NE 795 Assignment 6

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Question 1 Z-pinch

1(a) Sketch

The z-pinch sketch is shown in Figure 1.

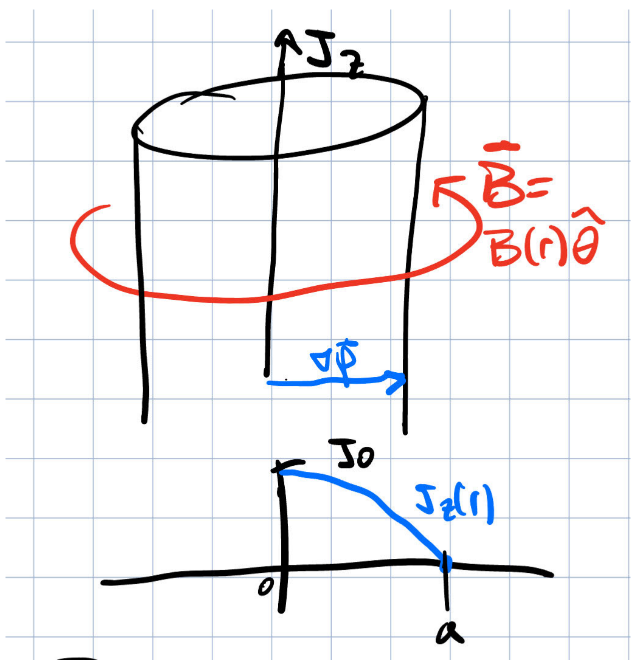

Figure 1: Z-pinch sketch

1(b) Field Strength

By Ampere's magnetic field induced by a current-carrying wire is

$$B_\theta = \frac{B_0 r}{a} \tag{1}$$

where

$$B_0 = \frac{\mu_0 I_{enc}}{2\pi a}. \tag{2}$$

The current is $I_z(r) = \int_{2\pi} \int_0^r J_z(r') r' dr d\theta$, or

$$\begin{aligned}
I_z &= 2\pi J_0 \int_0^r \left[1 - (\frac{r'}{a})^2 \right] r' dr' && (3)\\
&= 2\pi J_0 \int_0^r \left[r' - (\frac{r'^3}{a^2}) \right] dr' && (4)\\
&= 2\pi J_0 \left[\frac{r'^2}{2} - \frac{r'^4}{4a^2} \right]_0^r && (5)\\
I_z(r) = &= 2\pi J_0 \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right] && (6)
\end{aligned}$$

which is equal to $2\pi J_0 \left[\frac{1}{4} a^2\right]$ when $r \geq a$.
so,

$$\begin{aligned}
B_0(z) &= \frac{\mu_0}{2\pi a} I_z(r) && (7)\\
&= \frac{\mu_0}{2\pi a} \left(2\pi J_0 \left[\frac{r^2}{2} - \frac{r^4}{4a^2} \right] \right) && (8)\\
&= \mu_0 J_0 \left[\frac{r^2}{2a} - \frac{r^4}{4a^3} \right] && (9)
\end{aligned}$$

and

$$\begin{aligned}
B_\theta(r) &= \frac{B_0 r}{a} \hat{\theta} && (10)\\
&= \frac{r}{a} \mu_0 J_0 \left[\frac{r^2}{2a} - \frac{r^4}{4a^3} \right] \hat{\theta} && (11)\\
&= \mu_0 J_0 \left[\frac{r^3}{2a^2} - \frac{r^5}{4a^4} \right] \hat{\theta} && (12)
\end{aligned}$$

for $r \leq a$, and

$$
\begin{aligned}
B_\theta(r) &= \frac{B_0 r}{a}\hat{\theta} && (13)\\
&= \frac{r}{a}\mu_0 J_0\left[\frac{1}{4}a^2\right]\hat{\theta} && (14)\\
&= \frac{\mu_0 a J_0}{4} r\hat{\theta} && (15)
\end{aligned}
$$

for $r > a$.

1(c) Beta

Question 2 Bremsstrahlung

The given expression for heating power is

$$P_{fus} = n_D n_T \langle \sigma v \rangle_{DT} E_\alpha. \tag{16}$$

where $\langle \sigma v \rangle_{DT} = 10^{-34} T_i^3$. Assuming the reactivity is in units of m^3/s, then the heating power is in $\mathrm{MeV/sm}^3$. The approximation

$$P_{br} = 1.7 \times 10^{-38} z^2 n_e n_i \sqrt{T_e} \quad \left[\frac{\mathrm{W}}{\mathrm{m}^3}\right] \tag{17}$$

can be used for Bremsstrahlung loss power, or

$$P_{br} = 1.061 \times 10^{-25} z^2 n_e n_i \sqrt{T_e} \quad \left[\frac{\mathrm{MeV}}{\mathrm{m}^3\mathrm{s}}\right]. \tag{18}$$

for each species n_i. Then, assuming $n_D = n_T = \frac{1}{2} n_i = \frac{1}{2} n_e \gg n_\alpha$, this becomes

$$P_{br} = 2 * 1.061 \times 10^{-25} n_D^2 \sqrt{T_e} \quad \left[\frac{\mathrm{MeV}}{\mathrm{m}^3\mathrm{s}}\right] \tag{19}$$

and the ratio of heating power to Bremsstrahlung loss is:

