MA 580 Assignment 1

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Question 1

Given any vector norm $||\cdot||$ on $\mathbb{C}^n$:

1(a) Show that

$$|||x|| - ||y||| \leq ||x - y||; \quad \forall\, x, y \in \mathbb{C}^n \tag{1}$$

Using the definition of a vector norm $||\cdot||$:

(a) $||x|| \geq 0$ for all $x \in \mathbb{C}$; $||x|| = 0 \iff x = 0$

(b) $||x + y|| \leq ||x|| + ||y||$

(c) $||cx|| = |c| \times ||x||$ for all $c \in \mathbb{C}$

From (a):

$$||(x - y) + y|| \leq ||(x - y)|| + ||y|| \tag{2}$$

or

$$||(x - y) + y|| - ||y|| \leq ||(x - y)|| \tag{3}$$

$$||x|| - ||y|| \leq ||x - y|| \tag{4}$$

which satisfies Equation 1 for $||x|| \geq ||y||$ (when the L.H.S. is positive), but does not account for $||y|| > ||x||$, when the absolute value of the LHS may be greater than the RHS. This case may be shown starting from (b):

$$||x - y|| = |-1|\; ||(y - x)|| = ||(y - x)|| \tag{5}$$

then, similarly to Equation 2,

$$||(y-x)+x|| \leq ||(y-x)|| + ||x|| \tag{6}$$

$$\vdots$$

$$||y|| - ||x|| \leq ||y-x|| = ||x-y|| \tag{7}$$

Then, since

$$\pm(||x|| - ||y||) \leq ||x-y|| \tag{8}$$

or

$$|\ ||x|| - ||y||\ | \leq ||x-y|| \tag{9}$$

1(b) Given the function $f : \mathbb{C}^n \to \mathbb{C}^n$ defined by $f(x) = \mathbf{A}x$ where $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a fixed matrix. Show f is continuous.

For f to be continuous, $||f(x_n) - f(x)|| \to 0$ for every sequence $\{x_n\}$ that converges to x.

Since matrix operations are linear (and thus additive),

$$\|\mathbf{A}x_n - \mathbf{A}x\| = \|\mathbf{A}(x_n - x)\| \tag{10}$$

By Fact 2.18 in [**?**] and item (a) in the definition of a norm:

$$0 \leq \|\mathbf{A}(x_n - x)\| \leq \|\mathbf{A}\|\ \|(x_n - x)\| \tag{11}$$

By the definition of a Cauchy sequence, $\|(x_n - x)\| \to 0$. Since $\|\mathbf{A}\|$ is a constant, the right hand side of Equation 11 then also converges to zero. Since the left hand side is non-negative and less than the right hand side, it also converges to zero:

$$\|\mathbf{A}(x_n - x)\| \to 0 \tag{12}$$

and thus the function $f(x)$ is continuous.

1(c) Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{C}^n$. Define the function $f : \mathbb{C}^n \to \mathbb{C}$ by $f(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{u} \rangle$, where $\boldsymbol{u} \in \mathbb{C}^n$ if a fixed vector. Show f is continuous.

An inner product is defined as $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ such that:

(i) $\langle x, x \rangle \geq 0 \quad \forall\, x \in V; \qquad \langle x, x \rangle = 0 \iff x = 0$

(ii) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(iii) $\langle \lambda u, v\rangle = \lambda \langle u, v\rangle$

(iv) $\langle u, v\rangle = \overline{\langle v, u\rangle}$

Then, by item (ii),

$$\langle x_n, u\rangle - \langle x, u\rangle = \langle (x_n - x), u\rangle \tag{13}$$

By the Cauchy-Schwarz inequality,

$$|\langle (x_n - x), u\rangle| \leq \|(x_n - x)\| \times \|u\| \tag{14}$$

By the definition of a Cauchy Sequence converging to x, $\|(x_n - x)\| \to 0$. Then the RHS of Equation 14 also converges to zero. Since LHS $\leq$ RHS, the LHS must also converge to zero, and thus

$$|\langle (x_n - x), u\rangle| \to 0 \tag{15}$$

Then, by item (ii) and item (iii), an equivalent statement is:

$$|\, \langle x_n, u\rangle - \langle x, u\rangle \,| \to 0 \tag{16}$$

or

$$|f(x_n) - f(x)| \to 0 \tag{17}$$

and thus the function is continuous.

Question 2

Let $(V, \langle\cdot,\cdot\rangle)$ be a n-dimensional inner product space, over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with the norm $\|x\| = \langle x, x\rangle^{1/2}$. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis for V, i.e., $\langle v_i, v_j\rangle = \delta_{ij}$, $i, j \in \{1, \ldots, n\}$.

2(a) Show that any $x \in V$ can be written as the linear combination, $x = \sum_{i=1}^n \xi_i v_i$ with $\xi_i = \langle x, v_i\rangle$ (Fourier Expansion of x).

