Model of Super Spreading Events

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June 2021

Model

Let $x_t = y_t + z_t$

where

$y_t =$ Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t =$ Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P_{NSSE}(y_t) \circledast P_{SSE}(z_t)$$

where

$i.$ $\mathrm{P}_{NSSE}(y_t) = exp(-\lambda_t)\frac{1}{y_t!}\lambda_t^{y_t},$

with

$$\lambda_t = r0_1 \sum_{i=1}^{t-1} y_i \ (\ Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

ii. $P_{SSE}(z_t) = e^{-\lambda'_t}\frac{1}{z_t!}{\lambda'_t}^{z_t},$

with $\lambda_t \sim Poisson()$

Super-spreading events
For each super-spreading event there are;

- Poisson(μ) infecteds (For a given rate μ, e.g = 5)

- n_t = Number of super-spreading events

z_t

$z_t \sim Poisson(n_t\mu)$ = Number of infecteds by all Super-Spreading Events at time t

$n_t \sim Poisson(\lambda_t \cdot k)$ = Number of infecteds by a super-spreading person at time t

where

k = some multiplicative factor > 1

Probability(z_t)

$$
\begin{aligned}
\mathrm{P}(\mathrm{z}_t) &= \sum_{n_t=0}^{\infty} p(n_t = n_t) \times p(z_t|n_t) \\
\mathrm{P}(\mathrm{z}_t) &= \sum_{n_t=0}^{\infty} \frac{1}{n_t!} e^{-\lambda_t k} (\lambda_t k)^{n_t} \times \frac{1}{z_t!} e^{-n_t \mu} (n_t \mu)^{z_t} \\
\mathrm{P}(\mathrm{z}_t) &= \sum_{n_t=0}^{\infty} \frac{1}{n_t!} (\lambda_t k)^{n_t} e^{-n_t \mu} (n_t)^{z_t} \times \frac{1}{z_t!} e^{-\lambda_t k} \mu^{z_t}
\end{aligned}
$$

Other - Likelihood

$$L(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]$$

where

$$\lambda_{1t} = r0_1 \sum_{i=1}^{t-1} x_i\ (\ Gamma((t-i); k_1, \theta_1) - Gamma((t-i-1); k_1, \theta_1)$$
$$\lambda_{2t} = r0_2 \sum_{i=1}^{t-1} x_i\ (\ Gamma((t-i); k_2, \theta_2) - Gamma((t-i-1); k_2, \theta_2)$$

Log Likelihood

$$l(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \prod_{t=1}^{Ndays} log\left(p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}]\right) + log\left((1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]\right)$$
$$l(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) + log(exp(-\lambda_{1t})) + log(\frac{1}{x_t!}) + log(\lambda_{1t}^{x_t}) + log(1-p) + log(exp(-\lambda_{2t})) + log(\frac{1}{x_t!}) + log(\lambda_{2t}^{x_t})$$

$$l(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) - \lambda_{1t} + log(\frac{1}{x_t!}) + log(\lambda_{1t}^{x_t}) + log(1-p) - \lambda_{2t} + log(\frac{1}{x_t!}) + log(\lambda_{2t}^{x_t})$$

Terms that do not contain one of the model parameters are removed giving;

$$l(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \sum_{t=1}^{Ndays} log(p) - \lambda_{1t} + log(\lambda_{1t}^{x_t}) + log(1-p) - \lambda_{2t} + log(\lambda_{2t}^{x_t})$$

Bayesian Model

Likelihood

$$L(\lambda_1, \lambda_2, p|\ \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]$$

where

$$\lambda_{1t} = r0_1 \sum_{i=1}^{t-1} x_i\ (\ Gamma((t-i); k_1, \theta_1) - Gamma((t-i-1); k_1, \theta_1)$$
$$\lambda_{2t} = r0_2 \sum_{i=1}^{t-1} x_i\ (\ Gamma((t-i); k_2, \theta_2) - Gamma((t-i-1); k_2, \theta_2)$$

Priors

$$prior(r0_1) = Gamma(\alpha_1, \beta_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} r0^{\alpha_1 - 1} e^{-\beta_1 r0}$$

$$prior(r0_2) = Gamma(\alpha_2, \beta_2) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} r0^{\alpha_2 - 1} e^{-\beta_2 r0}$$

$$prior(p) = Beta(\alpha_3, \beta_3) = \frac{1}{\alpha, \beta} \cdot p^{\alpha_3 - 1}(1-x)^{\beta_3 - 1}$$

