

Exercise 3

Probability Theory 2020 Autumn

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1 Problem 1

No. A counterexample is given as below.

Let Ω be $(0, 1)$ and the probability measure $\mathbb{P} = \lambda$. Define X_n as

$$X_n = -\frac{1}{nx}, \quad x \in (0, 1), n \geq 1 \quad (1)$$

So $X_{n+1}(w) \geq X_n(w)$ for all $n \geq 1$ and $w \in \Omega$.

And $X = \lim_{n \rightarrow \infty} X_n = 0$. So $E[X] = 0$ but $E[X_n] = -\infty$, $\lim_{n \rightarrow \infty} E[X_n] \neq E[X]$

2 Problem 2

2.1 (a)

Using $\int_{\mathbb{R}} f_X(x) dx = 1$, we have

$$\int_{\mathbb{R}} f_X(x) dx = \int_{-2}^2 c \sqrt{4 - x^2} dx = 2\pi c = 1 \quad (2)$$

So $c = \frac{1}{2\pi}$

2.2 (b)

Because $f_X(-x) = f_X(x)$, for $k = 1, 3, \dots, 2n+1, \dots$, $E[X^k] = 0$.

For $k = 2n, n = 1, 2, \dots$

$$\begin{aligned} E[X^k] &= \int_{\mathbb{R}} x^k f_X(x) dx = \frac{1}{2\pi} \int_{-2}^2 x^k \sqrt{4 - x^2} dx \\ &= \frac{1}{\pi} \int_0^2 x^k \sqrt{4 - x^2} dx \quad (x = 2 \sin \theta) \\ &= \frac{2^{k+2}}{\pi} \int_0^{\frac{\pi}{2}} \sin^k \theta \cos^2 \theta d\theta \\ &= \frac{2^{k+2}}{\pi} \int_0^{\frac{\pi}{2}} [\sin^k \theta - \sin^{k+2} \theta] d\theta \end{aligned} \quad (3)$$

Denote $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$,

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(-\cos x) \\ &= -\cos x \sin^{n-1} x \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx \\ &= (n-1)(I_{n-2} - I_n) \end{aligned} \quad (4)$$

So $I_n = \frac{n-1}{n} I_{n-2}$ and $I_0 = \frac{\pi}{2}, I_2 = \frac{\pi}{4}$,

$$I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2} \frac{\pi}{2} = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}}$$

So, For $k = 2n, n = 1, 2, \dots$

$$\begin{aligned} E[X^k] &= \frac{2^{k+2}}{\pi} \int_0^{\frac{\pi}{2}} [\sin^k \theta - \sin^{k+2} \theta] d\theta \\ &= \frac{2^{k+2}}{\pi} (I_k - I_{k+2}) \\ &= \frac{2^{k+2}}{\pi} \left(\frac{(k)!}{((k/2)!)^2} \frac{\pi}{2^{k+1}} - \frac{(k+2)!}{((k/2+1)!)^2} \frac{\pi}{2^{k+3}} \right) \\ &= \frac{2(k)!}{((k/2)!)^2} - \frac{(k+2)!}{2((k/2+1)!)^2} \\ &= \frac{2}{k+2} \frac{(k)!}{((k/2)!)^2} \end{aligned} \tag{5}$$

In summary,

$$E[X^k] = \begin{cases} 0, & k = 2n-1, n = 1, 2, \dots \\ \frac{2}{k+2} \frac{(k)!}{((k/2)!)^2}, & k = 2n, n = 1, 2, \dots \end{cases} \tag{6}$$

3 Problem 3

Yes.

To prove this, we define a series of random variables by replacing X_i with Y_i once a time.

Define $Z_0 = X = X_1 + X_2 + \cdots + X_n, Z_1 = Y_1 + X_2 + \cdots + X_n, Z_j = \sum_{i=1}^j Y_i + \sum_{i=j+1}^n X_i, Z_n = Y = Y_1 + Y_2 + \cdots + Y_n$

We want to prove Z_j *stochastically dominates* Z_{j-1} , that is,

$$\forall k = 1, 2, \dots, n, P(Z_{j-1} \geq k) \geq P(Z_j \geq k)$$

Notice that the only difference between Z_{j-1} and Z_j is the j -th item, X_j or Y_j .

