Model of Super Spreading Events

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Model

A Poisson Mixture model with unknown rates λ_1 and λ_2 and mixture weight p is proposed to describe Super-spreading events in an epidemic.

Likelihood

$$L(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]$$

where

$$\lambda_{1t} = r0_1 \sum_{i=1}^{t-1} x_i \;(\; Gamma((t-i); k_1, \theta_1) - Gamma((t-i-1); k_1, \theta_1)$$
$$\lambda_{2t} = r0_2 \sum_{i=1}^{t-1} x_i \;(\; Gamma((t-i); k_2, \theta_2) - Gamma((t-i-1); k_2, \theta_2)$$

Log Likelihood

$$l(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \prod_{t=1}^{Ndays} log\left(p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}]\right) + log\left((1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]\right)$$

$$l(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) + log(exp(-\lambda_{1t})) + log(\frac{1}{x_t!}) + log(\lambda_{1t}^{x_t}) + log(1-p) + log(exp(-\lambda_{2t})) + log(\frac{1}{x_t!}) + log(\lambda_{2t}^{x_t})$$

$$l(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) - \lambda_{1t} + log(\frac{1}{x_t!}) + log(\lambda_{1t}^{x_t}) + log(1-p) - \lambda_{2t} + log(\frac{1}{x_t!}) + log(\lambda_{2t}^{x_t})$$

Terms that do not contain one of the model parameters are removed giving;

$$l(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) - \lambda_{1t} + log(\lambda_{1t}^{x_t}) + log(1-p) - \lambda_{2t} + log(\lambda_{2t}^{x_t})$$

Bayesian Model

Likelihood

$$L(\lambda_1, \lambda_2, p|\, \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]$$

where

$$\lambda_{1t} = r0_1 \sum_{i=1}^{t-1} x_i \;(\; Gamma((t-i); k_1, \theta_1) - Gamma((t-i-1); k_1, \theta_1)$$

$$\lambda_{2t} = r0_2 \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k_2, \theta_2) - Gamma((t-i-1); k_2, \theta_2)$$

Priors

$$prior(r0_1) = Gamma(\alpha_1, \beta_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} r0^{\alpha_1 - 1} e^{-\beta_1 r0}$$

$$prior(r0_2) = Gamma(\alpha_2, \beta_2) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} r0^{\alpha_2 - 1} e^{-\beta_2 r0}$$

$$prior(p) = Beta(\alpha_3, \beta_3) = \frac{1}{\alpha, \beta} \cdot p^{\alpha_3 - 1} (1-x)^{\beta_3 - 1}$$

Draft - Likelihood

$$L(r0_1, k_1, \theta_1, r0_2, k_2, \theta_2, p | \, \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t}) \frac{1}{x_t!} {\lambda_1}_t^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t}) \frac{1}{x_t!} {\lambda_2}_t^{x_t}]$$

Draft

Bayesian Epidemic Modelling

Mathematics

Likelihood

$$L(r0, k, \theta | \, \mathbf{x}) = \prod_{t=1}^{Ndays} exp(-\lambda_t) \frac{1}{x_t!} \lambda_t^{x_t}$$

where

$$\lambda_t = r0 \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Log likelihood

$$l = \sum_{t=1}^{Ndays} \; x_t ln(\lambda_t) - \lambda_t$$

$$l = \sum_{t=1}^{Ndays} \; x_t ln(r0\lambda_t) - r0\lambda_t$$

$$l = \sum_{t=1}^{Ndays} \; (x_t ln(r0) + x_t ln(\lambda_t) - r0\lambda_t)$$

where

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

$$l = \sum_{t=1}^{Ndays} \; (x_t ln(r0\lambda_t) - r0\lambda_t)$$

MLE

$$\frac{dl}{dr0} = \sum_{t=1}^{Ndays} \frac{x_t}{r0} - \sum_{t=1}^{Ndays} \lambda_t$$

$$\hat{r0} = \frac{\sum_{t=1}^{Ndays} x_t}{\sum_{t=1}^{Ndays} \lambda_t}$$

Bayesian Inference

Likelihood

$$L(r0, k, \theta | \, \mathbf{x}) = \prod_{t=1}^{Ndays} exp(-\lambda_t) \frac{1}{x_t!} \lambda_t^{x_t}$$

where

$$\lambda_t = r0 \sum_{i=1}^{t-1} x_i \ (\ Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Prior

$$p(r0) = Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} r0^{\alpha-1} e^{-\beta r0}$$

Bayesian Inference

- Shape, scale** To write out
Posterior $p(r0|x)$

Given Gamma(1, 1) prior on r0;

$$p(r0|x) \propto \prod_{t=1}^{Ndays} exp(-r0\lambda_t) \frac{1}{x_t!} (r0\lambda_t)^{x_t} \times r0^{\alpha-1} e^{-\beta r0}$$

$$\propto exp(-r0(\sum_{t=1}^{Ndays} \lambda_t + 1)) \times (r0\lambda_t)^{\sum_{t=1}^{Ndays} x_t}$$

$$\therefore p(r0|x) \propto Gamma(\sum_{t=1}^{Ndays} x_t + 1, \sum_{t=1}^{Ndays} \lambda_t + 1)$$

$$\propto Gamma(\sum_{t=1}^{Ndays} x_t + \alpha, \sum_{t=1}^{Ndays} \lambda_t + \beta)$$

where (*Above is shape, rate. Below is shape, scale)

$$\lambda_t = \sum_{i=1}^{t-1} x_i \ (\ Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Bayesian Inference v0

Posterior $p(r0|x)$

Given Gamma(1, 1) prior on r0;

$$p(r0|x) \propto \prod_{t=1}^{Ndays} exp(-r0\lambda_t) \frac{1}{x_t!} (r0\lambda_t)^{x_t} \times r0^{\alpha-1} e^{-\beta r0}$$

$$\propto exp(-r0(\sum_{t=1}^{Ndays} \lambda_t + 1)) \times (r0\lambda_t)^{\sum_{t=1}^{Ndays} x_t}$$

$$\therefore p(r0|x) \propto Gamma(\sum_{t=1}^{Ndays} x_t + 1, \sum_{t=1}^{Ndays} \lambda_t + 1)$$

$$\propto Gamma(\sum_{t=1}^{Ndays} x_t + \alpha, \sum_{t=1}^{Ndays} \lambda_t + \beta)$$

where (*Above is shape, rate. Below is shape, scale)

$$\lambda_t = \sum_{i=1}^{t-1} x_i \ (\ Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Prior

$$p(r0) = Gamma(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} r0^{\alpha-1} e^{-\beta r0}$$

Explanation of Mathematics

- Simulation of Epidemic - Infectiousness (Discrete gamma) - I.e 'Infectiousness Pressure' = Sum of all people - Explanation: Gamma is a continuous function so integrate over the density at that point in time (today - previous day) -Assumption: Number of daily cases follows a Poisson distribution. Reason why there is spikes from day to day

- Metropolis Hastings step - The logarithm of the acceptance probability λ includes; the sum of the log likelihood, prior and the proposal for the current time Y and the previous time step r0[t-1]

- Symmetrical proposal distribution so the proposal for the current time Y and the previous time step r0[t-1] cancel out

Assumptions

- Number of daily cases follows a Poisson distribution - Poisson-Gamma conjugacy leads to nice posterior - although not realistic

* Assume the following Poisson model of two regimes for n random variables