Online Inventory Constrained Trading in Monopoly and Oligopoly Markets

Paper #: 1570386480

***Abstract*—This paper is concerned with an inventory constrained trading problem: given an initial inventory, how to maximize the overall revenue in a trading period by dynamically adjusting the selling quantity in individual slots. We assume the trading price is a linear function of a slot-dependent base price and overall selling quantity. This problem is motivated by the possibility of trading spectrum with flexible use rights, although it may also arise in other trading scenarios. We first consider a monopoly market, in which a single firm sells goods with no close substitute. We propose optimal algorithms for both offline and online settings. The online algorithm is $4\theta^2/(4\theta-1)$-competitive, where θ is the ratio between the maximum and minimum base prices. We show that this competitive ratio is optimal among all deterministic stationary memoryless online algorithms. We then consider an oligopoly market, in which the selling firms compete to maximize their revenues. We model their strategic interactions as an N-person non-cooperative game. In the offline setting, we characterize the Nash Equilibrium and Loss of Revenue, defined as the largest achievable revenue ratio between the coordinated and strategic settings among all possible inputs. In the online setting, perhaps surprisingly, we show that the revenue in the online game can sometimes exceed that in the offline game. This effect is a type of Informational Braess' Paradox, and suggests that in game-theoretic setting, acquiring future information can make competitive sellers worse off.**

I. Introduction

Monopoly and oligopoly are market structures developed to model imperfect competition. In a monopoly market, a single firm sells goods with no close substitute. In an oligopoly market, a small number of relatively large firms produce undifferentiated products. In both markets, the firms' selling quantity has an impact on the market price.

We consider an *inventory constrained selling problem* in which a group of sellers (firms) wish to sell a given quantity of a particular good. Time is divided into slots of equal length, and the firms try to maximize their revenue over a given trading period (spanning multiple time slots) by dynamically adjusting the selling quantity in each slot. The sellers influence the market price, and for tractability we will assume that the market price is a linear function of a slot-dependent base price and total quantity sold. We consider scenarios with a single seller (monopoly market) and a small number of competitive sellers (oligopoly market).

A particular example that motivates our study is the trading of spectrum assets in a spectrum market. Such markets for allocating spectrum across different applications and locations would likely arise pending increased allocations of licensed spectrum with flexible use rights. The inventory constrained trading problem then arises when a spectrum licensee with a large amount of spectrum in a given region would like to sell, or make contracts to lease its spectrum at some designated future time. Here the time scale over which the trading occurs could conceivably vary dramatically, i.e., from seconds, corresponding to a spot market that responds to short-term traffic variations, to days corresponding to long-term traffic variations associated with special events (e.g., concert or sporting event). Another example is an energy storage unit participating in an electricity market, which obtains energy from renewables/the grid during a low-price period (e.g., nighttime) and tries to sell back the electricity during a high-price period (e.g., daytime) to maximize the revenue.

We study both "offline" and "online" versions of this problem. In the offline setting, the trading period and market parameters are common knowledge to market participants. In contrast, in the online setting, the trading period is unknown and the market parameters are revealed at the beginning of each individual time slot. We do not assume any stochastic knowledge of the trading period and market parameters in the online setting.

In the offline setting, we characterize the inventory constrained trading problem in a monopoly market as a convex optimization problem. An oligopoly market is cast as a non-cooperative game, and the problem is then to find the Nash equilibrium. These problems can be readily solved. Difficulty arises when we consider the online problem. In the monopoly market, the firm has to decide on the selling quantity without knowledge of future market parameters. This becomes even more challenging in an oligopoly market, where the problem becomes an online game – a setting which so far has not received much attention. The firms face uncertainty not only from the future market parameters but also from their competitors.

We conduct a comprehensive study on this problem and make the following contributions (summarized in Table I):

▷ In a monopoly market, we formulate the inventory constrained trading problem as a quadratic programming problem with packing constraints. In an oligopoly market, we model the strategic interaction among firms in the market as a game. This formulation is general and can be applied to various problem domains, including spectrum and electricity markets as the motivating examples for our study.

▷ For a monopoly market, in the offline setting where the trading period and market parameters are given ahead of time, we propose an efficient dual-based algorithm to solve the problem. In the online setting where neither the trading period nor market parameters are known beforehand, we propose a quasi-greedy online algorithm with competitive ratio of $4\theta^2/(4\theta-1)$, where θ is the ratio between the maximum and the minimum base prices. We show that this competitive ratio is optimal among all deterministic stationary memoryless online algorithms[1].

▷ For an oligopoly market, in the offline setting, we show the existence and uniqueness of Nash Equilibrium (NE) in the N-person non-cooperative game. We show that the Loss of Revenue (LoR, the largest achievable revenue ratio between the coordinated and strategic settings among all the possible inputs) is $(N+1)^2/4N$, where N is the number of firms. In the online setting, perhaps surprisingly, we show that revenue in the online game can exceed that in the offline game at the NE. We provide a set of sufficient conditions for having this counter-intuitive observation. This suggests that in the game-theoretic setting, knowing future information may not always be beneficial in terms of revenue, which is a type of Informational Braess' Paradox (IBP) [1].

▷ We evaluate the performance of the proposed online algorithm and illustrate our theoretical results by extensive numerical experiments. Simulation results show that our quasi-greedy online algorithm can achieve satisfactory performance as compared to the offline optimum in monopoly market. For example, when $\theta = 10$, our quasi-greedy online algorithm achieves an average empirical revenue ratio of 1.8, which is much smaller than its competitive ratio. This is reasonable as competitive ratio measures the worst case performance, thus we may observe a better revenue ratio in general cases. Simulation results in oligopoly market verify the IBP discussed in our theoretical study. While IBP appears in special cases, our simulation results show that typically the revenue in online game is less than that in offline game, especially for large θ. For example, when $\theta = 10$, the average revenue in the offline game is 5 times of that in the online game.

The rest of paper is organized as follows. Related work is presented in Sec. II. Sec. III presents the system model and problem formulation. Monopoly and oligopoly markets are studied in Secs. IV and V, respectively. Numerical results are presented in Sec. VI. Sec. VII concludes the paper.