Denote the sum of the left $n-1$ items as Z' , and $Z_{j-1} = Z' + X_j, Z_j = Z' + Y_j$, also denote $P(X_i = 1) = p_i \geq P(Y_i = 1) = q_i$.

$$\begin{aligned} P(Z_{j-1} \geq k) &= P(Z' + X_j \geq k) \\ &= P(Z' \geq k) + P(Z' = k-1, X_j = 1) \\ &= P(Z' \geq k) + P(Z' = k-1)p_j \end{aligned} \tag{7}$$

In the same way, $P(Z_j \geq k) = P(Z' \geq k) + P(Z' = k-1)q_j$.

Since $p_j \geq q_j$, we prove that $\forall k = 1, 2, \dots, n, P(Z_{j-1} \geq k) \geq P(Z_j \geq k)$

Therefore, $\forall k = 1, 2, \dots, n$

$$P(X \geq k) = P(Z_0 \geq k) \geq P(Z_1 \geq k) \geq \cdots \geq P(Z_n \geq k) = P(Y \geq k) \tag{8}$$

4 Problem 4

4.1 (a)

Suppose $V \sim \text{Exp}(\lambda)$, because U, V are independent, $E\left[\frac{V^2}{1+U}\right] = E[V^2] E\left[\frac{1}{1+U}\right]$.

$$E[V^2] = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = \int_0^\infty x^2 e^{-x} dx = -\frac{1}{\lambda^2} e^{-x} (2 + 2x + x^2) \Big|_0^\infty = \frac{2}{\lambda^2} \quad (9)$$

$$E\left[\frac{1}{1+U}\right] = \int_0^1 \frac{1}{1+x} dx = \ln 2 \quad (10)$$

So,

$$E\left[\frac{V^2}{1+U}\right] = E[V^2] E\left[\frac{1}{1+U}\right] = \frac{2 \ln 2}{\lambda^2} \quad (11)$$

4.2 (b)

$$\begin{aligned} P(U \leq V) &= \int_0^1 P(V \geq u) f_U(u) du \\ &= \int_0^1 \int_u^\infty \lambda e^{-\lambda v} dv du \\ &= \int_0^1 e^{-\lambda u} du \\ &= \frac{1}{\lambda} (1 - e^{-\lambda}) \end{aligned} \quad (12)$$

4.3 (c)

Because $U = \sqrt{Y}$, $V = \frac{Z}{\sqrt{Y}}$

$$f_{Y,Z}(y, z) = f_{U,V}(u(y, z), v(y, z)) \left| \frac{\partial(u, v)}{\partial(y, z)} \right| \quad (13)$$

And the Jacobian matrix is,

$$|J| = \left| \frac{\partial(u, v)}{\partial(y, z)} \right| = \left| \begin{array}{cc} \frac{1}{2\sqrt{Y}} & 0 \\ -\frac{1}{2Y\sqrt{Y}} & \frac{1}{\sqrt{Y}} \end{array} \right| = \frac{1}{2Y} \quad (14)$$

So the joint pdf of Y, Z is

$$f_{Y,Z}(y, z) = 1 \times \lambda e^{-\lambda \frac{z}{\sqrt{y}}} \times \frac{1}{2y} \quad (15)$$

The support set of Y, Z is $[0, 1] \times [0, \infty)$.

$$f_{Y,Z}(y,z) = \begin{cases} \frac{\lambda}{2y} e^{-\lambda \frac{z}{\sqrt{y}}}, & (y,z) \in [0,1] \times [0,\infty) \\ 0, & \text{else} \end{cases} \quad (16)$$

5 Problem 5

First $X_2 + X_3$ follows the Gamma distribution $\Gamma(2, \frac{1}{\lambda})$ with $\frac{1}{\lambda}$ as the scale parameter, to derive this, let $Z = X_2 + X_3$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X_2}(x) f_{X_3}(z-x) dx \\ &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \int_0^z \lambda^2 e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned} \quad (17)$$

