

# NE 795 Assignment 4

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## Question 1 Differential Fourier Analysis

**1(a) Iterative Error Equations** Using the definitions of iterative error,

$$\delta\psi^{(s+1/2)} = \psi^\star - \psi^{(s+1/2)} \quad (1)$$

$$\delta F^{(s+1/2)} = F^\star - F^{(s+1/2)} \quad (2)$$

$$\delta J^{(s+1)} = J^\star - J^{(s+1)} \quad (3)$$

$$\delta\phi^{(s+1)} = \phi^\star - \phi^{(s+1)}, \quad (4)$$

The equations for iterative error of the transport equation can be found by subtracting the equation for the  $(s + 1/2)$  iterate from the equation for the true (converged) solution:

$$\mu \frac{d\psi^\star}{dx} - \mu \frac{d\psi^{(s+1/2)}}{dx} + \sigma_t \psi^\star - \sigma_t \psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \phi^\star + q) - \frac{1}{2} (\sigma_s \phi^{(s)} + q) \quad (5)$$

$$\mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \delta\phi^{(s)}) \quad (6)$$

where

$$\delta\psi^{(s+1/2)}(0, \mu) = (\psi_{in}^+ - \psi_{in}^+) = 0 \quad \text{for } \mu > 0 \quad (7)$$

$$\delta\psi^{(s+1/2)}(X, \mu) = (\psi_{in}^- - \psi_{in}^-) = 0 \quad \text{for } \mu < 0 \quad (8)$$

and  $\star$  denotes the true (converged) solution. The equations for error in the closure terms are found in a similar way:

$$\delta F^{(s+1/2)} = \int_{-1}^1 \left( \frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu \quad (9)$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^1 \left( \frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(0, \mu) d\mu \quad (10a)$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^1 \left( \frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(X, \mu) d\mu. \quad (10b)$$

The equations for the errors in the SM equations are found the same way:

$$\frac{d\delta J^{(s+1)}}{dx} + (\sigma_t - \sigma_s) \delta \phi^{(s+1)} = 0 \quad (11)$$

$$\frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} + \sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} \quad (12)$$

$$\delta J^{(s+1)}(0) = -\frac{1}{2} \delta \phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (13)$$

$$\delta J^{(s+1)}(X) = \frac{1}{2} \delta \phi^{(s+1)}(X) - \delta P_R^{(s+1/2)} \quad (14)$$

These equations resemble the original iterative equations, with source (and incoming flux boundary conditions) set to zero. By taking the derivative of Equation 12 with respect to  $x$  and plugging into equation Equation 11, these equations can be reduced to a single equation for  $\delta \phi$  only:

$$\sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} \quad (15)$$

$$\delta J^{(s+1)} = \frac{1}{\sigma_t} \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3\sigma_t} \frac{d\delta \phi^{(s+1)}}{dx} \quad (16)$$

$$\frac{d\delta J^{(s+1)}}{dx} = \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2}, \quad (17)$$

evaluating Equation 11 using this equation for  $(\delta J)_x$ ,

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{d\delta J^{(s+1)}}{dx} = 0 \quad (18)$$

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2} = 0. \quad (19)$$

The boundary conditions  $J(0)$  and  $J(X)$  can be used to formulate axiliary conditions for the equation for  $\phi$ . Evaluating Equation 16 using Equations 13 and 14 yields the auxiliary conditions:

$$\frac{1}{\sigma_t(0)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=0} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (20a)$$

$$\frac{1}{\sigma_t(X)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=X} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \quad (20b)$$

Then the full set of equations for iterative error for the SM iterative algorithm is:

$$\begin{aligned} \mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} &= \frac{1}{2} (\sigma_s \delta\phi^{(s)}) & (21a) \\ \delta\psi^{(s+1/2)}(0, \mu) = 0 \quad \text{for } \mu > 0 \quad \text{and} \quad \delta\psi^{(s+1/2)}(X, \mu) = 0 \quad \text{for } \mu < 0 & (21b) \\ \delta F^{(s+1/2)} &= \int_{-1}^1 \left( \frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu & (21c) \\ \delta P_L^{(s+1/2)} &= \int_{-1}^1 \left( \frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(0, \mu) d\mu & (21d) \\ \delta P_R^{(s+1/2)} &= \int_{-1}^1 \left( \frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(X, \mu) d\mu & (21e) \\ (\sigma_t - \sigma_s) \delta\phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta\phi^{(s+1)}}{dx^2} &= 0 & (21f) \\ \frac{1}{\sigma_t(0)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=0} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=0} &= -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} & (21g) \\ \frac{1}{\sigma_t(X)} \frac{d\delta F^{(s+1/2)}}{dx} \Big|_{x=X} - \frac{1}{3\sigma_t} \frac{d\delta\phi^{(s+1)}}{dx} \Big|_{x=X} &= -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} & (21h) \end{aligned}$$

**1(b) Fourier Analysis** Validity of Fourier analysis depends on the conditions:

- Infinite spatial domain
- Constant coefficients in all equations ( $\sigma_t(x) = \sigma_t$  and  $\sigma_s(x) = \sigma_s$ ).

