Exercise 4
Probability Theory 2020 Autumn

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Contents

1 Problem 1

1.1 (i)

Claim that $M_X(s) = \infty$ for all $s \neq 0$. The proof is as below.

$$M_X(s) = \mathbb{E}[e^{sx}] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2+x^2} e^{sx} dx = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2+x^2} e^{-sx} dx \tag{1}$$

So $M_X(s) = M_X(-s)$, we just need to prove $M_X(s) = \infty$ for $s > 0$.

Notice that for $x > 0$, $e^x > \frac{x^3}{6}$.

$$\begin{aligned}
M_X(s) &= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2+x^2} e^{sx} dx > \int_{0}^{\infty} \frac{1}{\pi} \frac{t}{t^2+x^2} e^{sx} dx \\
&> \int_{0}^{\infty} \frac{1}{6\pi} \frac{ts^3x^3}{t^2+x^2} dx = \int_{0}^{\infty} \frac{t^3s^3}{6\pi} \frac{x^3}{1+x^2} dx \\
&= \frac{t^3s^3}{12\pi}(x^2 - \ln(1+x^2))\Big|_0^{\infty} = \infty
\end{aligned} \tag{2}$$

So $M_X(s) = \infty$ for all $s \neq 0$. And it is trivial that $M_X(0) = 1$.

$$M_X(s) = \begin{cases} 1, & s = 0 \\ \infty, & s \neq 0 \end{cases} \tag{3}$$

1.2 (ii)

Yes. An example is given as the symmetrized lognormal distribution.

For example, $Z \sim N(0,1)$ and $Y = e^Z \sim \text{lognormal}(0,1)$ and define X as

$$X = \begin{cases} Y, & \text{with probability } \dfrac{1}{2} \\ -Y, & \text{with probability } \dfrac{1}{2} \end{cases} \tag{4}$$

It is easy to verify that

$$f_Y(y) = f_Z(\ln y)\frac{1}{y} = \frac{1}{\sqrt{2\pi}y} e^{-\frac{1}{2}(\ln y)^2} \tag{5}$$

And the n-th moment of Y is

$$\begin{aligned}
\mathbb{E}[Y^n] &= \int_0^{\infty} \frac{1}{\sqrt{2\pi}y} e^{-\frac{1}{2}(\ln y)^2} * y^n dy = \int_0^{\infty} \frac{1}{\sqrt{2\pi}y} \exp\left(-\frac{1}{2}(\ln y)^2 + n\ln y\right) dy \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2 + \frac{n^2}{2}\right) dz \quad (z = \ln y - n) \\
&= e^{\frac{n^2}{2}}
\end{aligned} \tag{6}$$

And

$$f_X(x) = \frac{1}{2\sqrt{2\pi}|x|} e^{-\frac{1}{2}(\ln|x|)^2} \tag{7}$$

Then we will show that all the moments of X are finite. For all odd integers k,

$$\mathbb{E}[X^k] = \mathbb{E}[X_+^k] - \mathbb{E}[X_-^k] = \frac{1}{2}\mathbb{E}[Y^k] - \frac{1}{2}\mathbb{E}[Y^k] = 0 \tag{8}$$

For all even integers k,

$$\mathbb{E}[X^k] = \frac{1}{2}\mathbb{E}[Y^k] + \frac{1}{2}\mathbb{E}[Y^k] = e^{\frac{n^2}{2}} \tag{9}$$

So $\mathbb{E}[X^k] < \infty$ for all integers $k \geqslant 1$. But for the moment generating function $M_X(s)$, assuming $s > 0$,

$$\begin{aligned} M_X(s) = \mathbb{E}[e^{sx}] &= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}|x|} e^{-\frac{1}{2}(\ln|x|)^2} e^{sx} dx > \int_0^{\infty} \frac{1}{2\sqrt{2\pi}x} e^{-\frac{1}{2}(\ln x)^2} e^{sx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{se^z} dz > \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} e^{\frac{s}{6}z^3 - \frac{1}{2}z^2} \\ &> \int_{\frac{6}{s}}^{\infty} \frac{1}{2\sqrt{2\pi}} e^{\frac{1}{2}z^2} = \infty \quad (\frac{s}{6}z^3 > z^2 \text{ when } z > \frac{6}{s}) \end{aligned} \tag{10}$$

$M_X(s) = \infty$ for $s > 0$ and $M_X(s) = M_X(-s)$, so $M_X(s) = \infty$ for $s \neq 0$

2 Problem 2

2.1 (a)

Notice that the conclusion in (a) is a weaker version of the conclusion from (b) because $X^k \leqslant X^k e^{sX}$ when $s > 0$, we will just prove (b).

