

Exercise 7 & 8

Probability Theory 2020 Autumn

Hanmo Chen
2020214276

December 23, 2020

1 Problem 1

Denote the entries in U as u_{ij} and entries in Y as Y_{ij} , so

$$Y_{ij} = \sum_{r,s} u_{ri} u_{sj} X_{rs} \quad (1)$$

Also we have,

$$\text{Cov}(X_{ij}, X_{mn}) = \begin{cases} 2, & i = j = m = n \\ 1, & (i, j) = (m, n) \text{ or } (i, j) = (n, m), i \neq j \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Thus

$$\begin{aligned} \text{Cov}(Y_{ij}, Y_{mn}) &= \text{Cov} \left(\sum_{r,s} u_{ri} u_{sj} X_{rs}, \sum_{p,q} u_{pm} u_{qn} X_{pq} \right) \\ &= 2 \sum_{r=1}^n u_{ri} u_{rj} u_{rm} u_{rn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rm} u_{sn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rn} u_{sm} \\ &= \sum_{r,s} [u_{ri} u_{sj} u_{rm} u_{sn} + u_{ri} u_{sj} u_{rn} u_{sm}] \\ &= \left(\sum_r u_{ri} u_{rm} \right) \left(\sum_s u_{sj} u_{sn} \right) + \left(\sum_r u_{ri} u_{rn} \right) \left(\sum_s u_{sj} u_{sm} \right) \end{aligned} \quad (3)$$

Denote the column vectors in U as $\mathbf{u}_i, i = 1, 2, 3, \dots, N$, so $\mathbf{u}_i \cdot \mathbf{u}_j = \sum_r u_{ri} u_{rj} = \delta_{ij}$.

$$\text{Cov}(Y_{ij}, Y_{mn}) = \delta_{im} \delta_{jn} + \delta_{jm} \delta_{in} = \begin{cases} 2, & i = j = m = n \\ 1, & (i, j) = (m, n) \text{ or } (i, j) = (n, m), i \neq j \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

Because $X_{ij}, j \geq i$ are independent Gaussian variables, so the joint distribution of Y_{ij} is joint Gaussian distribution, which means,

$$\text{Cov}(Y_{ij}, Y_{mn}) = 0 \iff Y_{ij}, Y_{mn} \text{ are independent} \quad (5)$$

So all the entries on and above the diagonal of Y are independent, and $Y_{ii} \sim N(0, 2), i = 1, 2, 3, \dots, N$ and $Y_{ij} \sim N(0, 1), 1 \leq i < j \leq n$. (It is easy to see that $\mathbb{E}[Y_{ij}] = 0$)

2 Problem 2

Notice that

1. If $X_n = 1$, then X_{n+1}, X_{n+2}, \dots are independent of X_1, X_2, \dots, X_n
2. There is at least one 1 in any five-in-a-row X_i s as $\{X_n, X_{n+1}, \dots, X_{n+4}\}$

So we can split X_1, X_2, \dots, X_n into a series of epsisodes, each episode $L_j = [0, \dots, 0, 1]$ is consisted of n zeros (n can be $0, 1, 2, 3, 4$) and 1 one. And $L_j, j = 1, 2, \dots, m$ are independent. (For the last episode, if it is ended with 0, we can append 1 to its end and let $n = n + 1$.) Denote the length of each episode as l_j , so $\sum_{j=1}^m l_j = n$.

