Exercise 5
Probability Theory 2020 Autumn

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1 Problem 1

Yes. An example is given as below.

Let $X \sim N(0,1)$, denote $\alpha = \Phi^{-1}(0.75) = 0.674$. So $P(-\alpha \leqslant X \leqslant \alpha) = 0$. Define Y as,

$$Y = \begin{cases} X, & \text{if } -\alpha \leqslant X \leqslant \alpha \\ -X, & \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \tag{1}$$

And the cdf of Y is,

$$F_Y(y) = P(Y \leqslant y) = \begin{cases} P(X \geqslant -y) = \Phi(y), & y \leqslant -\alpha \\ P(Y \leqslant -\alpha) + P(-\alpha < Y \leqslant y) = \Phi(y), & -\alpha \leqslant y \leqslant \alpha \\ P(Y \leqslant \alpha) + P(\alpha < Y \leqslant y) = \Phi(y), & y > \alpha \end{cases} \tag{2}$$

So $Y \sim N(0,1)$. But

$$X + Y = \begin{cases} 2X, & \text{if } -\alpha \leqslant X \leqslant \alpha \\ 0, & \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \tag{3}$$

So the distribution of $X + Y$ is not Gaussian distribution.

2 Problem 2

$X \sim N(0,1)$. Proof is given as below.

Let $X_1, X_2, \cdots, X_n, \cdots$ be i.i.d. random variables of the same distribution with X.

Define $S_n = \sum_{i=1}^n X_i$. Let $T_1 = \frac{X_1+X_2}{\sqrt{2}} = \frac{S_2}{\sqrt{2}}$, $T_1' = \frac{X_3+X_4}{\sqrt{2}} = \frac{S_4-S_2}{\sqrt{2}}$, so T_1, T_1' follows the same distribution with X. Let $T_2 = \frac{T_1+T_1'}{\sqrt{2}} = \frac{S_4}{\sqrt{4}}$ also follows the same distribution of X. And define $T_n = \frac{S_{2^n}}{\sqrt{2^n}}$, and its distribution is also the same distribution of X.

According to CLT,

$$\frac{S_{2^n}}{\sqrt{2^n}} \xrightarrow{d} N(0,1) \tag{4}$$

So the distribution of X is $N(0,1)$.

Note: another possible method is to consider the characteristic function $\phi_X(t)$, we can get an equation $\phi_X(t) = [\phi_X(\frac{t}{\sqrt{2}})]^2$. Also we have $\phi_X(0) = 1, \phi'_X(0) = 0, \phi''_X(0) = 1$. By solving the function equation we can get $\phi_X(t) = \exp(-\frac{t^2}{2})$ and $X \sim N(0,1)$.

3 Problem 3

3.1 (i)

Denote $-S$ as the exponent of the density function.

$$\begin{aligned} S &= \frac{1}{2}\left(x_1^2 + \sum_{i=1}^{2n-2}(x_{i+1}-x_i)^2 + x_{2n-1}^2\right) \\ &= \sum_{i=1}^{2n-1} x_i^2 - \sum_{i=1}^{2n-2} x_{i+1}x_i \\ &= \sum_{i=1}^{2n-2} A_i(x_i - B_i x_{i+1})^2 + A_{2n-1}x_{2n-1}^2 \end{aligned} \tag{5}$$

To find A_i, B_i, compare the coefficents,

$$\begin{cases} 2A_iB_i = 1 \\ A_iB_i^2 + A_{i+1} = 1 \\ A_1 = 1 \end{cases} \tag{6}$$

By induction we have $B_n = \frac{n}{n+1}, A_n = \frac{n+1}{2n}$.

And let $Y_i = \sqrt{2A_i}(X_i - B_iX_{i+1}), i = 1, 2, \cdots, 2n-2, Y_{2n-1} = \sqrt{2A_{2n-1}}X_{2n-1}$.

$$f_{Y_1,\cdot,Y_n}(y_1, y_n) = \left(\prod_{i=1}^{2n-1}\sqrt{2A_i}\right)^{-1} c_n \exp\left(-\frac{1}{2}\left(\sum_{i=1}^{2n-1} y_i^2\right)\right) \tag{7}$$

Obviously $(Y_1, \cdots, Y_{2n-1})$ is a Gaussian random vector, so $(X_1, \cdots, X_{2n-1})$ as linear combinations of $(Y_1, \cdots, Y_{2n-1})$ is also a Gaussian vector.

3.2 (ii)

$$\left(\prod_{i=1}^{2n-1}\sqrt{2A_i}\right)^{-1} c_n = (\sqrt{2\pi})^{-(2n-1)} \tag{8}$$

So $c_n = \frac{\sqrt{2n}}{(\sqrt{2\pi})^{2n-1}}$

3.3 (iii)

To find $\mathrm{Var}(X_n)$ we need to find the inverse transform $X = M^{-1}Y$. However, since Y_i are independent standard Gaussian variables, and X_i are just linear combinations of $Y_i, Y_{i+1}, \cdots, Y_{2n-1}$, so X_{i+1} and Y_i are independent.