2.2 (b)

If $s < 0$, $X^k e^{sX} \leqslant X^k \leqslant X^k e^{s'X}, s' > 0$, so we will just prove the case when $s > 0$.

Because X is nonnegative,

$$e^{ax} = e^{(a-s)x} > \frac{(a-s)^k x^k}{k!} e^{sx} \tag{11}$$

So $x^k e^{sx} < e^{ax} \frac{k!}{(a-s)^k}$,

$$\mathbb{E}[X^k e^{sX}] \leqslant \mathbb{E}[\frac{k!}{(a-s)^k} e^{aX}] = \frac{k!}{(a-s)^k} M_X(a) < \infty \tag{12}$$

2.3 (c)

The inequality holds true only when $h > 0$. The following proof is given under the condition $h > 0$.

It's obvious when $X = 0$. And for $X > 0$, it is equivalent to $\frac{e^{hX}-1}{hX} \leqslant e^{hX}$, denote b as hX and $g(x)$ as e^X, using Lagrange's mean value theorm,

$$\frac{e^{hX}-1}{hX} = \frac{g(b)-g(0)}{b-0} = g'(c) \leqslant g'(b) = e^{hX}, c \in (0, b) \tag{13}$$

2.4 (d)

Using L'Hospital's Rule, $X = \lim_{h\to 0} \frac{e^{hX}-1}{h}$. Suppose $\{h_n\}$ is a sequence of nonnegative numbers and $\lim_{n\to\infty} h_n = 0$, so $h_n < \frac{a}{2}$ for sufficiently large n. And define $X_n = \frac{e^{h_n X}-1}{h_n}$.
Using the conclusion from (c)

$$0 < X_n \leqslant Xe^{h_n X} < Xe^{\frac{a}{2}X} \tag{14}$$

And $\mathbb{E}[Xe^{\frac{a}{2}X}] < \infty$, so according to the Dominant Convergence Theorm,

$$\mathbb{E}[X] = \mathbb{E}[\lim_{h\downarrow 0} \frac{e^{hX}-1}{h}] = \lim_{h\downarrow 0} \mathbb{E}[\frac{e^{hX}-1}{h}] = \lim_{h\downarrow 0} \frac{\mathbb{E}[e^{hX}]-1}{h} \tag{15}$$

3 Problem 3

Let $Z = \frac{X}{\sigma} \sim N(0, 1)$ and $z = \frac{x}{\sigma}$. So $xe^{x^2/(2\sigma^2)}P(X \geqslant x) = \sigma z e^{z^2/2} P(Z \geqslant z)$.
Define the Mills Ratio as

$$R(x) = \frac{1-\Phi(x)}{\phi(x)} = e^{\frac{x^2}{2}} \int_x^\infty e^{-\frac{t^2}{2}} dt, x > 0 \tag{16}$$

We will prove that $\frac{x}{x^+1} < R(x) < \frac{1}{x}, x > 0$, which is first given by Gorden(1941)[1]. The following proof is based on the blog[2]. Denote $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, then $\phi'(t) = -t\phi(t)$,

Let $h_1(t) = -\frac{1}{t}\phi(t)$,

$$\frac{dh_1(t)}{dt} = (\frac{1}{t^2} + 1)\phi(t) \tag{17}$$

$$\begin{aligned} 1 - \Phi(x) = \int_x^\infty \phi(t)dt &< \int_x^\infty (1 + \frac{1}{t^2})\phi(t)dt \\ &= h_1(t)\Big|_x^\infty = \frac{\phi(x)}{x} \end{aligned} \tag{18}$$

[1] Gordon, RD: Values of Mills' ratio of area to bounding ordinate and of the normal probability integral for large values of the argument. Ann. Math. Stat. 12, 364-366 (1941)
[2] `https://bowaggoner.com/blog/2018/03-17-gaussian-tails/index.html`

So $R(x) = \frac{1-\Phi(x)}{\phi(x)} < \frac{1}{x}$.

