

Exercise 5

Probability Theory 2020 Autumn

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November 21, 2020

1 Problem 1

Yes. An example is given as below.

Let $X \sim N(0, 1)$, denote $\alpha = \Phi^{-1}(0.75) = 0.674$. So $P(-\alpha \leq X \leq \alpha) = 0$. Define Y as,

$$Y = \begin{cases} X, & \text{if } -\alpha \leq X \leq \alpha \\ -X, & \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \quad (1)$$

And the cdf of Y is,

$$F_Y(y) = P(Y \leq y) = \begin{cases} P(X \geq -y) = \Phi(y), & y \leq -\alpha \\ P(Y \leq -\alpha) + P(-\alpha < Y \leq y) = \Phi(y), & -\alpha \leq y \leq \alpha \\ P(Y \leq \alpha) + P(\alpha < Y \leq y) = \Phi(y), & y > \alpha \end{cases} \quad (2)$$

So $Y \sim N(0, 1)$. But

$$X + Y = \begin{cases} 2X, & \text{if } -\alpha \leq X \leq \alpha \\ 0, & \text{if } X < -\alpha \text{ or } X > \alpha \end{cases} \quad (3)$$

So the distribution of $X + Y$ is not Gaussian distribution.

2 Problem 2

$X \sim N(0, 1)$. Proof is given as below.

Let $X_1, X_2, \dots, X_n, \dots$ be i.i.d. random variables of the same distribution with X .

Define $S_n = \sum_{i=1}^n X_i$. Let $T_1 = \frac{X_1+X_2}{\sqrt{2}} = \frac{S_2}{\sqrt{2}}$, $T'_1 = \frac{X_3+X_4}{\sqrt{2}} = \frac{S_4-S_2}{\sqrt{2}}$, so T_1, T'_1 follows the same distribution with X . Let $T_2 = \frac{T_1+T'_1}{\sqrt{2}} = \frac{S_4}{\sqrt{4}}$ also follows the same distribution of X . And define $T_n = \frac{S_{2^n}}{\sqrt{2^n}}$, and its distribution is also the same distribution of X .

According to CLT,

$$\frac{S_{2^n}}{\sqrt{2^n}} \xrightarrow{d} N(0, 1) \quad (4)$$

So the distribution of X is $N(0, 1)$.

Note: another possible method is to consider the characteristic function $\phi_X(t)$, we can get an equation $\phi_X(t) = [\phi_X(\frac{t}{\sqrt{2}})]^2$. Also we have $\phi_X(0) = 1, \phi'_X(0) = 0, \phi''_X(0) = 1$. By solving the function equation we can get $\phi_X(t) = \exp(-\frac{t^2}{2})$ and $X \sim N(0, 1)$.

3 Problem 3

3.1 (i)

Denote $-S$ as the exponent of the density function.

$$\begin{aligned} S &= \frac{1}{2} \left(x_1^2 + \sum_{i=1}^{2n-2} (x_{i+1} - x_i)^2 + x_{2n-1}^2 \right) \\ &= \sum_{i=1}^{2n-1} x_i^2 - \sum_{i=1}^{2n-2} x_{i+1} x_i \\ &= \sum_{i=1}^{2n-2} A_i (x_i - B_i x_{i+1})^2 + A_{2n-1} x_{2n-1}^2 \end{aligned} \quad (5)$$

To find A_i, B_i , compare the coefficients,

$$\begin{cases} 2A_i B_i = 1 \\ A_i B_i^2 + A_{i+1} = 1 \\ A_1 = 1 \end{cases} \quad (6)$$

By induction we have $B_n = \frac{n}{n+1}, A_n = \frac{n+1}{2n}$.

And let $Y_i = \sqrt{2A_i}(X_i - B_i X_{i+1}), i = 1, 2, \dots, 2n-2, Y_{2n-1} = \sqrt{2A_{2n-1}} X_{2n-1}$.

$$f_{Y_1, \dots, Y_n}(y_1, y_n) = \left(\prod_{i=1}^{2n-1} \sqrt{2A_i} \right)^{-1} c_n \exp \left(-\frac{1}{2} \left(\sum_{i=1}^{2n-1} y_i^2 \right) \right) \quad (7)$$

Obviously (Y_1, \dots, Y_{2n-1}) is a Gaussian random vector, so (X_1, \dots, X_{2n-1}) as linear combinations of (Y_1, \dots, Y_{2n-1}) is also a Gaussian vector.

3.2 (ii)

$$\left(\prod_{i=1}^{2n-1} \sqrt{2A_i} \right)^{-1} c_n = (\sqrt{2\pi})^{-(2n-1)} \quad (8)$$

So $c_n = \frac{\sqrt{2n}}{(\sqrt{2\pi})^{2n-1}}$

3.3 (iii)

To find $\text{Var}(X_n)$ we need to find the inverse transform $X = M^{-1}Y$. However, since Y_i are independent standard Gaussian variables, and X_i are just linear combinations of $Y_i, Y_{i+1}, \dots, Y_{2n-1}$, so X_{i+1} and Y_i are independent.

