NE 795 Assignment 2

Kyle Hansen

1 October 2025

Question 1

Show that

$$I_\nu = ch\nu\psi_\nu.$$

Intensity I is defined as the power (rate of energy delivery) by radiation per unit solid angle, equal to:

$$\text{Intensity} = \frac{\text{Energy}}{\text{particle}} \times \text{particle speed} \times \text{angular density} \tag{1}$$

Where, for photons, speed is always c and energy $E = h\nu$. Then

$$\text{Intensity} = I_\nu = h\nu \times c \times \text{angular density} \tag{2}$$

and the density is $\psi_\nu = \text{particles} \cdot \text{m}^{-3} \cdot \text{Sr}^{-1}$, and thus $I_\nu = ch\nu\psi_\nu$.

Question 2

Show that the equilibrium intensity is equal to

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}.$$

Given the radiative transfer equation:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I_\nu + \varkappa_\nu I\nu = \eta_\nu, \tag{3}$$

Where the function η_ν is the emission spectrum. Planck's and Kirchoff's laws, which describe the physics of photon emission, hold that the intensity of emitted radiation is

$$\varkappa_\nu B_\nu = \varkappa_\nu B(\nu, T) = (\varkappa_\nu)\frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}. \tag{4}$$

At equilibrium, the time and spatial derivatives both go to zero, so

$$\varkappa_\nu B_\nu = \varkappa_\nu I_\nu \tag{5}$$

or

$$\boxed{B_\nu(\vec{r}, \vec{\Omega}) = I_\nu(\vec{r}, \vec{\Omega})} \tag{6}$$

Question 3

Derive the form of the following moments of the specific intensity:

$$E_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\Omega, \qquad E = \int_0^\infty E_\nu d\nu,$$
$$F_\nu = \int_{4\pi} \Omega I_\nu d\Omega,$$
$$\mathbb{P}_\nu = \frac{1}{c}\int_{4\pi} \Omega \otimes \Omega I_\nu d\Omega, \qquad \mathbb{P} = \int_0^\infty \mathbb{P}_\nu d\nu$$

At equilibrium, since $I_\nu = B_\nu$, the radiative transfer equation can be reduced to

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I_\nu = \varkappa_\nu B\nu - \varkappa_\nu I\nu = 0. \tag{7}$$

Integrating over angle:

$$\int_{4\pi} \left\{\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I_\nu = 0\right\} d\Omega \tag{8}$$

becomes

$$\frac{\partial E_\nu}{\partial t} + \int \vec{\nabla} \cdot \left(\vec{\Omega} I_\nu\right) d\Omega = 0 \tag{9}$$

$$\boxed{\frac{\partial E_\nu}{\partial t} + \vec{\nabla} F_\nu = 0} \tag{10}$$

(this is the monochromatic equilibrium continuity equation)

The equation for $E = \int E_\nu d\nu$ is:

$$\int_0^\infty \left\{ \frac{\partial E_\nu}{\partial t} + \vec{\nabla} F_\nu = 0 \right\} d\nu \tag{11}$$

$$\boxed{\frac{\partial E}{\partial t} + \vec{\nabla} F + c = 0} \tag{12}$$

(this is the energy-integrated equilibrium continuity equation).

The equation for $F = \int \Omega I_\nu d\Omega$ is:

$$\int_{4\pi} \Omega \left\{ \frac{1}{c} \frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I_\nu = 0 \right\} d\Omega \tag{13}$$

$$\frac{1}{c} \frac{\partial F_\nu}{\partial t} + \vec{\nabla} \int_{4\pi} \Omega \otimes \Omega I_\nu d\Omega = 0 \tag{14}$$

$$\boxed{\frac{1}{c} \frac{\partial F_\nu}{\partial t} + c\vec{\nabla} \mathbb{P}_\nu = 0} \tag{15}$$

and the same equation integrated over frequency is

$$\boxed{\frac{1}{c} \frac{\partial F}{\partial t} + c\vec{\nabla} \mathbb{P} = 0} \tag{16}$$

