NE 795 Assignment 2

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Question 1 Plasma Conditions

1(a) Conditions

List the three conditions that describe the plasma state.

Conditions of a plasma:

- Sufficient ionization ($\omega_p \tau > 1$)
- Collective behavior ($L \gg \lambda_D$)
- Quasi-neutrality ($N_D \gg 1$)

Where plasma frequency is calculated with

$$\omega_p = \left(\frac{n_\mathrm{e} \mathrm{e}^2}{\epsilon_0 m_\mathrm{e}} \right)^{1/2} \tag{1}$$

and the Debye length is

$$\lambda_d = \sqrt{\frac{\epsilon_0 k_B T}{n_i \mathrm{e}^2}} \tag{2}$$

1(b)

Given

$$\sqrt{\frac{\mathrm{e}^2}{\epsilon_0 m_\mathrm{e}}} = 56.41 \ \mathrm{m}^{3/2} \mathrm{s}^- 1$$

$$\sqrt{\frac{\epsilon_0}{\mathrm{e}^2}} = 7433.94 \ \mathrm{eV}^{-1/2} \mathrm{m}^{-1/2}$$

Evaluate these conditions for the following systems given the electron density (n_e), electron temperature (T_e) and plasma-neutral collision time (τ):

- The ionosphere ($n_{\mathrm{e}} = 1 \cdot 10^{12}$ m$^{-3}, T_{\mathrm{e}} = 0.1$ eV$, \tau = 1 \cdot 10^{-5}$ s)
  - **Sufficient degree of ionization**: the plasma frequency for the ionosphere is

$$\omega_p = \sqrt{n_{\mathrm{e}}}\sqrt{\frac{\mathrm{e}^2}{\epsilon_0 m_{\mathrm{e}}}} = 5.641 \cdot 10^7 \mathrm{s}^-1 \tag{3}$$

    Then the product $\omega_p \tau = (5.641 \cdot 10^7)(1 \cdot 10^{-5}) = 564.1$ *does* satisty the ionization condition for a plasma.

  - **Collective behavior**: The Debye length for this system, assuming that $n_i = n_{\mathrm{e}}$, is

$$\lambda_D = \sqrt{\frac{\epsilon_0}{\mathrm{e}^2}}\sqrt{\frac{k_B T}{n_{\mathrm{e}}}} = (7433.94\ \mathrm{eV}^{-1/2}\mathrm{m}^{-1/2})(\sqrt{\frac{0.1\ \mathrm{eV}}{1 \times 10^{12}\ \mathrm{m}^{-3}}}) = 2.351 \cdot 10^{-3}\ \mathrm{m} \tag{4}$$

    Which is much smaller than the length scale for the ionosphere (km), satisfying the condition for plasma.

  - **Quasi-neutrality**: The number of particles in a Debye sphere must be large, given by the formula

$$N_D = n_{\mathrm{e}}\frac{4}{3}\pi\lambda_D^3 = (1 \times 10^{12})(\frac{4}{3}\pi)(2.351 \cdot 10^{-3})^3 = 54431.0 \gg 1 \tag{5}$$

    so the quasineutrality condition is satisfied.

- A candle flame ($n_{\mathrm{e}} = 1 \cdot 10^{14}$ m$^{-3}, T_{\mathrm{e}} = 0.1$ eV$, \tau = 1 \cdot 10^{-10}$ s)
  - **Sufficient degree of ionization**: the plasma frequency for the candle is

$$\omega_p = \sqrt{n_{\mathrm{e}}}\sqrt{\frac{\mathrm{e}^2}{\epsilon_0 m_{\mathrm{e}}}} = 5.641 \cdot 10^8 \mathrm{s}^-1 \tag{6}$$

    Then the product $\omega_p \tau = (5.641 \cdot 10^8)(1 \cdot 10^{-10}) = 0.0564$ does *not* satisty the ionization condition for a plasma.

