Exercise 5
Probability Theory 2020 Autumn

Hanmo Chen
2020214276

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1 Problem 1

Yes. An example is given as below.

Let $X \sim N(0,1)$, denote $\alpha = \Phi^{-1}(0.75) = 0.674$. So $P(-\alpha \leqslant X \leqslant \alpha) = 0$. Define Y as,

$$Y = \begin{cases} X, \quad \text{if } -\alpha \leqslant X \leqslant \alpha \\ -X, \quad \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \tag{1}$$

And the cdf of Y is,

$$F_Y(y) = P(Y \leqslant y) = \begin{cases} P(X \geqslant -y) = \Phi(y), & y \leqslant -\alpha \\ P(Y \leqslant -\alpha) + P(-\alpha < Y \leqslant y) = \Phi(y), & -\alpha \leqslant y \leqslant \alpha \\ P(Y \leqslant \alpha) + P(\alpha < Y \leqslant y) = \Phi(y), & y > \alpha \end{cases} \tag{2}$$

So $Y \sim N(0,1)$. But

$$X + Y = \begin{cases} 2X, \quad \text{if } -\alpha \leqslant X \leqslant \alpha \\ 0, \quad \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \tag{3}$$

So the distribution of $X + Y$ is not Gaussian distribution.

2 Problem 2

$X \sim N(0,1)$. Proof is given as below.

Let $X_1, X_2, \cdots, X_n, \cdots$ be i.i.d. random variables of the same distribution with X.

Define $S_n = \sum_{i=1}^n X_i$. Let $T_1 = \frac{X_1+X_2}{\sqrt{2}} = \frac{S_2}{\sqrt{2}}$, $T_1' = \frac{X_3+X_4}{\sqrt{2}} = \frac{S_4-S_2}{\sqrt{2}}$, so T_1, T_1' follows the same distribution with X. Let $T_2 = \frac{T_1+T_1'}{\sqrt{2}} = \frac{S_4}{\sqrt{4}}$ also follows the same distribution of X. And define $T_n = \frac{S_{2^n}}{\sqrt{2^n}}$, and its distribution is also the same distribution of X.

According to CLT,

$$\frac{S_{2^n}}{\sqrt{2^n}} \xrightarrow{d} N(0,1) \tag{4}$$

So the distribution of X is $N(0,1)$.

Note: another possible method is to consider the characteristic function $\phi_X(t)$, we can get an equation $\phi_X(t) = [\phi_X(\frac{t}{\sqrt{2}})]^2$. Also we have $\phi_X(0) = 1, \phi'_X(0) = 0, \phi''_X(0) = 1$. By solving the function equation we can get $\phi_X(t) = \exp(-\frac{t^2}{2})$ and $X \sim N(0,1)$.

3 Problem 3

3.1 (1)

Denote $-S$ as the exponent of the density function.

$$\begin{aligned} S &= \frac{1}{2}\left(x_1^2 + \sum_{i=1}^{2n-2}(x_{i+1}-x_i)^2 + x_{2n-1}^2\right) \\ &= \sum_{i=1}^{2n-1} x_i^2 - \sum_{i=1}^{2n-2} x_{i+1}x_i \\ &= \sum_{i=1}^{2n-2} A_i(x_i - B_i x_{i+1})^2 + A_{2n-1}x_{2n-1}^2 \end{aligned} \tag{5}$$

To find A_i, B_i, compare the coefficents,

$$\begin{cases} 2A_iB_i = 1 \\ A_iB_i^2 + A_{i+1} = 1 \\ A_1 = 1 \end{cases} \tag{6}$$

By induction we have $B_n = \frac{n}{n+1}, A_n = \frac{n+1}{2n}$.

And let $Y_i = \sqrt{2A_i}(X_i - B_iX_{i+1}), i = 1, 2, \cdots, 2n-2, Y_{2n-1} = \sqrt{2A_{2n-1}}X_{2n-1}$.

$$f_{Y_1,\cdot,Y_n}(y_1, y_n) = \left(\prod_{i=1}^{2n-1}\sqrt{2A_i}\right)^{-1} c_n \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{2n-1} y_i^2\right)\right) \tag{7}$$

Obviously $(Y_1, \cdots, Y_{2n-1})$ is a Gaussian random vector, so $(X_1, \cdots, X_{2n-1})$ as linear combinations of $(Y_1, \cdots, Y_{2n-1})$ is also a Gaussian vector.

3.2 (2)

$$\left(\prod_{i=1}^{2n-1}\sqrt{2A_i}\right)^{-1} c_n = (\sqrt{2\pi})^{-(2n-1)} \tag{8}$$

So $c_n = \frac{\sqrt{2n}}{(\sqrt{2\pi})^{2n-1}}$

3.3 (3)

To find $\mathrm{Var}(X_n)$ we need to find the inverse transform $X = M^{-1}Y$. However, since Y_i are independent standard Gaussian variables, and X_i are just linear combinations of $Y_i, Y_{i+1}, \cdots, Y_{2n-1}$, so X_{i+1} and Y_i are independent.

By $X_i = \sqrt{\frac{i}{i+1}} Y_i + \frac{i}{i+1} X_{i+1}$,

$$\mathrm{Var}(X_i) = \frac{i^2}{(i+1)^2} \mathrm{Var}(X_{i+1}) + \frac{i}{i+1} \tag{9}$$

And $\mathrm{Var}(X_{2n-1}) = \frac{2n-1}{2n}$. By induction,

$$\mathrm{Var}(X_i) = \frac{(2n-i)i}{2n} \tag{10}$$

So $\mathrm{Var}(X_n) = \frac{n}{2}$.

Note: another tricky method. Notice that X_i are symmetric about n, so $\mathrm{Var}(X_{n-1}) = \mathrm{Var}(X_{n+1})$.

And by letting $i = n, n-1$ in (9),

$$\begin{cases} \mathrm{Var}(X_n) = \dfrac{n^2}{(n+1)^2} \mathrm{Var}(X_{n+1}) + \dfrac{n}{n+1} \\ \mathrm{Var}(X_{n-1}) = \dfrac{(n-1)^2}{n^2} \mathrm{Var}(X_n) + \dfrac{n-1}{n} \end{cases} \tag{11}$$

Solving the equation, we also get $\mathrm{Var}(X_n) = \frac{n}{2}$.