Due to the space limitation, all proofs are in our technical report [2], unless specified otherwise.

[1] An algorithm is *stationary and memoryless* if each seller's decision is a time-invariant function of the market parameters at the current epoch.

TABLE I
Summary of Contributions

	Monopoly Market	Oligopoly Market
Offline setting	Propose optimal dual-based algorithm	Establish uniqueness of NE; show LoR is $\frac{(N+1)^2}{4N}$
Online setting	Propose online algorithm with CR of $\frac{4\theta^2}{4\theta-1}$	Characterize sufficient conditions to observe IBP

II. Related work

This paper connects to, and builds on, existing work in four related areas: (i) Dynamic Pricing in Revenue Management; (ii) Online One-way Trading; (iii) Network Cournot Competition; (iv) Competition with Incomplete Information.

The inventory constrained trading problem belongs to the class of single-product dynamic pricing problems, which have been extensively studied in the operations research and management science [3], [4]. In dynamic pricing, given initial inventories, firms dynamically adjust the price over a finite horizon to maximize their expected revenue, where the demand is modelled through a demand function. There are extensive studies on monopolistic and oligopolistic dynamic pricing models [5]. In contrast to the common assumption on the distributional information of the demand in the dynamic pricing literature, we do not assume any stochastic model for the demand/price in our study of the online setting. There is also an extensive literature on dynamic pricing when the underlying demand function is unknown and the policy is designed to strike a balance between demand learning and pricing, see [6] and the related work therein. Our paper, however, assume that the demand function is common knowledge to market participants.

Our online problem in a monopoly market is closely related to the one-way trading problem[2] studied by El-Yaniv et al. [7], where they propose a threshold-based online algorithm with competitive ratio $O(\ln\theta)$. In that problem, any remaining items must be sold at the last epoch. In contrast, we do not impose such a constraint and the online trading period ends without advance notice to the firms in the online setting. Beside [7], variants of the one-way trading problem are studied in the literature. Zhang et al. [8] study the problem when every two consecutive prices are interrelated and propose optimal deterministic online algorithm. Fujiwara et al. [9] study the problem using average-case competitive analysis under the assumption that the distribution of the maximum exchange rate is known. We note that our problem differs from the original one-way trading problem in two aspects: firstly, the trading period is not known to the firm; secondly, the market price is affected by the total quantity sold.

[2] In the one-way trading problem, a firm is asked to exchange some initial wealth given in some currency, for some asserts or other currency within a certain time period.

The offline game considered in an oligopoly market is similar to Network Cournot Competition (NCC), which has gained substantial attention recently [10], [11], [12], [13]. The NCC is developed to model oligopolistic competition among firms across several different markets. Abolhassani et al. [13] propose algorithms for finding the NE of NCC games in different situations. Pang et. al. [12] study the Price of Anarchy in the context of NCC. Our offline problem in oligopoly market can be viewed as the NCC where each epoch corresponds to one market and firms participate in every market. We note that the inventory constraint in the problem distinguishes our game from the original NCC, which can be viewed as adding another constraint to the production limit of one firm across different markets.

The closest work to the online game in oligopoly market is on games with incomplete information, where the players do not know with certainty some parameters of the game, such as the other players' strategy sets and utility functions. Rather, prior distributions are usually assumed and the game is analysed by determining Bayesian Nash Equilibria [14]. Other related work is on the IBP studied in congestion games [1], where the authors find that in some networks, players receiving additional information can become worse off. This observation shares some similarity with the counterintuitive effect we observed in the online game.

III. System Model and Problem Formulation

Model and Settings. We first consider the monopoly scenario in which the firm has an inventory Δ to sell within a trading period T. The trading period is divided into slots with equal length, denoted as $\mathcal{T} = \{1, 2, ..., T\}$. In each time slot, or epoch t, the firm decides on the selling quantity, denoted as $v(t)$. For ease of discussion, we denote $\boldsymbol{v} \triangleq (v(1), v(2), ..., v(T))^T$ as the decision vector of the firm during the trading period.

In oligopoly market, there are N firms in the market and each firm has some inventory Δ_i to sell. The selling quantity for firm i at epoch t is denoted as $v_i(t)$. As before, we denote $\boldsymbol{v}_i \triangleq (v_i(1), v_i(2), ..., v_i(T))^T$ as the decision vector of firm i during the trading period. Denote $v_{-i}(t)$ as the aggregate supply from all the firms except firm i at epoch t. Similarly, we denote $\boldsymbol{v}_{-i} \triangleq (v_{-i}(1), v_{-i}(2), ..., v_{-i}(T))^T$. Note that $\boldsymbol{v}, \boldsymbol{v}_i, \boldsymbol{v}_{-i} \in \mathbb{R}^T$.

Note that with the definition of

$$v(t) \triangleq \sum_{i=1}^{N} v_i(t), \quad \Delta \triangleq \sum_{i=1}^{N} \Delta_i,$$

the problem in monopoly market is equivalent to the problem in oligopoly market when firms coordinate to maximize their total revenue.

Price Function. The *effective price* at epoch t, denoted as $p_e(t)$, is a linear function of the *base price* $p_b(t)$ and the aggregate supply at epoch t, denoted by $v(t)$:

$$p_e(t) \triangleq p_b(t) - \alpha_t v(t), \forall t \in \mathcal{T}.$$

Here $\alpha_t \geq 0$ is the *market coefficient*, which captures the linear relation between the effective selling price of the goods on the market and the firms' selling quantities. When $\alpha_t = 0$, the firms are price takers, which models the market with perfect competition as the market participants (firms) have no power over the market price. When $\alpha_t > 0$, the firms are price makers, which is characteristic of monopoly and oligopoly markets. We call $p_b(t)$ and α_t the *market parameters*. Note that this price function belongs to the class of affine inverse demand function, which is commonly studied in economics [15].

