Exercise 7 & 8
Probability Theory 2020 Autumn

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1 Problem 1

Denote the entries in U as u_{ij} and entries in Y as Y_{ij},so

$$Y_{ij} = \sum_{r,s} u_{ri} u_{sj} X_{ij} \tag{1}$$

Also we have,

$$\mathrm{Cov}(X_{ij}, X_{mn}) = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Thus

$$\begin{aligned} \mathrm{Cov}(Y_{ij}, Y_{mn}) &= \mathrm{Cov}\left(\sum_{r,s} u_{ri} u_{sj} X_{ij}, \sum_{p,q} u_{pm} u_{qn} X_{mn} \right) \\ &= 2 \sum_{r=1}^{n} u_{ri} u_{rj} u_{rm} u_{rn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rm} u_{sn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rn} u_{sm} \\ &= \sum_{r,s} [u_{ri} u_{sj} u_{rm} u_{sn} + u_{ri} u_{sj} u_{rn} u_{sm}] \\ &= \left(\sum_r u_{ri} u_{rm} \right) \left(\sum_s u_{sj} u_{rn} \right) + \left(\sum_r u_{ri} u_{rn} \right) \left(\sum_s u_{sj} u_{sm} \right) \end{aligned} \tag{3}$$

Denote the column vectors in U as $\boldsymbol{u}_i, i = 1, 2, 3, \cdots, N$, so $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \sum_r u_{ri} u_{rj} = \delta_{ij}$.

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Because $X_{ij}, j \geqslant i$ are independent Gaussian variables, so the joint distribution of Y_{ij} is joint Gaussian distribution, which means,

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = 0 \Longleftrightarrow Y_{ij}, Y_{mn} \text{ are independent} \tag{5}$$

So all the entries on and above the diagonal of Y are independent, and $Y_{ii} \sim N(0,2), i = 1,2,3,\cdots,N$ and $Y_{ij} \sim N(0,1), 1 \leqslant i < j \leqslant n$. (It is easy to see that $\mathbb{E}[Y_{ij}] = 0$)

2 Problem 2

Notice that

1. If $X_n = 1$, then $X_{n+1}, X_{n+2}, \cdots$ are independent of $X_1, X_2, \cdots, X_n$
2. There is at least one 1 in any five-in-a-row X_is as $\{X_n, X_{n+1}, \cdots, X_{n+4}\}$

So we can split $X_1, X_2, \cdot, X_n$ into a series of epsisodes, each epsisode $L_j = [0, \cdots, 0, 1]$ is consisted of n zeros (n can be $0,1,2,3,4$) and 1 one. And $L_j, j = 1,2,\cdots,m$ are independent. (For the last epsisode, if it is ended with 0, we can append 1 to its end and let $n = n+1$.) Denote the length of each epsisode as l_j, so $\sum_{j=1}^m l_j = n$.

Consider the distribution of l_j, it can only take values in $1,2,3,4,5$,

- $P(l_j = 1) = P(X_1 = 1) = 0.2$
- $P(l_j = 2) = P(X_1 = 0, X_2 = 1) = 0.16$
- $P(l_j = 3) = P(X_1 = 0, X_2 = 0, X_3 = 1) = 0.128$
- $P(l_j = 4) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1) = 0.1024$
- $P(l_j = 5) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0) = 0.4096$

So

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{m\to\infty} \frac{m}{l_1 + l_2 + \cdots + l_m} = \lim_{m\to\infty} \frac{1}{\frac{1}{m}\sum_{j=1}^m l_j} \tag{6}$$

According to Strong Law of Large Numbers,

$$\frac{1}{m}\sum_{j=1}^m l_j \xrightarrow{a.s} E[l_j] = 3.3616 \tag{7}$$

So

$$\lim_{n\to\infty} \frac{S_n}{n} \xrightarrow{a.s} \frac{1}{3.3616} \tag{8}$$

3 Problem 3

3.1 (i)

Suppose the corresponding k of X_n is k_n, i.e. $\sum_{i=1}^{k_n} Y_i = X_n + n$. If $X_n \geqslant 1$, $\sum_{i=1}^{k_n} Y_i \geqslant n+1$, so $k_{n+1} = k_n, X_{n+1} = X_n - 1$. If $X_n = 0$, $\sum_{i=1}^{k_n} Y_i = n, \sum_{i=1}^{k_n+1} Y_i = n + Y_{n+1} \geqslant n+1$, so $k_{n+1} = k_n, X_{n+1} = Y_{n+1} - 1$.

