Exercise 2
Probability Theory 2020 Autumn

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Contents

1 Problem 1

Define the number of each dice as X_1, X_2, X_3 and event A as $X_1 > X_2 > X_3$ and B as X_1, X_2, X_3. Obviously there are $6^3 = 216$ possible combinations of (X_1, X_2, X_3) with equal probability. So we just need to find the size of $A = \{(X_1, X_2, X_3) | X_1 > X_2 > X_3, X_1, X_2, X_3 \in \{1, 2, \cdots 6\}\}$

Fix $X_1 = i$, $X_2 = j < i$, there are $j - 1$ possible choices of X_3. So the size of A is

$$\sum_{i=1}^{6} \left(\sum_{j=1}^{i-1} (j-1) \right) = \sum_{i=1}^{6} \left(\frac{(i-1)(i-2)}{2} \right) = 20 \tag{1}$$

Symmetrically, $|A| = |B| = 20$. So $P(X_1 > X_2 > X_3) = P(X_1 < X_2 < X_3) = \frac{20}{216} = \frac{5}{54}$

2 Problem 2

2.1 (i)

Using the Lagrange Multiplier method,

$$L = H(X) + \lambda_1 (\sum_{k=1}^{n} p_k - 1) + \lambda_2 (\sum_{k=1}^{n} p_k x_k - \mu) \tag{2}$$

To maximize L,

$$\begin{cases} \frac{\partial L}{\partial p_k} = 0 \quad (k = 1, 2, \cdots, n) \\ \frac{\partial L}{\partial \lambda_1} = 0 \\ \frac{\partial L}{\partial \lambda_2} = 0 \end{cases} \tag{3}$$

And

$$\frac{\partial L}{\partial p_k} = -1 - \log(p_k) + \lambda_1 + \lambda_2 x_k = 0 \Longleftrightarrow p_k = e^{\lambda_2 x_k + \lambda_1 - 1} = C r^{x_k} \tag{4}$$

where $C = e^{\lambda_1 - 1}, r = e^{\lambda_2}$ which are constants determined by $\sum_{k=1}^{n} p_k = 1$ and $\sum_{k=1}^{n} x_k p_k = \mu$.

2.2 (ii)

For a countable support set, let $n \to \infty$,

$$L = H(X) + \lambda_1 (\sum_{k=1}^{\infty} p_k - 1) + \lambda_2 (\sum_{k=1}^{\infty} p_k x_k - \mu) \tag{5}$$

To maximize L,

$$\begin{cases} \dfrac{\partial L}{\partial p_k} = 0 \quad (k = 1, 2, \cdots, \infty) \\ \dfrac{\partial L}{\partial \lambda_1} = 0 \\ \dfrac{\partial L}{\partial \lambda_2} = 0 \end{cases} \tag{6}$$

And

$$\frac{\partial L}{\partial p_k} = -1 - \log(p_k) + \lambda_1 + \lambda_2 x_k = 0 \Longleftrightarrow p_k = e^{\lambda_2 x_k + \lambda_1 - 1} = Cr^{x_k} \tag{7}$$

where $C = e^{\lambda_1 - 1}, r = e^{\lambda_2}$ which are constants determined by $\sum_{k=1}^{\infty} p_k = 1$ and $\sum_{k=1}^{\infty} x_k p_k = \mu$. For the case of $x_k = k$,

$$\begin{cases} \displaystyle\sum_{i=1}^{\infty} p_k = \sum_{i=1}^{\infty} Cr^k = \frac{Cr}{1-r} = 1 \\ \displaystyle\sum_{k=1}^{\infty} x_k p_k = \sum_{i=1}^{\infty} Ckr^k = \frac{Cr}{(1-r)^2} = \mu \end{cases} \tag{8}$$

So $C = \mu - 1, r = \frac{\mu - 1}{\mu}$ and $P(X = k) = Cr^k$ is a geometric distribution.

3 Problem 3 (Conditionally convergent series)

3.1 (i)

According to Leibniz's test, $S_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+k/n} = \int_0^1 \frac{1}{1+x} dx = \ln 2 \tag{9}$$

3.2 (ii)

Notice that $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{2}(1 - \frac{1}{2})$, $\frac{1}{3} - \frac{1}{6} - \frac{1}{8} = \frac{1}{2}(\frac{1}{3} - \frac{1}{4})$ and so on. Thus,

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{\ln 2}{2} \tag{10}$$

3.3 (iii)

$\frac{1}{n} > \ln(1 + \frac{1}{n}) = \ln(n+1) - \ln(n)$, so $\sum_{k=1}^{n} \frac{1}{k} > \ln(n+1)$, and the infinite sum converges. So the series $\frac{(-1)^{(n+1)}}{n}$ is conditionally convergent but not absolutely convergent.

