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Super-duper Thesis Title

Diploma Thesis

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Budapest, 2018

Contents

Introduction

TODO

Chapter 1

Coupon coloring in arbitrary graphs

In this chapter we will examine the so-called total domatic number of graphs.

Let $G = (V, E)$ be a graph without isolated vertices.

Definition 1.0.1. *$S \subseteq V$ is a total dominating set if every vertex has a neighbor in S. The total domatic number of G is the maximum number of disjoint total dominating sets.*

Sometimes it is more convenient to look at total dominating sets as color classes.

Definition 1.0.2. *A coloring of the vertices is called a k-coupon coloring if every vertex has a neighbor from each color class. The coupon coloring number of G is the maximum k for which a k-coupon coloring exists. The coupon coloring number is denoted by $\chi_c(G)$.*

1.1 Complexity

It turns out that determining the total domatic number (or equivalently the coupon coloring number) of a graph is rather hard. Even determining if the total domatic number of a graph is at least 2 is NP-complete. We prove this by showing that a variant of 3SAT is reducible to this question in polynomial time.

Definition 1.1.1. *An instance of the not-all-equal 3-satisfiability (NAE-3SAT) problem consists of a set C of clauses on a finite set X of Boolean variables, where each clause contains three literals. The question is whether there is a truth assignment for X that satisfies all the clauses in C such that each clause contains a false literal.*

Theorem 1.1.1. *NAE-3SAT is NP-complete.*

Proof. It can be checked in polynomial time whether a given truth assignment meets the requirement, so NAE-3SAT is in NP.

To prove NP completeness, we show first a reduction from 3SAT to NAE-4SAT. Let C be the set of clauses and X be the set of variables in an instance of 3SAT. Let $X' = X \cup y$, and $C' = \{(x_1 \vee x_2 \vee x_3 \vee y) \mid (x_1 \vee x_2 \vee x_3) \in C\}$. We claim that the NAE-4SAT problem defined by (C', X') is satisfiable if and only if the 3SAT problem defined by (C, X) is satisfiable. If the 3SAT formula is satisfied by a truth assignment then the same assignment with assigning the value false to y satisfies the NAE-4SAT problem. Now suppose a truth assignment satisfies the NAE-4SAT problem. If y has value false, then the same assignment of X satisfies the 3SAT problem. If y has value true, then changing every truth value in X to its opposite gives a truth assignment satisfying the 3SAT formula.

Finally, the reduction from NAE-4SAT to NAE-3SAT is by adding clauses $(x_1 \vee x_2 \vee z)$ and $(\bar{z} \vee x_3 \vee y)$ instead of each clause $(x_1 \vee x_2 \vee x_3 \vee y) \in C'$. $\square$

Theorem 1.1.2. *It is NP-complete to decide whether the total domatic number of a graph is at least* 2*.*

Proof. Given a partition of the vertices into 2 sets, it can be checked in polynomial time whether these sets are total dominating sets. So the problem is a member of NP.

For proving NP-completeness, we will show that NAE-3SAT is reducible to this problem in polynomial time. Let C be the set of clauses and X be the set of variables in an instance of NAE-3SAT. We can assume that every variable x appears in at least one clause. Otherwise we add a new clause containing x and $\bar{x}$ to the formula. Now we construct the corresponding graph G. For each variable x, introduce 3 vertices x_1, x_2, x_3, and 2 edges x_1x_2, x_2x_3. For each clause c, introduce a vertex c. If x is a literal in c, then add the edge cx_1 to the graph. If $\bar{x}$ is a literal in c, then add the edge cx_3.

Suppose G has a partition into 2 disjoint total dominating sets: T and F. Assign the value true for each variable x with $x_1 \in T$ and assign the value false otherwise. For any variable x, x_1 and x_3 are the only neighbors of x_2, so x_1 and x_3 must be in different sets of the partition. If c is a vertex corresponding to a clause, then it must have neighbors both in T and F, and so the literals in c cannot be all true nor false.

Suppose now that the variables have a truth assignment such that each clause contains both true and false literals. Define T and F as follows. Put all the vertices corresponding to clauses into T. For each variable x put x_2 into F. Furthermore, if true was assigned to x, then put x_1 into T, x_3 into F, and conversely otherwise. $\square$

Let us note that the constructed graph in the proof is always a bipartite graph.

