NE 795 Assignment 4

Kyle Hansen

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Question 1 Differential Fourier Analysis

1(a) Iterative Error Equations Using the definitions of iterative error,

$$\delta\psi^{(s+1/2)} = \psi^\star - \psi^{(s+1/2)} \tag{1}$$
$$\delta F^{(s+1/2)} = F^\star - F^{(s+1/2)} \tag{2}$$
$$\delta J^{(s+1)} = J^\star - J^{(s+1)} \tag{3}$$
$$\delta\phi^{(s+1)} = \phi^\star - \phi^{(s+1)}, \tag{4}$$

The equations for iterative error of the transport equation can be found by subtracting the equation for the $(s+1/2)$ iterate from the equation for the true (converged) solution:

$$\mu\frac{d\psi^\star}{dx} - \mu\frac{d\psi^{(s+1/2)}}{dx} + \sigma_t\psi^\star - \sigma_t\psi^{(s+1/2)} = \frac{1}{2}\left(\sigma_s\phi^\star + q\right) - \frac{1}{2}\left(\sigma_s\phi^{(s)} + q\right) \tag{5}$$
$$\mu\frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t\delta\psi^{(s+1/2)} = \frac{1}{2}\left(\sigma_s\delta\phi^{(s)}\right) \tag{6}$$

where

$$\delta\psi^{(s+1/2)}(0,\mu) = (\psi^+_{in} - \psi^+_{in}) = 0 \quad \text{for } \mu > 0 \tag{7}$$
$$\delta\psi^{(s+1/2)}(X,\mu) = (\psi^-_{in} - \psi^-_{in}) = 0 \quad \text{for } \mu < 0 \tag{8}$$

and * denotes the true (converged) solution. The equations for error in the closure terms are found in a similar way:

$$\delta F^{(s+1/2)} = \int_{-1}^{1}\left(\frac{1}{3} - \mu^2\right)\delta\psi^{(s+1/2)}d\mu \tag{9}$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^{1} \left(\frac{1}{2} - |\mu|\right) \delta\psi^{(s+1/2)}(0,\mu) d\mu \tag{10a}$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^{1} \left(\frac{1}{2} - |\mu|\right) \delta\psi^{(s+1/2)}(X,\mu) d\mu. \tag{10b}$$

The equations for the errors in the SM equations are found the same way:

$$\frac{d\delta J^{(s+1)}}{dx} + (\sigma_t - \sigma_s)\,\delta\phi^{(s+1)} = 0 \tag{11}$$

$$\frac{1}{3}\frac{d\delta\phi^{(s+1)}}{dx} + \sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} \tag{12}$$

$$\delta J^{(s+1)}(0) = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \tag{13}$$

$$\delta J^{(s+1)}(X) = \frac{1}{2}\delta\phi^{(s+1)}(X) - \delta P_R^{(s+1/2)} \tag{14}$$

These equations resemble the original iterative equations, with source (and incoming flux boundary conditions) set to zero. By taking the derivative of Equation 12 with respect to x and plugging into equation Equation 11, these equations can be reduced to a single equation for $\delta\phi$ only:

$$\sigma_t \delta J^{(s+1)} = \frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3}\frac{d\delta\phi^{(s+1)}}{dx} \tag{15}$$

$$\delta J^{(s+1)} = \frac{1}{\sigma_t}\frac{d\delta F^{(s+1/2)}}{dx} - \frac{1}{3\sigma_t}\frac{d\delta\phi^{(s+1)}}{dx} \tag{16}$$

$$\frac{d\delta J^{(s+1)}}{dx} = \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2}, \tag{17}$$

evaluating Equation 11 using this equation for $(\delta J)_x$,

$$(\sigma_t - \sigma_s)\,\delta\phi^{(s+1)} + \frac{d\delta J^{(s+1)}}{dx} = 0 \tag{18}$$

$$(\sigma_t - \sigma_s)\,\delta\phi^{(s+1)} + \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2} = 0. \tag{19}$$

The boundary conditions $J(0)$ and $J(X)$ can be used to formulate axiliary conditions for the equation for ϕ. Evaluating Equation 16 using Equations 13 and 14 yields the auxiliary conditions:

$$\frac{1}{\sigma_t(0)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=0} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \tag{20a}$$

$$\frac{1}{\sigma_t(X)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=X} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \tag{20b}$$