Draft - Likelihood

$$L(r0_1, k_1, \theta_1, r0_2, k_2, \theta_2, p|\, \mathbf{x}) = \prod_{t=1}^{Ndays} p \cdot [exp(-\lambda_{1t})\frac{1}{x_t!}\lambda_{1t}^{x_t}] + (1-p) \cdot [exp(-\lambda_{2t})\frac{1}{x_t!}\lambda_{2t}^{x_t}]$$

Draft

Bayesian Epidemic Modelling

Mathematics

Likelihood

$$L(r0, k, \theta | \, \mathbf{x}) = \prod_{t=1}^{Ndays} exp(-\lambda_t) \frac{1}{x_t!} \lambda_t^{x_t}$$

where

$$\lambda_t = r0 \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Log likelihood

$$l = \sum_{t=1}^{Ndays} x_t ln(\lambda_t) - \lambda_t$$

$$l = \sum_{t=1}^{Ndays} x_t ln(r0\lambda_t) - r0\lambda_t$$

$$l = \sum_{t=1}^{Ndays} (x_t ln(r0) + x_t ln(\lambda_t) - r0\lambda_t)$$

where

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

$$l = \sum_{t=1}^{Ndays} (x_t ln(r0\lambda_t) - r0\lambda_t)$$

MLE

$$\frac{dl}{dr0} = \sum_{t=1}^{Ndays} \frac{x_t}{r0} - \sum_{t=1}^{Ndays} \lambda_t$$

$$\hat{r0} = \frac{\sum_{t=1}^{Ndays} x_t}{\sum_{t=1}^{Ndays} \lambda_t}$$

Bayesian Inference

Likelihood

$$L(r0, k, \theta | \, \mathbf{x}) = \prod_{t=1}^{Ndays} exp(-\lambda_t) \frac{1}{x_t!} \lambda_t^{x_t}$$

where

$$\lambda_t = r0 \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Prior

$$p(r0) = Gamma(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} r0^{\alpha-1} e^{-\beta r0}$$

Bayesian Inference

- Shape, scale** To write out
Posterior $p(r0|x)$

Given Gamma(1, 1) prior on r0;

$$p(r0|x) \propto \prod_{t=1}^{Ndays} exp(-r0\lambda_t) \frac{1}{x_t!} (r0\lambda_t)^{x_t} \times r0^{\alpha-1} e^{-\beta r0}$$

$$\propto exp(-r0(\sum_{t=1}^{Ndays} \lambda_t + 1)) \times (r0\lambda_t)^{\sum_{t=1}^{Ndays} x_t}$$

$$\therefore p(r0|x) \propto Gamma(\sum_{t=1}^{Ndays} x_t + 1, \sum_{t=1}^{Ndays} \lambda_t + 1)$$

$$\propto Gamma(\sum_{t=1}^{Ndays} x_t + \alpha, \sum_{t=1}^{Ndays} \lambda_t + \beta)$$

where (*Above is shape, rate. Below is shape, scale)

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Bayesian Inference v0

Posterior $p(r0|x)$

Given Gamma(1, 1) prior on r0;

$$p(r0|x) \propto \prod_{t=1}^{Ndays} exp(-r0\lambda_t) \frac{1}{x_t!} (r0\lambda_t)^{x_t} \times r0^{\alpha-1} e^{-\beta r0}$$

$$\propto exp(-r0(\sum_{t=1}^{Ndays} \lambda_t + 1)) \times (r0\lambda_t)^{\sum_{t=1}^{Ndays} x_t}$$

$$\therefore p(r0|x) \propto Gamma(\sum_{t=1}^{Ndays} x_t + 1, \sum_{t=1}^{Ndays} \lambda_t + 1)$$

$$\propto Gamma(\sum_{t=1}^{Ndays} x_t + \alpha, \sum_{t=1}^{Ndays} \lambda_t + \beta)$$

where (*Above is shape, rate. Below is shape, scale)

$$\lambda_t = \sum_{i=1}^{t-1} x_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

Prior

$$p(r0) = Gamma(\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} r0^{\alpha-1} e^{-\beta r0}$$

Explanation of Mathematics

- Simulation of Epidemic - Infectiousness (Discrete gamma) - I.e 'Infectiousness Pressure' = Sum of all people - Explanation: Gamma is a continuous function so integrate over the density at that point in time (today - previous day) -Assumption: Number of daily cases follows a Poisson distribution. Reason why there is spikes from day to day

- Metropolis Hastings step - The logarithm of the acceptance probability λ includes; the sum of the log likelihood, prior and the proposal for the current time Y and the previous time step r0[t-1]

- Symmetrical proposal distribution so the proposal for the current time Y and the previous time step r0[t-1] cancel out

Assumptions

- Number of daily cases follows a Poisson distribution - Poisson-Gamma conjugacy leads to nice posterior - although not realistic

* Assume the following Poisson model of two regimes for n random variables