Payoff Function. In monopoly market, the firm determines its selling quantity $v(t)$ at epoch t and receives a revenue of $p_e(t)v(t)$. Thus the payoff function for the firm is the sum of revenue in all epochs, which is given by

$$\pi(\boldsymbol{v}) \triangleq \sum_{t=1}^{T} (p_b(t) - \alpha_t v(t))\, v(t).$$

In oligopoly market, firm i sells $v_i(t)$ amounts of goods at epoch t and receives a revenue of $p_e(t)v_i(t)$. Thus the payoff function for firm i is given by

$$\pi_i(\boldsymbol{v}_i, \boldsymbol{v}_{-i}) \triangleq \sum_{t=1}^{T} (p_b(t) - \alpha_t v(t)) \cdot v_i(t).$$

Note that the payoff function for firm i also depends on the decision vector of the other firms $\boldsymbol{v}_{-i}$, since the market price is affected by the aggregate supply in the market.

Problem Formulation. In monopoly market, there is no competition and the single firm wishes to solve the following optimization problem:

$$\begin{aligned} \mathit{OptSel}: \max \quad & \pi(\boldsymbol{v}), & \text{(1a)} \\ s.t. \quad & \sum_{t=1}^{T} v(t) \leq \Delta, & \text{(1b)} \\ var. \quad & v(t) \geq 0, \forall t \in \mathcal{T}, & \text{(1c)} \end{aligned}$$

where (1b) is the inventory constraint. It is straightforward to verify that this is a convex optimization problem.

In oligopoly market, for each firm, for example firm i, given the other firms' decision vector $\boldsymbol{v}_{-i}$, firm i aims to maximize its revenue by solving the following optimization problem:

$$\begin{aligned} \mathit{GameSel}_i: \max \quad & \pi_i(\boldsymbol{v}_i, \boldsymbol{v}_{-i}), & \text{(2a)} \\ s.t. \quad & \sum_{t=1}^{T} v_i(t) \leq \Delta_i, & \text{(2b)} \\ var. \quad & v_i(t) \geq 0, \forall t \in \mathcal{T}. & \text{(2c)} \end{aligned}$$

Note that $\mathit{GameSel}_i$, $i = 1, 2, ..., N$, defines an N-person non-cooperative game among the firms. Hereafter we denote the game as $\mathcal{G}$. In the following, we study problem *OptSel* and game $\mathcal{G}$ in Sec. IV and Sec. V, respectively.

IV. Monopoly Market

In this section, we focus on monopoly market and study problem *OptSel* in both offline and online settings.

Algorithm 1 Binary search algorithm for λ^*

if $\sum_{t\in\mathcal{T}} \frac{p_b(t)}{2\alpha_t} \leq \Delta$ **then return** $\lambda^* = 0$
else
  Pick $\lambda_L = 0$, $\lambda_H = \max_{t\in\mathcal{T}}(p_b(t))$
  while $|\lambda_L - \lambda_H| > \epsilon$ **do**
    $\lambda_M = \frac{\lambda_L+\lambda_H}{2}, v(t) = 0, \forall t \in \mathcal{T}$
    for each $t \in \mathcal{T}$ **do**
      Compute $v(t)$ according to (3)
    end for
    if $\sum_{t\in\mathcal{T}} v > \Delta$ **then**
      $\lambda_L = \lambda_M$
    else
      $\lambda_H = \lambda_M$
    end if
  end while
  return $\lambda^* = \lambda_M$
end if

A. Offline Setting

In the offline setting, $T, p_b(t), \alpha_t, \forall t \in \mathcal{T}$ are known in advance to the single firm. We propose a dual-base algorithm to obtain the optimal revenue and study how the revenue changes as the initial inventory increases.

We observe that in the offline setting, problem *OptSel* is a convex problem with packing constraint. By investigating the KKT conditions of problem *OptSel* and exploring the structure of the optimal solution, we propose a binary-search based algorithm in Alg. 1 to obtain the dual solution with optimized convergence rate. The optimality of the solution is stated in the following theorem.

Theorem 1. *Denote λ^* as the optimal dual solution of problem OptSel, then λ^* can be obtained by the binary search algorithm in Algorithm 1 and the optimal primal solution can be expressed as*

$$v^*(t) = \frac{[p_b(t) - \lambda^*]^+}{2\alpha_t}, \forall t \in \mathcal{T}. \tag{3}$$

Here the Lagrange multiplier λ^* can be interpreted as the marginal cost of the inventory. Note that when $v(t) > 0$,

$$\frac{\partial \pi(\boldsymbol{v})}{\partial v(t)} = p_b(t) - 2\alpha_t v(t)$$

is the marginal revenue at epoch t. Thus (3) essentially implies that the marginal revenue equals to the marginal cost (i.e., λ^*) in epochs that the selling quantity is positive.

Define $Obj(\Delta)$ as the optimal objective value of problem *OptSel* given the total inventory Δ. The following corollary states that $Obj(\Delta)$ is concave in Δ.

Corollary 1. *$Obj(\Delta)$ is a concave increasing function when $\Delta \in [0, \Delta_0)$, and it is constant when $\Delta \in [\Delta_0, +\infty)$, where $\Delta_0 = \sum_{t\in\mathcal{T}} \frac{p_b(t)}{2\alpha_t}$.*

Corollary 1 implies that there is a diminishing return in the revenue w.r.t. Δ. Intuitively, this is because that the additional revenue gained by increasing the inventory becomes less significant if the inventory is large enough.

B. Online Setting

In this section, we consider the problem in the online setting. We propose two online algorithms and study their performance in terms of competitive ratio. To demonstrate the superiority of our proposed algorithms, we compare their performance against all deterministic stationary memoryless online (DSMO) algorithms and deterministic threshold-based online (DTO) algorithms.