Then the full set of equations for iterative error for the SM iterative algorithm is:

$$\mu\frac{d\delta\psi^{(s+1/2)}}{dx} + \sigma_t\delta\psi^{(s+1/2)} = \frac{1}{2}\left(\sigma_s\delta\phi^{(s)}\right) \tag{21a}$$

$$\delta\psi^{(s+1/2)}(0,\mu) = 0 \quad \text{for } \mu > 0 \qquad \text{and} \qquad \delta\psi^{(s+1/2)}(X,\mu) = 0 \quad \text{for } \mu < 0 \tag{21b}$$

$$\delta F^{(s+1/2)} = \int_{-1}^{1}\left(\frac{1}{3} - \mu^2\right)\delta\psi^{(s+1/2)}d\mu \tag{21c}$$

$$\delta P_L^{(s+1/2)} = \int_{-1}^{1}\left(\frac{1}{2} - |\mu|\right)\delta\psi^{(s+1/2)}(0,\mu)d\mu \tag{21d}$$

$$\delta P_R^{(s+1/2)} = \int_{-1}^{1}\left(\frac{1}{2} - |\mu|\right)\delta\psi^{(s+1/2)}(X,\mu)d\mu \tag{21e}$$

$$\left(\sigma_t - \sigma_s\right)\delta\phi^{(s+1)} + \frac{1}{\sigma_t}\frac{d^2\delta F^{(s+1/2)}}{dx^2} - \frac{1}{3\sigma_t}\frac{d^2\delta\phi^{(s+1)}}{dx^2} = 0 \tag{21f}$$

$$\frac{1}{\sigma_t(0)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=0} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=0} = -\frac{1}{2}\delta\phi^{(s+1)}(0) + \delta P_L^{(s+1/2)} \tag{21g}$$

$$\frac{1}{\sigma_t(X)}\left.\frac{d\delta F^{(s+1/2)}}{dx}\right|_{x=X} - \frac{1}{3\sigma_t}\left.\frac{d\delta\phi^{(s+1)}}{dx}\right|_{x=X} = -\frac{1}{2}\delta\phi^{(s+1)}(X) + \delta P_R^{(s+1/2)} \tag{21h}$$

1(b) Fourier Analysis Validity of Fourier analysis depends on the conditions:

- Infinite spatial domain
- Constant coefficients in all equations ($\sigma_t(x) = \sigma_t$ and $\sigma_s(x) = \sigma_s$).

If these conditions are true, the Fourier Ansatz is used to formulate an assumed form of the solutions to the system defined in 21:

$$\delta\psi^{(s)} = \int_{-\infty}^{\infty} f^{(s)}(\lambda,\mu)e^{i\lambda\sigma_t x}d\lambda \tag{22a}$$

$$\delta F^{(s)} = \int_{-\infty}^{\infty} g^{(s)}(\lambda)e^{i\lambda\sigma_t x}d\lambda \tag{22b}$$

$$\delta\phi^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda)e^{i\lambda\sigma_t x}d\lambda \tag{22c}$$

where λ is the Fourier mode wavenumber. Substituting the Fourier Ansatz into the system 21:

$$\int_{-\infty}^{\infty}\left[\sigma_t(i\mu\lambda+1)f^{(s+1/2)}(\lambda,\mu)e^{i\lambda\sigma_t x}=\frac{\sigma_s}{2}h^{(s)}(\lambda)e^{i\lambda\sigma_t x}\right]d\lambda \tag{23a}$$

$$\int_{-\infty}^{\infty}\left[g^{(s+1/2)}(\lambda)e^{i\lambda\sigma_t x}=\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)f^{(s+1/2)}(\lambda,\mu)e^{i\lambda\sigma_t x}d\mu\right]d\lambda \tag{23b}$$

$$\int_{-\infty}^{\infty}\left[\left(\sigma_t-\sigma_s\right)h^{(s+1)}(\lambda)e^{i\lambda\sigma_t x}-\lambda^2\sigma_t g^{(s+1/2)}(\lambda)e^{i\lambda\sigma_t x}+\frac{1}{3}\lambda^2\sigma_t h^{(s+1)}(\lambda)e^{i\lambda\sigma_t x}=0\right]d\lambda \tag{23c}$$

