Problem Set 4

Issued: Monday 16th November, 2020 **Due:** Monday 30th November, 2020

Notations: We use $\mathsf{x}, \mathsf{y}, \mathsf{w}$ and $\underline{\mathsf{x}}, \underline{\mathsf{y}}, \underline{\mathsf{w}}$ to denote random variables and random vectors.

4.1. Please review Chapter 12 in Cover's book, then you can get some ideas on how to find the K-L divergence in Sanov's Theorem. Let x_i be i.i.d. $\sim \mathcal{N}(0, \sigma^2)$:

(a) Find the behavior of $-\frac{1}{n} \log \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n \mathsf{x}_i^2 \geq \alpha^2\right)$. This can be done from the first principles (since the normal distribution is nice) or by using Sanov's theorem.

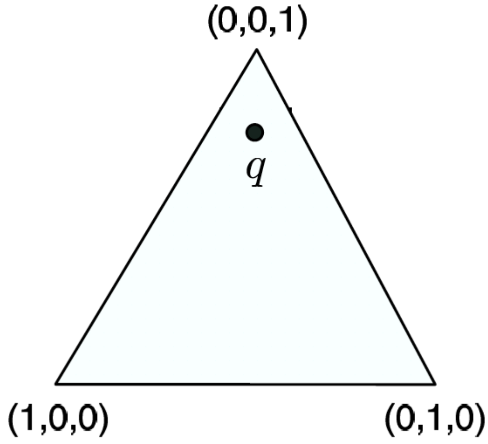
(b) What does the data look like if $\frac{1}{n}\sum_{i=1}^n \mathsf{x}_i^2 \geq \alpha^2$. That is, what is the distribution that minimizes the K-L divergence in the Sanov's theorem.

4.2. We hope to derive an asymptotic value of $\begin{pmatrix} n \\ k \end{pmatrix}$.

(a) Firstly, let's prove the lemma about Stirling's approximation of factorials, which we have used before.
$$\left(\frac{n}{e}\right)^n \leq n! \leq n\left(\frac{n}{e}\right)^n$$
Please justify the following steps:
$$\ln(n!) = \sum_{i=2}^{n-1} \ln i + \ln n \leq \cdots$$
$$\ln(n!) = \sum_{i=1}^{n} \ln i \geq \cdots$$

(b) If $0 < p < 1$, and $k = \lfloor np \rfloor$, i.e., k is the largest integer less than or equal to np, then please find
$$\lim_{n\to\infty} \frac{1}{n} \log \begin{pmatrix} n \\ k \end{pmatrix}$$
Could you explain it without Stirling's Approximation?
Now let p_i's be a probability distribution on m symbols. Guess what is
$$\lim_{n\to\infty} \frac{1}{n} \log \begin{pmatrix} & & n & & \\ \lfloor np_1 \rfloor & \lfloor np_2 \rfloor & \cdots & \lfloor np_{m-1} \rfloor & (n - \sum_{i=1}^{m-1} \lfloor np_i \rfloor) \end{pmatrix}$$

4.3. Consider the set of distributions on $\Omega = \{0, 1, 2\}$ and note that they lie on the 2-simplex
$$\{p = (p_0, p_1, p_2) : p_0 + p_1 + p_2 = 1, p_0 \geq 0, p_1 \geq 0, p_2 \geq 0\}$$
represented by the triangular figure. Let y be a random variable such that $p_{\mathsf{y}}(i) = p_i, i \in \{0, 1, 2\}$. Let $q = (1/6, 1/6, 2/3)$ be a particular probability mass function.



(a) Draw on the simplex the linear family corresponding to the expectation $\mathbb{E}[\mathsf{y}] = 0$, i.e. draw $\mathcal{L}_0 = \{p : \mathbb{E}_p[\mathsf{y}] = 0\}$.

