Exercise 7 & 8
Probability Theory 2020 Autumn

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December 24, 2020

1 Problem 1

Denote the entries in U as u_{ij} and entries in Y as Y_{ij},so

$$Y_{ij} = \sum_{r,s} u_{ri}u_{sj}X_{ij} \tag{1}$$

Also we have,

$$\mathrm{Cov}(X_{ij}, X_{mn}) = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Thus

$$\begin{aligned} \mathrm{Cov}(Y_{ij}, Y_{mn}) &= \mathrm{Cov}\left(\sum_{r,s} u_{ri}u_{sj}X_{ij}, \sum_{p,q} u_{pm}u_{qn}X_{mn}\right) \\ &= 2\sum_{r=1}^{n} u_{ri}u_{rj}u_{rm}u_{rn} + \sum_{r\neq s} u_{ri}u_{sj}u_{rm}u_{sn} + \sum_{r\neq s} u_{ri}u_{sj}u_{rn}u_{sm} \\ &= \sum_{r,s} [u_{ri}u_{sj}u_{rm}u_{sn} + u_{ri}u_{sj}u_{rn}u_{sm}] \\ &= \left(\sum_r u_{ri}u_{rm}\right)\left(\sum_s u_{sj}u_{rn}\right) + \left(\sum_r u_{ri}u_{rn}\right)\left(\sum_s u_{sj}u_{sm}\right) \end{aligned} \tag{3}$$

Denote the column vectors in U as $\boldsymbol{u}_i, i = 1, 2, 3, \cdots, N$, so $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \sum_r u_{ri}u_{rj} = \delta_{ij}$.

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Because $X_{ij}, j \geqslant i$ are independent Gaussian variables, so the joint distribution of Y_{ij} is joint Gaussian distribution, which means,

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = 0 \Longleftrightarrow Y_{ij}, Y_{mn} \text{ are independent} \tag{5}$$

So all the entries on and above the diagonal of Y are independent, and $Y_{ii} \sim N(0,2), i = 1,2,3,\cdots,N$ and $Y_{ij} \sim N(0,1), 1 \leqslant i < j \leqslant n$. (It is easy to see that $\mathbb{E}[Y_{ij}] = 0$)

2 Problem 2

Notice that

1. If $X_n = 1$, then $X_{n+1}, X_{n+2}, \cdots$ are independent of $X_1, X_2, \cdots, X_n$
2. There is at least one 1 in any five-in-a-row X_is as $\{X_n, X_{n+1}, \cdots, X_{n+4}\}$

So we can split $X_1, X_2, \cdot, X_n$ into a series of epsisodes, each epsisode $L_j = [0, \cdots, 0, 1]$ is consisted of n zeros (n can be $0,1,2,3,4$) and 1 one. And $L_j, j = 1,2,\cdots,m$ are independent. (For the last epsisode, if it is ended with 0, we can append 1 to its end and let $n = n+1$.) Denote the length of each epsisode as l_j, so $\sum_{j=1}^m l_j = n$.

Consider the distribution of l_j, it can only take values in $1,2,3,4,5$,

- $P(l_j = 1) = P(X_1 = 1) = 0.2$
- $P(l_j = 2) = P(X_1 = 0, X_2 = 1) = 0.16$
- $P(l_j = 3) = P(X_1 = 0, X_2 = 0, X_3 = 1) = 0.128$
- $P(l_j = 4) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1) = 0.1024$
- $P(l_j = 5) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0) = 0.4096$

So

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{m\to\infty} \frac{m}{l_1 + l_2 + \cdots + l_m} = \lim_{m\to\infty} \frac{1}{\frac{1}{m}\sum_{j=1}^m l_j} \tag{6}$$

According to Strong Law of Large Numbers,

$$\frac{1}{m}\sum_{j=1}^m l_j \xrightarrow{a.s} E[l_j] = 3.3616 \tag{7}$$

So

$$\lim_{n\to\infty} \frac{S_n}{n} \xrightarrow{a.s} \frac{1}{3.3616} \tag{8}$$

3 Problem 3

3.1 (i)

Suppose the corresponding k of X_n is k_n, i.e. $\sum_{i=1}^{k_n} Y_i = X_n + n$. If $X_n \geqslant 1$, $\sum_{i=1}^{k_n} Y_i \geqslant n+1$, so $k_{n+1} = k_n, X_{n+1} = X_n - 1$. If $X_n = 0$, $\sum_{i=1}^{k_n} Y_i = n, \sum_{i=1}^{k_n+1} Y_i = n + Y_{n+1} \geqslant n+1$, so $k_{n+1} = k_n, X_{n+1} = Y_{n+1} - 1$.