The statements remain true when the integral is removed (the terms are differentiated with respect to λ). All equations can also be divided by $e^{i\lambda\sigma_t x}$,

$$\sigma_t(i\mu\lambda+1)f^{(s+1/2)}(\lambda,\mu)=\frac{\sigma_s}{2}h^{(s)}(\lambda) \tag{24a}$$

$$g^{(s+1/2)}(\lambda)=\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)f^{(s+1/2)}(\lambda,\mu)d\mu \tag{24b}$$

$$\left(\sigma_t-\sigma_s\right)h^{(s+1)}(\lambda)-\lambda^2\sigma_t g^{(s+1/2)}(\lambda)+\frac{1}{3}\lambda^2\sigma_t h^{(s+1)}(\lambda)=0. \tag{24c}$$

Solving these equations yields the expression:

$$h^{(s+1)}(\lambda)=h^{(s)}(\lambda)\cdot\frac{\lambda^2}{1-c+\frac{1}{3\sigma_t}\lambda^2}\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)\frac{c}{2(1+i\mu\lambda)}d\mu, \tag{25}$$

where c is the scattering ratio σ_s/σ_t. Then the iteration eigenvalue $\omega(\lambda)$ is

$$\omega(\lambda)=\frac{\lambda^2}{1-c+\frac{1}{3\sigma_t}\lambda^2}\left[\frac{c}{2}\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)\frac{1}{(1+i\mu\lambda)}d\mu\right] \tag{26}$$

where the integral is equal to

$$\left[\frac{c}{2}\int_{-1}^{1}\left(\frac{1}{3}-\mu^2\right)\frac{1}{(1+i\mu\lambda)}d\mu\right]=\frac{\left(2\left(\lambda^2+3\right)\tan^{-1}\lambda\right)-6\lambda}{3\lambda^3} \tag{27}$$

1(c) Plots and Spectral Radius

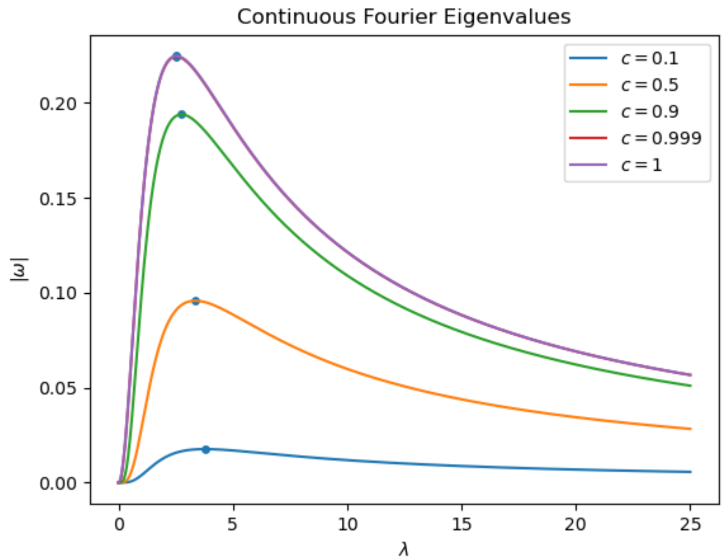


Figure 1: Continuous Fourier Eigenvalue Spectrum

Table 1: Continuous Spectral Radii

c	0.1	0.5	0.9	0.999	1
ρ	0.0177	0.0958	0.1939	0.2243	0.2247

Question 2 Discretized Fourier Analysis

2(a) Iterative Error Equations Subtracting the $(s+1/2)$ transport iteration equation from the equation for the true solution $\psi^\star$ yields