Assume that x can be expanded as a linear combination of the basis vectors:

$$x = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_i v_i \tag{18}$$

Where x_i is the component of x in the v_i direction, with a scalar magnitude of a_i. This projection always exists and is unique. Then

$$v_i = \frac{1}{a_i} x_i \tag{19}$$

The definition of the orthonormal basis can then be rewritten:

$$\left\langle \frac{1}{a_i} x_i, v_j \right\rangle = \delta_{ij} \tag{20}$$

$$\langle x_i, v_j\rangle = \delta_{ij} a_i \tag{21}$$

then the coefficients are given by

$$a_i = \langle x_i, v_i\rangle = \xi_i \tag{22}$$

The coefficient ξ_i is always defined and is unique, so x may always be expanded uniquely in this way.

2(b) Show that for $x \in V$, $\|x\|^2 = \sum_{i=1}^n |\langle x, v_i\rangle|^2$. This is known as Parseval's identity.

From the problem definition,

$$\|x\|^2 = \langle x, x\rangle \tag{23}$$

Following subsection 2(a), this can then be rewritten as

$$\|x\|^2 = \left\langle \sum_{i=1}^{n} \xi_i v_i, \sum_{j=1}^{n} \xi_j v_j \right\rangle \tag{24}$$

This term can be simplified using another inner product property implicit in item (iii) and item (iv):

$$\|x\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \overline{\xi_j} \langle v_i, v_j \rangle \tag{25}$$

Since the set of basis vectors $\{v\}$ are orthonormal (by the problem definition), this can further simplify, as all terms where $i \neq j$ become zero, and the inner product $\langle v_i, v_j \rangle$ becomes 1 for all terms where $i = j$:

$$\|x\|^2 = \sum_{i=1}^{n} \xi_i \overline{\xi_i} \tag{26}$$

And, using a property of the complex conjugate,

$$\|x\|^2 = \sum_{i=1}^{n} |\xi_i|^2 \tag{27}$$

Where $\xi_i = \langle x, v_i \rangle$:

$$\|x\|^2 = \sum_{i=1}^{n} | \langle x, v_i \rangle |^2 \tag{28}$$

Question 3

Consider the vector norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ on $\mathbb{R}^n$ and their induced matrix norms.

3(a) Show that for all $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n}\,\|\boldsymbol{x}\|_2 \leq n\,\|\boldsymbol{x}\|_\infty .$$

The first inequality can be proven intuitively; since $\|\boldsymbol{x}\|_\infty = \max_{i=1\ldots n}(|x_i|) = \sqrt{\max(x_i^2)}$. The L2 norm is defined as:

$$\|\boldsymbol{x}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i)^2} \tag{29}$$

Since all terms x_i^2 are non-negative, it is certain that $\max(x_i^2) \leq \sum x_i^2$, and $\sqrt{\max(x_i^2)} \leq \sqrt{\sum x_i^2}$. Thus

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 . \tag{30}$$

The L2 norm can be re-written:

$$\|\boldsymbol{x}\|_2^2 = \sum_{i=1}^{n}(x_i^2) = \left\|\boldsymbol{x}^\top\boldsymbol{x}\right\|_1 \tag{31}$$

and by the Cauchy-Schwarz inequality,

$$\left\|\boldsymbol{x}^\top\boldsymbol{x}\right\|_1 \leq \left\|\boldsymbol{x}^\top\right\|_1 \cdot \|\boldsymbol{x}\|_1 = \|\boldsymbol{x}\|_1^2 \tag{32}$$

Then

$$\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{x}\|_1^2 \tag{33}$$

or

$$\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \tag{34}$$

Using the Cauchy-Schwarz inequality,

$$\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n}(x_i \cdot 1) = \langle x, \mathbf{1}\rangle \leq \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle \langle \mathbf{1}, \mathbf{1}\rangle} \tag{35}$$

and since $\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \|\boldsymbol{x}\|_2$ and $\langle \mathbf{1}, \mathbf{1} \rangle = n$,

$$\|\boldsymbol{x}\|_1 \leq \sqrt{n} \, \|\boldsymbol{x}\|_2 \tag{36}$$

The fourth term of the inequality in the problem definition can be rewritten

$$\left(\sqrt{n} \, \|\boldsymbol{x}\|_2\right)^2 = n \sum_{i=1}^{n} x_i^2 \tag{37}$$

And the last term in the inequality can be rewritten in a similar way:

$$(n \, \|\boldsymbol{x}\|_\infty)^2 = n^2(\max(|x_i|))^2 = n(n \max(x_i^2)) = n \sum_{i=1}^{n} \max(x_i^2) \tag{38}$$

