

# Lecture 4: Model-Free Prediction

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# Outline

- 1 Introduction
- 2 Monte-Carlo Learning
- 3 Temporal-Difference Learning
- 4  $TD(\lambda)$

# Model-Free Reinforcement Learning

- Last lecture:
  - Planning by dynamic programming
  - Solve a *known* MDP
- This lecture:
  - Model-free prediction
  - Estimate the value function of an *unknown* MDP
- Next lecture:
  - Model-free control
  - Optimise the value function of an *unknown* MDP

# Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is *model-free*: no knowledge of MDP transitions / rewards
- MC learns from *complete* episodes: no bootstrapping
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to *episodic* MDPs
  - All episodes must terminate

# Monte-Carlo Policy Evaluation

- Goal: learn  $v_\pi$  from episodes of experience under policy  $\pi$

$$S_1, A_1, R_2, \dots, S_k \sim \pi$$

- Recall that the *return* is the total discounted reward:

$$G_t = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T$$

- Recall that the value function is the expected return:

$$v_\pi(s) = \mathbb{E}_\pi [G_t \mid S_t = s]$$

- Monte-Carlo policy evaluation uses *empirical mean* return instead of *expected* return

# First-Visit Monte-Carlo Policy Evaluation

- To evaluate state  $s$
- The **first** time-step  $t$  that state  $s$  is visited in an episode,
- Increment counter  $N(s) \leftarrow N(s) + 1$
- Increment total return  $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return  $V(s) = S(s)/N(s)$
- By law of large numbers,  $V(s) \rightarrow v_\pi(s)$  as  $N(s) \rightarrow \infty$

# Every-Visit Monte-Carlo Policy Evaluation

- To evaluate state  $s$
- **Every** time-step  $t$  that state  $s$  is visited in an episode,
- Increment counter  $N(s) \leftarrow N(s) + 1$
- Increment total return  $S(s) \leftarrow S(s) + G_t$
- Value is estimated by mean return  $V(s) = S(s)/N(s)$
- Again,  $V(s) \rightarrow v_{\pi}(s)$  as  $N(s) \rightarrow \infty$

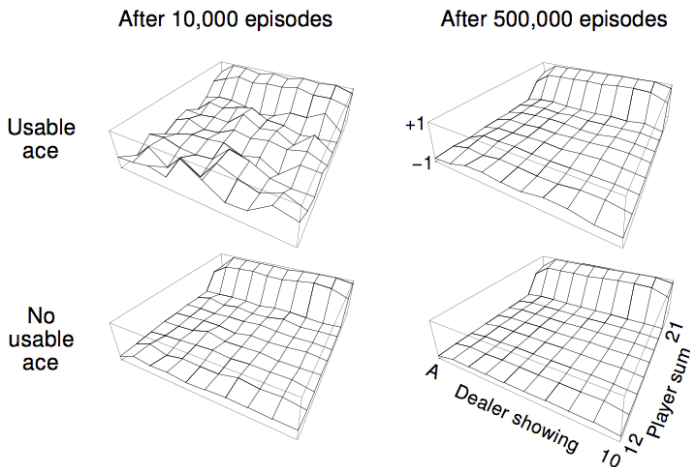
# Blackjack Example

- States (200 of them):
  - Current sum (12-21)
  - Dealer's showing card (ace-10)
  - Do I have a "useable" ace? (yes-no)
- Action **stick**: Stop receiving cards (and terminate)
- Action **twist**: Take another card (no replacement)
- Reward for **stick**:
  - +1 if sum of cards  $>$  sum of dealer cards
  - 0 if sum of cards = sum of dealer cards
  - -1 if sum of cards  $<$  sum of dealer cards
- Reward for **twist**:
  - -1 if sum of cards  $>$  21 (and terminate)
  - 0 otherwise
- Transitions: automatically **twist** if sum of cards  $<$  12