$$\mu_m\left(\psi^\star_{m,i+1/2} - \psi^\star_{m,i-1/2}\right) - \mu_m\left(\psi^{(s+1/2)}_{m,i+1/2} - \psi^{(s+1/2)}_{m,i-1/2}\right) + \sigma_{t,i}\Delta x_i(\psi^\star_{m,i} - \psi^{(s+1/2)}_{m,i}) = \frac{1}{2}\left(\sigma_{s,i}\phi^\star_i - \sigma_{s,i}\phi^{(s)}_i + \cancel{(q_i - q_i)}\right)\Delta x_i \quad (28)$$

$$\mu_m\left(\delta\psi^{(s+1/2)}_{m,i+1/2} - \delta\psi^{(s+1/2)}_{m,i-1/2}\right) + \sigma_{t,i}\Delta x_i(\delta\psi^{(s+1/2)}_{m,i}) = \frac{1}{2}\left(\sigma_{s,i}\delta\phi^{(s)}_i\right)\Delta x_i, \quad (29)$$

with boundary conditions $\delta\psi^{(s+1/2)}_{m,1/2} = 0$ for $\mu_m > 0$ and $\delta\psi^{(s+1/2)}_{m,N+1/2} = 0$ for $\mu_m < 0$. The error in scalar flux is

$$\delta\phi_{i+1/2}^{(s)} = \sum_{m=1}^{M} \delta\psi_{m,i+1/2}^{(s)} w_m, \tag{30}$$

and the SC cell-center values are

$$\psi_{m,i}^{\star} - \psi_{m,i}^{(s+1/2)} = \alpha_{m,i}\left(\psi_{m,i-1/2}^{\star} - \psi_{m,i-1/2}^{(s+1/2)}\right) + (1-\alpha_{m,i})\left(\psi_{m,i+1/2}^{\star} - \psi_{m,i+1/2}^{(s+1/2)}\right) \tag{31}$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_{m,i}\left(\delta\psi_{m,i-1/2}^{(s+1/2)}\right) + (1-\alpha_{m,i})\left(\delta\psi_{m,i+1/2}^{(s+1/2)}\right). \tag{32}$$

The error in the second-moment closure term is

$$\delta F_i^{(s+1/2)} = \sum_{m=1}^{M} \left(\frac{1}{3} - \mu_m^2\right) \delta\psi_{m,i}^{(s+1/2)} w_m \tag{33}$$

$$\delta F_{i+1/2}^{(s+1/2)} = \sum_{m=1}^{M} \left(\frac{1}{3} - \mu_m^2\right) \delta\psi_{m,i+1/2}^{(s+1/2)} w_m, \tag{34}$$

and the error in d_i is

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2}\Delta x_{i-1/2}}\left(F_i^{\star} - F_i^{(s+1/2)} - F_{i-1}^{\star} + F_{i-1}^{(s+1/2)}\right) \tag{35}$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2}\Delta x_{i-1/2}}\left(\delta F_i^{(s+1/2)} - \delta F_{i-1}^{(s+1/2)}\right). \tag{36}$$

The boundary term error is

$$\delta P_L = \sum_{m=1}^{M} \left(\frac{1}{2} - |\mu_m|\right) \delta\psi_{m,1/2}^{(s+1/2)} w_m \tag{37}$$

$$\delta P_R = \sum_{m=1}^{M} \left(\frac{1}{2} - |\mu_m|\right) \delta\psi_{m,N+1/2}^{(s+1/2)} w_m. \tag{38}$$

The error in the SM equation is given by

$$\begin{aligned} &- a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} \\ &\qquad = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} + \Delta x_i \cancel{q_i - q_i}, \end{aligned} \tag{39}$$

where $\sigma_{a,i} = \sigma_{t,i} - \sigma_{s,i}$. The errors for the boundary conditions are given by

$$\left(\frac{1}{2} + a_1\right) \delta\phi_0^{(s+1)} - a_1 \delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} + \cancel{2J_{in}^+ - 2J_{in}^+} \tag{40a}$$

$$\left(\frac{1}{2} + a_{N+1}\right) \delta\phi_{N+1}^{(s+1)} - a_{N+1} \delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} - \cancel{2J_{in}^- + 2J_{in}^-}. \tag{40b}$$

