Exercise 7 & 8
Probability Theory 2020 Autumn

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1 Problem 1

Denote the entries in U as u_{ij} and entries in Y as Y_{ij},so

$$Y_{ij} = \sum_{r,s} u_{ri} u_{sj} X_{ij} \tag{1}$$

Also we have,

$$\mathrm{Cov}(X_{ij}, X_{mn}) = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Thus

$$\begin{aligned} \mathrm{Cov}(Y_{ij}, Y_{mn}) &= \mathrm{Cov}\left(\sum_{r,s} u_{ri} u_{sj} X_{ij}, \sum_{p,q} u_{pm} u_{qn} X_{mn}\right) \\ &= 2\sum_{r=1}^{n} u_{ri} u_{rj} u_{rm} u_{rn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rm} u_{sn} + \sum_{r \neq s} u_{ri} u_{sj} u_{rn} u_{sm} \\ &= \sum_{r,s} [u_{ri} u_{sj} u_{rm} u_{sn} + u_{ri} u_{sj} u_{rn} u_{sm}] \\ &= \left(\sum_r u_{ri} u_{rm}\right)\left(\sum_s u_{sj} u_{rn}\right) + \left(\sum_r u_{ri} u_{rn}\right)\left(\sum_s u_{sj} u_{sm}\right) \end{aligned} \tag{3}$$

Denote the column vectors in U as $\boldsymbol{u}_i, i = 1, 2, 3, \cdots, N$, so $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \sum_r u_{ri} u_{rj} = \delta_{ij}$.

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Because $X_{ij}, j \geqslant i$ are independent Gaussian variables, so the joint distribution of Y_{ij} is joint Gaussian distribution, which means,

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = 0 \Longleftrightarrow Y_{ij}, Y_{mn} \text{ are independent} \tag{5}$$

So all the entries on and above the diagonal of Y are independent, and $Y_{ii} \sim N(0,2), i = 1,2,3,\cdots,N$ and $Y_{ij} \sim N(0,1), 1 \leqslant i < j \leqslant n$. (It is easy to see that $\mathbb{E}[Y_{ij}] = 0$)

2 Problem 2

Notice that

1. If $X_n = 1$, then $X_{n+1}, X_{n+2}, \cdots$ are independent of $X_1, X_2, \cdots, X_n$
2. There is at least one 1 in any five-in-a-row X_is as $\{X_n, X_{n+1}, \cdots, X_{n+4}\}$

So we can split $X_1, X_2, \cdot, X_n$ into a series of epsisodes, each epsisode $L_j = [0, \cdots, 0, 1]$ is consisted of n zeros (n can be $0,1,2,3,4$) and 1 one. And $L_j, j = 1,2,\cdots,m$ are independent. (For the last epsisode, if it is ended with 0, we can append 1 to its end and let $n = n+1$.) Denote the length of each epsisode as l_j, so $\sum_{j=1}^m l_j = n$.

Consider the distribution of l_j, it can only take values in $1,2,3,4,5$,

- $P(l_j = 1) = P(X_1 = 1) = 0.2$
- $P(l_j = 2) = P(X_1 = 0, X_2 = 1) = 0.16$
- $P(l_j = 3) = P(X_1 = 0, X_2 = 0, X_3 = 1) = 0.128$
- $P(l_j = 4) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1) = 0.1024$
- $P(l_j = 5) = P(X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 0) = 0.4096$

So

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{m\to\infty} \frac{m}{l_1 + l_2 + \cdots + l_m} = \lim_{m\to\infty} \frac{1}{\frac{1}{m}\sum_{j=1}^m l_j} \tag{6}$$

According to Strong Law of Large Numbers,

$$\frac{1}{m}\sum_{j=1}^m l_j \xrightarrow{a.s} E[l_j] = 3.3616 \tag{7}$$

So

$$\lim_{n\to\infty} \frac{S_n}{n} \xrightarrow{a.s} \frac{1}{3.3616} \tag{8}$$