NE 795 Assignment 4

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Question 1 Differential Fourier Analysis

1(a) Iterative Error Equations Using the definitions of iterative error,

$$
\begin{aligned}
\delta\psi^{(s+1/2)} &= \psi^{\star} - \psi^{(s+1/2)} && (1)\\
\delta F^{(s+1/2)} &= F^{\star} - F^{(s+1/2)} && (2)\\
\delta J^{(s+1)} &= J^{\star} - J^{(s+1)} && (3)\\
\delta\phi^{(s+1)} &= \phi^{\star} - \phi^{(s+1)}, && (4)
\end{aligned}
$$

The equations for iterative error of the transport equation can be found by subtracting the equation for the $(s+1/2)$ iterate from the equation for the true (converged) solution:

$$
\begin{aligned}
\mu\frac{d\psi^{\star}}{dx} - \mu\frac{d\psi^{(s+1/2)}}{dx} + \sigma_t\psi^{\star} - \sigma_t\psi^{(s+1/2)} &= \frac{1}{2}\left(\sigma_s\phi^{\star} + q\right) - \frac{1}{2}\left(\sigma_s\phi^{(s)} + q\right) && (5)\\
\mu\frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t\delta\psi^{(s+1/2)} &= \frac{1}{2}\left(\sigma_s\delta\phi^{(s)}\right) && (6)
\end{aligned}
$$

where

$$
\begin{aligned}
\delta\psi^{(s+1/2)}(0,\mu) &= (\psi_{in}^{+} - \psi_{in}^{+}) = 0 \quad \text{for } \mu > 0 && (7)\\
\delta\psi^{(s+1/2)}(X,\mu) &= (\psi_{in}^{-} - \psi_{in}^{-}) = 0 \quad \text{for } \mu < 0 && (8)
\end{aligned}
$$

and * denotes the true (converged) solution. The equations for error in the closure terms are found in a similar way:

$$
\delta F^{(s+1/2)} = \int_{-1}^{1}\left(\frac{1}{3} - \mu^2\right)\delta\psi^{(s+1/2)}d\mu \qquad (9)
$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^{1} \left(\frac{1}{2} - |\mu|\right) \delta\psi^{(s+1/2)}(0,\mu) d\mu \tag{10a}$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^{1} \left(\frac{1}{2} - |\mu|\right) \delta\psi^{(s+1/2)}(X,\mu) d\mu. \tag{10b}$$

The equations for the error in the second moment equations is found the same way:

$$\frac{d\delta J^{(s+1)}}{dx} + (\sigma_t - \sigma_s)\, \delta\phi^{(s+1)} = 0 \tag{11}$$

$$\frac{1}{3}\frac{d\delta\phi^{(s+1)}}{dx} + \sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} \tag{12}$$

$$\delta J^{(s+1)}(0) = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \tag{13}$$

$$\delta J^{(s+1)}(X) = \frac{1}{2}\delta\phi^{(s+1)}(X) - \delta P_R^{(s+1/2)} \tag{14}$$

These equations resemble the original iterative equations, with source (and incoming flux boundary conditions) set to zero. By taking the derivative of **??** with respect to x and plugging into equation **??**, these equations can be reduced to a single equation for $\delta\phi$ only:

$$\sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3}\frac{d\delta\phi^{(s+1)}}{dx} \tag{15}$$

$$\delta J^{(s+1)} = \frac{1}{\sigma_t}\frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3\sigma_t}\frac{d\delta\phi^{(s+1)}}{dx} \tag{16}$$

$$\frac{d\delta J^{(s+1)}}{dx} = \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2}, \tag{17}$$

evaluating **??** using this equation for $(\delta J)_x$,

$$(\sigma_t - \sigma_s)\, \delta\phi^{(s+1)} + \frac{d\delta J^{(s+1)}}{dx} = 0 \tag{18}$$

$$(\sigma_t - \sigma_s)\, \delta\phi^{(s+1)} + \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2} = 0. \tag{19}$$

The boundary conditions $J(0)$ and $J(X)$ can be used to formulate axiliary conditions for the equation for ϕ. Evaluating **??** using Equations **??** and **??** yields the auxiliary conditions:

$$\frac{1}{\sigma_t(0)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=0} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \tag{20a}$$

$$\frac{1}{\sigma_t(X)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=X} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \tag{20b}$$

Then the full set of equations for iterative error for the SM iterative algorithm is:

$$\boxed{\begin{aligned}
\mu\frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t\delta\psi^{(s+1/2)} &= \frac{1}{2}\left(\sigma_s\delta\phi^{(s)}\right) && \text{(21a)}\\
\delta\psi^{(s+1/2)}(0,\mu) = 0 \quad \text{for } \mu > 0 \qquad &\text{and} \qquad \delta\psi^{(s+1/2)}(X,\mu) = 0 \quad \text{for } \mu < 0 && \text{(21b)}\\
\delta F^{(s+1/2)} &= \int_{-1}^{1}\left(\frac{1}{3} - \mu^2\right)\delta\psi^{(s+1/2)}d\mu && \text{(21c)}\\
\delta P_L^{(s+1/2)} &= \int_{-1}^{1}\left(\frac{1}{2} - |\mu|\right)\delta\psi^{(s+1/2)}(0,\mu)d\mu && \text{(21d)}\\
\delta P_R^{(s+1/2)} &= \int_{-1}^{1}\left(\frac{1}{2} - |\mu|\right)\delta\psi^{(s+1/2)}(X,\mu)d\mu && \text{(21e)}\\
\left(\sigma_t - \sigma_s\right)\delta\phi^{(s+1)} + \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2} &= 0 && \text{(21f)}\\
\frac{1}{\sigma_t(0)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=0} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=0} &= -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} && \text{(21g)}\\
\frac{1}{\sigma_t(X)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=X} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=X} &= -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} && \text{(21h)}
\end{aligned}}$$

1(b) Fourier Analysis

Validity of Fourier analysis depends on the conditions:

- Infinite spatial domain
- Constant coefficients in all equations ($\sigma_t(x) = \sigma_t$ and $\sigma_s(x) = \sigma_s$).

If these conditions are true, the Fourier Ansatz is used to formulate an assumed form of the solutions to the system defined in **??**:

where λ is the Fourier mode wavenumber.

1(c) Plots

1(d) Spectral Radius

Question 2 Discretized Fourier Analysis

2(a) Iterative Error Equations

2(b) Fourier Analysis

2(c) Plots

2(d) Spectral Radius