And, since all terms x_i^2 are less than or equal to $\max(x_i^2)$, the sum in Equation 37 must be less than the sum in Equation 38, and thus

$$\left(\sqrt{n} \, \|\boldsymbol{x}\|_2\right)^2 \leq (n \, \|\boldsymbol{x}\|_\infty)^2 \tag{39}$$

or

$$\sqrt{n} \, \|\boldsymbol{x}\|_2 \leq n \, \|\boldsymbol{x}\|_\infty \tag{40}$$

Combining Equations 30, 34, 36, and 40 yields the original inequality from the problem definition,

$$\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 \leq \sqrt{n} \, \|\boldsymbol{x}\|_2 \leq n \, \|\boldsymbol{x}\|_\infty \, . \tag{41}$$

3(b) Use the result of the previous part to show that for all $\mathbf{A} \in \mathbb{R}^{n \times n}$,

$$\|\mathbf{A}\|_1 \leq \sqrt{n} \, \|\mathbf{A}\|_2 \leq n \, \|\mathbf{A}\|_1 \quad \text{and} \quad \|\mathbf{A}\|_\infty \leq \sqrt{n} \, \|\mathbf{A}\|_2 \leq n \, \|\mathbf{A}\|_\infty \, .$$

By the previous part, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\mathbf{A}\boldsymbol{x}\|_1 \leq \sqrt{n} \, \|\mathbf{A}\boldsymbol{x}\|_2 \tag{42}$$

By submultiplicativaty,

$$\|\mathbf{A}\boldsymbol{x}\|_1 \leq \sqrt{n} \, \|\mathbf{A}\|_2 \, \|\boldsymbol{x}\|_2 \tag{43}$$

Let $\|\boldsymbol{x}\|_1 = 1$. Again, using the result of the previous part, $\|\boldsymbol{x}\|_2 \leq \|\boldsymbol{x}\|_1 = 1$:

$$\|\mathbf{A}\|_1 \leq \sqrt{n}\,\|\mathbf{A}\|_2\,\|\boldsymbol{x}\|_2 \leq \sqrt{n}\,\|\mathbf{A}\|_2\,\cancelto{1}{\|\boldsymbol{x}\|_1} \tag{44}$$

The next portion of the inequality can be proved in a similar way. By the previous part, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\mathbf{A}\boldsymbol{x}\|_2 \leq \|\mathbf{A}\boldsymbol{x}\|_1 \tag{45}$$

By the submultiplicative property, $\|\mathbf{A}\boldsymbol{x}\|_1 \leq \|\mathbf{A}\|_1\,\|\boldsymbol{x}\|_1$, so

$$\|\mathbf{A}\boldsymbol{x}\|_2 \leq \|\mathbf{A}\|_1\,\|\boldsymbol{x}\|_1 \tag{46}$$

By the inequality in the previous part, $\|\boldsymbol{x}\|_1 \leq \sqrt{n}\,\|\boldsymbol{x}\|_2$, so

$$\|\mathbf{A}\boldsymbol{x}\|_2 \leq \|\mathbf{A}\|_1\,\sqrt{n}\,\|\boldsymbol{x}\|_2 \tag{47}$$

Letting $\|\boldsymbol{x}\|_2 = 1$, this gives:

$$\|\mathbf{A}\|_2 \leq \sqrt{n}\,\|\mathbf{A}\|_1 \tag{48}$$

Multiplying this statement by $\sqrt{n}$ yields

$$\sqrt{n}\,\|\mathbf{A}\|_2 \leq n\,\|\mathbf{A}\|_1 \tag{49}$$

Combining Equation 44 and Equation 49 yields the first inequality in the problem definition: $\|\mathbf{A}\|_1 \leq \sqrt{n}\,\|\mathbf{A}\|_2 \leq n\,\|\mathbf{A}\|_1$.

The second statement in the problem definition is proven in a similar way. From the previous part, for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\|\mathbf{A}\boldsymbol{x}\|_\infty \leq \|\mathbf{A}\boldsymbol{x}\|_2 \tag{50}$$

By submultiplicativity, $\|\mathbf{A}\boldsymbol{x}\|_2 \leq \|\mathbf{A}\|_2\,\|\boldsymbol{x}\|_2$, so

$$\|\mathbf{A}\boldsymbol{x}\|_\infty \leq \|\mathbf{A}\|_2\,\|\boldsymbol{x}\|_2 \tag{51}$$

From the previous part, $\sqrt{n}\,\|\boldsymbol{x}\|_2 \leq n\,\|\boldsymbol{x}\|_\infty$, or $\|\boldsymbol{x}\|_2 \leq \sqrt{n}\,\|\boldsymbol{x}\|_\infty$, so

$$\|\mathbf{A}\boldsymbol{x}\|_\infty \leq \|\mathbf{A}\|_2\,\sqrt{n}\,\|\boldsymbol{x}\|_\infty \tag{52}$$

