The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well

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January 23, 2007

Abstract

State-of-the-art stochastic volatility models generate a "volatility smirk" that explains why out-of-the-money index puts have high prices relative to the Black-Scholes benchmark. These models also adequately explain how the volatility smirk moves up and down in response to changes in risk. However, the data indicate that the slope and the level of the smirk fluctuate largely independently. While single-factor stochastic volatility models can capture the slope of the smile, they cannot explain such largely independent fluctuations in its level and slope over time. We propose to model these movements using a two-factor stochastic volatility model. Because the factors have distinct correlations with market returns, and because the weights of the factors vary over time, the model generates stochastic correlation between volatility and stock returns. Besides providing more flexible modeling of the time variation in the smirk, the model also provides more flexible modeling of the volatility term structure. Our empirical results indicate that the model improves on the benchmark Heston model by 15% in-sample and 11% out-of-sample. The better fit results from improvements in the modeling of the term structure dimension as well as the moneyness dimension. We also find that a two-factor model with one persistent factor provides the best out-of-sample fit, and improves on the Heston model by 19%. This finding illustrates that parameter parsimony is critical to ensuring satisfactory out-of-sample performance.

JEL Classification: G12

Keywords: Stochastic correlation; stochastic volatility; equity index options; multifactor model; persistence; affine; out-of-sample.

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1 Introduction

An extensive empirical literature has documented the empirical biases of the Black-Scholes (1973) option valuation model for the purpose of the valuation of equity index options. Most prominently amongst these biases, observed market prices for out-of-the-money put prices (and in-the-money call prices) are higher than Black-Scholes prices. This stylized fact is known as the volatility "smirk" or the volatility "smile". Implied volatilities for at-the-money options also contain a term structure effect that cannot be explained by the Black-Scholes model.

Perhaps the most popular approach to modeling the smirk is the use of stochastic volatility models that allow for negative correlation between the level of the stock return and its volatility.[1] This negative correlation captures the stylized fact that decreases in the stock price are associated with larger increases in volatility than similar stock price increases (Black (1976), Christie (1982)). This stylized fact is known as the leverage effect. The leverage effect is important for equity index option valuation, because it increases the probability of a large loss and consequently the value of out-of-the-money put options. The leverage effect induces negative skewness in stock returns, which in turn yields a volatility smirk.

The stochastic volatility models of Hull and White (1988), Melino and Turnbull (1990), and Heston (1993) allow for nonzero correlation between the level of the stock return and its volatility. Several papers have documented that these stochastic volatility models are helpful in modeling the smirk, and that the modeling of the leverage effect is critical in this regard (e.g., see Bakshi, Cao, and Chen (1997), Bates (2000), Chernov and Ghysels (2000), Jones (2003), Nandi (1998) and Pan (2002)). Stochastic volatility models can also address term structure effects by modeling mean reversion in the variance dynamic. Consequently many papers use a single-factor stochastic volatility model as the starting point for more complex models.[2]

Single-factor stochastic volatility models can generate smiles and smirks. However, these models are overly restrictive in their modeling of the relationship between the volatility level and the slope of the smirk. The data suggest that the shape of the smile is largely independent of the volatility level. There are low volatility days with a steep volatility slope as well as a flat volatility slope, and also high volatility days with steep and flat volatility slopes. A single-factor stochastic volatility model can generate steep smirks or flat smirks at a given volatility level, but cannot generate both for a given parameterization. In a purely cross-sectional analysis, this is not a problem, because we can estimate different parameter values for the one-factor model to calibrate the time-varying nature of the cross-section. However, when estimating model parameters using multiple cross-

[1] For early stochastic volatility models see for example Hull and White (1987), Scott (1987) and Wiggins (1987).

[2] See for instance the extensive literature on jump models. Bakshi, Cao and Chen (1997), Bates (1996, 2000), Broadie, Chernov and Johannes (2004), Eraker (2004) and Pan (2002) compare the empirical fit of the Heston (1993) model with more complex models that contain different types of jumps in returns and volatility.

sections of option contracts, a one-factor model has a structural problem. If its parameters are geared towards explaining a slope of the smirk that is on average high over the sample, it will result in large model error on those days that the slope of the smirk is relatively flat, and vice versa. Another way to understand this restrictiveness is that in single factor stochastic volatility models, the correlation between volatility and stock returns is constant over time, and this limits the model's ability to capture the time-varying nature of the smirk. To date, we do not have a good understanding of how incorporating stochastic correlation can improve the performance of benchmark stochastic volatility models such as Heston (1993).

This paper uses a straightforward way to incorporate a stochastic correlation by using multiple stochastic volatility factors. We demonstrate that two-factor models have much more flexibility in controlling the level and slope of the smile. An additional advantage is that two-factor models also provide more flexibility to model the volatility term structure. In our empirical estimates, one of the factors has high mean reversion and determines the correlation between short-term returns and volatility. The other factor has lower mean reversion and determines the correlation between long-term returns and volatility. We implement and test a two-factor stochastic volatility model that builds on the valuation results in Heston (1993) to maintain a closed-form solution for option prices and remain computationally tractable. We test this model in- and out-of-sample, and we pay particular attention to model parsimony in order to improve out-of-sample performance.

We implement and test the multifactor volatility model using 1990-1995 data on European call options on the S&P500. We split up our data set into six samples that each contain one year of options data. We therefore perform six in-sample exercises and then evaluate all six sets of parameter estimates using the five other datasets, leading to a total of thirty out-of-sample exercises. We repeat this exercise using a more parsimonious variation of the two-factor model which contains one fully persistent factor. We find that in-sample the dollar RMSE of the multifactor model is 15% lower than that of the one-factor Heston (1993) model, while the dollar RMSE of the two-factor model with persistent factor is 14% lower. Out-of-sample, the two-factor model improves on the one-factor model by 11% on average, while the RMSE of the persistent two-factor model is 19% lower on average. These results clearly emphasize the importance of model parsimony, as well as the need to compare models out-of-sample.

To provide more insight into the differences in pricing performance, we extensively investigate along which dimensions the estimated two-factor model differs from the one-factor model. We find that the two-factor model substantially improves on the one-factor model in the term structure dimension as well as in the moneyness dimension. We also demonstrate that the modeling of conditional skewness and kurtosis in the standard one-factor model is extremely restrictive, and that estimated conditional higher moments are highly correlated with the estimated conditional variance. In contrast, the two-factor model allows for more flexibility in modeling conditional skewness and kurtosis for given levels of conditional variance, which is consistent with the finding that the slope of the smirk evolves largely independently from the level of the volatility.

In the option literature, Taylor and Xu (1994) use a two-factor model to uncover short-run and long-run variance expectations in foreign exchange markets. Bates (2000) investigated the empirical fit of two-factor models, and compared it with the performance of stochastic volatility models augmented with Poisson-normal jumps, but did not emphasize the model's mechanics for capturing moneyness effects. [3]

Our focus is therefore on explaining why a two-factor model works better than a one-factor model, by emphasizing its features for modeling the moneyness and term structure dimensions. While we do compare the statistical fit of the models, we put more emphasis on the underlying stylized facts that result in an improved statistical fit. In line with this approach, we conduct an extensive out-of-sample exercise to corroborate that the improved in-sample fit is due to the improved modeling of these stylized facts, rather than to a simple increase in the number of model parameters.[4]

It is perhaps somewhat surprising that multifactor models have not yet become more popular in the option valuation literature. In the yield curve literature, which uses models with a mathematical structure similar to those in the option valuation literature, the use of multifactor models of the short rate is widespread. In fact, it is widely accepted in the literature that one factor is not sufficient to capture the time variation and cross-sectional variation in the term structure. The consensus seems to be that a minimum of three factors are needed.[5] Option valuation and term structure modeling have a lot in common: in both cases one faces the demanding task of providing a good empirical fit to the time-series as well as the cross-sectional dimension using tractable, parsimonious models. We therefore believe that the use of multifactor models is as critical for the equity option valuation literature as it is for the term structure literature, and that in the future multifactor models may become as widespread in the option valuation literature as they now are in the term structure literature.[6]

[3] Analyzing different but related models, Huang and Wu (2004) consider a two-factor volatility model driven by Levy processes, and Santa-Clara and Yan (2006) consider a model with separate diffusions for the volatility and the Poisson jump intensity.

[4] See also Alizadeh, Brandt and Diebold (2002), Chernov, Gallant, Ghysels and Tauchen (2003), and Christoffersen, Jacobs and Wang (2005) for related work. Eraker (2004) and Duffie, Pan and Singleton (2000) suggest the potential usefulness of our approach. Carr and Wu (2005) model stochastic skewness in currency options using a different approach.

[5] See Litterman and Scheinkman (1991) for a characterization of the number of factors needed to model the term structure. See Pearson and Sun (1994) for an early example of multifactor term structure models, and Duffee (1999) and Dai and Singleton (2000, 2002, 2003) for further applications. Duffie and Kan (1996) and Dai and Singleton (2000) provide widely used classifications of multifactor term structure models.

[6] If anything, modeling equity options is probably more challenging than modeling the term structure because modeling the cross-sectional dimension for equity options requires the modeling of moneyness effects as well as

In the equity option valuation literature, the deficiencies of the one-factor stochastic volatility model have traditionally been addressed by adding a jump process to the return dynamic.[7] This paper does not question the usefulness of this approach. Instead, we surmise that adding additional factors to the volatility process is a dynamic way of addressing model deficiencies.[8] Our paper does not investigate whether multifactor models or jump processes are more appropriate for modeling option data, or if indeed both features are useful. That particular empirical question has to be decided by an out-of-sample comparison between jump models and multifactor stochastic volatility models, and such a comparison is beyond the scope of this paper.

The paper proceeds as follows: In Section 2 we discuss the empirical regularities of the option data and present a principal component analysis to motivate the use of multiple volatility factors. In Section 3 we develop a two-factor stochastic volatility process which has the potential to match the empirical regularities found in Section 2. We first derive a closed form option valuation formula for the model and then illustrate some critical differences between one- and two-factor models. In Section 4 we present the estimation strategy and assess the empirical fit of the model using S&P500 index options in-and out-of-sample. Section 5 further explores the empirical results, and Section 6 concludes.

2 Exploring the Index Option Data

2.1 Data Description

For our empirical investigation, we use data on European S&P500 call options. We record option quotes on the Chicago Board Options Exchange each Wednesday within 30 minutes of closing. From the bid and ask quotes, mid-quotes are calculated as simple averages. Each option quote is matched with the underlying index level which is adjusted for dividends, by subtracting the present value of the future realized flow of dividends between the quote date and the maturity date of the option.[9] T-bill rates are used for this purpose. The risk-free rate for each option maturity is calculated via interpolation of available T-bill rates. Options with less than seven days to maturity

maturity effects. On the other hand, the multifactor stochastic volatility models considered in this paper differ from multifactor term structure models in the sense that the one factor stochastic volatility model can itself be considered as a two-factor model, with the first factor being provided by the stock return.

[7] See for example Andersen, Benzoni and Lund (2002), Bakshi, Cao and Chen (1997), Bates (1996, 2000), Broadie, Chernov and Johannes (2004) Carr and Wu (2004),Chernov, Gallant, Ghysels and Tauchen (2003), Eraker, Johannes and Polson (2003), Eraker (2004), Pan (2002) and Huang and Wu (2004).

[8] Interestingly, recent research has investigated the importance of jump processes for modeling the term structure of interest rates. See for example Johannes (2004). This paper complements this line of research by using an approach that is more typical of the existing empirical research on term structure models and applying it to the valuation of equity options.

[9] This procedure follows Harvey and Whaley (1992a and b).

are omitted from the sample, as are options with extreme moneyness, and options which violate various no-arbitrage conditions. These filtering rules follow Bakshi, Cao and Chen (1997).

Table 1 summarizes our data set of 21,752 contracts, which covers every Wednesday during the January 1, 1990 through December 31, 1995 period.[10] Panels A through C in Table are split up into four (calendar) days-to-maturity (DTM) categories and six moneyness (S/X) categories. Panel A reports the number of contracts in each category, Panel B reports the average call price in each category, and Panel C gives the average Black-Scholes implied volatility in each category. The systematic and well-known volatility "smirk" across moneyness is evident from each column in Panel C. While the smirk is most dramatic for the short-maturity options, it is present in each maturity category.

In our estimation exercise, we split the 1990-1995 option data up in six sample periods, on a year-by-year basis. Panel D in Table 1 reports the number of contracts, the average call price and the average Black-Scholes implied volatility for these six year-by-year samples. Note that the number of available contracts per year increases steadily over time, due to increasing volume over time in the options market. The average call price per year is increasing over time, due to increases in the S&P500 index. The average implied Black-Scholes volatility starts out relatively high in 1990, falls to a low in 1993 and then increases slightly in 1994 and 1995.

Figure 1 presents our option data from a different perspective. For each of the 313 Wednesdays in the 1990-1995 sample, the top panel presents the average implied Black-Scholes volatility. The average is taken across maturities and strike prices. For comparison, the bottom panel presents the one-month, at-the-money VIX volatility index. It is clear that our sample adequately captures the time-variation in the overall market, and that S&P500 index options experience a sharp increase in implied volatility in the second half of 1990 and the beginning of 1991, around the time of the First Gulf War.

2.2 Principal Component Analysis

Our objective in this section is to investigate if the data support multiple variance factors, without relying on any particular option valuation model. This is not straightforward, because by definition the stock return variance is a latent factor. We circumvent the unobservability of the return variance by using an observable proxy. In particular, we investigate the number of factors in the implied Black-Scholes variance.[11] While this approach clearly has some limitations, it is meant to provide

[10] More recent CBOE data is available via www.optionmetrics.com but the syncronization of options quotes and the underlying index value is unfortunately not transparent. Futures options are available from the Chicago Mercantile Exchange, but these options are American style and so must be adjusted for early exercise premium. We therefore rely on an older but arguably more reliable data set.

[11] While Black-Scholes implied standard deviation is more extensively used as a measure of stock return variability than Black-Scholes implied variance, we report a principal component analysis of variance because the latent factors

a first indication of the need for multiple factors, and is not meant to substitute for a more detailed statistical analysis.

Table 2 reports the results of a principal component analysis of Black-Scholes implied variances. To facilitate the interpretation of the principal component analysis, it is not performed on the raw data but on a standardized variance surface. This variance surface is constructed as follows. In a first stage, we fit a quadratic polynomial in maturity and moneyness for each day in the dataset. In the second stage, we use these estimates to generate a standardized variance surface using five different levels of moneyness ($0.9, 0.95, 1, 1.05$ and 1.1) and five different maturities (30 days, 60 days, 90 days, 180 days and 270 days).

Table 2 reports the loadings on the first four principal components and the fraction of the variance explained by each of these four components. The most important conclusion is that the first component explains 88.486% of the variation in the data, and that the first two components together explain over 95% of the variation in the data. The results therefore seem to suggest that a two-factor model may be a good representation of the data.[12]

The first principal component has relatively similar weights for all 25 data series. The top panel of Figure 2 represents this component over time, and it can be clearly seen that this component is closely related to the average implied volatility represented in the top panel of Figure 1. Indeed, the correlation between the first principal component and the level of the implied variance is 79%. Recall that the level of the implied variance is simply the average of the implied variances across moneyness and maturity on a given day. The loadings of the second principal component on the twenty-five data series are not as uniform as is the case for the first principal component. Table 2 shows that it has large positive loadings on in-the-money calls with short maturities and negative loadings on most other options. It is therefore to be expected that the second principal component combines maturity and moneyness effects.

In summary, a principal component analysis of implied Black-Scholes variances reveals that a stochastic volatility model with two factors is likely to capture a lot of the variation in the data. Empirically, the question is whether such a richer model results in a better fit than a one-factor model? It is especially important to consider out-of-sample exercises. After all, the two-factor model can simply be seen as a model that nests the one-factor model, and therefore it will provide a better in-sample fit. Whether this more richly parameterized model provides reliable enough improvements to increase the out-of-sample fit is a much more stringent test of the model. Before we turn to a detailed empirical analysis, we now present the two-factor model and provide some intuition for the model.

in the subsequent model are variances and not standard deviations. An analysis of Black-Scholes implied volatilities yielded very similar results.

