Model of Super Spreading Events

Hannah Craddock

June 2021

Model

Let $x_t = y_t + z_t$

where

y_t = Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t$ = Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

i. y_t - Non super-spreading events

$y_t \sim Poisson(\alpha \cdot \lambda_t)$,

$$p(y_t) = exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t},$$

with

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

ii. z_t - super-spreading events

For each SSE event we get Poisson(γ) infections

$z_t | n_t \sim Poisson(\gamma \cdot n_t)$ = Number of infecteds by all Super-Spreading Events at time t

$n_t \sim Poisson(\beta \cdot \lambda_t)$ = Number of super-spreading events

$$p(z_t) = \sum_{n_t=0}^{\infty} p(n_t = n_t) \cdot p(z_t | n_t = n_t)$$

$\therefore p(z_t)$ is a Poisson-Poisson Compound

Determining the Mean and Variance of $\mathbf{z}_t$

$$\mathbb{E}(z_t) = \mathbb{E}(\mathbb{E}(z_t|n_t)\,)$$

$$= \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma \, \mathbb{E}(n_t)$$

$$= \gamma \cdot \beta \cdot \lambda_t$$

Var(z_t)= $\mathrm{Var}(\mathbb{E}(z_t|n_t)) + \mathbb{E}(\mathrm{Var}(z_t|n_t))$

As $z_t|n_t$ and n_t are poisson rvs the mean and variance are equivalent giving;

$$= \text{Var}(\gamma \cdot n_t) + \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Poisson-Poisson Compound z_t in the form of a Negative Binomial RV

The mean and variance of the Poisson-Poisson Compound z_t is equated to that of a Negative Binomial distribution of size $\boldsymbol{n}$ with probability of success $\boldsymbol{p}$ and density given by;

$$\frac{\Gamma(x+n)}{\Gamma(n) \cdot x!} \cdot p^n (1-p)^x$$

for $x = 0, 1, 2, ..., n > 0$ and $0 < p <= 1$.

This represents the number of failures which occur in a sequence of Bernoulli trials before a target number of successes is reached. The mean and variance are respectfully;

$$\mu = \frac{n(1-p)}{p}, \qquad Var = \frac{n(1-p)}{p^2}$$

Equating the poisson-poisson compound to that of the negative binomial distribution gives;

I. $\mu = \dfrac{n(1-p)}{p} = \gamma \cdot \beta \cdot \lambda_t$

II $Var = \dfrac{n(1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$

From **I** we can say that the size

$$n = \frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p} \qquad \textbf{III.}$$

Substituting **III** into **II** gives;

$$\frac{(\frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p}) \cdot (1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Cancelling like terms gives;

$$\frac{(\frac{\cancel{p} \cdot \cancel{\gamma \cdot \beta \cdot \lambda_t}}{\cancel{1-p}}) \cdot (\cancel{1-p})}{p^{\cancel{2}}} = \gamma^{\cancel{2}} \cdot \cancel{\beta \cdot \lambda_t} + \cancel{\gamma \cdot \beta \cdot \lambda_t}$$

$$\frac{1}{p} = \gamma + 1$$

$$1 = p \cdot \gamma + p$$

$$1 = p \cdot \gamma + p$$

$$\therefore p = \frac{1}{\gamma + 1} \qquad \textbf{IV.}$$

Substituting **IV** into **III** gives;

$$n = \frac{\frac{1}{\gamma+1} \cdot \gamma \cdot \beta \cdot \lambda_t}{1 - \frac{1}{\gamma+1}}$$

$$n = \frac{\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma+1}}{\frac{\gamma+1-1}{\gamma+1}}$$

$$n = \frac{\frac{\cancel{\gamma} \cdot \beta \cdot \lambda_t}{\cancel{\gamma+1}}}{\frac{\cancel{\gamma}}{\cancel{\gamma+1}}}$$

$$\therefore n = \beta \cdot \lambda_t$$

z_t can thus be written as a Negative Binomial of size n with probability of success p, i.e

$$z_t \sim \mathrm{NB}(\beta \cdot \lambda_t, \ \ \frac{1}{\gamma+1})$$

$$p(z_t) = \frac{\Gamma(x+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!} \cdot \frac{1}{\gamma+1}^{\beta\cdot\lambda_t} \cdot (1-\frac{1}{\gamma+1})^{z_t} \qquad \text{for } z_t = 0,1,2,..., n > 0 \text{ and } 0 < p <= 1$$