Then the pressure tensor can be found similarly:

$$\int_{4\pi} \Omega \otimes \Omega \left\{ \frac{1}{c} \frac{\partial I_\nu}{\partial t} + \vec{\Omega} \cdot \vec{\nabla} I_\nu = 0 \right\} d\Omega \tag{17}$$

$$\boxed{\frac{\partial \mathbb{P}_\nu}{\partial t} + \int_{4\pi} \Omega \otimes \Omega (\vec{\Omega} \cdot \vec{\nabla} I_\nu) d\Omega = 0} \tag{18}$$

and the frequency-integrated equation:

$$\boxed{\frac{\partial \mathbb{P}}{\partial t} + \int_{4\pi} \Omega \otimes \Omega (\vec{\Omega} \cdot \vec{\nabla} I) d\Omega = 0} \tag{19}$$

In the case of the P_1 equations, the pressure equation is closed by an approximation.

Question 4

Derive the speed of radiation wave in vacuum in the radiative transfer (RT) model defined by

- The gray time-dependent P_1 equations
- The gray time-dependent $P_{1/3}$ equations

In a vacuum, the gray P_1 equations are:

$$\frac{\partial E}{\partial t} + \vec{\nabla} F = 0 \tag{20}$$
$$\frac{1}{c}\frac{\partial F}{\partial t} + \frac{c}{3}\vec{\nabla} E = 0 \tag{21}$$

and the gray $P_{1/3}$ equations are:

$$\frac{\partial E}{\partial t} + \vec{\nabla} F = 0 \tag{22}$$
$$\frac{1}{3c}\frac{\partial F}{\partial t} + \frac{c}{3}\vec{\nabla} E = 0 \tag{23}$$

4(a) The grey time-dependent P_1 equations:

The speed of the wave can be found using the standard wave equation:

$$\frac{\partial^2 u}{\partial t^2} = v^2 \nabla^2 u \tag{24}$$

For some quantity u traveling at speed v. In this case, $u = E$. Then, from Equation 20, the second time derivative is

$$\frac{\partial^2 E}{\partial t^2} = -\nabla \frac{\partial F}{\partial t} \tag{25}$$

where ∇F can be found from the second line of Equation 20:

$$\frac{1}{c}\frac{\partial F}{\partial t} = -\frac{c}{3}\vec{\nabla} E \tag{26}$$
$$\frac{\partial F}{\partial t} = -\frac{c^2}{3}\vec{\nabla} E \tag{27}$$
$$\nabla \frac{\partial F}{\partial t} = -\nabla \frac{c^2}{3}\nabla E = -\frac{c^2}{3}\nabla^2 E \tag{28}$$

plugging back into Equation 25,

$$\frac{\partial^2 E}{\partial t^2} = -\nabla \frac{c^2}{3}\nabla^2 E \tag{29}$$

which takes the form of the wave equation (Equation 24), where $v^2 = c^2/3$ or

$$\boxed{v = \frac{c}{\sqrt{3}}} \tag{30}$$

4(b) The grey time-dependent $P_{1/3}$ equations:

Since the first equation of the two approximations are the same, the second time derivative of E still isotropic

$$\frac{\partial^2 E}{\partial t^2} = -\nabla \frac{\partial F}{\partial t} \tag{31}$$

the ∇F term can be found using the second line of Equation 22:

$$\frac{1}{3c}\frac{\partial F}{\partial t} = -\frac{c}{3}\vec{\nabla} E \tag{32}$$

$$\frac{\partial F}{\partial t} = -c^2 \nabla E \tag{33}$$

$$\nabla \frac{\partial F}{\partial t} = -c^2 \nabla^2 E \tag{34}$$

Then, plugging into Equation 31,

$$\frac{\partial^2 E}{\partial t^2} = c^2 \nabla^2 E \tag{35}$$

This equation again takes the form of the wave equation, where

$$\boxed{v = c} \tag{36}$$

Question 5

Derive the system of the time-dependent P_1 and MEB equations in multigroup form from the spectral P_1 and MEB equations given by

$$\begin{aligned}
\frac{\partial E_\nu}{\partial t} + \nabla F_\nu + c\varkappa_\nu E\nu &= 4\pi \varkappa_\nu B_\nu \\
\frac{\partial F_\nu}{\partial t} + \frac{1}{3}\nabla E_\nu + \varkappa_\nu F_\nu &= 0 \\
\frac{\partial \varepsilon(T)}{\partial t} &= \varkappa_\nu (cE_\nu - 4\pi B_\nu)
\end{aligned}$$