# Blackjack Value Function after Monte-Carlo Learning



Policy: **stick** if sum of cards  $\geq 20$ , otherwise **twist**

# Incremental Mean

The mean  $\mu_1, \mu_2, \dots$  of a sequence  $x_1, x_2, \dots$  can be computed incrementally,

$$\begin{aligned}\mu_k &= \frac{1}{k} \sum_{j=1}^k x_j \\ &= \frac{1}{k} \left( x_k + \sum_{j=1}^{k-1} x_j \right) \\ &= \frac{1}{k} (x_k + (k-1)\mu_{k-1}) \\ &= \mu_{k-1} + \frac{1}{k} (x_k - \mu_{k-1})\end{aligned}$$

# Incremental Monte-Carlo Updates

- Update  $V(s)$  incrementally after episode  $S_1, A_1, R_2, \dots, S_T$
- For each state  $S_t$  with return  $G_t$

$$N(S_t) \leftarrow N(S_t) + 1$$

$$V(S_t) \leftarrow V(S_t) + \frac{1}{N(S_t)} (G_t - V(S_t))$$

- In non-stationary problems, it can be useful to track a running mean, i.e. forget old episodes.

$$V(S_t) \leftarrow V(S_t) + \alpha (G_t - V(S_t))$$

# Temporal-Difference Learning

- TD methods learn directly from episodes of experience
- TD is *model-free*: no knowledge of MDP transitions / rewards
- TD learns from *incomplete* episodes, by *bootstrapping*
- TD updates a guess towards a guess

# MC and TD

- Goal: learn  $v_\pi$  online from experience under policy  $\pi$
- Incremental every-visit Monte-Carlo
  - Update value  $V(S_t)$  toward *actual* return  $G_t$

$$V(S_t) \leftarrow V(S_t) + \alpha (G_t - V(S_t))$$

- Simplest temporal-difference learning algorithm: TD(0)
  - Update value  $V(S_t)$  toward *estimated* return  $R_{t+1} + \gamma V(S_{t+1})$

$$V(S_t) \leftarrow V(S_t) + \alpha (R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$

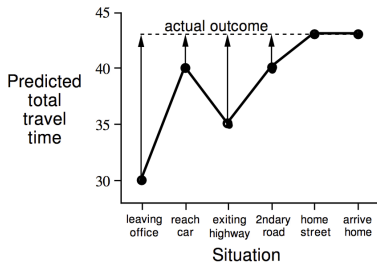
- $R_{t+1} + \gamma V(S_{t+1})$  is called the *TD target*
- $\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$  is called the *TD error*

# Driving Home Example

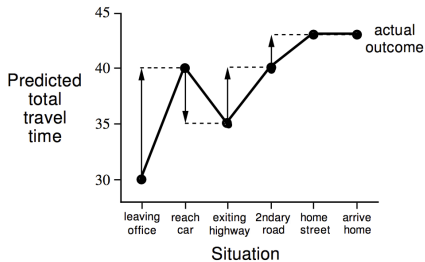
<b>State</b>	<b>Elapsed Time (minutes)</b>	<b>Predicted Time to Go</b>	<b>Predicted Total Time</b>
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

# Driving Home Example: MC vs. TD

Changes recommended by  
Monte Carlo methods ( $\alpha=1$ )



Changes recommended  
by TD methods ( $\alpha=1$ )



# Advantages and Disadvantages of MC vs. TD

- TD can learn *before* knowing the final outcome
  - TD can learn online after every step
  - MC must wait until end of episode before return is known
- TD can learn *without* the final outcome
  - TD can learn from incomplete sequences
  - MC can only learn from complete sequences
  - TD works in continuing (non-terminating) environments
  - MC only works for episodic (terminating) environments