[12]Interestingly, Skiadopoulos, Hodges and Clewlow (1999) reaches a similar conclusion when analyzing changes rather than levels in implied volatility.

3 The Model

3.1 Return Dynamics

Suppose the volatility of the risk-neutral, ex-dividend stock price process is determined by two factors[13]

$$
\begin{aligned}
dS &= rSdt + \sqrt{V_1}Sdz_1 + \sqrt{V_2}Sdz_2 && (1)\\
dV_1 &= (a_1 - b_1V_1)dt + \sigma_1\sqrt{V_1}dz_3 && (2)\\
dV_2 &= (a_2 - b_2V_2)dt + \sigma_2\sqrt{V_2}dz_4 && (3)
\end{aligned}
$$

We assume z_1 and z_2 are uncorrelated. Note that the variance of the stock return is the sum of the two variance factors

$$Var_t[dS/S] = (V_1 + V_2)dt = Vdt \tag{4}$$

In addition, we assume the following stochastic structure: z_1 has correlation ρ_1 with z_3, and z_2 has correlation ρ_2 with z_4, but z_1 is uncorrelated with z_4, z_2 is uncorrelated with z_3, and z_3 is uncorrelated with z_4. In other words, the variance is the sum of two uncorrelated factors that may be individually correlated with stock returns. For each factor, the covariance with the stock return is given by

$$Cov_t[dS/S, dV_j] = \sigma_j\rho_jV_jdt \tag{5}$$

The covariance of stock returns with overall variance is given by

$$Cov_t[dS/S, dV] = (\sigma_1\rho_1V_1 + \sigma_2\rho_2V_2)dt \tag{6}$$

The correlation between the stock return and variance is determined by ρ_1 and ρ_2, and depends on the current levels of the factors. Note that this implies that the leverage correlation in the two-factor model is given by.

$$Corr_t[dS/S, dV] = \frac{\sigma_1\rho_1V_1 + \sigma_2\rho_2V_2}{\sqrt{\sigma_1^2V_1 + \sigma_2^2V_2}\sqrt{V_1 + V_2}} \tag{7}$$

While this model is conceptually fairly straightforward, it holds promise to resolve existing biases in option valuation because of its flexibility. For the purpose of modeling moneyness effects, note that the correlation of stock returns with overall variance depends on the current levels of the factors. Hence, this model displays not only stochastic variance, but also *stochastic correlation* between stock return and variance, and this feature potentially enables the model to capture fluctuations in option skewness. For the purpose of modeling term structure effects, one of the factors can

[13] Due to our choice of estimation method, we only require the risk-neutral process. Risk-neutralization in this model can be motivated in the usual way by specifying a representative agent with logarithmic utility. See for instance Lewis (2000).

have relatively "fast" mean-reversion (high b) to determine short-run variance, while the other factor can have relatively "slow" mean-reversion (low b) to determine long-run variance. The different implications of these factors can also be influenced by interactions between mean reversion parameters and the parameters that determine the third and fourth moment of returns (ρ and σ).

3.2 Special Cases

While the two-factor model introduced above is already very parsimonious, in the empirical section below we analyze a special case of the two-factor model which turns out to work remarkably well out-of-sample. For this special case, we restrict the first volatility factor to be fully persistent, that is, we set $a_1 = b_1 = 0$, which results in the following extremely simple "random walk" variance factor dynamic

$$dV_1 = \sigma_1 \sqrt{V_1} dz_3. \tag{8}$$

As before, z_1 has correlation ρ_1 with z_3, but is uncorrelated with the other shocks. Henceforth, we will refer to this model as the "persistent factor" model.

The one-factor Heston (1993) model is one of the most popular models in the option valuation literature. This model is also a special case given by

$$dS = rSdt + \sqrt{V} S dz_1 \tag{9}$$

$$dV = (a - bV)dt + \sigma \sqrt{V} dz_2 \tag{10}$$

where the correlation between z_1 and z_2 is ρ.

3.3 Option Valuation

For option valuation, we need to determine the characteristic function of the log-spot price, $x = \ln(S)$. Generalizing the results in Heston (1993), the characteristic function satisfies

$$E_t[\exp(i\phi x(t+\tau))] = S(t)^{i\phi} f(V_1, V_2, \tau, \phi), \tag{11}$$

where

$$f(V_1, V_2, \tau, \phi) = \exp(A(\tau, \phi) + B_1(\tau, \phi)V_1 + B_2(\tau, \phi)V_2),$$

$$\begin{aligned} A(\tau, \phi) &= r\phi i\tau + \frac{a_1}{\sigma_1^2}\left[(b_1 - \rho_1\sigma_1\phi i + d_1)\tau - 2\ln\left[\frac{1 - g_1\exp(d_1\tau)}{1 - g_1}\right]\right] \\ &\quad + \frac{a_2}{\sigma_2^2}\left[(b_2 - \rho_2\sigma_2\phi i + d_2)\tau - 2\ln\left[\frac{1 - g_2\exp(d_2\tau)}{1 - g_2}\right]\right], \end{aligned}$$

$$B_j(\tau, \phi) = \frac{b_j - \rho_j\sigma_j\phi i + d_j}{\sigma_j^2}\left[\frac{1 - \exp(d_j\tau)}{1 - g_j\exp(d_j\tau)}\right],$$

$$g_j = \frac{b_j - \rho_j \sigma_j \phi i + d_j}{b_j - \rho_j \sigma_j \phi i - d_j},$$

$$d_j = \sqrt{(\rho_j \sigma_j \phi i - b_j)^2 + \sigma_j^2(\phi i + \phi^2)}.$$

Note that the $B_j(\tau, \phi)$ terms are identical to their one-dimensional counterpart in Heston (1993), and $A(\tau, \phi) = r\phi i\tau$ plus the sum of two terms that are identical to their one-dimensional counterpart.

Using these results, European call options can be valued via Fourier inversion by inserting the probabilities

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi \ln(S(t)/K)} f(V_1, V_2, \tau, \phi + 1)}{i\phi S(t) e^{r\tau}}\right] \tag{12}$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{i\phi \ln(S(t)/K)} f(V_1, V_2, \tau, \phi)}{i\phi}\right] \tag{13}$$

into the option valuation formula

$$C(t) = S(t)P_1 - Ke^{-r\tau}P_2. \tag{14}$$

3.4 Expected Future Spot Variances

The two-factor model has the potential to improve on the fit of the one-factor model by allowing for richer modeling of maturity and moneyness effects. While the improvements in the moneyness dimension are a bit more subtle, the improvements in the maturity dimension are relatively easy to understand. Because the one-factor model has only one parameter to capture mean reversion to the unconditional variance, the patterns for the term structure of conditional variance are rather limited. The two-factor model has two parameters capturing the mean reversion of each of the factors. Dependent on the size of each of the two factors, this can lead to very rich patterns in the term structure of the conditional variance.

The formula for expected future spot variance in the two-factor model is

$$E_t\left[V\left(t+\tau\right)\right] = \frac{a_1}{b_1} + \left(V_1 - \frac{a_1}{b_1}\right)\exp(-b_1\tau) + \frac{a_2}{b_2} + \left(V_2 - \frac{a_2}{b_2}\right)\exp(-b_2\tau)$$

Figure 3 presents a parametric example where we plot the expected future variance over a one-year horizon using a number of different combinations of initial spot variances, V_1 and V_2 in the two-factor model.

For the one-factor model, we use a long run variance a/b of 0.04 and a mean reversion coefficient b of 2. For this particular example, the spot variance is taken to be 0.03. For the one-factor model, the expected future variance converges monotonically to the long-run variance. To make the comparison with the two-factor model as straightforward as possible, we let the mean reversion coefficient for the first variance factor b be 0.02 and for the second variance factor b_2 is 8, but we take the long run variance to be the .02 for each factor so that the total long run variance is the same as in the one-factor model.

We generate different term structures of expected future variances for the two-factor model by simply varying the levels of the two spot variances. The spot variance for the first factor V_1 is 0, .01, .02, and .03 respectively, and the second spot variance V_2 is $0.03 - V_1$ in all cases so as to fix the overall spot variance at .03 as in the one-factor model. It can be seen that the two-factor model can lead to many different patterns for expected future variances, including monotonically increasing expected variances, but also expected variances that first increase and subsequently decrease and vice versa.

3.5 The Level of Volatility and the Slope of the Smirk

In order to illustrate the advantages offered by the two-factor model, we start by characterizing a simple stylized fact of the volatility smirk: the slope of the smirk is largely independent of the level of volatility. This was first noticed by Derman (1999), who documents that the slope of the smile changes little when volatility changes. However, to the best of our knowledge this stylized fact is not extensively documented elsewhere, and we therefore perform a simple empirical exercise to provide additional evidence.

For each of the 313 Wednesdays in the 1990-1995 sample, we regress the implied volatilities of all option contracts on that Wednesday on a constant and the natural logarithm of the contract's moneyness. The estimate of the intercept can be interpreted as the volatility level for that day and the estimated coefficient on the moneyness can be interpreted as the steepness of the slope. Figure 4 presents the results of this analysis. First, note from comparing the time series of the intercepts in the second panel with Figure 1 that the estimated volatility levels are very reliable. A visual inspection of the second and third panel suggests that the slope evolves quite independently from the volatility level. The estimated slope coefficients vary significantly over time, but these changes are not necessarily related to sharp increases or decreases in the volatility level. Moreover, when the index falls in late 1990, the variance increases but there does not seem to be a change in the estimated slope coefficients. The fourth panel computes the correlation between the volatility level (in the second panel) and the slope (in the third panel). We report the rolling correlation based on an expanding window. We first compute the correlation in week 31 using the first thirty observations, and subsequently expand the window. The absolute value of the correlation is never very large, and it is at times positive and at other times negative. The maximum over the six years in the sample is 21% and the minimum is -18%.

3.6 Can Stochastic Volatility Models Capture this Stylized Fact?

Not if it has only one stochastic factor. We perform a few simple simulation exercises to demonstrate that one-factor stochastic volatility models have difficulty to model the simple stylized fact that the slope is largely independent from the volatility level. The model's problems can be illustrated in several ways, but a particularly simple example is the following. Consider the volatility smirks on four different days in the sample in Figure 5. The smirk is subject to strong term structure effects, but the smirks in Figure 5 are for options with either 23 or 29 days to maturity and are therefore comparable. The figure includes two relatively low volatility days: August 16, 1995 and February 15, 1995. October 24, 1990 and September 26, 1990 are relatively high volatility days. Figure 5 therefore illustrates that to fit these cross-sections of option contracts simultaneously, a model has to be able to accommodate both high and low smirk slopes on low volatility days, as well as high and low smirk slopes on high volatility days.

Figure 6 demonstrates that the two-factor volatility model has the potential to provide this flexibility, whereas the one-factor model does not. The dashed lines in Figure 6 represent implied volatilities for the two-factor model. In each case, the parameterization for the model is the following: $b_1 = 2$, $a_1/b_1 = 0.04$, $\sigma_1 = 0.1$, $\rho_1 = -0.1$, $b_2 = 2$, $a_2/b_2 = 0.04$, $\sigma_2 = 0.8$, $\rho_2 = -0.6$. The only difference for the two-factor model is that for the two pictures on the left the factor spot variances are $V_1 = V$ and $V_2 = 0$, whereas in the two pictures on the right they are $V_2 = V$ and $V_1 = 0$. We conduct two experiments for each parameterization: in the two top pictures of each panel, the initial variance V is 0.03, and in the two bottom pictures of each panel the initial variance V is 0.07.

The one-factor model cannot capture these moneyness patterns. In Panel A, the one-factor model is calibrated on the two-factor data represented in the pictures on the left, in which the first component drives the results. In Panel B, the one-factor model is calibrated on the two-factor data represented in the pictures on the right, in which the second component drives the results. When the one-factor model is calibrated to capture a steep smirk, it cannot capture a flat smirk and vice versa. In other words, the one-factor model can generate steep smirks or flat smirks at a given volatility level, but cannot generate both for a given parameterization. In a purely cross-sectional analysis, this is not a problem, because we can estimate different parameter values for the one-factor model to calibrate the time-varying nature of the cross-section. However, in most recent empirical exercises in the academic finance literature, the emphasis is (justifiably) on demonstrating the ability of the model to capture a variety of different cross-sections with fixed model parameters. Parameters are estimated using multiple cross-sections of option contracts, while iterating on the underlying return data to link the cross-sections. A one-factor model has a structural problem in this type of exercise. If its parameters are geared towards explaining a slope of the smirk that is on average high over the sample, it will result in large model error on those days that the slope of the smirk is relatively flat, and vice versa.

4 Empirical Methodology and Results

In this section we implement the two-factor stochastic volatility models using our data set on S&P500 options, and we compare the models' empirical performance with that of the standard one-factor Heston (1993) model. Section 4.1 details the estimation methodology, Sections 4.2-4.3 discuss the in- and out-of-sample results, Section 4.4 compares the empirical results with ad-hoc benchmarks, and Section 4.5 discusses patterns in model errors.

4.1 Estimation Methodology

When implementing the SV models, one is confronted with the challenge of jointly estimating the model's structural parameters, $\Theta = \{a_i, b_i, \sigma_i, \rho_i\}_{i=1,2}$, as well as the spot volatilities $\{V_i(t)\}_{i,t}$. Various approaches are available in the literature. One popular approach treats the spot volatilities as just another parameter which is re-estimated daily. This approach is followed for example by Bakshi, Cao, and Chen (1997). Another approach consists of filtering volatility using the time series of underlying returns, which is done in a Bayesian setting in Jones (2003) and Eraker (2004). Andersen, Benzoni and Lund (2002) and Chernov and Ghysels (2000) use an Efficient Method of Moments approach, Pan (2002) uses the Generalized Method of Moments, and Carr and Wu (2005) use a Kalman filter approach.

In this paper, we follow the approach taken by Bates (2000), who estimates the structural parameters and spot volatilities using option data only in an iterative two-step procedure. This approach is also used by Huang and Wu (2004). Consider a sample of T Wednesdays of options data. In our implementation we choose $T = 52$, which corresponds to a calendar year. Given a set of starting values for Θ and $\{V_i(t)\}$, the iterative procedure proceeds as follows.

Step 1: For a given set of structural parameters, Θ, solve T sum of squared pricing error optimization problems of the form:

$$\left\{\hat{V}_1(t), \hat{V}_2(t)\right\} = \arg\min \sum_{j=1}^{N_t} (C_{j,t} - C_j(\Theta, V_1(t), V_2(t)))^2, \ t = 1, 2, ..., T. \tag{15}$$

where $C_{j,t}$ is the quoted price of contract j on day t and is the corresponding model price. N_t is the number of contracts available on day t.

Step 2: For a given set of spot variances $\{V_1(t), V_2(t)\}$ obtained in Step 1, solve one aggregate sum of squared pricing error optimization problem of the form:

$$\hat{\Theta} = \arg\min \sum_{j,t}^{N} (C_{j,t} - C_j(\Theta, V_1(t), V_2(t)))^2. \tag{16}$$

where $N = \sum_{t=1}^{T} N_t$.

The procedure iterates between Step 1 and Step 2 until no further significant decreases in the overall objectives in Step 2 are obtained. Note that while each function evaluation requires recomputing the model option price for every option involved, the closed-form characteristic functions above guarantee that these calculations can be done in an expedient fashion. Furthermore, the two-step procedure is remarkably well-behaved. Convergence is achieved in just a few iterations within each step and overall convergence also requires only few iterations between Step 1 and Step 2.

4.2 Parameter Estimates and In-Sample Results

We present parameter estimates and in-sample results for three models: the one-factor, two- factor, and persistent-factor models. The iterative two-step estimation routine is applied to each of these three models for each of the six calendar years in our sample. The resulting 18 sets of parameter estimates are reported in Table 3. Note that for expositional reasons Table 3 reports on the ratio of coefficients a/b rather than a. The ratio a/b is equal to the unconditional annual variance for the one-factor model and to the mean of the volatility factor in the two-factor case.