Consider the distribution of l_j , it can only take values in $1, 2, 3, 4, 5$,

- $P(l_j = 1) = P(X_1 = 1) = 0.2$
- $P(l_j = 2) = P(X_1 = 0, X_2 = 1) = 0.16$
- $P(l_j = 3) = P(X_1 = 0, X_2 = 0, X_3 = 1) = 0.128$
- $P(l_j = 4) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1) = 0.1024$
- $P(l_j = 5) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0) = 0.4096$

So

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} = \lim_{m \rightarrow \infty} \frac{m}{l_1 + l_2 + \dots + l_m} = \lim_{m \rightarrow \infty} \frac{1}{\frac{1}{m} \sum_{j=1}^m l_j} \quad (6)$$

According to Strong Law of Large Numbers,

$$\frac{1}{m} \sum_{j=1}^m l_j \xrightarrow{a.s.} E[l_j] = 3.3616 \quad (7)$$

So

$$\lim_{n \rightarrow \infty} \frac{S_n}{n} \xrightarrow{a.s.} \frac{1}{3.3616} \quad (8)$$

3 Problem 3

3.1 (i)

Suppose the corresponding k of X_n is k_n , i.e. $\sum_{i=1}^{k_n} Y_i = X_n + n$. If $X_n \geq 1$, $\sum_{i=1}^{k_n} Y_i \geq n + 1$, so $k_{n+1} = k_n, X_{n+1} = X_n - 1$. If $X_n = 0$, $\sum_{i=1}^{k_n} Y_i = n, \sum_{i=1}^{k_n+1} Y_i = n + Y_{n+1} \geq n + 1$, so $k_{n+1} = k_n, X_{n+1} = Y_{n+1} - 1$.

So given X_n , X_{n+1} is independent of X_{n-1}, \dots, X_1 . $\{X_n\}_{n=1}^\infty$ forms a Markov Chain. And the transition probability is,

$$P(X_{n+1} = i | X_n = 0) = p_{i+1}, i = 0, 1, \dots \quad (9)$$

$$P(X_{n+1} = i | X_n = j, j \geq 1) = \begin{cases} 1, & i = j - 1 \\ 0, & \text{otherwise} \end{cases} \quad (10)$$

3.2 (ii)

Notice that $f(n) = P(X_n = 0)$, so $\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} P(X_n = 0)$. If we want $\lim_{n \rightarrow \infty} f(n)$ exists, the Markov chain must be irreducible, aperiodic and positive recurrent.

It is irreducible obviously. Consider the support set $\mathcal{Y} = \{i : p_i > 0\}$ of Y , if $\inf \mathcal{Y} = N < \infty$, the state space \mathcal{S} of the Markov Chain is finite $\{0, 1, \dots, N\}$. Obviously N can be reached from 0. And because $N - 1, N - 2, \dots, 0$ can be reached from N , so it is irreducible. If $\inf \mathcal{Y} = \infty$, for any state n , there exists a state $m > n$, and m can be reached from 0, so n can be reached from 0. In that case, the Markov chain is also irreducible.

For it to be aperiodic, if it comes from 0 to i , it will return to 0 in i steps. So if $\mathcal{Y} = \{i : p_i > 0\}$ is like $\{2, 4, \dots, 2k, \dots\}$ or $\{3, 6, 9, \dots, 3k, \dots\}$, for certain steps it will not arrive at 0. So the Markov chain is aperiodic if and only if $\gcd(\mathcal{Y}) = 1$

And it is positive recurrent if and only if $\mathbb{E}[T_0] < \infty$. It is easy to see that $P(T_0 = i + 1) = p_i, i \geq 1$, so

$$\mathbb{E}[T_0] = \sum_{i=1}^{\infty} i p_i = \mathbb{E}[Y_1] \quad (11)$$

So the necessary and sufficient condition for $\lim_{n \rightarrow \infty} f(n)$ to exist is $\gcd(\{i + 1 : p_i > 0\}) = 1$ and $\sum_{i=1}^{\infty} i p_i < \infty$

3.3 (iii)

The limit equals to the steady-state probability,

$$\pi_0 = \lim_{n \rightarrow \infty} f(n) = \frac{1}{\mathbb{E}[T_0]} = \frac{1}{\mu} \quad (12)$$

4 Problem 4

4.1 (i)

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geq 1 | X_0 = n)$ So the probability that the chain never returns to zero is $f(0) = f(1)$. When $X_0 = 0$, $X_1 = 1$, and X_2 must be 2. So $f(1) = \frac{4}{5}f(2)$.

