infty} \left[g^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x} = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) f^{(s+1/2)}(\lambda, \mu) e^{i\lambda\sigma_t x} d\mu \right] d\lambda \quad (23b)$$

$$\int_{-\infty}^{\infty} \left[(\sigma_t - \sigma_s) h^{(s+1)}(\lambda) e^{i\lambda\sigma_t x} - \lambda^2 \sigma_t g^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x} + \frac{1}{3} \lambda^2 \sigma_t h^{(s+1)}(\lambda) e^{i\lambda\sigma_t x} = 0 \right] d\lambda \quad (23c)$$

The statements remain true when the integral is removed (the terms are differentiated with respect to λ). All equations can also be divided by $e^{i\lambda\sigma_t x}$,

$$\sigma_t(i\mu\lambda + 1) f^{(s+1/2)}(\lambda, \mu) = \frac{\sigma_s}{2} h^{(s)}(\lambda) \quad (24a)$$

$$g^{(s+1/2)}(\lambda) = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) f^{(s+1/2)}(\lambda, \mu) d\mu \quad (24b)$$

$$(\sigma_t - \sigma_s) h^{(s+1)}(\lambda) - \lambda^2 \sigma_t g^{(s+1/2)}(\lambda) + \frac{1}{3} \lambda^2 \sigma_t h^{(s+1)}(\lambda) = 0. \quad (24c)$$

Solving these equations yields the expression:

$$h^{(s+1)}(\lambda) = h^{(s)}(\lambda) \cdot \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t} \lambda^2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{c}{2(1 + i\mu\lambda)} d\mu, \quad (25)$$

where c is the scattering ratio σ_s/σ_t . Then the iteration eigenvalue $\omega(\lambda)$ is

$$\omega(\lambda) = \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t} \lambda^2} \left[\frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] \quad (26)$$

where the integral is equal to

$$\left[\frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] = \frac{(2(\lambda^2 + 3) \tan^{-1} \lambda) - 6\lambda}{3\lambda^3} \quad (27)$$

1(c) Plots

1(d) Spectral Radius

Question 2 Discretized Fourier Analysis

2(a) Iterative Error Equations Subtracting the $(s + 1/2)$ transport iteration equation from the equation for the true solution ψ^* yields

$$\begin{aligned} \mu_m (\psi_{m,i+1/2}^* - \psi_{m,i-1/2}^*) - \mu_m (\psi_{m,i+1/2}^{(s+1/2)} - \psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)}) = \\ \frac{1}{2} (\sigma_{s,i} \phi_i^* - \sigma_{s,i} \phi_i^{(s)} + (q_i - \overline{q}_i)) \Delta x_i \end{aligned} \quad (28)$$

$$\mu_m (\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2} (\sigma_{s,i} \delta\phi_i^{(s)}) \Delta x_i, \quad (29)$$

with boundary conditions $\delta\psi_{m,1/2}^{(s+1/2)} = 0$ for $\mu_m > 0$ and $\delta\psi_{m,N+1/2}^{(s+1/2)} = 0$ for $\mu_m < 0$. The error in scalar flux is

$$\delta\phi_{i+1/2}^{(s)} = \sum_{m=1}^M \delta\psi_{m,i+1/2}^{(s)} w_m, \quad (30)$$

and the SC cell-center values are

$$\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\psi_{m,i-1/2}^* - \psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\psi_{m,i+1/2}^* - \psi_{m,i+1/2}^{(s+1/2)}) \quad (31)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\delta\psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\delta\psi_{m,i+1/2}^{(s+1/2)}). \quad (32)$$

The error in the second-moment closure term is

$$\delta F_i^{(s+1/2)} = \sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \delta\psi_{m,i}^{(s+1/2)} w_m \quad (33)$$

$$\delta F_{i+1/2}^{(s+1/2)} = \sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \delta\psi_{m,i+1/2}^{(s+1/2)} w_m, \quad (34)$$

and the error in d_i is

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} (F_i^* - F_i^{(s+1/2)} - F_{i-1}^* + F_{i-1}^{(s+1/2)}) \quad (35)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} (\delta F_i^{(s+1/2)} - \delta F_{i-1}^{(s+1/2)}). \quad (36)$$

The boundary term error is

$$\delta P_L = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,1/2}^{(s+1/2)} w_m \quad (37)$$

$$\delta P_R = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,N+1/2}^{(s+1/2)} w_m. \quad (38)$$

