Model of Super Spreading Events

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June 2021

Model

Let $x_t = y_t + z_t$

where

y_t = Number of infecteds not by a super-spreading event (NSSE)

$z_t = x_t - y_t$ = Number of infecteds infected by a super-spreading event (SSE)

Probability

$$P(x_t) = \sum_{y_t=0}^{x_t} P_{NSSE}(y_t) \cancel{\circledast} P_{SSE}(z_t)$$

where

i. $P_{NSSE}(y_t) = exp(-\lambda_t)\frac{1}{y_t!}\lambda_t^{y_t}$,

$y_t \sim Poisson(\alpha \lambda_t)$.

with

$$\lambda_t = \cancel{R_0} \sum_{i=1}^{t-1} y_i \; (\; Gamma((t-i); k, \theta) - Gamma((t-i-1); k, \theta)$$

ii. $\cancel{P_{SSE}(z_t) = e^{-\lambda'_t}\frac{1}{z_t!}\lambda_t'^{z_t}}$,

For each SSE we get Poisson(γ) infecteds

Super-spreading events

$z_t | n_t$

$\cancel{z_t} \sim Poisson(n_t \cdot \cancel{s_{it}})$ = Number of infecteds by all Super-Spreading Events at time t

$\boxed{n_t \sim Poisson(\cancel{\lambda_{2t}})}$ = Number of ~~super-spreaders~~ /super-spreading events

$\beta \lambda_t$

~~$s_{it} \sim Poisson(r0_2 \cdot \lambda_t)$ = Number of infecteds by a super-spreader at time t~~

I.e

~~$z_t \sim Poisson(n_t \cdot s_{it}) = s_{1t} + s_{2t} + ... + s_{nt} = z_t = \sum_{i=0}^{n_t} s_{it}$~~

Probability(z_t)

$$p(z_t) = \sum_{n_t=0}^{\infty} p(n_t)\,p(z_t|n_t).$$

Poisson-Poisson compound.

$$\cancel{\text{P}(z_t) = \sum_{n_t=0}^{\infty}\sum_{i=0}^{n_t} p(s_{it} = s_{it})(n_t = n_t) \times p(z_t|s_{it}, n_t)}$$

$$\cancel{\text{P}(z_t) = \sum_{n_t=0}^{\infty}\sum_{i=0}^{n_t} \frac{1}{s_{it}!}e^{-\lambda_t \cdot r0_2}(\lambda_t \cdot r0_2)^{s_{it}} \times \frac{1}{n_t!}e^{-\lambda_{2t}}\lambda_2^{n_t} \times \frac{1}{z_t!}e^{-s_{it}\cdot n_t}(s_{it}n_t)^{z_t}}$$

Prove of Compound Poisson Distribution & Negative Binomial Equivalence

Prove z_t, the number of infecteds by all Super-Spreading Events at time t, is distributed according to a Negative Binomial Distribution

Proof
To prove this the probability generating function (PGF) of z_t, G_{zt}, is calculated. This is the composition of the PGFs G_{nt} and G_{st}, i.e;

$G_{zt}(\omega) = G_{nt}(G_{st}(\omega))$

where G_{nt} and G_{st} are the PGFs of a Poisson distribution, i.e;

$G_{nt} = exp(\lambda_2(\omega - 1))$

$G_{st} = exp(r0_2 \cdot \lambda_t(\omega - 1))$

This gives;

$G_{zt}(\omega) = exp(\lambda_2(exp(r0_2 \cdot \lambda_t(\omega - 1)) - 1))$

This needs to be of the form;

$\left(\dfrac{1-p}{1-p\omega}\right)^r$ which is the PGF of the Negative Binomial Distribution, NB(r,p), whereby;

r is the number of 'successes' i.e the number of superspreaders and p is the probability of each success.

*Prove of Poisson Compound Distribution & Negative Binomial Equivalence

Let $n_t \sim Poisson(-rln(1-p))$,

$s_{i_t} \sim Log(p)$ with pmf $f(k; r, p) = \dfrac{-p^k}{kln(1-p)}$,

where

k is the number of failures (non super-spreading events).

Then the random sum;

$\sum_{i=0}^{n_t} s_{i_t}$ is distributed according to a Negative Binomial Distribution, NB(r, p)

Proof

To prove this the PGF G_{zt} of X is calculated, the composition of G_{nt} and G_{si_t}, i.e

$G_{zt}(\omega) = G_{nt}(G_{si_t}(\omega))$

where

$G_{nt} = exp(\lambda(\omega - 1))$

$G_{si_t} = \dfrac{ln(1-p\omega)}{ln(1-p)}, |\omega| < \dfrac{1}{p}$

This gives;

$G_{zt}(\omega) = G_{nt}(s_{i_t}(\omega))$

$= exp\left(\lambda\left(\frac{ln(1-p\omega)}{ln(1-p)} - 1\right)\right)$

$= exp\left(-r(ln(1-p\omega) - ln(1-p))\right)$

$\left(\frac{1-p}{1-p\omega}\right)^r, |\omega| < \frac{1}{p}$ which is the PGF of the Negative Binomial Distribution;