Consider first the one-factor model. The parameters are intuitively plausible and quite stable across estimation years. The mean-reversion of volatility parameter b is roughly between 2 and 3, the unconditional variance a/b between 0.025 and 0.05, the volatility of volatility σ is between 0.6 and 0.7, and the correlation ρ between returns and return volatility is between -0.7 and -0.6. The overall root-mean-squared valuation error (RMSE) is between \$0.466 (in 1992) and \$0.717 (in 1994). The overall in-sample RMSE is \$0.594 across the six years, which is quite impressive in a data set with an average observed market price of \$27.91.[14]

Consider next the two-factor model. A consistent finding across estimation years is that one of the factors is slowly mean-reverting, with b between 0.15 and 0.30, and that the second factor mean-reverts much quicker with b roughly between 2 and 8. Notice also that the slowly mean-reverting first factor has a higher volatility of volatility than the second factor in each of the estimation years. The in-sample fit of the two-factor model is impressive. The penultimate column in Table 3 reports the RMSE in dollars and the last column normalizes the dollar number by the RMSE of the one-factor model. The RMSE percentage improvement is between 8.6% in 1993 and 28.8% in 1991. The overall average RMSE of the two-factor model is \$0.502 or 15.6% below the overall RMSE of the one-factor model.

The finding of a volatility factor with mean-reverting parameter close to zero suggests that it may prove worthwhile to investigate a model with a fully persistent volatility factor. This model is obtained by setting $a_1 = b_1 = 0$, and is referred to as the persistent-factor model in Table 3. The parameters of the persistent-factor model are very similar to those of the two-factor model. The

[14] The overall RMSE is reported in Table 5.

RMSE column indicates that the in-sample RMSE of the persistent-factor model is very close to that of the two-factor model as well. The overall persistent-factor RMSE (reported in Table 5) is \$0.5106, which is only marginally worse than the RMSE of the two-factor model. The finding of a higher RMSE for the persistent-factor model is of course not surprising, because this model is nested by the two-factor model. The fact that the constraint of imposing a fully persistent first volatility factor worsens the in-sample fit only marginally holds substantial promise for the model's out-of-sample performance. We turn to this topic next.

4.3 Out-of-Sample Results

Our out-of-sample results are obtained as follows. We fix the vector of structural parameters Θ at its in-sample value and use Step 1 described in Section 4.1 to compute the out-of-sample spot volatilities. The overall sum of squared pricing errors is then simply calculated as the sum of the T sums of squares from Step 1. This out-of-sample implementation follows Huang and Wu (2004).

Table 4 shows the out-of-sample RMSE year-by-year for the three models. The structural parameters reported in Table 3, which are estimated on a year-by-year basis, are used for out-of-sample valuation in each of the five other years in the sample. This gives a total of thirty out-of-sample results. Each panel in Table 4 represents a model. Each column in each of these panels represents the evaluation results for a parameter vector estimated in a given estimation year. The diagonal terms (in grey cells) are in-sample RMSEs, because the estimation year and evaluation year coincide. All non-diagonal terms represent our-of-sample model evaluations. The rightmost three columns in Table 4 report the year-by-year average out-of-sample performance across the five sets of estimates ("Average Out of S."), the ratio of the "Average Out of S." to the in-sample RMSE in the grey cell ("Out by In Ratio"), and finally the "Average Out of S." normalized by that of the one-factor model ("Average Out vs. 1F").

Consider first the benchmark one-factor model in Panel A of Table 4. Note first that the in-sample RMSEs on the diagonal are always lower than any of the corresponding out-of-sample RMSEs on the same row. This is reassuring, because it demonstrates that the estimation routine is satisfactory. More interestingly, note that the out-of-sample RMSEs in cells adjacent to the diagonal in-sample cells are often quite close to the in-sample value. However, when the estimation year and evaluation year are further apart, the difference between the in-sample and out-of-sample RMSEs are usually more substantial. For example, when the 1990 estimates are used for the 1995 evaluation year, the RMSE is \$1.416, whereas the in-sample RMSE for 1995 is only \$0.514. Conversely when the 1995 estimates are used for the 1990 evaluation year, the RMSE is \$1.002 versus an in-sample RMSE for 1990 of \$0.621. The relative out-of-sample performance of the 1990 estimates is thus worse than that of the 1995 estimates, when considering a sample five years removed in time. The seemingly lower quality of the 1990 estimates is also evidenced by the "Average Out of S." column, which shows an increase in the out-of-sample RMSE towards the end of the sample, and by the "Out by In Ratio" column which shows a particularly large deterioration in the 1995 evaluation year. One possible reason for this finding may be that the number of contracts per year increases over time, which may improve the precision of the parameter estimates.

Comparing the one-factor model with the two-factor model in Panel B of Table 4, we see that the two-factor model performs better than the one-factor model in almost every case. The only exceptions occur when using the 1990 and 1991 estimates for the 1995 evaluation year. This may indicate that the more richly specified two-factor model requires a richer data set which is not available in the early years. Conversely, note that the two-factor model performance for the 1990 and 1991 evaluation years is particularly impressive regardless of the estimation year. The two-factor model is thus better able to capture option price dynamics in the highly volatile markets surrounding the 1990-1991 recession and first Gulf War. It is particularly impressive that the two-factor out-of-sample RMSEs for 1990 in the first row of Panel B are better than or close to the in-sample RMSE of the one-factor model in the top left cell of Panel A.

The rightmost column of Panel B reports the average out-of-sample RMSE of the two-factor model versus the average out-of-sample RMSE for the one-factor model for each evaluation year. Notice that the in-sample improvements from Table 3 are retained out-of-sample, albeit most impressively so in the early years of the sample.

The persistent-factor model in Panel C is motivated by the estimates of the two-factor model, which indicate that one of the factors is very persistent. A comparison of the entries in Panel C with those in Panel A reveals that the out-of-sample performance of the persistent factor model is better than that of the one-factor model in every single cell. The out-of-sample deterioration of the estimates obtained in the early sample years is much less dramatic in Panel C and the impressive out-of-sample performance in the early evaluation years (across estimation years) is retained. The last column of Panel C reveals that the persistent factor model offers an improvement of between 13 and 28% over the one factor model. These numbers are very impressive, because robust out-of-sample improvements on the one-factor Heston (1993) model are difficult to come by.

The out-of-sample performance of the persistent-factor model is clearly better than that of the one-factor model. Furthermore, a comparison of the cells in Panel C with those in Panel B shows that the persistent-factor model has a lower RMSE than the two-factor model in Panel B in most (but not all) of the cases. The persistent factor model significantly improves on the two-factor model in some cases, and is never significantly outperformed by the two-factor model. Consistent with this observation, a comparison of the "Average Out of S." columns in Panels C and B indicates that on a year-by-year comparison the persistent factor model dominates the two factor model in each of the years considered.

4.4 Ad-Hoc Benchmark Models

While the out-of-sample performance of the two-factor model, and in particular that of the persistent-factor model, is impressive relative to the one-factor model, the question arises whether the one-factor model is a good choice of benchmark. There are two ways to address this question. First, there are not many structural option pricing models available in the literature that robustly improve upon the out-of-sample performance of the Heston (1993) model. Models that contain Poisson jumps in returns and/or volatility may significantly improve on the in-sample performance of the Heston (1993) model, but the out-of-sample improvements are modest or non-existing (see for example Bakshi, Cao and Chen (1997) and Eraker (2004)).[15] Option valuation models that are based on Levy processes for the underlying seem to be more successful out-of-sample (see Huang and Wu (2004) and Carr and Wu (2005)). While it is always difficult to compare the performance of models estimated using different techniques, as well as across data samples, our simple two-factor model seems to improve on the performance of the one-factor Heston (1993) model by at least the same amount as the most sophisticated jump models.

Table 5 addresses the appropriateness of the benchmark in a different way, by comparing the average out-of-sample RMSEs from Table 4 with two often used ad-hoc benchmark models. The first ad-hoc benchmark, labeled "Black-Scholes" is implemented by finding the spot volatility each Wednesday which minimizes the mean-squared Black-Scholes pricing errors. Thus, while retaining the structure of the Black-Scholes (BS) pricing formula, it allows for time-varying volatility via weekly re-estimation. Thus it can be viewed as a Hull and White (1987) model. Clearly, all three stochastic volatility models we consider perform well in comparison with this benchmark.

The second benchmark labeled, "Ad Hoc OLS" is the model used in Dumas, Fleming and Whaley (1998), which regresses implied BS volatilities on a second order polynomial in strike price and maturity. The model option prices are then calculated using the fitted values from the regressions as the volatility parameter for each contract. Dumas, Fleming and Whaley (1998) found that this method outperformed the deterministic volatility models they considered in their paper. Notice that the stochastic volatility models we consider all outperform the ad-hoc BS benchmark.

Finally, Christoffersen and Jacobs (2004) note that when the ad hoc model is implemented via minimization of the mean-squared valuation errors using non-linear least squares (NLS) rather than OLS regression on implied volatilities, the fit of the ad hoc model improves drastically. We find that this finding is confirmed for the sample used in this paper. As far as we know, no model in the literature has been shown to outperform such a modified ad hoc NLS benchmark which is reported in the rightmost column of Table 5. We speculate that when implemented as above, it will prove extremely difficult or impossible for a structural model such as the one considered in this paper to

[15] Broadie, Chernov and Johannes (2004) show that when restricting certain parameters to be equal to estimates from historical returns, adding Poisson jumps to a stochastic volatility model improves option fit.

improve on the benchmark in Christoffersen and Jacobs (2004).

Note that the benchmark models are implemented in-sample, meaning that all their parameters are estimated using current week option price information. Arguably, a fairer comparison between these models and the stochastic volatility models may therefore be to use the in-sample RMSE for the latter. The last two lines of Table 5 do exactly this. Notice that the two-factor stochastic volatility models which use structural parameters estimated on a year of data and spot volatilities on the current week of data provide a fit that is close to the fit of the ad hoc NLS model, for which all parameters are estimated on the current week of data.

For completeness, Table 5 also reports the overall out-of-sample RMSE in dollars as well as expressed as a percentage of the one-factor model RMSE. Notice that the two-factor model leads to an 11% out-of-sample improvement and the persistent factor model leads to an impressive 19% out-of-sample improvement. These numbers are in line with their in-sample counterparts of 15% and 14% respectively, which are reported at the bottom of Table 5 for reference.

4.5 Patterns in Model Errors Over Time

Table 6 and Figure 7 provide more insight into the in-sample differences between the one-factor, two-factor and persistent-factor models. Figure 7 further documents the differences between the in-sample RMSEs of the three models by graphing the average weekly RMSE over the 1990-1995 sample for the one-factor model and the ratio of the RMSE to the one-factor RMSE for the two-factor and persistent-factor models. The analysis is in-sample: for each year, the corresponding parameters from Table 3 are used. Notice that the ratio RMSEs in the middle and lower panel are almost always less than one so that the improvement in the two-factor and persistent-factor model is not confined to a particular time period in our sample. Figure 7 does confirm the finding in Table 3 that the differences between the models are consistently largest in 1990 and 1991. To understand these differences, it is instructive to remember the volatility patterns in Figures 1 and 2. The volatility is higher and more variable in 1990 and 1991 compared to subsequent years, and presumably this is driving the large RMSE differences during those years. However, episodes of up to 50% improvements in the RMSE over the one-factor model occur throughout the sample. While the relative improvement over the one-factor model is similar in the two-factor and the persistent-factor models, the two-factor model appears to fare relatively better in 1995.

Table 6 reports on in-sample RMSE by moneyness and maturity for the three models. Again we report the RMSE for the one-factor model (Panel A) and the RMSE ratios to the one-factor model for the two-factor (Panel B) and persistent-factor (Panel C) models. Notice that the ratio RMSEs are never above one. The table thus indicates that the two-factor and persistent-factor models improve on the one-factor model for all moneyness and maturity bins. For some short-maturity options the improvements are rather modest, but this is not surprising because those options are relatively cheap and therefore do not receive a large weight in the objective function (16).

5 Model Properties

In Section 4, we compared the pricing performance of the three models. While these types of comparisons are critical, they do not highlight the model features that enable the model to fit the data better. In this section, we explore the empirical results in more detail and try to provide more intuition for the differences in empirical performance.

5.1 Conditional Dynamics

We start by plotting the conditional dynamics for $Cov_t(dS/S, dV)$ and $Var_t(dV)$, which can be easily related to the parameter estimates. For the two-factor model, $Cov_t(dS/S, dV)$ is given by (6) and $Var_t(dV)$ is given by

$$Var_t(dV) = (\sigma_1^2 V_1 + \sigma_2^2 V_2)dt \tag{17}$$

For the one-factor model we have

$$Cov_t(dS/S, dV) = \sigma\rho V dt \tag{18}$$

$$Var_t(dV) = \sigma^2 V dt \tag{19}$$

From (18) it is clear that the correlation between the variance and $Cov_t(dS/S, dV)$ is equal to minus one, and the correlation between the variance and $Var_t(dV)$ is equal to plus one. The two-factor model on the other hand is not as restrictive, as is evident from (6) and (17).

In Figure 8, we report on the conditional variance path for the one-factor model, as well as the path of $Cov_t(dS/S, dV)$ given by (18) and the path of $Var_t(dV)$ given by (19). This analysis is in-sample: to compute the conditional moments in a given year, the corresponding parameters from Table 3 for that year are used. Because the variance is high in 1990-1991, the absolute value of $Cov_t(dS/S, dV)$ and the value of $Var_t(dV)$ are also high in that period.

Figure 9 presents the variance as well as $Cov_t(dS/S, dV)$ and $Var_t(dV)$ for the two-factor model.[16] Figure 9 also plots the paths of the two variance factors. The plots of the variance paths illustrate clearly that the first variance factor is much more persistent than the second variance factor. Figure 9 nicely highlights the mechanics of the two-factor model. First, consider the conditional variance of variance. The results in Table 3 indicate that for all years, the estimate of the σ_1 parameter is much larger than that of the σ_2 parameter. It can be seen from (17) that as

[16] Because the in-sample performance of the persistent-factor model is very similar to that of the two-factor model, the plots for the persistent-factor model look very similar to the ones in Figure 9. We omitted these plots to save space.

a result, the conditional variance of variance will be high when the first variance factor is high. In such cases high overall variance is accompanied by high overall variance of variance. However, it is also possible that high variance is accompanied by relatively low variance of variance, namely in those cases where the high variance results from a peak in the second variance factor. A similar logic applies to the path of $Cov_t(dS/S, dV)$. If the first volatility factor is more negatively correlated with stock returns than the second factor, the model is capable of generating more flexible paths for $Cov_t(dS/S, dV)$. If the first variance factor is high, the covariance between returns and volatility is high. If the second factor is high, the variance is high but the covariance with returns is reduced (in absolute value). This mechanism is empirically important because the relative size of the factors significantly varies through time.

When comparing the paths for $Cov_t(dS/S, dV)$ and $Var_t(dV)$ for the two-factor model in Figure 9 with those for the one-factor model in Figure 8, it is clear that the increase in (the absolute value of) $Cov_t(dS/S, dV)$ and $Var_t(dV)$ in 1994 is much more pronounced for the two-factor model. In general, $Cov_t(dS/S, dV)$ and especially $Var_t(dV)$ display much more variation in the two-factor model.

5.2 Stochastic Correlation

Figure 10 provides additional intuition for the two-factor model's properties by documenting how the correlation between stock returns and volatility changes over time, given the estimates obtained in Table 3. Each panel uses a different set of parameter estimates, corresponding to the six sets of results reported in Table 3. For each set of estimates, which are obtained using one year of options data, we graph the time varying-correlation in (7) for the 1990-1995 sample. In each panel, we also graph the constant correlation for the one-factor model.

Figure 10 clearly demonstrates that time-varying correlation is an important model feature. For instance, using the 1990 estimates, the correlation fluctuates between -0.75 and -0.83 in the 1990-1995 sample. Using the 1994 estimates, the correlation fluctuates between -0.63 and -0.84. Moreover, the correlation paths suggest that correlation does not follow a very persistent process, and that it is important to allow correlation to change quickly over time. Note from (7) that the correlation between stock returns and volatility is not restricted to lie in between the correlations of stock returns and the respective volatility factors. For example, using the 1994 estimates, the correlations between returns and the first and second volatility factors are -0.80 and -0.79 respectively, but the stochastic correlation varies between -0.63 and -0.84. Finally, it must be noted that although the correlation coefficients ρ_1 and ρ_2 may not differ much for certain parameter sets, these relatively small differences yield large differences in the correlations between the respective variance paths on the one hand and $Cov_t(dS/S, dV)$ and $Var_t(dV)$ on the other hand. For instance, while the in-sample correlation of $Cov_t(dS/S, dV)$ with the first (more persistent) variance factor is -0.89, the correlation between $Cov_t(dS/S, dV)$ and the second variance factor is only -0.33 (not reported). Likewise, the correlation between $Var_t(dV)$ and the first variance factor is 0.92 but the correlation between $Var_t(dV)$ and the second variance factor is 0.16.