$$\frac{P_{fus}}{P_{br}} = \frac{n_D^2\,(10^{-34}T_i^3)\,E_\alpha}{1.061\times10^{-25}\,(2)\,n_d^2\sqrt{T_e}} \tag{20}$$

$$1 = \frac{\cancel{n_D^2}\,(10^{-34}T_i^3)\,E_\alpha}{1.061\times10^{-25}\cancel{n_D^2}\cdot 4\sqrt{T_e}} \tag{21}$$

$$T_i^{-5/2} = \frac{(10^{-34})\,E_\alpha}{2\cdot 1.061\times10^{-25}} \tag{22}$$

$$T_i^{-5/2} = 1.6494\times10^{-9} \tag{23}$$

$$T_i = 3258.9 \tag{24}$$

for P_{br} to equal P_{fus}, and thus at temperatures below 3.26 keV, the Bremsstrahlung power loss exceeds the fusion heating power.

Question 3 Helium Fraction

Alpha impurities would lead to decreased D and T population, which decreases fusion power, as well as a greater Bremsstrahlung power loss than for either D or T, since P_{br} is proportional to z^2. Even a small impurity of helium may have a large effect on this power ratio.

3(a) Fusion Power Ratio Introducing α impurity fraction f, Equation 20 becomes

$$\frac{P_{fus}}{P_{br}} = \frac{\frac{(1-f)^2}{4}n_e^2\langle\sigma v\rangle_{DT}E_\alpha}{1.061\times10^{-25}n_e\sqrt{T_e}\,[4fn_e+(1-f)n_e]}, \tag{25}$$

or, using the value for $\langle\sigma v\rangle_{DT}$,

$$\frac{P_{fus}}{P_{br}} = \frac{\frac{(1-f)^2}{4}n_e^2 10^{-34}T_i^3E_\alpha}{1.061\times10^{-25}n_e\sqrt{T_e}\,[(3f+1)n_e]}. \tag{26}$$

3(b) 10% helium fraction At 3.26 keV, the balance temperature from the previous problem, the ratio with impurity $f = 0.1$ is

$$\frac{P_{fus}}{P_{br}} = \frac{\frac{0.9^2}{4}(10^{-34})(3258.9)^3(3.5)}{1.061\times10^{-25}\sqrt{3258.9}(1.3)} \tag{27}$$

$$\frac{P_{fus}}{P_{br}} = 0.3115 \tag{28}$$

where clearly even a small helium fraction can greatly decrease net power.

3(c) Balance with confinement time The volumetric helium production rate is

$$\left(\frac{d}{dt}n_{He}\right)_+ = \frac{P_{fus}}{E_\alpha}\left[\mathrm{m}^{-3}\mathrm{s}^{-1}\right] \tag{29}$$

so, for plasma volume V and given n_e, the gain in f is

$$\left(\frac{df}{dt}\right)_+ = \frac{\frac{(1-f)^2}{4}n_e(10^{-34})E_\alpha T_i^3}{E_\alpha} = k(1-2f+f^2), \tag{30}$$

and the loss in He or f (over the entire volume) is

$$\left(\frac{d}{dt}n_{He}\right)_- = -\frac{n_He}{\tau_{p,He}} \tag{31}$$

$$\left(\frac{df}{dt}\right)_- = -\frac{f}{\tau_{p,He}} \tag{32}$$

so the Helium concentration f is found by solving

$$\frac{df}{dt} = k(1-2f+f^2) - \frac{f}{\tau_{p,He}} \tag{33}$$

$$\frac{df}{dt} = k - (2k+1/\tau)f + kf^2 \tag{34}$$

where

$$k = \frac{10^{-34}n_e V T_i^3}{4} \tag{35}$$

From the next subquestion (where fusion power is a given constant), the simplification can be made that fusion power is constant (no $1-f$ term in fusion power), where the differential equation becomes

$$\frac{df}{dt} = \frac{P_{fus}V}{E_\alpha n_e V} - \frac{1}{\tau}f, \tag{36}$$

which has the solution

$$f(t) = \frac{P_{fus} V \tau}{E_\alpha n_e V} \left(1 - e^{-t/\tau}\right) \tag{37}$$

which is

$$\frac{P_{fus} V \tau}{E_\alpha n_e V} \tag{38}$$

at equilibrium.

3(d) Numerical Results From before, $E_\alpha = 3.5$ MeV $= 5.6076 \times 10^{-19}$ MJ, using the given values,

$$\frac{P_{fus} V \tau}{E_\alpha n_e V} = \frac{(500 \text{ MW})\tau}{(5.6076 \times 10^{-19} \text{ MJ})(10^{20} \text{ m}^{-3})(840 \text{ m}^3)} = 0.010615 \text{ s} \times \tau. \tag{39}$$

Then the equilibrium values of f, assuming no effect on fusion power, are

$$f(\tau = 1 \text{ s}) = 0.010615 \tag{40}$$
$$f(\tau = 5 \text{ s}) = 0.053074 \tag{41}$$
$$f(\tau = 10 \text{ s}) = 0.106148. \tag{42}$$

Then a higher confinment time for helium can lead to greater equilibrium helium concentration and therefore higher heat loss due to Bremsstrahlung. As shown in part (b), a helium fraction of $f = 0.1$ leads to significant heat loss, which a helium confinement time of 10s leads to in this example.