Probability Theory Exercise 5

Issued: 2020/11/19 Due: 2020/12/3

1. Suppose that the marginal distributions of both X and Y are standard normal distribution $N(0,1)$, and we do not make any other assumptions on the joint distribution of X and Y. Is it possible that the distribution of $X+Y$ is not Gaussian distribution? (We view the zero random variable as a special case of Gaussian distribution, i.e., if a random variable is equal to 0 with probability 1 then we say that it has Gaussian distribution $N(0,0)$.) If your answer is yes, give such an example. If your answer is no, please explain why.

2. Let X and Y be i.i.d. random variables with mean 0 and variance 1. Suppose that $(X+Y)/\sqrt{2}$ has the same distribution as X. Find **all** possible distributions of X that satisfy the above conditions. Prove that the distributions you find are the **only** distributions that satisfy the above conditions.

3. Let $(X_1, X_2, \dots, X_{2n-1})$ be a random vector with density function

$$f_{X_1,\dots,X_{2n-1}}(x_1,\dots,x_{2n-1}) = c_n \exp\left(-\frac{1}{2}\left(x_1^2 + \sum_{i=1}^{2n-2}(x_{i+1}-x_i)^2 + x_{2n-1}^2\right)\right),$$

where c_n is the normalizing constant. (Notice that there are $2n$ **square terms, not** $2n-1$ **square terms** in the exponent of the density function.) Prove that $(X_1, X_2, \dots, X_{2n-1})$ is a Gaussian random vector and find the value of c_n. Also find the variance $\mathrm{Var}(X_n)$. (You only need to find $\mathrm{Var}(X_n)$. You don't need to calculate the variance of every X_i.)

4. Let $X_1, X_2, \dots$ be i.i.d. Cauchy random variables with PDF $f(x) = \frac{1}{\pi(1+x^2)}$. Let $S_n = X_1+X_2+\dots+X_n$.
(i) Does $\{S_n/n\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If your answer is yes, what is the limit distribution?
(ii) Does $\{S_n/n^2\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If your answer is yes, what is the limit distribution?
(iii) Does $\{S_n/\sqrt{n}\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If your answer is yes, what is the limit distribution?

5. Let $X_1, X_2, X_3, \dots$ be independent random variables with distribution $P(X_i = i) = P(X_i = -i) = 1/2$ for all i. (Note that the distribution of each X_i is **different**!) Define $S_n = X_1 + X_2 + \dots + X_n$ for every positive integer n.
(i) Does $\{S_n/n^2\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If your answer is yes, what is the limit distribution?
(ii) Does $\{S_n/n^{3/2}\}_{n=1}^{\infty}$ converge in distribution? Please explain why. If your answer is yes, what is the limit distribution?
(iii) For every real number $x \in \mathbb{R}$, find the limit $\lim_{n\to\infty} P\left(\frac{S_n}{n} \le x\right)$.

6. Let $X_1, X_2, X_3, \dots$ be i.i.d. random variables with distribution $P(X_i = 1) = P(X_i = 1/2) = 1/2$ for all i. For all positive integer n, define $Y_n = \prod_{i=1}^{n} X_i$ and $S_n = \sum_{i=1}^{n} Y_i$.
(i) Does $\{S_n\}_{n=1}^{\infty}$ converge almost surely to some limit random variable? Please explain why. If your answer is yes, find the mean and variance of the limit random variable.
(ii) Now suppose that the distribution of X_i is $P(X_i = 2) = P(X_i = 1/4) = 1/2$ for all i, and the definition of Y_n and S_n remain the same. In this case, does $\{S_n\}_{n=1}^{\infty}$ converge almost surely to some limit random variable? Please explain why.