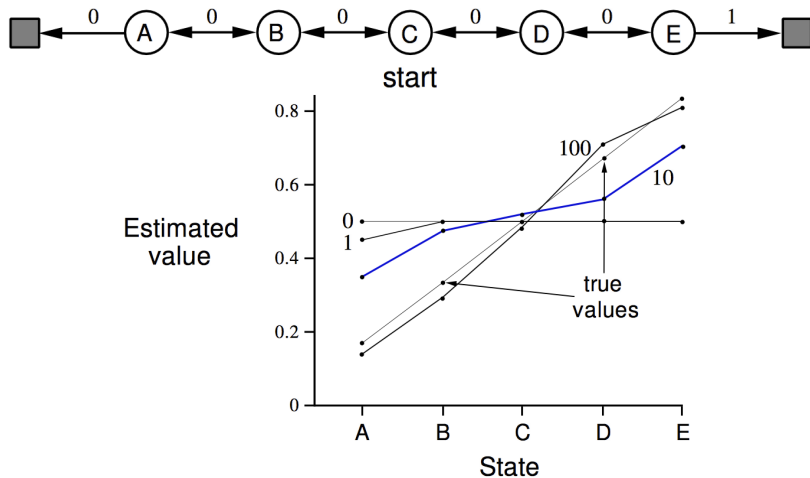
# Bias/Variance Trade-Off

- Return  $G_t = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T$  is *unbiased* estimate of  $v_\pi(S_t)$
- True TD target  $R_{t+1} + \gamma v_\pi(S_{t+1})$  is *unbiased* estimate of  $v_\pi(S_t)$
- TD target  $R_{t+1} + \gamma V(S_{t+1})$  is *biased* estimate of  $v_\pi(S_t)$
- TD target is much lower variance than the return:
  - Return depends on *many* random actions, transitions, rewards
  - TD target depends on *one* random action, transition, reward

## Advantages and Disadvantages of MC vs. TD (2)

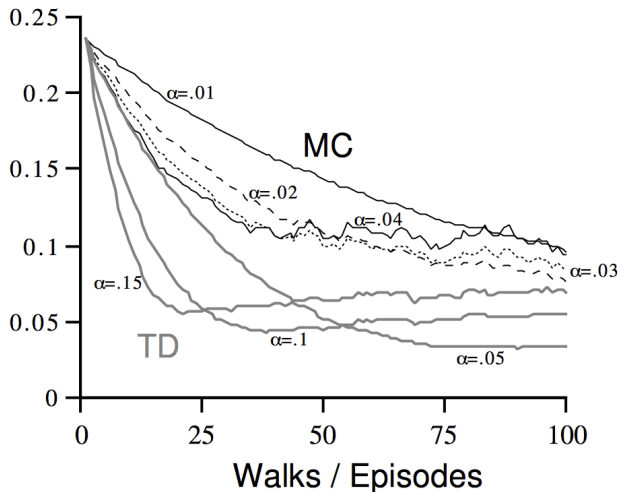
- MC has high variance, zero bias
  - Good convergence properties
  - (even with function approximation)
  - Not very sensitive to initial value
  - Very simple to understand and use
- TD has low variance, some bias
  - Usually more efficient than MC
  - TD(0) converges to  $v_{\pi}(s)$
  - (but not always with function approximation)
  - More sensitive to initial value

# Random Walk Example



# Random Walk: MC vs. TD

RMS error,  
averaged  
over states



# Batch MC and TD

- MC and TD converge:  $V(s) \rightarrow v_{\pi}(s)$  as experience  $\rightarrow \infty$
- But what about batch solution for finite experience?

$$s_1^1, a_1^1, r_2^1, \dots, s_{T_1}^1$$

$$\vdots$$

$$s_1^K, a_1^K, r_2^K, \dots, s_{T_K}^K$$

- e.g. Repeatedly sample episode  $k \in [1, K]$
- Apply MC or TD(0) to episode  $k$

# AB Example

Two states  $A, B$ ; no discounting; 8 episodes of experience

$A, 0, B, 0$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 0$

What is  $V(A), V(B)$ ?

# AB Example

Two states  $A, B$ ; no discounting; 8 episodes of experience

$A, 0, B, 0$

$B, 1$

$B, 1$

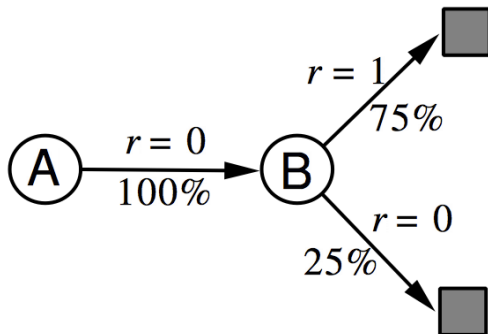
$B, 1$

$B, 1$

$B, 1$

$B, 1$

$B, 0$



What is  $V(A), V(B)$ ?

# Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - Best fit to the observed returns

$$\sum_{k=1}^K \sum_{t=1}^{T_k} (G_t^k - V(s_t^k))^2$$

- In the AB example,  $V(A) = 0$
- TD(0) converges to solution of max likelihood Markov model
  - Solution to the MDP  $\langle \mathcal{S}, \mathcal{A}, \hat{\mathcal{P}}, \hat{\mathcal{R}}, \gamma \rangle$  that best fits the data

$$\hat{\mathcal{P}}_{s,s'}^a = \frac{1}{N(s,a)} \sum_{k=1}^K \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k, s_{t+1}^k = s, a, s')$$

$$\hat{\mathcal{R}}_s^a = \frac{1}{N(s,a)} \sum_{k=1}^K \sum_{t=1}^{T_k} \mathbf{1}(s_t^k, a_t^k = s, a) r_t^k$$

- In the AB example,  $V(A) = 0.75$

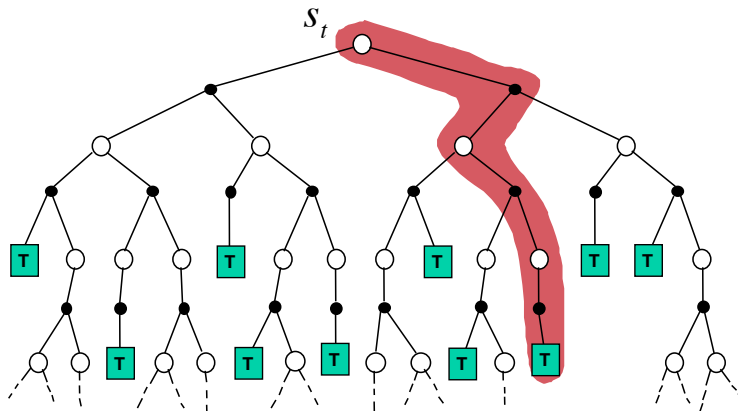


## Advantages and Disadvantages of MC vs. TD (3)

- TD exploits Markov property
  - Usually more efficient in Markov environments
- MC does not exploit Markov property
  - Usually more effective in non-Markov environments

