

NE 795 Assignment 4

Kyle Hansen

10 November 2025

Question 1 Differential Fourier Analysis

1(a) Iterative Error Equations Using the definitions of iterative error,

$$\delta\psi^{(s+1/2)} = \psi^* - \psi^{(s+1/2)} \quad (1)$$

$$\delta F^{(s+1/2)} = F^* - F^{(s+1/2)} \quad (2)$$

$$\delta J^{(s+1)} = J^* - J^{(s+1)} \quad (3)$$

$$\delta\phi^{(s+1)} = \phi^* - \phi^{(s+1)}, \quad (4)$$

The equations for iterative error of the transport equation can be found by subtracting the equation for the $(s + 1/2)$ iterate from the equation for the true (converged) solution:

$$\mu \frac{d\psi^*}{dx} - \mu \frac{d\psi^{(s+1/2)}}{dx} + \sigma_t \psi^* - \sigma_t \psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \phi^* + q) - \frac{1}{2} (\sigma_s \phi^{(s)} + q) \quad (5)$$

$$\mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \delta\phi^{(s)}) \quad (6)$$

where

$$\delta\psi^{(s+1/2)}(0, \mu) = (\psi_{in}^+ - \psi_{in}^+) = 0 \quad \text{for } \mu > 0 \quad (7)$$

$$\delta\psi^{(s+1/2)}(X, \mu) = (\psi_{in}^- - \psi_{in}^-) = 0 \quad \text{for } \mu < 0 \quad (8)$$

and $*$ denotes the true (converged) solution. The equations for error in the closure terms are found in a similar way:

$$\delta F^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu \quad (9)$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(0, \mu) d\mu \quad (10a)$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta \psi^{(s+1/2)}(X, \mu) d\mu. \quad (10b)$$

The equations for the errors in the SM equations are found the same way:

$$\frac{d\delta J^{(s+1)}}{dx} + (\sigma_t - \sigma_s) \delta \phi^{(s+1)} = 0 \quad (11)$$

$$\frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} + \sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} \quad (12)$$

$$\delta J^{(s+1)}(0) = -\frac{1}{2} \delta \phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (13)$$

$$\delta J^{(s+1)}(X) = \frac{1}{2} \delta \phi^{(s+1)}(X) - \delta P_R^{(s+1/2)} \quad (14)$$

These equations resemble the original iterative equations, with source (and incoming flux boundary conditions) set to zero. By taking the derivative of Equation 12 with respect to x and plugging into equation Equation 11, these equations can be reduced to a single equation for $\delta \phi$ only:

$$\sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3} \frac{d\delta \phi^{(s+1)}}{dx} \quad (15)$$

$$\delta J^{(s+1)} = \frac{1}{\sigma_t} \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3\sigma_t} \frac{d\delta \phi^{(s+1)}}{dx} \quad (16)$$

$$\frac{d\delta J^{(s+1)}}{dx} = \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2}, \quad (17)$$

evaluating Equation 11 using this equation for $(\delta J)_x$,

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{d\delta J^{(s+1)}}{dx} = 0 \quad (18)$$

$$(\sigma_t - \sigma_s) \delta \phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta \phi^{(s+1)}}{dx^2} = 0. \quad (19)$$

The boundary conditions $J(0)$ and $J(X)$ can be used to formulate auxiliary conditions for the equation for ϕ . Evaluating Equation 16 using Equations 13 and 14 yields the auxiliary conditions:

$$\frac{1}{\sigma_t(0)} \left. \frac{d\delta F^{(s+1/2)}}{dx} \right|_{x=0} - \frac{1}{3\sigma_t} \left. \frac{d\delta\phi^{(s+1)}}{dx} \right|_{x=0} = -\frac{1}{2} \delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (20a)$$

$$\frac{1}{\sigma_t(X)} \left. \frac{d\delta F^{(s+1/2)}}{dx} \right|_{x=X} - \frac{1}{3\sigma_t} \left. \frac{d\delta\phi^{(s+1)}}{dx} \right|_{x=X} = -\frac{1}{2} \delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \quad (20b)$$

