# Energy-based Associative Memories

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# Outline of Presentation

# Associative Memories

## Definition (Associative memory)

An *associative memory* is a 3-tuple $\langle f, A, C \rangle$ obeying the following properties[a]:

(i) The *address* matrix $A$ is an $N \times D$ matrix of patterns we wish to learn as *cues* or *queries*;

(ii) The *content* matrix $C$ is an $N \times M$ matrix of patterns we wish to learn as *responses*, or associate with each $a_i$, $i = 1, 2, \ldots, N$; and,

(iii) The *recall* function $f_{A,C} : \mathbb{R}^D \to \mathbb{R}^M$ maps $D$-dimensional query or cue patterns $x$ to memorized $M$-dimensional patterns $y$.

[a] Following the definition from (Kanerva, 1993)

(a) Associative memories learn to associate patterns $(x, y)$ in the learning phase

(b) Typically the task is to recall $y$ based on noisy, masked, or degraded forms of $x$

(c) An associative memory is said to be *auto-associative* if $x = y$ (therefore $A = C$)

(d) It is said to be *hetero-associative* otherwise

# Correlation Matrix based AMs

(a) There are many different kinds of associative memories

(b) The kind of memory that we will be focusing on here are *correlation matrix* based memories (Amari, 1972; J. J. Hopfield, 1984; J. Hopfield, 1982; Kohonen, 1988; Nakano, 1972)

## Definition (Correlation Matrix AM)

A *correlation matrix* AM is a tuple $\langle f, A, C \rangle$ with:

(i) Standard $N \times D$ address matrix,

(ii) Standard $N \times M$ content matrix,

(iii) Recall function:

$$f_{A,C}(x) = g(Wx),$$
$$\text{with } W = C^\top A.$$

The function $g$ in (iii) is either:

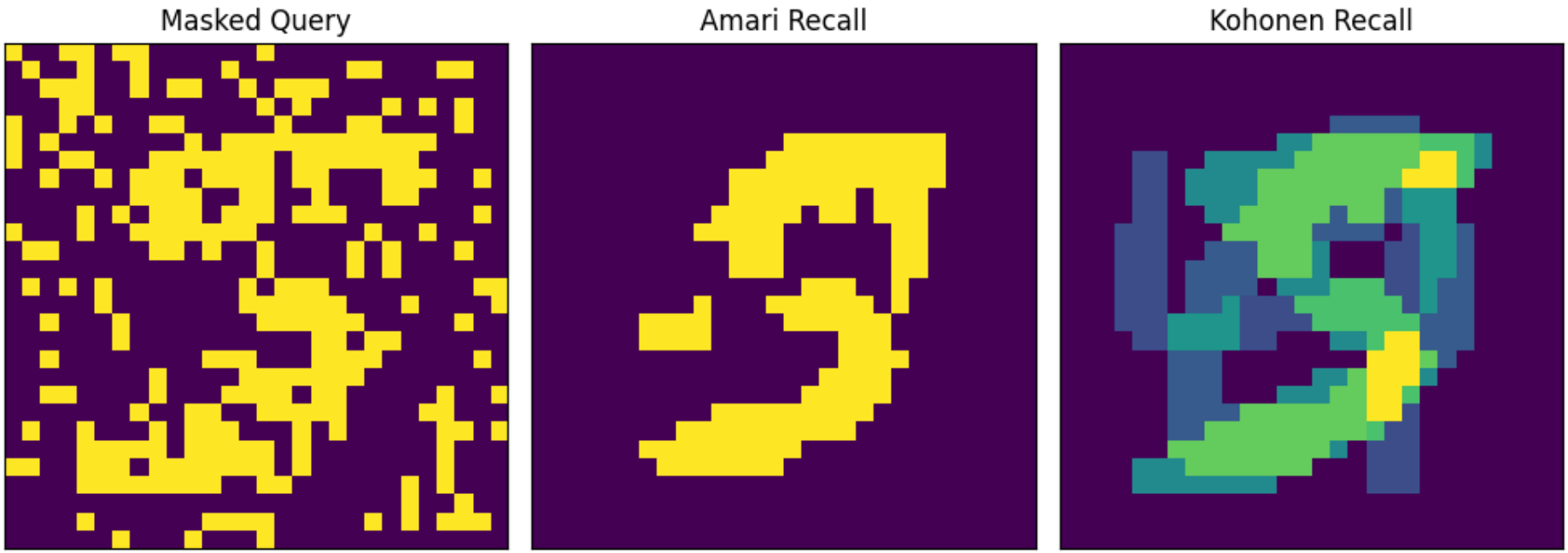
(a) In Kohonen (1988) the identity function $g(x) = x$

