Exercise 4
Probability Theory 2020 Autumn

Hanmo Chen
Student ID 2020214276

November 8, 2020

Contents

1 Problem 1

1.1 (i)

Claim that $M_X(s) = \infty$ for all $s \neq 0$. The proof is as below.

$$M_X(s) = \mathbb{E}[e^{sx}] = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2 + x^2} e^{sx} dx = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2 + x^2} e^{-sx} dx \tag{1}$$

So $M_X(s) = M_X(-s)$, we just need to prove $M_X(s) = \infty$ for $s > 0$.

Notice that for $x > 0$, $e^x > \frac{x^3}{6}$.

$$\begin{aligned} M_X(s) &= \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{t}{t^2 + x^2} e^{sx} dx > \int_{0}^{\infty} \frac{1}{\pi} \frac{t}{t^2 + x^2} e^{sx} dx \\ &> \int_{0}^{\infty} \frac{1}{6\pi} \frac{ts^3x^3}{t^2 + x^2} dx = \int_{0}^{\infty} \frac{t^3s^3}{6\pi} \frac{x^3}{1 + x^2} dx \\ &= \frac{t^3s^3}{12\pi} (x^2 - \ln(1 + x^2))\big|_0^{\infty} = \infty \end{aligned} \tag{2}$$

So $M_X(s) = \infty$ for all $s \neq 0$. And it is trivial that $M_X(0) = 1$.

$$M_X(s) = \begin{cases} 1, & s = 0 \\ \infty, & s \neq 0 \end{cases} \tag{3}$$

1.2 (b)

Yes. An example is given as the symmetrized lognormal distribution.

For example, $Z \sim N(0, 1)$ and $Y = e^Z \sim \text{lognormal}(0, 1)$ and define X as

$$X = \begin{cases} Y, & \text{with probability } \dfrac{1}{2} \\ -Y, & \text{with probability } \dfrac{1}{2} \end{cases} \tag{4}$$

It is easy to verify that

$$f_Y(y) = f_Z(\ln y) \frac{1}{y} = \frac{1}{\sqrt{2\pi} y} e^{-\frac{1}{2}(\ln y)^2} \tag{5}$$

And the n-th moment of Y is

$$\begin{aligned} \mathbb{E}[Y^n] &= \int_0^{\infty} \frac{1}{\sqrt{2\pi} y} e^{-\frac{1}{2}(\ln y)^2} * y^n dy = \int_0^{\infty} \frac{1}{\sqrt{2\pi} y} \exp\left(-\frac{1}{2}(\ln y)^2 + n \ln y\right) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2 + \frac{n^2}{2}\right) dz \quad (z = \ln y - n) \\ &= e^{\frac{n^2}{2}} \end{aligned} \tag{6}$$

And

$$f_X(x) = \frac{1}{2\sqrt{2\pi}|x|} e^{-\frac{1}{2}(\ln|x|)^2} \tag{7}$$

Then we will show that all the moments of X are finite. For all odd integers k,

$$\mathbb{E}[X^k] = \mathbb{E}[X_+^k] - \mathbb{E}[X_-^k] = \frac{1}{2}\mathbb{E}[Y^k] - \frac{1}{2}\mathbb{E}[Y^k] = 0 \tag{8}$$

For all even integers k,

$$\mathbb{E}[X^k] = \frac{1}{2}\mathbb{E}[Y^k] + \frac{1}{2}\mathbb{E}[Y^k] = e^{\frac{n^2}{2}} \tag{9}$$

So $\mathbb{E}[X^k] < \infty$ for all integers $k \geqslant 1$. But for the moment generating function $M_X(s)$, assuming $s > 0$,

$$\begin{aligned} M_X(s) = \mathbb{E}[e^{sx}] &= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}|x|} e^{-\frac{1}{2}(\ln|x|)^2} e^{sx} dx > \int_0^{\infty} \frac{1}{2\sqrt{2\pi}x} e^{-\frac{1}{2}(\ln x)^2} e^{sx} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}z^2} e^{se^z} dz > \int_0^{\infty} \frac{1}{2\sqrt{2\pi}} e^{\frac{s}{6}z^3 - \frac{1}{2}z^2} \\ &> \int_{\frac{6}{s}}^{\infty} \frac{1}{2\sqrt{2\pi}} e^{\frac{1}{2}z^2} = \infty \quad (\frac{s}{6}z^3 > z^2 \text{ when } z > \frac{6}{s}) \end{aligned} \tag{10}$$