Then the full set of equations for iterative error for the discretized SM Method is

$$\mu_m \left(\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)}\right) + \sigma_{t,i}\Delta x_i (\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2}\left(\sigma_{s,i}\delta\phi_i^{(s)}\right)\Delta x_i \tag{41a}$$

$$\delta\psi_{m,1/2}^{(s+1/2)} = 0 \text{ for } \mu_m > 0, \qquad \delta\psi_{m,N+1/2}^{(s+1/2)} = 0 \text{ for } \mu_m < 0 \tag{41b}$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_{m,i}\left(\delta\psi_{m,i-1/2}^{(s+1/2)}\right) + (1 - \alpha_{m,i})\left(\delta\psi_{m,i+1/2}^{(s+1/2)}\right) \tag{41c}$$

$$\delta d_i^{(s+1/2)} = \frac{1}{\sigma_{t,i-1/2}\Delta x_{i-1/2}}\left[\sum_{m=1}^{M}\left(\frac{1}{3} - \mu_m^2\right)\left(\delta\psi_{m,i}^{(s+1/2)} - \delta\psi_{m,i-1}^{(s+1/2)}\right) w_m\right] \tag{41d}$$

$$\delta P_L = \sum_{m=1}^{M}\left(\frac{1}{2} - |\mu_m|\right)\delta\psi_{m,1/2}^{(s+1/2)} w_m \tag{41e}$$

$$\delta P_R = \sum_{m=1}^{M}\left(\frac{1}{2} - |\mu_m|\right)\delta\psi_{m,N+1/2}^{(s+1/2)} w_m \tag{41f}$$

$$-a_i(\delta\phi_{i-1}^{(s+1)}) + (a_{i+1} + a_i + (\sigma_{a,i})\Delta x_i)\delta\phi_i^{(s+1)} - a_{i+1}\delta\phi_{i+1}^{(s+1)} = \delta d_i^{(s+1/2)} - \delta d_{i+1}^{(s+1/2)} \tag{41g}$$

$$\left(\frac{1}{2} + a_1\right)\delta\phi_0^{(s+1)} - a_1\delta\phi_1^{(s+1)} = \delta P_L^{(s+1/2)} - \delta d_1^{(s+1/2)} \tag{41h}$$

$$\left(\frac{1}{2} + a_{N+1}\right)\delta\phi_{N+1}^{(s+1)} - a_{N+1}\delta\phi_{N+1}^{(s+1)} = \delta P_R^{(s+1/2)} + \delta d_{N+1}^{(s+1/2)} \tag{41i}$$

where

$$a_i = \frac{1}{3\sigma_{t,i-1/2}\Delta x_{i-1/2}} \tag{42}$$

$$\phi_0 = \phi_{1/2}, \qquad \phi_{N+1} = \phi_{N+1/2} \tag{43}$$

$$\psi_{m,0} = \psi_{m,1/2}, \qquad \psi_{m,N+1} = \psi_{m,N+1/2}. \tag{44}$$

2(b) Fourier Analysis Like the continuous form of the SM method, Fourier Analysis of the discretized form requires that these assumptions are satisfied:

- Infinite spatial domain

- Constant coefficients $\sigma_a, \sigma_t, \theta, \tau$
- Uniform spatial grid Δx

First, the iterative equations are rewritten to apply the assumptions, and eliminating the variable δd_i for simplicity. The new iterative system is

$$\mu_m \left(\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)}\right) + \sigma_t \Delta x (\delta\psi_{m,i}^{(s+1/2)}) = \frac{1}{2}\left(\sigma_s \delta\phi_i^{(s)}\right) \Delta x \tag{45}$$

$$\delta\psi_{m,i}^{(s+1/2)} = \alpha_m \left(\delta\psi_{m,i-1/2}^{(s+1/2)}\right) + (1-\alpha_m)\left(\delta\psi_{m,i+1/2}^{(s+1/2)}\right) \tag{46}$$

$$(2a + \sigma_a \Delta x)\delta\phi_i^{(s+1)} - a(\delta\phi_{i-1}^{(s+1)} + \delta\phi_{i+1}^{(s+1)}) = \tag{47}$$

$$\frac{1}{\sigma_t \Delta x}\left[\sum_{m=1}^{M} w_m \left(\frac{1}{3} - \mu_m^2\right)\left(2\delta\psi_{m,i}^{(s+1/2)} - \delta\psi_{m,i+1}^{(s+1/2)} - \delta\psi_{m,i-1}^{(s+1/2)}\right)\right]. \tag{48}$$