If these conditions are true, the Fourier Ansatz is used to formulate an assumed form of the solutions to the system defined in 21:

$$\delta\psi^{(s)} = \int_{-\infty}^{\infty} f^{(s)}(\lambda, \mu) e^{i\lambda\sigma_t x} d\lambda \quad (22a)$$

$$\delta F^{(s)} = \int_{-\infty}^{\infty} g^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22b)$$

$$\delta\phi^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22c)$$

where  $\lambda$  is the Fourier mode wavenumber. Substituting the Fourier Ansatz into the system 21:

$$\sigma_t(i\mu\lambda + 1)f^{(s+1/2)}(\lambda, \mu) = \frac{\sigma_s}{2}h^{(s)}(\lambda) \quad (23a)$$

$$g^{(s+1/2)}(\lambda) = \int_{-1}^1 \left(\frac{1}{3} - \mu^2\right) f^{(s+1/2)}(\lambda, \mu) d\mu \quad (23b)$$

$$(\sigma_t - \sigma_s) h^{(s+1)}(\lambda) - \lambda^2 \sigma_t g^{(s+1/2)}(\lambda) + \frac{1}{3} \lambda^2 \sigma_t h^{(s+1)}(\lambda) = 0 \quad (23c)$$

Solving these equations yields the expression:

$$h^{(s+1)}(\lambda) = h^{(s)}(\lambda) \cdot \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t}\lambda^2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2\right) \frac{c}{2(1 + i\mu\lambda)} d\mu, \quad (24)$$

where  $c$  is the scattering ratio  $\sigma_s/\sigma_t$ . Then the iteration eigenvalue  $\omega(\lambda)$  is

$$\omega(\lambda) = \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t}\lambda^2} \left[ \frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2\right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] \quad (25)$$

where the integral is equal to

$$\left[ \frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2\right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] = \frac{(2(\lambda^2 + 3) \tan^{-1} \lambda) - 6\lambda}{3\lambda^3} \quad (26)$$

## 1(c) Plots

## 1(d) Spectral Radius

# Question 2 Discretized Fourier Analysis

**2(a) Iterative Error Equations** Subtracting the  $(s + 1/2)$  transport iteration equation from the equation for the true solution  $\psi^*$  yields

$$\mu_m (\psi_{m,i+1/2}^* - \psi_{m,i-1/2}^*) - \mu_m (\psi_{m,i+1/2}^{(s+1/2)} - \psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)}) = \frac{1}{2} (\sigma_{s,i} \phi_i^* - \sigma_{s,i} \phi_i^{(s)} + \cancel{(q_i - q_i)}) \Delta x_i \quad (27)$$

$$\mu_m (\delta \psi_{m,i+1/2}^{(s+1/2)} - \delta \psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\delta \psi_{m,i}^{(s+1/2)}) = \frac{1}{2} (\sigma_{s,i} \delta \phi_i^{(s)}) \Delta x_i, \quad (28)$$

with boundary conditions  $\delta \psi_{m,1/2}^{(s+1/2)} = 0$  for  $\mu_m > 0$  and  $\delta \psi_{m,N+1/2}^{(s+1/2)} = 0$  for  $\mu_m < 0$ . The error in scalar flux is

$$\delta \phi_{i+1/2}^{(s)} = \sum_{m=1}^M \delta \psi_{m,i+1/2}^{(s)} w_m, \quad (29)$$

and the SC cell-center values are

$$\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\psi_{m,i-1/2}^* - \psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\psi_{m,i+1/2}^* - \psi_{m,i+1/2}^{(s+1/2)}) \quad (30)$$

$$\delta \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\delta \psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\delta \psi_{m,i+1/2}^{(s+1/2)}). \quad (31)$$

The error in the second-moment closure term is

$$\delta F_i^{(s+1/2)} = \sum_{m=1}^M \left( \frac{1}{3} - \mu_m^2 \right) \delta \psi_{m,i}^{(s+1/2)} w_m \quad (32)$$

$$\delta F_{i+1/2}^{(s+1/2)} = \sum_{m=1}^M \left( \frac{1}{3} - \mu_m^2 \right) \delta \psi_{m,i+1/2}^{(s+1/2)} w_m, \quad (33)$$

and the error in  $d_i$  is

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} \left( F_i^* - F_i^{(s+1/2)} - F_{i-1}^* + F_{i-1}^{(s+1/2)} \right) \quad (34)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} \left( \delta F_i^{(s+1/2)} - \delta F_{i-1}^{(s+1/2)} \right). \quad (35)$$

The boundary term error is

$$\delta P_L = \sum_{m=1}^M \left( \frac{1}{2} - |\mu_m| \right) \delta \psi_{m,1/2}^{(s+1/2)} w_m \quad (36)$$

$$\delta P_R = \sum_{m=1}^M \left( \frac{1}{2} - |\mu_m| \right) \delta \psi_{m,N+1/2}^{(s+1/2)} w_m. \quad (37)$$

