

Model of Super Spreading Events

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Model

Let $x_t = y_t + z_t$

where

y_t = Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t$ = Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

i. y_t - Non super-spreading events

$y_t \sim \text{Poisson}(\alpha \cdot \lambda_t)$,

$$p(y_t) = \exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t},$$

with

$$\lambda_t = \sum_{i=1}^{t-1} y_i \cdot (\text{Gamma}((t-i); k, \theta) - \text{Gamma}((t-i-1); k, \theta))$$

ii. z_t - super-spreading events

For each SSE event we get $\text{Poisson}(\gamma)$ infections

$z_t | n_t \sim \text{Poisson}(\gamma \cdot n_t)$ = Number of infecteds by all Super-Spreading Events at time t

$n_t \sim \text{Poisson}(\beta \cdot \lambda_t)$ = Number of super-spreading events

$$p(z_t) = \sum_{n_t=0}^{\infty} p(n_t = n_t) \cdot p(z_t | n_t = n_t)$$

$\therefore p(z_t)$ is a Poisson-Poisson Compound

Determining the Mean and Variance of z_t

$$\mathbb{E}(z_t) = \mathbb{E}(\mathbb{E}(z_t | n_t))$$

$$= \mathbb{E}(\gamma \cdot n_t)$$

$$= \gamma \mathbb{E}(n_t)$$

$$= \gamma \cdot \beta \cdot \lambda_t$$

$$\text{Var}(z_t) = \text{Var}(\mathbb{E}(z_t | n_t)) + \mathbb{E}(\text{Var}(z_t | n_t))$$

As $z_t|n_t$ and n_t are poisson rvs the mean and variance are equivalent giving;

$$\begin{aligned} &= \text{Var}(\gamma \cdot n_t) + \mathbb{E}(\gamma \cdot n_t) \\ &= \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t \end{aligned}$$

Poisson-Poisson Compound z_t in the form of a Negative Binomial RV

The mean and variance of the Poisson-Poisson Compound z_t is equated to that of a Negative Binomial distribution of size n with probability of success p and density given by;

$$\frac{\Gamma(x+n)}{\Gamma(n) \cdot x!} \cdot p^n (1-p)^x$$

for $x = 0, 1, 2, \dots, n > 0$ and $0 < p \leq 1$.

This represents the number of failures which occur in a sequence of Bernoulli trials before a target number of successes is reached. The mean and variance are respectfully;

$$\mu = \frac{n(1-p)}{p}, \quad \text{Var} = \frac{n(1-p)}{p^2}$$

Equating the poisson-poisson compound to that of the negative binomial distribution gives;

$$\text{I. } \mu = \frac{n(1-p)}{p} = \gamma \cdot \beta \cdot \lambda_t$$

$$\text{II } \text{Var} = \frac{n(1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

From **I** we can say that the size

$$n = \frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p} \quad \text{III.}$$

Substituting **III** into **II** gives;

$$\frac{\left(\frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p}\right) \cdot (1-p)}{p^2} = \gamma^2 \cdot \beta \cdot \lambda_t + \gamma \cdot \beta \cdot \lambda_t$$

Cancelling like terms gives;

$$\frac{\cancel{\left(\frac{p \cdot \gamma \cdot \beta \cdot \lambda_t}{1-p}\right)} \cdot \cancel{(1-p)}}{p^{\cancel{2}}} = \gamma^{\cancel{2}} \cdot \beta \cdot \cancel{\lambda_t} + \gamma \cdot \beta \cdot \cancel{\lambda_t}$$

$$\frac{1}{p} = \gamma + 1$$

$$1 = p \cdot \gamma + p$$

$$1 = p \cdot \gamma + p$$

$$\therefore p = \frac{1}{\gamma + 1} \quad \text{IV.}$$

Substituting **IV** into **III** gives;

$$n = \frac{\frac{1}{\gamma + 1} \cdot \gamma \cdot \beta \cdot \lambda_t}{1 - \frac{1}{\gamma + 1}}$$

