Tsinghua-Berkeley Shenzhen Institute
Information Theory and Statistical Learning
Fall 2020

Problem Set 1

Issued: Monday 14th September, 2020 **Due:** Monday 21st September, 2020

Notations: We use $\mathsf{x}, \mathsf{y}, \mathsf{w}$ and $\underline{\mathsf{x}}, \underline{\mathsf{y}}, \underline{\mathsf{w}}$ to denote random variables and random vectors.

1.1. *Mathematical Expectation, Variance, and Covariance Matrix* Prove the following properties, where $\underline{\mathsf{x}}$ is a random vector in $\mathbb{R}^k$.

(a) i. $\mathbb{E}[\mathsf{x}|\mathsf{y}] = \mathbb{E}[\mathbb{E}[\mathsf{x}|\mathsf{yz}]|\mathsf{y}]$.

ii. $\mathbb{E}[\mathsf{x}g(\mathsf{y})|\mathsf{y}] = g(\mathsf{y})\,\mathbb{E}[\mathsf{x}|\mathsf{y}]$, for all functions $g\colon \mathcal{Y} \to \mathbb{R}$.

iii. $\mathbb{E}[\mathsf{x}\,\mathbb{E}[\mathsf{x}|\mathsf{y}]] = \mathbb{E}[(\mathbb{E}[\mathsf{x}|\mathsf{y}])^2]$.

iv. $\operatorname{var}(\mathsf{x}) = \mathbb{E}[\operatorname{var}(\mathsf{x}|\mathsf{y})] + \operatorname{var}(\mathbb{E}[\mathsf{x}|\mathsf{y}])$.

(b) i. $\operatorname{cov}(\underline{\mathsf{x}}) = \mathbb{E}[\operatorname{cov}(\underline{\mathsf{x}}|\mathsf{y})] + \operatorname{cov}(\mathbb{E}[\underline{\mathsf{x}}|\mathsf{y}])$.

ii. $\det(\operatorname{cov}(\underline{\mathsf{x}})) = 0 \iff \exists\, \underline{c} \in \mathbb{R}^k \setminus \{\underline{0}\}$, such that $\underline{c}^{\mathrm{T}}\underline{\mathsf{x}}$ is a constant.

1.2. The Pearson correlation coefficient $\rho(\mathsf{x}, \mathsf{y})$ of two random variables x and y is defined as

$$\rho(\mathsf{x}, \mathsf{y}) \triangleq \frac{\mathbb{E}[(\mathsf{x} - \mathbb{E}[\mathsf{x}])(\mathsf{y} - \mathbb{E}[\mathsf{y}])]}{\sqrt{\operatorname{var}(\mathsf{x}) \operatorname{var}(\mathsf{y})}}. \tag{1}$$

(a) When $\operatorname{var}(\mathsf{x}) = \operatorname{var}(\mathsf{y})$, prove that $\rho = a^*$ where a^* is the coefficient in the linear regression problem:

$$(a^*, b^*) \triangleq \operatorname*{arg\,min}_{(a,b)\in\mathbb{R}^2} \mathbb{E}[(\mathsf{y} - a\mathsf{x} - b)^2].$$

(b) Prove that

$$\mathsf{x} \perp\!\!\!\perp \mathsf{y} \iff \forall f, g,\ \rho(f(\mathsf{x}), g(\mathsf{y})) = 0.$$

1.3. Suppose x has a normal distribution $\mathsf{x} \sim \mathcal{N}(0, 1)$. Please compute the density of $\exp(\mathsf{x})$. (The answer is called the **lognormal distribution**.)