So

$$\begin{aligned} P(X_1 > X_2 + X_3) &= P(X_1 > Z) = \int_0^{\infty} f_{X_1}(x) P(Z < x) dx \\ &= \int_0^{\infty} \left(\int_0^x \lambda^2 z e^{-\lambda z} dz \right) \lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} (1 - (1 + \lambda x) e^{-\lambda x}) \lambda e^{-\lambda x} dx \\ &= 1 - \int_0^{\infty} \lambda e^{-2\lambda x} dx - \int_0^{\infty} \lambda^2 x e^{-2\lambda x} dx \\ &= 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned} \quad (18)$$

6 Problem 6

The eigenvalues Y_1, Y_2 are the roots of the following equation,

$$(\lambda - X_1)(\lambda - X_2) = X_3^2 \quad (19)$$

Part 1

First, let $U = \frac{X_1 + X_2}{2}$, $V = \frac{X_1 - X_2}{2}$, we prove that $U, V \stackrel{i.i.d.}{\sim} N(0, 1)$.

$$\begin{aligned}
f_{U,V}(u,v) &= f_{X_1,X_2}(x_1,x_2) \left| \frac{\partial(x_1,x_2)}{\partial(u,v)} \right| \\
&= \frac{1}{4\pi} e^{-\frac{x_1^2+x_2^2}{4}} \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \\
&= \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}
\end{aligned} \tag{20}$$

So U, V are independent and follow the standard Gaussian distribution.

Part 2

By solving the equation 19 we have

$$\begin{cases} Y_1 + Y_2 = X_1 + X_2 = 2U \\ |Y_1 - Y_2| = 2\sqrt{\left(\frac{X_1 - X_2}{2}\right)^2 + X_3^2} = 2\sqrt{V^2 + X_3^2} \end{cases} \tag{21}$$

Denote $Z_1 = \frac{Y_1+Y_2}{2}$, $Z_2 = \frac{|Y_1-Y_2|}{2}$, so $Z_1 = U \sim N(0, 1)$, $Z_2 = \sqrt{V^2 + X_3^2}$.

Because X_1, X_2, X_3 are independent, U, V, X_3 are also independent, Z_1 and Z_2 are independent. Because V and X_3 are standard Gaussian random variables, $Z_2 = \sqrt{V^2 + X_3^2} \sim \chi_2$ (also known as the Rayleigh distribution)

To derive $f_{Z_2}(z)$,

$$\begin{aligned}
F_{Z_2}(z) &= P(Z_2 \leq z) = P(X_3^2 + V^2 \leq z^2) = \iint_{x^2+y^2 \leq z^2} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx dy \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^z r e^{-\frac{r^2}{2}} dr d\theta \\
&= \int_0^z r e^{-\frac{r^2}{2}} dr \\
&= 1 - e^{-\frac{z^2}{2}}
\end{aligned} \tag{22}$$

And $f_{Z_2}(z) = F'_{Z_2}(z) = z e^{-\frac{z^2}{2}}$, $z \geq 0$

Part 3

Assume $Y_1 \geq Y_2$,

$$\begin{aligned}
f_{Y_1, Y_2}(y_1, y_2) &= f_{Z_1, Z_2}(z_1, z_2) \left| \frac{\partial(z_1, z_2)}{\partial(y_1, y_2)} \right| \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{z_1^2}{2}} z_2 e^{-\frac{z_2^2}{2}} \left\| \begin{matrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{matrix} \right\| \\
&= \frac{1}{4\sqrt{2\pi}} (y_1 - y_2) e^{-\frac{y_1^2 + y_2^2}{4}}, y_1 \geq y_2
\end{aligned} \tag{23}$$

Symmetrically, when $Y_1 \leq Y_2$,

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{4\sqrt{2\pi}} (y_2 - y_1) e^{-\frac{y_1^2 + y_2^2}{4}}, y_1 \leq y_2 \tag{24}$$

So, the joint pdf of Y_1 and Y_2 is,

$$f_{Y_1, Y_2}(y_1, y_2) \propto |y_1 - y_2| e^{-\frac{y_1^2 + y_2^2}{4}} \tag{25}$$

By $\int_{\mathbb{R}^2} f_{Y_1, Y_2}(y_1, y_2) dy_1 dy_2 = 1$

$$f_{Y_1, Y_2}(y_1, y_2) = \frac{1}{8\sqrt{2\pi}} |y_1 - y_2| e^{-\frac{y_1^2 + y_2^2}{4}} \tag{26}$$