5.3 The Term Structure of Higher Moments

The investigation of $Cov_t(dS/S, dV)$ and $Var_t(dV)$ above is interesting because these moments are related to skewness and kurtosis, and the simple expressions in (6) and (17) provide ample intuition for the sample paths. This section provides a more thorough investigation of conditional skewness and kurtosis. The intuition underlying our empirical findings in this section is that because conditional skewness and kurtosis are determined by $Cov_t(dS/S, dV)$ and $Var_t(dV)$ as well as by a rescaling of these objects which is also related to the conditional variance, the paths for conditional skewness and kurtosis in the one-factor model are too strongly linked to the variance path.

From (11) it is clear that the moment generating function of the log return $x(t+\tau) - x(t)$ is given by $f(V_1, V_2, \tau, \phi)$. In general we have that the i'th conditional cumulant can be computed from the moment generating function as

$$\kappa_{i,t,\tau} = \left. \frac{\partial^i \ln(f(V_1, V_2, \tau, \phi))}{\partial \phi^i} \right|_{\phi=0}$$

For the first four moments we have the following relationship between cumulants and moments

$$\begin{aligned}
E_t\left[x(t+\tau) - x(t)\right] &= \kappa_{1,t,\tau} \\
Var_t\left[x(t+\tau) - x(t)\right] &= \kappa_{2,t,\tau} \\
Skew_t\left[x(t+\tau) - x(t)\right] &= \kappa_{3,t,\tau}/\kappa_{2,t,\tau}^{3/2} \\
Kurt_t\left[x(t+\tau) - x(t)\right] &= \kappa_{4,t,\tau}/\kappa_{2,t,\tau}^{2}
\end{aligned}$$

Where $Kurt$ is defined as excess kurtosis. Unfortunately, the analysis of conditional skewness and kurtosis is rather complex because no simple expressions are available for the cumulants. We use closed-form expressions for conditional cumulants that are derived using Mathematica. These expressions are rather lengthy and are available from the authors on request.

We now present empirical results for the conditional moments using the estimates for the one-factor and two-factor models presented in Table 3. Figures 11-12 plot the conditional moments over time. We implement this by using the time series of spot variances $\{V_1(t), V_2(t)\}$ for the two-factor model and $\{V(t)\}$ for the one-factor model, obtained in Step 1 of the optimization algorithm in Section 4.1. Figure 11 presents results using the 1990 parameter estimates, and Figure 12 is obtained using the 1993 parameters. This analysis therefore contains an out-of-sample as well as an in-sample component. Because the conditional moments depend on maturity, we present results for two different horizons. In each figure, Panel A presents the conditional moments for one-month returns and Panel B for three-month returns.

A number of important conclusions obtain. The one-factor and two-factor model do not generate substantially different in-sample and out-of-sample paths for the conditional mean and the conditional variance. These paths are also remarkably similar across years. However, in some cases important differences obtain for the conditional skewness and kurtosis paths generated by the one-factor and two-factor models, and these differences strongly depend on which parameter estimates are used.

The conditional skewness and kurtosis paths generated by the one-factor and two-factor models differ in two important respects. First, using the 1993 estimates the (absolute value of) the conditional skewness and the conditional kurtosis is larger for the two-factor model. Second, it can be easily seen from the figures that the trend and the variation in the conditional skewness and kurtosis for the one-factor model are very strongly linked to the trend and variation in the conditional variance. While some relationship between these three conditional moments is also evident in the two-factor model, the relationship with the trend and especially the changes in the conditional variance seems much weaker. Consider the conditional skewness and kurtosis obtained using the 1990 parameters in Figure 11. In this case the average level of the conditional skewness and kurtosis for the two-factor model is similar to that of the one-factor model, but while there is a pronounced trend in the skewness and kurtosis paths for the one-factor model which is driven by the trend in the conditional variance, this is not the case for the two-factor model.

In summary, we conclude that the one-factor and two-factor models differ little in terms of the conditional variance paths. However, the models differ substantially in terms of the flexibility offered to model the conditional skewness and kurtosis, and the one-factor model seems more constrained in this respect.

5.4 What do the Variance Factors Capture?

We have argued that two-factor models are more flexible than one-factor models for the purpose of modeling moneyness effects as well as the modeling of the volatility term structure. Our empirical results confirm that the two-factor models provide a better fit, and the conditional dynamics suggest that this improved fit is partly due to the improved modeling of higher conditional moments. However, we have not yet directly related the improvement in fit to the modeling of the smirk and the volatility term structure.

The first row of Panel A in Table 7 reports the correlation between the volatility level and the slope of the volatility smirk, as computed in Figure 4, by regressing implied volatilities on a given day on an intercept and log moneyness. The second and third rows repeat the correlation based on the same regression analysis, except that the implied volatilities used in the regressions are not those based on the data, but are based on the option prices predicted by the one-factor and two-factor models respectively. This analysis therefore indicates whether the two-factor model better matches the correlations in the data. While the raw data yields negative as well as positive correlations, the one-factor model always yields sizeable negative correlations. The two-factor model performs better in some years than in others, but overall it matches the patterns in the data much better.

It could be argued that regressing implied volatilities on log moneyness may lead to noisy results, because significant maturity effects are not filtered out. Panel B of Table 7 repeats the analysis of Panel A, but implied data or model volatilities are regressed on $\ln\left[S/(X\exp(-rT))\right]/\sqrt{T}$, to remove maturity effects. While the correlations are somewhat different, the two-factor model again captures the patterns in the data much better than the one-factor model. The one-factor model seems to consistently generate substantial negative correlation between volatility level and slope, regardless of the patterns in the data.

In Table 8 we compute the absolute correlation between the variance factors and the time series for the level and slope of the smirk, as computed by regressing the implied volatilities on a given Wednesday on moneyness corrected for maturity $\ln\left[S/(X\exp(-rT))\right]/\sqrt{T}$. We report the results on a year-by-year basis, because the parameter estimates are on a year-by-year basis.

The first two rows of Table 8 report results for the one-factor model. While the first row indicates that the variance factor is highly correlated with the time series for the level, the second row indicates that the correlation with the slope factor is rather small. Rows 3-8 of Table 8 report on the two-factor model. Rows 3-6 report simple univariate correlations, and rows 7-8 report the multiple correlation from regressing either the level or the slope on both variance factors. The results indicate that the level factor is partly captured by both variance factors. When considering both factors jointly in row 7, the multiple correlation coefficient is very close to one, and at least as high as the correlation coefficient in row 1. The slope of the smirk is also captured by both variance factors. The multiple correlation coefficient in row 8 is always higher than the correlation coefficient in row 2, but in some years, such as 1990, the difference is very small, whereas in other years, such as 1995, the difference is large. Altogether, Table 8 indicates that the two-factor model is better at capturing the slope of the smirk, even if the multiple correlation coefficient indicates that a large part of the variation in the smirk remains unexplained

Table 9 reports on a similar analysis. Instead of regressing implied volatilities on moneyness, we regress on a constant and maturity. Some important conclusions obtain. The absolute correlations between the variance factor and the term structure slope in row 2 are much higher than the corresponding correlations in row 2 of Table 8. The same conclusion obtains when comparing the multiple correlation coefficients in the bottom rows of Tables 8 and 9. Finally, in 1995 neither the one-factor nor the two factor model manage to capture the term structure slope.

When combining the findings from Tables 8 and 9, we arrive at the following conclusions. First, whereas the two-factor model offers more flexibility than the one-factor model for modeling the maturity dimension as well as the moneyness dimension, the data and the loss function put relatively more emphasis on the modeling of the volatility term structure. Second, although the two-factor model substantially outperforms the one-factor model, the two-factor model seems to encounter some difficulty in capturing both moneyness and term structure effects. In 1995, when the multiple correlation between the two variance factors and the slope of the smirk is 74%, the multiple correlation between the variance factors and the term structure slope is only 5%, much lower than in any of the other years. There seems to be a trade-off between capturing both dimensions of the data, which suggests that an even richer model may be needed.

6 Summary and Conclusion

This paper investigates a tractable model for equity index option valuation that allows for rich modeling of term structure and moneyness effects. It is important to have simple yet robust models that are relevant from a theoretical as well as a practical perspective, and we believe that the two-factor SV model satisfies this criterion. Adding volatility factors to an existing framework and exploiting the pricing results of Heston (1993), greatly improves the model's flexibility to capture the volatility term structure. Moreover, we demonstrate that the two-factor model is more flexible in capturing largely independent fluctuations in the level and the slope of the volatility smirk, which are inextricably linked in the one-factor Heston model.

We are not the first to suggest the use of multiple volatility factors, but our discussion of the role of multiple factors in capturing term structure and moneyness effects is novel. This insight explains why one-factor models are unable to account for some important stylized facts in index option data. The in-sample and out-of-sample performance of the two-factor model, which is on par with the most sophisticated models currently available in the literature, forcefully illustrates the power of the multifactor approach. We also show that one particular variation of the two-factor model, which we label the persistent-factor model, improves on the more general two-factor model out-of-sample.

The paucity of multifactor volatility models in the option valuation literature is remarkable when one considers the related empirical literature on yield curve modeling. The theoretical models and empirical techniques used in the option valuation literature are closely related to those used in the yield curve literature. Interestingly, almost every paper in the yield curve literature uses a multifactor model, and in fact three-factor models have become the standard. We speculate that in the future, multifactor models may become as important for the equity option literature as they are for the term structure literature.

A number of extensions to the analysis in this paper may prove worthwhile. First, it may prove interesting to compare the relative value of adding jump components and additional variance factors to a stochastic volatility model. The resulting models may have different implications for the modeling of term structure and moneyness effects, and their performance may differ in-sample as well as out-of-sample. Second, an integrated analysis of multifactor models using options data as well as underlying returns ought to be done. Following the observations of Bates (1996) and Broadie, Chernov and Johannes (2004), it will be of particular interest to investigate the model's pricing performance when imposing consistency between physical and risk-neutral estimates. Third, the focus of our empirical analysis is to convince the reader that a second factor allows for more realistic modeling of conditional higher moments, which improves the modeling of term structure and moneyness effects. We leave open the question if additional factors are needed, and how they would improve pricing performance. The out-of-sample performance of such models is of particular interest, because often richly parameterized models perform poorly out-of-sample. Fourth, our analysis also does not address the interesting question of how the dynamics of the different factors ought to be specified. We intentionally choose a simple specification to obtain a closed-form solution. In the term structure literature, recent empirical studies have demonstrated that multifactor models with some square-root factors and some Gaussian factors outperform multifactor models with multiple square-root factors. Also, while the variance factors in the model are assumed to be uncorrelated in order to obtain a closed form solution, correlated factors may improve model fit. Finally, checking the robustness of our results using more recent data sets would be interesting.

In summary, our paper argues that we need multifactor models to capture some of the most salient stylized facts in index option prices. We hope that our results will lead to a more extensive search for an even better multifactor model.

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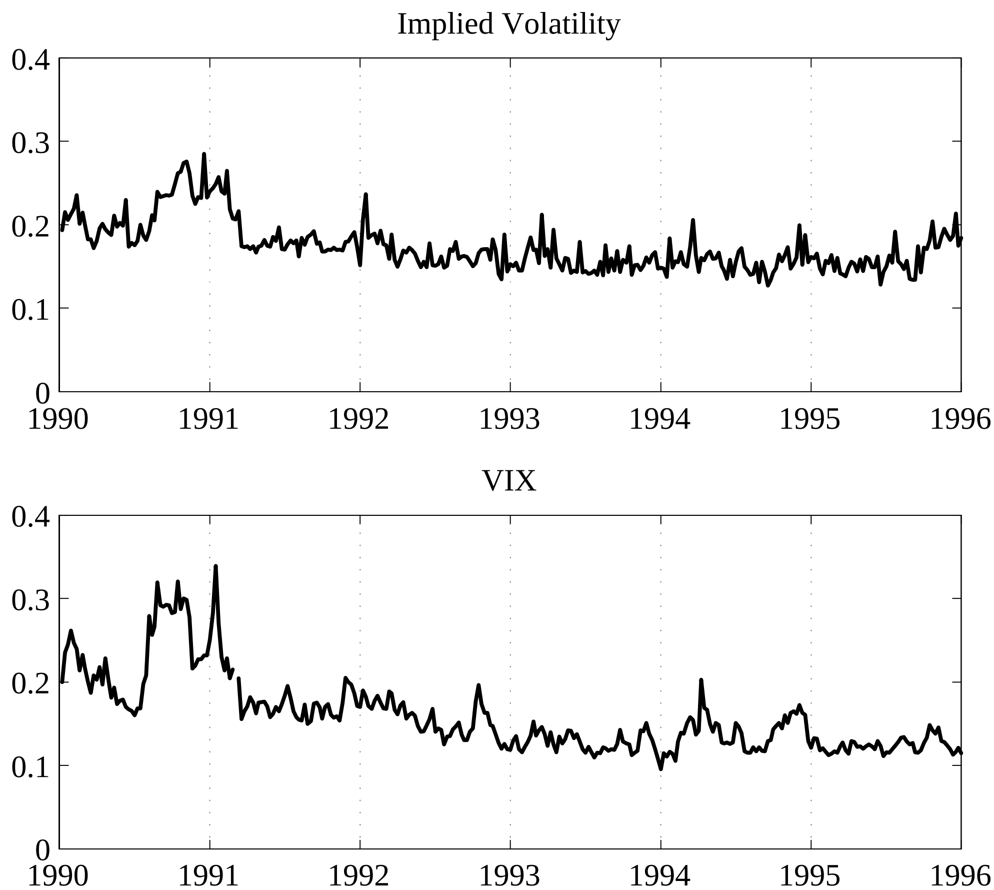
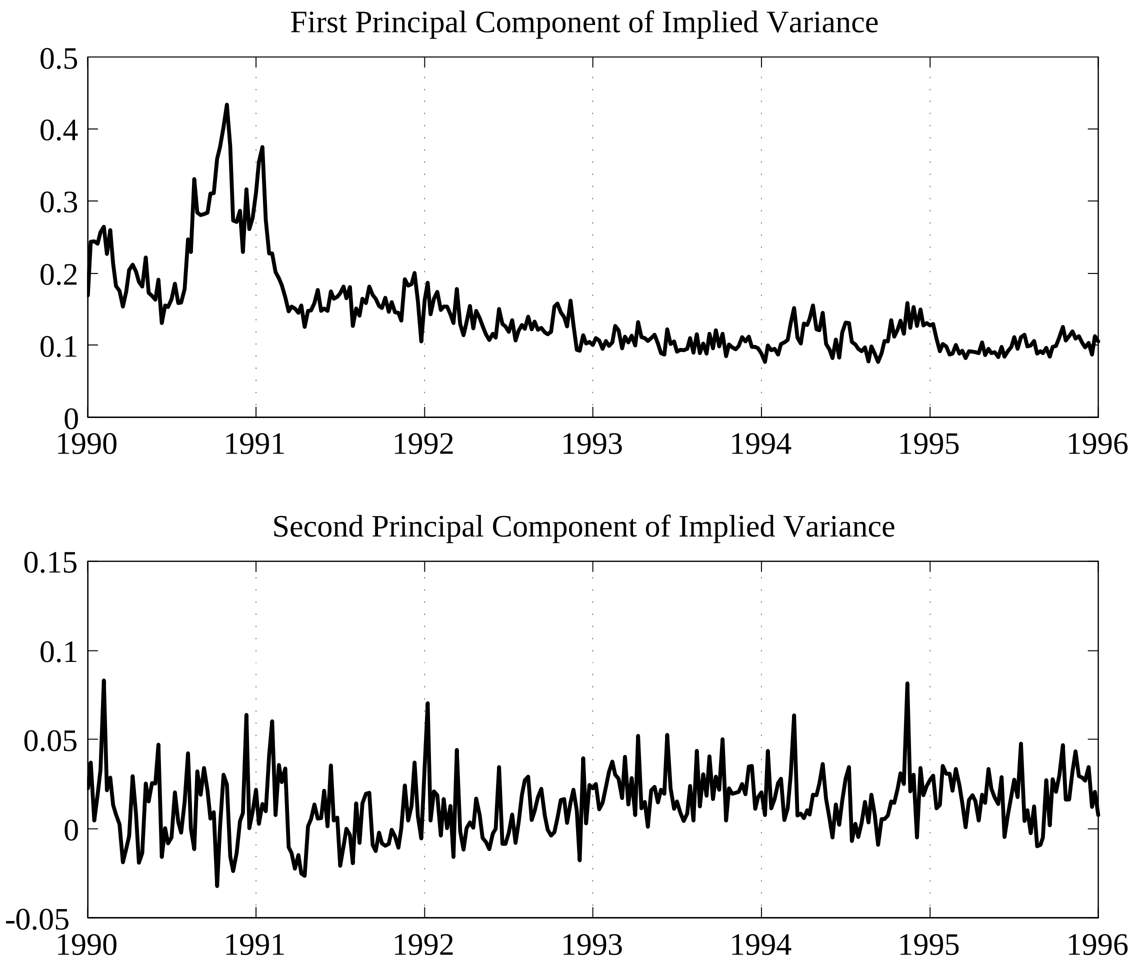
Figure 1: Average Implied Volatility in S&P500 Option Data and the CBOE VIX.