So given X_n, X_{n+1} is independent of $X_{n-1}, \cdots, X_1$. $\{X_n\}_{n=1}^{\infty}$ forms a Markov Chain. And the transition probability is,

$$P(X_{n+1} = i | X_n = 0) = p_{i+1}, i = 0, 1, \cdots \tag{9}$$

$$P(X_{n+1} = i | X_n = j, j \geqslant 1) = \begin{cases} 1, & i = j-1 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

3.2 (ii)

Notice that $f(n) = P(X_n = 0)$, so $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} P(X_n = 0)$. If we want $\lim_{n\to\infty} f(n)$ exists, the Markov chain must be irreducible, aperiodic and positive recurrent.

It is irreducible obviously. Consider the support set $\mathcal{Y} = \{i : p_i > 0\}$ of Y, if $\inf \mathcal{Y} = N < \infty$, the state space $\mathcal{S}$ of the Markov Chain is finite $\{0, 1, \cdots, N\}$. Obviously N can be reached from 0. And because $N-1, N-2, \cdots, 0$ can be reached from N, so it is irreducible. If $\inf \mathcal{Y} = \infty$, for any state n, there exists a state $m > n$, and m can be reached from 0, so n can be reached from 0. In that case, the Markov chain is also irreducible.

For it to be aperiodic, if it comes from 0 to i, it will return to 0 in i steps. So if $\mathcal{Y} = \{i : p_i > 0\}$ is like $\{2, 4, \cdots, 2k, \cdots\}$ or $\{3, 6, 9, \cdots, 3k, \cdots,\}$, for certian steps it will not arrive at 0. So the Markov chain is aperiodic if and only if $\gcd(\mathcal{Y}) = 1$

And it is positive recurrent if and only if $\mathbb{E}[T_0] < \infty$. It is easy to see that $P(T_0 = i+1) = p_i, i \geqslant 1$, so

$$\mathbb{E}[T_0] = \sum_{i=1}^{\infty} i p_i = \mathbb{E}[Y_1] \tag{11}$$

So the necessary and sufficient condition for $\lim\limits_{n\to\infty} f(n)$ to exist is $\gcd(\{i+1 : p_i > 0\}) = 1$ and $\sum_{i=1}^{\infty} i p_i < \infty$

3.3 (iii)

The limits equals to the steady-state probability,

$$\pi_0 = \lim_{n\to\infty} f(n) = \frac{1}{\mathbb{E}[T_0]} = \frac{1}{\mu} \tag{12}$$

4 Problem 4

4.1 (i)

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geqslant 1 | X_0 = n)$ So the probability that the chain never returns to zero is $f(0) = f(1)$. When $X_0 = 0$, $X_1 = 1$, and X_2 must be 2. So $f(1) = \frac{4}{5} f(2)$.

Consider $f(2)$,

$$
\begin{aligned}
f(2) &= P(X_n > 0, \forall n \geqslant 1 | X_0 = 2) \\
&= P(X_1 = 1, X_n > 0, \forall n \geqslant 2 | X_0 = 2) + P(X_1 = 3, X_n > 0, \forall n \geqslant 2 | X_0 = 2) \\
&= p_{21} P(X_n > 0, \forall n \geqslant 1 | X_0 = 1) + p_{23} P(X_n > 0, \forall n \geqslant 1 | X_0 = 3) \\
&= \frac{4}{13} f(1) + \frac{9}{13} f(3)
\end{aligned} \tag{13}
$$

Because $f(2) = \frac{5}{4} f(1)$

$$
f(3) - f(2) = \frac{4}{9}(f(2) - f(1)]) = \frac{4}{9} \times \frac{1}{4} f(1) = \frac{1}{9} f(1) \tag{14}
$$

In general, we have

$$
f(n+1) - f(n) = \frac{n^2}{((n+1)^2}(f(n) - f(n-1)) = \frac{1}{(n+1)^2} f(1) \tag{15}
$$

So

$$
f(n) = \sum_{i=1}^{n} \frac{1}{i^2} f(1) \tag{16}
$$

And note that $f(n) \to 1$ as $n \to \infty$, using the famous lemma

$$
\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \tag{17}
$$

So $f(1) = \frac{6}{\pi^2}$. And the probability that the chain never returns to zero is $\frac{6}{\pi^2}$.

4.2 (ii)

Using the same method, we have

$$
f(n) = \sum_{i=1}^{n} \frac{1}{i^\alpha} f(1) \tag{18}
$$

When $\alpha > 1$, $\sum_{i=1}^{n} \frac{1}{i^\alpha}$ converges, so $0 < f(1) < 1$. The markov chain is transient (because it is irreducible and state 0 is transient) .