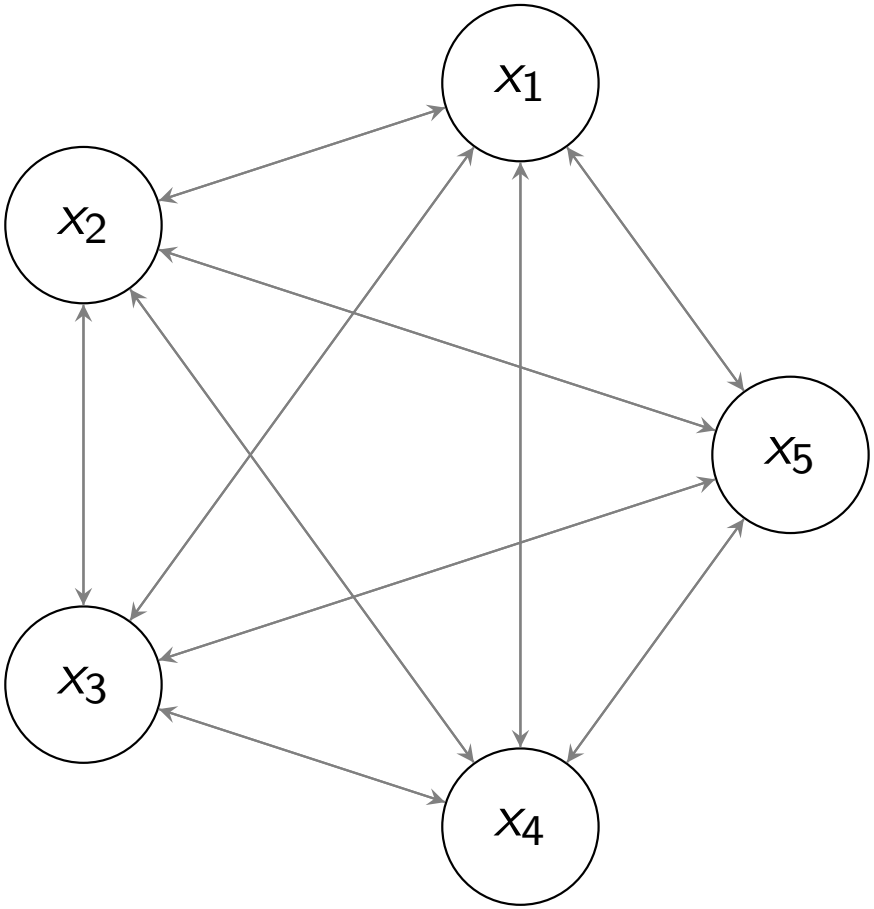
(b) In Amari (1972), J. Hopfield (1982), and Nakano (1972), $g(x) = \operatorname{sgn}\{x\}$, with

$$\operatorname{sgn}\{n\} = \begin{cases} -1, & \text{if } n < 0, \\ 0, & \text{if } n = 0, \\ 1, & \text{if } n > 0. \end{cases}$$

called the *signum* function.

# Kohonen Network versus Amari Network






Where each connection is $w_{ij} = w_{ji}$.

# Hebbian Update Rules

## Definition (Hebbian update-rule; Hebbian learning)

Let $x_n$, $n = 1, 2, \ldots, L$ be a *layer* of computational units, up to $L$ layers. Let the weights $W_{ij}$ be a matrix representing the weights of full connections between layers $x_i$ and $x_j$. Then, we say that a weight update rule is *Hebbian* iff

$$\Delta W_{ij} = f(x_i, x_j).$$

## Example (Activity Product Rule)

The simplest form of Hebbian learning is the following update rule[a]:

$$\Delta W_{ij} = \eta x_j x_i^\top,$$

where $\eta$ is the *learning rate*.