Then the full set of equations for iterative error for the SM iterative algorithm is:

$$\mu \frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t \delta\psi^{(s+1/2)} = \frac{1}{2} (\sigma_s \delta\phi^{(s)}) \quad (21a)$$

$$\delta\psi^{(s+1/2)}(0, \mu) = 0 \quad \text{for } \mu > 0 \quad \text{and} \quad \delta\psi^{(s+1/2)}(X, \mu) = 0 \quad \text{for } \mu < 0 \quad (21b)$$

$$\delta F^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \delta\psi^{(s+1/2)} d\mu \quad (21c)$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(0, \mu) d\mu \quad (21d)$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^1 \left(\frac{1}{2} - |\mu| \right) \delta\psi^{(s+1/2)}(X, \mu) d\mu \quad (21e)$$

$$(\sigma_t - \sigma_s) \delta\phi^{(s+1)} + \frac{1}{\sigma_t} \frac{d^2 \delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t} \frac{d^2 \delta\phi^{(s+1)}}{dx^2} = 0 \quad (21f)$$

$$\frac{1}{\sigma_t(0)} \left. \frac{d\delta F^{(s+1/2)}}{dx} \right|_{x=0} - \frac{1}{3\sigma_t} \left. \frac{d\delta\phi^{(s+1)}}{dx} \right|_{x=0} = -\frac{1}{2} \delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \quad (21g)$$

$$\frac{1}{\sigma_t(X)} \left. \frac{d\delta F^{(s+1/2)}}{dx} \right|_{x=X} - \frac{1}{3\sigma_t} \left. \frac{d\delta\phi^{(s+1)}}{dx} \right|_{x=X} = -\frac{1}{2} \delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \quad (21h)$$

1(b) Fourier Analysis

Validity of Fourier analysis depends on the conditions:

- Infinite spatial domain
- Constant coefficients in all equations ($\sigma_t(x) = \sigma_t$ and $\sigma_s(x) = \sigma_s$).

If these conditions are true, the Fourier Ansatz is used to formulate an assumed form of the solutions to the system defined in 21:

$$\delta\psi^{(s)} = \int_{-\infty}^{\infty} f^{(s)}(\lambda, \mu) e^{i\lambda\sigma_t x} d\lambda \quad (22a)$$

$$\delta F^{(s)} = \int_{-\infty}^{\infty} g^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22b)$$

$$\delta\phi^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\lambda\sigma_t x} d\lambda \quad (22c)$$

where λ is the Fourier mode wavenumber. Substituting the Fourier Ansatz into the system 21:

$$\sigma_t(i\mu\lambda + 1)f^{(s+1/2)}(\lambda, \mu) = \frac{\sigma_s}{2}h^{(s)}(\lambda) \quad (23a)$$

$$g^{(s+1/2)}(\lambda) = \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) f^{(s+1/2)}(\lambda, \mu) d\mu \quad (23b)$$

$$(\sigma_t - \sigma_s)h^{(s+1)}(\lambda) - \lambda^2 \sigma_t g^{(s+1/2)}(\lambda) + \frac{1}{3}\lambda^2 \sigma_t h^{(s+1)}(\lambda) = 0 \quad (23c)$$

Solving these equations yields the expression:

$$h^{(s+1)}(\lambda) = h^{(s)}(\lambda) \cdot \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t}\lambda^2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{c}{2(1 + i\mu\lambda)} d\mu, \quad (24)$$

where c is the scattering ratio σ_s/σ_t . Then the iteration eigenvalue $\omega(\lambda)$ is

$$\omega(\lambda) = \frac{\lambda^2}{1 - c + \frac{1}{3\sigma_t}\lambda^2} \left[\frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] \quad (25)$$

where the integral is equal to

$$\left[\frac{c}{2} \int_{-1}^1 \left(\frac{1}{3} - \mu^2 \right) \frac{1}{(1 + i\mu\lambda)} d\mu \right] = \frac{(2(\lambda^2 + 3)\tan^{-1}\lambda) - 6\lambda}{3\lambda^3} \quad (26)$$