When $\alpha \leqslant 1$, $\sum_{i=1}^{n} \frac{1}{i^\alpha}$ goes to ∞, so $f(1) = 0$. The markov chain is recurrent (because it is irreducible and state 0 is recurrent) .

And to determine it is positive recurrent or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^{\infty}$. Obviously we have $\pi_0 = \frac{1}{2^\alpha + 1}\pi_1$ and for $n \geqslant 1$, we have

$$
\pi_n = p_{n-1,n}\pi_{n-1} + p_{n+1,n}\pi_{n+1} \tag{19}
$$

Thus we have

$$
p_{n+1,n}\pi_{n+1} - p_{n,n+1}\pi_n = p_{n,n-1}\pi_n - p_{n-1,n}\pi_{n-1} = p_{1,0}\pi_1 - \pi_0 = 0 \tag{20}
$$

So

$$\pi_{n+1} = \frac{p_{n,n+1}}{p_{n+1,n}}\pi_n = \frac{(n+2)^\alpha + (n+1)^\alpha}{(n+1)^\alpha + n^\alpha}\pi_n = ((n+2)^\alpha + (n+1)^\alpha)\pi_0 \tag{21}$$

And $\sum_{n=0}^{\infty} \pi_n = 1$

$$\pi_0 \sum_{n=0}^{\infty} (n^\alpha + (n+1)^\alpha) = 1 \tag{22}$$

When $\alpha \geqslant -1$, $\sum_{n=0}^{\infty} n^\alpha$ goes to ∞, $\pi_0 = 0$, the Markov chain is null recurrent.

When $\alpha < -1$, $\sum_{n=0}^{\infty} n^\alpha$ converges, $0 < \pi_0 < 1$, the Markov chain is positive recurrent. And $\pi_0 = \frac{1}{2\zeta(-\alpha)}$, where $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$ is the Riemann function.

In summary,

- When $\alpha > 1$, the Markov chain is transient.
- When $-1 \leqslant \alpha \leqslant 1$, the Markov chain is null recurrent.
- When $\alpha < 1$, the Markov chain is positive recurrent.

5 Problem 5

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geqslant 1 | X_0 = n)$. The probability that the Markov chain never returns to zero is $f(1)$. Also we have $f(2) = pf(3), f(1) = pf(3) + (1-p)f(2)$. So $f(2) = \frac{1}{2-p}f(1), f(3) = \frac{1}{(2-p)p}f(1)$. And for $n \geqslant 2$,

$$\begin{cases} f(2n-1) = pf(2n+1) + (1-p)f(2n) \\ f(2n) = pf(2n+1) + (1-p)f(2n-2) \end{cases} \tag{23}$$

$$\begin{cases} f(2n) = \dfrac{1}{2-p}((1-p)f(2n-2) + f(2n-1)) \\ f(2n+1) = \dfrac{1}{(2-p)p}(f(2n-1) - (1-p)^2 f(2n-2)) \end{cases} \tag{24}$$

Solving the equation $f(2n+1) + \lambda f(2n) = C(f(2n-1) + \lambda f(2n-2))$, we have $\lambda = -1$ and $\frac{-(1-p)^2}{p}$. Thus,

$$\begin{cases} f(2n+1) - f(2n) = \dfrac{(1-p)}{(2-p)p}(f(2n-1) - f(2n-2)) = \left[\dfrac{(1-p)}{(2-p)p}\right]^n f(1) \\ f(2n+1) - \dfrac{(1-p)^2}{p}f(2n) = f(2n-1) - \dfrac{(1-p)^2}{p}f(2n-2) = f(1) \end{cases} \tag{25}$$

When $\frac{(1-p)}{(2-p)p} > 1, p < \frac{3-\sqrt{5}}{2}$, $f(1)$ must be 0. So the chain is recurrent.

When $\frac{(1-p)}{(2-p)p} = 1, p = \frac{3-\sqrt{5}}{2}$, $f(2n+1) - f(2n) = f(1)$. Using equation (23), we have $f(2n+1) = f(2n-1) + (1-p)f(1)$, because $1 - p > 0$, $f(1)$ must be zero and the chain is recurrent.

And when $\frac{(1-p)}{(2-p)p} < 1, p > \frac{3-\sqrt{5}}{2}$, $f(1) = 1 - \frac{(1-p)^2}{p} > 0$, the chain is transient.

To determine whether it is positive or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^{\infty}$.