Consider $f(2)$,

$$\begin{aligned}
f(2) &= P(X_n > 0, \forall n \geq 1 | X_0 = 2) \\
&= P(X_1 = 1, X_n > 0, \forall n \geq 2 | X_0 = 2) + P(X_1 = 3, X_n > 0, \forall n \geq 2 | X_0 = 2) \\
&= p_{21}P(X_n > 0, \forall n \geq 1 | X_0 = 1) + p_{23}P(X_n > 0, \forall n \geq 1 | X_0 = 3) \\
&= \frac{4}{13}f(1) + \frac{9}{13}f(3)
\end{aligned} \tag{13}$$

Because $f(2) = \frac{5}{4}f(1)$

$$f(3) - f(2) = \frac{4}{9}(f(2) - f(1)) = \frac{4}{9} \times \frac{1}{4}f(1) = \frac{1}{9}f(1) \tag{14}$$

In general, we have

$$f(n+1) - f(n) = \frac{n^2}{(n+1)^2}(f(n) - f(n-1)) = \frac{1}{(n+1)^2}f(1) \tag{15}$$

So

$$f(n) = \sum_{i=1}^n \frac{1}{i^2}f(1) \tag{16}$$

And note that $f(n) \rightarrow 1$ as $n \rightarrow \infty$, using the famous lemma

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \tag{17}$$

So $f(1) = \frac{6}{\pi^2}$. And the probability that the chain never returns to zero is $\frac{6}{\pi^2}$.

4.2 (ii)

Using the same method, we have

$$f(n) = \sum_{i=1}^n \frac{1}{i^\alpha}f(1) \tag{18}$$

When $\alpha > 1$, $\sum_{i=1}^n \frac{1}{i^\alpha}$ converges, so $0 < f(1) < 1$. The markov chain is transient (because it is irreducible and state 0 is transient).

When $\alpha \leq 1$, $\sum_{i=1}^n \frac{1}{i^\alpha}$ goes to ∞ , so $f(1) = 0$. The markov chain is recurrent (because it is irreducible and state 0 is recurrent).

And to determine it is positive recurrent or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^\infty$. Obviously we have $\pi_0 = \frac{1}{2^\alpha+1}\pi_1$ and for $n \geq 1$, we have

$$\pi_n = p_{n-1,n}\pi_{n-1} + p_{n+1,n}\pi_{n+1} \tag{19}$$

Thus we have

$$p_{n+1,n}\pi_{n+1} - p_{n,n+1}\pi_n = p_{n,n-1}\pi_n - p_{n-1,n}\pi_{n-1} = p_{1,0}\pi_1 - \pi_0 = 0 \tag{20}$$

So

$$\pi_{n+1} = \frac{p_{n,n+1}}{p_{n+1,n}} \pi_n = \frac{(n+2)^\alpha + (n+1)^\alpha}{(n+1)^\alpha + n^\alpha} \pi_n = ((n+2)^\alpha + (n+1)^\alpha) \pi_0 \quad (21)$$

And $\sum_{n=0}^{\infty} \pi_n = 1$

$$\pi_0 \sum_{n=0}^{\infty} (n^\alpha + (n+1)^\alpha) = 1 \quad (22)$$

When $\alpha \geq -1$, $\sum_{n=0}^{\infty} n^\alpha$ goes to ∞ , $\pi_0 = 0$, the Markov chain is null recurrent.

When $\alpha < -1$, $\sum_{n=0}^{\infty} n^\alpha$ converges, $0 < \pi_0 < 1$, the Markov chain is positive recurrent. And $\pi_0 = \frac{1}{2\zeta(-\alpha)}$, where $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{i^s}$ is the Riemann function.