By $X_i = \sqrt{\frac{i}{i+1}}Y_i + \frac{i}{i+1}X_{i+1}$,

$$\mathrm{Var}(X_i) = \frac{i^2}{(i+1)^2}\,\mathrm{Var}(X_{i+1}) + \frac{i}{i+1} \tag{9}$$

And $\mathrm{Var}(X_{2n-1}) = \frac{2n-1}{2n}$. By induction,

$$\mathrm{Var}(X_i) = \frac{(2n-i)i}{2n} \tag{10}$$

So $\mathrm{Var}(X_n) = \frac{n}{2}$.

Note: another tricky method. Notice that X_i are symmetric about n, so $\mathrm{Var}(X_{n-1}) = \mathrm{Var}(X_{n+1})$.

And by letting $i = n, n-1$ in (9),

$$\begin{cases} \mathrm{Var}(X_n) = \dfrac{n^2}{(n+1)^2}\,\mathrm{Var}(X_{n+1}) + \dfrac{n}{n+1} \\ \mathrm{Var}(X_{n-1}) = \dfrac{(n-1)^2}{n^2}\,\mathrm{Var}(X_n) + \dfrac{n-1}{n} \end{cases} \tag{11}$$

Solving the equation, we also get $\mathrm{Var}(X_n) = \frac{n}{2}$.

4 Problem 4

For Cauchy random variable $f_X(x) = \frac{1}{\pi}\frac{1}{x^2+1}$, the characteristic function is

$$\phi_X(t) = e^{-|t|} \tag{12}$$

For $\frac{S_n}{n^k}$, the characteristic function is

$$\phi_k(t) = \left[\phi_X\left(\frac{t}{n^k}\right)\right]^n = e^{-\frac{|t|}{n^{k-1}}} \tag{13}$$

4.1 (i)

When $k = 1$, $\phi_1(t) = e^{-|t|}$. So $\frac{S_n}{n^k}$ is also a Cauchy random variable and converges in distribution.

4.2 (ii)

When $k = 2$, $\phi_2(t) = e^{-\frac{|t|}{n}}$.

$$\lim_{n\to\infty} \phi_2(t) = \lim_{n\to\infty} e^{-\frac{|t|}{n}} = 1 \tag{14}$$

Using Fourier inverse transform, we know that the pdf of $\frac{S_n}{n^2}$ converges to the Dirac function $\delta(x)$.

$$\frac{S_n}{n^2} \xrightarrow{d.} 0 \tag{15}$$

4.3 (iii)

When $k = \frac{1}{2}$, $\lim_{n\to\infty} \phi_{0.5}(t) = \lim_{n\to\infty} e^{-|t|\sqrt{n}}$ doesn't converge for $t \neq 0$. So $\frac{S_n}{\sqrt{n}}$ doesn't converge in distribution.

5 Problem 5

For X_k, $\phi_{X_k}(t) = \frac{e^{ikt}+e^{-ikt}}{2} = \cos(kt)$. So

$$\phi_{\frac{S_n}{n^k}}(t) = \prod_{i=1}^{n} \cos(\frac{i}{n^k}t) \tag{16}$$

5.1 (i)

When $k = 2$, $\lim_{n\to\infty} \frac{i}{n^k} t = 0$ for $i = 1, 2, \cdots, n$.

Consider the taylor series of $\ln(\cos(x)) = -\frac{x^2}{2} + O(x^3)$, as $x \to 0$,

$$\cos(x) \sim \exp\left(-\frac{x^2}{2} + O(x^3)\right) \tag{17}$$

$$\begin{aligned}
\lim_{n\to\infty} \phi_{\frac{S_n}{n^2}}(t) &= \lim_{n\to\infty} \prod_{i=1}^{n} \exp\left(-\frac{i^2t^2}{2n^4} + O(\frac{i^3t^3}{n^3})\right) \\
&= \lim_{n\to\infty} \exp\left(-\frac{t^2}{2n^4}\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i^3t^3O(\frac{1}{n^6})\right) \\
&= \lim_{n\to\infty} \exp\left(-\frac{t^2(n+1)(2n+1)}{12n^3} + O(\frac{1}{n^2})\right) \\
&= 1
\end{aligned} \tag{18}$$

So $S_n/n^2 \xrightarrow{d.} 0$

5.2 (ii)

Using the same method,

$$
\begin{aligned}
\lim_{n\to\infty}\phi_{\frac{S_n}{n^{3/2}}}(t) &= \lim_{n\to\infty}\prod_{i=1}^{n}\exp\left(-\frac{i^2t^2}{2n^3}+O(\frac{i^3t^3}{n^{9/2}})\right) \\
&= \lim_{n\to\infty}\exp\left(-\frac{t^2}{2n^3}\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^3t^3O(\frac{1}{n^{9/2}})\right) \\
&= \lim_{n\to\infty}\exp\left(-\frac{t^2(n+1)(2n+1)}{12n^2}+O(\frac{1}{\sqrt{n}})\right) \\
&= \exp\left(-\frac{t^2}{6}\right)
\end{aligned} \tag{19}
$$

So $S_n/n^{\frac{3}{2}} \xrightarrow{d} N(0,\frac{1}{3})$ (which can also be concluded from Lyapunov CLT).