Then, letting $\|\boldsymbol{x}\|_\infty = 1$, it can be shown that

$$\|\mathbf{A}\|_\infty \leq \sqrt{n}\,\|\mathbf{A}\|_2 \tag{53}$$

Thus proving the first portion of the second inequality in the problem definition. The second portion can be proven in a similar way. For any $\boldsymbol{x} \in \mathbb{R}^n$,

$$\sqrt{n}\,\|\mathbf{A}\boldsymbol{x}\|_2 \leq n\,\|\mathbf{A}\boldsymbol{x}\|_\infty \tag{54}$$

By submultiplicativity, $\|\mathbf{A}\boldsymbol{x}\|_\infty \leq \|\mathbf{A}\|_\infty \|\boldsymbol{x}\|_\infty$, so

$$\sqrt{n}\,\|\mathbf{A}\boldsymbol{x}\|_2 \leq n\,\|\mathbf{A}\|_\infty \|\boldsymbol{x}\|_\infty \tag{55}$$

By the inequality in the previous part, $\|\boldsymbol{x}\|_\infty \leq \|\boldsymbol{x}\|_2$, so

$$\sqrt{n}\,\|\mathbf{A}\boldsymbol{x}\|_2 \leq n\,\|\mathbf{A}\|_\infty \|\boldsymbol{x}\|_2 \tag{56}$$

Letting $\|\boldsymbol{x}\|_2 = 1$,

$$\sqrt{n}\,\|\mathbf{A}\|_2 \leq n\,\|\mathbf{A}\|_\infty \tag{57}$$

Then from Equation 44, 49 53, and 57,

$$\|\mathbf{A}\|_1 \leq \sqrt{n}\,\|\mathbf{A}\|_2 \leq n\,\|\mathbf{A}\|_1 \quad \text{and} \quad \|\mathbf{A}\|_\infty \leq \sqrt{n}\,\|\mathbf{A}\|_2 \leq n\,\|\mathbf{A}\|_\infty . \tag{58}$$

Question 4

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

4(a) Show that, $\lambda_1 \boldsymbol{x}^\top \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{A} \boldsymbol{x} \leq \lambda_n \boldsymbol{x}^\top \boldsymbol{x}$, for all $\boldsymbol{x} \in \mathbb{R}^n$.

Since $\mathbf{A}$ is real and symmetric, by the spectral theorem for real symmetric matrices, there exists a matrix $\mathbf{U}$ such that $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, where $\mathbf{\Lambda}$ is a diagonal matrix composed of the eigenvalues of $\mathbf{A}$, and $\mathbf{U}$ is orthogonal and composed of the eigenvectors of $\mathbf{A}$.

Since $\mathbf{U}$ is orthonormal, it has the useful property that $\mathbf{U}^\top = \mathbf{U}^{-1}$, or $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$.

Since $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, it is also true that:

$$\boldsymbol{x}^\top \mathbf{A} \boldsymbol{x} = \boldsymbol{x}^\top \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top \boldsymbol{x} \tag{59}$$

$$\boldsymbol{x}^\top \mathbf{A} \boldsymbol{x} = (\mathbf{U}^\top \boldsymbol{x})^\top \mathbf{\Lambda}\mathbf{U}^\top \boldsymbol{x} \tag{60}$$

or

$$\boldsymbol{x}^\top \mathbf{A} \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i (\mathbf{U}^\top \boldsymbol{x})_i^2 \tag{61}$$

The term $\boldsymbol{x}^\top \boldsymbol{x}$ can be rewritten using $\mathbf{U}^\top\mathbf{U} = \mathbf{I}$:

$$\boldsymbol{x}^\top \boldsymbol{x} = \boldsymbol{x}^\top \mathbf{I} \boldsymbol{x} = \boldsymbol{x}^\top \mathbf{U}\mathbf{U}^\top \boldsymbol{x} = (\mathbf{U}^\top \boldsymbol{x})^\top \mathbf{U}^\top \boldsymbol{x} \tag{62}$$

$$\boldsymbol{x}^\top \boldsymbol{x} = \sum_{i=1}^{n} (\mathbf{U}^\top \boldsymbol{x})_i^2 \tag{63}$$

Implicit from the inequality in the problem definition is:

$$\sum_{i=1}^{n} \lambda_1 \leq \sum_{i=1}^{n} \lambda_i \leq \sum_{i=1}^{n} \lambda_n \tag{64}$$

Multiplying this inequality by $(\mathbf{U}^\top \boldsymbol{x})_i^2$ (and factoring out the constant λ terms) gives:

$$\lambda_1 \sum_{i=1}^{n} (\mathbf{U}^\top \boldsymbol{x})_i^2 \leq \sum_{i=1}^{n} \lambda_i (\mathbf{U}^\top \boldsymbol{x})_i^2 \leq \lambda_n \sum_{i=1}^{n} (\mathbf{U}^\top \boldsymbol{x})_i^2 \tag{65}$$