The error in the SM equation is given by

$$\begin{aligned} -a_i(\delta \phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i}) \Delta x_i) \delta \phi_i^{(s+1)} - a_{i+1} \delta \phi_{i+1}^{(s+1)} \\ = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} + \Delta x_i q_i - q_i, \end{aligned} \quad (39)$$

where $\sigma_{a,i} = \sigma_{t,i} - \sigma_{s,i}$. The errors for the boundary conditions are given by

$$\left(\frac{1}{2} + a_1 \right) \delta \phi_0^{(s+1)} - a_1 \delta \phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} + \cancel{2J_{in}^+} - \cancel{2J_{in}^-} \quad (40a)$$

$$\left(\frac{1}{2} + a_{N+1} \right) \delta \phi_{N+1}^{(s+1)} - a_{N+1} \delta \phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} - \cancel{2J_{in}^-} + \cancel{2J_{in}^+}. \quad (40b)$$

Then the full set of equations for iterative error for the discretized SM Method is

$$\mu_m \left(\delta \psi_{m,i+1/2}^{(s+1/2)} - \delta \psi_{m,i-1/2}^{(s+1/2)} \right) + \sigma_{t,i} \Delta x_i (\delta \psi_{m,i}^{(s+1/2)}) = \frac{1}{2} \left(\sigma_{s,i} \delta \phi_i^{(s)} \right) \Delta x_i \quad (41a)$$

$$\delta \psi_{m,1/2}^{(s+1/2)} = 0 \text{ for } \mu_m > 0, \quad \delta \psi_{m,N+1/2}^{(s+1/2)} = 0 \text{ for } \mu_m < 0 \quad (41b)$$

$$\delta \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} \left(\delta \psi_{m,i-1/2}^{(s+1/2)} \right) + (1 - \alpha_{m,i}) \left(\delta \psi_{m,i+1/2}^{(s+1/2)} \right) \quad (41c)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} \left[\sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \left(\delta \psi_{m,i}^{(s+1/2)} - \delta \psi_{m,i-1}^{(s+1/2)} \right) w_m \right] \quad (41d)$$

$$\delta P_L = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,1/2}^{(s+1/2)} w_m \quad (41e)$$

$$\delta P_R = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,N+1/2}^{(s+1/2)} w_m \quad (41f)$$

$$-a_i(\delta \phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i}) \Delta x_i) \delta \phi_i^{(s+1)} - a_{i+1} \delta \phi_{i+1}^{(s+1)} = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} \quad (41g)$$

$$\left(\frac{1}{2} + a_1 \right) \delta \phi_0^{(s+1)} - a_1 \delta \phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} \quad (41h)$$

$$\left(\frac{1}{2} + a_{N+1} \right) \delta \phi_{N+1}^{(s+1)} - a_{N+1} \delta \phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} \quad (41i)$$

where

$$a_i = \frac{1}{3\sigma_{t,i-1/2}\Delta x_{i-1/2}} \quad (42)$$

$$\phi_0 = \phi_{1/2}, \quad \phi_{N+1} = \phi_{N+1/2} \quad (43)$$

$$\psi_{m,0} = \psi_{m,1/2}, \quad \psi_{m,N+1} = \psi_{m,N+1/2}. \quad (44)$$

2(b) Fourier Analysis Like the continuous form of the SM method, Fourier Analysis of the discretized form requires that these assumptions are satisfied:

- Infinite spatial domain
- Constant coefficients $\sigma_a, \sigma_t, \theta, \tau$
- Uniform spatial grid Δx

First, the iterative equations are rewritten to apply the assumptions, and eliminating the variable δd_i for simplicity. The new iterative system is

$$\mu_m \left(\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} \right) + \sigma_t \Delta x (\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2} \left(\sigma_s \delta\phi_i^{(s)} \right) \Delta x \quad (45)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_m \left(\delta\psi_{m,i-1/2}^{(s+1/2)} \right) + (1 - \alpha_m) \left(\delta\psi_{m,i+1/2}^{(s+1/2)} \right) \quad (46)$$

(47)

$$(2a + \sigma_a \Delta x) \delta\phi_i^{(s+1)} - a(\delta\phi_{i-1}^{(s+1)} + \delta\phi_{i+1}^{(s+1)}) = \frac{1}{\sigma_t \Delta x} \left[\sum_{m=1}^M w_m \left(\frac{1}{3} - \mu_m^2 \right) \left(2\delta\psi_{m,i}^{(s+1/2)} - \delta\psi_{m,i+1}^{(s+1/2)} - \delta\psi_{m,i-1}^{(s+1/2)} \right) \right]. \quad (48)$$