Online Settings: In the online setting, the trading period is unknown to the firms and the market parameters $(p_b(t), \alpha_t)$ in a particular epoch are not known until the beginning of that epoch. Upon observing the market parameters, the firm has to make an irrevocable decision on the selling quantity, with the objective of maximizing its overall revenue while respecting the inventory constraint. We assume that $p_b(t) \in [p_{min}, p_{max}], \forall t \in \mathcal{T}$, where p_{min} and p_{max} are the minimum and maximum base prices, respectively, and they are known to the firm beforehand. We denote $\theta = p_{max}/p_{min}$ as the ratio between the maximum and minimum base prices and later we will use θ to analyze our algorithms.

Competitive Ratio: Denote a deterministic online algorithm as $\mathcal{A}$. Then $\mathcal{A}$ is called c-competitive if

$$c = \max_{\sigma\in\Sigma} \frac{\eta_{OPT}(\sigma)}{\eta_{\mathcal{A}}(\sigma)},$$

where Σ is the set of all possible inputs $(T, p_b(t), \alpha_t, t \in \mathcal{T})$, and $\eta_{OPT}(\sigma)$ and $\eta_{\mathcal{A}}(\sigma)$ are the revenues generated by the optimal algorithm OPT and the online algorithm $\mathcal{A}$, respectively. We call c as the competitive ratio (CR) of algorithm $\mathcal{A}$. Online algorithm design is often thought of as a two-player game between an adversary who controls σ to maximize the ratio $\frac{\eta_{OPT}(\sigma)}{\eta_{\mathcal{A}}(\sigma)}$ and the algorithm designer who tries to minimize the ratio.

Remark: When $\alpha_t = 0, \forall t \in \mathcal{T}$, the firm becomes a price taker. This price-taker scenario is known as the one-way trading problem and has been well studied in the literature. In [16], [7], the authors propose threshold-based online algorithm with CR of $O(\ln\theta)$ and they show that threshold-based online algorithm is optimal among all deterministic online algorithms. Since the price-taker scenario is a special case of our problem, we note that the CR of any deterministic online algorithm in monopoly market is at least $O(\ln\theta)$. Next, we focus on the general case where $\alpha_t \geq 0, \forall t \in \mathcal{T}$.

Greedy Online Algorithm: Note that in each epoch, $v(t) = \frac{p_b(t)}{2\alpha_t}$ maximizes the revenue in the current epoch and the effective price is

$$p_e(t) = p_b(t) - \alpha_t v(t) = \frac{p_b(t)}{2}.$$

Inspired by this observation, we design a simple greedy online algorithm: in each epoch $t \in \mathcal{T}$, the algorithm sells an amount of

$$\begin{cases} \min\{\frac{p_b(t)}{2\alpha_t}, r(t)\}, & \alpha_t > 0 \\ r(t), & \alpha_t = 0 \end{cases} \tag{4}$$

where $r(t) \geq 0$ is the residual inventory at epoch t. Since the selling quantity greedily optimizes the revenue in each epoch, we call this algorithm as "greedy" online algorithm.

When $\Delta \geq \sum_{t=1}^{T} \frac{p_b(t)}{2\alpha_t}$, it is straightforward to check that the greedy online algorithm is optimal, i.e., it behaves exactly the same as the offline algorithm. For the general case, the following theorem characterizes the CR of this greedy algorithm.

Theorem 2. *The greedy online algorithm has a CR of* 2θ*.*

Intuitively, as θ increases, there is more uncertainty in the base price, thus we may expect the online algorithm to be more conservative. However, the selling decisions in the greedy online algorithm are not related to θ, which suggests that this algorithm is suboptimal. In the following, we propose another online algorithm whose selling decisions are proportionate to $1/\theta$ and it outperforms the greedy online algorithm in terms of CR.

Quasi-Greedy Online Algorithm: At each epoch $t \in \mathcal{T}$, the online algorithm sells the following amount

$$\begin{cases} \min\{\frac{p_b(t)}{4\theta\alpha_t}, r(t)\}, & \alpha_t > 0 \\ r(t), & \alpha_t = 0 \end{cases} \tag{5}$$

Since this algorithm shares similar structure with the greedy online algorithm, we call this algorithm as "quasi-greedy" online algorithm. The following theorem characterizes the CR of the quasi-greedy online algorithm.

Theorem 3. *The quasi-greedy online algorithm has a CR of* $4\theta^2/(4\theta-1)$*.*

The quasi-greedy online algorithm achieves a better CR than the greedy online algorithm since

$$\theta < \frac{4\theta^2}{4\theta - 1} < 2\theta.$$

When $\theta = 1$ and $p_b(t) = p_b, \forall t \in \mathcal{T}$, i.e., the base price is time-invariant[3], the quasi-greedy online algorithm is 4/3-competitive. This observation suggests that the quasi-greedy online algorithm can be close to optimal in this special case. Meanwhile, the CR is increasing w.r.t. θ, which suggests that larger price fluctuations will degrade the performance of the online algorithm.

In the following, we compare the performance of the quasi-greedy online algorithm with two classes of conceivable online algorithms: DSMO and DTO algorithms. DSMO algorithms make decisions solely based on the market parameters at the current epoch, and does not use any past information. DTO algorithms make decisions according to a threshold function, which has been shown to be optimal among all the deterministic online algorithms when $\alpha_t = 0, \forall t \in \mathcal{T}$ [16].

Deterministic Stationary Memoryless Online Algorithm (DSMO): A DSMO algorithm can be described by a time-invariant function $f : [p_{min}, p_{max}] \times [0, \infty] \to [0, \Delta]$. Namely, the DSMO algorithm takes the current market parameters $p_b(t), \alpha_t$ as inputs, and outputs the selling quantity. Note that both greedy and quasi-greedy online algorithms belong to the class of DSMO algorithms. In the following, we show the optimality of our quasi-greedy online algorithm among all the DSMO algorithms in terms of CR.

