Model of Super Spreading Events

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Model

Let $x_t = y_t + z_t$

where

y_t = Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t$ = Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

i. y_t - Non super-spreading events

$y_t \sim Poisson(\alpha \cdot \lambda_t)$,

$$p(y_t) = exp(-\alpha \cdot \lambda_t)\frac{1}{y_t!}(\alpha \cdot \lambda_t)^{y_t},$$

this should be x_i

with

$$\lambda_t = \sum_{i=1}^{t-1} y_i \left(Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta) \right)$$

ii. z_t - super-spreading events

For each SSE event we get Poisson(γ) infections

$z_t|n_t \sim Poisson(\gamma \cdot n_t)$ = Number of infecteds by all Super-Spreading Events at time t

$n_t \sim Poisson(\beta \cdot \lambda_t)$ = Number of super-spreading events

$$p(z_t) = \sum_{n_t=0}^{\infty} p(n_t = n_t) \cdot p(z_t|n_t = n_t)$$

$\therefore p(z_t)$ is a Poisson-Poisson Compound

Determining the Mean and Variance of $\mathbf{z}_t$

$\mathbb{E}(z_t) = \mathbb{E}(\mathbb{E}(z_t|n_t))$

$= \mathbb{E}(\gamma \cdot n_t)$

$= \gamma\,\mathbb{E}(n_t)$

$= \gamma \cdot \beta \cdot \lambda_t$

$\text{Var}(z_t) = \text{Var}(\mathbb{E}(z_t|n_t)) + \mathbb{E}(\text{Var}(z_t|n_t))$

As $z_t|n_t$ and n_t are poisson rvs the mean and variance are equivalent giving;

$$= \mathrm{Var}(\gamma \cdot n_t) + \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Poisson-Poisson Compound z_t in the form of a Negative Binomial RV

The mean and variance of the Poisson-Poisson Compound z_t is equated to that of a Negative Binomial distribution of size $\boldsymbol{n}$ with probability of success $\boldsymbol{p}$ and density given by;

$$\frac{\Gamma(x+n)}{\Gamma(n)x!}p^n(1-p)^x$$

for $x = 0, 1, 2, ..., n > 0$ and $0 < p <= 1$.

This represents the number of failures which occur in a sequence of Bernoulli trials before a target number of successes is reached. The mean and variance are respectfully;

$$\mu = \frac{n(1-p)}{p}, \qquad Var = \frac{n(1-p)}{p^2}$$

Equating the poisson-poisson compound to that of the negative binomial distribution gives;

I. $\mu = \dfrac{n(1-p)}{p} = \gamma \cdot \beta \cdot \lambda_t$

II $Var = \dfrac{n(1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$

From **I** we can say that the size

$$n = \frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p} \qquad \textbf{III.}$$

Substituting **III** into **II** gives;

$$\frac{(\frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p}) \cdot (1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Cancelling like terms gives;

$$\frac{(\frac{\cancel{p} \cdot \cancel{\gamma \cdot \beta \cdot \lambda_t}}{\cancel{1-p}}) \cdot (\cancel{1-p})}{p^{\cancel{2}}} = \overbrace{\gamma^{\cancel{2}} \cdot \cancel{\beta \cdot \lambda_t}}^{\gamma} + \overbrace{\cancel{\gamma \cdot \beta \cdot \lambda_t}}^{1} \Rightarrow \frac{1}{p} = \gamma + 1$$

$$\cancel{\frac{1}{p} = \gamma}$$

NO

$$\therefore p = \cancel{\frac{1}{\gamma}} \qquad \textbf{IV.}$$

$$p = \frac{1}{\gamma + 1}$$

Substituting **IV** into **III** gives;

$$n = \frac{\frac{1}{\gamma} \cdot \gamma \cdot \beta \cdot \lambda_t}{1 - \frac{1}{\gamma}}$$

$$n = \frac{\frac{1}{\cancel{\gamma}} \cdot \cancel{\gamma} \cdot \beta \cdot \lambda_t}{1 - \frac{1}{\gamma}}$$

$$n = \frac{\beta \cdot \lambda_t}{\frac{\gamma - 1}{\gamma}}$$

$n = \dfrac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1}$

z_t can be written as a Negative Binomial of size n with probability of success p, i.e

$z_t \sim \mathrm{NB}(\dfrac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1}, \dfrac{1}{\gamma})$

$$p(z_t) = \frac{\Gamma(x + \frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1})}{\Gamma(\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1})x!} \cdot \frac{1}{\gamma}^{\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1}} (1 - \frac{1}{\gamma})^x \qquad \text{for } x = 0, 1, 2, ..., n > 0 \text{ and } 0 < p <= 1$$

Likelihood

$$?^{**}\ P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t) \qquad \text{or} \qquad \sum_{y_t=0}^{x_t} \sum_{z_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

$$\begin{aligned} L(\alpha, \beta, \gamma, \lambda | \, \mathbf{x_t}) &= \prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t) \\ &= \prod_{t=1}^{Ndays} \sum_{y_t=0}^{x_t} exp(-\alpha \cdot \lambda_t) \frac{1}{y_t!} (\alpha \cdot \lambda_t)^{y_t} \cdot \frac{\Gamma(x + \frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1})}{\Gamma(\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1})x!} \cdot \frac{1}{\gamma}^{\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma - 1}} (1 - \frac{1}{\gamma})^x \end{aligned}$$