Then, applying the Fourier Ansatz, the assumed form of the two solution functions are

$$\delta\psi_{m,i+1/2}^{(s+1/2)} = \int_{-\infty}^{\infty} f_m^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x_{i+1/2}} d\lambda \quad (49)$$

$$\delta\phi_i^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\lambda\sigma_t x_i} d\lambda \quad (50)$$

where $f_m(\lambda)$ denotes $f(\lambda, \mu_m)$. The cell-edge and neighboring cell terms can be expanded in

the assumed form to simplify:

$$\begin{aligned}
\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)} (e^{i\lambda\sigma_t x_{i+1/2}} - e^{i\lambda\sigma_t x_{i+1/2}}) \\
&= f_m^{(s+1/2)} (e^{i\lambda\sigma_t(x_i + \frac{1}{2}\Delta x)} - e^{i\lambda\sigma_t(x_i - \frac{1}{2}\Delta x)}) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} (e^{\frac{1}{2}i\lambda\sigma_t\Delta x} - e^{-\frac{1}{2}i\lambda\sigma_t\Delta x}) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(\cos \frac{1}{2}i\lambda\sigma_t\Delta x + i \sin \frac{1}{2}i\lambda\sigma_t\Delta x - \cos \frac{-1}{2}i\lambda\sigma_t\Delta x - i \sin \frac{-1}{2}i\lambda\sigma_t\Delta x \right) \\
&= f_m^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x_i} \left(2i \sin \left(\frac{1}{2}i\lambda\sigma_t\Delta x \right) \right)
\end{aligned} \tag{51}$$

which uses the identities $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. The SC differencing scheme is similarly rewritten:

$$\begin{aligned}
(\alpha_m)\delta\psi_{m,i+1/2}^{(s+1/2)} + (1 - \alpha_m)\delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)} (\alpha_m e^{i\lambda\sigma_t x_{i-1/2}} + (1 - \alpha_m) e^{i\lambda\sigma_t x_{i+1/2}}) \\
&= f_m^{(s+1/2)} \left(\alpha_m \left(e^{i\lambda\sigma_t(x_i - \frac{1}{2}\Delta x)} - e^{i\lambda\sigma_t(x_i + \frac{1}{2}\Delta x)} \right) + e^{i\lambda\sigma_t(x_i + \frac{1}{2}\Delta x)} \right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(\alpha_m \left(e^{-\frac{1}{2}i\lambda\sigma_t\Delta x} - e^{\frac{1}{2}i\lambda\sigma_t\Delta x} \right) + e^{\frac{1}{2}i\lambda\sigma_t\Delta x} \right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(-2i\alpha_m \sin \left(\frac{1}{2}\lambda\sigma_t\Delta x \right) + e^{\frac{1}{2}i\lambda\sigma_t\Delta x} \right)
\end{aligned} \tag{52}$$

The cell-centered scalar flux term from the SM equation becomes

$$\begin{aligned}
\delta\phi_{i+1}^{(s+1)} + \delta\phi_{i-1}^{(s+1)} &= h^{(s+1)} [e^{i\lambda\sigma_t x_{i+1}} + e^{i\lambda\sigma_t x_{i-1}}] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} [e^{i\lambda\sigma_t\Delta x} + e^{-i\lambda\sigma_t\Delta x}] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} [\cos(\lambda\sigma_t\Delta x) + i \sin(\lambda\sigma_t\Delta x) + \cos(-\lambda\sigma_t\Delta x) + i \sin(-\lambda\sigma_t\Delta x)] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} [2 \cos(\lambda\sigma_t\Delta x)]
\end{aligned} \tag{53}$$

Next, Equation 45 is evaluated using using the Fourier Ansatz forms, plugging in Equation 46 for $\delta\psi_{m,i}$ and using 51 and 52 for the cell-edge terms:

$$\begin{aligned}
2i\mu_m f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \sin \left(\frac{1}{2}\lambda\sigma_t\Delta x \right) + \sigma_t\Delta x f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(-2i\alpha_m \sin \left(\frac{1}{2}\lambda\sigma_t\Delta x \right) + e^{\frac{1}{2}i\lambda\sigma_t\Delta x} \right) \\
= \frac{\sigma_s\Delta x}{2} h^{(s)} e^{i\lambda\sigma_t x_i}
\end{aligned} \tag{54}$$

$$f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[(\mu_m - \sigma_t \Delta x \alpha_m) 2i \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) + \sigma_t \Delta x e^{\frac{1}{2} i \lambda \sigma_t \Delta x} \right] = \frac{\sigma_s \Delta x}{2} h^{(s)} e^{i\lambda\sigma_t x_i} \quad (55)$$

$$f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[\left(\frac{\mu_m}{\sigma_t \Delta x} - \alpha_m \right) 2i \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) + e^{\frac{1}{2} i \lambda \sigma_t \Delta x} \right] = \frac{c}{2} h^{(s)} e^{i\lambda\sigma_t x_i} \quad (56)$$

$$f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[\left(\frac{1}{e^{\tau_m} - 1} \right) 2i \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) + e^{\frac{1}{2} i \lambda \sigma_t \Delta x} \right] = \frac{c}{2} h^{(s)} e^{i\lambda\sigma_t x_i} \quad (57)$$