So $\pi_0 = (1-p)\pi_2, \pi_1 = \pi_0$. For $n \geqslant 1$,

$$\begin{cases} \pi_{2n+1} = p\pi_{2n-1} + p\pi_{2n} \\ \pi_{2n} = (1-p)\pi_{2n-1} + (1-p)\pi_{2n+2} \end{cases} \tag{26}$$

Solving the equation $\pi_{2n+1} + \lambda\pi_{2n+2} = C(\pi_{2n-1} + \lambda\pi_{2n})$, $\lambda = -(1-p)$ or $-\frac{p}{1-p}$.

So

$$\begin{cases} \pi_{2n+1} - (1-p)\pi_{2n+2} = \pi_{2n-1} - (1-p)\pi_{2n} = 0 \\ \pi_{2n+1} - \dfrac{p}{1-p}\pi_{2n+2} = \dfrac{p(2-p)}{1-p}(\pi_{2n-1} - \dfrac{p}{1-p}\pi_{2n}) = \left[\dfrac{p(2-p)}{1-p}\right]^n (\pi_1 - \dfrac{p}{1-p}\pi_2) \end{cases} \tag{27}$$

So

$$\frac{p^2-3p+1}{1-p}\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \frac{p^2-3p+1}{1-p}\pi_2 \tag{28}$$

When $p = \frac{3-\sqrt{5}}{2}, p^2-3p+1 = 0$, in equation (26) we have that $\pi_{2n+1} = (1-p)\pi_{2n} = \pi_{2n-1}$, by $\sum_{n=0}^{\infty} \pi_n = 1$ we have $\pi_0 = 0$, the Markov chain is null recurrent.

When $p < \frac{3-\sqrt{5}}{2}$, $\frac{p(2-p)}{1-p} < 1$,

$$\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \pi_2 \tag{29}$$

$$\sum_{n=0}^{\infty} \pi_n = (1-p)\pi_2 + (2-p)\pi_2 \sum_{n=0}^{\infty} \left[\frac{p(2-p)}{1-p}\right]^n = \frac{(1-p)^2(3-p)}{p^2-3p+1}\pi_2 = 1 \tag{30}$$

In summary,

- When $p > \frac{3-\sqrt{5}}{2}$, the Markov chain is transient.
- When $p = \frac{3-\sqrt{5}}{2}$, the Markov chain is null recurrent.
- When $p < \frac{3-\sqrt{5}}{2}$, the Markov chain is positive recurrent. And the stationary distribution is,

$$\begin{cases} \pi_0 = \dfrac{p^2-3p+1}{(1-p)(3-p)} \\ \pi_{2n} = \left[\dfrac{p(2-p)}{1-p}\right]^{n-1} \dfrac{p^2-3p+1}{(1-p)^2(3-p)}, n \geqslant 1 \\ \pi_{2n-1} = \left[\dfrac{p(2-p)}{1-p}\right]^{n-1} \dfrac{p^2-3p+1}{(1-p)(3-p)}, n \geqslant 1 \end{cases} \tag{31}$$

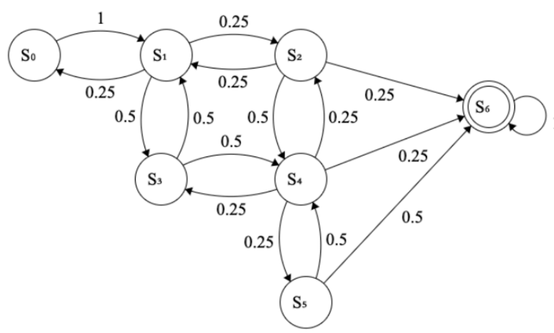
6 Problem 6

6.1 (i)

Define $M_n = \max(|X_n|, |Y_n|), m_n = \min(|X_n|, |Y_n|)$, we can sort (X_n, Y_n) into different states by (M_n, m_n),

- $S_0 = \{M_n = m_n = 0\} = \{(0,0)\}$
- $S_1 = \{m_n = 0, M_n = 1\} = \{(0,1), (1,0), (0,-1), (-1,0)\}$
- $S_2 = \{m_n = 0, M_n = 2\} = \{(0,2), (2,0), (0,-2), (-2,0)\}$
- $S_3 = \{m_n = 1, M_n = 1\} = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$
- $S_4 = \{m_n = 1, M_n = 2\} = \{(2,1), (1,2), (2,-1), (-1,2), (-2,1), (-1,2), (-2,-1), (-1,-2)\}$
- $S_5 = \{m_n = 2, M_n = 2\} = \{(2,-2), (-2,2), (2,2), (-2,-2)\}$
- $S_6 = \{M_n = 3\}$

And we have an absorbing markov chain as follow.