(b) Draw $\mathcal{L}_{1/2} = \{p : \mathbb{E}_p[\mathsf{y}] = 1/2\}$

(c) Specify the exponential family $\mathcal{E}$ that passes through q and is orthogonal to $\mathcal{L}_{1/2}$, and draw the entire family on the 2-simplex.
Hint: Remember we introduced two versions of the exponential family, which are Lagrange-Multiplier induced one and parameterized one. You might be confused when you are facing cardinality-3 distributions, especially the Lagrange-Multiplier induced one. It is good if you can think about the equivalency of the two versions. Let's do the problem firstly under the parameterized version. That is $\mathcal{E} = \{\tilde{q} : \tilde{q} = qe^{sf(\mathsf{y})-\alpha(s)}\}$. Following the definition above, $f(\mathsf{y}) = \mathsf{y}$.

(d) Calculate the I-projection $p^\star$ of q onto $\mathcal{L}_{1/2}$ and mark it on the simplex.

(e) Draw $\mathcal{P} = \{p : \mathbb{E}_p[\mathsf{y}] \leq 1/2\}$.

(f) Calculate the I-projection $p^\star$ of q onto $\mathcal{P}$ and mark it.
Hint: $D(\cdot\|q)$ is convex in its first argument.

4.4. Let $q(y) > 0$ $(y = 0, 1, \cdots)$ be a probability mass function for a random variable y and let $\mathcal{P}$ be the set of all PMFs defined over $\{0, \cdots, M-1\}$ for a known constant M:

$$\mathcal{P} \triangleq \{p : p(y) = 0,\ \forall y \geq M\}.$$

We can represent each element p of $\mathcal{P}$ as a M-dimensional vector $[p_0, \cdots, p_{M-1}]^{\mathrm{T}}$ that lies on a $(M-1)$-dimensional simplex, i.e., $\sum_{m=0}^{M-1} p_m = 1$.

(a) Show that, for all $p \in \mathcal{P}$, $D(q\|p) = \infty$

(b) Show that, for all $p \in \mathcal{P}$, $D(p\|q) < \infty$

(c) Find the I-projection of q onto $\mathcal{P}$, $p^\star = \arg\min_{p\in\mathcal{P}} D(p\|q)$, and the corresponding divergence $D(p^\star\|q)$ in terms of $Q(y) \triangleq \mathbb{P}(\mathsf{y} \leq y)$, the CDF of the random variable y.
Let $\mathcal{P}_\epsilon$ be the space of all PMFs with weight of ϵ on values M and above:

$$\mathcal{P}_\epsilon \triangleq \left\{p : \sum_{y=M}^{\infty} p(y) = \epsilon\right\}$$

We can think of $\mathcal{P}_\epsilon$ as an extension of $\mathcal{P}$ to the distributions defined for all integers that only allows limited weight to be allocated to the values outside $\{0, \cdots, M-1\}$.

(d) Find the I-projection of q onto $\mathcal{P}_\epsilon$, $p_\epsilon^\star = \arg\min_{p \in \mathcal{P}_\epsilon} D(p\|q)$ and the corresponding divergence $D(p_\epsilon^\star\|q)$ in terms of $Q(y)$. Show that $\lim_{\epsilon \to 0} D(p_\epsilon^\star\|q) = D(p^\star\|q)$.

(e) Show that $\mathcal{P}_\epsilon$ can be represented as a linear family of PMFs.

(f) Show that $p_\epsilon^\star$ belongs to an exponential family through q and find the value of the parameter that corresponds to $p_\epsilon^\star$.

