Exercise 3
Probability Theory 2020 Autumn

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Contents

1 Problem 1

No. A counterexample is given as below.

Let Ω be $(0,1)$ and the probability measure $\mathbb{P} = \lambda$. Define X_n as

$$X_n = -\frac{1}{nx}, \quad x \in (0,1), n \geqslant 1 \tag{1}$$

So $X_{n+1}(w) \geqslant X_n(w)$ for all $n \geqslant 1$ and $w \in \Omega$.

And $X = \lim\limits_{n\to\infty} X_n = 0$. So $\mathrm{E}[X] = 0$ but $\mathrm{E}[X_n] = -\infty$, $\lim\limits_{n\to\infty} \mathrm{E}[X_n] \neq \mathrm{E}[X]$

2 Problem 2

2.1 (a)

Using $\int_{\mathbb{R}} f_X(x)dx = 1$, we have

$$\int_{\mathbb{R}} f_X(x)dx = \int_{-2}^{2} c\sqrt{4-x^2}dx = 2\pi c = 1 \tag{2}$$

So $c = \frac{1}{2\pi}$

2.2 (b)

Because $f_X(-x) = f_X(x)$, for $k = 1, 3, \cdots, 2n+1, \cdots$, $\mathrm{E}[X^k] = 0$.

For $k = 2n, n = 1, 2, \cdots$

$$\begin{aligned}
\mathrm{E}[X^k] &= \int_{\mathbb{R}} x^k f_X(x)dx = \frac{1}{2\pi}\int_{-2}^{2} x^k\sqrt{4-x^2}dx \\
&= \frac{1}{\pi}\int_{0}^{2} x^k\sqrt{4-x^2}dx \quad (x = 2\sin\theta) \\
&= \frac{2^{k+2}}{\pi}\int_0^{\frac{\pi}{2}} \sin^k\theta\cos^2\theta d\theta \\
&= \frac{2^{k+2}}{\pi}\int_0^{\frac{\pi}{2}} \left[\sin^k\theta - \sin^{k+2}\theta\right] d\theta
\end{aligned} \tag{3}$$

Denote $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$,

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x d(-\cos x) \\
&= -\cos x \sin^{n-1} x\Big|_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\cos^2 x dx \\
&= (n-1)(I_{n-2} - I_n)
\end{aligned} \tag{4}$$

So $I_n = \frac{n-1}{n} I_{n-2}$ and $I_0 = \frac{\pi}{2}, I_2 = \frac{\pi}{4}$,

$I_{2n} = \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\frac{\pi}{2} = \frac{(2n)!}{(n!)^2}\frac{\pi}{2^{2n+1}}$

So, For $k = 2n, n = 1, 2, \cdots$

$$\begin{aligned}
\mathrm{E}[X^k] &= \frac{2^{k+2}}{\pi}\int_0^{\frac{\pi}{2}}\left[\sin^k\theta - \sin^{k+2}\theta\right]d\theta \\
&= \frac{2^{k+2}}{\pi}\left(I_k - I_{k+2}\right) \\
&= \frac{2^{k+2}}{\pi}\left(\frac{(k)!}{((k/2)!)^2}\frac{\pi}{2^{k+1}} - \frac{(k+2)!}{((k/2+1)!)^2}\frac{\pi}{2^{k+3}}\right) \\
&= \frac{2(k)!}{((k/2)!)^2} - \frac{(k+2)!}{2((k/2+1)!)^2} \\
&= \frac{2}{k+2}\frac{(k)!}{((k/2)!)^2}
\end{aligned} \tag{5}$$

In summary,

$$\mathrm{E}[X^k] = \begin{cases} 0, \quad k = 2n-1, n = 1, 2, \cdots \\ \dfrac{2}{k+2}\dfrac{(k)!}{((k/2)!)^2}, \quad k = 2n, n = 1, 2, \cdots \end{cases} \tag{6}$$

3 Problem 3

Yes.

To prove this, we define a series of random variables by replacing X_i with Y_i once a time.

Define $Z_0 = X = X_1 + X_2 + \cdots + X_n, Z_1 = Y_1 + X_2 + \cdots + X_n, Z_j = \sum_{i=1}^{j} Y_j + \sum_{i=J+1}^{n} X_i, Z_n = Y = Y_1 + Y_2 + \cdots + Y_n$

We want to prove Z_j *stochastically dominates* Z_{j-1}, that is,

$\forall k = 1, 2, \cdots, n,\ P(Z_{j-1} \geqslant k) \geqslant P(Z_j \geqslant k)$

Notice that the only difference between Z_{j-1} and Z_j is the j-th item, X_j or Y_j.