By $X_i = \sqrt{\frac{i}{i+1}}Y_i + \frac{i}{i+1}X_{i+1}$,

$$\text{Var}(X_i) = \frac{i^2}{(i+1)^2} \text{Var}(X_{i+1}) + \frac{i}{i+1} \quad (9)$$

And $\text{Var}(X_{2n-1}) = \frac{2n-1}{2n}$. By induction,

$$\text{Var}(X_i) = \frac{(2n-i)i}{2n} \quad (10)$$

So $\text{Var}(X_n) = \frac{n}{2}$.

Note: another tricky method. Notice that X_i are symmetric about n , so $\text{Var}(X_{n-1}) = \text{Var}(X_{n+1})$.

And by letting $i = n, n-1$ in (9),

$$\begin{cases} \text{Var}(X_n) = \frac{n^2}{(n+1)^2} \text{Var}(X_{n+1}) + \frac{n}{n+1} \\ \text{Var}(X_{n-1}) = \frac{(n-1)^2}{n^2} \text{Var}(X_n) + \frac{n-1}{n} \end{cases} \quad (11)$$

Solving the equation, we also get $\text{Var}(X_n) = \frac{n}{2}$.

4 Problem 4

For Cauchy random variable $f_X(x) = \frac{1}{\pi} \frac{1}{x^2+1}$, the characteristic function is

$$\phi_X(t) = e^{-|t|} \quad (12)$$

For $\frac{S_n}{n^k}$, the characteristic function is

$$\phi_k(t) = \left[\phi_X\left(\frac{t}{n^k}\right) \right]^n = e^{-\frac{|t|}{n^{k-1}}} \quad (13)$$

4.1 (i)

When $k = 1$, $\phi_1(t) = e^{-|t|}$. So $\frac{S_n}{n^k}$ is also a Cauchy random variable and converges in distribution.

4.2 (ii)

When $k = 2$, $\phi_2(t) = e^{-\frac{|t|}{n}}$.

$$\lim_{n \rightarrow \infty} \phi_2(t) = \lim_{n \rightarrow \infty} e^{-\frac{|t|}{n}} = 1 \quad (14)$$

Using Fourier inverse transform, we know that the pdf of $\frac{S_n}{n^2}$ converges to the Dirac function $\delta(x)$.

$$\frac{S_n}{n^2} \xrightarrow{d} 0 \quad (15)$$

4.3 (iii)

When $k = \frac{1}{2}$, $\lim_{n \rightarrow \infty} \phi_{0.5}(t) = \lim_{n \rightarrow \infty} e^{-|t|\sqrt{n}}$ doesn't converge for $t \neq 0$. So $\frac{S_n}{\sqrt{n}}$ doesn't converge in distribution.

5 Problem 5

For X_k , $\phi_{X_k}(t) = \frac{e^{ikt} + e^{-ikt}}{2} = \cos(kt)$. So

$$\phi_{\frac{S_n}{n^k}}(t) = \prod_{i=1}^n \cos\left(\frac{i}{n^k}t\right) \quad (16)$$

5.1 (i)

When $k = 2$, $\lim_{n \rightarrow \infty} \frac{i}{n^k}t = 0$ for $i = 1, 2, \dots, n$.

Consider the Taylor series of $\ln(\cos(x)) = -\frac{x^2}{2} + O(x^3)$, as $x \rightarrow 0$,

$$\cos(x) \sim \exp\left(-\frac{x^2}{2} + O(x^3)\right) \quad (17)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \phi_{\frac{S_n}{n^2}}(t) &= \lim_{n \rightarrow \infty} \prod_{i=1}^n \exp\left(-\frac{i^2 t^2}{2n^4} + O\left(\frac{i^3 t^3}{n^3}\right)\right) \\ &= \lim_{n \rightarrow \infty} \exp\left(-\frac{t^2}{2n^4} \sum_{i=1}^n i^2 + \sum_{i=1}^n i^3 t^3 O\left(\frac{1}{n^6}\right)\right) \\ &= \lim_{n \rightarrow \infty} \exp\left(-\frac{t^2(n+1)(2n+1)}{12n^3} + O\left(\frac{1}{n^2}\right)\right) \\ &= 1 \end{aligned} \quad (18)$$

So $S_n/n^2 \xrightarrow{d} 0$

5.2 (ii)

Using the same method,

$$\begin{aligned}
\lim_{n \rightarrow \infty} \phi_{\frac{S_n}{n^{3/2}}}(t) &= \lim_{n \rightarrow \infty} \prod_{i=1}^n \exp \left(-\frac{i^2 t^2}{2n^3} + O\left(\frac{i^3 t^3}{n^{9/2}}\right) \right) \\
&= \lim_{n \rightarrow \infty} \exp \left(-\frac{t^2}{2n^3} \sum_{i=1}^n i^2 + \sum_{i=1}^n i^3 t^3 O\left(\frac{1}{n^{9/2}}\right) \right) \\
&= \lim_{n \rightarrow \infty} \exp \left(-\frac{t^2(n+1)(2n+1)}{12n^2} + O\left(\frac{1}{\sqrt{n}}\right) \right) \\
&= \exp \left(-\frac{t^2}{6} \right)
\end{aligned} \tag{19}$$

So $S_n/n^{3/2} \xrightarrow{d} N(0, \frac{1}{3})$ (which can also be concluded from Lyapunov CLT).