Then, applying the Fourier Ansatz, the asssumed form of the two solution functions are

$$\delta\psi_{m,i+1/2}^{(s+1/2)} = \int_{-\infty}^{\infty} f_m^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x_{i+1/2}} d\lambda \tag{49}$$

$$\delta\phi_i^{(s)} = \int_{-\infty}^{\infty} h^{(s)}(\lambda) e^{i\lambda\sigma_t x_i} d\lambda \tag{50}$$

where $f_m(\lambda)$ denotes $f(\lambda, \mu_m)$. The cell-edge and neighboring cell terms can be expanded in the assumed form to simplify. Introducing $\beta = \frac{1}{2}\lambda\sigma_t\Delta x$,

$$\begin{aligned}
\delta\psi_{m,i+1/2}^{(s+1/2)} - \delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)}\left(e^{i\lambda\sigma_t x_{i+1/2}} - e^{i\lambda\sigma_t x_{i+1/2}}\right) \\
&= f_m^{(s+1/2)}\left(e^{i\lambda\sigma_t (x_i + \frac{1}{2}\Delta x)} - e^{i\lambda\sigma_t (x_i - \frac{1}{2}\Delta x)}\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(e^{i\beta} - e^{-i\beta}\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(\cos\beta + i\sin\beta - \cos(-\beta) - i\sin(-\beta)\right) \\
&= f_m^{(s+1/2)}(\lambda) e^{i\lambda\sigma_t x_i}\left(2i\sin\beta\right)
\end{aligned} \tag{51}$$

which uses the identities $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$. The SC differencing scheme is similarly rewritten:

$$\begin{aligned}
(\alpha_m)\delta\psi_{m,i+1/2}^{(s+1/2)} + (1-\alpha_m)\delta\psi_{m,i-1/2}^{(s+1/2)} &= f_m^{(s+1/2)}\left(\alpha_m e^{i\lambda\sigma_t x_{i-1/2}} + (1-\alpha_m)\, e^{i\lambda\sigma_t x_{i+1/2}}\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(\alpha_m e^{-i\beta} + (1-\alpha_m)\, e^{i\beta}\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(i\sin\beta(1-\alpha_m-\alpha_m) + \cos\beta(\alpha + 1 - \alpha)\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(\cos\beta - (1-2\alpha_m)\sin\beta\right) \\
&= f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i}\left(\cos\beta + \tilde{\alpha}_m \sin\beta\right)
\end{aligned} \tag{52}$$

Where $\tilde{\alpha}_m = (1 - 2\alpha_m)$. The cell-centered scalar flux term from the SM equation becomes

$$\begin{aligned}
\delta\phi_{i+1}^{(s+1)} + \delta\phi_{i-1}^{(s+1)} &= h^{(s+1)} \left[e^{i\lambda\sigma_t x_{i+1}} + e^{i\lambda\sigma_t x_{i-1}}\right] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} \left[e^{2i\beta} + e^{-2i\beta}\right] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} \left[\cos(2\beta) + i\sin(2\beta) + \cos(-2\beta) + i\sin(-2\beta)\right] \\
&= h^{(s+1)} e^{i\lambda\sigma_t x_i} \left[2\cos(2\beta)\right]
\end{aligned} \tag{53}$$

and the center angular flux term from the SM equation similarly becomes

$$\delta\psi_{m,i+1}^{(s+1/2)} + \delta\psi_{m,i-1}^{(s+1/2)} = f_m^{(s+1/2)} e^{i\lambda\sigma_t x_i} \left(2\cos(2\beta)\right)\left(\cos\beta + \tilde{\alpha}_m \sin\beta\right) \tag{54}$$