In summary,

- When $\alpha > 1$, the Markov chain is transient.
- When $-1 \leq \alpha \leq 1$, the Markov chain is null recurrent.
- When $\alpha < -1$, the Markov chain is positive recurrent.

5 Problem 5

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geq 1 | X_0 = n)$. The probability that the Markov chain never returns to zero is $f(1)$. Also we have $f(2) = pf(3)$, $f(1) = pf(3) + (1-p)f(2)$. So $f(2) = \frac{1}{2-p}f(1)$, $f(3) = \frac{1}{(2-p)p}f(1)$. And for $n \geq 2$,

$$\begin{cases} f(2n-1) = pf(2n+1) + (1-p)f(2n) \\ f(2n) = pf(2n+1) + (1-p)f(2n-2) \end{cases} \quad (23)$$

$$\begin{cases} f(2n) = \frac{1}{2-p}((1-p)f(2n-2) + f(2n-1)) \\ f(2n+1) = \frac{1}{(2-p)p}(f(2n-1) - (1-p)^2 f(2n-2)) \end{cases} \quad (24)$$

Solving the equation $f(2n+1) + \lambda f(2n) = C(f(2n-1) + \lambda f(2n-2))$, we have $\lambda = -1$ and $\frac{-(1-p)^2}{p}$. Thus,

$$\begin{cases} f(2n+1) - f(2n) = \frac{(1-p)}{(2-p)p}(f(2n-1) - f(2n-2)) = \left[\frac{(1-p)}{(2-p)p} \right]^n f(1) \\ f(2n+1) - \frac{(1-p)^2}{p} f(2n) = f(2n-1) - \frac{(1-p)^2}{p} f(2n-2) = f(1) \end{cases} \quad (25)$$

When $\frac{(1-p)}{(2-p)p} > 1, p < \frac{3-\sqrt{5}}{2}$, $f(1)$ must be 0. So the chain is recurrent.

When $\frac{(1-p)}{(2-p)p} = 1, p = \frac{3-\sqrt{5}}{2}$, $f(2n+1) - f(2n) = f(1)$. Using equation (23), we have $f(2n+1) = f(2n-1) + (1-p)f(1)$, because $1-p > 0$, $f(1)$ must be zero and the chain is recurrent.

And when $\frac{(1-p)}{(2-p)p} < 1, p > \frac{3-\sqrt{5}}{2}$, $f(1) = 1 - \frac{(1-p)^2}{p} > 0$, the chain is transient.

To determine whether it is positive or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^\infty$.

So $\pi_0 = (1-p)\pi_2, \pi_1 = \pi_0$. For $n \geq 1$,

$$\begin{cases} \pi_{2n+1} = p\pi_{2n-1} + p\pi_{2n} \\ \pi_{2n} = (1-p)\pi_{2n-1} + (1-p)\pi_{2n+2} \end{cases} \quad (26)$$

Solving the equation $\pi_{2n+1} + \lambda\pi_{2n+2} = C(\pi_{2n-1} + \lambda\pi_{2n})$, $\lambda = -(1-p)$ or $-\frac{p}{1-p}$.

So

$$\begin{cases} \pi_{2n+1} - (1-p)\pi_{2n+2} = \pi_{2n-1} - (1-p)\pi_{2n} = 0 \\ \pi_{2n+1} - \frac{p}{1-p}\pi_{2n+2} = \frac{p(2-p)}{1-p}(\pi_{2n-1} - \frac{p}{1-p}\pi_{2n}) = \left[\frac{p(2-p)}{1-p}\right]^n (\pi_1 - \frac{p}{1-p}\pi_2) \end{cases} \quad (27)$$

So

$$\frac{p^2 - 3p + 1}{1-p}\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \frac{p^2 - 3p + 1}{1-p}\pi_2 \quad (28)$$

When $p = \frac{3-\sqrt{5}}{2}$, $p^2 - 3p + 1 = 0$, in equation (26) we have that $\pi_{2n+1} = (1-p)\pi_{2n} = \pi_{2n-1}$, by $\sum_{n=0}^\infty \pi_n = 1$ we have $\pi_0 = 0$, the Markov chain is null recurrent.