So given X_n, X_{n+1} is independent of $X_{n-1}, \cdots, X_1$. $\{X_n\}_{n=1}^{\infty}$ forms a Markov Chain. And the transition probability is,

$$P(X_{n+1} = i | X_n = 0) = p_{i+1}, i = 0, 1, \cdots \tag{9}$$

$$P(X_{n+1} = i | X_n = j, j \geqslant 1) = \begin{cases} 1, & i = j - 1 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

3.2 (ii)

Notice that $f(n) = P(X_n = 0)$, so $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} P(X_n = 0)$. If we want $\lim_{n\to\infty} f(n)$ exists, the Markov chain must be irreducible, aperiodic and positive recurrent.

It is irreducible obviously. Consider the support set $\mathcal{Y} = \{i : p_i > 0\}$ of Y, if $\inf \mathcal{Y} = N < \infty$, the state space $\mathcal{S}$ of the Markov Chain is finite $\{0, 1, \cdots, N\}$. Obviously N can be reached from 0. And because $N-1, N-2, \cdots, 0$ can be reached from N, so it is irreducible. If $\inf \mathcal{Y} = \infty$, for any state n, there exists a state $m > n$, and m can be reached from 0, so n can be reached from 0. In that case, the Markov chain is also irreducible.

For it to be aperiodic, if it comes from 0 to i, it will return to 0 in i steps. So if $\mathcal{Y} = \{i : p_i > 0\}$ is like $\{2, 4, \cdots, 2k, \cdots\}$ or $\{3, 6, 9, \cdots, 3k, \cdots, \}$, for certian steps it will not arrive at 0. So the Markov chain is aperiodic if and only if $\gcd(\mathcal{Y}) = 1$

And it is positive recurrent if and only if $\mathbb{E}[T_0] < \infty$. It is easy to see that $P(T_0 = i+1) = p_i, i \geqslant 1$, so

$$\mathbb{E}[T_0] = \sum_{i=1}^{\infty} i p_i = \mathbb{E}[Y_1] \tag{11}$$

So the necessary and sufficient condition for $\lim\limits_{n\to\infty} f(n)$ to exist is $\gcd(\{i+1 : p_i > 0\}) = 1$ and $\sum_{i=1}^{\infty} i p_i < \infty$

3.3 (iii)

The limis equals to the steady-state probability,

$$\pi_0 = \lim_{n\to\infty} f(n) = \frac{1}{\mathbb{E}[T_0]} = \frac{1}{\mu} \tag{12}$$

4 Problem 4

4.1 (i)

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geqslant 1 | X_0 = n)$ So the probability that the chain never returns to zero is $f(0) = f(1)$. When $X_0 = 0$, $X_1 = 1$, and X_2 must be 2. So $f(1) = \frac{4}{5} f(2)$.

Consider $f(2)$,

$$
\begin{aligned}
f(2) &= P(X_n > 0, \forall n \geqslant 1 | X_0 = 2) \\
&= P(X_1 = 1, X_n > 0, \forall n \geqslant 2 | X_0 = 2) + P(X_1 = 3, X_n > 0, \forall n \geqslant 2 | X_0 = 2) \\
&= p_{21} P(X_n > 0, \forall n \geqslant 1 | X_0 = 1) + p_{23} P(X_n > 0, \forall n \geqslant 1 | X_0 = 3) \\
&= \frac{4}{13} f(1) + \frac{9}{13} f(3)
\end{aligned} \tag{13}
$$

Because $f(2) = \frac{5}{4} f(1)$

$$
f(3) - f(2) = \frac{4}{9}(f(2) - f(1)]) = \frac{4}{9} \times \frac{1}{4} f(1) = \frac{1}{9} f(1) \tag{14}
$$

In general, we have

$$
f(n+1) - f(n) = \frac{n^2}{((n+1)^2}(f(n) - f(n-1)) = \frac{1}{(n+1)^2} f(1) \tag{15}
$$

So

$$
f(n) = \sum_{i=1}^{n} \frac{1}{i^2} f(1) \tag{16}
$$

And note that $f(n) \to 1$ as $n \to \infty$, using the famous lemma

$$
\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \tag{17}
$$

So $f(1) = \frac{6}{\pi^2}$. And the probability that the chain never returns to zero is $\frac{6}{\pi^2}$.