$M_X(s) = \infty$ for $s > 0$ and $M_X(s) = M_X(-s)$, so $M_X(s) = \infty$ for $s \neq 0$

2 Problem 2

2.1 (a)

Notice that the conclusion in (a) is a weaker version of the conclusion from (b) because $X^k \leqslant X^k e^{sX}$ when $s > 0$, we will just prove (b).

2.2 (b)

If $s < 0$, $X^k e^{sX} \leqslant X^k \leqslant X^k e^{s'X}, s' > 0$, so we will just prove the case when $s > 0$.

Because X is nonnegative,

$$e^{ax} = e^{(a-s)x} > \frac{(a-s)^k x^k}{k!} e^{sx} \tag{11}$$

So $x^k e^{sx} < e^{ax} \frac{k!}{(a-s)^k}$,

$$\mathbb{E}[X^k e^{sX}] \leqslant \mathbb{E}[\frac{k!}{(a-s)^k} e^{aX}] = \frac{k!}{(a-s)^k} M_X(a) < \infty \tag{12}$$

2.3 (c)

The inequality holds true only when $h > 0$. The following proof is given under the condition $h > 0$.

It's obvious when $X = 0$. And for $X > 0$, it is equivalent to $\frac{e^{hX}-1}{hX} \leqslant e^{hX}$, denote b as hX and $g(x)$ as e^X, using Lagrange's mean value theorm,

$$\frac{e^{hX}-1}{hX} = \frac{g(b)-g(0)}{b-0} = g'(c) \leqslant g'(b) = e^{hX}, c \in (0,b) \tag{13}$$

2.4 (d)

Using L'Hospital's Rule, $X = \lim_{h\to 0} \frac{e^{hX}-1}{h}$. Suppose $\{h_n\}$ is a sequence of nonnegative numbers and $\lim_{n\to\infty} h_n = 0$, so $h_n < \frac{a}{2}$ for sufficiently large n. And define $X_n = \frac{e^{h_n X}-1}{h_n}$.
Using the conclusion from (c)

$$0 < X_n \leqslant X e^{h_n X} < X e^{\frac{a}{2}X} \tag{14}$$

And $\mathbb{E}[Xe^{\frac{a}{2}X}] < \infty$, so according to the Dominant Convergence Theorm,

$$\mathbb{E}[X] = \mathbb{E}[\lim_{h\downarrow 0} \frac{e^{hX}-1}{h}] = \lim_{h\downarrow 0} \mathbb{E}[\frac{e^{hX}-1}{h}] = \lim_{h\downarrow 0} \frac{\mathbb{E}[e^{hX}]-1}{h} \tag{15}$$

3 Problem 3

Yes.

To prove this, we define a series of random variables by replacing X_i with Y_i once a time.

Define $Z_0 = X = X_1 + X_2 + \cdots + X_n, Z_1 = Y_1 + X_2 + \cdots + X_n, Z_j = \sum_{i=1}^{j} Y_j + \sum_{i=J+1}^{n} X_i, Z_n = Y = Y_1 + Y_2 + \cdots + Y_n$

We want to prove Z_j *stochastically dominates* Z_{j-1}, that is,

$\forall k = 1, 2, \cdots, n,\ P(Z_{j-1} \geqslant k) \geqslant P(Z_j \geqslant k)$

Notice that the only difference between Z_{j-1} and Z_j is the j-th item, X_j or Y_j.

Denote the sum of the left $n-1$ items as Z', and $Z_{j-1} = Z' + X_j, Z_j = Z' + Y_j$, also denote $P(X_i = 1) = p_i \geqslant P(Y_i = 1) = q_i$.

$$\begin{aligned} P(Z_{j-1} \geqslant k) &= P(Z' + X_j \geqslant k) \\ &= P(Z' \geqslant k) + P(Z' = k-1, X_j = 1) \\ &= P(Z' \geqslant k) + P(Z' = k-1)p_j \end{aligned} \tag{16}$$

In the same way, $P(Z_j \geqslant k) = P(Z' \geqslant k) + P(Z' = k-1)q_j$.