When $p < \frac{3-\sqrt{5}}{2}$, $\frac{p(2-p)}{1-p} < 1$,

$$\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \pi_2 \quad (29)$$

$$\sum_{n=0}^\infty \pi_n = (1-p)\pi_2 + (2-p)\pi_2 \sum_{n=0}^\infty \left[\frac{p(2-p)}{1-p}\right]^n = \frac{(1-p)^2(3-p)}{p^2 - 3p + 1}\pi_2 = 1 \quad (30)$$

In summary,

- When $p > \frac{3-\sqrt{5}}{2}$, the Markov chain is transient.
- When $p = \frac{3-\sqrt{5}}{2}$, the Markov chain is null recurrent.
- When $p < \frac{3-\sqrt{5}}{2}$, the Markov chain is positive recurrent. And the stationary distribution is,

$$\begin{cases} \pi_0 = \frac{p^2 - 3p + 1}{(1-p)(3-p)} \\ \pi_{2n} = \left[\frac{p(2-p)}{1-p}\right]^{n-1} \frac{p^2 - 3p + 1}{(1-p)^2(3-p)}, n \geq 1 \\ \pi_{2n-1} = \left[\frac{p(2-p)}{1-p}\right]^{n-1} \frac{p^2 - 3p + 1}{(1-p)(3-p)}, n \geq 1 \end{cases} \quad (31)$$

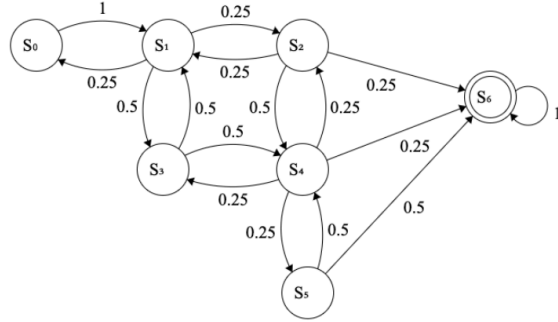
6 Problem 6

6.1 (i)

Define $M_n = \max(|X_n|, |Y_n|)$, $m_n = \min(|X_n|, |Y_n|)$, we can sort (X_n, Y_n) into different states by (M_n, m_n) ,

- $S_0 = \{M_n = m_n = 0\} = \{(0, 0)\}$
- $S_1 = \{m_n = 0, M_n = 1\} = \{(0, 1), (1, 0), (0, -1), (-1, 0)\}$
- $S_2 = \{m_n = 0, M_n = 2\} = \{(0, 2), (2, 0), (0, -2), (-2, 0)\}$
- $S_3 = \{m_n = 1, M_n = 1\} = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$
- $S_4 = \{m_n = 1, M_n = 2\} = \{(2, 1), (1, 2), (2, -1), (-1, 2), (-2, 1), (-1, 2), (-2, -1), (-1, -2)\}$
- $S_5 = \{m_n = 2, M_n = 2\} = \{(2, -2), (-2, 2), (2, 2), (-2, -2)\}$
- $S_6 = \{M_n = 3\}$

And we have an absorbing markov chain as follow.