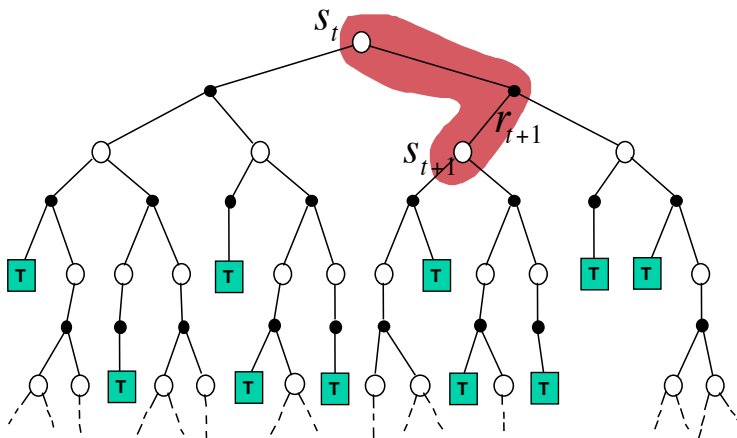
# Monte-Carlo Backup

$$V(S_t) \leftarrow V(S_t) + \alpha (G_t - V(S_t))$$



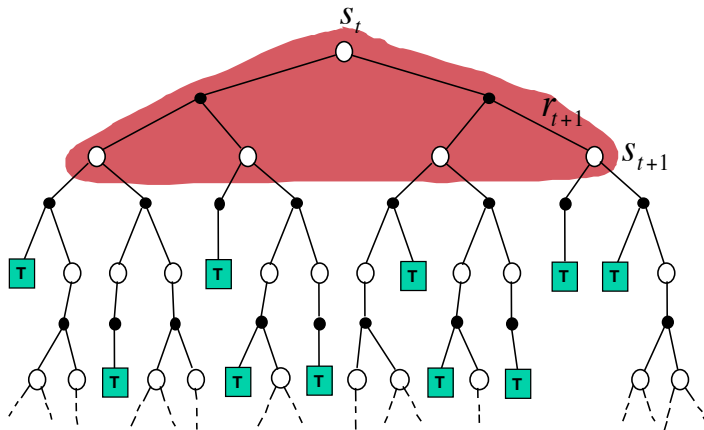
# Temporal-Difference Backup

$$V(S_t) \leftarrow V(S_t) + \alpha (R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$$



# Dynamic Programming Backup

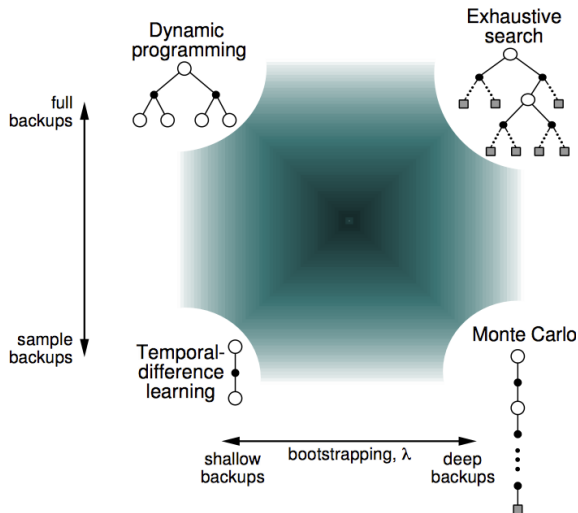
$$V(S_t) \leftarrow \mathbb{E}_{\pi} [R_{t+1} + \gamma V(S_{t+1})]$$



# Bootstrapping and Sampling

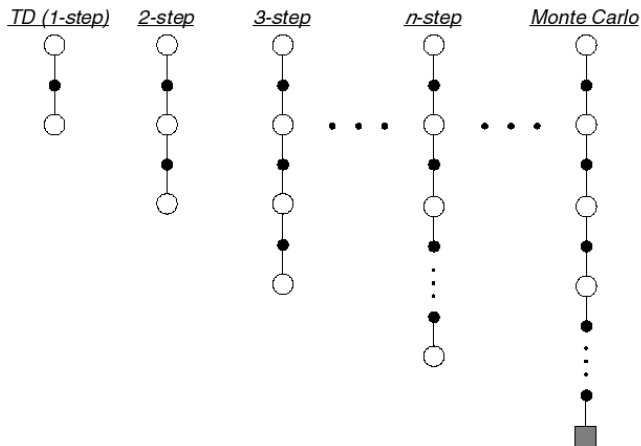
- **Bootstrapping**: update involves an estimate
  - MC does not bootstrap
  - DP bootstraps
  - TD bootstraps
- **Sampling**: update samples an expectation
  - MC samples
  - DP does not sample
  - TD samples

# Unified View of Reinforcement Learning



# $n$ -Step Prediction

- Let TD target look  $n$  steps into the future



# $n$ -Step Return

- Consider the following  $n$ -step returns for  $n = 1, 2, \infty$ :

$$n = 1 \quad (TD) \quad G_t^{(1)} = R_{t+1} + \gamma V(S_{t+1})$$

$$n = 2 \quad G_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2})$$

$$\vdots \quad \vdots$$

$$n = \infty \quad (MC) \quad G_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T$$