Using these terms, the transport equation can be written in terms of the Fourier Ansatz (eliminating the common $e^{i\lambda\sigma_t x_i}$ term):

$$2f_m^{(s+1/2)}\mu_m i\sin\beta + \sigma_t\Delta x\left(\cos\beta + \tilde{\alpha}_m i\sin\beta\right) f_m^{(s+1/2)} = \frac{\sigma_s\Delta x}{2} h^{(s)} \tag{55}$$

$$f_m^{(s+1/2)} = h^{(s)} \frac{\sigma_s\Delta x}{2\left[(i\sin\beta)(2\mu_m + \sigma_t\Delta x\tilde{\alpha}_m) + \sigma_t\Delta x\cos\beta\right]}. \tag{56}$$

The SM equation can be rewritten in the same way

$$\begin{gathered}
(2a + \sigma_a\Delta x)h^{(s+1)} - a(2\cos(2\beta))h^{(s+1)} = \\
\frac{1}{\sigma_t\Delta x}\sum_{m=1}^{M} w_m\left(\frac{1}{3} - \mu_m^2\right)\left(2 - 2\cos(2\beta)\right)\left(\cos\beta - 2\alpha_m i\sin\beta\right)
\end{gathered} \tag{57}$$

$$h^{(s+1)} = \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(2 - 2\cos 2\beta\right)\left(\cos\beta + \tilde{\alpha}_m i\sin\beta\right) f_m^{(s+1/2)}}{\left(2a - 2a\cos 2\beta + \sigma_a\Delta x\right)\left(\sigma_t\Delta x\right)} \tag{58}$$

Then plugging in Equation 55, the expression for $h^{(s+1)} = \omega(\lambda)h^{(s)}$ is found:

$$h^{(s+1)} = h^{(s)}\sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(2 - 2\cos 2\beta\right)\left(\cos\beta + \tilde{\alpha}_m i\sin\beta\right)\left(\sigma_s\Delta x\right)}{2\left[(i\sin\beta)(2\mu_m + \sigma_t\Delta x\tilde{\alpha}_m) + \sigma_t\Delta x\cos\beta\right]\left(2a - 2a\cos 2\beta + \sigma_a\Delta x\right)\left(\sigma_t\Delta x\right)} \tag{59}$$

$$h^{(s+1)} = h^{(s)}\frac{c}{2}\frac{\left(2 - 2\cos 2\beta\right)}{\left(2a - 2a\cos 2\beta + \sigma_a\Delta x\right)}\sum_{m=1}^{M}\frac{\left(\frac{1}{3} - \mu_m^2\right) w_m\left(\cos\beta + \tilde{\alpha}_m i\sin\beta\right)}{\left[(i\sin\beta)(2\mu_m + \sigma_t\Delta x\tilde{\alpha}_m) + \sigma_t\Delta x\cos\beta\right]} \tag{60}$$

using $1 - \cos 2\beta = 2\sin\beta$,

$$h^{(s+1)} = h^{(s)} \frac{c}{2} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(\cos\beta + \tilde{\alpha}_m i \sin\beta\right)}{(i\sin\beta)(2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m) + \sigma_t \Delta x \cos\beta} \tag{61}$$

then, multiplying both sides of the summation term by the complex cojugate of the denominator,

$$h^{(s+1)} = h^{(s)} \frac{c}{2} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \times \\ \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(\sigma_t \Delta x \cos^2\beta + (\sigma_t \Delta x \tilde{\alpha}_m - (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m)) i \sin\beta\cos\beta + \tilde{\alpha}_m (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m) \sin^2\beta\right)}{(\sin^2\beta)(2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m)^2 + (\sigma_t \Delta x)^2 \cos^2\beta} \tag{62}$$

If $\{\tilde{\alpha}_m\}$ and $\{\mu_m\}$, then the imaginary term becomes zero in this sum, so this expression is equivalent to:

$$h^{(s+1)} = h^{(s)} \frac{c}{2} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \times \\ \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(\sigma_t \Delta x \cos^2\beta + \tilde{\alpha}_m (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m) \sin^2\beta\right)}{\sin^2\beta (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m)^2 + (\sigma_t \Delta x)^2 \cos^2\beta}. \tag{63}$$