Which can be rewritten using Equation 61 and Equation 63 as:

$$\lambda_1 \boldsymbol{x}^\top \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{A} \boldsymbol{x} \leq \lambda_n \boldsymbol{x}^\top \boldsymbol{x} \tag{66}$$

Question 5

5(a) Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be an orthonormal basis of $\mathbb{R}^n$. Show that for every $\mathbf{A} \in \mathbb{R}^{n \times n}$,

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \boldsymbol{v}_i^\top \mathbf{A} \boldsymbol{v}_i.$$

The orthonormal vectors v_i can be used to construct an orthogonal matrix $\mathbf{V} = \{v_1, v_2 \ldots, v_n\}$. Since $\mathbf{V}$ is orthogonal, $\mathbf{V}\mathbf{V}^\top = I$. Then $\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A}\mathbf{V}\mathbf{V}^\top$. One of the properties of the trace operator is:

$$\text{trace}(\mathbf{A}\mathbf{B}^\top) = \text{trace}(\mathbf{B}^\top \mathbf{A}) \tag{67}$$

since terms along the diagonal of the product are unaffected by the order being exchanged. Then,

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{A}\mathbf{V}\mathbf{V}^\top) = \text{trace}(\mathbf{V}^\top \mathbf{A}\mathbf{V}) \tag{68}$$

Which can be expanded to:

$$\text{trace}(\mathbf{V}^\top \mathbf{A}\mathbf{V}) = \sum_{i=1}^{n} (\mathbf{V}^\top \mathbf{A}\mathbf{V})_{ii} \tag{69}$$

5(b) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric. Show that $\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\mathbf{A}$.

By the spectral theorem for real symmetric matrices, $\mathbf{A}$ can be expressed as $\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top$ with orthogonal matrix $\mathbf{U}$ and diagonal matrix $\boldsymbol{\Lambda}$, with eigenvalues along the diagonal. Then:

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^\top) \tag{70}$$

Considering only the diagonal of $\mathbf{A}$, the terms are found with:

$$(\mathbf{A})_{ii} = \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbf{U}_{ij} \boldsymbol{\Lambda}_{jk} \mathbf{U}_{ki}^\top \tag{71}$$

Then the trace can be calculated with:

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n} \mathbf{A}_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \mathbf{U}_{ij}\mathbf{\Lambda}_{jk}\mathbf{U}_{ki}^{\top} \tag{72}$$

Which can be rewritten as:

$$\text{trace}(\mathbf{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \mathbf{\Lambda}_{jk}\mathbf{U}_{ki}^{\top}\mathbf{U}_{ij} \tag{73}$$

Which is equal to $\text{trace}(\mathbf{\Lambda}\mathbf{U}^{\top}\mathbf{U})$. Because $\mathbf{U}$ is orthogonal, $\mathbf{U}^{\top}\mathbf{U} = \mathbf{I}$,

$$\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{\Lambda}\mathbf{U}^{\top}\mathbf{U}) = \text{trace}(\mathbf{\Lambda}\mathbf{I}) = \sum_{i=1}^{n} \lambda_i \tag{74}$$

5(c) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices and suppose $\mathbf{B}$ is symmetric positive semidefinite. Prove the following:

$$|\text{trace}(\mathbf{AB})| \leq \|\mathbf{A}\|_2 \, \text{trace}(\mathbf{B}).$$

Let $\mathbf{C} = \mathbf{AB}$. Since $\mathbf{B}$ is symmetric, using the spectral theorem for real symmetric matrices, it can be expressed as $\mathbf{B} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}$, where $\mathbf{U}$ is orthonormal, its columns form a basis for $\mathbb{R}^n$, and $\mathbf{\Lambda}$ is a diagonal vector containing the eigenvalues of $\mathbf{B}$. Then

$$\text{trace}(\mathbf{AB}) = \text{trace}(\mathbf{C}) = \text{trace}(\mathbf{A}\mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\top}) \tag{75}$$

Using Equation 72, which shows demonstrates that the trace operator of a product of matrices is invariant under circular transposition, this can be rewritten as:

$$\text{trace}(\mathbf{C}) = \text{trace}(\mathbf{U}^{\top}\mathbf{A}\mathbf{U}\mathbf{\Lambda}) \tag{76}$$

The trace can be expanded as the sum of the diagonal the product of two vectors:

$$\text{trace}(\mathbf{C}) = \sum_{i=1}^{n}\sum_{k=1}^{n} (\mathbf{U}^{\top}\mathbf{A}\mathbf{U})_{ik}(\mathbf{\Lambda})_{ki} \tag{77}$$

Since $(\mathbf{\Lambda})_{ki} = 0$ for $k \neq i$, this can then be simplified:

$$\text{trace}(\mathbf{C}) = \sum_{i=1}^{n} (\mathbf{U}^{\top}\mathbf{A}\mathbf{U})_{ii}\lambda_i \tag{78}$$