Since $p_j \geqslant q_j$, we prove that $\forall k = 1, 2, \cdots, n, \quad P(Z_{j-1} \geqslant k) \geqslant P(Z_j \geqslant k)$

Therefore, $\forall k = 1, 2, \cdots, n$

$$P(X \geqslant k) = P(Z_0 \geqslant k) \geqslant P(Z_1 \geqslant k) \geqslant \cdots \geqslant P(Z_n \geqslant k) = P(Y \geqslant k) \tag{17}$$

4 Problem 4

4.1 (a)

Suppose $V \sim \text{Exp}(\lambda)$, because U, V are independent, $\mathrm{E}\left[\frac{V^2}{1+U}\right] = \mathrm{E}[V^2]\,\mathrm{E}\left[\frac{1}{1+U}\right]$.

$$\mathrm{E}[V^2] = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = \int_0^\infty x^2 e^{-x} dx = -\frac{1}{\lambda^2} e^{-x}(2 + 2x + x^2)\Big|_0^\infty = \frac{2}{\lambda^2} \tag{18}$$

$$\mathrm{E}\left[\frac{1}{1+U}\right] = \int_0^1 \frac{1}{1+x} dx = \ln 2 \tag{19}$$

So,

$$\mathrm{E}\left[\frac{V^2}{1+U}\right] = \mathrm{E}[V^2]\,\mathrm{E}\left[\frac{1}{1+U}\right] = \frac{2\ln 2}{\lambda^2} \tag{20}$$

4.2 (b)

$$\begin{aligned} P(U \leqslant V) &= \int_0^1 P(V \geqslant u) f_U(u) du \\ &= \int_0^1 \int_u^\infty \lambda e^{-\lambda v} dv du \\ &= \int_0^1 e^{-\lambda u} du \\ &= \frac{1}{\lambda}(1 - e^{-\lambda}) \end{aligned} \tag{21}$$

4.3 (c)

Because $U = \sqrt{Y}, V = \frac{Z}{\sqrt{Y}}$

$$f_{Y,Z}(y,z) = f_{U,V}(u(y,z), v(y,z)) \left|\frac{\partial(u,v)}{\partial(y,z)}\right| \tag{22}$$

And the Jacobian matrix is,

$$|J| = \left|\frac{\partial(u,v)}{\partial(y,z)}\right| = \begin{vmatrix} \frac{1}{2\sqrt{Y}} & 0 \\ -\frac{Z}{2Y\sqrt{Y}} & \frac{1}{\sqrt{Y}} \end{vmatrix} = \frac{1}{2Y} \tag{23}$$

So the joint pdf of Y, Z is

$$f_{Y,Z}(y,z) = 1 \times \lambda e^{-\lambda \frac{z}{\sqrt{y}}} \times \frac{1}{2y} \tag{24}$$

The support set of Y, Z is $[0,1] \times [0, \infty)$.

$$f_{Y,Z}(y,z) = \begin{cases} \dfrac{\lambda}{2y} e^{-\lambda \frac{z}{\sqrt{y}}}, & (y,z) \in [0,1] \times [0,\infty) \\ 0, & \text{else} \end{cases} \tag{25}$$

5 Problem 5

First $X_2 + X_3$ follows the Gamma distribution $\Gamma(2, \frac{1}{\lambda})$ with $\frac{1}{\lambda}$ as the scale parameter, to derive this, let $Z = X_2 + X_3$,

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X_2}(x)\, f_{X_3}(z-x)\, dx \\ &= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)} dx \\ &= \int_0^z \lambda^2 e^{-\lambda z} dx \\ &= \lambda^2 z e^{-\lambda z} \end{aligned} \tag{26}$$

