Probability Theory Exercise 7 and 8

Issued: 2020/12/10 Due: 2020/12/31

1. Let X be an $n \times n$ **symmetric** matrix, whose entries are denoted as $X_{ij}, 1 \le i, j \le n$. Suppose that all the entries on and above the diagonal are independent, i.e., the entries $X_{ij}, 1 \le i \le j \le n$ are independent. Further assume that X_{ii} has $N(0,2)$ distribution for every $1 \le i \le n$, and X_{ij} has $N(0,1)$ distribution for every $1 \le i < j \le n$. Now let U be an $n \times n$ orthogonal matrix, and let $Y = U^T X U$. Clearly Y is symmetric. Prove that all the entries on and above the diagonal of Y are independent, and find the distributions of all these entries.

2. Let X_1, X_2, X_3, X_4 be i.i.d. Bernoulli-$1/5$ random variables. When $i \ge 5$, we produce X_i based on $X_1, \dots, X_{i-1}$ as follows: If $X_{i-1} = X_{i-2} = X_{i-3} = X_{i-4} = 0$, then we set $X_i = 1$; and in all the other cases we set X_i to be a Bernoulli-$1/5$ random variable, independent of $X_1, \dots, X_{i-1}$. Prove that $\frac{1}{n}\sum_{i=1}^n X_i$ converges almost surely when $n \to \infty$ and find the distribution of the limit random variable.

3. Let $Y_1, Y_2, \dots$ be i.i.d. random variables that only take **positive integer** values. For $i = 1, 2, \dots$, let $p_i := \mathbb{P}(Y_1 = i)$. Suppose that $\mu := \mathbb{E}[Y_1] < \infty$. Define a sequence of random variables X_n as follows:

$$X_n = \inf\{m \ge n : m = Y_1 + \ldots + Y_k \text{ for some } k \ge 0\} - n.$$

For every $n = 1, 2, \dots$, further define a function

$$f(n) := \mathbb{P}\left(n = Y_1 + \ldots + Y_k \text{ for some } k \ge 0\right).$$

(i) Prove that $\{X_n\}_{n=0}^\infty$ forms a Markov chain and find the transition probabilities.
(ii) Find the necessary and sufficient condition for the limit $\lim_{n\to\infty} f(n)$ to exist. Please state the necessary and sufficient condition in terms of $p_1, p_2, \dots$ and prove your statement.
(iii) When the limit $\lim_{n\to\infty} f(n)$ exists, find the limit. Express it in terms of μ and/or $p_1, p_2, \dots$

4. Let $\{X_n\}_{n=0}^\infty$ be a Markov chain with state space $\{0, 1, 2, \dots\}$. The transition probabilities are

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \ge 1$$

(i) Suppose that $X_0 = 0$. Find the probability that the chain never returns to state 0.
(ii) Now suppose that the transition probabilities are

$$p_{0,1} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^\alpha p_{i,i-1}, \quad i \ge 1$$

for some constant α. For every $\alpha \in (-\infty, \infty)$, indicate whether the chain is positive recurrent, null recurrent, or transient. Prove your conclusion.

5. Let $\{X_n\}_{n=0}^\infty$ be a Markov chain with state space $\{0, 1, 2, \dots\}$. The transition probabilities are

$$p_{0,1} = 1, \quad p_{2n-1,2n+1} = p, \quad p_{2n-1,2n} = 1-p, \quad p_{2n,2n+1} = p, \quad p_{2n,2n-2} = 1-p \quad \text{for all } n \ge 1,$$

where $p \in (0,1)$ is some constant. For every $p \in (0,1)$, indicate whether the Markov chain is transient, null recurrent, or positive recurrent. Prove your conclusion. When the Markov chain is positive recurrent, calculate the stationary distribution.

6. Let $\{(X_n, Y_n)\}_{n=0}^{\infty}$ be a 2-dimensional symmetric random walk. Namely, this is a Markov chain where (X_{n+1}, Y_{n+1}) takes one of the following 4 values with equal probability: $(X_n+1, Y_n), (X_n-1, Y_n), (X_n, Y_n+1), (X_n, Y_n-1)$. Suppose that $X_0 = Y_0 = 0$.
(i) Define $T := \inf\{n \geq 0 : \max(|X_n|, |Y_n|) = 3\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = 3, Y_T = 0)$.
(ii) Now define $T := \inf\{n \geq 0 : |X_n| + |Y_n| = 3\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = 3, Y_T = 0)$.
(iii) Now define $T := \inf\{n \geq 0 : \max(-X_n, |Y_n|) = 2\}$. Find the value of $\mathbb{E}[T]$ and $\mathbb{P}(X_T = -2, Y_T = 0)$.
(iv) Now define $T := \inf\{n \geq 0 : \max(X_n, Y_n) = 2\}$. Find the value of $\mathbb{E}[T]$.

7. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. Let $X_1, X_2, X_3, \ldots$ be i.i.d. random variables with distribution $P(X_i = 1) = P(X_i = -1) = 1/2$ for all i.
(i) Suppose that $\sum_{i=1}^{\infty} a_i^2 < \infty$. Find the probability $\mathbb{P}\Big(\big|\sum_{i=1}^{\infty} a_i X_i\big| < \infty\Big)$.
(ii) Suppose that $\sum_{i=1}^{\infty} a_i^2 = \infty$. Find the probability $\mathbb{P}\Big(\big|\sum_{i=1}^{\infty} a_i X_i\big| < \infty\Big)$.

8. Produce a sequence of random variables $\{X_n\}_{n\geq 0}$ as follows: Let X_0 and X_1 be some fixed constants. For $i > 1$, let $X_i = X_{i-1} + X_{i-2}$ with probability $1/2$ and $X_i = |X_{i-1} - X_{i-2}|$ with probability $1/2$.
(i) Suppose that $X_0 = 0$ and $X_1 = 1$. Find the probability

$$\mathbb{P}(\exists n \text{ such that } X_n = 3 \text{ and } X_i \neq 0 \text{ for all } 1 \leq i < n).$$

(This is the probability of the sequence $\{X_n\}$ reaching 3 before returning to the starting point 0.)
(ii) Now suppose that $X_0 = 1$ and $X_1 = 2$. Find the probability

$$\mathbb{P}(\exists n \text{ such that } X_n = X_{n+1} = 1).$$

9. Produce a sequence of random variables $\{X_n\}_{n\geq 0}$ as follows: Let $X_0 = q$ with probability 1, where $q \in (0,1)$ is some constant. For $n \geq 1$, let $X_n = X_{n-1}^2$ with probability $1/2$ and $X_n = 2X_{n-1} - X_{n-1}^2$ with probability $1/2$. Prove that $\{X_n\}_{n\geq 0}$ converges almost surely, and find the distribution of the limit random variable.

10. Given an integer $n \geq 1$, define $(X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)})$ as a random vector uniformly distributed in the ball

$$\big(X_1^{(n)}\big)^2 + \big(X_2^{(n)}\big)^2 + \cdots + \big(X_n^{(n)}\big)^2 \leq n.$$

Find the limit joint distribution of the random vector $(X_1^{(n)}, X_2^{(n)}, X_3^{(n)})$ as $n \to \infty$.