$$n = \frac{\frac{\gamma \cdot \beta \cdot \lambda_t}{\gamma + 1}}{\frac{\gamma + 1 - 1}{\gamma + 1}}$$

$$n = \frac{\cancel{\gamma} \cdot \beta \cdot \lambda_t}{\frac{\cancel{\gamma + 1}}{\cancel{\gamma}}}$$

$$\therefore n = \beta \cdot \lambda_t$$

z_t can thus be written as a Negative Binomial of size n with probability of success p , i.e

$$z_t \sim \text{NB}(\beta \cdot \lambda_t, \frac{1}{\gamma + 1})$$

$$p(z_t) = \frac{\Gamma(x + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (1 - \frac{1}{\gamma + 1})^{z_t} \quad \text{for } z_t = 0, 1, 2, \dots, n > 0 \text{ and } 0 < p \leq 1$$

Likelihood

$$P(x_t) = \sum_{y_t=0}^{x_t} P(y_t) \cdot P(z_t)$$

$$\begin{aligned} L(\alpha, \beta, \gamma, \lambda | \mathbf{x}_t) &= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} P(y_t) \cdot P(z_t) \\ &= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} \exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (1 - \frac{1}{\gamma + 1})^{z_t} \\ &= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} \exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t} \end{aligned}$$

Log Likelihood

$$\begin{aligned} l(\alpha, \beta, \gamma, \lambda | \mathbf{x}_t) &= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} \ln(P(y_t) \cdot P(z_t)) \\ &= \sum_{y_t=0}^{x_t} \prod_{t=1}^{Ndays} \ln \left(\exp(-\alpha \cdot \lambda_t) \cdot \frac{1}{y_t!} \cdot (\alpha \cdot \lambda_t)^{y_t} \times \frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!} \cdot \frac{1}{\gamma + 1}^{\beta \cdot \lambda_t} \cdot (\frac{\gamma}{\gamma + 1})^{z_t} \right) \end{aligned}$$

Using the fact that the logarithm of a product is the sum of the logarithms gives;

$$\begin{aligned} l(\alpha, \beta, \gamma, \lambda | \mathbf{x}_t) &= \sum_{y_t=0}^{x_t} \left(- \sum_{t=1}^{Ndays} \alpha \cdot \lambda_t + \sum_{t=1}^{Ndays} \ln\left(\frac{1}{y_t!}\right) + \sum_{t=1}^{Ndays} \ln(y_t \cdot \alpha \cdot \lambda_t) + \sum_{t=1}^{Ndays} \ln\left(\frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!}\right) \right. \\ &\quad \left. + \sum_{y_t=0}^{x_t} \beta \cdot \lambda_t \cdot \ln\left(\frac{1}{\gamma + 1}\right) + \sum_{t=1}^{Ndays} z_t \cdot \ln\left(\frac{\gamma}{\gamma + 1}\right) \right) \end{aligned}$$

Simplifying this gives;

$$\begin{aligned}
l(\alpha, \beta, \gamma, \lambda_t | \mathbf{x}_t) = & \sum_{y_t=0}^{x_t} \left(-\alpha \sum_{t=1}^{Ndays} \lambda_t + \sum_{t=1}^{Ndays} \ln\left(\frac{1}{y_t!}\right) + \sum_{t=1}^{Ndays} \ln(y_t \cdot \alpha \cdot \lambda_t) + \sum_{t=1}^{Ndays} \ln\left(\frac{\Gamma(z_t + \beta \cdot \lambda_t)}{\Gamma(\beta \cdot \lambda_t) \cdot z_t!}\right) \right. \\
& \left. + \beta \cdot \ln\left(\frac{1}{\gamma + 1}\right) \sum_{t=1}^{Ndays} \lambda_t + \ln\left(\frac{\gamma}{\gamma + 1}\right) \sum_{t=1}^{Ndays} z_t \right)
\end{aligned}$$