So

$$\begin{aligned} P(X_1 > X_2 + X_3) = P(X_1 > Z) &= \int_0^{\infty} f_{X_1}(x) P(Z < x) dx \\ &= \int_0^{\infty} \left(\int_0^x \lambda^2 z e^{-\lambda z} dz \right) \lambda e^{-\lambda x} dx \\ &= \int_0^{\infty} \left(1 - (1+\lambda x) e^{-\lambda x} \right) \lambda e^{-\lambda x} dx \\ &= 1 - \int_0^{\infty} \lambda e^{-2\lambda x} dx - \int_0^{\infty} \lambda^2 x e^{-2\lambda x} dx \\ &= 1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \end{aligned} \tag{27}$$

6 Problem 6

The eigenvalues Y_1, Y_2 are the roots of the following equation,

$$(\lambda - X_1)(\lambda - X_2) = X_3^2 \tag{28}$$

Part 1

First, let $U = \frac{X_1+X_2}{2}, V = \frac{X_1-X_2}{2}$, we prove that $U, V \overset{i.i.d.}{\sim} N(0,1)$.

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X_1,X_2}(x_1,x_2)\left|\frac{\partial(x_1,x_2)}{\partial(u,v)}\right| \\
&= \frac{1}{4\pi}e^{-\frac{x_1^2+x_2^2}{4}}\left\|\begin{matrix}1 & 1\\ 1 & -1\end{matrix}\right\| \\
&= \frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}} \times \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}
\end{aligned}
\tag{29}
$$

So U, V are independent and follow the standard Gaussian distribution.

Part 2

By solving the equation 28 we have

$$
\begin{cases}
Y_1 + Y_2 = X_1 + X_2 = 2U \\
|Y_1 - Y_2| = 2\sqrt{\left(\dfrac{X_1 - X_2}{2}\right)^2 + X_3^2} = 2\sqrt{V^2 + X_3^2}
\end{cases}
\tag{30}
$$

Denote $Z_1 = \frac{Y_1+Y_2}{2}$, $Z_2 = \frac{|Y_1-Y_2|}{2}$, so $Z_1 = U \sim N(0,1), Z_2 = \sqrt{V^2 + X_3^2}$.

Because X_1, X_2, X_3 are independent, U, V, X_3 are also independent, Z_1 and Z_2 are independent. Because V and X_3 are standard Gaussian random variables , $Z_2 = \sqrt{V^2 + X_3^2} \sim \chi_2$ (also known as the Rayleigh distribution)

To derive $f_{Z_2}(z)$,

$$
\begin{aligned}
F_{Z_2}(z) = P(Z_2 \leqslant z) = P(X_3^2 + V^2 \leqslant z^2) &= \iint_{x^2+y^2\leqslant z^2} \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^z re^{-\frac{r^2}{2}}drd\theta \\
&= \int_0^z re^{-\frac{r^2}{2}}dr \\
&= 1 - e^{-\frac{z^2}{2}}
\end{aligned}
\tag{31}
$$

And $f_{Z_2}(z) = F'_{Z_2}(z) = ze^{-\frac{z^2}{2}}, z \geqslant 0$

Part 3

Assume $Y_1 \geqslant Y_2$,

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= f_{Z_1,Z_2}(z_1,z_2)\left|\frac{\partial(z_1,z_2)}{\partial(y_1,y_2)}\right| \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{z_1^2}{2}} z_2 e^{-\frac{z_2^2}{2}} \left\| \begin{matrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{matrix} \right\| \\
&= \frac{1}{4\sqrt{2\pi}}(y_1-y_2)e^{-\frac{y_1^2+y_2^2}{4}}, y_1 \geqslant y_2
\end{aligned} \tag{32}
$$

Symmetrically, when $Y_1 \leqslant Y_2$,

$$
f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{4\sqrt{2\pi}}(y_2-y_1)e^{-\frac{y_1^2+y_2^2}{4}}, y_1 \leqslant y_2 \tag{33}
$$

So, the joint pdf of Y_1 and Y_2 is,

$$
f_{Y_1,Y_2}(y_1,y_2) \propto |y_1-y_2|\, e^{-\frac{y_1^2+y_2^2}{4}} \tag{34}
$$

By $\int_{\mathbb{R}^2} f_{Y_1,Y_2}(y_1,y_2)dy_1dy_2 = 1$

$$
f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{8\sqrt{2\pi}}\,|y_1-y_2|\, e^{-\frac{y_1^2+y_2^2}{4}} \tag{35}
$$