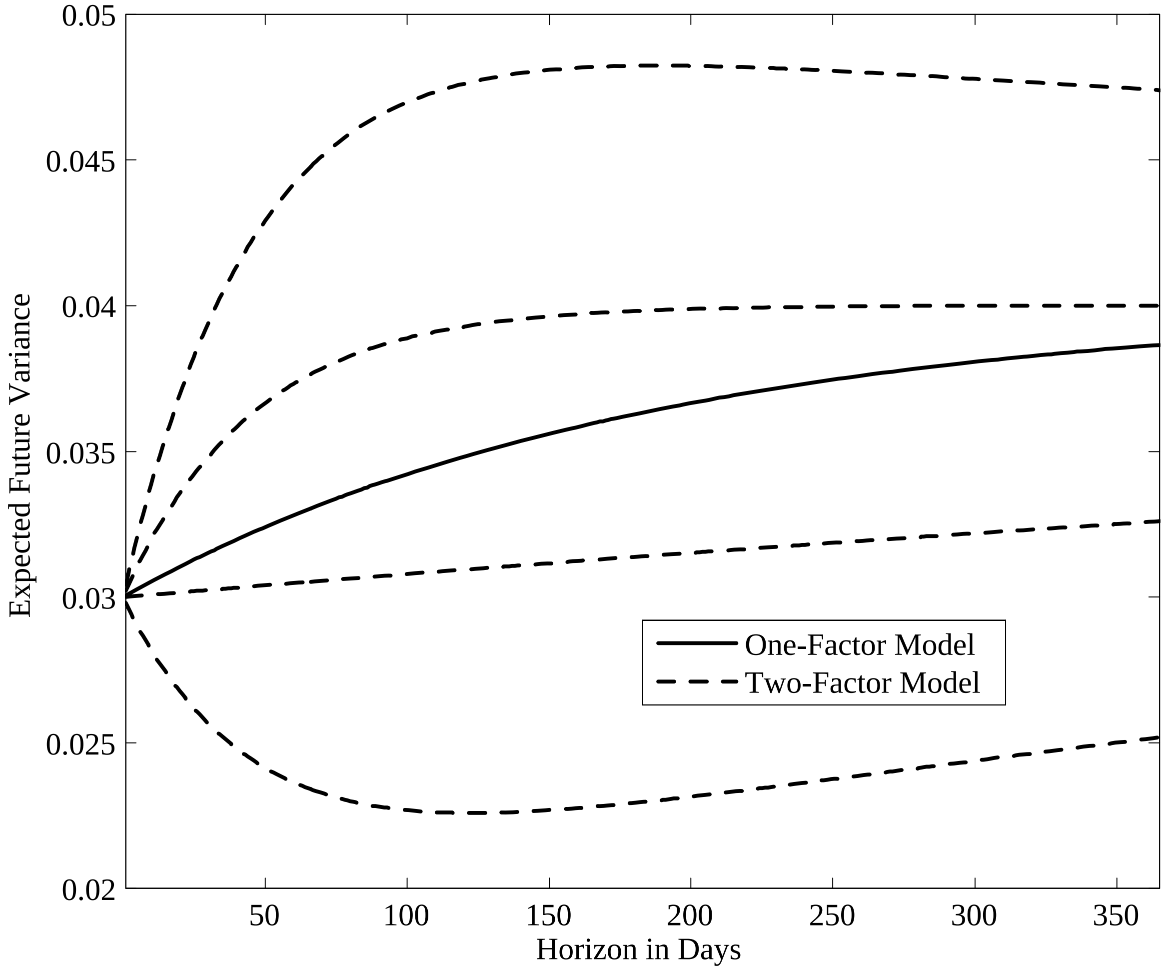
Notes to figure: The top panel plots the average implied Black-Scholes volatility each Wednesday during 1990-1996. The average is taken across maturities and strike prices using the call options in our data set. For comparison, the bottom panel shows the one-month, at-the-money VIX volatility index retrieved from www.cboe.com.

Figure 2: First Two Principal Components for the Implied Index Variance.



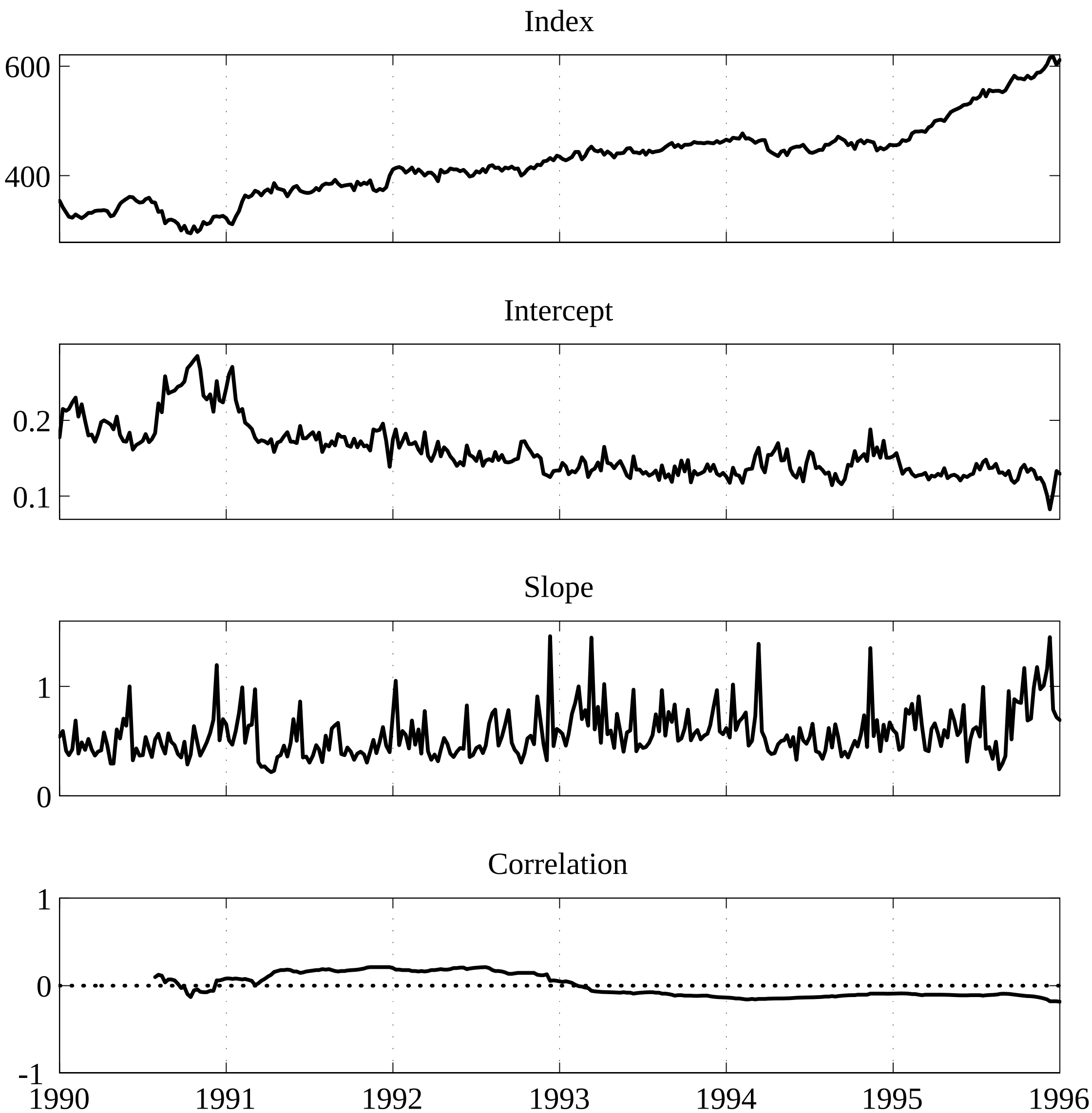
Notes to figure: The two panels plot the first and second principal component for the implied index variance during 1990-1995. The principal component analysis is performed using implied variances for each Wednesday during 1990-1995. In a first stage, a quadratic polynomial in maturity and moneyness is fit for each Wednesday, using data for all available moneyness and maturity. In a second stage, these estimates are used to generate a variance surface with standardized moneyness and maturity. The principal component analysis is performed on this standardized variance surface.

Figure 3: Expected Future Variance in the One-Factor and Two-Factor Models.



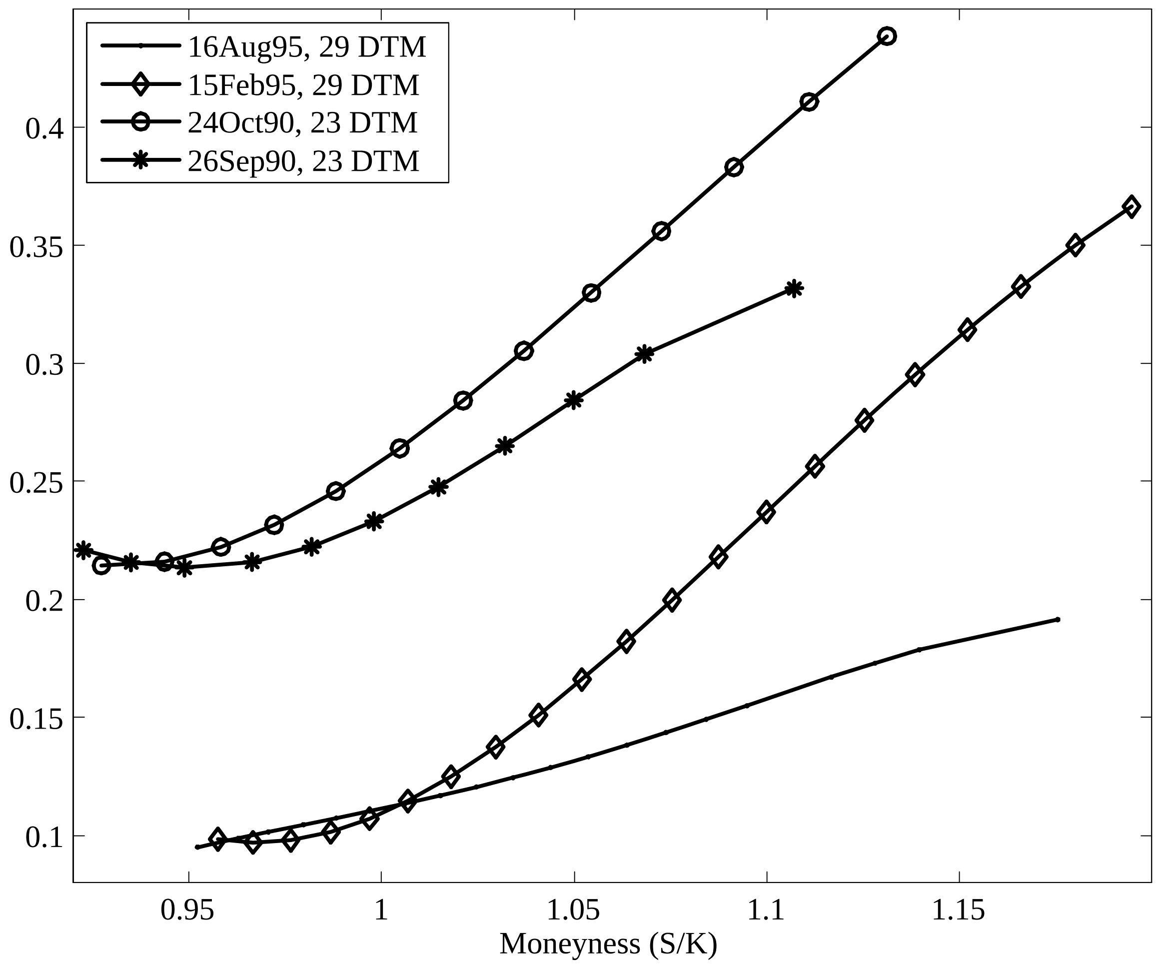
Notes to figure: We plot the expected future spot variance over a one-year horizon using parameterizations for the one-factor and two-factor models. For the one-factor model, the long run variance a/b is 0.04, the mean reversion coefficient b is 2, and the spot variance V is 0.03. For the two-factor model, the long run variance is also 0.04, the mean reversion coefficient for the first variance factor b_1 is 0.3 and for the second variance component b_2 is 8. The long run mean is .02 for both factors. The spot variance for the first factor V_1 is 0, .01, .02, and .03 respectively, and the second spot variance V_2 is $0.03 - V_1$ in all cases.

Figure 4: Implied Volatility Regressions 1990-1995.



Notes to figure: The first panel plots the underlying index for 1990-1995. The time series in the second and third panel are obtained by regressing implied volatilities for all available contracts on a given day on an intercept and log moneyness. The resulting coefficients can be interpreted as the volatility level and the slope of the smirk on that day. The time series in the fourth panel is the rolling correlation between the volatility level and the slope of the smirk. We first compute the correlation in week 31 using the previous 30 observations and subsequently expand the window.

Figure 5: Volatility Smirks for Selected Days and Maturities.



Notes to figure: We plot the volatility smirk for call options with either 23 or 29 days to maturity on four different days: August 16, 1995, a low volatility day with a flat smirk, February 15, 1995, a low volatility day with a steep smirk, September 26, 1990, a high volatility day with a flat smirk, and October 24, 1990, a high volatility day with a steep smirk.

Figure 6.A: Volatility Smirks for the One-Factor (solid) and Two-Factor (dashed) Models.