5.3 (iii)

As $S_n/n^{3/2} \xrightarrow{d} N(0, \frac{1}{3})$,

$$\lim_{n \rightarrow \infty} P\left(\frac{S_n}{n} \leq x\right) = \lim_{n \rightarrow \infty} P\left(\frac{S_n}{n^{3/2}} \leq \frac{x}{\sqrt{n}}\right) = \frac{1}{2} \tag{20}$$

6 Problem 6

6.1 (i)

According to SLLN, $\frac{1}{n} \sum_{i=1}^n \log(X_i) \xrightarrow{a.s.} E[\log(X_1)] = -\frac{1}{2} \log 2$ and $Y_n = \prod_{i=1}^n X_i = \exp \left(\sum_{i=1}^n \log(X_i) \right)$, so

$$P \left(\left\{ w : \lim_{n \rightarrow \infty} \sqrt[n]{Y_n(w)} = \frac{1}{\sqrt{2}} \right\} \right) = P \left(\left\{ w : \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log(X_i(w)) = -\frac{1}{2} \log 2 \right\} \right) = 1 \tag{21}$$

According to root criterion for convergence,

$$P \left(\left\{ w : \lim_{n \rightarrow \infty} S_n(w) \text{ converges} \right\} \right) = P \left(\left\{ w : \lim_{n \rightarrow \infty} \sqrt[n]{Y_n(w)} = \frac{1}{\sqrt{2}} \right\} \right) = 1 \tag{22}$$

So S_n converges almost surely. Denote the limit random variable as S .

Define $T_n^{(1)} = 1 + X_n, T_n^{(2)} = 1 + X_{n-1}T_n^{(1)}, T_n^{(i+1)} = 1 + X_{n-i}T_n^{(i)}, i = 1, 2, \dots, n-1$, by definition,

$$T_n^{(n)} = 1 + X_1 + X_1X_2 + \dots + X_1X_2 \dots X_n = S_n - 1 \tag{23}$$

Notice that $T_n^{(i)}$ are just functions of (X_{n-i+1}, \dots, X_n) , so $T_n^{(i)}$ is independent of X_{n-i} .

$$E[T_n^{(i+1)}] = E[1 + X_{n-i}T_n^{(i)}] = 1 + E[X_{n-i}]E[T_n^{(i)}] = 1 + \frac{3}{4}E[T_n^{(i)}] \tag{24}$$

And $E[T_n^{(1)}] = \frac{7}{4}$ so $E[T_n^{(n)}] = 4 - 3 \times (\frac{3}{4})^n, E[S_n] = E[T_n^{(n)}] - 1 = 3(1 - (\frac{3}{4})^n)$,

$$E[S] = \lim_{n \rightarrow \infty} 3(1 - (\frac{3}{4})^n) = 3 \quad (25)$$

Because

$$\begin{aligned} \text{Var}(XY) &= E[X^2Y^2] - (E[XY])^2 = E[X^2]E[Y^2] - (E[X]E[Y])^2 \\ &= \text{Var}(X) \text{Var}(Y) + \text{Var}(X)(E[Y])^2 + \text{Var}(Y)(E[X])^2 \end{aligned} \quad (26)$$

$$\text{Var}(T_n^{(i+1)}) = \frac{5}{8} \text{Var}(T_n^{(i)}) + \frac{1}{16} (4 - 3 \times (\frac{3}{4})^i)^2 \quad (27)$$

$$\text{And } \text{Var}(T_n^{(1)}) = \frac{1}{16}, \text{Var}(T_n^{(n)}) = (\frac{5}{8})^n \sum_{i=0}^{n-1} (\frac{8}{5})^{i+1} (1 - (\frac{3}{4})^{i+1})^2.$$

$$\text{Var}(S) = \lim_{n \rightarrow \infty} \text{Var}(T_n^{(n)}) = \frac{8}{3} \quad (28)$$

Note: let $n, i \rightarrow \infty$ in (24) and (27), we can get

$$\begin{cases} E[S] = 1 + \frac{3}{4} E[S] \\ \text{Var}(S) = \frac{5}{8} \text{Var}(S) + 1 \end{cases} \quad (29)$$

which also leads to $E[S] = 3, \text{Var}(S) = \frac{8}{3}$

6.2 (ii)

Also we have $E[\log(X_1)] = -\frac{1}{2} \log 2 < 0$, thus

$$P\left(\left\{w : \lim_{n \rightarrow \infty} S_n(w) \text{ converges} \right\}\right) = P\left(\left\{w : \lim_{n \rightarrow \infty} \sqrt[n]{Y_n(w)} = \frac{1}{\sqrt{2}} \right\}\right) = 1 \quad (30)$$

So S_n converges almost surely. But $E[S_n]$ and (S_n) increases to ∞ !