1(c) Plots

1(d) Spectral Radius

Question 2 Discretized Fourier Analysis

2(a) Iterative Error Equations Subtracting the $(s + 1/2)$ transport iteration equation from the equation for the true solution ψ^* yields

$$\begin{aligned} \mu_m (\psi_{m,i+1/2}^* - \psi_{m,i-1/2}^*) - \mu_m (\psi_{m,i+1/2}^{(s+1/2)} - \psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)}) = \\ \frac{1}{2} (\sigma_{s,i} \phi_i^* - \sigma_{s,i} \phi_i^{(s)} + (q_i - \bar{q}_i)) \Delta x_i \end{aligned} \quad (27)$$

$$\mu_m (\delta \psi_{m,i+1/2}^{(s+1/2)} - \delta \psi_{m,i-1/2}^{(s+1/2)}) + \sigma_{t,i} \Delta x_i (\delta \psi_{m,i}^{(s+1/2)}) = \frac{1}{2} (\sigma_{s,i} \delta \phi_i^{(s)}) \Delta x_i, \quad (28)$$

with boundary conditions $\delta \psi_{m,1/2}^{(s+1/2)} = 0$ for $\mu_m > 0$ and $\delta \psi_{m,N+1/2}^{(s+1/2)} = 0$ for $\mu_m < 0$. The error in scalar flux is

$$\delta \phi_{i+1/2}^{(s)} = \sum_{m=1}^M \delta \psi_{m,i+1/2}^{(s)} w_m, \quad (29)$$

and the SC cell-center values are

$$\psi_{m,i}^* - \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\psi_{m,i-1/2}^* - \psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\psi_{m,i+1/2}^* - \psi_{m,i+1/2}^{(s+1/2)}) \quad (30)$$

$$\delta \psi_{m,i}^{(s+1/2)} = \alpha_{m,i} (\delta \psi_{m,i-1/2}^{(s+1/2)}) + (1 - \alpha_{m,i}) (\delta \psi_{m,i+1/2}^{(s+1/2)}). \quad (31)$$

The error in the second-moment closure term is

$$\delta F_i^{(s+1/2)} = \sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \delta \psi_{m,i}^{(s+1/2)} w_m \quad (32)$$

$$\delta F_{i+1/2}^{(s+1/2)} = \sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \delta \psi_{m,i+1/2}^{(s+1/2)} w_m, \quad (33)$$

and the error in d_i is

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} (F_i^* - F_i^{(s+1/2)} - F_{i-1}^* + F_{i-1}^{(s+1/2)}) \quad (34)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} (\delta F_i^{(s+1/2)} - \delta F_{i-1}^{(s+1/2)}). \quad (35)$$

The boundary term error is

$$\delta P_L = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,1/2}^{(s+1/2)} w_m \quad (36)$$

$$\delta P_R = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta \psi_{m,N+1/2}^{(s+1/2)} w_m. \quad (37)$$

The error in the SM equation is given by

$$\begin{aligned} -a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} \\ = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} + \Delta x_i q_i - q_i, \end{aligned} \quad (38)$$

where $\sigma_{a,i} = \sigma_{t,i} - \sigma_{s,i}$. The errors for the boundary conditions are given by

$$\left(\frac{1}{2} + a_1\right) \delta\phi_0^{(s+1)} - a_1 \delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} + \cancel{2J_{in}^+} - \cancel{2J_{in}^-} \quad (39a)$$

$$\left(\frac{1}{2} + a_{N+1}\right) \delta\phi_{N+1}^{(s+1)} - a_{N+1} \delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} - \cancel{2J_{in}^-} + \cancel{2J_{in}^+}. \quad (39b)$$