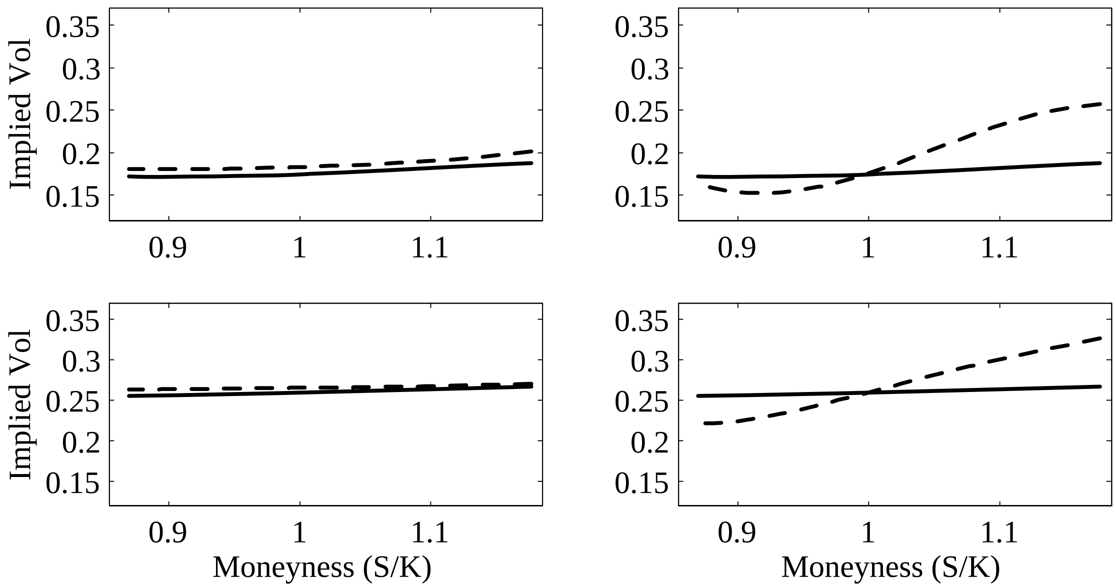
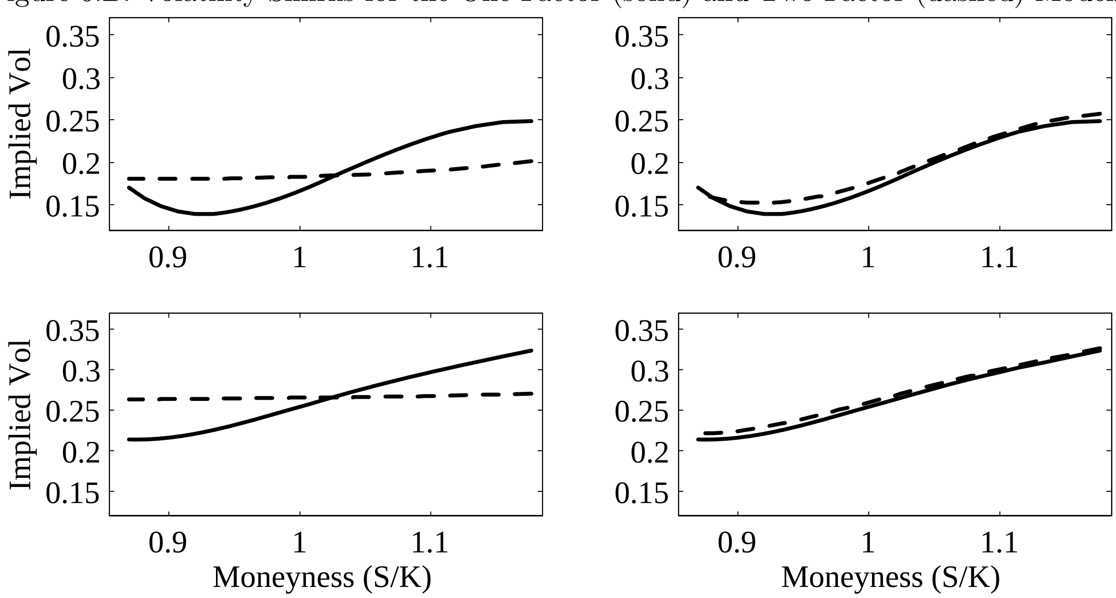
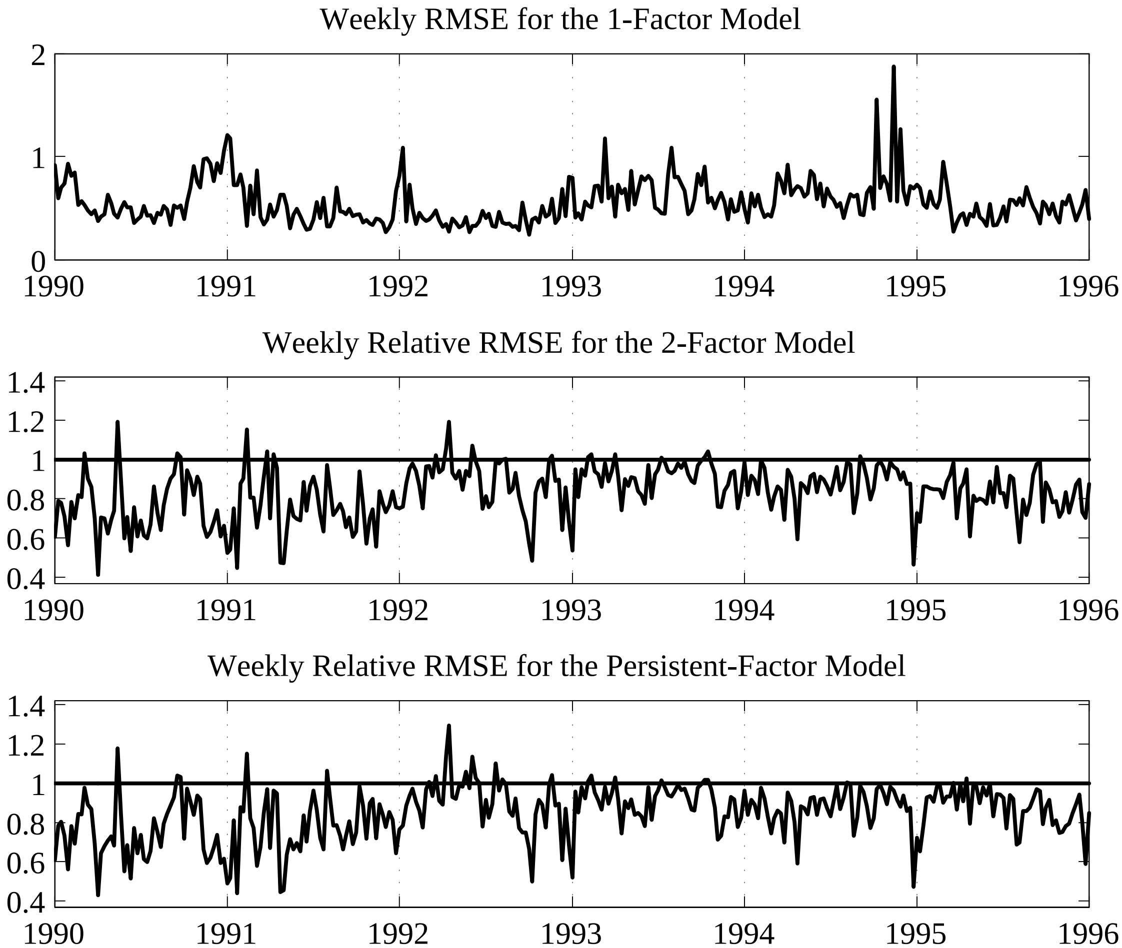


Figure 6.B: Volatility Smirks for the One-Factor (solid) and Two-Factor (dashed) Models.



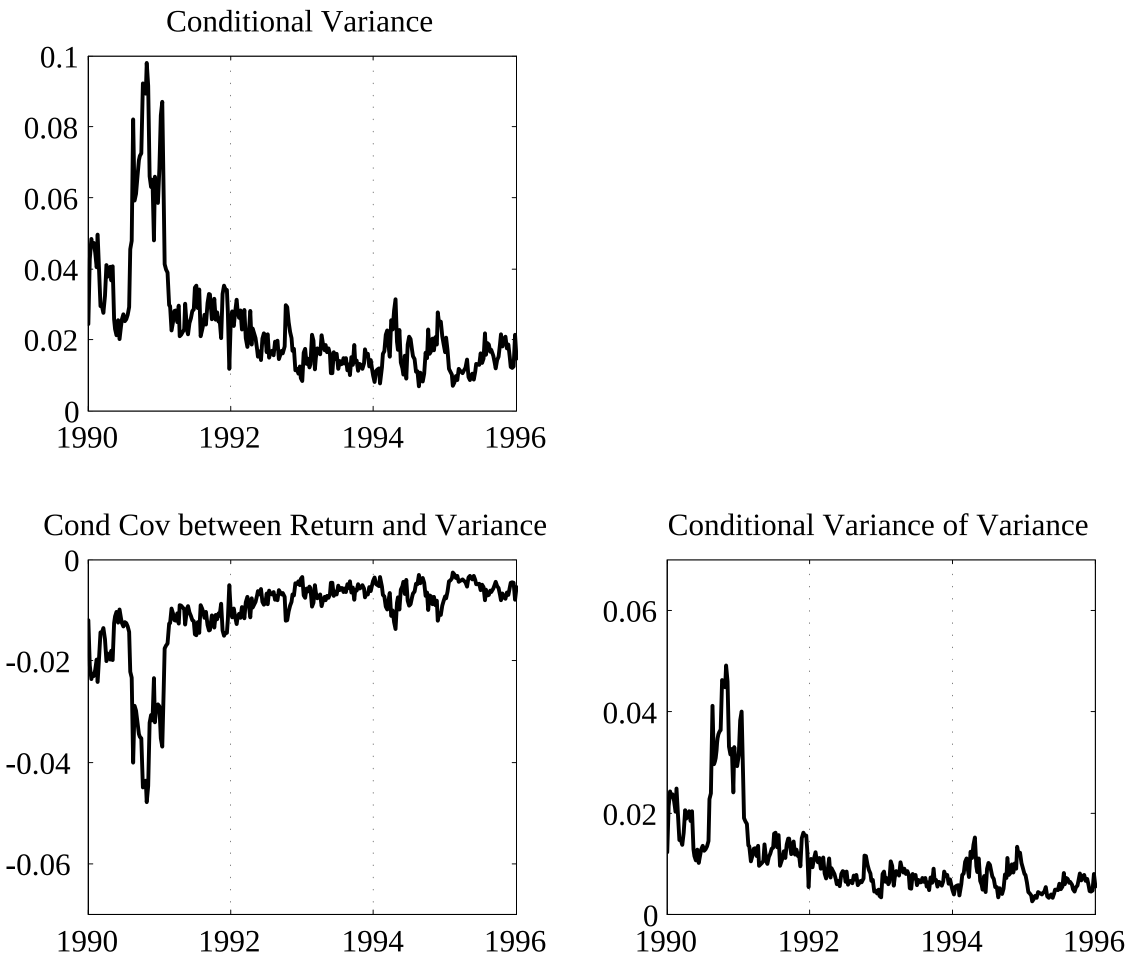
Notes to figure: The dashed lines represent the volatility smirk for options with 30 days to maturity using the following parameterization of the two-factor model: $b_1 = 2$, $a_1/b_1 = 0.04$, $\sigma_1 = 0.1$, $\rho_1 = -0.1$, $b_2 = 2$, $a_2/b_2 = 0.04$, $\sigma_2 = 0.8$, $\rho_2 = -0.6$. In the top pictures of each panel, the initial variance is 0.03, in the bottom pictures of each panel the initial variance is 0.07. In the pictures on the left the spot variance factors are $V_1 = V$ and $V_2 = 0$, in the pictures on the right we have $V_2 = V$ and $V_1 = 0$. In Panel A, the one-factor model is calibrated on the two-factor data represented in the pictures on the left. In Panel B, the one-factor model is calibrated on the two-factor data represented in the pictures on the right.

Figure 7: Weekly In-Sample Root Mean Squared Error (RMSE). 1990-1995.



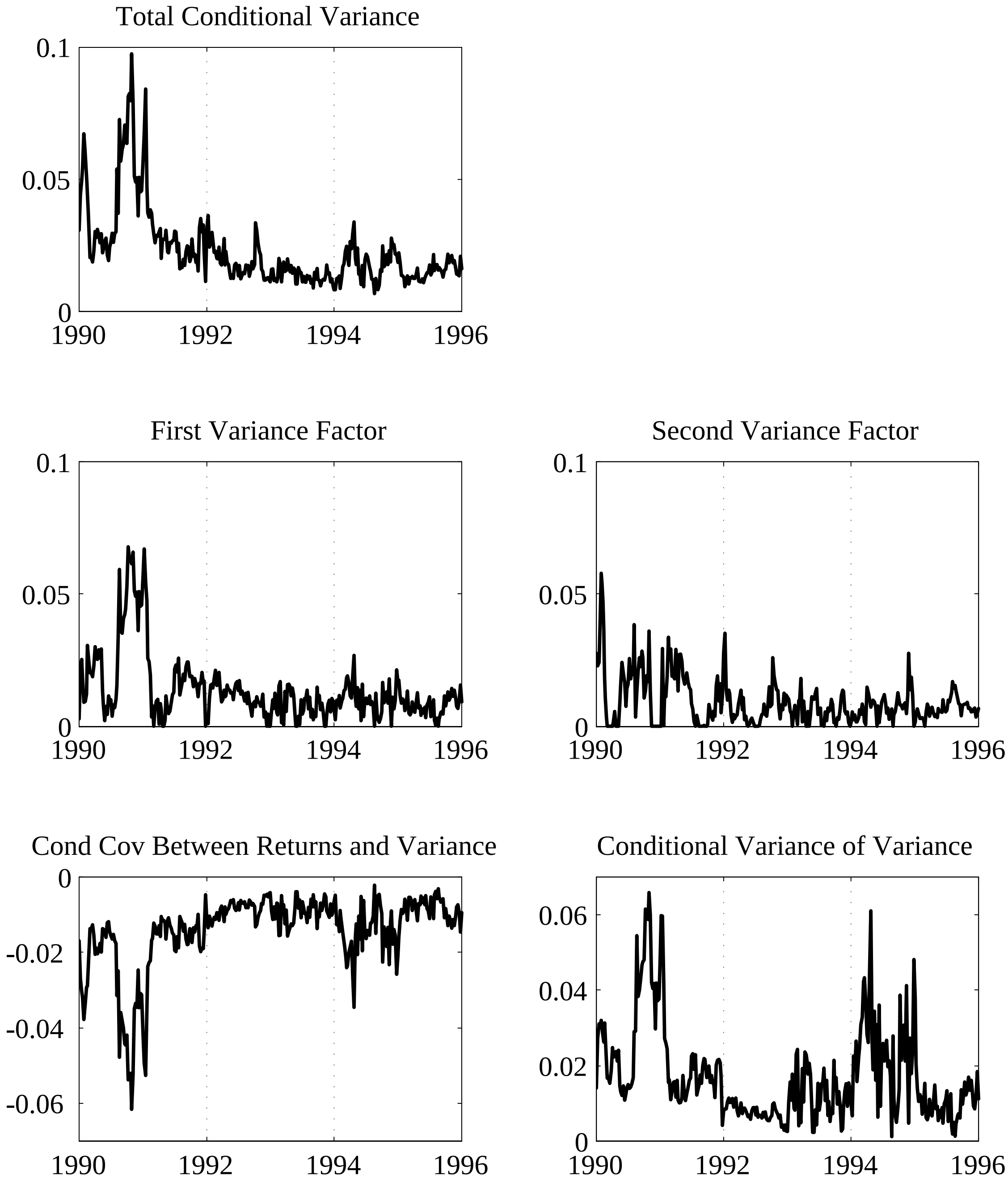
Notes to figure: The top panel shows the weekly root mean squared option valuation error (RMSE) for the one-factor model. The middle panel shows the ratio of the RMSEs from the two-factor and one-factor models. The bottom panel shows the ratio of the RMSEs from the persistent-factor and the one-factor models.

Figure 8: Conditional Dynamics for the One-Factor Model. In-Sample. 1990-1995.