Then let $\boldsymbol{d}$ and $\mathbf{D}$ be the vector and corresponding diagonal matrix containing only the diagonal entries of the matrix product, or $\boldsymbol{d} \stackrel{\text{def}}{=} \boldsymbol{e}_i^\top(\mathbf{U}^\top\mathbf{A}\mathbf{U})\boldsymbol{e}_i$ and $\mathbf{D} = \operatorname{diag}(\boldsymbol{d})$, where $\boldsymbol{e}_i$ is the ith standard basis vector in $\mathbb{R}^n$. Using this notation,

$$\operatorname{trace}(\mathbf{C}) = \sum_{i=1}^{n}(D)_{ii}\lambda_i = \sum_{i=1}^{n}\boldsymbol{d}_i\lambda_i \tag{79}$$

$$|\operatorname{trace}(\mathbf{C})| = |\sum_{i=1}^{n}\boldsymbol{d}_i\lambda_i| = \sum_{i=1}^{n}|\boldsymbol{d}_i|\lambda_i \tag{80}$$

Where the eigenvalues λ_i are known to be non-negative since the matrix $\mathbf{B}$ is positive semidefinite. The value $|\boldsymbol{d}_i|$ can be bounded using submultiplicativity:

$$|\boldsymbol{d}_i| = \left\|\boldsymbol{e}_i^\top(\mathbf{U}^\top\mathbf{A}\mathbf{U})\boldsymbol{e}_i\right\|_2 \leq \left\|\mathbf{U}^\top\mathbf{A}\mathbf{U}\right\|_2 \cancelto{1}{\left\|\boldsymbol{e}_i^\top\right\|_2 \left\|\boldsymbol{e}_i\right\|_2} \leq \|\mathbf{A}\|_2 \cancelto{1}{\left\|\mathbf{U}^\top\right\|_2 \|\mathbf{U}\|_2} \tag{81}$$

Since $\mathbf{U}$ is an orthonormal matrix (which has the property that $\|\mathbf{U}\boldsymbol{x}\| = \|\boldsymbol{x}\|$), it (and its transpose) have a norm of 1. Since $|\boldsymbol{d}_i| \leq \|\mathbf{A}\|_2$,

$$|\operatorname{trace}(\mathbf{C})| = \sum_{i=1}^{n}|\boldsymbol{d}_i|\lambda_i \leq \sum_{i=1}^{n}\|\mathbf{A}\|_2\,\lambda_i = \|\mathbf{A}\|_2\sum_{i=1}^{n}\lambda_i = \|\mathbf{A}\|_2\operatorname{trace}(\mathbf{B}) \tag{82}$$

$$|\operatorname{trace}(\mathbf{C})| \leq \|\mathbf{A}\|_2\operatorname{trace}(\mathbf{B}) \tag{83}$$

Question 6

Computer problem, written in Python

6(a) Let $\boldsymbol{u} = [u_1 \; u_2 \; \dots \; u_{n-1}]^\top$ be the vector of unknowns. Write the system in matrix vector form, $\mathbf{A}\boldsymbol{u} = \boldsymbol{b}$. Describe the entries of the matrix $\mathbf{A}$ and the vector $\boldsymbol{b}$.

The matrix $\mathbf{A}$ is a tridiagonal matrix. The boundary conditions are given by $\mathbf{A}_{0,0} = \mathbf{A}_{n,n} = 1$. For all $i = 1, 2, \dots, n-1$, all $\mathbf{A}_{i,i} = 2/h^2$, and $\mathbf{A}_{i,i+1} = \mathbf{A}_{i,i-1} = -1/h^2$. The values of the vector $\boldsymbol{b}$ (the 'source' term) are given by $\boldsymbol{b}_i = f(x_i)$ for all $i = 1, 2, \dots, n-1$, and 0 for $i = 0, n$.

6(b) Computer code and results.

```python
import numpy
import matplotlib.pyplot as plt

class solution:
    def __init__(self, x, u):
            self.mesh = x
            self.values = u

pi = numpy.pi

def sourceFunc(x):
      return 2*(pi**2)*((17*(numpy.cos(3*pi*x))*(numpy.sin(5*pi*x))
                        + (15*(numpy.sin(3*pi*x))*(numpy.cos(5*pi*x)))

def analyticSolution(x):
      return numpy.cos(3*pi*x)*numpy.sin(5*pi*x)

def elliptic_solve_1d(f,n):
       h = 1/n
       x = numpy.linspace(0, 1, n+1)

       b = f(x)
       b[0] = 0
       b[n] = 0

       A = numpy.zeros((n+1, n+1))
       A[0][0] = 1
       A[n][n] = 1
```

```
        for i in range(1, n):
            A[i][i] = 2/(h**2)
            A[i][i-1] = -1/(h**2)
            A[i][i+1] = -1/(h**2)
        u = numpy.linalg.inv(A) @ b
        return solution(x, u)

mesh_analytic = numpy.linspace(0, 1, 200)
soln_analytic = analyticSolution(mesh_analytic)

k = numpy.linspace(3, 10, 8).astype('int')
print(k)
n = (2**k).astype('int')

# plot all values of k
plt.figure(dpi=250)
for i in n:
        s = elliptic_solve_1d(sourceFunc,i)
        plt.plot(s.mesh, s.values, label=i)

plt.plot(mesh_analytic, soln_analytic, "--", label="Analytic")
plt.legend()
plt.xlabel("x")
plt.ylabel("u")
plt.title("All numerical Solutions")
# plt.savefig("all_num_solns")
# plt.show()
plt.close()

###
# other plots omitted...
###
```

