Exercise 7 & 8
Probability Theory 2020 Autumn

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December 10, 2020

1 Problem 1

Denote the entries in U as u_{ij} and entries in Y as Y_{ij},so

$$Y_{ij} = \sum_{r,s} u_{ri}u_{sj}X_{ij} \tag{1}$$

Also we have,

$$\mathrm{Cov}(X_{ij}, X_{mn}) = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Thus

$$\begin{aligned} \mathrm{Cov}(Y_{ij}, Y_{mn}) &= \mathrm{Cov}\left(\sum_{r,s} u_{ri}u_{sj}X_{ij}, \sum_{p,q} u_{pm}u_{qn}X_{mn}\right) \\ &= 2\sum_{r=1}^{n} u_{ri}u_{rj}u_{rm}u_{rn} + \sum_{r\neq s} u_{ri}u_{sj}u_{rm}u_{sn} + \sum_{r\neq s} u_{ri}u_{sj}u_{rn}u_{sm} \\ &= \sum_{r,s} [u_{ri}u_{sj}u_{rm}u_{sn} + u_{ri}u_{sj}u_{rn}u_{sm}] \\ &= \left(\sum_r u_{ri}u_{rm}\right)\left(\sum_s u_{sj}u_{rn}\right) + \left(\sum_r u_{ri}u_{rn}\right)\left(\sum_s u_{sj}u_{sm}\right) \end{aligned} \tag{3}$$

Denote the column vectors in U as $\boldsymbol{u}_i, i = 1, 2, 3, \cdots, N$, so $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = \sum_r u_{ri}u_{rj} = \delta_{ij}$.

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} = \begin{cases} 2, & i = j = m = n \\ 1, & (i,j) = (m,n) \text{ or } (i,j) = (n,m), i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Because $X_{ij}, j \geqslant i$ are independent Gaussian variables, so the joint distribution of Y_{ij} is joint Gaussian distribution, which means,

$$\mathrm{Cov}(Y_{ij}, Y_{mn}) = 0 \Longleftrightarrow Y_{ij}, Y_{mn} \text{ are independent} \tag{5}$$

So all the entries on and above the diagonal of Y are independent, and $Y_{ii} \sim N(0,2), i = 1,2,3,\cdots,N$ and $Y_{ij} \sim N(0,1), 1 \leqslant i < j \leqslant n$. (It is easy to see that $\mathbb{E}[Y_{ij}] = 0$)

2 Problem 2