4.5. *Joint Gaussian Distribution.* Suppose $\underline{\mathsf{x}} = (\mathsf{x}_1, \mathsf{x}_2)^{\mathrm{T}}$ is a Gaussian random vector with $\mathbb{E}[\mathsf{x}_1] = \mathbb{E}[\mathsf{x}_2] = 0, \mathrm{var}(\mathsf{x}_1) = \mathrm{var}(\mathsf{x}_2) = \sigma^2$, and $\rho_{\mathsf{x}} \triangleq \rho(\mathsf{x}_1, \mathsf{x}_2)$ denoting the correlation coefficient between x_1 and x_2. Let $\underline{\mathsf{y}} = (\mathsf{y}_1, \mathsf{y}_2)^{\mathrm{T}} \triangleq \mathbf{A}\underline{\mathsf{x}}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & -\rho_{\mathsf{x}} \\ 0 & 1 \end{bmatrix}.$$

Then, $\underline{\mathsf{y}}$ is also a Gaussian random vector, since it is a linear transformation of $\underline{\mathsf{x}}$.

(a) Calculate $\mathbf{K}_{\mathsf{x}} \triangleq \mathrm{cov}(\underline{\mathsf{x}})$ and $\mathbf{K}_{\mathsf{y}} \triangleq \mathrm{cov}(\underline{\mathsf{y}})$.

(b) Prove that $\rho(\mathsf{y}_1, g(\mathsf{y}_2)) = 0$, for all functions[1] $g(\cdot)$. *Hint:* First prove that $\mathsf{y}_1 \perp\!\!\!\perp \mathsf{y}_2$.

(c) Prove that $\mathbb{E}[(\mathsf{x}_1 - \rho_{\mathsf{x}}\mathsf{x}_2)^2] \leq \mathbb{E}[(\mathsf{x}_1 - g(\mathsf{x}_2))^2]$, for all functions $g \colon \mathbb{R} \to \mathbb{R}$. *Hint:* Rewrite the inequality using y_1 and y_2.

4.6. *Mathematical expectation and variance in estimation.* Suppose we want to estimate the value of y using an estimator $\hat{\mathsf{y}}$, and using its MSE (Mean Square Error) to evaluate the goodness of estimate, defined as

$$\mathrm{MSE}(\hat{\mathsf{y}}) \triangleq \mathbb{E}[(\mathsf{y} - \hat{\mathsf{y}})^2].$$

The estimator $\hat{\mathsf{y}}$ could be chosen from a set $\mathcal{A}$, and our goal is to find the best estimator in $\mathcal{A}$ which achieves the least MSE. Then the best estimator is called the MMSE (Minimum Mean Square Error) estimator.

(a) Assume we want to use a real number to estimate y, i.e., $\mathcal{A} = \mathbb{R}$.

i. Prove that $\mathbb{E}[\mathsf{y}]$ is the MMSE estimator:

$$\mathbb{E}[\mathsf{y}] = \arg\min_{\alpha \in \mathbb{R}} \mathbb{E}[(\mathsf{y} - \alpha)^2].$$

ii. Evaluate this estimator's MSE.

(b) Now you are allowed to use a function of x to estimate y, i.e., $\mathcal{A} = \{f(\cdot) : \mathcal{X} \mapsto \mathbb{R}\}$. Prove that:

i. $\mathbb{E}[\mathsf{y}|\mathsf{x}]$ is the MMSE estimator:

$$\mathbb{E}[\mathsf{y}|\mathsf{x}] = \arg\min_{f \colon \mathcal{X} \mapsto \mathbb{R}} \mathbb{E}[(\mathsf{y} - f(\mathsf{x}))^2],$$

[1] Strictly speaking, $g(\cdot)$ is required to be measurable.

ii. The MSE of estimator $\mathbb{E}[\mathsf{y}|\mathsf{x}]$ is

$$\mathrm{MSE}(\mathbb{E}[\mathsf{y}|\mathsf{x}]) = \mathbb{E}[\mathrm{var}(\mathsf{y}|\mathsf{x})].$$

(c) Compare these two estimators. First, prove that

$$\mathsf{x} \perp\!\!\!\perp \mathsf{y} \implies \mathrm{MSE}(\mathbb{E}[\mathsf{y}]) = \mathrm{MSE}(\mathbb{E}[\mathsf{y}|\mathsf{x}]) \implies \forall f,\ \rho(f(\mathsf{x}), \mathsf{y}) = 0,$$

where $\rho(\cdot,\cdot)$ is the Pearson correlation coefficient. In general, which one of these two estimators would have less MSE than the other?

