Model of Super Spreading Events

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June 2021

Model

Let $x_t = y_t + z_t$

where

$y_t =$ Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t =$ Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

i. y_t - Non super-spreading events

$y_t \sim Poisson(\alpha \cdot \lambda_t)$,

$$p(y_t) = exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t},$$

with

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

ii. z_t - super-spreading events

For each SSE event we get Poisson(γ) infections

$z_t|n_t \sim Poisson(\gamma \cdot n_t) =$ Number of infecteds by all Super-Spreading Events at time t

$n_t \sim Poisson(\beta \cdot \lambda_t) =$ Number of super-spreading events

$$p(z_t) = \sum_{n_t=0}^{\infty} p(n_t = n_t) \cdot p(z_t|n_t = n_t)$$

$\therefore p(z_t)$ is a Poisson-Poisson Compound

Determining the Mean and Variance of $\mathbf{z}_t$

$$\mathbb{E}(z_t) = \mathbb{E}(\mathbb{E}(z_t|n_t)\,)$$

$$= \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma \, \mathbb{E}(n_t)$$

$$= \gamma \cdot \beta \cdot \lambda_t$$

Var(z_t)= Var($\mathbb{E}(z_t|n_t)) + \mathbb{E}(Var(z_t|n_t))$

As $z_t|n_t$ and n_t are poisson rvs the mean and variance are equivalent giving;

$$= \mathrm{Var}(\gamma \cdot n_t) + \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Poisson-Poisson Compound z_t in the form of a Negative Binomial RV

The mean and variance of the Poisson-Poisson Compound z_t is equated to that of a Negative Binomial distribution of size $\boldsymbol{n}$ with probability of success $\boldsymbol{p}$ and density given by;

$$\frac{\Gamma(x+n)}{\Gamma(n) \cdot x!} \cdot p^n (1-p)^x$$

for $x = 0, 1, 2, ..., n > 0$ and $0 < p <= 1$.

This represents the number of failures which occur in a sequence of Bernoulli trials before a target number of successes is reached. The mean and variance are respectfully;

$$\mu = \frac{n(1-p)}{p}, \qquad Var = \frac{n(1-p)}{p^2}$$

Equating the poisson-poisson compound to that of the negative binomial distribution gives;

I. $\mu = \dfrac{n(1-p)}{p} = \gamma \cdot \beta \cdot \lambda_t$

II $Var = \dfrac{n(1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$

From **I** we can say that the size

$$n = \frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p} \qquad \textbf{III.}$$

Substituting **III** into **II** gives;

$$\frac{(\dfrac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p}) \cdot (1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Cancelling like terms gives;

$$\frac{(\dfrac{\cancel{p} \cdot \cancel{\gamma \cdot \beta \cdot \lambda_t}}{\cancel{1-p}}) \cdot (\cancel{1-p})}{p^{\cancel{2}}} = \gamma^{\cancel{2}} \cdot \cancel{\beta \cdot \lambda_t} + \cancel{\gamma \cdot \beta \cdot \lambda_t}$$

$$\frac{1}{p} = \gamma + 1$$

$$1 = p \cdot \gamma + p$$

$$1 = p \cdot \gamma + p$$

$$\therefore p = \frac{1}{\gamma + 1} \qquad \textbf{IV.}$$

Substituting **IV** into **III** gives;

$$n = \frac{\dfrac{1}{\gamma+1} \cdot \gamma \cdot \beta \cdot \lambda_t}{1 - \dfrac{1}{\gamma+1}}$$

$$n = \frac{\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma + 1}}{\frac{\gamma + 1 - 1}{\gamma + 1}}$$

$$n = \frac{\frac{\cancel{\gamma} \cdot \beta \cdot \lambda_t}{\cancel{\gamma + 1}}}{\frac{\cancel{\gamma}}{\cancel{\gamma + 1}}}$$

$$\therefore n = \beta \cdot \lambda_t$$

z_t can thus be written as a Negative Binomial of size n with probability of success p, i.e

$$z_t \sim \text{NB}(\beta \cdot \lambda_t, \ \frac{1}{\gamma + 1})$$

$$p(z_t) = \frac{\Gamma(x + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (1 - \frac{1}{\gamma + 1})^{z_t} \qquad \text{for } z_t = 0, 1, 2, ..., n > 0 \text{ and } 0 < p <= 1$$

Likelihood

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

$$L(\alpha, \beta, \gamma, \lambda | \ \mathbf{x_t}) = \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} P(y_t) \cdot P(z_t)$$

$$= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (1 - \frac{1}{\gamma + 1})^{z_t}$$

$$= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t}$$

Log Likelihood

$$l(\alpha, \beta, \gamma, \lambda | \ \mathbf{x_t}) = ln\left(\prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)) \right)$$

$$= \sum_{t=1}^{Ndays} ln\left(\sum_{y_t=0}^{x_t} (exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t} \right)$$

Log exp sum implementation

I. $LSE(log(x_1), ..., log(x_n)) = log(x_1 + ... + x_n)$

Also;

II. $LSE(x_1, ..., x_n) = x^* + log\left(exp(x_1 - x^*) + ... + exp(x_n - x^*) \right)$

where

$x^* = max(x_1, ..., x_n)$

and x_i is underlined in the likelihood, i.e;

$$l(\alpha,\beta,\gamma,\lambda|\,\mathbf{x_t}) = \sum_{t=1}^{Ndays} ln\left(\sum_{y_t=0}^{x_t}\underline{(exp(-\alpha\cdot\lambda_t)\cdot\frac{1}{y_t!}\cdot(\alpha\cdot\lambda_t)^{y_t}\times\frac{\Gamma(z_t+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!}\cdot\frac{1}{\gamma+1}^{\beta\cdot\lambda_t}\cdot(\frac{\gamma}{\gamma+1})^{z_t}}\right)$$

$$x_i = exp(-\alpha\cdot\lambda_t)\cdot\frac{1}{y_t!}\cdot(\alpha\cdot\lambda_t)^{y_t}\times\frac{\Gamma(z_t+\beta\cdot\lambda_t)}{\Gamma(\beta\cdot\lambda_t)\cdot z_t!}\cdot\frac{1}{\gamma+1}^{\beta\cdot\lambda_t}\cdot(\frac{\gamma}{\gamma+1})^{z_t} \text{ for a given } \mathbf{y}_t$$