And the corresponding transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{R} \\ \boldsymbol{0} & 1 \end{bmatrix} \tag{32}$$

So the expectation of absorbing time is,

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1} \cdot \boldsymbol{1} = (\frac{135}{13}, \frac{122}{13}, \frac{80}{13}, \frac{17}{2}, \frac{73}{13}, \frac{99}{26}) \tag{33}$$

$\mathbb{E}[T] = \frac{135}{13}$.

To find $\mathbb{P}(X_T = 3, Y_T = 0)$, first notice that $\mathbb{P}(X_T = 3, Y_T = 0) = \mathbb{P}(X_T = 0, Y_T = 3) = \mathbb{P}(X_T = -3, Y_T = 0) = \mathbb{P}(X_T = 0, Y_T = -3) = \frac{1}{4}\,\mathbb{P}(M_T = 3, m_T = 0)$. Then we can divide $S_6 = \{M_n = 3\}$ into 3 states which are also absorbing,

- $S_6^{(0)} = \{M_n = 3, m_n = 0\}$
- $S_6^{(1)} = \{M_n = 3, m_n = 1\}$
- $S_6^{(2)} = \{M_n = 3, m_n = 2\}$

Thus the corresponding matrix $\boldsymbol{R}$ is,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{34}$$

The absorbing probability is,

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1} \cdot \boldsymbol{R} = \begin{bmatrix} \frac{4}{13} & \frac{6}{13} & \frac{3}{13} \\ \frac{4}{13} & \frac{6}{13} & \frac{3}{13} \\ \frac{11}{26} & \frac{5}{13} & \frac{5}{26} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{5}{26} & \frac{7}{13} & \frac{7}{26} \\ \frac{5}{52} & \frac{7}{26} & \frac{33}{52} \end{bmatrix} \tag{35}$$

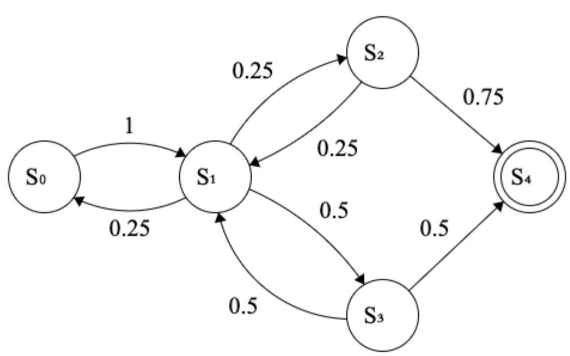
So $\mathbb{P}(X_T = 3, Y_T = 0) = \frac{1}{4} \times \frac{4}{13} = \frac{1}{13}$

6.2 (ii)

Using the same method in (i), define the states as follow,

- $S_0 = \{|X_n| + |Y_n| = 0\}$
- $S_1 = \{|X_n| + |Y_n| = 1\}$
- $S_2 = \{|X_n| + |Y_n| = 2, \big||X_n| - |Y_n|\big| = 2\}$
- $S_3 = \{|X_n| + |Y_n| = 2, \big||X_n| - |Y_n|\big| = 0\}$
- $S_4 = \{|X_n| + |Y_n| = 3\}$

And we have an absorbing markov chain as follow.



We define the expectation of steps from S_i to reach S_3 as $f_i, i = 0, 1, 2$.

$$\begin{cases} f_0 = f_1 + 1 \\ f_1 = 0.25 f_0 + 0.25 f_2 + 0.5 f_3 + 1 \\ f_2 = 0.25 f_1 + 1 \\ f_3 = 0.5 f_1 + 1 \end{cases} \tag{36}$$

Solving the equation we have $\mathbb{E}[T] = f_0 = \frac{39}{7}$. (It is equivalent to $(\boldsymbol{I} - \boldsymbol{Q})^{-1} \cdot \mathbf{1}$.)

Also we can split S_4 into two sub states,

- $S_4^{(0)} = \{|X_n| + |Y_n| = 3, \big||X_n| - |Y_n|\big| = 3\}$
- $S_4^{(1)} = \{|X_n| + |Y_n| = 3, \big||X_n| - |Y_n|\big| = 1\}$

And the corresponding matrix $\boldsymbol{R}$ is

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.25 & 0.5 \\ 0 & 0.5 \end{bmatrix} \tag{37}$$

The absorbing probability is,

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1} \cdot \boldsymbol{R} = \begin{bmatrix} \frac{1}{7} & \frac{6}{7} \\ \frac{1}{7} & \frac{6}{7} \\ \frac{2}{7} & \frac{5}{7} \\ \frac{1}{14} & \frac{13}{14} \end{bmatrix} \tag{38}$$

So $P(X_T = 3, Y_T = 0) = \frac{1}{4} P(|X_T| + |Y_T| = 3, \big||X_T| - |Y_T|\big| = 3) = \frac{1}{28}$.