Using the multigroup photon frequency convention:

$$I_p = \int_{\nu_p}^{\nu_{p+1}} I_\nu d\nu$$
$$E_p = \int_{\nu_p}^{\nu_{p+1}} E_\nu d\nu$$
$$F_p = \int_{\nu_p}^{\nu_{p+1}} F_\nu d\nu$$

with the frequency range groups

$$\omega_p = [\nu_p, \nu_{p+1}];\ p = 1, 2, \ldots, N_p$$
$$\nu_1 = 0;\ \nu_{N_p+1} = \infty.$$

The zeroth moment is equal exactly to:

$$\frac{\partial E_p}{\partial t} + \nabla F_p + c \int_p d\nu\ \varkappa_\nu E\nu = 4\pi \int_p d\nu\ \varkappa_\nu B_\nu. \tag{37}$$

with the shorthand notation $\int_p = \int_{\nu_p}^{\nu_{p+1}}$. Substituting the opacities for their group-averaged opacities (weighted with the appropriate distribution) leaves the exact equation intact:

$$\boxed{\frac{\partial E_p}{\partial t} + \nabla F_p + c\varkappa_p^E E_p = 4\pi \varkappa_p^B B_p} \tag{38}$$

where the weighted opacities are:

$$\varkappa_p^B = \frac{\int_p d\nu\ \varkappa_\nu B_\nu}{\int_p d\nu\ B_\nu} \tag{39}$$

$$\varkappa_p^E = \frac{\int_p d\nu\ \varkappa_\nu E_\nu}{\int_p d\nu\ E_\nu} \tag{40}$$

Integrating the first moment over group frequency, and using the group opacity approximation defined above, is:

$$\boxed{\frac{\partial F_p}{\partial t} + \frac{1}{3}\nabla E_p + \varkappa_p^F F_\nu = 0} \tag{41}$$

with the weighted opacity:

$$\varkappa_p^F = \frac{\int_p d\nu \ \varkappa_\nu F_\nu}{\int_p d\nu \ F_\nu}. \tag{42}$$

Integrating the MEB equation and using the group opacities from before yields:

$$\boxed{\frac{\partial \varepsilon(T)}{\partial t} = c\varkappa_p^E E_\nu - 4\pi \varkappa_p^B B_\nu} \tag{43}$$

This leaves the group opacities:

$$\varkappa_p^B = \frac{\int_p d\nu \ \varkappa_\nu B_\nu}{\int_p d\nu \ B_\nu} \tag{44}$$

$$\varkappa_p^E = \frac{\int_p d\nu \ \varkappa_\nu E_\nu}{\int_p d\nu \ E_\nu} \tag{45}$$

$$\varkappa_p^F = \frac{\int_p d\nu \ \varkappa_\nu F_\nu}{\int_p d\nu \ F_\nu}. \tag{46}$$

Since $B_\nu(T)$ is a known distribution, the group opacity $\varkappa_p^B$ can be evaluated exactly, but the other two cannot, and must be approximated. Near equilibrium, when $E_\nu \approx B_\nu$, then the same $\varkappa_p^B$ can be used– this method offers the advantage that group opacities are set once, and do not need to be calcualted iteratively. Another approximation would be averaging this opacity over $B_\nu(T_{rad})$ using the equivalent temperature of the radiation at this point in phase space, rather than the physical temperature, Where

$$T_{rad} = \left(\frac{1}{a_R}\int_0^\infty E_\nu\right)^{1/4} = \left(\frac{1}{a_R}\sum_{p=1}^{N_p} E_p\right)^{1/4}. \tag{47}$$

Using this approximation,

$$\varkappa_p^E \approx \frac{\int_p d\nu \ \varkappa_\nu B_\nu(T_{rad})}{\int_p d\nu \ B_\nu(T_{rad})}. \tag{48}$$