In the same way, let $h_2(t) = -\frac{t}{1+t^2}\phi(t)$,

$$\frac{dh_2(t)}{dt} = \frac{x^4+2x^2-1}{(x^2+1)^2}\phi(t) = (1-\frac{2}{(x^2+1)^2})\phi(t) \tag{19}$$

$$\begin{aligned} 1-\Phi(x) = \int_x^\infty \phi(t)dt > \int_x^\infty ((1-\frac{2}{(x^2+1)^2})\phi(t)dt \\ = h_2(t)\Big|_x^\infty = \frac{x}{x^2+1}\phi(x) \end{aligned} \tag{20}$$

So $\frac{x}{x^2+1} < R(x) < \frac{1}{x}$. Thus $\lim_{x\to\infty} xR(x) = 1$.

$$\lim_{x\to\infty} xe^{x^2/(2\sigma^2)}P(X \geqslant x) = \lim_{z\to\infty} \sigma z e^{z^2/2} P(Z \geqslant z) = \frac{\sigma}{\sqrt{2\pi}} \lim_{x\to\infty} xR(x) = \frac{\sigma}{\sqrt{2\pi}} \tag{21}$$

4 Problem 4

4.1 (i)

Suppose the pdf of $X_1, X_2, \cdots, X_n$ is $f_X(x)$ and the cdf is $F_X(x)$. And $P(\min(X_1, X_2, \cdots, X_n) = X_1) = P(X_1 \leqslant \min(X_2, \cdots, X_n))/$. Define $Y = \min(X_2, \cdots, X_n)$,

$$F_Y(y) = P(\min(X_2, \cdots, X_n) \leqslant y) = 1 - \prod_{i=2}^{n} P(X_i > y) = 1 - (1 - F_X(y))^{n-1} \tag{22}$$

Because X_1 and $X_2, \cdots, X_n$ are independent, X_1 and $Y = g(X_2, \cdots, X_n)$ are also independent.

$$\begin{aligned} P(X_1 \leqslant Y) = \int_\infty^\infty f_X(x)P(Y \geqslant x)dx = \int_\infty^\infty f_X(x)(1-F_X(x))^{n-1}dx \\ = \int_0^1 (1-F_X(y))^{n-1}d(1-F_X(x)) = \frac{1}{n} \end{aligned} \tag{23}$$

So $P(\min(X_1, X_2, \cdots, X_n) = X_1) = \frac{1}{n}$.

4.2 (ii)

$$P(\min(X_1, X_2, \cdots, X_n) = X_1) = P(X_1 = 0) + P(X_1 = 1, X_2 = 1, \cdots, X_n = 1) = 1 - p + p^n \tag{24}$$

4.3 (iii)

$P(\min(X_1, X_2, \cdots, X_n) = X_1) = P(X_1 \leqslant \min(X_2, \cdots, X_n))$. Define $Y = \min(X_2, \cdots, X_n)$,

$$F_Y(y) = P(\min(X_2, \cdots, X_n) \leqslant y) = 1 - \prod_{i=2}^{n} P(X_i > y) = 1 - \exp(-\sum_{i=2}^{n} \lambda_i y), x \geqslant 0 \tag{25}$$

So $Y \sim Exponential(\sum_{i=2}^n \lambda_i)$ and Y, X_1 are independent.

$$P(X_1 \leqslant Y) = \int_0^\infty f_X(x)P(Y \geqslant x)dx = \int_0^\infty \lambda_1 \exp(-\lambda_1 x)\exp(-\sum_{i=2}^n \lambda_i x)dx$$
$$= \frac{\lambda_1}{\sum_{i=1}^n \lambda_i} \tag{26}$$

So $P(\min(X_1, X_2, \cdots, X_n) = X_1) = \frac{\lambda_1}{\sum_{i=1}^n \lambda_i}$.

5 Problem 5

Denote $\Phi(x)$ and $\phi(x)$ as the pdf and cdf of standard Gaussian variable. So

$$P(X \geq \beta\sqrt{\log(n)}) = 1 - [\Phi(\beta\sqrt{\log(n)})]^n \tag{27}$$

According to Problem 3,

$$1 - \frac{1}{x}\phi(x) < \Phi(x) < 1 - \frac{x}{x^2+1}\phi(x) \tag{28}$$

So

$$\left(1 - \frac{1}{\sqrt{2\pi\log(n)}\beta n^{\beta^2/2}}\right)^n < [\Phi(\beta\sqrt{\log(n)})]^n < \left(1 - \frac{\beta\sqrt{\log(n)}}{\sqrt{2\pi}(\beta^2\log(n)+1)n^{\beta^2/2}}\right)^n \tag{29}$$

Denote $L_n(\beta) = \left(1 - \frac{1}{\sqrt{2\pi\log(n)}\beta n^{\beta^2/2}}\right)^n, U_n(\beta) = \left(1 - \frac{\beta\sqrt{\log(n)}}{\sqrt{2\pi}(\beta^2\log(n)+1)n^{\beta^2/2}}\right)^n$.