Denote the sum of the left $n-1$ items as Z', and $Z_{j-1} = Z' + X_j, Z_j = Z' + Y_j$, also denote $P(X_i = 1) = p_i \geqslant P(Y_i = 1) = q_i$.

$$\begin{aligned}
P(Z_{j-1} \geqslant k) &= P(Z' + X_j \geqslant k) \\
&= P(Z' \geqslant k) + P(Z' = k-1, X_j = 1) \\
&= P(Z' \geqslant k) + P(Z' = k-1)p_j
\end{aligned} \tag{7}$$

In the same way, $P(Z_j \geqslant k) = P(Z' \geqslant k) + P(Z' = k-1)q_j$.

Since $p_j \geqslant q_j$, we prove that $\forall k = 1, 2, \cdots, n, \quad P(Z_{j-1} \geqslant k) \geqslant P(Z_j \geqslant k)$

Therefore, $\forall k = 1, 2, \cdots, n$

$$P(X \geqslant k) = P(Z_0 \geqslant k) \geqslant P(Z_1 \geqslant k) \geqslant \cdots \geqslant P(Z_n \geqslant k) = P(Y \geqslant k) \tag{8}$$

4 Problem 4

4.1 (a)

Suppose $V \sim \text{Exp}(\lambda)$, because U, V are independent, $\mathrm{E}\left[\frac{V^2}{1+U}\right] = \mathrm{E}[V^2]\,\mathrm{E}\left[\frac{1}{1+U}\right]$.

$$\mathrm{E}[V^2] = \int_0^{\infty} \lambda x^2 e^{-\lambda x} dx = \int_0^{\infty} x^2 e^{-x} dx = -\frac{1}{\lambda^2} e^{-x}(2 + 2x + x^2)\Big|_0^{\infty} = \frac{2}{\lambda^2} \tag{9}$$

$$\mathrm{E}\left[\frac{1}{1+U}\right] = \int_0^1 \frac{1}{1+x} dx = \ln 2 \tag{10}$$

So,

$$\mathrm{E}\left[\frac{V^2}{1+U}\right] = \mathrm{E}[V^2]\,\mathrm{E}\left[\frac{1}{1+U}\right] = \frac{2\ln 2}{\lambda^2} \tag{11}$$

4.2 (b)

$$\begin{aligned} P(U \leqslant V) &= \int_0^1 P(V \geqslant u) f_U(u) du \\ &= \int_0^1 \int_u^{\infty} \lambda e^{-\lambda v} dv du \\ &= \int_0^1 e^{-\lambda u} du \\ &= \frac{1}{\lambda}(1 - e^{-\lambda}) \end{aligned} \tag{12}$$

4.3 (c)

Because $U = \sqrt{Y}, V = \frac{Z}{\sqrt{Y}}$

$$f_{Y,Z}(y, z) = f_{U,V}(u(y, z), v(y, z)) \left| \frac{\partial(u, v)}{\partial(y, z)} \right| \tag{13}$$

And the Jacobian matrix is,

$$|J| = \left| \frac{\partial(u, v)}{\partial(y, z)} \right| = \begin{vmatrix} \frac{1}{2\sqrt{Y}} & 0 \\ -\frac{Z}{2Y\sqrt{Y}} & \frac{1}{\sqrt{Y}} \end{vmatrix} = \frac{1}{2Y} \tag{14}$$

So the joint pdf of Y, Z is

$$f_{Y,Z}(y, z) = 1 \times \lambda e^{-\lambda \frac{z}{\sqrt{y}}} \times \frac{1}{2y} \tag{15}$$

The support set of Y, Z is $[0, 1] \times [0, \infty)$.

$$f_{Y,Z}(y,z) = \begin{cases} \dfrac{\lambda}{2y} e^{-\lambda \frac{z}{\sqrt{y}}}, & (y,z) \in [0,1] \times [0,\infty) \\ 0, & \text{else} \end{cases} \tag{16}$$

