

# Probability Theory Exercise 6

Issued: 2020/12/3

Due: 2020/12/17

1. Let  $X_1, X_2, \dots, X_{2n}$  be i.i.d. random variables with the distribution  $P(X_i = 1) = P(X_i = -1) = 1/2$  for all  $1 \leq i \leq 2n$ . For every  $1 \leq i \leq 2n$ , define  $S_i = X_1 + X_2 + \dots + X_i$ . What is the conditional probability  $P(S_i \geq 0 \text{ for all } 1 \leq i \leq 2n \mid S_{2n} = 0)$ ?

2. Let  $X_1, X_2, X_3, \dots$  be i.i.d. Bernoulli- $p$  random variables, i.e.,  $P(X_i = 1) = p \in (0, 1)$  and  $P(X_i = 0) = 1 - p$  for all  $i$ . Define another sequence  $\{L_n\}_{n=1}^\infty$  of integer-valued random variables as follows: If  $X_n = 0$ , then we define  $L_n = 0$ . If  $X_n = 1$ , then we define  $L_n$  as the unique positive integer such that  $X_n = X_{n+1} = X_{n+2} = \dots = X_{n+L_n-1} = 1, X_{n+L_n} = 0$ . Define  $L_{\max}^{(n)} := \max(L_1, \dots, L_n)$ . Prove that there is a function of  $p$ , which we denote as  $f(p)$ , such that

$$\lim_{n \rightarrow \infty} P(L_{\max}^{(n)} \geq c \log(n)) = 1 \text{ for any constant } c < f(p),$$

$$\lim_{n \rightarrow \infty} P(L_{\max}^{(n)} \geq c \log(n)) = 0 \text{ for any constant } c > f(p).$$

Also solve the exact expression of  $f(p)$ . (We use natural logarithm in this problem.)

3. Consider a random walk on the integers. We start at  $X_0 = 0$ . In each step, we have  $P(X_n = X_{n-1} + 1) = P(X_n = X_{n-1} - 1) = 1/2$ . Define the random variable  $U$  as the unique positive integer such that  $X_U = 0$  and  $X_i \neq 0$  for all  $0 < i < U$ . In other words, in step  $U$ , this random walk returns to the origin for the first time. Now let  $m > 0$  be a positive integer. Define another random variable  $N_m$  as the number of times this random walk visits  $m$  before step  $U$ . More precisely, we have

$$N_m := |\{i : 0 < i < U, X_i = m\}|.$$

(For a set  $A$ ,  $|A|$  denotes the size of  $A$ .) Note that  $N_m$  is simply the number of times this random walk visits  $m$  before returning to 0.

(i) Find  $P(N_m \geq 1)$ .

(ii) For every positive integer  $n$ , find  $P(N_m = n)$ .

4. Passengers arrive at a train station as a Poisson process with rate  $\lambda$ .

(i) Suppose that there is only one train, and it departs at a deterministic time  $T$ . Let  $W$  be the sum of waiting time of all the passengers. Compute  $\mathbb{E}[W]$ .

(ii) Now suppose that there are two trains. One departs at  $T$ , and the other departs at  $S < T$ . Compute  $\mathbb{E}[W]$  in this case.

5. Let  $G \sim \mathbb{G}(n, p)$  with  $p = \frac{\log(n)}{n} + \frac{c}{n}$  for some constant  $c$ . ( $c$  can be negative.) Find

$$\lim_{n \rightarrow \infty} P(G \text{ is connected}).$$

6. Let  $G \sim \mathbb{G}(n, p)$  with  $p = n^{-\delta}$  for some constant  $\delta > 0$ . Prove that there exists a constant  $\delta_0 > 0$  such that

$$\lim_{n \rightarrow \infty} P(G \text{ contains 4 vertices that are pairwise connected}) = 0 \text{ if } \delta > \delta_0,$$

$$\lim_{n \rightarrow \infty} P(G \text{ contains 4 vertices that are pairwise connected}) = 1 \text{ if } \delta < \delta_0.$$

Also find the value of  $\delta_0$ . (Note that  $G$  contains a triangle means that  $G$  contains 3 vertices that are pairwise connected.)