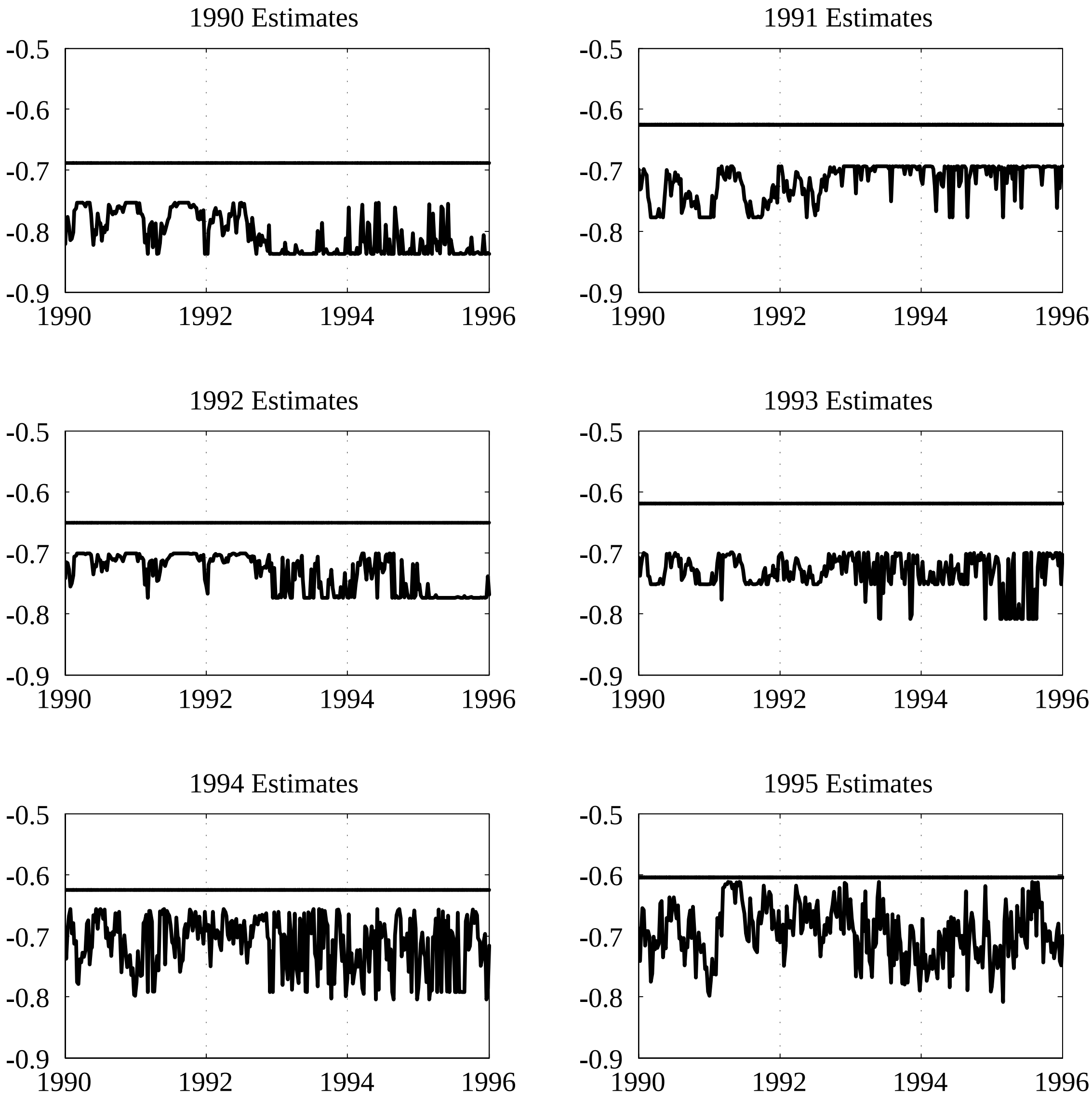
Notes to figure: The top panel on the left plots the conditional variance for the one-factor model. The bottom panel on the left plots the conditional covariance between returns and variance, according to (18). The bottom panel on the right plots the conditional variance of the variance, according to (19). For every year, we use the corresponding parameter estimates from Table 3 in each panel. This generates in-sample conditional dynamics. Note that the correlation between the conditional variance path and the path of the conditional covariance between returns and variance is -1 by construction, and that the correlation between the conditional variance path and the conditional variance of variance path is +1 by construction.

Figure 9: Conditional Dynamics for the Two-Factor Model. In-Sample. 1990-1995.



Notes to figure: The top panel on the left plots the conditional variance for the two-factor model. The middle panels plot the two variance factors. The bottom panel on the left plots the conditional covariance between returns and variance, according to (6). The bottom panel on the right plots the conditional variance of the variance, according to (17). For every year, we use the corresponding parameter estimates from Table 3 in each panel. This generates in-sample conditional dynamics.

Figure 10: Correlation between Stock Returns and Volatility



Notes to figure: We plot correlation between stock returns and volatility. Each panel uses a different set of parameter estimates. We use six sets of parameter estimates, corresponding to the results reported in Table 3. In each panel, we report the time-varying correlation for the two-factor model, as well as the constant correlation for the one-factor model.

Figure 11.A: Conditional 1-Month Return Moments, 1990 Estimates.

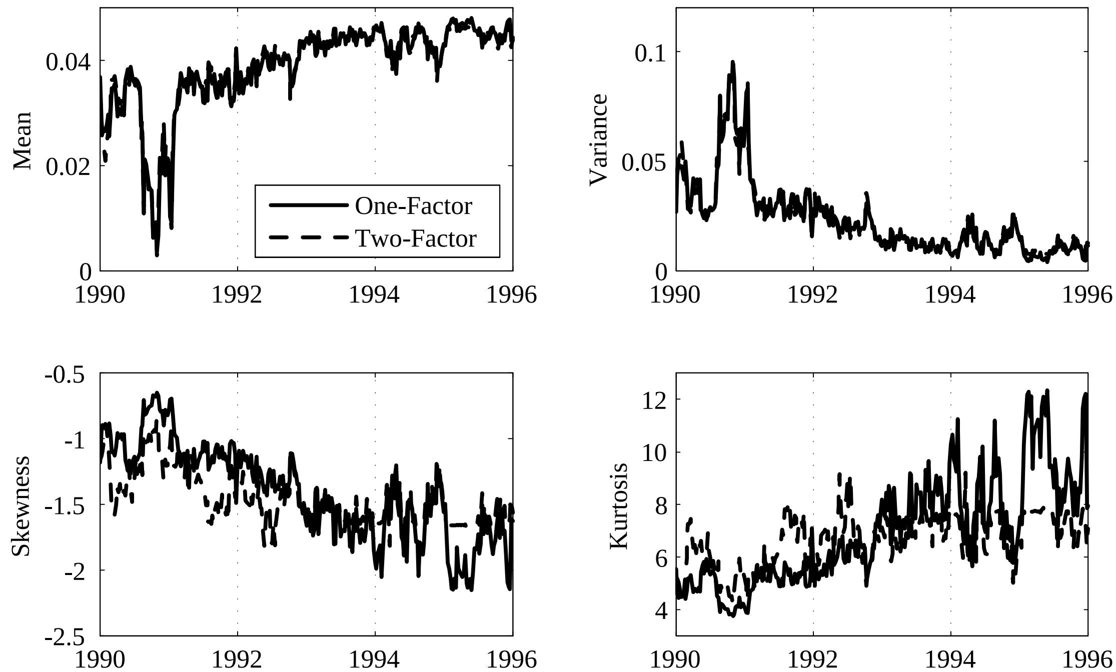
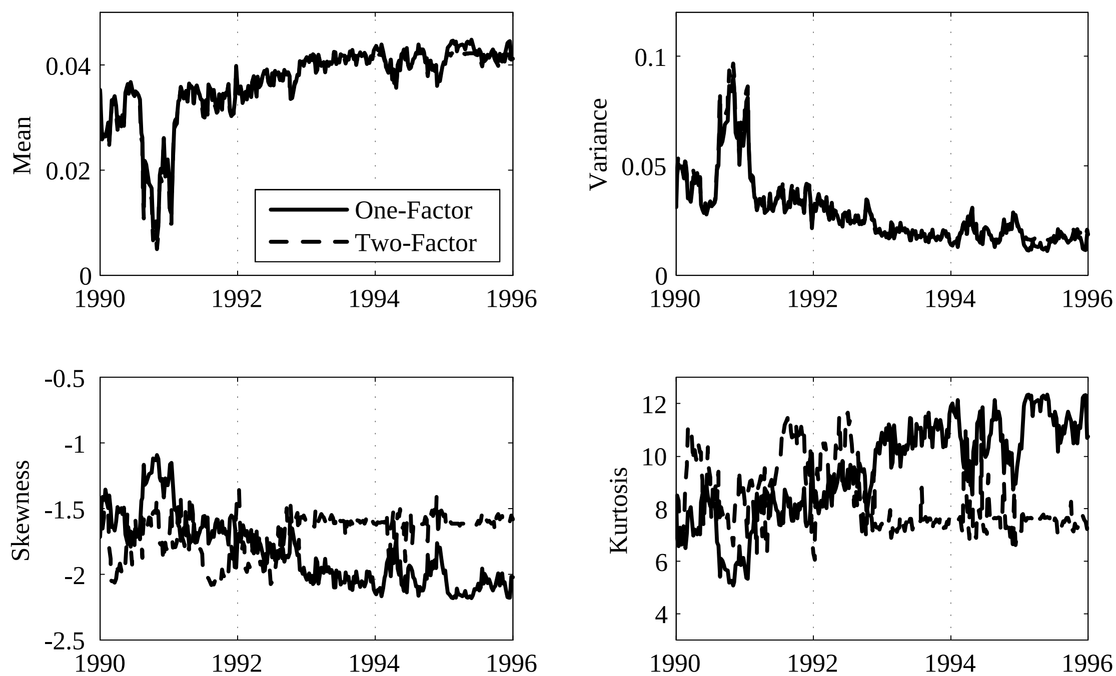


Figure 11.B: Conditional 3-Month Return Moments, 1990 Estimates.



Notes to figure: We plot the first four conditional moments of the log return on the underlying asset in the 1-factor and 2-factor models using the 1990 parameter estimates from Table 3. Panel A shows the 1-month moments and Panel B shows the 3-month moments.

Figure 12.A: Conditional 1-Month Return Moments, 1993 Estimates.

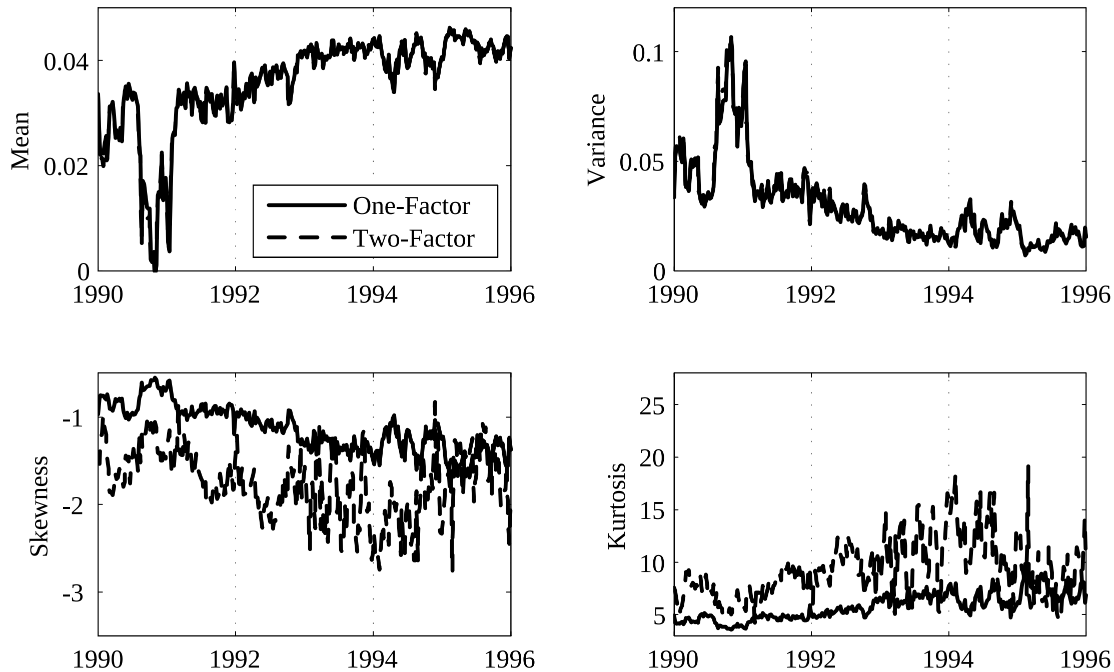
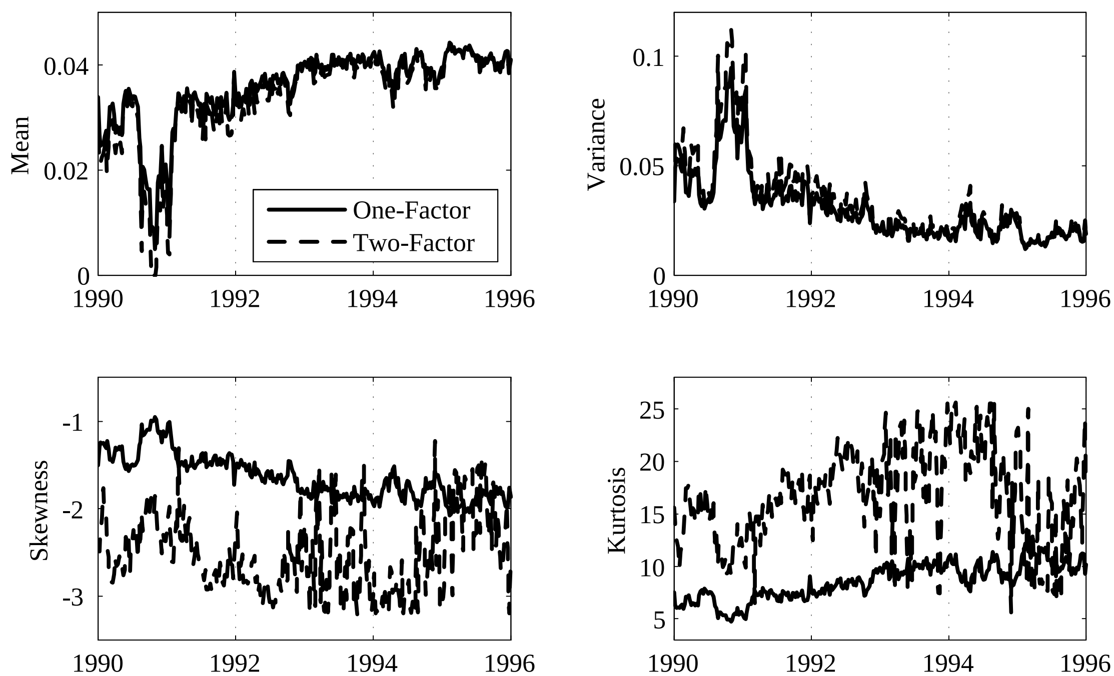


Figure 12.B: Conditional 3-Month Return Moments, 1993 Estimates.



Notes to figure: We plot the first four conditional moments of the log return on the underlying asset in the 1-factor and 2-factor models using the 1993 parameter estimates from Table 3. Panel A shows the 1-month moments and Panel B shows the 3-month moments.

Table 1: S&P500 Index Call Option Data. 1990-1995.

Panel A. Number of Call Option Contracts

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	101	1,884	1,931	1,765	5,681
0.975<S/X<1	283	1,272	706	477	2,738
1<S/X<1.025	300	1,212	726	523	2,761
1.025<S/X<1.05	261	1,167	654	406	2,488
1.05<S/X<1.075	245	1,039	582	390	2,256
S/X>1.075	554	2,353	1,679	1,242	5,828
All	1,744	8,927	6,278	4,803	21,752

Panel B. Average Call Price

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.88	2.30	6.25	11.93	6.61
0.975<S/X<1	2.29	6.83	15.19	27.50	12.12
1<S/X<1.025	8.35	13.60	22.48	34.34	19.29
1.025<S/X<1.05	17.57	22.00	30.11	42.03	26.93
1.05<S/X<1.075	27.11	30.84	38.14	48.83	35.43
S/X>1.075	50.73	52.82	58.98	68.30	57.70
All	24.41	23.69	28.68	36.03	27.91

Panel C. Average Implied Volatility from Call Options

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.1625	0.1266	0.1348	0.1394	0.1340
0.975<S/X<1	0.1308	0.1296	0.1448	0.1562	0.1383
1<S/X<1.025	0.1527	0.1459	0.1558	0.1607	0.1520
1.025<S/X<1.05	0.1914	0.1647	0.1665	0.1657	0.1682
1.05<S/X<1.075	0.2429	0.1828	0.1775	0.1739	0.1864
S/X>1.075	0.3919	0.2359	0.1961	0.1868	0.2288
All	0.2442	0.1700	0.1620	0.1607	0.1716

Panel D. Call Option Characteristics Across Sample Years

	Contracts	Average Price	Average IV
1990	2,857	22.03	0.2153
1991	2,974	22.02	0.1878
1992	3,345	23.13	0.1681
1993	3,578	26.84	0.1578
1994	4,297	29.06	0.1584
1995	4,701	38.40	0.1597

Notes to Table: Our sample consists of European call options written on the S&P500 index. We select contracts quoted within 30 minutes from closing on every Wednesday during the January 1, 1990 to December 31, 1995 period. The moneyness and maturity filters used by Bakshi, Cao and Chen (1997) are applied here as well. The implied volatilities are extracted using the Black-Scholes formula.

