Exercise 2
Probability Theory 2020 Autumn

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Contents

1 Problem 1

Define the number of each dice as X_1, X_2, X_3 and event A as $X_1 > X_2 > X_3$ and B as X_1, X_2, X_3. Obviously there are $6^3 = 216$ possible combinations of (X_1, X_2, X_3) with equal probability. So we just need to find the size of $A = \{(X_1, X_2, X_3)|X_1 > X_2 > X_3, X_1, X_2, X_3 \in \{1, 2, \cdots 6\}\}$

Fix $X_1 = i$, $X_2 = j < i$, there are $j - 1$ possible choices of X_3. So the size of A is

$$\sum_{i=1}^{6}\left(\sum_{j=1}^{i-1}(j-1)\right) = \sum_{i=1}^{6}\left(\frac{(i-1)(i-2)}{2}\right) = 20 \tag{1}$$

Symmetrically, $|A| = |B| = 20$. So $P(X_1 > X_2 > X_3) = P(X_1 < X_2 < X_3) = \frac{20}{216} = \frac{5}{54}$

2 Problem 2

2.1 (i)

Using the Lagrange Multiplier method,

$$L = H(X) + \lambda_1(\sum_{k=1}^{n} p_k - 1) + \lambda_2(\sum_{k=1}^{n} p_k x_k - \mu) \tag{2}$$

To maximize L,

$$\begin{cases} \dfrac{\partial L}{\partial p_k} = 0 \quad (k = 1, 2, \cdots, n) \\ \dfrac{\partial L}{\partial \lambda_1} = 0 \\ \dfrac{\partial L}{\partial \lambda_2} = 0 \end{cases} \tag{3}$$

And

$$\frac{\partial L}{\partial p_k} = -1 - \log(p_k) + \lambda_1 + \lambda_2 x_k = 0 \Longleftrightarrow p_k = e^{\lambda_2 x_k + \lambda_1 - 1} = Cr^{x_k} \tag{4}$$

where $C = e^{\lambda_1 - 1}, r = e^{\lambda_2}$ which are constants determined by $\sum_{k=1}^{n} p_k = 1$ and $\sum_{k=1}^{n} x_k p_k = \mu$.

2.2 (ii)

For a countable support set, let $n \to \infty$,

$$L = H(X) + \lambda_1(\sum_{k=1}^{\infty} p_k - 1) + \lambda_2(\sum_{k=1}^{\infty} p_k x_k - \mu) \tag{5}$$

To maximize L,

$$
\begin{cases}
\dfrac{\partial L}{\partial p_k} = 0 \quad (k = 1, 2, \cdots, \infty) \\
\dfrac{\partial L}{\partial \lambda_1} = 0 \\
\dfrac{\partial L}{\partial \lambda_2} = 0
\end{cases} \tag{6}
$$

And

$$
\frac{\partial L}{\partial p_k} = -1 - \log(p_k) + \lambda_1 + \lambda_2 x_k = 0 \Longleftrightarrow p_k = e^{\lambda_2 x_k + \lambda_1 - 1} = C r^{x_k} \tag{7}
$$

where $C = e^{\lambda_1 - 1}, r = e^{\lambda_2}$ which are constants determined by $\sum_{k=1}^{\infty} p_k = 1$ and $\sum_{k=1}^{\infty} x_k p_k = \mu$. For the case of $x_k = k$,

$$
\begin{cases}
\displaystyle\sum_{i=1}^{\infty} p_k = \sum_{i=1}^{\infty} C r^k = \frac{Cr}{1-r} = 1 \\
\displaystyle\sum_{k=1}^{\infty} x_k p_k = \sum_{i=1}^{\infty} C k r^k = \frac{Cr}{(1-r)^2} = \mu
\end{cases} \tag{8}
$$

So $C = \mu - 1, r = \frac{\mu - 1}{\mu}$ and $P(X = k) = C r^k$ is a geometric distribution.

3 Problem 3 (Conditionally convergent series)

3.1 (i)

According to Leibniz's test, $S_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, thus

$$
\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1 + k/n} = \int_0^1 \frac{1}{1+x} dx = \ln 2 \tag{9}
$$

3.2 (ii)

Notice that $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{2}(1 - \frac{1}{2})$, $\frac{1}{3} - \frac{1}{6} - \frac{1}{8} = \frac{1}{2}(\frac{1}{3} - \frac{1}{4})$ and so on. Thus,

$$
1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{\ln 2}{2} \tag{10}
$$

3.3 (iii)

$\frac{1}{n} > \ln(1 + \frac{1}{n}) = \ln(n+1) - \ln(n)$, so $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$, and the infinite sum converges. So the series $\frac{(-1)^{(n+1)}}{n}$ is conditionally convergent but not absolutely convergent.

4 Problem 4

4.1 (i)

For $X \sim \text{Geometric}(p)$,we have $E(X) = \frac{1}{p}$ and $E(X^2) = \frac{2-p}{p^2}$.

Because $P(X-1=k|X>1) = P(X=k)$,

$$\begin{aligned} E[X^3|X>1] &= E[(X-1)^3 + 3(X-1)^2 + 3(X-1) + 1|X>1] \\ &= E[X^3] + 3E[X^2] + 3E[X] + 1 \\ &= E[X^3] + \frac{6-3p}{p^2} + \frac{3}{p} + 1 \end{aligned} \tag{11}$$

Also, $E[X^3] = E[X^3|X>1](1-p) + p$.

$$E[X^3] = \frac{(\frac{6}{p^2}+1)(1-p)+p}{p} = \frac{6-6p+p^2}{p^3} \tag{12}$$