6.3 (iii)

Define the expectation steps from $(X_n = x, Y_n = y)$ to first arrive at the boundary as $f(x, y)$.

$$\begin{cases} f(-2, y) = f(x, 2) = f(x, -2) = 0 \\ f(x, y) = \dfrac{1}{4}[f(x-1, y) + f(x+1, y) + f(x, y-1) + f(x, y+1)] + 1 \end{cases} \tag{39}$$

According to symmetry we have $f(x, -1) = f(x, 1)$.

Define $g_i = f(i, 0), h_i = f(i, -1) = f(i, 1)$. We have

$$\begin{cases} g_{-2} = h_{-2} = 0 \\ g_i = 0.5 h_i + 0.25 g_{i-1} + 0.25 g_{i+1} + 1 \\ h_i = 0.25 h_{i-1} + 0.25 h_{i+1} + 0.25 g_i + 1 \end{cases} \tag{40}$$

Because $\lim\limits_{n \to \infty} g_{n+1} - g_n = 0$, we have $\lim\limits_{n \to \infty} g_n = 8$, $\lim\limits_{n \to \infty} h_n = 6$. Denote $\alpha_i = g_i - 8, \beta_i = h_i - 6$.

$$\alpha_i + \sqrt{2}\beta_i = \frac{4+\sqrt{2}}{14}(\alpha_{i-1} + \sqrt{2}\beta_{i-1}) + \frac{4+\sqrt{2}}{14}(\alpha_{i+1} + \sqrt{2}\beta_{i+1}) \tag{41}$$

$$\alpha_i - \sqrt{2}\beta_i = \frac{4-\sqrt{2}}{14}(\alpha_{i-1} - \sqrt{2}\beta_{i-1}) + \frac{4-\sqrt{2}}{14}(\alpha_{i+1} - \sqrt{2}\beta_{i+1}) \tag{42}$$

We have

$$\begin{aligned} \alpha_n + \sqrt{2}\beta_i &= A_1\lambda_1^n + A_2\lambda_2^n \\ \alpha_n - \sqrt{2}\beta_i &= B_1\mu_1^n + A_2\mu_2^n \end{aligned} \tag{43}$$

where $\lambda_{1,2} = \frac{1}{2}(4-\sqrt{2} \pm \sqrt{14-8\sqrt{2}}), \mu_{1,2} = \frac{1}{2}(4+\sqrt{2} \pm \sqrt{14+8\sqrt{2}})$. Because lim_$n \to \infty \alpha_n = \lim_{n\to\infty} \beta_n = 0$, we have $A_1 = B_1 = 0$, thus

$$\begin{aligned} \alpha_n &= \frac{1}{2}\left(A\lambda^n + B\mu^n\right) \\ \beta_n &= \frac{1}{2\sqrt{2}}\left(A\lambda^n - B\mu^n\right) \end{aligned} \tag{44}$$

where $\lambda = \frac{1}{2}(4-\sqrt{2} - \sqrt{14-8\sqrt{2}}), \mu = \frac{1}{2}(4+\sqrt{2} - \sqrt{14+8\sqrt{2}})$.

And $\alpha_{-2} = g_{-2} - 8 = -8, \beta_{-2} = h_{-2} - 6 = -6$, we have $A = -\lambda^2(8+6\sqrt{2}), B = \mu^2(6\sqrt{2}-8)$.

$$g_0 = \alpha_0 + 8 = \frac{1}{2}(A+B) + 8 = 3\sqrt{2}(\mu^2 - \lambda^2) - 4(\mu^2 + \lambda^2) + 8 = 10\sqrt{7+\sqrt{17}} - 8\sqrt{14-2\sqrt{17}} - 8 = 6.1617 \tag{45}$$

To solve $P(X_T = 0, Y_T = 0)$, similarly we need to find $(\boldsymbol{I} - \boldsymbol{Q})^{-1} \cdot \boldsymbol{R}$. However, since the Markov chain is infinite, we can define $p(x,y)$ as $P(X_T = 0, Y_T = 0 | X_0 = x, Y_0 = y)$ and solve the following equation,

$$\begin{cases} p(-2,0) = 1 \\ p(-2,-1) = p(-2,1) = p(x,2) = p(x,-2) = 0 \\ p(x,y) = \dfrac{1}{4}[p(x-1,y) + p(x+1,y) + p(x,y-1) + p(x,y+1)] \end{cases} \tag{46}$$

Follows the same step, we have $P(X_T = -2, Y_T = 0) = p(0,0) = \frac{1}{2}(\lambda^2 + \mu^2) = 16 - \sqrt{2(95+7\sqrt{17})} = 0.1304$.