And the corresponding transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \quad (32)$$

So the expectation of absorbing time is,

$$(\mathbf{I} - \mathbf{Q})^{-1} \cdot \mathbf{1} = \left(\frac{135}{13}, \frac{122}{13}, \frac{80}{13}, \frac{17}{2}, \frac{73}{13}, \frac{99}{26} \right) \quad (33)$$

$$\mathbb{E}[T] = \frac{135}{13}.$$

To find $\mathbb{P}(X_T = 3, Y_T = 0)$, first notice that $\mathbb{P}(X_T = 3, Y_T = 0) = \mathbb{P}(X_T = 0, Y_T = 3) = \mathbb{P}(X_T = -3, Y_T = 0) = \mathbb{P}(X_T = 0, Y_T = -3) = \frac{1}{4} \mathbb{P}(M_T = 3, m_T = 0)$. Then we can divide $S_6 = \{M_n = 3\}$ into 3 states which are also absorbing,

- $S_6^{(0)} = \{M_n = 3, m_n = 0\}$
- $S_6^{(1)} = \{M_n = 3, m_n = 1\}$
- $S_6^{(2)} = \{M_n = 3, m_n = 2\}$

Thus the corresponding matrix \mathbf{R} is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad (34)$$

The absorbing probability is,

$$(\mathbf{I} - \mathbf{Q})^{-1} \cdot \mathbf{R} = \begin{bmatrix} \frac{4}{13} & \frac{6}{13} & \frac{3}{13} \\ \frac{4}{13} & \frac{6}{13} & \frac{3}{13} \\ \frac{11}{13} & \frac{5}{13} & \frac{5}{13} \\ \frac{26}{52} & \frac{13}{26} & \frac{26}{52} \\ \frac{1}{4} & \frac{2}{7} & \frac{4}{7} \\ \frac{5}{26} & \frac{13}{7} & \frac{33}{52} \end{bmatrix} \quad (35)$$

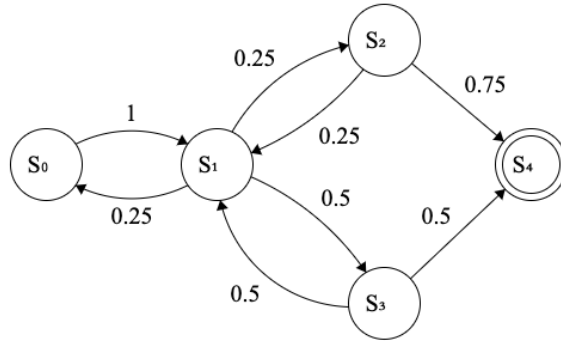
So $\mathbb{P}(X_T = 3, Y_T = 0) = \frac{1}{4} \times \frac{4}{13} = \frac{1}{13}$

6.2 (ii)

Using the same method in (i), define the states as follow,

- $S_0 = \{|X_n| + |Y_n| = 0\}$
- $S_1 = \{|X_n| + |Y_n| = 1\}$
- $S_2 = \{|X_n| + |Y_n| = 2, ||X_n| - |Y_n|| = 2\}$
- $S_3 = \{|X_n| + |Y_n| = 2, ||X_n| - |Y_n|| = 0\}$
- $S_4 = \{|X_n| + |Y_n| = 3\}$

And we have an absorbing markov chain as follow.



We define the expectation of steps from S_i to reach S_3 as $f_i, i = 0, 1, 2$.

$$\begin{cases} f_0 = f_1 + 1 \\ f_1 = 0.25f_0 + 0.25f_2 + 0.5f_3 + 1 \\ f_2 = 0.25f_1 + 1 \\ f_3 = 0.5f_1 + 1 \end{cases} \quad (36)$$

Solving the equation we have $\mathbb{E}[T] = f_0 = \frac{39}{7}$. (It is equivalent to $(\mathbf{I} - \mathbf{Q})^{-1} \cdot \mathbf{1}$.)

Also we can split S_4 into two sub states,

- $S_4^{(0)} = \{|X_n| + |Y_n| = 3, ||X_n| - |Y_n|| = 3\}$
- $S_4^{(1)} = \{|X_n| + |Y_n| = 3, ||X_n| - |Y_n|| = 1\}$

And the corresponding matrix \mathbf{R} is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.25 & 0.5 \\ 0 & 0.5 \end{bmatrix} \quad (37)$$

The absorbing probability is,

$$(\mathbf{I} - \mathbf{Q})^{-1} \cdot \mathbf{R} = \begin{bmatrix} \frac{1}{7} & \frac{6}{7} \\ \frac{1}{7} & \frac{6}{7} \\ \frac{2}{7} & \frac{5}{7} \\ \frac{1}{14} & \frac{13}{14} \end{bmatrix} \quad (38)$$

So $P(X_T = 3, Y_T = 0) = \frac{1}{4}P(|X_T| + |Y_T| = 3, ||X_T| - |Y_T|| = 3) = \frac{1}{28}$.