Likelihood

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

$$L(\alpha,\beta,\gamma,\lambda|\ \mathbf{x_t}) = \prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} P(y_t)\cdot P(z_t)$$

$$= \prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} exp(-\alpha\cdot\lambda_t)\cdot\frac{1}{y_t!}\cdot(\alpha\cdot\lambda_t)^{y_t} \times \frac{\Gamma(z_t+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!}\cdot\frac{1}{\gamma+1}^{\beta\cdot\lambda_t}\cdot(1-\frac{1}{\gamma+1})^{z_t}$$

$$= \prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} exp(-\alpha\cdot\lambda_t)\cdot\frac{1}{y_t!}\cdot(\alpha\cdot\lambda_t)^{y_t} \times \frac{\Gamma(z_t+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!}\cdot\frac{1}{\gamma+1}^{\beta\cdot\lambda_t}\cdot(\frac{\gamma}{\gamma+1})^{z_t}$$

Log Likelihood

$$l(\alpha,\beta,\gamma,\lambda|\ \mathbf{x_t}) = ln\left(\prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} P(y_t)\cdot P(z_t))\right)$$

$$= \sum_{t=1}^{Ndays} ln\left(\sum_{y_t=0}^{x_t}(exp(-\alpha\cdot\lambda_t)\cdot\frac{1}{y_t!}\cdot(\alpha\cdot\lambda_t)^{y_t} \times \frac{\Gamma(z_t+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!}\cdot\frac{1}{\gamma+1}^{\beta\cdot\lambda_t}\cdot(\frac{\gamma}{\gamma+1})^{z_t}\right)$$

Log exp sum implementation

Simplest case x_1, x_2

Let $L_i = log(x_i)$

$$log(x_1 + x_2) = log\Big(exp(L_1) + exp(L_2)\Big)$$

$$= log\Big(\big(exp(L_1) + exp(L_2)\big) * \frac{exp(M)}{exp(M)}\Big)$$

$$= log\big(exp(M)\big) \quad + \quad log\Big(\frac{exp(L_2)}{exp(M)} + \frac{exp(L_2)}{exp(M)}\Big)$$

$$= M + log\Big(exp(L_1 - M) + exp(L_2 - M)\Big)$$

In general;

$$LSE(x_1, ..., x_n) = x^* + log\Big(exp(x_1 - x^*) + ... + exp(x_n - x^*)\Big)$$

where

$x^* = max(x_1, ..., x_n)$

For this super-spreading events model x_i is underlined in the likelihood;

$$l(\alpha, \beta, \gamma, \lambda |\, \mathbf{x_t}) = \sum_{t=1}^{Ndays} ln\Bigg(\sum_{y_t=0}^{x_t} \underline{(exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t}}\Bigg)$$

$$x_i = exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t} \text{ for a given } y_t$$

and

$L_i = log(x_i)$

$$= -(\alpha \cdot \lambda_t) \;-\; lfactorial(y_t) \;+\; y_t \cdot log(\alpha \cdot \lambda_t) \;+\; lgamma(z_t + \beta \cdot \lambda_t) \;-\; lgamma(\beta \cdot \lambda_t)$$

$$- lfactorial(z_t) \;\; - \beta \cdot \lambda_t \cdot log(\gamma + 1) \;\; + z_t \cdot log(\gamma) \;-\; z_t \cdot log(\gamma + 1)$$

Replacing z_t with $x_t - y_t$ gives;

$$L_i = -(\alpha \cdot \lambda_t) \;-\; lfactorial(y_t) \;+\; y_t \cdot log(\alpha \cdot \lambda_t) \;+ lgamma((x_t - y_t) + \beta \cdot \lambda_t) - \; lgamma(\beta \cdot \lambda_t)$$

$$-lfactorial(x_t - y_t) \;\; - \beta \cdot \lambda_t \cdot log(\gamma + 1) \;\; + (x_t - y_t) \cdot log(\gamma) \;\; - (x_t - y_t) \cdot log(\gamma + 1)$$