Exercise 6
Probability Theory 2020 Autumn

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1 Problem 1

$$P(S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n | S_{2n} = 0) = \frac{P(S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0)}{P(S_{2n} = 0)} \tag{1}$$

Consider the event $\{S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0\}$ partitioned by its first return to zero,

$$\{S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0\} = \bigcup_{j=1}^{n} \{S_i > 0, \forall 1 \leqslant i \leqslant 2j-1, S_{2j} = 0, S_i \geqslant 0, \forall 2j+1 \leqslant i \leqslant 2n-1, S_{2n} = 0\} \tag{2}$$

Denote $p_{2n} = P(S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0)$ and $q_{2n} = P(S_i > 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0) = \frac{1}{2}P(S_i \neq 0, \forall 1 \leqslant i \leqslant 2n-1, S_{2n} = 0) = \frac{1}{2}f_{2n}$,

$$p_{2n} = \sum_{i=1}^{n} p_{2n-2i} q_{2i} \tag{3}$$

Define $P(z) = \sum_{n=0}^{\infty} p_{2n} z^n, Q(z) = \sum_{n=1}^{\infty} q_{2n} z^n = \frac{1}{2}F(z)$

$$P(z) = \frac{1}{1-Q(z)} = \frac{2}{1+\sqrt{1-z}} = \frac{2}{z}(1-\sqrt{1-z}) = \frac{2}{z}F(z) \tag{4}$$

So $p_{2n} = 2f_{2n+2} = \frac{2}{2n+1}\binom{2n+2}{n+1}\frac{1}{2^{2n+2}}$. Thus,

$$P(S_i \geqslant 0, \forall 1 \leqslant i \leqslant 2n | S_{2n} = 0) = \frac{p_{2n}}{u_{2n}} = \frac{1}{n+1} \tag{5}$$

2 Problem 2

$L_{\max}^{(n)}$ is the longest length of success runs in $X_1, X_2, \cdots, X_n$.[1]

[1] Reference: Erdös, P., & Rényi, A. (1970). On a new law of large numbers. Journal d'analyse mathématique, 23(1), 103-111.

Denote $S_0 = 0, S_n = \sum_{i=1}^{n} X_i$ and define

$$\Theta(n,k) = \max_{0 \leqslant i \leqslant n-k} \frac{S_{i+k} - S_i}{k} \tag{6}$$

So

$$L_{\max}^{(n)} \geqslant k \Longleftrightarrow \Theta(n,k) \geqslant 1 \tag{7}$$

$$P(\frac{S_{i+k} - S_i}{k} \geqslant 1) = p^k = e^{k \log p} \tag{8}$$

And

$$\begin{aligned} P(L_{\max}^{(n)} \geqslant k) = P(\Theta(n,k) \geqslant 1) &= P\left(\max_{0 \leqslant i \leqslant n-k} \frac{S_{i+k} - S_i}{k} \geqslant 1\right) \\ &= 1 - P\left(\max_{0 \leqslant i \leqslant n-k} \frac{S_{i+k} - S_i}{k} < 1\right) = 1 - P\left(\frac{S_{i+k} - S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right) \end{aligned} \tag{9}$$

Now we are trying to find the upper and lower bound of $P\left(\frac{S_{i+k}-S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right)$. First,

$$\begin{aligned} P\left(\frac{S_{i+k} - S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right) &\leqslant P\left(\frac{S_{(i+1)k} - S_{ik}}{k} < 1, i = 0, 1 \cdots [\frac{n-k}{k}]\right) \\ &\leqslant (1 - e^{k \log p})^{n/k} = (1 - n^{c \log p})^{n/k} \\ &\leqslant \exp\left(-\frac{n^{1 - c \log \frac{1}{p}}}{c \log n}\right) \end{aligned} \tag{10}$$

Then denote A_i as the event $\left\{\frac{S_{i+k}-S_i}{k} < 1\right\}$

$$\begin{aligned} P\left(\frac{S_{i+k} - S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right) &= P(\bigcap_{i=0}^{n-k} A_i) = 1 - P(\bigcup_{i=0}^{n-k} \overline{A_i}) \\ &\geqslant 1 - \sum_{i=1}^{n} P(\overline{A_i}) = 1 - np^k = 1 - n^{1 - c \log \frac{1}{p}} \end{aligned} \tag{11}$$

If $c < \frac{1}{\log \frac{1}{p}}$,

$$\lim_{n \to \infty} P\left(\frac{S_{i+k} - S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right) \leqslant \lim_{n \to \infty} \exp\left(-\frac{n^{1 - c \log \frac{1}{p}}}{c \log n}\right) = 0 \tag{12}$$

Thus $\forall c < \frac{1}{\log \frac{1}{p}}, \lim_{n \to \infty} P(L_{\max}^{(n)} \geqslant c \log n) = 1$

If $c > \frac{1}{\log \frac{1}{p}}$,

$$\lim_{n \to \infty} P\left(\frac{S_{i+k} - S_i}{k} < 1, 0 \leqslant i \leqslant n-k\right) \geqslant \lim_{n \to \infty} 1 - n^{1 - c \log \frac{1}{p}} = 1 \tag{13}$$

Thus $\forall c > \frac{1}{\log \frac{1}{p}}$, $\lim_{n\to\infty} P(L_{\max}^{(n)} \geqslant c \log n) = 0$. $f(p) = \frac{1}{\log \frac{1}{p}}$.