5 Problem 5

First $X_2 + X_3$ follows the Gamma distribution $\Gamma(2, \frac{1}{\lambda})$ with $\frac{1}{\lambda}$ as the scale parameter, to derive this, let $Z = X_2 + X_3$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X_2}(x) f_{X_3}(z-x)\,dx \\ &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \int_0^z \lambda^2 e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned} \tag{17}$$

So

$$\begin{aligned} P(X_1 > X_2 + X_3) = P(X_1 > Z) &= \int_0^{\infty} f_{X_1}(x) P(Z < x) dx \\ &= \int_0^{\infty} \left(\int_0^x \lambda^2 z e^{-\lambda z} dz \right) \lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} \left(1 - (1+\lambda x) e^{-\lambda x}\right) \lambda e^{-\lambda x} dx \\ &= 1 - \int_0^{\infty} \lambda e^{-2\lambda x} dx - \int_0^{\infty} \lambda^2 x e^{-2\lambda x} dx \\ &= 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned} \tag{18}$$

6 Problem 6

The eigenvalues Y_1, Y_2 are the roots of the following equation,

$$(\lambda - X_1)(\lambda - X_2) = X_3^2 \tag{19}$$

Part 1

First, let $U = \frac{X_1+X_2}{2}, V = \frac{X_1-X_2}{2}$, we prove that $U, V \overset{i.i.d.}{\sim} N(0,1)$.

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X_1,X_2}(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right| \\
&= \frac{1}{4\pi}e^{-\frac{x_1^2+x_2^2}{4}}\left\|\begin{matrix}1 & 1\\ 1 & -1\end{matrix}\right\| \\
&= \frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}
\end{aligned} \tag{20}
$$

So U, V are independent and follow the standard Gaussian distribution.

Part 2

By solving the equation 19 we have

$$
\begin{cases}
Y_1 + Y_2 = X_1 + X_2 = 2U \\
|Y_1 - Y_2| = 2\sqrt{\left(\dfrac{X_1 - X_2}{2}\right)^2 + X_3^2} = 2\sqrt{V^2 + X_3^2}
\end{cases} \tag{21}
$$

Denote $Z_1 = \frac{Y_1+Y_2}{2}$, $Z_2 = \frac{|Y_1-Y_2|}{2}$, so $Z_1 = U \sim N(0,1), Z_2 = \sqrt{V^2 + X_3^2}$.

Because X_1, X_2, X_3 are independent, U, V, X_3 are also independent, Z_1 and Z_2 are independent. Because V and X_3 are standard Gaussian random variables , $Z_2 = \sqrt{V^2 + X_3^2} \sim \chi_2$ (also known as the Rayleigh distribution)

To derive $f_{Z_2}(z)$,

$$
\begin{aligned}
F_{Z_2}(z) = P(Z_2 \leqslant z) = P(X_3^2 + V^2 \leqslant z^2) &= \iint_{x^2+y^2\leqslant z^2} \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^z re^{-\frac{r^2}{2}}drd\theta \\
&= \int_0^z re^{-\frac{r^2}{2}}dr \\
&= 1 - e^{-\frac{z^2}{2}}
\end{aligned} \tag{22}
$$

And $f_{Z_2}(z) = F'_{Z_2}(z) = ze^{-\frac{z^2}{2}}, z \geqslant 0$

Part 3

Assume $Y_1 \geqslant Y_2$,

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Z_1,Z_2}(z_1,z_2)\left|\frac{\partial(z_1,z_2)}{\partial(y_1,y_2)}\right| \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}}z_2e^{-\frac{z_2^2}{2}}\left\|\begin{matrix}0.5 & 0.5\\ 0.5 & -0.5\end{matrix}\right\| \\
&= \frac{1}{2\sqrt{2\pi}}(y_1-y_2)e^{-\frac{y_1^2+y_2^2}{4}}, y_1 \geqslant y_2
\end{aligned} \tag{23}
$$

Symmetrically, when $Y_1 \leqslant Y_2$,

$$
f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\sqrt{2\pi}}(y_2-y_1)e^{-\frac{y_1^2+y_2^2}{4}}, y_1 \leqslant y_2 \tag{24}
$$

So, the joint pdf of Y_1 and Y_2 is,

$$
f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\sqrt{2\pi}}\left|y_1-y_2\right|e^{-\frac{y_1^2+y_2^2}{4}} \tag{25}
$$