Tsinghua-Berkeley Shenzhen Institute
INFORMATION THEORY AND STATISTICAL LEARNING
Fall 2020

Homework 1

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- *I certify that all solutions are entirely in my words and that I have not looked at another student's solutions. I have credited all external sources in this write up.*

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1.1. *Mathematical Expectation, Variance, and Covariance Matrix*

(a) i. $\mathbb{E}[x|y] = \mathbb{E}[\mathbb{E}[x|yz]|y]$

Proof:

$$\mathbb{E}[x|yz] = \sum_x xP_{X|YZ}(x|yz) \tag{1}$$

Denote $\mathbb{E}[x|yz]$ as $g(y,z)$

$$\begin{aligned}
E[g(y,z)|y] &= \sum_z g(y,z)P_{Z|Y}(z|y) \\
&= \sum_z \left(\sum_x xP_{X|YZ}(x|yz) \right) P_{Z|Y}(z|y) \\
&= \sum_{x,z} xf_{X|YZ}(x|yz)P_{Z|Y}(z|y) \\
&= \sum_{x,z} xP_{X,Z|Y}(x,z|y) \\
&= \sum_x xP_{X|Y}(x|y) \\
&= \mathbb{E}[x|y]
\end{aligned} \tag{2}$$

So we have $\mathbb{E}[x|y] = \mathbb{E}[\mathbb{E}[x|yz]|y]$

ii. $\mathbb{E}[xg(y)|y] = g(y)\mathbb{E}[x|y]$

Proof:

$$\begin{aligned}
\mathbb{E}[xg(y)|y] &= \sum_x xg(y)P_{X|Y}(x|y) \\
&= g(y)\sum_x xP_{X|Y}(x|y) \quad (g(y) \text{ is constant with } y \text{ fixed}) \\
&= g(y)\mathbb{E}[x|y]
\end{aligned} \tag{3}$$

iii. $\mathbb{E}[x\mathbb{E}[x|y]] = \mathbb{E}\left[(\mathbb{E}[x|y])^2\right]$

Proof:
Denote $\mathbb{E}[x|y]$ as $g(y)$, using the conclusion from Equation [3], considering the following expression $\mathbb{E}[xg(y)|y]$

$$\mathbb{E}[xg(y)|y] = g(y)\mathbb{E}[x|y] = (\mathbb{E}[x|y])^2 \tag{4}$$

Considering the expectation of each side in the above equation,

$$\begin{aligned} \mathbb{E}[\mathbb{E}[xg(y)|y]] &= \mathbb{E}[xg(y)] \quad \text{(Total Expectation Rule)} \\ &= \mathbb{E}[x\mathbb{E}(x|y)] \\ &= \mathbb{E}[(\mathbb{E}[x|y])^2] \end{aligned} \tag{5}$$