4 Problem 4

4.1 (i)

For $X \sim \text{Geometric}(p)$,we have $E(X) = \frac{1}{p}$ and $E(X^2) = \frac{2-p}{p^2}$.

Because $P(X-1=k|X>1) = P(X=k)$,

$$\begin{aligned} E[X^3|X>1] &= E[(X-1)^3 + 3(X-1)^2 + 3(X-1) + 1|X>1] \\ &= E[X^3] + 3E[X^2] + 3E[X] + 1 \\ &= E[X^3] + \frac{6-3p}{p^2} + \frac{3}{p} + 1 \end{aligned} \tag{11}$$

Also, $E[X^3] = E[X^3|X>1](1-p) + p$.

$$E[X^3] = \frac{(\frac{6}{p^2}+1)(1-p)+p}{p} = \frac{6-6p+p^2}{p^3} \tag{12}$$

For $E[X^4]$

$$\begin{aligned} E[X^4|X>1] &= E[(X-1)^4 + 4(X-1)^3 + 6(X-1)^2 + 4(X-1) + 1|X>1] \\ &= E[(X-1)^4|X>1] + 4 * \frac{6-6p+p^2}{p^3} + 6 * \frac{2-p}{p^2} + 4 * \frac{1}{p} + 1 \\ &= E[X^4] + \frac{24-12p+2p^2+p^3}{p^3} \end{aligned} \tag{13}$$

Also, $E[X^4] = E[X^4|X>1](1-p) + p$.

$$E[X^4] = \frac{(\frac{24-12p+2p^2+p^3}{p^3})(1-p)+p}{p} = \frac{24-36p+14p^2-p^3}{p^4} \tag{14}$$

4.2 (ii)

First we prove that $E[X(X-1)\cdots(X-k)] = \lambda^{k+1}$,

$$\begin{aligned} E[X(X-1)\cdots(X-k)] &= \sum_{n=0}^{\infty} n(n-1)\cdots(n-k)e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \sum_{n=k+1}^{\infty} n(n-1)\cdots(n-k)e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \sum_{m=0}^{\infty} m(m+1)\cdots(m+k+1)e^{-\lambda}\frac{\lambda^{(m+k+1)}}{(m+k+1)!} \quad (m=n-k-1) \\ &= \lambda^{k+1}\sum_{m=0}^{\infty} e^{-\lambda}\frac{\lambda^m}{m!} = \lambda^{k+1} \end{aligned} \tag{15}$$

Thus, $E[X] = \lambda, E[X^2] = E[X(X-1)] + E[X] = \lambda^2 + \lambda$.

$$
\begin{aligned}
E[X^3] &= E[X(X-1)(X-2)] + 3E[X^2] - 2E[X] \\
&= \lambda^3 + 3(\lambda^2+\lambda) - 2\lambda = \lambda^3 + 3\lambda^2 + \lambda
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
E[X^4] &= E[X(X-1)(X-2)(X-3)] + 6E[X^3] - 11E[X^2] + 6E[X] \\
&= \lambda^4 + 6(\lambda^3 + 3\lambda^2 + \lambda) - 11(\lambda^2 + \lambda) + 6\lambda \\
&= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda
\end{aligned} \tag{17}
$$

Note: *This problem can also be solved using the moment generating function.*

5 Problem 5

If $\beta > 1$, there must be a collision.

If $0 \leqslant \beta \leqslant 1$, consider the probability that there is no collision.

Let us start from the first person, who can choose any number(n choices). To avoid collision, the second person can only choose from the left $n-1$ numbers, and the third person $n-2$ and so on. So there are $n(n-1)\cdots(n-k+1)$ possible choices of numbers without collision. So the probability that a collision happens is

$$
P(n,k) = 1 - \frac{n(n-1)\cdots(n-k+1)}{n^k} = 1 - \frac{n!}{n^k(n-k)!} \tag{18}
$$

When $\beta = 1$, $P(n,k) = 1 - \frac{n!}{n^n}$, using Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, $\lim\limits_{n\to\infty} P(n,k) = 1$.

When $0 \leqslant \beta < 1$, $n \to \infty$, $k/n \to 0$, so $1 - \frac{i}{n} \sim e^{-i/n}$,

$$
\frac{n(n-1)\cdots(n-k+1)}{n^k} = 1(1-\frac{1}{n})\cdots(1-\frac{k-1}{n}) \approx \prod_{i=0}^{k-1} e^{-\frac{i}{n}} \approx e^{-\frac{k^2}{2n}} \tag{19}
$$

So $\beta_0 = \frac{1}{2}$.