6.4 (iv)

7 Problem 7

7.1 (i)

First, for a random variable $X, \mathbb{E}[X] = 0$ and there exists some constant α, such that $\mathbb{E}[e^{tx}] \leqslant e^{\alpha^2 t^2/2}$, then X is sub-Gaussian, and we have that

$$\mathbb{P}(X \geqslant \lambda) = P(e^{tX} \geqslant e^{t\lambda}) \leqslant \frac{E[e^{tX}]}{e^{t\lambda}} \leqslant e^{\alpha^2 t^2/2 - t\lambda} \tag{47}$$

Let $t = \frac{\lambda}{\alpha}$, we have

$$\mathbb{P}(X \geqslant \lambda) \leqslant e^{-\frac{\lambda^2}{2\alpha^2}} \tag{48}$$

$$\mathbb{P}(|X| \geqslant \lambda) \leqslant 2e^{-\frac{\lambda^2}{2\alpha^2}} \tag{49}$$

For X_i, we have $\mathbb{E}[\exp(tX_i)] = \frac{1}{2}(e^t + e^{-t}) \leqslant \exp(t^2/2)$. When $\sum\limits_{i=1}^{\infty} a_i^2 = S < \infty$,

$$\mathbb{E}[\exp(\sum_{i=1}^{\infty} a_i X_i t)] = \prod_{i=1}^{\infty} \mathbb{E}[\exp(t a_i X_i)] \leqslant \prod_{i=1}^{\infty} \exp(t^2 a_i^2) = e^{St^2} \tag{50}$$

Denote $X = \sum_{i=1}^{\infty} a_i X_i$,

$$\mathbb{P}(|X| \geqslant \lambda) \leqslant 2e^{-\frac{\lambda^2}{S}} \tag{51}$$

Thus we have X converges almost surely, i.e $\mathbb{P}(|X| \leqslant \infty) = \mathbb{P}(|\sum_{i=1}^{\infty} a_i X_i| < \infty) = 1$.

7.2 (ii)

By the corollary of Kolmogorov three-series theorem, given independent variables $X_1, X_2, \cdots, X_n$,

$$\mathbb{P}\left(\sum_{k=1}^{\infty} X_k \text{ converges }\right) = 0 \text{ or } 1 \tag{52}$$

And because $\mathbb{E}[(\sum_{i=1}^{\infty} a_i X_i)^2] = \sum_{i=1}^{\infty} a_i^2 = \infty$, $\mathbb{P}(|\sum_{i=1}^{\infty} a_i X_i| = \infty) > 0$, thus

$$\mathbb{P}\left(|\sum_{i=1}^{\infty} a_i X_i| < \infty\right) = 0 \tag{53}$$

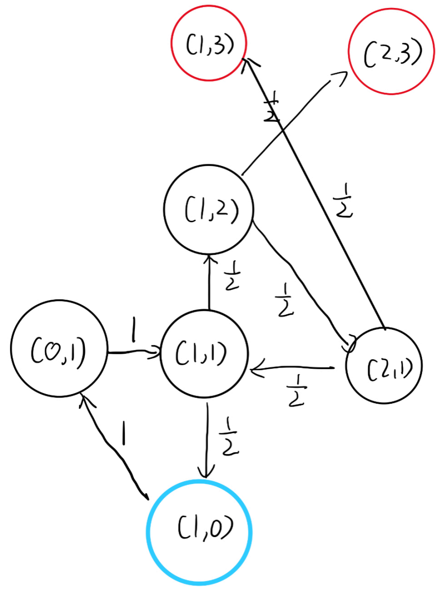
8 Problem 8

8.1 (i)

It is easy to see that $Y_n = (X_{n-1}, X_n)$ is a Markov Chain. And $Y_1 = (0, 1)$. Thus $Y_2 = (1, 1)$. And the state transition graph is as follow.