If $\beta > \sqrt{2}$, $\beta^2/2 > 1$, $\lim_{n\to\infty} L_n(\beta) = \lim_{n\to\infty} Un(\beta) = 1$. So $\lim_{n\to\infty} P(X \geq \beta\sqrt{\log(n)}) = 0$

If $\beta < \sqrt{2}$, $\beta^2/2 < 1$, $\lim_{n\to\infty} L_n(\beta) = \lim_{n\to\infty} Un(\beta) = 0$. So $\lim_{n\to\infty} P(X \geq \beta\sqrt{\log(n)}) = 1$

So $\beta_0 = \sqrt{2}$. And when $\beta = \sqrt{2}$, using Bernoulli's inequality $(1+x)^n \geqslant 1 + nx$ for $x \geqslant -1$,

$\lim_{n\to\infty} L_n(\sqrt{2}) = \lim_{n\to\infty} \left(1 - \frac{1}{2\sqrt{\pi\log(n)}n}\right)^n \geqslant \lim_{n\to\infty} 1 - \frac{1}{2\sqrt{\pi\log(n)}} = 1$

Also we have $L_n(\sqrt{2}) \leqslant 1$ so $\lim_{n\to\infty} L_n(\sqrt{2}) = 1$

And $U_n(\beta) \leqslant 1$, so for $\beta = \sqrt{2}$, $\lim_{n\to\infty} P(X \geq \beta\sqrt{\log(n)}) = 1$

6 Problem 6

Denote V_n as $Var(X_n)$, we have

$$X_n = \sum_{i=1}^{X_{n-1}} W_i \tag{30}$$

with $\mathbb{E}[W_i] = \mu, \mathrm{Var}(W_i) = \sigma^2$.

First we need to use the Law of Total Variance

$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbb{E}[X \mid Y]) \tag{31}$$

The proof is given as below.

Because $\mathrm{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2$,

$$\begin{aligned}\mathbb{E}[\mathrm{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2]\end{aligned} \tag{32}$$

Meanwhile,

$$\begin{aligned}\mathrm{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[\mathbb{E}[X|Y]])^2 \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2\end{aligned} \tag{33}$$

Therefore,

$$\begin{aligned}\mathbb{E}[\mathrm{Var}(X|Y)] + \mathrm{Var}(\mathbb{E}[X|Y]) =& \mathbb{E}[x^2] - (\mathbb{E}[x])^2 \\ =& \mathrm{Var}(X)\end{aligned} \tag{34}$$

Using the Law of Total Variance,

$$\mathrm{Var}(X_n) = \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]) \tag{35}$$

For the first item,

$$\mathbb{E}\left[\mathrm{Var}\left(X_n|X_{n-1}\right)\right] = \mathbb{E}\left[\mathrm{Var}\left(\sum_{i=1}^{X_{n-1}} W_i|X_{n-1}\right)\right] = \mathbb{E}\left[X_{n-1}\sigma^2\right] = \sigma^2\mathbb{E}[X_{n-1}] \tag{36}$$

For the second item,

$$\mathrm{Var}\left(\mathbb{E}\left[X_n|X_{n-1}\right]\right) = \mathrm{Var}\left(\mathbb{E}\left[\sum_{i=1}^{X_{n-1}} W_i|X_{n-1}\right]\right) = \mathrm{Var}\left(X_{n-1}\mu\right) = \mu^2\,\mathrm{Var}(X_{n-1}) \tag{37}$$

Also we have

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}X_n|X_{n-1}] = \mu\mathbb{E}[X_{n-1}] = \mu^n \tag{38}$$

So

$$V_n = \sigma^2\mu^{n-1} + \mu^2 V_{n-1} \tag{39}$$

And $V_1 = \mathrm{Var}(W_1) = \sigma^2$. If $\mu = 1$, $V_n = n\sigma^2$.

If $\mu \neq 1$,

$$\frac{V_n}{\mu^{2n}} = \frac{V_{n-1}}{\mu^{2n-2}} + \frac{\sigma^2}{\mu^{n+1}} = \sum_{i=1}^{n} \frac{\sigma^2}{\mu^{i+1}} = \frac{\sigma^2}{\mu^{n+1}} \frac{\mu^n - 1}{\mu - 1} \tag{40}$$

So $\mathrm{Var}(X_n) = \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}$.

In summary,

$$\mathrm{Var}(X_n) = \begin{cases} n\sigma^2, & \text{if } \mu = 1, \\ \sigma^2 \mu^{n-1} \dfrac{\mu^n - 1}{\mu - 1}, & \text{if } \mu \neq 1. \end{cases} \tag{41}$$