4.7. Consider the estimation of one-hot encoded vectors, where the settings are similar to those of Problem 3.3. In particular, suppose y takes values from $\mathcal{Y} = \{1, 2, \cdots, k\}$, then its one hot encoding is a k-dimensional vector defined as $\underline{\mathsf{y}} \triangleq (\mathbb{1}_{\mathsf{y}=1}, \mathbb{1}_{\mathsf{y}=2}, \cdots, \mathbb{1}_{\mathsf{y}=k})^{\mathrm{T}}$, i.e., $\underline{\mathsf{y}}$ is the i-th vector of the standard basis if $\mathsf{y} = i$.

Now, we would use $\underline{\hat{\mathsf{y}}}$ to estimate $\underline{\mathsf{y}}$, and use its MSE to evaluate the goodness of estimate. The MSE is defined similarly as the scalar case, except that the scalar quadratic operator is replaced by the ℓ_2 norm squared:

$$\mathrm{MSE}(\underline{\hat{\mathsf{y}}}) \triangleq \mathbb{E}[\|\underline{\mathsf{y}} - \underline{\hat{\mathsf{y}}}\|_2^2].$$

Again, the estimator $\underline{\hat{\mathsf{y}}}$ could be chosen from a set $\mathcal{A}$.

(a) Suppose we want to use a vector to estimate $\underline{\mathsf{y}}$, i.e., $\mathcal{A} = \mathbb{R}^k$. Prove that $\underline{P}_{\mathsf{y}}(\cdot)$ is the MMSE estimator:

$$\underline{P}_{\mathsf{y}}(\cdot) = \operatorname*{arg\,min}_{\underline{\alpha} \in \mathbb{R}^k} \mathbb{E}[\|\underline{\mathsf{y}} - \underline{\alpha}\|_2^2],$$

where $\underline{P}_{\mathsf{y}}(\cdot) \triangleq [P_{\mathsf{y}}(1), P_{\mathsf{y}}(2), \cdots, P_{\mathsf{y}}(k)]^{\mathrm{T}}$.

(b) Now you are allowed to use a multivariant function of x to estimate $\underline{\mathsf{y}}$, i.e., $\mathcal{A} = \{\underline{f} : \mathcal{X} \mapsto \mathbb{R}^k\}$. Prove that the MMSE estimator is $\underline{P}_{\mathsf{y}|\mathsf{x}}(\cdot|\mathsf{x})$:

$$\underline{P}_{\mathsf{y}|\mathsf{x}}(\cdot|\mathsf{x}) = \operatorname*{arg\,min}_{\underline{f}:\ \mathcal{X} \mapsto \mathbb{R}^k} \mathbb{E}[\|\underline{\mathsf{y}} - \underline{f}(\mathsf{x})\|_2^2],$$

where $\underline{P}_{\mathsf{y}|\mathsf{x}}(\cdot|\mathsf{x}) \triangleq [P_{\mathsf{y}|\mathsf{x}}(1|\mathsf{x}), P_{\mathsf{y}|\mathsf{x}}(2|\mathsf{x}), \cdots, P_{\mathsf{y}|\mathsf{x}}(k|\mathsf{x})]^{\mathrm{T}}$.

4.8. The data $\mathsf{x}[n] = ar^n + \mathsf{w}[n]$ for $n = 0, \cdots, N-1$ are observed. The random variables $\mathsf{w}[0], \cdots, \mathsf{w}[N-1]$ are i.i.d. Gaussian random variables with zero mean and variance σ^2. r is a non-zero constant. Find the Cramér-Rao bound for a. Does an efficient estimator exist? If so, what is it and what is its variance?