The SM equation (48) can be expanded by plugging in 46, and using 53:

$$\begin{aligned} h^{(s+1)} e^{i\lambda\sigma_t x_i} [(2a + \sigma_a \Delta x) - a (2 \cos(\lambda \sigma_t \Delta x))] &= \\ \frac{1}{\sigma_t \Delta x} \left[\sum_{m=1}^M w_m \left(\frac{1}{3} - \mu_m^2 \right) \left\{ 2 f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right. \right. \\ \left. \left. - f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} e^{i\lambda\sigma_t \Delta x} \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right. \right. \\ \left. \left. - f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} e^{-i\lambda\sigma_t \Delta x} \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right\} \right] \end{aligned} \quad (58)$$

$$\begin{aligned} h^{(s+1)} e^{i\lambda\sigma_t x_i} [(2a + \sigma_a \Delta x) - a (2 \cos(\lambda \sigma_t \Delta x))] &= \\ \frac{1}{\sigma_t \Delta x} \left[\sum_{m=1}^M w_m \left(\frac{1}{3} - \mu_m^2 \right) \left\{ (2 - (e^{i\lambda\sigma_t \Delta x} + e^{-i\lambda\sigma_t \Delta x})) \times \right. \right. \\ \left. \left. f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right\} \right] \end{aligned} \quad (59)$$

and using an intermediate step from Equation 53,

$$\begin{aligned} h^{(s+1)} e^{i\lambda\sigma_t x_i} [(2a + \sigma_a \Delta x) - a (2 \cos(\lambda \sigma_t \Delta x))] &= \\ \frac{e^{i\lambda\sigma_t x_i}}{\sigma_t \Delta x} \left[\sum_{m=1}^M w_m f_m^{(s+1/2)} \left(\frac{1}{3} - \mu_m^2 \right) \left\{ (2 - \cos(\lambda \sigma_t \Delta x)) \times \right. \right. \\ \left. \left. \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right\} \right] \end{aligned} \quad (60)$$

$$\begin{aligned}
h^{(s+1)} e^{i \lambda \sigma_t x_i} & \left[\left(\frac{2}{3} + \sigma_a \sigma_t (\Delta x)^2 \right) - \frac{2}{3} (\cos(\lambda \sigma_t \Delta x)) \right] = \\
& e^{i \lambda \sigma_t x_i} \left[\sum_{m=1}^M w_m f_m^{(s+1/2)} \left(\frac{1}{3} - \mu_m^2 \right) \left\{ (2 - \cos(\lambda \sigma_t \Delta x)) \times \right. \right. \\
& \left. \left. \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right\} \right] \quad (61)
\end{aligned}$$

Then, the system for f and h is, eliminating the common $e^{i \lambda \sigma_t x_i}$ similarly to the continuous Fourier Analysis,

$$f_m^{(s+1/2)} \left[\left(\frac{1}{e^{\tau_m} - 1} \right) 2i \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) + e^{\frac{1}{2} i \lambda \sigma_t \Delta x} \right] = \frac{c}{2} h^{(s)} \quad (62)$$

$$\begin{aligned}
h^{(s+1)} & \left[\left(\frac{2}{3} + \sigma_a \sigma_t (\Delta x)^2 \right) - \frac{2}{3} (\cos(\lambda \sigma_t \Delta x)) \right] = \\
& \frac{1}{\sigma_t \Delta x} \left[\sum_{m=1}^M w_m f_m^{(s+1/2)} \left(\frac{1}{3} - \mu_m^2 \right) \left\{ (2 - \cos(\lambda \sigma_t \Delta x)) \left(e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right) \right\} \right] \quad (63)
\end{aligned}$$

Then the expression for $h^{(s+1)}$ is

$$\begin{aligned}
h^{(s+1)} &= h^{(s)} \sum_{m=1}^M \left\{ w_m c \left[e^{\frac{1}{2} i \lambda \sigma_t \Delta x} - 2i \alpha_m \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) \right] \left[\frac{1}{3} - \mu_m^2 \right] [2 - \cos(\lambda \sigma_t \Delta x)] \right. \\
&\left. \left(2\sigma_t \Delta x \left[\left(\frac{2}{3} + \sigma_a \sigma_t (\Delta x)^2 \right) - \frac{2}{3} (\cos(\lambda \sigma_t \Delta x)) \right] \left[\left(\frac{1}{e^{\tau_m} - 1} \right) 2i \sin \left(\frac{1}{2} \lambda \sigma_t \Delta x \right) + e^{\frac{1}{2} i \lambda \sigma_t \Delta x} \right] \right) \right\} \quad (64)
\end{aligned}$$

2(c) Plots

2(d) Spectral Radius