- Define the  $n$ -step return

$$G_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

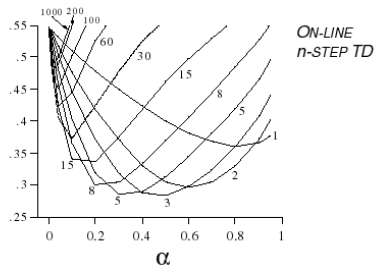
- $n$ -step temporal-difference learning

$$V(S_t) \leftarrow V(S_t) + \alpha \left( G_t^{(n)} - V(S_t) \right)$$

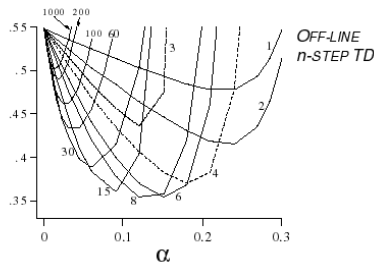


# Large Random Walk Example

*RMS error,  
averaged over  
first 10 episodes*



*RMS error,  
averaged over  
first 10 episodes*



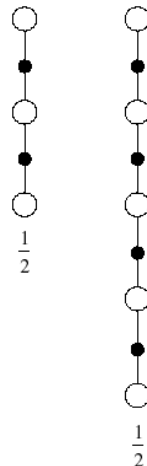
# Averaging $n$ -Step Returns

- We can average  $n$ -step returns over different  $n$
- e.g. average the 2-step and 4-step returns

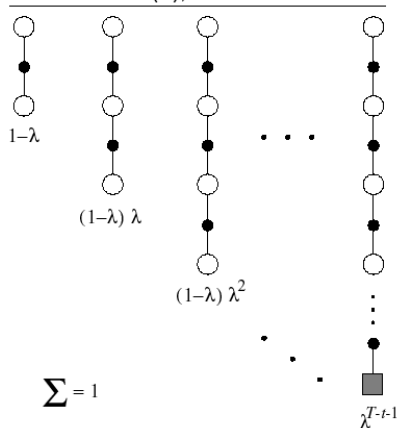
$$\frac{1}{2}G^{(2)} + \frac{1}{2}G^{(4)}$$

- Combines information from two different time-steps
- Can we efficiently combine information from all time-steps?

One backup



# $\lambda$ -return

TD( $\lambda$ ),  $\lambda$ -return

- The  $\lambda$ -return  $G_t^\lambda$  combines all  $n$ -step returns  $G_t^{(n)}$
- Using weight  $(1-\lambda)\lambda^{n-1}$

$$G_t^\lambda = (1-\lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

- Forward-view TD( $\lambda$ )

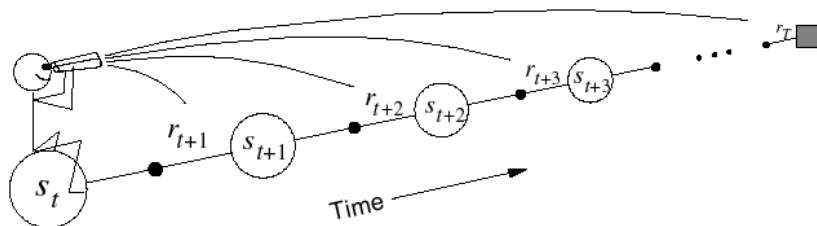
$$V(S_t) \leftarrow V(S_t) + \alpha (G_t^\lambda - V(S_t))$$

# TD( $\lambda$ ) Weighting Function

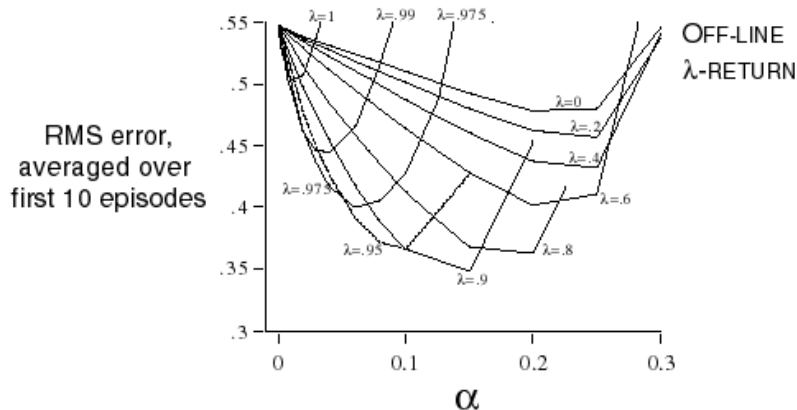


$$G_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

# Forward-view TD( $\lambda$ )



- Update value function towards the  $\lambda$ -return
- Forward-view looks into the future to compute  $G_t^\lambda$
- Like MC, can only be computed from complete episodes