6.3 (iii)

Define the expectation steps from $(X_n = x, Y_n = y)$ to first arrive at the boundary as $f(x, y)$.

$$\begin{cases} f(-2, y) = f(x, 2) = f(x, -2) = 0 \\ f(x, y) = \frac{1}{4}[f(x-1, y) + f(x+1, y) + f(x, y-1) + f(x, y+1)] + 1 \end{cases} \quad (39)$$

According to symmetry we have $f(x, -1) = f(x, 1)$.

Define $g_i = f(i, 0), h_i = f(i, -1) = f(i, 1)$. We have

$$\begin{cases} g_{-2} = h_{-2} = 0 \\ g_i = 0.5h_i + 0.25g_{i-1} + 0.25g_{i+1} + 1 \\ h_i = 0.25h_{i-1} + 0.25h_{i+1} + 0.25g_i + 1 \end{cases} \quad (40)$$

Because $\lim_{n \rightarrow \infty} g_{n+1} - g_n = 0$, we have $\lim_{n \rightarrow \infty} g_n = 8, \lim_{n \rightarrow \infty} h_n = 6$. Denote $\alpha_i = g_i - 8, \beta_i = h_i - 6$.

$$\alpha_i + \sqrt{2}\beta_i = \frac{4 + \sqrt{2}}{14}(\alpha_{i-1} + \sqrt{2}\beta_{i-1}) + \frac{4 + \sqrt{2}}{14}(\alpha_{i+1} + \sqrt{2}\beta_{i+1}) \quad (41)$$

$$\alpha_i - \sqrt{2}\beta_i = \frac{4 - \sqrt{2}}{14}(\alpha_{i-1} - \sqrt{2}\beta_{i-1}) + \frac{4 - \sqrt{2}}{14}(\alpha_{i+1} - \sqrt{2}\beta_{i+1}) \quad (42)$$

We have

$$\begin{aligned} \alpha_n + \sqrt{2}\beta_i &= A_1\lambda_1^n + A_2\lambda_2^n \\ \alpha_n - \sqrt{2}\beta_i &= B_1\mu_1^n + A_2\mu_2^n \end{aligned} \quad (43)$$

where $\lambda_{1,2} = \frac{1}{2}(4 - \sqrt{2} \pm \sqrt{14 - 8\sqrt{2}})$, $\mu_{1,2} = \frac{1}{2}(4 + \sqrt{2} \pm \sqrt{14 + 8\sqrt{2}})$. Because $\lim_{n \rightarrow \infty} \alpha_n = \lim_{n \rightarrow \infty} \beta_n = 0$, we have $A_1 = B_1 = 0$, thus

$$\begin{aligned} \alpha_n &= \frac{1}{2}(A\lambda^n + B\mu^n) \\ \beta_n &= \frac{1}{2\sqrt{2}}(A\lambda^n - B\mu^n) \end{aligned} \quad (44)$$

where $\lambda = \frac{1}{2}(4 - \sqrt{2} - \sqrt{14 - 8\sqrt{2}})$, $\mu = \frac{1}{2}(4 + \sqrt{2} - \sqrt{14 + 8\sqrt{2}})$.