[a] (Haykin, 2009)

# Correlation Matrix Memories are Hebbian

We can formulate Correlation Matrix AM learning of patterns as Hebbian. Let $\xi_\mu$, $\mu = 1, 2, \ldots, N$ be the $N$ patterns of $D$ dimension that we want the system to memorize. For each time step, suppose we present a new pattern $\xi_{t=\mu}$:

$$W^{(0)} \leftarrow [0]_{D\times D},$$
$$W^{(t+1)} \leftarrow W^{(t)} + \eta \xi_i \xi_i^\top.$$

Formulated as a difference,

$$\Delta W = \eta \xi_i \xi_i^\top.$$

At time $t = \mu$, this is equivalent to the matrix:

$$W = \sum_{\mu=1}^{N} \xi_\mu \xi_\mu^\top = \Xi^\top \Xi,$$

with $N \times D$ pattern matrix $\Xi = [\xi_1, \xi_2, \ldots, \xi_N]$.

# Energy and Correlation Matrix (Hopfield!)

Correlation Matrix AMs can be understood as performing *energy minimization*. In the same way that we can specify the *weight dynamics* with the update rule, we can also specify the *neural dynamics* of the Hopfield network; i.e., asynchronous update rules.

## Definition (Hopfield energy)

Let $\Xi = [\xi_1, \xi_2, \ldots, \xi_N]$ be the $N \times D$ pattern matrix. Let $W = \Xi^T \Xi$. Let $\sigma$ be a $D$-dimensional query vector. The *energy* (J. J. Hopfield, 1984; Krotov & Hopfield, 2016) of the AM given the query vector $\sigma$ is the *dot product correlations* between the weights and the query vector:

$$E_{\text{Hopfield}} = -\frac{1}{2}\sum_{i,j}^{N} W_{ij}\sigma_i\sigma_j = -\frac{1}{2}\sum_{\mu=1}^{N}\left(\sum_{i=1}^{D}\Xi_{\mu i}\sigma_i\right)^2 .$$

## Energy intuition

The **energy** of a Hopfield network is the *weighted* dot product between each computational unit and every other. Low energy means there is *high* correlation between the value of each comptuational unit and the others. High energy means that there is *low* correlation between each computational unit and every other.

# Asynchronous update rule

In the energy-based framework, the AM is updated *asynchronously*. At each time step $t$, we pick a random neuron and flip its value. If this value *lowers* the energy, then we keep the flip. Otherwise, we pick another random value.

## Definition (Asynchronous Hopfield update rule)

$$\sigma_i^{(t+1)} = \text{argmin}_{b \in \{-1,1\}} [E_{\text{Hopfield}}(\sigma_i')]$$

with $\sigma' = \sigma^{(t)}$, but with the $i$'th element set to $b$.

## Fact (One-step rule relation to asynchronous rule)

*The asynchronous rule defined above relates to the single-pass update rule as you can derive the single-pass rule from the difference of the energies of the query buffer with the i'th unit flipped and unflipped. (Krotov et al., 2025; Mimura et al., 2025).*