The error in the SM equation is given by

$$-a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} + \Delta x_i \cancel{q_i} - \cancel{q_i}, \quad (38)$$

where  $\sigma_{a,i} = \sigma_{t,i} - \sigma_{s,i}$ . The errors for the boundary conditions are given by

$$\left(\frac{1}{2} + a_1\right)\delta\phi_0^{(s+1)} - a_1\delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} + \cancel{2J_{in}^+} - \cancel{2J_{in}^+} \quad (39a)$$

$$\left(\frac{1}{2} + a_{N+1}\right)\delta\phi_{N+1}^{(s+1)} - a_{N+1}\delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} - \cancel{2J_{in}^-} + \cancel{2J_{in}^-}. \quad (39b)$$

Then the full set of equations for iterative error for the discretized SM Method is

$$\mu_m \left( \delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} \right) + \sigma_{t,i}\Delta x_i(\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2} \left( \sigma_{s,i}\delta\phi_i^{(s)} \right) \Delta x_i \quad (40a)$$

$$\delta\psi_{m,1/2}^{(s+1/2)} = 0 \text{ for } \mu_m > 0, \quad \delta\psi_{m,N+1/2}^{(s+1/2)} = 0 \text{ for } \mu_m < 0 \quad (40b)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_{m,i} \left( \delta\psi_{m,i-1/2}^{(s+1/2)} \right) + (1 - \alpha_{m,i}) \left( \delta\psi_{m,i+1/2}^{(s+1/2)} \right) \quad (40c)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2}\Delta x_{i-1/2}} \left[ \sum_{m=1}^M \left( \frac{1}{3} - \mu_m^2 \right) \left( \delta\psi_{m,i}^{(s+1/2)} - \delta\psi_{m,i-1}^{(s+1/2)} \right) w_m \right] \quad (40d)$$

$$\delta P_L = \sum_{m=1}^M \left( \frac{1}{2} - |\mu_m| \right) \delta\psi_{m,1/2}^{(s+1/2)} w_m \quad (40e)$$

$$\delta P_R = \sum_{m=1}^M \left( \frac{1}{2} - |\mu_m| \right) \delta\psi_{m,N+1/2}^{(s+1/2)} w_m \quad (40f)$$

$$-a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} \quad (40g)$$

$$\left(\frac{1}{2} + a_1\right)\delta\phi_0^{(s+1)} - a_1\delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} \quad (40h)$$

$$\left(\frac{1}{2} + a_{N+1}\right)\delta\phi_{N+1}^{(s+1)} - a_{N+1}\delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} \quad (40i)$$

where

$$a_i = \frac{1}{3\sigma_{t,i-1/2}\Delta x_{i-1/2}} \quad (41)$$

$$\phi_0 = \phi_{1/2}, \quad \phi_{N+1} = \phi_{N+1/2} \quad (42)$$

$$\psi_{m,0} = \psi_{m,1/2}, \quad \psi_{m,N+1} = \psi_{m,N+1/2}. \quad (43)$$

**2(b) Fourier Analysis** Like the continuous form of the SM method, Fourier Analysis of the discretized form requires that these assumptions are satisfied:

- Infinite spatial domain
- Constant coefficients  $\sigma_a, \sigma_t, \theta, \tau$
- Uniform spatial grid  $\Delta x$

Then, using the Fourier Ansatz, the assumed form of the errors are

$$\delta\psi_{m,i+1/2}^{(s+1/2)} = \int_{-\infty}^{\infty} f_m^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_{i+1/2}} d\lambda \quad (44)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \int_{-\infty}^{\infty} \Psi_m^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (45)$$

$$\delta d_i^{(s+1/2)} = \int_{-\infty}^{\infty} g^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (46)$$

$$\delta\phi_i^{(s+1)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (47)$$

$$(48)$$

where  $f_m(\lambda)$  denotes  $f(\lambda, \mu_m)$ . Substituting these into the system 40 yields the system to find  $\omega(\lambda) = h^{(s+1)}/h^{(s)}$ . First, some common terms can be simplified:

$$\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} = f_m^{(s+1/2)} [e^{i\lambda\sigma_t x_{i+1/2}} - e^{i\lambda\sigma_t x_{i-1/2}}] \quad (49)$$

$$= f_m^{(s+1/2)} [e^{i\lambda\sigma_t (x_i + \frac{1}{2}\Delta x)} - e^{i\lambda\sigma_t (x_i - \frac{1}{2}\Delta x)}] \quad (50)$$

$$= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} [e^{\frac{1}{2}i\lambda\sigma_t \Delta x} - e^{-\frac{1}{2}i\lambda\sigma_t \Delta x}] \quad (51)$$

then

$$\begin{aligned} \delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[ \cos\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) + i \sin\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) \right. \\ &\quad \left. - \cos\left(-\frac{1}{2}\lambda\sigma_t \Delta x\right) - i \sin\left(-\frac{1}{2}\lambda\sigma_t \Delta x\right) \right]. \end{aligned} \quad (52)$$

Since  $\cos(-\theta) = \cos(\theta)$  and  $\sin(-\theta) = -\sin(\theta)$ ,

$$\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} = f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[ 2i \sin\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) \right]. \quad (53)$$

**2(c) Plots**

**2(d) Spectral Radius**