And $\alpha_{-2} = g_{-2} - 8 = -8$, $\beta_{-2} = h_{-2} - 6 = -6$, we have $A = -\lambda^2(8 + 6\sqrt{2})$, $B = \mu^2(6\sqrt{2} - 8)$.

$$g_0 = \alpha_0 + 8 = \frac{1}{2}(A+B) + 8 = 3\sqrt{2}(\mu^2 - \lambda^2) - 4(\mu^2 + \lambda^2) + 8 = 10\sqrt{7 + \sqrt{17}} - 8\sqrt{14 - 2\sqrt{17}} - 8 = 6.1617 \quad (45)$$

To solve $P(X_T = 0, Y_T = 0)$, similarly we need to find $(\mathbf{I} - \mathbf{Q})^{-1} \cdot \mathbf{R}$. However, since the Markov chain is infinite, we can define $p(x, y)$ as $P(X_T = 0, Y_T = 0 | X_0 = x, Y_0 = y)$ and solve the following equation,

$$\begin{cases} p(-2, 0) = 1 \\ p(-2, -1) = p(-2, 1) = p(x, 2) = p(x, -2) = 0 \\ p(x, y) = \frac{1}{4}[p(x-1, y) + p(x+1, y) + p(x, y-1) + p(x, y+1)] \end{cases} \quad (46)$$

Follows the same step, we have $P(X_T = -2, Y_T = 0) = p(0, 0) = \frac{1}{2}(\lambda^2 + \mu^2) = 16 - \sqrt{2(95 + 7\sqrt{17})} = 0.1304$.

6.4 (iv)

7 Problem 7

7.1 (i)

First, for a random variable X , $\mathbb{E}[X] = 0$ and there exists some constant α , such that $\mathbb{E}[e^{tx}] \leq e^{\alpha^2 t^2 / 2}$, then X is sub-Gaussian, and we have that

$$\mathbb{P}(X \geq \lambda) = P(e^{tX} \geq e^{t\lambda}) \leq \frac{E[e^{tX}]}{e^{t\lambda}} \leq e^{\alpha^2 t^2 / 2 - t\lambda} \quad (47)$$

Let $t = \frac{\lambda}{\alpha}$, we have

$$\mathbb{P}(X \geq \lambda) \leq e^{-\frac{\lambda^2}{2\alpha^2}} \quad (48)$$

$$\mathbb{P}(|X| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{2\alpha^2}} \quad (49)$$

For X_i , we have $\mathbb{E}[\exp(tX_i)] = \frac{1}{2}(e^t + e^{-t}) \leq \exp(t^2/2)$. When $\sum_{i=1}^{\infty} a_i^2 = S < \infty$,

$$\mathbb{E}[\exp(\sum_{i=1}^{\infty} a_i X_i t)] = \prod_{i=1}^{\infty} \mathbb{E}[\exp(ta_i X_i)] \leq \prod_{i=1}^{\infty} \exp(t^2 a_i^2) = e^{St^2} \quad (50)$$

Denote $X = \sum_{i=1}^{\infty} a_i X_i$,

$$\mathbb{P}(|X| \geq \lambda) \leq 2e^{-\frac{\lambda^2}{S}} \quad (51)$$

Thus we have X converges almost surely, i.e $\mathbb{P}(|X| < \infty) = \mathbb{P}(|\sum_{i=1}^{\infty} a_i X_i| < \infty) = 1$.

7.2 (ii)

By the corollary of Kolmogorov three-series theorem, given independent variables X_1, X_2, \dots, X_n ,

$$\mathbb{P}\left(\sum_{k=1}^{\infty} X_k \text{ converges}\right) = 0 \text{ or } 1 \quad (52)$$

And because $\mathbb{E}[(\sum_{i=1}^{\infty} a_i X_i)^2] = \sum_{i=1}^{\infty} a_i^2 = \infty$, $\mathbb{P}(|\sum_{i=1}^{\infty} a_i X_i| = \infty) > 0$, thus

$$\mathbb{P}\left(|\sum_{i=1}^{\infty} a_i X_i| < \infty\right) = 0 \quad (53)$$