4.2 (ii)

Using the same method, we have

$$
f(n) = \sum_{i=1}^{n} \frac{1}{i^\alpha} f(1) \tag{18}
$$

When $\alpha > 1$, $\sum_{i=1}^{n} \frac{1}{i^\alpha}$ converges, so $0 < f(1) < 1$. The markov chain is transient (because it is irreducible and state 0 is transient) .

When $\alpha \leqslant 1$, $\sum_{i=1}^{n} \frac{1}{i^\alpha}$ goes to ∞, so $f(1) = 0$. The markov chain is recurrent (because it is irreducible and state 0 is recurrent) .

And to determine it is positive recurrent or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^{\infty}$. Obviously we have $\pi_0 = \frac{1}{2^\alpha+1}\pi_1$ and for $n \geqslant 1$, we have

$$
\pi_n = p_{n-1,n}\pi_{n-1} + p_{n+1,n}\pi_{n+1} \tag{19}
$$

Thus we have

$$
p_{n+1,n}\pi_{n+1} - p_{n,n+1}\pi_n = p_{n,n-1}\pi_n - p_{n-1,n}\pi_{n-1} = p_{1,0}\pi_1 - \pi_0 = 0 \tag{20}
$$

So

$$\pi_{n+1} = \frac{p_{n,n+1}}{p_{n+1,n}}\pi_n = \frac{(n+2)^\alpha + (n+1)^\alpha}{(n+1)^\alpha + n^\alpha}\pi_n = ((n+2)^\alpha + (n+1)^\alpha)\pi_0 \tag{21}$$

And $\sum_{n=0}^{\infty} \pi_n = 1$

$$\pi_0 \sum_{n=0}^{\infty}(n^\alpha + (n+1)^\alpha) = 1 \tag{22}$$

When $\alpha \geqslant -1$, $\sum_{n=0}^{\infty} n^\alpha$ goes to ∞, $\pi_0 = 0$, the Markov chain is null recurrent.

When $\alpha < -1$, $\sum_{n=0}^{\infty} n^\alpha$ converges, $0 < \pi_0 < 1$, the Markov chain is positive recurrent. And $\pi_0 = \frac{1}{2\zeta(-\alpha)}$, where $\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s}$ is the Riemann function.

In summary,

- When $\alpha > 1$, the Markov chain is transient.
- When $-1 \leqslant \alpha \leqslant 1$, the Markov chain is null recurrent.
- When $\alpha < 1$, the Markov chain is positive recurrent.

5 Problem 5

Denote the function $f(n)$ as $P(X_n > 0, \forall n \geqslant 1 | X_0 = n)$. The probability that the Markov chain never returns to zero is $f(1)$. Also we have $f(2) = pf(3), f(1) = pf(3) + (1-p)f(2)$. So $f(2) = \frac{1}{2-p}f(1), f(3) = \frac{1}{(2-p)p}f(1)$. And for $n \geqslant 2$,

$$\begin{cases} f(2n-1) = pf(2n+1) + (1-p)f(2n) \\ f(2n) = pf(2n+1) + (1-p)f(2n-2) \end{cases} \tag{23}$$

$$\begin{cases} f(2n) = \dfrac{1}{2-p}((1-p)f(2n-2) + f(2n-1)) \\ f(2n+1) = \dfrac{1}{(2-p)p}(f(2n-1) - (1-p)^2 f(2n-2)) \end{cases} \tag{24}$$

Solving the equation $f(2n+1) + \lambda f(2n) = C(f(2n-1) + \lambda f(2n-2))$, we have $\lambda = -1$ and $\frac{-(1-p)^2}{p}$. Thus,

$$\begin{cases} f(2n+1) - f(2n) = \dfrac{(1-p)}{(2-p)p}(f(2n-1) - f(2n-2)) = \left[\dfrac{(1-p)}{(2-p)p}\right]^n f(1) \\ f(2n+1) - \dfrac{(1-p)^2}{p}f(2n) = f(2n-1) - \dfrac{(1-p)^2}{p}f(2n-2) = f(1) \end{cases} \tag{25}$$

When $\frac{(1-p)}{(2-p)p} > 1, p < \frac{3-\sqrt{5}}{2}$, $f(1)$ must be 0. So the chain is recurrent.