Theorem 4. *The quasi-greedy online algorithm achieves the best CR among all DSMO algorithms.*

Deterministic Threshold-based Online Algorithm (DTO): In the price-taker scenario ($\alpha_t = 0, \forall t \in \mathcal{T}$), Yang et al. [16] propose an optimal DTO algorithm with CR $O(\ln\theta)$. Recall that in [16], the threshold function is an non-decreasing function $g(\delta)$: $[0, \Delta] \to [p_{min,} p_{max}]$, where $\delta \in [0, \Delta]$ is the quantity of the items that has already been sold. At epoch t, if the market price $p_b(t)$ is larger than the threshold, the firm sells the quantity $v(t)$ that satisfies: $p_b(t) = g(\delta + v(t))$. It is straightforward to adapt the approach in [16] to design a DTO algorithm when the firm is price maker, i.e., $\alpha_t \geq 0, \forall t \in \mathcal{T}$. However, this seemingly minor difference in the setting, i.e., price maker ($\alpha_t \geq 0$) vs price taker ($\alpha_t = 0$), leads to a substantial difference in the CR, in particular $O(\theta)$ vs $O(\ln\theta)$, as shown in the following theorem.

Theorem 5. *When* $\alpha_t \geq 0, \forall t \in \mathcal{T}$*, the CR of any DTO algorithm is at least* $O(\theta)$*.*

Theorem 5 leads to two observations. Firstly, it reveals the fact that in the price-maker scenario, DTO algorithms perform worse than in the price-taker scenario. Intuitively, this is because in the price-maker scenario, the adversary has more variables (i.e., α_t) to control as compared to the price-taker case, thus it has more degree of freedom to construct worst case input to degrade the performance of online algorithms. Secondly, Theorems 3 and 5 suggest that no DTO algorithms can achieve a CR better than our quasi-greedy algorithm (in the order sense).

V. Oligopoly Market

In this section, we study the strategic behaviors of firms in oligopoly market and consider the game $\mathcal{G}$ formulated in (2) in both offline and online settings.

A. Offline Game

In the offline setting where $T, p_b(t), \alpha_t, \forall t \in \mathcal{T}$ are common knowledge, we establish the existence and uniqueness of Nash Equilibrium (NE) and determine the Loss of Revenue (LoR) of game $\mathcal{G}$.

[3]Note that when $\theta = 1$, the firm still face uncertainty from α_t.

Algorithm 2 Gauss-Seidel best response-based algorithm

```
Set n = 0
Choose any feasible point v^(0) = (v_1^(0), v_2^(0), ..., v_N^(0))
while ||v^{n+1} − v^n||^2 > ϵ do
    for every firm i do
        Compute v_i^(n+1) by solving GameSel_i
        Set v^(n+1) = (v_1^(n+1), ..., v_i^(n+1), v_{i+1}^(n) ..., v_N^(n))
    end for
    n = n + 1
end while
```

1) Existence and Uniqueness of NE: We say $\boldsymbol{v}^* \triangleq (\boldsymbol{v}_1^*, \boldsymbol{v}_2^*, ..., \boldsymbol{v}_N^*)$ constitutes an NE in $\mathcal{G}$, if

$$\pi_i(\boldsymbol{v}_i^*, \boldsymbol{v}_{-i}^*) \geq \pi_i(\boldsymbol{v}_i, \boldsymbol{v}_{-i}^*), \forall i = 1, 2, ..., N.$$

Namely, no firm has the incentive to deviate from the NE.

Remark: We note that game $\mathcal{G}$ is similar to the Network Cournot Competition (NCC) game [11], [12], where each epoch can be considered as a Cournot market and the firms participate in all T markets. However, in our game, every firm has an inventory constraint across different markets, which is not studied in NCC. This unique constraint makes our game more challenging to analyze.

Theorem 6. *The game $\mathcal{G}$ always has a unique NE.*

Theorem 6 is proved by mapping game $\mathcal{G}$ to a variational inequality (VI) problem, and applying VI theory to show the existence and uniqueness of the NE. Since the game has a unique NE, we can use the Gauss-Seidel best response algorithm to obtain the NE. At each iteration of the algorithm, every firm, given the strategies of the others, updates his own strategy by solving its own optimization problem *GameSel*$_i$. The algorithm is formally stated in Alg. 2 for completeness. Building on the VI framework, one can prove that the Gauss-Seidel best response algorithm globally converges to the NE [17]. We skip the details of this standard procedure.

2) Loss of Revenue: In oligopoly market, each firm is selfish and seeks to maximize its own revenue and we expect the total revenue to be smaller than that in the coordinated setting. It is then interesting to examine how the total revenue decreases due to the selfish behaviors of the firms. Towards this end, we define the Loss of Revenue (LoR) as the largest achievable revenue ratio between the coordinated and strategic settings among all the possible inputs.

Theorem 7. *When all N firms in the market have the same inventory, i.e., $\Delta_i = \frac{\Delta}{N}$, the LoR is $\frac{(N+1)^2}{4N}$.*

Theorem 7 characterizes the LoR in the special case when the inventory of firms are the same. This is also the LoR when firms have different inventories with trading period $T = 1$. Note that simulations in Sec. VI suggest that $(N+1)^2/(4N)$ is also the LoR for general cases.

Corollary 2. *For the single-slot N-firm game defined in GameSel$_i$, the LoR is $\frac{(N+1)^2}{4N}$.*

We observe that LoR is $O(N)$. As N goes to infinity, the LoR is unbounded. The intuition can be explained as follows. Each firm in the market has certain amount of items to sell. In the strategic setting, as long as the market price is positive, all firms have incentive to sell since otherwise their payoff is zero. When the number of firms becomes large, the effective price approaches zero as it is linearly decreasing in the aggregate supply. Meanwhile, the total selling quantity in each epoch is upper bounded since the effective price must be positive. Thus the total revenue in the market will be close to zero as well. In contrast, in the coordinated setting, as the number of firms increases, there are more and more inventory in the market. From Corollary 1 we know that the total revenue is non-decreasing in the inventory. Thus the LoR is unbounded when the number of firms goes to infinity.

B. Online Game

Now we consider the online setting of the oligopoly market. In this setting, the firms do not know the trading period T or the market parameters $p_b(t), \alpha_t, \forall t \in \mathcal{T}$ in advance. They only observe the current pricing parameters $(p_b(t), \alpha_t)$ at the beginning of each epoch t. Moreover, each firm knows nothing about their opponents' strategies except their initial inventories.