Table 2: Principal Component Analysis of Implied Variance.

S/K	Maturity (days)	First	Second	Third	Fourth
		Principal Components			
0.90	30	0.122	-0.083	-0.241	0.162
0.90	60	0.130	-0.105	-0.232	0.129
0.90	90	0.136	-0.124	-0.216	0.106
0.90	180	0.148	-0.147	-0.085	0.136
0.90	270	0.140	-0.056	0.258	0.561
0.95	30	0.161	-0.055	-0.246	0.143
0.95	60	0.167	-0.095	-0.231	0.081
0.95	90	0.171	-0.127	-0.206	0.039
0.95	180	0.172	-0.176	-0.047	0.031
0.95	270	0.150	-0.110	0.285	0.362
1.00	30	0.204	0.031	-0.209	0.110
1.00	60	0.207	-0.043	-0.192	0.007
1.00	90	0.209	-0.101	-0.161	-0.062
1.00	180	0.195	-0.196	0.012	-0.097
1.00	270	0.157	-0.158	0.311	0.181
1.05	30	0.248	0.231	-0.096	0.089
1.05	60	0.249	0.086	-0.090	-0.085
1.05	90	0.246	-0.024	-0.066	-0.189
1.05	180	0.215	-0.200	0.092	-0.237
1.05	270	0.162	-0.197	0.329	0.025
1.10	30	0.289	0.651	0.145	0.164
1.10	60	0.288	0.355	0.104	-0.143
1.10	90	0.280	0.146	0.097	-0.309
1.10	180	0.232	-0.179	0.188	-0.368
1.10	270	0.165	-0.226	0.332	-0.089
Explained by PC:		88.486%	7.122%	2.231%	1.412%

Notes to Table: We report factor loadings and percentage of variance explained by the first four principal components. The principal component analysis is performed using implied variances for each Wednesday during 1990-1995. In a first stage, a quadratic polynomial in maturity and moneyness is fit for each Wednesday, using data for all available moneyness and maturity. In a second stage, these estimates are used to generate a variance surface with standardized moneyness and maturity. The principal component analysis is performed on this standardized variance surface.

Table 3: Parameter Estimates and In-sample Fit.

Estimation Year and Model	First Volatility Factor				Second Volatility Factor				In Sample Fit	
	b_1	a_1/b_1	σ_1	ρ_1	b_2	a_2/b_2	σ_2	ρ_2	RMSE	Ratio
1990										
One Factor	1.9970	0.0492	0.6828	-0.6935					0.621	1.000
Two Factors	0.2736	0.0228	0.9071	-0.7532	8.2885	0.0256	0.6500	-0.8370	0.463	0.746
Persistent Factor			0.9371	-0.7430	7.2191	0.0263	0.6164	-0.8096	0.464	0.747
1991										
One Factor	2.4278	0.0447	0.6785	-0.6259					0.541	1.000
Two Factors	0.2980	0.0185	0.9440	-0.7772	6.9815	0.0266	0.6097	-0.6941	0.385	0.712
Persistent Factor			0.8852	-0.8051	5.4347	0.0271	0.5391	-0.5908	0.385	0.712
1992										
One Factor	2.6952	0.0393	0.6271	-0.6502					0.466	1.000
Two Factors	0.1548	0.0450	0.7177	-0.7010	5.0379	0.0224	0.4759	-0.7734	0.400	0.857
Persistent Factor			0.7025	-0.7605	4.7821	0.0245	0.4828	-0.6852	0.408	0.875
1993										
One Factor	2.8153	0.0299	0.6973	-0.6189					0.654	1.000
Two Factors	0.2997	0.0209	1.1956	-0.7510	5.8516	0.0190	0.4714	-0.8081	0.598	0.914
Persistent Factor			0.9834	-0.7419	6.4326	0.0193	0.5008	-0.8304	0.606	0.926
1994										
One Factor	2.8420	0.0311	0.6970	-0.6245					0.717	1.000
Two Factors	0.2498	0.0057	1.4967	-0.8047	3.3116	0.0255	0.4093	-0.7920	0.643	0.897
Persistent Factor			1.4967	-0.8047	3.3079	0.0254	0.4083	-0.7908	0.643	0.897
1995										
One Factor	2.9598	0.0243	0.6087	-0.5991					0.514	1.000
Two Factors	0.2563	0.0040	1.0720	-0.8084	1.8817	0.0233	0.2875	-0.6997	0.421	0.818
Persistent Factor			0.9425	-0.8456	3.0306	0.0205	0.3515	-0.7167	0.453	0.881

Notes to Table: Each model is estimated year-by-year using the Wednesday closing option quotes from Table 1. The structural parameters reported above and the weekly spot volatilities are estimated using the iterative two-step method outlined in Section 3. The in sample root mean squared errors (RMSE) are calculated on the dollar difference between the market price and model price for each option. The Ratio RMSE is calculated by normalizing each RMSE with the RMSE from the one-factor model.

Table 4: In and Out of Sample RMSE.

A. One Factor

Evaluation Year	Estimation Year 1990	1991	1992	1993	1994	1995	Average Out of S.	Out by In Ratio	Average Out vs. 1F
1990	0.621	0.657	0.698	0.871	0.828	1.002	0.820	1.322	1.000
1991	0.580	0.541	0.575	0.846	0.781	0.990	0.771	1.425	1.000
1992	0.561	0.516	0.466	0.685	0.611	0.852	0.656	1.406	1.000
1993	0.998	1.033	0.895	0.654	0.680	0.708	0.875	1.337	1.000
1994	0.970	1.004	0.859	0.740	0.717	0.884	0.896	1.249	1.000
1995	1.416	1.459	1.222	0.634	0.743	0.514	1.147	2.231	1.000

B. Two Factors

Evaluation Year	Estimation Year 1990	1991	1992	1993	1994	1995	Average Out of S.	Out by In Ratio	Average Out vs. 1F
1990	0.463	0.479	0.508	0.673	0.703	0.675	0.615	1.328	0.749
1991	0.412	0.385	0.426	0.703	0.678	0.695	0.598	1.552	0.776
1992	0.450	0.470	0.400	0.606	0.569	0.687	0.563	1.410	0.859
1993	0.972	1.042	0.744	0.598	0.605	0.617	0.816	1.365	0.933
1994	0.923	1.005	0.740	0.657	0.643	0.773	0.829	1.289	0.925
1995	1.445	1.478	0.877	0.575	0.664	0.421	1.078	2.563	0.940

C. Persistent Factor

Evaluation Year	Estimation Year 1990	1991	1992	1993	1994	1995	Average Out of S.	Out by In Ratio	Average Out vs. 1F
1990	0.464	0.486	0.502	0.568	0.689	0.666	0.588	1.269	0.717
1991	0.414	0.385	0.422	0.615	0.676	0.729	0.586	1.521	0.760
1992	0.425	0.439	0.408	0.554	0.575	0.704	0.549	1.346	0.837
1993	0.861	0.912	0.735	0.606	0.599	0.606	0.754	1.243	0.861
1994	0.823	0.917	0.751	0.659	0.643	0.723	0.779	1.211	0.870
1995	1.196	1.148	0.811	0.564	0.635	0.453	0.909	2.006	0.792

Notes to Table: For each model we use the year-by-year parameter estimates from Table 3 to compute model option prices for each of the six years in our sample. The resulting in-sample RMSEs are reported on the diagonal in grey boxes. The off-diagonal numbers are out-of-sample RMSEs. The "Average Out of S." column reports the overall average out-of-sample RMSE for each estimate. The "Out by In Ratio" column reports the out-of-sample over in-sample RMSE. The "Average Out vs 1F." column reports the average out-of-sample RMSE normalized by the one-factor model.

Table 5: Average Out of Sample RMSE: SV Versus Ad Hoc Models.

		Stochastic Volatility Models			Benchmarks		
		1 Factor	**2 Factors**		Black-	Ad Hoc	Ad Hoc
			General	Persistent	Scholes	OLS	NLS
Out of Sample	1990	0.8204	0.6149	0.5882	1.5828	0.7827	0.3161
	1991	0.7709	0.5979	0.5859	1.4408	0.8555	0.3002
	1992	0.6555	0.5634	0.5489	1.4724	1.1172	0.3348
	1993	0.8748	0.8164	0.7536	1.6027	1.6959	0.5731
	1994	0.8961	0.8293	0.7795	1.9282	1.4616	0.5801
	1995	1.1471	1.0783	0.9087	1.5081	1.6155	0.3901
	Overall	0.8970	0.8026	0.7284	1.6108	1.3534	0.4428
	Ratio to 1F	1.0000	0.8947	0.8121	1.7958	1.5088	0.4937
In Sample	Overall	0.5943	0.5018	0.5106	1.6108	1.3534	0.4428
	Ratio to 1F	1.0000	0.8444	0.8590	2.7103	2.2771	0.7450

Notes to Table: Each row reports the out of sample RMSE in a particular evaluation year for each model averaging across five estimation years. The "Overall" row reports the average across the six evaluation years. The "Ratio" row normalizes the Overall RMSE for each model by the Overall RMSE for the one-factor model. The Black-Scholes benchmark is calculated using a different volatility each week but keeping that volatility constant across the contracts observed in a given week. The Ad Hoc OLS benchmark is calculated as in Dumas, Fleming and Whaley (1998) by regressing implied volatility on a second order polynomial in the strike price and maturity. The fitted values from the regression are plugged into the Black-Scholes formula to calculate the model price. The Ad Hoc NLS implements the Ad Hoc model using NLS as in Christoffersen and Jacobs (2004).

Table 6: RMSE and Ratios by Moneyness and Maturity. 1990-1995. In-Sample.

Panel A. RMSE from One-Factor Model.

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.4755	0.5000	0.4032	0.6124	0.5086
0.975<S/X<1	0.4912	0.5036	0.3919	0.6293	0.5013
1<S/X<1.025	0.4603	0.4767	0.3905	0.5942	0.4793
1.025<S/X<1.05	0.5192	0.4469	0.3693	0.6678	0.4802
1.05<S/X<1.075	0.7012	0.5116	0.4212	0.8173	0.5790
S/X>1.075	0.8811	0.7807	0.5679	0.9886	0.7881
All	0.6669	0.5805	0.4492	0.7485	0.5966

Panel B. RMSE Ratio: Two-Factor Model over One-Factor Model

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.6848	0.6318	0.7696	0.8379	0.7601
0.975<S/X<1	0.8453	0.7605	0.8523	0.7740	0.7882
1<S/X<1.025	0.9122	0.8544	0.8822	0.7997	0.8500
1.025<S/X<1.05	0.9270	0.9112	0.9307	0.8744	0.9048
1.05<S/X<1.075	0.9440	0.9161	0.8792	0.8003	0.8775
S/X>1.075	0.9922	0.8759	0.8445	0.8189	0.8674
All	0.9523	0.8350	0.8402	0.8214	0.8436

Panel C. RMSE Ratio: Persistent-Factor Model over One-Factor Model

	DTM<20	20<DTM<80	80<DTM<180	DTM>180	All
S/X<0.975	0.6761	0.6174	0.7894	0.8674	0.7755
0.975<S/X<1	0.8496	0.7558	0.8706	0.7923	0.7945
1<S/X<1.025	0.9279	0.8704	0.8932	0.8164	0.8652
1.025<S/X<1.05	0.9584	0.9403	0.9542	0.8771	0.9252
1.05<S/X<1.075	0.9638	0.9398	0.8879	0.8026	0.8917
S/X>1.075	0.9955	0.8968	0.8556	0.8164	0.8772
All	0.9616	0.8496	0.8546	0.8305	0.8558

Notes to Table: We use the parameter estimates in Table 3 to compute the root mean squared option valuation error (RMSE) for various moneyness and maturity bins. Panel A reports the RMSE for the One-Factor model. Panel B reports the ratio of the RMSE from the Two-factor model to the RMSE from the One-Factor Model. Panel C reports the ratio of the RMSE from the Persistent Factor model to the RMSE from the One-Factor model.

Table 7: Correlation Between the Volatility Level and the Slope of the Volatility Smirk.

Panel A: Raw Moneyness

	1990	1991	1992	1993	1994	1995
Data	16%	45%	14%	26%	31%	-36%
One-Factor Model	-54%	-37%	-49%	-19%	-36%	-56%
Two-Factor Model	33%	30%	-36%	18%	-6%	-45%

Panel B: Moneyness Normalized by Maturity

	1990	1991	1992	1993	1994	1995
Data	-21%	19%	-8%	-35%	3%	-50%
One-Factor Model	-76%	-70%	-67%	-61%	-71%	-84%
Two-Factor Model	-40%	1%	-55%	-29%	-37%	-67%

Notes to Table: We compute the correlation between the volatility level and the slope of the smirk on a year-by-year basis. The volatility level and the slope are obtained by regressing implied volatilities on a measure of moneyness. In Panel A, implied volatilities are regressed on simple log moneyness. In Panel B, implied volatilities are regressed on log moneyness normalized by maturity. In the first row of each panel, the regression is performed on the raw data. In the second row, model option prices from the one-factor model are used. In the third row, model option prices from the two-factor model are used.

Table 8: Absolute Correlation of Variance Factors with Volatility Level and Slope of the Smirk.

One-Factor Model

	1990	1991	1992	1993	1994	1995
Level	99%	98%	98%	87%	97%	79%
Smirk	19%	23%	6%	5%	14%	7%

Two-Factor Model, Factor 1

	1990	1991	1992	1993	1994	1995
Level	90%	84%	53%	50%	45%	8%
Smirk	20%	29%	23%	26%	47%	66%

Two-Factor Model, Factor 2

	1990	1991	1992	1993	1994	1995
Level	17%	5%	38%	1%	65%	61%
Smirk	5%	18%	25%	30%	28%	66%

Two-Factor Model, Multiple Correlation

	1990	1991	1992	1993	1994	1995
Level	99%	99%	98%	88%	98%	83%
Smirk	20%	29%	28%	30%	48%	74%

Notes to Table: On a year-by-year basis, we compute the absolute correlation between the time-series of the variance factors and the level and slope of the smirk obtained by regressing implied volatilities on moneyness. Moneyness is normalized by maturity. For the two-factor model, we compute absolute correlations with each of the variance factors, and we also compute the multiple correlation coefficient obtained by regressing either level or slope of the smirk on both factors.

Table 9: Absolute Correlation of Variance Factors with Volatility Level and At-the-Money Term Structure Slope.

One-Factor Model

	1990	1991	1992	1993	1994	1995
Level	92%	92%	92%	79%	92%	70%
Term Structure Slope	28%	44%	55%	47%	66%	5%

Two-Factor Model, Factor 1

	1990	1991	1992	1993	1994	1995
Level	76%	72%	24%	18%	33%	34%
Term Structure Slope	1%	11%	23%	10%	15%	5%

Two-Factor Model, Factor 2

	1990	1991	1992	1993	1994	1995
Level	38%	22%	64%	35%	70%	26%
Term Structure Slope	77%	70%	79%	48%	59%	4%

Two-Factor Model, Multiple Correlation

	1990	1991	1992	1993	1994	1995
Level	98%	97%	97%	89%	94%	67%
Term Structure Slope	81%	86%	84%	70%	71%	5%

Notes to Table: On a year-by-year basis, we compute the absolute correlation between the time-series of the variance factors and the level and slope of the at-the-money term structure obtained by regressing implied volatilities on maturity. Only contracts with moneyness between 0.97 and 1.03 are used in the regressions. For the two-factor model, we compute absolute correlations with each of the variance factors, and we also compute the multiple correlation coefficient obtained by regressing either level or slope of the term structure on both factors.