So if X_n reaches 0 before 3, the path is consisted of three parts,

- $(0, 1) \to (1, 1)$
- m (can be zero) circles of $(1, 1) \to (2, 1) \to (1, 2) \to (1, 1)$
- $(1, 1) \to (1, 0)$



So

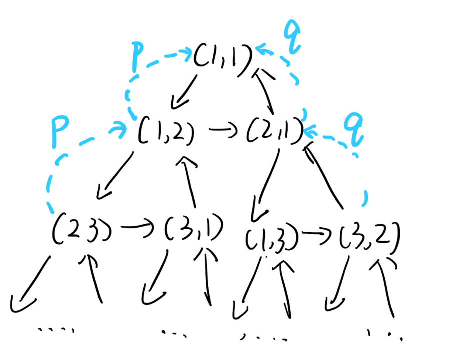
$$\mathbb{P}(X_n \text{ reaches 0 before 3}) = \sum_{m=0}^{\infty} \frac{1}{2}(\frac{1}{8})^m = \frac{4}{7} \tag{54}$$

Thus

$$\mathbb{P}(X_n \text{ reaches 3 before 0}) = 1 - \frac{4}{7} = \frac{3}{7} \tag{55}$$

8.2 (ii)

$Y_1 = (1,2)$. Denote p as the hitting probability of $(1,1)$ from $(1,2)$ and q as the hitting probability of $(1,1)$ from $(2,1)$. Consider the Markov chain starting from $(1,2)$ and $(2,1)$, it can be expressed as a binary tree with recursive structure.



So we have

$$\begin{cases} p = \dfrac{1}{2}q + \dfrac{1}{2}p^2 \\ q = \dfrac{1}{2} + \dfrac{1}{2}pq \end{cases} \tag{56}$$

We have $p(2-p)^2 = 1$. Because $p \leqslant 1$ we have $p = 1$ or $\frac{3-\sqrt{5}}{2}$.

Notice that

$$\mathbb{P}\left(\exists n, Y_n = (1,1) | Y_0 = (1,2)\right) = \mathbb{P}\left(Y_n = (1,1) | Y_{n-1} = (2,1)\right) \mathbb{P}\left(\exists n, Y_{n-1} = (2,1) | Y_0 = (1,2)\right) \leqslant \frac{1}{2} \tag{57}$$

So $p = \frac{3-\sqrt{5}}{2}$.

9 Problem 9

$\{X_n\}_{n\geqslant 0}$ is a martingle because $\mathbb{E}[X_n|X_{n-1}] = X_{n-1}$. Also we have $0 < X_n < 1$ by induction. Thus by Martingale Convergence Theorem, X_n converges almost surely.

Denote the limit random variable as X_∞. First $E[X_\infty] = q$. Also

$$\mathrm{Var}(X_n) = \sum_{i=1}^{n} \mathbb{E}[(X_i - X_{i-1})^2] = \sum_{i=1}^{n} \mathbb{E}[(X_{i-1} - X_{i-1}^2)^2] \tag{58}$$

Given $X_{i-1} = \lambda$, we have

$$\mathbb{E}[(X_i - X_i^2)^2] = \frac{1}{2}[(\lambda(2-\lambda) - \lambda^2(2-\lambda)^2)^2 + (\lambda^2 - \lambda^4)^2] = \lambda^2(1-\lambda)^2(\lambda^2(\lambda-1)^2 - 6\lambda(1-\lambda) + 2) \tag{59}$$

Because $\lambda^2(\lambda-1)^2 - 6\lambda(1-\lambda) + 2 \geqslant (\lambda^2 - \lambda + 1)^2$, we have

$$\mathrm{Var}(X_n) \geqslant q(1-q)\left[1 - (1 - q + q^2)^n\right] \tag{60}$$

And

$$\lim_{n\to\infty} \mathrm{Var}(X_n) \geqslant q(1-q) \tag{61}$$

Also we have

$$\mathrm{Var}(X_\infty) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \leqslant \mathbb{E}[X] - (\mathbb{E}[X])^2 = q - q^2 \tag{62}$$

So the limit distribution is Bernoulli(q).

10 Problem 10

For n-dimention sphere,

$$V_n = \frac{\pi^{\frac{n}{2}} R^n}{\Gamma\left(\frac{n}{2}+1\right)} \tag{63}$$

Denote $S = X_1^2 + X_2^2 + X_3^2$,

$$\begin{aligned} f(x_1, x_2, x_3) &= \lim_{n\to\infty} \frac{\pi^{\frac{n-3}{2}} (n-S)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-3}{2}+1\right)} \times \frac{1}{V_n} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{S}{2}} = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{x_1^2+x_2^2+x_3^2}{2}} \end{aligned} \tag{64}$$

So $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ follows the standard Gaussian distribution as $n \to \infty$.