Since the firms have incomplete information about both the market parameters (i.e., T and $(p_b(t), \alpha_t)$) and their opponents' strategies, the concept of standard NE in the offline game is not appropriate here. We need to define a new equilibrium concept for the online game. We remark that there are few results for such online game settings in the literature. Our contribution in this part is to propose a new type of equilibrium for this online game and show its existence. Next, we introduce our new equilibrium concept.

We first describe each firm's *beliefs* about their opponents, in order to deal with the incomplete information. There are many approaches to form the belief. For example, one approach adopts the worst-case principle, where each firm believes that their opponents try to minimize its own payoff. The resulting equilibrium is called robust NE [18]. Since robust NE is pessimistic, we adopt another concept called *conjectural equilibrium* [19]. In conjectural equilibrium (CE), each firm forms a belief about their opponents' actions based on its own action. Formally, firm i's belief about firm j's action is $B_{ij}(v_i)$. We write $B_i(v_i) = (B_{ij}(v_i))_{j \neq i}$ as i's belief about its opponents' action profile. Here we implicitly assume that the belief does not change over time, and the belief depends only on current actions. We could consider more complicated beliefs that change over time and/or depend on all past actions, but we prefer simpler beliefs. Under the belief, firm i's *conjectural payoff* is then

$$\begin{aligned}\widetilde{\pi}_i(\boldsymbol{v}_i) &\triangleq \pi_i(\boldsymbol{v}_i, B_i(\boldsymbol{v}_i)) \qquad (6)\\ &= \sum_{t=1}^{T}\left[p_b(t) - \alpha_t\left(v_i(t) + \sum_{j\neq i} B_{ij}\left(v_i(t)\right)\right)\right] v_i(t).\end{aligned}$$

Note that under the belief, firm i's conjectural payoff depends only on its own action, which resolves the issue of incomplete information about its opponents.

Next, we describe the firms' *online strategies*, in order to deal with incomplete information about the market parameters. Similar to the monopoly case, we define firm i's DSMO strategy (online strategy hereafter) as a function $f_i : [p_{min}, p_{max}] \times [0,\infty) \times [0,\Delta_i] \rightarrow [0,\Delta_i]$, which determines how much to sell based on the current pricing parameters $(p_b(t), \alpha_t)$ and the residual inventory $r_i(t)$. We define the CR of online strategy f_i as

$$c(B_i) \triangleq \max_{\sigma\in\Sigma} \quad \frac{\widetilde{\eta}^{B_i}_{OPT}(\sigma)}{\widetilde{\eta}^{B_i}_{f_i}(\sigma)}, \qquad (7)$$

where $\widetilde{\eta}^{B_i}_{OPT}(\sigma)$ and $\widetilde{\eta}^{B_i}_{f_i}(\sigma)$ are the *conjectural revenues* generated by the optimal strategy OPT and the online strategy f_i, respectively. Note that the conjectural revenue is the sum of the conjectural payoffs under the belief B_i. We use the belief B_i as the superscript of the conjectural revenue to emphasize this dependence. Note that the optimal strategy OPT here maximizes the conjectural revenue.

We are now ready to formally define our new equilibrium concept.

Definition *(Online Conjectural Equilibrium)* A conjectural equilibrium in the online setting is a collection of strategies and beliefs $\left(\{f_i^*\}_{i=1}^N, \{B_i^*\}_{i=1}^N\right)$ that satisfies the following two properties:

- (Optimality) for each firm i, the online strategy f_i^* achieves the optimal CR among all the online strategies, with respect to the conjectural revenue under belief B_i;
- (Consistency) for each firm i, its belief is consistent with its opponents' actions, i.e., $B_{ij}\left(f_i^*\left(p_b(t), \alpha_t, r_i(t)\right)\right) = f_j^*\left(p_b(t), \alpha_t, r_i(t)\right)$.

In our definition of equilibrium, the first requirement of optimality is natural as in any equilibrium definition. However, we note that the optimality is with respect to the CR, not the absolute payoff, for the following reason. For the absolute payoff, the worst-case input is $p_b(t) = p_{min}$ and α_t arbitrarily large. Under this input, the revenue can be made arbitrarily small, and the optimal strategy can only guarantee a nonnegative revenue. Therefore, any strategy that sells no unit under p_{min} and large α_t maximizes the worst-case revenue. In other words, optimality on the worst-case absolute payoff admits too many optimal strategies, and does not confine the firms' behaviors under meaningful inputs. Hence, we define optimality with respect to the CR.

The second requirement of consistency is essential. It ensures that the firms' beliefs are "correct", in the sense that the opponents' actual actions are indeed as they believed. It also ensures that the conjectural payoff and revenue are the actual payoff and revenue at the equilibrium.

Next, we show that there exists a natural combination of strategy and belief that constitutes an online conjectural equilibrium.

Theorem 8. *An online conjectural equilibrium always exists, and a particular example is as follows:*

- *the belief is that the opponents' actions are proportional to the inventory, namely*

$$B_{ij}(v_i) = v_i \cdot \frac{\Delta_j}{\Delta_i}, \qquad (8)$$

- *the strategy is the quasi-greedy online algorithm with a modified pricing parameter α_t, namely*

$$\min\left\{\frac{p_b(t)}{4\theta\left(\alpha_t \frac{\Delta}{\Delta_i}\right)}, r_i(t)\right\}, \forall \alpha_t > 0$$

In general, there may be multiple equilibria, depending on the set of beliefs. The proposed equilibrium has two appealing properties. First, the strategy is a natural extension of the online algorithm in the monopoly setting. Therefore, the structure of the strategy remains the same in both monopoly and oligopoly settings. Second, the belief is simple, and follows the intuition that if one firm sells more, the others will sell more as well (because of favorable pricing parameters).

Finally, we conclude this section by presenting an informational Braess' paradox for our online inventory constrained trading game. Intuitively, we may expect higher total revenue in the offline game since the firms have full future information. However, as shown in the following theorem, the revenue in offline game of the NE can be smaller than that in online game at the NE previously defined.