- $\beta < \beta_0$: $\lim\limits_{n\to\infty} P(n,k) = 0$
- $\beta = \beta_0$: $\lim\limits_{n\to\infty} P(n,k) = \frac{1}{\sqrt{e}}$
- $\beta > \beta_0$: $\lim\limits_{n\to\infty} P(n,k) = 1$

6 Problem 6

6.1 (i)

Define the event A_m as $\exists n$ s.t. $S_n = m$, and $f(m) = \mathbb{P}(A_m)$.

It can be decomposed into one of those 6 situations,

1. $\exists n$ s.t. $S_n = n-1$, and $X_{n+1} = 1$
2. $\exists n$ s.t. $S_n = n-2$, and $X_{n+1} = 2$ (Note that when $X_{n+1} = 1$, it is the first situation)
3. $\exists n$ s.t. $S_n = n-3$, and $X_{n+1} = 3$

4. $\exists n$ s.t. $S_n = n - 4$, and $X_{n+1} = 4$
5. $\exists n$ s.t. $S_n = n - 5$, and $X_{n+1} = 5$
6. $\exists n$ s.t. $S_n = n - 6$, and $X_{n+1} = 6$

$$f(m) = \mathbb{P}(A_m) = \frac{1}{6}\sum_{i=1}^{6} f(m-i), \quad m > 6 \tag{20}$$

For convenience, define $f(0) = 1$ and $f(n) = 0$ for $n < 0$ so $f(m) = \frac{1}{6}\sum_{i=1}^{6} f(m-i)$ for $m > 0$.

The computer program in *Python* is as follows.

```python
res = [1,1/6]

def f(n):
    if len(res)>n:
        return res[n]
    else:
        while len(res) <= n:
            res.append(sum(res[-6:])/6)
        return res[-1]
```

Figure 1: Source Code to compute $f(m)$

$$f(2020) = 0.28571428571428575 \tag{21}$$

6.2 (ii)

Consider the event A_m^c. It can be decomposed into one of those situations,

1. $\exists n$ s.t. $S_n = n - 1$, but $X_{n+1} = 2, 3, 4, 5, 6$
2. $\exists n$ s.t. $S_n = n - 2$, but $X_{n+1} = 3, 4, 5, 6$ (Note that when $X_{n+1} = 1$, it is the first situation)
3. $\exists n$ s.t. $S_n = n - 3$, but $X_{n+1} = 4, 5, 6$
4. $\exists n$ s.t. $S_n = n - 4$, but $X_{n+1} = 5, 6$
5. $\exists n$ s.t. $S_n = n - 5$, but $X_{n+1} = 6$

So

$$1 - f(m) = \mathbb{P}(A_n^c) = \sum_{i=1}^{5} f(m-i)\frac{6-i}{6} \tag{22}$$

Suppose $f(m)$ converges as $m \to \infty$ and $p^* = \lim_{m\to\infty} f(m)$, let $m \to \infty$ in the equation above (22),

$$1 - p^* = p^* \sum_{i=1}^{5} \frac{6-i}{6} = \frac{5}{2} p^* \tag{23}$$

So $p^* = \frac{2}{7}$.

6.3 (iii)

Denote $a_m = \max\limits_{m-5 \leqslant n \leqslant m} f(n)$ and $b_n = \min\limits_{m-5 \leqslant n \leqslant m} f(n)$.

Part 1

First we prove that a_m is non-increasing and b_n is non-decreasing.

Because $f(m+1) = \frac{1}{6} \sum\limits_{n=m-5}^{m} f(n) \leqslant \max\limits_{m-5 \leqslant n \leqslant m} f(n) = a_m$,

$a_{m+1} = \max\limits_{m-4 \leqslant n \leqslant m+1} f(n) \leqslant \max(f(m+1), a_m) = a_m$

Symmetrically, $b_{m+1} \geqslant b_m$.

Part 2

Because $0 \leqslant b_m \leqslant b_{m+1} \leqslant a_{m+1} \leqslant a_m \leqslant 1$, $\{a_m\}, \{b_m\}$ converges.

Denote $\lim\limits_{m\to\infty} a_m = a$ and $\lim\limits_{m\to\infty} b_m = b$, we will prove that $a = b$.

If $a > b$, let $\epsilon < \frac{1}{12}(a-b)$, $\exists N, \forall n \geqslant N$, $a - \epsilon < a_n < a + \epsilon, b - \epsilon < b_n < b + \epsilon$.

Thus $\forall n \geqslant N + 6, f(n) \leqslant \frac{5}{6}(a+\epsilon) + \frac{1}{6}(b+\epsilon) \leqslant a - \epsilon$, so $a_n \leqslant a - \epsilon$, which is contradictory.

So $a = b$ and $f(m)$ converges.