Forward-View TD( $\lambda$ ) on Large Random Walk

## Backward View TD( $\lambda$ )

- Forward view provides theory
- Backward view provides mechanism
- Update online, every step, from incomplete sequences

# Eligibility Traces



- Credit assignment problem: did bell or light cause shock?
- **Frequency heuristic**: assign credit to most frequent states
- **Recency heuristic**: assign credit to most recent states
- *Eligibility traces* combine both heuristics

$$E_0(s) = 0$$

$$E_t(s) = \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s)$$





# Backward View TD( $\lambda$ )

- Keep an eligibility trace for every state  $s$
- Update value  $V(s)$  for every state  $s$
- In proportion to TD-error  $\delta_t$  and eligibility trace  $E_t(s)$

$$\delta_t = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$$

$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$



# TD( $\lambda$ ) and TD(0)

- When  $\lambda = 0$ , only current state is updated

$$E_t(s) = \mathbf{1}(S_t = s)$$
$$V(s) \leftarrow V(s) + \alpha \delta_t E_t(s)$$

- This is exactly equivalent to TD(0) update

$$V(S_t) \leftarrow V(S_t) + \alpha \delta_t$$

# TD( $\lambda$ ) and MC

- When  $\lambda = 1$ , credit is deferred until end of episode
- Consider episodic environments with offline updates
- Over the course of an episode, total update for TD(1) is the same as total update for MC

## Theorem

*The sum of offline updates is identical for forward-view and backward-view TD( $\lambda$ )*

$$\sum_{t=1}^T \alpha \delta_t E_t(s) = \sum_{t=1}^T \alpha \left( G_t^\lambda - V(S_t) \right) \mathbf{1}(S_t = s)$$

# MC and TD(1)

- Consider an episode where  $s$  is visited once at time-step  $k$ ,
- TD(1) eligibility trace discounts time since visit,

$$\begin{aligned} E_t(s) &= \gamma E_{t-1}(s) + \mathbf{1}(S_t = s) \\ &= \begin{cases} 0 & \text{if } t < k \\ \gamma^{t-k} & \text{if } t \geq k \end{cases} \end{aligned}$$

- TD(1) updates accumulate error *online*

$$\sum_{t=1}^{T-1} \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^{T-1} \gamma^{t-k} \delta_t = \alpha (G_k - V(S_k))$$

- By end of episode it accumulates total error

$$\delta_k + \gamma \delta_{k+1} + \gamma^2 \delta_{k+2} + \dots + \gamma^{T-1-k} \delta_{T-1}$$

# Telescoping in TD(1)

When  $\lambda = 1$ , sum of TD errors telescopes into MC error,

$$\begin{aligned}
 & \delta_t + \gamma \delta_{t+1} + \gamma^2 \delta_{t+2} + \dots + \gamma^{T-1-t} \delta_{T-1} \\
 &= R_{t+1} + \gamma V(S_{t+1}) - V(S_t) \\
 &+ \gamma R_{t+2} + \gamma^2 V(S_{t+2}) - \gamma V(S_{t+1}) \\
 &+ \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3}) - \gamma^2 V(S_{t+2}) \\
 &\quad \vdots \\
 &+ \gamma^{T-1-t} R_T + \gamma^{T-t} V(S_T) - \gamma^{T-1-t} V(S_{T-1}) \\
 &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} \dots + \gamma^{T-1-t} R_T - V(S_t) \\
 &= G_t - V(S_t)
 \end{aligned}$$