# Energy Dynamics

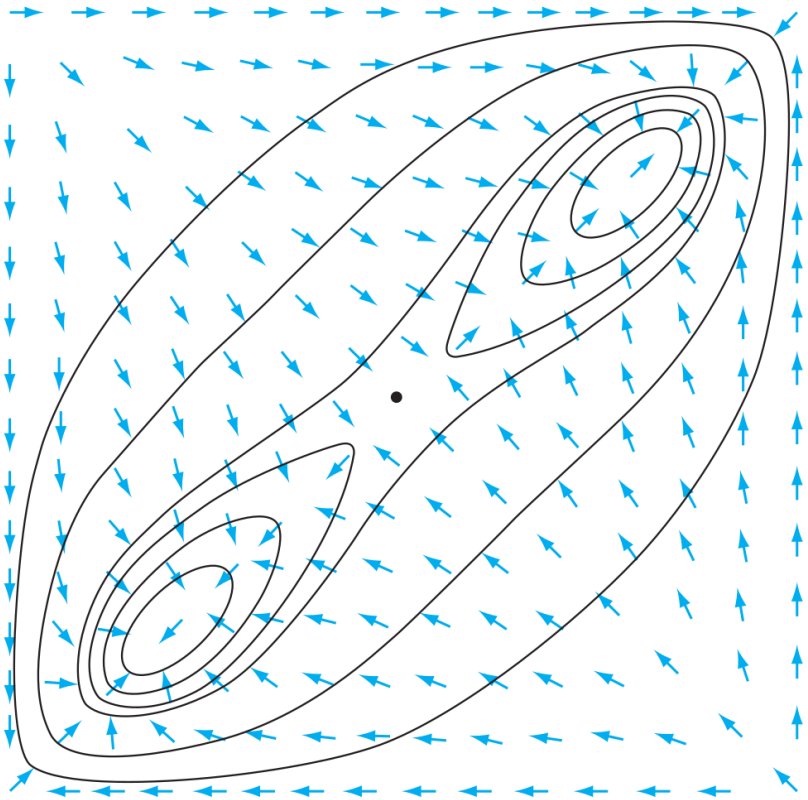
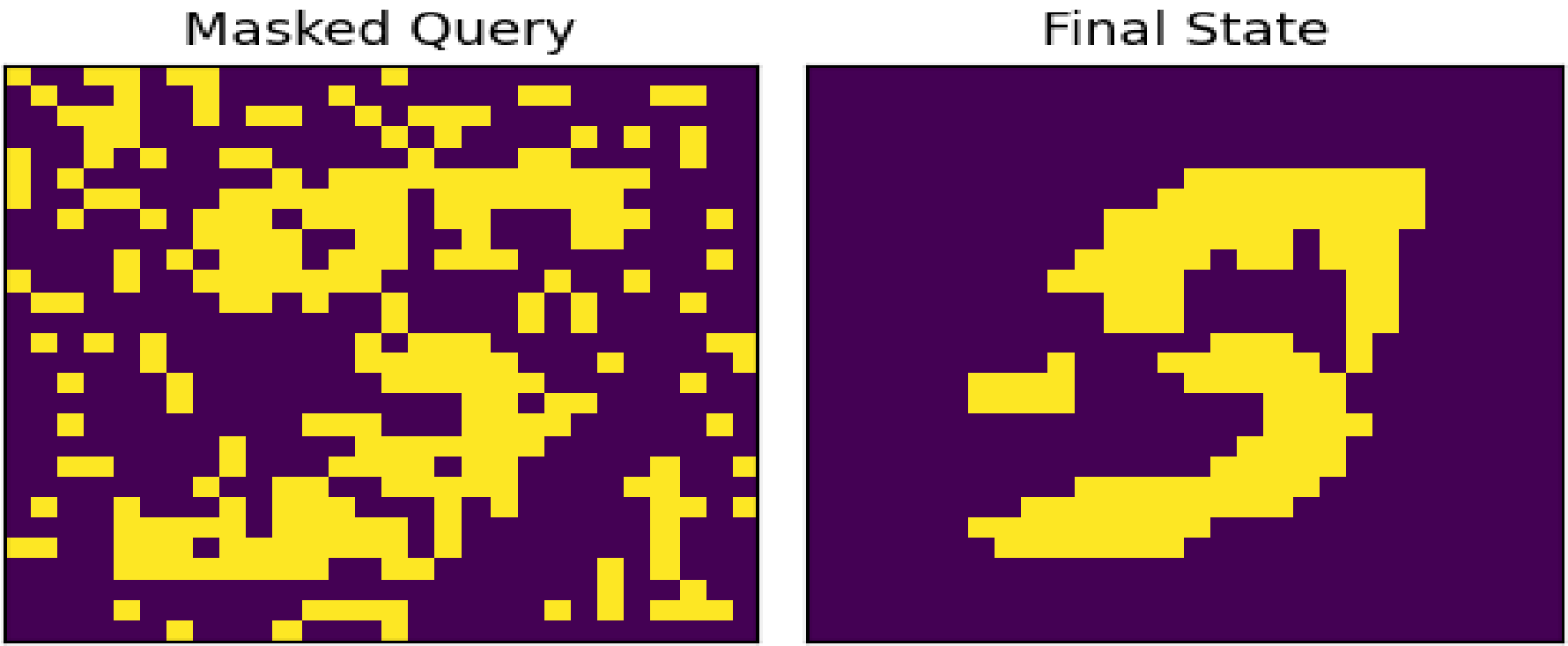
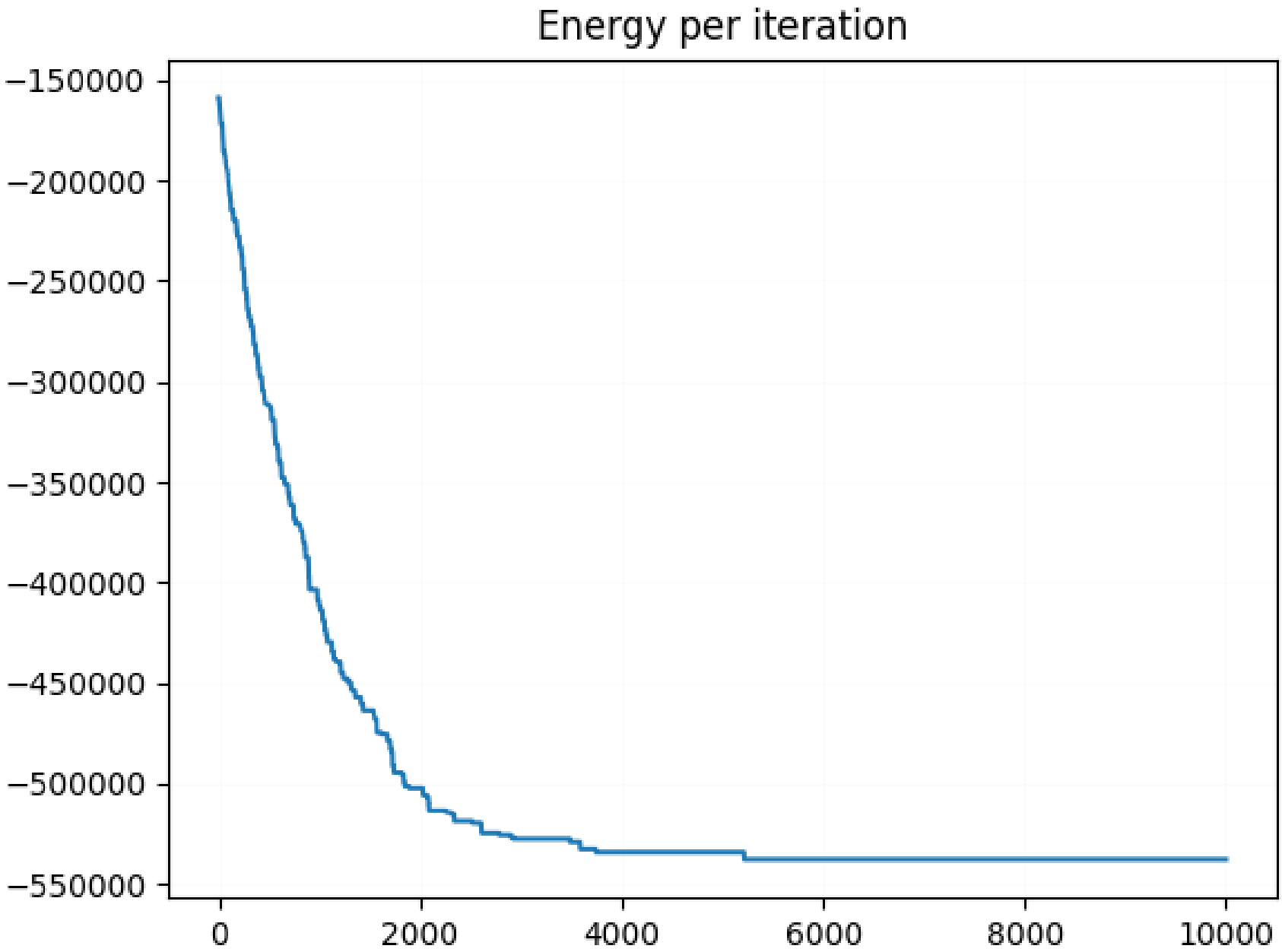