The code, as described in subsection 6(a), assigns values $(n+1)\times(n+1)$ matrix $\mathbf{A}$. Nonzero values only exist along the tridiagonal, where the corner values maintain the boundary condition ($u(0) = u(1) = 0$), and the inner values maintain the differential equation ($-u_{xx} = f(x)$).

The primary results of the Python code can be seen in Figure 1. The results using the extreme values of k can be seen in Figure 2, and the intermediate solution from $k = 7$ can be seen in Figure 3.

Without quantitative analysis, it can be seen that the numerical solution converges relatively quickly to the analytic solution with regards to k– in Figure 1, the numerical solutions are visually identical to the analytical solution for $n \geq 64$ or $k \geq 6$.

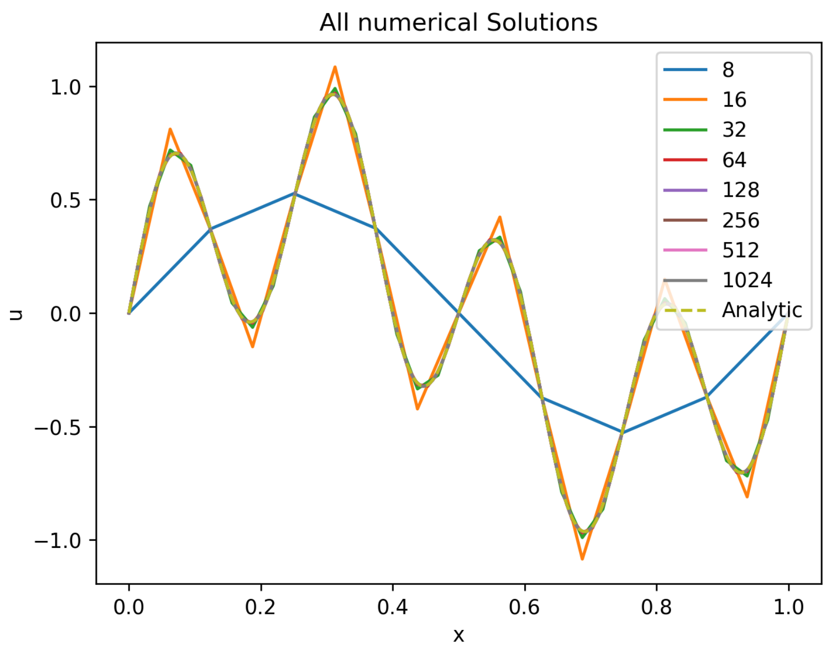
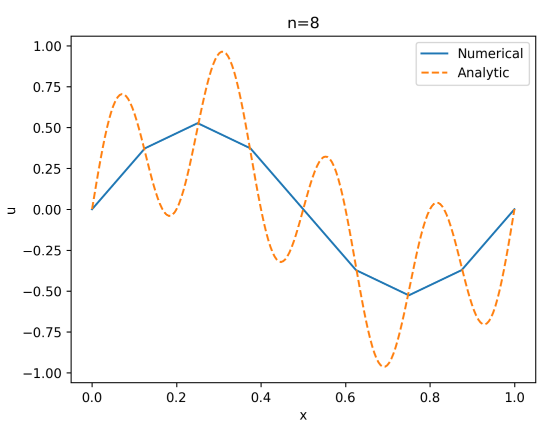
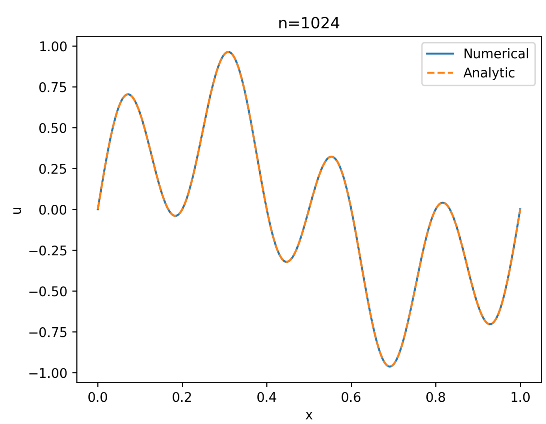


Figure 1: Numerical solutions for $k = 3, \dots 10$



(a) $k = 3$



(b) $k = 10$

Figure 2: Numerical and analytic solutions for extreme values of k.

