MA 580 Assignment 4

Kyle Hansen

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AI Use Statement: ChatGPT was used to help recall the name and properties of the quadratic form in problem 2

Question 1 GMRES Residual Estimate

An estimate for GMRES residual is given by

$$\frac{\|r_k\|}{\|r_0\|} \leq \|p_k(A)\| \tag{1}$$

for every $p_k \in \mathcal{P}_k$ ($p_k(0) = 1$). Then, for diagonizable matrix $A = V\Lambda V^{-1}$,

$$\frac{\|r_k\|}{\|r_0\|} \leq \left\|Vp_k(\Lambda)V^T\right\| \leq \|V\| \, \|p_k(\Lambda)\| \, \left\|V^{-1}\right\| \tag{2}$$

and since $\|V\| \, \|V^{-1}\| = \kappa_2(V)$,

$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa_2(V) \, \|p_k(\Lambda)\| = \kappa_2(V) \left\| \begin{bmatrix} p_k(\lambda_1) & & & \\ & p_k(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_n) \end{bmatrix} \right\| \tag{3}$$

Then, since the norm of a diagonal matrix is its largest eigenvalue,

$$\frac{\|r_k\|}{\|r_0\|} \leq \kappa_2(V) \max_{z\in\sigma(A)} |p_k(z)|. \tag{4}$$

Then an appropriate choice of polynomial p_k would be one that minimizes the the value about the eigenvalues of A. Here $(\frac{10-z}{10}\frac{20-z}{20})^{k/2}$ is chosen, since $p_k = 0$ at the center of the range of possible eivenvalues for this problem. For this choice of p_k, the maximum absolute value $|\max p_k(z)|$ for $z \subset (9, 11) \cup (19, 21)$ occurs at $z = 9$ and is equal to $(11/200)^{k/2}$. Then, for the given properties of A, the residual at iteration k can be bounded by

$$\frac{\|r_k\|}{\|r_0\|} \leq 100 \cdot \left(\frac{11}{200}\right)^{\frac{k}{2}}. \tag{5}$$

Using a relative residual of 10^{-6}, the number of iterations required is

$$10^{-6} = 100 \cdot \left(\frac{11}{200}\right)^{\frac{k}{2}} \tag{6}$$

$$\frac{10^{-6}}{100} = \left(\frac{11}{200}\right)^{\frac{k}{2}} \tag{7}$$

$$\ln 10^{-8} = \frac{k}{2} \ln \frac{11}{200} \tag{8}$$

$$2\frac{\ln 10^{-8}}{\ln \frac{11}{200}} = k = 12.702 \tag{9}$$

And thus about 13 iterations are required for the desired level of convergence.

Question 2 CG Error Estimate

The CG iterative error is

$$e_k = x^\star - x_k \tag{10}$$

$$= x^\star - x_0 - \sum_{j=0}^{k-1} r_j A^j r_0 \tag{11}$$

$$= e_0 - \sum_{j=0}^{k-1} r_j A^j r_0 \leq p(A) e_0, \qquad \forall p \in \mathcal{P}_k, \tag{12}$$

and its norm is bounded by

$$\|e_k\|_A \leq \|p(A)e_0\|_A \tag{13}$$

$$\|e_k\|_A^2 \leq \|p(A)e_0\|_A^2 = (p(A)e_0)^T A p(A) e_0. \tag{14}$$

Since $A = U \Lambda U^T$, where Λ is a diagonal matrix composed of the eigenvalues of A and U is an orthonormal matrix composed of the corresponding eigenvectors. Then $p(A) = U p(\Lambda) U^T$ and

$$
\begin{aligned}
\|e_k\|_A^2 &\leq (p(A)e_0)^T U \Lambda U^T (p(A)e_0) && (15)\\
&\leq (Up(\Lambda)U^T e_0)^T U \Lambda U^T (Up(\Lambda)U^T e_0) && (16)\\
&\leq e_0^T U p(\Lambda) \cancelto{I}{U^T U} \Lambda \cancelto{I}{U^T U} p(\Lambda) U^T e_0 && (17)\\
&\leq e_0^T U p(\Lambda) \Lambda p(\Lambda) U^T e_0 && (18)
\end{aligned}
$$

This product is of quadratic form and can be represented as a sum

$$
\begin{aligned}
\|e_k\|_A^2 &\leq (U^T e_0)^T p(\Lambda)\Lambda p(\Lambda) U^T e_0 && (19)\\
&\leq \sum_{i=0}^{n} \sum_{j=1}^{n} (U^T e_0)_i (p(\Lambda)\Lambda p(\Lambda))_{i,j} (U^T e_0)_j && (20)
\end{aligned}
$$

where, since $p(\Lambda)\Lambda p(\Lambda)$ is a diagonal matrix,

$$
\begin{aligned}
\|e_k\|_A^2 &\leq \sum_{j=1}^{n} (U^T e_0)_j \lambda_j p(\lambda_j)^2 (U^T e_0)_j && (21)\\
\|e_k\|_A^2 &\leq \sum_{j=1}^{n} \lambda_j p(\lambda_j)^2 \langle u_j, e_0 \rangle , && (22)
\end{aligned}
$$

where u_j are the columns of U, equal to the eigenvectors of A.

Question 3 Convergence Proof

Here $x^\star$ denotes the exact solution $x^\star = A^{-1}b$.

The CG iterative error can be estimated by

$$
\frac{\|e_k\|_A}{\|e_0\|_A} \leq \|p_k(A)\|_A \,. \tag{23}
$$

Then one valid choice for the residual polynomial is

$$
\bar{p}_k(z) = \prod_{i=1}^{k} \frac{\lambda_i - z}{\lambda_i} \tag{24}
$$

where $p_k(0) = 1$ and $p_k(\lambda_i) = 0$ for $1 \leq i \leq k$. Then, with matrix V composed of the eigenvectors of A and diagonal matrix Λ where the diagonal entries are the corresponsing eigenvalues of A, $A = V\Lambda V^{-1}$, and $p_k(A) = V p_K(\Lambda) V^{-1}$:

$$p_k(A) = \sum_{j=0}^{k} c_j A^j \tag{25}$$

$$p_k(A) = \sum_{j=0}^{k} c_j V D^j V^{-1} \tag{26}$$

$$p_k(A) = V \left(\sum_{j=0}^{k} c_j D^j \right) V^{-1} \tag{27}$$

$$p_k(A) = V p_K(\Lambda) V^{-1}, \tag{28}$$

and since $\overline{p}_k(\lambda_i) = 0$ for $1 \leq i \leq k$,

$$\overline{p}_n(\lambda_l) = \frac{\lambda_1 - \lambda_l}{\lambda_1} \cdots \frac{\cancelto{0}{\lambda_l - \lambda_l}}{\lambda_l} \cdots \frac{\lambda_n - \lambda_l}{\lambda_n} = 0 \text{ for } 1 \leq l \leq n \tag{29}$$

and

$$\frac{\|e_k\|_A}{\|e_0\|_A} \leq \|\overline{p}_n(A)\|_A = \left\| V \overline{p}_n(\Lambda) V^{-}1 \right\|_A = 0 \tag{30}$$

Then the error of the nth iterate is zero, so the algorithm converges in at most n iterations.

Question 4 CG Inverse Problem