Figure: "Neurodynamics" of the Hopfield energy function. From Haykin (2009), originally from J. J. Hopfield (1984).

# Energy-based Recall (Demonstration)




Energy per iteration
−150000
−200000
−250000
−300000
−350000
−400000
−450000
−500000
−550000
0
2000
4000
6000
8000
10000

# Increasing storage capacity with Dense Associative Memories

Convergent investigation into Dense Associative Memory-like models began soon after J. Hopfield (1982) with Hintzman (1984). Tensor-based energy functions and weights with Chen et al. (1986) and Psaltis and Cheol Hoon Park (1986) (see also, Kelly et al. (2017)).

Hopfield networks famously have a capacity for $\sim 0.14D$ memories, where $D$ is the dimension of the patterns to store; note that this is can be even further diminished by correlated patterns, introducing "cross-talk" (note even in Kohonen (1988))

# Generalized Energy Function

## Definition (Generalized Energy Function)

Consider the following abstraction of $E_{\text{Hopfield}}$:

$$E_{F_n} = -\sum_{\mu=1}^{N} F\left(\sum_{i=1}^{D} \xi_{\mu i}\sigma_i\right)$$

We call $F_n$ the *Lagrangian* function (Krotov, 2021). $F_n$ is of the form $\frac{1}{n}(x)^n$.