5.3 (iii)

As $S_n/n^{\frac{3}{2}} \xrightarrow{d} N(0,\frac{1}{3})$,

$$
\lim_{n\to\infty}P(\frac{S_n}{n}\leqslant x)=\lim_{n\to\infty}P(\frac{S_n}{n^{3/2}}\leqslant\frac{x}{\sqrt{n}})=\frac{1}{2} \tag{20}
$$

6 Problem 6

6.1 (i)

According to SLLN, $\frac{1}{n}\sum_{i=1}^{n}\log(X_i)\xrightarrow{a.s.}E[\log(X_1)]=-\frac{1}{2}\log 2$ and $Y_n=\prod_{i=1}^{n}X_i=\exp\left(\sum_{i=1}^{n}\log(X_i)\right)$, so

$$
P\left(\left\{w:\lim_{n\to\infty}\sqrt[n]{Y_n(w)}=\frac{1}{\sqrt{2}}\right\}\right)=P\left(\left\{w:\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\log(X_i(w))=-\frac{1}{2}\log 2\right\}\right)=1 \tag{21}
$$

According to root criterion for convergence,

$$
P\left(\left\{w:\lim_{n\to\infty}S_n(w)\text{ converges}\right\}\right)=P\left(\left\{w:\lim_{n\to\infty}\sqrt[n]{Y_n(w)}=\frac{1}{\sqrt{2}}\right\}\right)=1 \tag{22}
$$

So S_n converges almost surely. Denote the limit random variable as S.

Define $T_n^{(1)}=1+X_n, T_n^{(2)}=1+X_{n-1}T_n^{(1)}, T_n^{(i+1)}=1+X_{n-i}T_n^{(i)}, i=1,2,\cdots,n-1$, by definition,

$$
T_n^{(n)}=1+X_1+X_1X_2+\cdots+X_1X_2\cdots X_n=S_n+1 \tag{23}
$$

Notice that $T_n^{(i)}$ are just functions of $(X_{n-i+1},\cdots,X_n)$, so $T_n^{(i)}$ is independent of X_{n-i}.

$$
E[T_n^{(i+1)}]=E[1+X_{n-i}T_n^{(i)}]=1+E[X_{n-i}]E[T_n^{(i)}]=1+\frac{3}{4}E[T_n^{(i)}] \tag{24}
$$

And $E[T_n^{(1)}]=\frac{7}{4}$ so $E[T_n^{(n)}]=4-3\times(\frac{3}{4})^n, E[S_n]=E[T_n^{(n)}]-1=3(1-(\frac{3}{4})^n)$,

$$E[S] = \lim_{n\to\infty} 3(1-(\frac{3}{4})^n) = 3 \tag{25}$$

Because

$$\begin{aligned}\mathrm{Var}(XY) &= E[X^2Y^2] - (E[XY])^2 = E[X^2]E[Y^2] - (E[X]E[Y])^2 \\ &= \mathrm{Var}(X)\,\mathrm{Var}(Y) + \mathrm{Var}(X)(E[Y])^2 + \mathrm{Var}(Y)(E[X])^2\end{aligned} \tag{26}$$

$$\mathrm{Var}(T_n^{(i+1)}) = \frac{5}{8}\,\mathrm{Var}(T_n^{(i)}) + \frac{1}{16}(4 - 3\times(\frac{3}{4})^i)^2 \tag{27}$$

And $\mathrm{Var}(T_n^{(1)}) = \frac{1}{16}, \mathrm{Var}(T_n^{(n)}) = (\frac{5}{8})^n \sum_{i=0}^{n-1} (\frac{8}{5})^{i+1}(1-(\frac{3}{4})^{i+1})^2$.

$$\mathrm{Var}(S) = \lim_{n\to\infty} = \mathrm{Var}(T_n^{(n)}) = \frac{8}{3} \tag{28}$$

Note: let $n, i \to \infty$ in (24) and (27), we can get

$$\begin{cases} E[S] = 1 + \frac{3}{4}E[S] \\ \mathrm{Var}(S) = \frac{5}{8}\,\mathrm{Var}(S) + 1 \end{cases} \tag{29}$$

which also leads to $E[S] = 3, \mathrm{Var}(S) = \frac{8}{3}$

6.2 (ii)

Also we have $E[\log(X_1)] = -\frac{1}{2}\log 2 < 0$, thus

$$P\left(\left\{w : \lim_{n\to\infty} S_n(w) \text{ converges}\right\}\right) = P\left(\left\{w : \lim_{n\to\infty} \sqrt[n]{Y_n(w)} = \frac{1}{\sqrt{2}}\right\}\right) = 1 \tag{30}$$

So S_n converges almost surely. But $E[S_n]$ and $(\ S_n)$ increases to ∞!