Theorem 9. *Assume that the firms play the equilibrium proposed in Theorem 8. When $T, p_b(t), \alpha_t, \forall t \in \mathcal{T}$ and θ satisfy the following sufficient conditions:*

$$\Delta_i \geq \sum_{t=1}^{T} \frac{p_b(t)}{(N+1)\alpha_t}, \qquad (9)$$

$$\frac{4\theta-1}{16\theta^2} > \frac{N}{(N+1)^2}, \qquad (10)$$

the revenue in online game is larger than that in offline game.

A toy example to observe IBP can be constructed as follows. Consider an oligopoly market with $N = 10$ firms, each with an inventory of $\Delta_i = 100$. The trading period is $T = 2$ and base-price peak-to-valley ratio is $\theta = 2$. The market parameters are

$$p_b(1) = 150, \alpha_1 = 5,$$

$$p_b(2) = 120, \alpha_2 = 4.$$

Under these settings, the revenue ratio between offline game and online game is 0.76.

Although counter-intuitive, this result can be explained as follows. In the game settings, the firms aim to maximize their own payoffs. With their opponents' actions fixed, each firm will get higher individual payoff when it has complete information. However, the firms play differently in the online and offline settings. Therefore, each firm may receive lower individual payoff in the offline setting, because of the different actions taken by their opponents. In fact, we can observe that in the online setting, the firms play less aggressively by selling less, resulting in higher selling prices. As a result, the revenue in the online setting will be higher, if the firms can play aggressively in the offline setting without the restriction from their inventory (the first sufficient condition), and if the market parameters are not so volatile that the advantage of knowledge about environment offsets the aggressive plays (the second sufficient condition). This suggests that in game-theoretic setting, knowing future information may not always be beneficial.

VI. Simulation

In this section we carry out simulations using synthetic traces to evaluate the performance of the proposed online algorithms and verify our theoretical results under various settings.

A. Setup and trace generation

Unless otherwise specified, in the simulation, we fix the trading period as $T = 20$. The number of firms is set as $N = 1$ in monopoly market and $N = 10$ in oligopoly market. We assume that $p_{min} = 100$ and $p_{max} = \theta p_{min}$. The base prices $p_b(t), \forall t \in \mathcal{T}$ are generated from $[p_{min}, p_{max}]$ uniformly at random. Similarly, the market coefficients $\alpha_t, \forall t \in \mathcal{T}$ are chosen as i.i.d. random variables from [0, 10], and inventory Δ is generated uniformly from [100,200] for each run.

B. Performance of online algorithm

Purpose: The competitive analysis in Sec. IV-B suggests that both greedy and quasi-greedy online algorithms are $O(\theta)$-competitive and DTO algorithms are at least $O(\theta)$-competitive. As CR measures the worst case performance, in practice, we may expect better revenue ratio for these online algorithms. In this subsection, we vary the base price peak-to-valley ratio θ and compare the performance of these three online algorithms[4] in monopoly market with $N = 1$. We note that each point in the following figures is averaged over 1000 executions under different realizations of $p_b(t), \alpha_t$ and Δ.

Observations: As shown in Fig. 1, all three online algorithms have a revenue ratio that is much smaller than

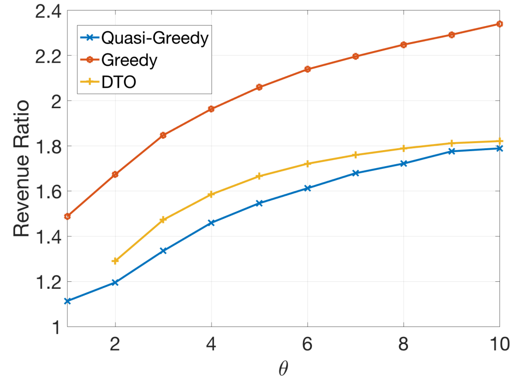

Fig. 1. The average empirical revenue ratio of three online algorithms under different price fluctuations.

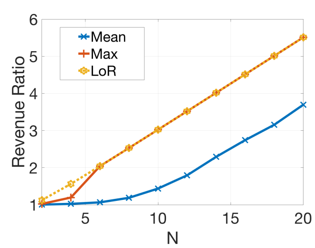

Fig. 2. The revenue ratio and LoR under different number of firms.

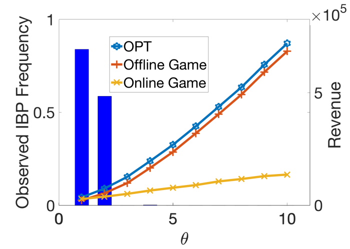

Fig. 3. The revenue and frequency of observed IBP under different price fluctuations.

the corresponding CR. We can also observe that increasing θ degrades the performance of all three online algorithms. This is intuitive since larger price fluctuation indicates higher uncertainty. Among all the three algorithms, quasi-greedy online algorithm has the best revenue ratio. However, as shown in IV-B, the greedy online algorithm is optimal when $\Delta \geq \sum_{t=1}^{T} \frac{p_b(t)}{2\alpha_t}$. Thus under proper conditions, the greedy online algorithm can outperform the quasi-greedy online algorithm. Note that when $\theta = 1$, the revenue ratio of DTO algorithm is infinite since the first threshold of the DTO algorithm considered in [16] is p_{min} and it will not sell any items when $p_b(t) = p_{min}, \alpha_t > 0$.

C. LoR in the offline game

Purpose: Due to the strategic behaviors of firms in oligopoly market, we can observe the loss in revenue as compared to the coordinated setting. Analysis in Sec. V-A2 shows that the LoR is $(N+1)^2/(4N)$ under some special cases, i.e., either $\Delta_i = \Delta/N, i = 1, 2, ..., N$ (Theorem 7) or $T = 1$ (Corollary 2). A natural question is will this result still hold more generally? In this subsection, we study the LoR in the general case to answer this question. In the following simulations, we fix $\theta = 5$ and uniformly generate the inventory of each firm $\Delta_i, i = 1, 2, ..., N$ independently from $[100, 200]$.

