ST407 Monte Carlo Methods 2019/20

Assignment 1

Issued: Friday, October 18th, 2019

Deadline: 11am on Thursday, October 31st, 2019

Note: Completed work should be submitted to the student support office no later than the above deadline, by posting it in the appropriate year box. You *must* attach and sign an "Assignment Cover Sheet", which can be found beside the postboxes outside the student support office. The cover sheet includes a section where you declare whether the work is your own. In Warwick assessment is taken very seriously and the work you submit must be your own, it must be written in your own words and any computer code which is included must be your own original work. This does not preclude you from *discussing* the assignment with others. If it is unclear what the difference is, please consult prior to completing any work a copy of the "Assignment Cover Sheet" to understand fully what you are declaring on submission.

1. Consider the following density with parameters $\alpha > 0$ and $\beta > 0$:

$$f(x;\alpha,\beta) \propto x^{\alpha-1}(1-x^{\alpha})^{\beta-1}, \quad \text{where } x \in [0,1]. \tag{1}$$

(a) Write an `R` function to evaluate this density (up to its normalising constant), and produce a plot to illustrate this density with the parameterisations $\alpha = \beta = 1/5$, $\alpha = \beta = 1$ and $\alpha = \beta = 5$.

(b) i. Write a function in `R` which uses an inversion sampling approach to return samples from $f(x;\alpha,\beta)$ (for user-specified parameters α, β), using only randomness generated from the `runif` command. Clearly show your working for any computations used in your approach, and clearly state any assumptions made.

ii. Generate 1,000,000 samples from the density $f(x;2,1/5)$, using directly the function given in *(b)i.*, and provide an appropriate visualisation of the results.

(c) i. Write a function in `R` which uses a rejection sampling approach to return samples from $f(x;\alpha,\beta)$ (for user-specified parameters α, β), using only randomness generated from the `runif` command. Clearly show your working for any computations used in your approach, and clearly state any assumptions made.

ii. Generate 1,000,000 samples from the density $f(x;2,5)$, using directly the function given in *(c)i.*, and provide an appropriate visualisation of the results.

(d) Comment on your approaches and results in *(b)* and *(c)*, with specific consideration to their efficiency for both the parameterisations given, and with other possible user-specified α, β.

(e) Consider the following density, with parameter $\gamma > 0$ and where f is as given in (1):

$$h(x;\gamma) \propto \max\{f(x;2,1/\gamma), f(x;2,\gamma)\}, \quad \text{where } x \in [0,1]. \tag{2}$$

i. Write a function in `R` which returns samples from $h(x;\gamma)$. You may wish to use your function in *(b)* and/or *(c)*, and/or additional uniform randomness (generated from the `runif` command). Clearly show your working for any computations used in your approach, and clearly state any assumptions made.

ii. Generate 1,000,000 samples from the density $h(x;5)$, using directly the function given in *(e)i.*, and provide an appropriate visualisation of the results.

iii. Comment on your approach, with specific consideration to the efficiency under different parameterisations, γ.

2. Let $\mathcal{A} := \left\{(u,v) : u \in \left[0, \sqrt{\pi(v/u)}\right]\right\}$ where $\pi \geq 0$ is an integrable function.

   (a) Show that if $(U,V) \sim \mathrm{Unif}[\mathcal{A}]$, then V/U has density $f(y) \propto \pi(y)$ where $\int_{-\infty}^{\infty} \pi(y)\mathrm{d}y = 2\cdot\mathrm{area}(\mathcal{A})$. *(Hint: You may wish to consider the transformation $(X,Y) := (U, V/U)$)*

   (b) Provide conditions on f such that $\mathcal{A}$ can be enclosed within a rectangle of finite area, $\mathcal{B}$. Qualitatively describe the conditions imposed. *(Hint: The boundary of $\mathcal{A}$ can be found parametrically by $\{(u(z), v(z)) : z \in \mathbb{R}\}$ for some appropriate $u(z)$ and $v(z)$)*

   (c) Consider the following density,

$$f(y) \propto \pi(y) = \exp\left\{-(y-2)^2\right\} + \exp\left\{-(y+2)^2\right\}. \tag{3}$$

      i. Write an `R` function to evaluate this density (up to its normalising constant), and produce a plot to illustrate it.

      ii. Write an `R` function to evaluate and produce a plot to visualise the region $\mathcal{A}$. Analytically, or otherwise, find appropriate a_-, a_+, b_- and b_+ such that $\mathcal{A} \subset \mathcal{B} := [a_-, a_+] \times [b_-, b_+]$.

   (d) By application of *(a)* and your answer to *(c)ii.*, write an `R` function which returns samples from the density in (3), using only randomness generated from the `runif` command.

   (e) Generate 1,000,000 samples from the density in (3) using directly the function given in *(d)*, and provide an appropriate visualisation of the results (which should include a comparison with your answer to *(c)i.*). Comment on your results, with particular attention to both the efficiency and the practicality of your approach (for which you may wish to reconsider your answer to *(b)*).

Some general guidance:

- You don't need to write more than a few sides of text with some accompanying figures and code snippets to score very highly. Clear and concise answers are positively encouraged! Remember that it is worth just 10% of the module marks, and so use your time efficiently.
- If you write computer code in order to answer these questions then include that code as part of your answer. Clearly annotate / document your code – it's your responsibility to convey what you are doing.