  - **Collective behavior**: The Debye length for this system, assuming that $n_i = n_{\mathrm{e}}$, is

$$\lambda_D = \sqrt{\frac{\epsilon_0}{\mathrm{e}^2}}\sqrt{\frac{k_B T}{n_{\mathrm{e}}}} = (7433.94\ \mathrm{eV}^{-1/2}\mathrm{m}^{-1/2})(\sqrt{\frac{0.1\ \mathrm{eV}}{1 \times 10^{14}\ \mathrm{m}^{-3}}}) = 2.351 \cdot 10^{-4}\ \mathrm{m} \tag{7}$$

    Which is much smaller than the length scale for a candle (about 1 cm), satisfying the condition for plasma.

- **Quasi-neutrality**: The number of particles in a Debye sphere must be large, given by the formula

$$N_D = n_e \frac{4}{3}\pi\lambda_D^3 = (1 \times 10^{14})(\frac{4}{3}\pi)(2.351 \cdot 10^{-4})^3 = 5441.8 \gg 1 \tag{8}$$

so the quasineutrality condition is satisfied, since the population is not 'sufficiently ionized.'

with elementary charge (e) $1.602 \cdot 10^{-19}$ C, electron mass (m_e) $9.109 \cdot 10^{-31}$ kg and vacuum permittivity (ϵ_0) $8.854 \cdot 10^{-12}$ Fm-1

1(c) Conclude for both systems if they are a plasma (or not) and provide a reasoning for your conclusion.

- Ionosphere: all three conditions are satisfied– the system is quasineutral, sufficiently ionized, and behaves collectively, so the system can be considered a plasma.
- Candle flame: the system cannot be considered a plasma, since the electron collision frequency is too low, and the system is not 'sufficiently ionized.'

Question 2 Debye Length

Consider two infinite, parallel plates located at x = -d and x = d. Both plates are kept at a potential of $\Phi = 0$V. The space between the plates is uniformly filled with a gas at density N of particles with charge q (note this is not a plasma, all the particles have the same charge).

2(a) Using Poisson's equation, show that the potential distribution between the plates is given by the equation:

$$\Phi(x) = \frac{Nq}{2\epsilon_0}(d^2 - x^2)$$

Possion's equation for electric potential is

$$\nabla^2\Phi = \frac{-\rho}{\epsilon_0} \tag{9}$$

where ρ is the charge carrier density and ϵ_0 is the permittivity of free space. In one dimension this becomes:

$$\frac{d^2}{dx^2}\Phi = \frac{-\rho}{\epsilon_0} \tag{10}$$

Integrating both sides gives:

$$\frac{d}{dx}\Phi = \frac{-\rho}{\epsilon_0}x + c_1 \tag{11}$$

and, integrating again:

$$\Phi(x) = \frac{-\rho}{2\epsilon_0}x^2 + c_1 x + c_2 = \tag{12}$$

Evaluating this equation at the boundary conditions gives $c_2 = \frac{-\rho}{2\epsilon_0}d^2$ and $c_1 = 0$. Expanding the form of $\rho = Nq$ then gives the equation:

$$\Phi(x) = \frac{Nq}{2\epsilon_0}(d^2 - x^2). \tag{13}$$

2(b) The Debye length λ_D describes the scale length at which the balance between electrostatic potential energy and thermal kinetic energy is established in a plasma. Show that for $d > \lambda_D$ the electrostatic potential energy needed to transport a particle from one of the plates to the midpoint at $x = 0$ exceeds the average thermal kinetic energy of particles in a one-dimensional system. Assume a one-dimensional Maxwellian distribution with $< E_{kinetic} >= \frac{1}{2}k_B T$

The Debye length is given by:

$$\lambda_D = \sqrt{\frac{\epsilon_0 kT}{Nq^2}} = \sqrt{\frac{E}{q}\frac{2\epsilon_0}{Nq}} \tag{14}$$

Then, setting the length $d = \lambda_D$, the potential difference between the centerline and one plate, $\Phi(0) - \Phi(d)$, is

$$\frac{Nq}{2\epsilon_0}(\frac{E}{q}\frac{2\epsilon_0}{Nq}) = \frac{E}{q} \tag{15}$$

Which is to say that the potential can be overcome by a particle of charge q and energy E. Increasing the distance would then increase the energy required– if $d > \lambda_D$, then the energy must be greater than E to overcome this difference.