Observations: Fig. 2 demonstrates that in the general case, the maximum revenue ratio between the coordinated

[4] We adapt the DTO algorithm studied in [16] ($\alpha_t = 0$) to the settings ($\alpha_t \geq 0$) studied in this paper.

and strategic settings is consistent with the LoR. This suggests that in the general case, LoR is also $(N+1)^2/(4N)$. Meanwhile, we can observe that the average of the revenue ratio is small and increases w.r.t. N, which coincides with the intuition that increasing competition in the market leads to lower overall revenue for the firms.

D. Revenue in the online game

Purpose: Under the conditions indicated in Theorem 9, the revenue in the online game can be larger than that in the offline game. We also construct one example for such effect to appear in the discussion below Theorem 9. However, in general cases these conditions may not hold, thus we expect a lower revenue in the online game as compared to the offline game. It is then interesting to evaluate how the revenue (on average) differs in online and offline games in general cases. Towards this end, we vary the price peak-to-valley ratio θ and measure the mean of revenue in both settings.

Observations: We show the frequency of observed IBP under different price fluctuation in Fig. 3. The results illustrate the frequency with which IBP occurs and show that IBP is more likely to be observed when θ is small. In Fig. 3, OPT is the optimal revenue when firms coordinate to maximize their total revenue with complete information, which serves as the benchmark. Results in Fig. 3 depict that on average the revenue in the online game is much less than that in the offline game, especially when θ is large. In particular, when $\theta = 10$, the average revenue in the offline game is 5 times of that in the online game. This is reasonable as larger θ indicates that there is more uncertainty in the market, and the selling quantities in online game are more conservative as they are proportional to $1/\theta$.

VII. Conclusions

In this paper, we conduct a comprehensive study on the inventory constrained trading problem in monopoly and oligopoly markets. In monopoly market, we propose optimal algorithms for both offline and online settings. Our quasi-greedy online algorithm is $4\theta^2/(4\theta-1)$-competitive and it is optimal among all the DSMO algorithms and no DTO algorithm can achieve a competitive ratio better than our algorithm (in order sense). In oligopoly market, we characterize the uniqueness of NE of game $\mathcal{G}$ and show that LoR is $(N+1)^2/(4N)$ in the offline setting. Perhaps surprisingly, in the online setting, we show that the revenue in the online game can sometimes exceed that in the offline game at NE. This counterintuitive result suggests that in game-theoretic setting, knowing future information may not always be beneficial.

References

[1] D. Acemoglu, A. Makhdoumi, A. Malekian, and A. Ozdaglar, "Informational braess' paradox: The effect of information on traffic congestion," *arXiv preprint arXiv:1601.02039*, 2016.

[2] Technical Report. Internet: https://goo.gl/oEdhE5.

[3] K. T. Talluri and G. J. Van Ryzin, *The theory and practice of revenue management*. Springer Science & Business Media, 2006.

[4] W. Elmaghraby and P. Keskinocak, "Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions," *Management science*, vol. 49, no. 10, pp. 1287–1309, 2003.

[5] Y. Levin, J. McGill, and M. Nediak, "Dynamic pricing in the presence of strategic consumers and oligopolistic competition," *Management science*, vol. 55, no. 1, pp. 32–46, 2009.

[6] O. Besbes and A. Zeevi, "Dynamic pricing without knowing the demand function: Risk bounds and near-optimal algorithms," *Operations Research*, vol. 57, no. 6, pp. 1407–1420, 2009.

[7] R. El-Yaniv, A. Fiat, R. M. Karp, and G. Turpin, "Optimal search and one-way trading online algorithms," *Algorithmica*, vol. 30, no. 1, pp. 101–139, 2001.

[8] W. Zhang, Y. Xu, F. Zheng, and Y. Dong, "Optimal algorithms for online time series search and one-way trading with interrelated prices," *Journal of combinatorial optimization*, vol. 23, no. 2, pp. 159–166, 2012.

[9] H. Fujiwara, K. Iwama, and Y. Sekiguchi, "Average-case competitive analyses for one-way trading," *Journal of Combinatorial Optimization*, vol. 21, no. 1, pp. 83–107, 2011.

[10] K. Bimpikis, S. Ehsani, and R. Ilkilic, "Cournot competition in networked markets." in *Proceedings of ACM EC*, 2014.

[11] S. Bose, D. W. Cai, S. Low, and A. Wierman, "The role of a market maker in networked cournot competition," in *IEEE CDC*, 2014.

[12] J. Pang, H. Fu, W. Lee, and A. Wierman, "The efficiency of open access in platforms for networked cournot markets," in *IEEE INFOCOM*, 2017.

[13] M. Abolhassani, M. H. Bateni, M. Hajiaghayi, H. Mahini, and A. Sawant, "Network cournot competition," in *International Conference on Web and Internet Economics*, 2014.

[14] J. C. Harsanyi, "Games with incomplete information played by bayesian players, i–iii: part i. the basic model," *Management science*, vol. 50, no. 12, pp. 1804–1817, 2004.

[15] H. R. Varian, *Microeconomic analysis*. Norton & Company, 1992.

[16] L. Yang, M. H. Hajiesmaili, H. Yi, and M. Chen, "Online offering strategies for storage-assisted renewable power producer in hour-ahead market," in *Proceedings of ACM SIGMETRICS (poster paper)*, 2017.

[17] G. Scutari, D. P. Palomar, F. Facchinei, and J.-s. Pang, "Convex optimization, game theory, and variational inequality theory," *IEEE Signal Processing Magazine*, vol. 27, no. 3, pp. 35–49, 2010.

[18] M. Aghassi and D. Bertsimas, "Robust game theory," *Mathematical Programming*, vol. 107, no. 1, pp. 231–273, 2006.

[19] A. Rubinstein and A. Wolinsky, "Rationalizable conjectural equilibrium: between nash and rationalizability," *Games and Economic Behavior*, vol. 6, no. 2, pp. 299–311, 1994.