Then the full set of equations for iterative error for the discretized SM Method is

$$\mu_m \left(\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} \right) + \sigma_{t,i} \Delta x_i (\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2} \left(\sigma_{s,i} \delta\phi_i^{(s)} \right) \Delta x_i \quad (40a)$$

$$\delta\psi_{m,1/2}^{(s+1/2)} = 0 \text{ for } \mu_m > 0, \quad \delta\psi_{m,N+1/2}^{(s+1/2)} = 0 \text{ for } \mu_m < 0 \quad (40b)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_{m,i} \left(\delta\psi_{m,i-1/2}^{(s+1/2)} \right) + (1 - \alpha_{m,i}) \left(\delta\psi_{m,i+1/2}^{(s+1/2)} \right) \quad (40c)$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2} \Delta x_{i-1/2}} \left[\sum_{m=1}^M \left(\frac{1}{3} - \mu_m^2 \right) \left(\delta\psi_{m,i}^{(s+1/2)} - \delta\psi_{m,i-1}^{(s+1/2)} \right) w_m \right] \quad (40d)$$

$$\delta P_L = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta\psi_{m,1/2}^{(s+1/2)} w_m \quad (40e)$$

$$\delta P_R = \sum_{m=1}^M \left(\frac{1}{2} - |\mu_m| \right) \delta\psi_{m,N+1/2}^{(s+1/2)} w_m \quad (40f)$$

$$-a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} \quad (40g)$$

$$\left(\frac{1}{2} + a_1\right) \delta\phi_0^{(s+1)} - a_1 \delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} \quad (40h)$$

$$\left(\frac{1}{2} + a_{N+1}\right) \delta\phi_{N+1}^{(s+1)} - a_{N+1} \delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} \quad (40i)$$

where

$$a_i = \frac{1}{3\sigma_{t,i-1/2} \Delta x_{i-1/2}} \quad (41)$$

$$\phi_0 = \phi_{1/2}, \quad \phi_{N+1} = \phi_{N+1/2} \quad (42)$$

$$\psi_{m,0} = \psi_{m,1/2}, \quad \psi_{m,N+1} = \psi_{m,N+1/2}. \quad (43)$$

2(b) Fourier Analysis Like the continuous form of the SM method, Fourier Analysis of the discretized form requires that these assumptions are satisfied:

- Infinite spatial domain
- Constant coefficients $\sigma_a, \sigma_t, \theta, \tau$
- Uniform spatial grid Δx

Then, using the Fourier Ansatz, the assumed form of the errors are

$$\delta\psi_{m,i+1/2}^{(s+1/2)} = \int_{-\infty}^{\infty} f_m^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_{i+1/2}} d\lambda \quad (44)$$

$$\delta\psi_{m,i}^{(s+1/2)} = \int_{-\infty}^{\infty} \Psi_m^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (45)$$

$$\delta d_i^{(s+1/2)} = \int_{-\infty}^{\infty} g^{(s+1/2)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (46)$$

$$\delta\phi_i^{(s+1)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\sigma_t \lambda x_i} d\lambda \quad (47)$$

$$(48)$$

where $f_m(\lambda)$ denotes $f(\lambda, \mu_m)$. Substituting these into the system 40 yields the system to find $\omega(\lambda) = h^{(s+1)}/h^{(s)}$. First, some common terms can be simplified:

$$\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} = f_m^{(s+1/2)} [e^{i\lambda\sigma_t x_{i+1/2}} - e^{i\lambda\sigma_t x_{i-1/2}}] \quad (49)$$

$$= f_m^{(s+1/2)} [e^{i\lambda\sigma_t (x_i + \frac{1}{2}\Delta x)} - e^{i\lambda\sigma_t (x_i - \frac{1}{2}\Delta x)}] \quad (50)$$

$$= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} [e^{\frac{1}{2}i\lambda\sigma_t \Delta x} - e^{-\frac{1}{2}i\lambda\sigma_t \Delta x}] \quad (51)$$

then

$$\begin{aligned} \delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[\cos\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) + i \sin\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) \right. \\ &\quad \left. - \cos\left(-\frac{1}{2}\lambda\sigma_t \Delta x\right) - i \sin\left(-\frac{1}{2}\lambda\sigma_t \Delta x\right) \right]. \end{aligned} \quad (52)$$

Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$,

$$\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} = f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left[2i \sin\left(\frac{1}{2}\lambda\sigma_t \Delta x\right) \right]. \quad (53)$$

2(c) Plots

2(d) Spectral Radius