When $\frac{(1-p)}{(2-p)p} = 1, p = \frac{3-\sqrt{5}}{2}$, $f(2n+1) - f(2n) = f(1)$. Using equation (23), we have $f(2n+1) = f(2n-1) + (1-p)f(1)$, because $1-p > 0$, $f(1)$ must be zero and the chain is recurrent.

And when $\frac{(1-p)}{(2-p)p} < 1, p > \frac{3-\sqrt{5}}{2}$, $f(1) = 1 - \frac{(1-p)^2}{p} > 0$, the chain is transient.

To determine whether it is positive or null recurrent, we assume the stationary distribution is $\pi^* = (\pi_i)_{i=0}^{\infty}$.

So $\pi_0 = (1-p)\pi_2, \pi_1 = \pi_0$. For $n \geqslant 1$,

$$\begin{cases} \pi_{2n+1} = p\pi_{2n-1} + p\pi_{2n} \\ \pi_{2n} = (1-p)\pi_{2n-1} + (1-p)\pi_{2n+2} \end{cases} \tag{26}$$

Solving the equation $\pi_{2n+1} + \lambda\pi_{2n+2} = C(\pi_{2n-1} + \lambda\pi_{2n})$, $\lambda = -(1-p)$ or $-\frac{p}{1-p}$.

So

$$\begin{cases} \pi_{2n+1} - (1-p)\pi_{2n+2} = \pi_{2n-1} - (1-p)\pi_{2n} = 0 \\ \pi_{2n+1} - \dfrac{p}{1-p}\pi_{2n+2} = \dfrac{p(2-p)}{1-p}(\pi_{2n-1} - \dfrac{p}{1-p}\pi_{2n}) = \left[\dfrac{p(2-p)}{1-p}\right]^n (\pi_1 - \dfrac{p}{1-p}\pi_2) \end{cases} \tag{27}$$

So

$$\frac{p^2-3p+1}{1-p}\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \frac{p^2-3p+1}{1-p}\pi_2 \tag{28}$$

When $p = \frac{3-\sqrt{5}}{2}, p^2 - 3p + 1 = 0$, in equation (26) we have that $\pi_{2n+1} = (1-p)\pi_{2n} = \pi_{2n-1}$, by $\sum_{n=0}^{\infty} \pi_n = 1$ we have $\pi_0 = 0$, the Markov chain is null recurrent.

When $p < \frac{3-\sqrt{5}}{2}$, $\frac{p(2-p)}{1-p} < 1$,

$$\pi_{2n+2} = \left[\frac{p(2-p)}{1-p}\right]^n \pi_2 \tag{29}$$

$$\sum_{n=0}^{\infty} \pi_n = (1-p)\pi_2 + (2-p)\pi_2 \sum_{n=0}^{\infty} \left[\frac{p(2-p)}{1-p}\right]^n = \frac{(1-p)^2(3-p)}{p^2-3p+1}\pi_2 = 1 \tag{30}$$

In summary,

- When $p > \frac{3-\sqrt{5}}{2}$, the Markov chain is transient.
- When $p = \frac{3-\sqrt{5}}{2}$, the Markov chain is null recurrent.
- When $p < \frac{3-\sqrt{5}}{2}$, the Markov chain is positive recurrent. And the stationary distribution is,

$$\begin{cases} \pi_0 = \dfrac{p^2-3p+1}{(1-p)(3-p)} \\ \pi_{2n} = \left[\dfrac{p(2-p)}{1-p}\right]^{n-1} \dfrac{p^2-3p+1}{(1-p)^2(3-p)}, n \geqslant 1 \\ \pi_{2n-1} = \left[\dfrac{p(2-p)}{1-p}\right]^{n-1} \dfrac{p^2-3p+1}{(1-p)(3-p)}, n \geqslant 1 \end{cases} \tag{31}$$