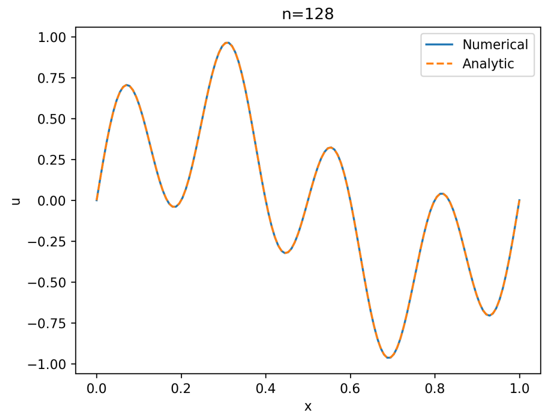


Figure 3: Numerical and analytic solution for intermediate value of k. Note numerical solutions for $k = 7, 10$ are visually equivalent

6(c) The absolute L_∞ errors for each solution can be seen in Figure 4. The error decreases linearly with k. Since $n = 2^k$, this indicates logarithmic convergence with regards to n, where E is proportional to $-\ln_2(n)$.

It should be noted that the first mesh refinement actually increases the error, since the mesh boundaries roughly coincide with local maxima/minima on the analytical solution, where a linear approximation is a poor model of the real behavior of the solution. The mesh boundaries for $k = 3$ fully enclose these local extrema, so their effect is not seen as much as in $k = 4$.

The linear convergence is further indicated in Figure 5, where the ratio of each error to the error at $k - 1$ converges to 0.25, which again would indicate logarithmic convergence with regards to n – doubling the number of meshpoints would improve the error by a factor of one-fourth.

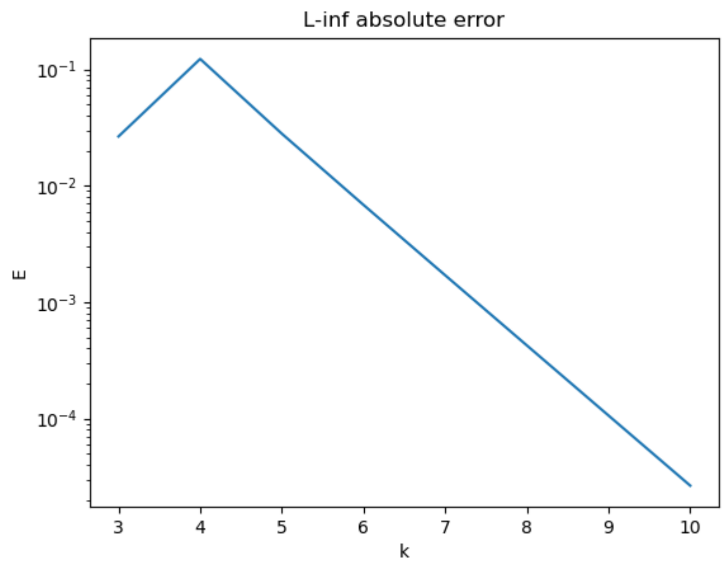


Figure 4: Absolute L_∞ error for all solutions

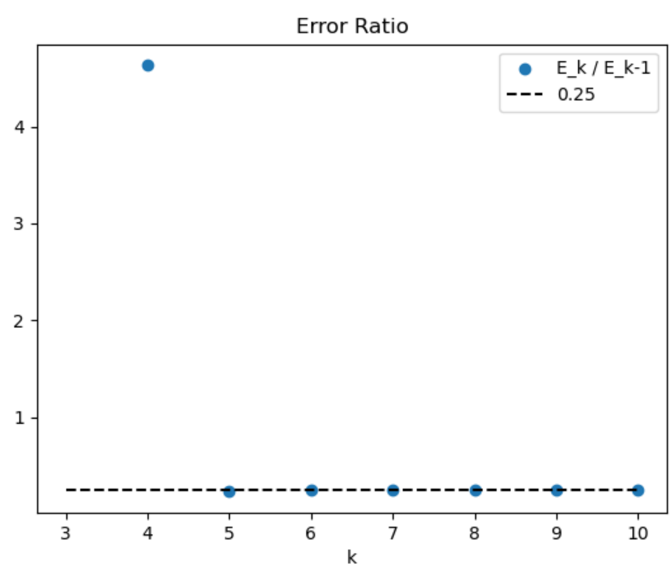


Figure 5: Convergence ratio. Note $E_k/E_{k-1} \to 0.25$