Dividing both sides by $\cos^2\beta$ yields

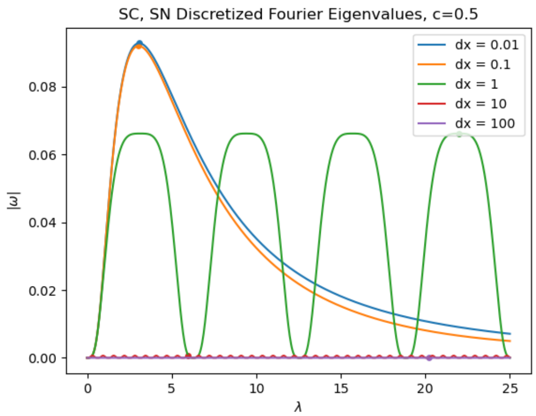
$$h^{(s+1)} = h^{(s)} \frac{c}{2} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \times \\ \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(\sigma_t \Delta x + \tilde{\alpha}_m (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m) \tan^2\beta\right)}{\tan^2\beta (2\mu_m + \sigma_t \Delta x \tilde{\alpha}_m)^2 + (\sigma_t \Delta x)^2} \tag{64}$$

and dividing both sides by $(\sigma_t \Delta x)^2$ yields:

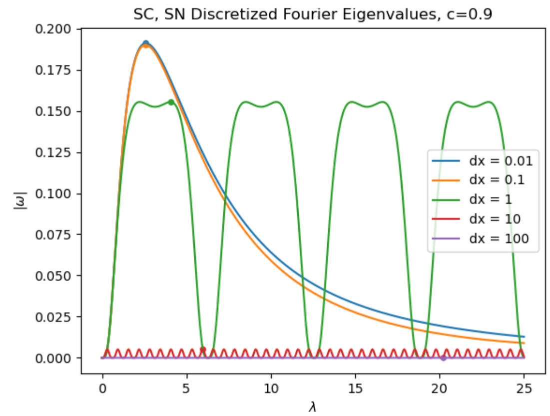
$$h^{(s+1)} = h^{(s)} \frac{c}{2} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \sum_{m=1}^{M} \frac{1}{\sigma_t \Delta x} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(1 + \tilde{\alpha}_m (\frac{2\mu_m}{\sigma_t \Delta x} + \tilde{\alpha}_m) \tan^2\beta\right)}{1 + (\frac{2\mu_m}{\sigma_t \Delta x} + \tilde{\alpha}_m)^2 \tan^2\beta}, \tag{65}$$

and

$$\omega(\lambda) = \frac{c}{2\sigma_t \Delta x} \frac{4\sin^2\beta}{4a\sin^2\beta + \sigma_a \Delta x} \sum_{m=1}^{M} \frac{\left(\frac{1}{3} - \mu_m^2\right) w_m \left(1 + \tilde{\alpha}_m (\frac{2\mu_m}{\sigma_t \Delta x} + \tilde{\alpha}_m) \tan^2\beta\right)}{1 + (\frac{2\mu_m}{\sigma_t \Delta x} + \tilde{\alpha}_m)^2 \tan^2\beta}. \tag{66}$$



(a)



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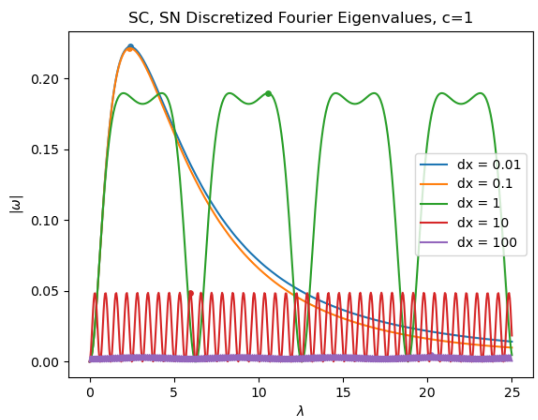

(c)

Figure 2: Discretized Fourier Eigenvalues

Table 2: Discretized Spectral Radii

c	$\Delta x = 0.01$	$\Delta x = 0.1$	$\Delta x = 1$	$\Delta x = 10$	$\Delta x = 100$
0.5	0.0928	0.0920	0.0662	6.3×10^{-4}	6.5×10^{-7}
0.9	0.1910	0.1899	0.1553	6.3×10^{-3}	5.8×10^{-8}
1.0	0.2224	0.2213	0.1896	0.0488	0.0049