# TD( $\lambda$ ) and TD(1)

- TD(1) is roughly equivalent to every-visit Monte-Carlo
- Error is accumulated online, step-by-step
- If value function is only updated offline at end of episode
- Then total update is exactly the same as MC

# Telescoping in TD( $\lambda$ )

For general  $\lambda$ , TD errors also telescope to  $\lambda$ -error,  $G_t^\lambda - V(S_t)$

$$\begin{aligned}
 G_t^\lambda - V(S_t) &= -V(S_t) + (1-\lambda)\lambda^0 (R_{t+1} + \gamma V(S_{t+1})) \\
 &\quad + (1-\lambda)\lambda^1 (R_{t+1} + \gamma R_{t+2} + \gamma^2 V(S_{t+2})) \\
 &\quad + (1-\lambda)\lambda^2 (R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 V(S_{t+3})) \\
 &\quad + \dots \\
 &= -V(S_t) + (\gamma\lambda)^0 (R_{t+1} + \gamma V(S_{t+1}) - \gamma\lambda V(S_{t+1})) \\
 &\quad + (\gamma\lambda)^1 (R_{t+2} + \gamma V(S_{t+2}) - \gamma\lambda V(S_{t+2})) \\
 &\quad + (\gamma\lambda)^2 (R_{t+3} + \gamma V(S_{t+3}) - \gamma\lambda V(S_{t+3})) \\
 &\quad + \dots \\
 &= (\gamma\lambda)^0 (R_{t+1} + \gamma V(S_{t+1}) - V(S_t)) \\
 &\quad + (\gamma\lambda)^1 (R_{t+2} + \gamma V(S_{t+2}) - V(S_{t+1})) \\
 &\quad + (\gamma\lambda)^2 (R_{t+3} + \gamma V(S_{t+3}) - V(S_{t+2})) \\
 &\quad + \dots \\
 &= \delta_t + \gamma\lambda\delta_{t+1} + (\gamma\lambda)^2\delta_{t+2} + \dots
 \end{aligned}$$

## Forwards and Backwards TD( $\lambda$ )

- Consider an episode where  $s$  is visited once at time-step  $k$ ,
- TD( $\lambda$ ) eligibility trace discounts time since visit,

$$\begin{aligned} E_t(s) &= \gamma\lambda E_{t-1}(s) + \mathbf{1}(S_t = s) \\ &= \begin{cases} 0 & \text{if } t < k \\ (\gamma\lambda)^{t-k} & \text{if } t \geq k \end{cases} \end{aligned}$$

- Backward TD( $\lambda$ ) updates accumulate error *online*

$$\sum_{t=1}^T \alpha \delta_t E_t(s) = \alpha \sum_{t=k}^T (\gamma\lambda)^{t-k} \delta_t = \alpha \left( G_k^\lambda - V(S_k) \right)$$

- By end of episode it accumulates total error for  $\lambda$ -return
- For multiple visits to  $s$ ,  $E_t(s)$  accumulates many errors



# Offline Equivalence of Forward and Backward TD

## Offline updates

- Updates are accumulated within episode
- but applied in batch at the end of episode

# Online Equivalence of Forward and Backward TD

## Online updates

- TD( $\lambda$ ) updates are applied online at each step within episode
- Forward and backward-view TD( $\lambda$ ) are slightly different
- **NEW**: Exact online TD( $\lambda$ ) achieves perfect equivalence
- By using a slightly different form of eligibility trace
- Sutton and von Seijen, ICML 2014

# Summary of Forward and Backward TD( $\lambda$ )

Offline updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0) 	TD( $\lambda$ ) 	TD(1) 
Forward view	TD(0)	Forward TD( $\lambda$ )	MC
Online updates	$\lambda = 0$	$\lambda \in (0, 1)$	$\lambda = 1$
Backward view	TD(0) 	TD( $\lambda$ ) ≠	TD(1) ≠
Forward view	TD(0) 	Forward TD( $\lambda$ ) 	MC 
Exact Online	TD(0)	Exact Online TD( $\lambda$ )	Exact Online TD(1)

= here indicates equivalence in total update at end of episode.