We can also get an update rule from this:

## Definition (Generalized Update Function)

For energy function with Lagrangian $F_n$, $E_{F_n}$, the single-pass update function is (Demircigil et al., 2017; Krotov, 2021; Krotov & Hopfield, 2016):

$$\sigma^{(t+1)} = \text{sgn}\left\{\sum_{\mu=1}^{N} \xi_{\mu i} F_n'\left(\sum_{j\neq i}^{D} \xi_{\mu j}\sigma_j\right)\right\},$$

equivalently,

$$\sigma^{(t+1)} = \text{sgn}\left\{\Xi^T F'\left(\Xi\sigma\right)\right\}$$

## Example (Classical Hopfield)

The above definition gets us the classical Hopfield energy function immediately, with energy,

$$E_{\text{Hopfield}} = E_{F_2} = -\frac{1}{2}\sum_{\mu=1}^{N}\left(\sum_{i=1}^{D}\xi_{\mu i}\sigma_i\right)^2;$$

and update:

$$\sigma^{(t+1)} = \text{sgn}\left\{\Xi^T\Xi\sigma^{(t)}\right\}.$$

## Example (Minerva2)

We can also get Minerva2 (and more generally, every Minerva model) (Hintzman, 1984; Kelly et al., 2017): [a]

$$E_{\text{Minerva2}} = E_{F_4} = -\frac{1}{4}\sum_{\mu=1}^{N}\left(\sum_{i=1}^{D}\xi_{\mu i}\sigma_i\right)^4;$$

and update rule:

$$\sigma^{(t+1)} = \text{sgn}\left\{\Xi^T\left(\Xi\sigma^{(t)}\right)^3\right\}.$$

[a] To see further corresp. with Memory Tesseract, note that $F_4$ gives 4-way interaction term in the weights.

# Capacity

There are many ways to characterize the capacity of an AM; note that since it is totally defined, we won't get "out-of-bounds" memory access or null-pointers

The trade off is we get a "crap-in-crap-out" phenomenon: if the memory is filled with too many items, we enter into *spurious* energy states

Dense Associative Memory theory gives us the following relation:

**Theorem (Capacity scaling laws)**

*We say that $N^{max}$ is the number of patterns the AM can store ( literature uses $K$, see Krotov and Hopfield (2016) ). With Lagrangian function $F_n$, $N^{max}$ scales exponentially with $n$:*

$$N^{max} \propto N^{n-1}.$$

Proof is ommitted. See Demircigil et al. (2017) and Krotov and Hopfield (2016).

## Lemma (Capacity of Minerva2)

*We immediately get from the above that the capacity of Minerva2 is:*

$$N^{max} \propto N^3.$$

# Modern Hopfield Networks

So far we have been dealing with bipolar values $x \in \{-1, 1\}$: but this perspective can be expanded to real values. Namely, Modern Hopfield Networks (Ramsauer et al., 2021) which are equivalent to Multi-head Attention in Transformers (Vaswani et al., 2023).

## Definition (Modern Hopfield)

Energy of multi-head attention relies on the Lagrangian function *logsumexp*, defined as:

$$\mathsf{lse}(\beta, x) = \frac{1}{\beta} \log \left( \sum_{i=1}^{D} \exp(\beta x_i) \right).$$

This gives us an update rule

$$\sigma^{(t+1)} = \Xi^T \mathsf{softmax} \left( \beta \Xi \sigma^{(t)} \right).$$

# If you're interested in this

(a) This was just the summary of general results

(b) There is a new framework for architecture agnostic energy-descent networks, called HAMUX (Krotov, 2021)

(c) HAMUX gets you compositional energy functions, and an energy-descent rule which is in terms of all of the different parts of the network

(d) I'm not too familiar with it, so I didn't talk about it here.

# DAM Not Hebbian

(a) Closing remarks: DAM is not Hebbian. This is because the weight update rule requires multiple interaction terms, again see Kelly et al. (2017).

(b) Training usually proceeds using conventional methods: backpropagation or even predictive coding (Millidge et al., 2022; Salvatori et al., 2021)

(c) can we make a Hebbian network which has similar scaling laws and capacity as DAMs?

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