Corollary 1.1.1. *It is NP-complete to decide whether the total domatic number of a bipartite graph is at least* 2*.*

1.2 Degree restrictions

A natural question is whether graphs with an appropriately big minimum degree always have a total domatic number of at least 2.

Theorem 1.2.1. *For every d there exists a graph with minimum degree d and without 2 disjoint total dominating sets.*

Proof. TODO $\square$

TODO: Other degree stuff (k-regular, maxdeg-mindeg small enough, etc)

Chapter 2

Coupon coloring in planar graphs

From now on we will focus on 2-coupon colorings and planar graphs.

2.1 Introducing the Goddard-Henning conjecture

A conjecture of Goddard and Henning is the following.

Conjecture 2.1.1. *If G is a simple triangulated planar graph of order at least 4, then the total domatic number of G is at least 2.*

Remark 2.1.1. *The simplicity of the graph is necessary. Suppose the graph on figure 2.1 has a 2-coupon coloring. Then A and C must have different colors, because they are the only neighbors of B. Similarly, C and E must have different colors, as well as E and A. That is a contradiction, since A, C and E form a triangular.*

Remark 2.1.2. *Allowing triangulated disks (i.e. planar graphs with at most one face greater than 3), the conjecture does not hold. For example, the graph on figure 2.2 does not have a 2-coupon coloring from similar reasons as the previous one. We will show later that this graph is a member of a bigger graph family without 2 disjoint dominating sets.*

There are some sufficient conditions known for having a total domatic number of at least 2. We will cover some of them along the way. The first example is a graph family for which an easy induction shows that they are 2-coupon colorable.

Definition 2.1.1. *A graph is called a stacked graph if it can be constructed from a triangle by repeatedly putting a new vertex in a face and connecting it with the vertices on the boundary of that face.*

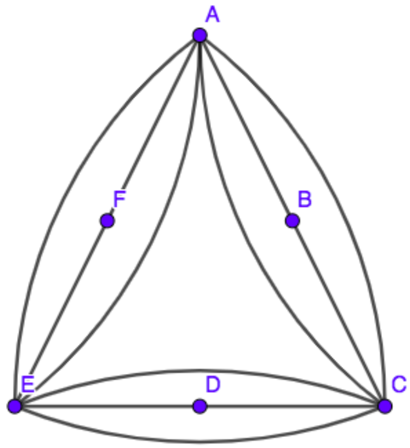

Figure 2.1: Simplicity is necessary

Remark 2.1.3. *Stacked graphs are triangulated.*

Claim 2.1.1. *Stacked graphs with at least* 4 *vertices are* 2*-coupon colorable.*

Proof. We can determine the colors of the vertices as the graph is constructed. The current coloring will maintain two following two properties.

1. It is a 2-coupon coloring of the current graph.

2. Every face has vertices from both color classes.

The construction of the graph starts with a simple triangle. Color two vertices of the triangle to blue, and the remaining vertex to red. Color the vertex added to the graph in the first step to red. This coloring has the desired properties. When a vertex is inserted into a face, color the new vertex to red, if there is only one red vertex on the face's boundary, and blue otherwise. This trivially maintains the desired properties. □

2.2 Outerplanar and Hamiltonian graphs

Zoltán Lóránt Nagy showed that the conjecture of Goddard and Henning holds for Hamiltonian graphs. For this, he characterized the 2-coupon colorable maximal outerplanar graphs.

Definition 2.2.1. *A graph is outerplanar if it has a planar drawing for which all vertices belong to the outer face. A maximal outerplanar graph is an outerplanar graph such that adding any edge results in a not outerplanar graphs.*

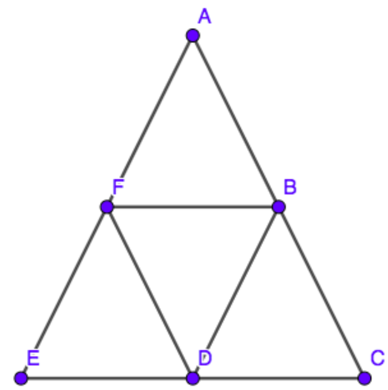

Figure 2.2: The conjecture does not hold for triangulated disks

Remark 2.2.1. *Te outer face of a maximal outerplanar graph is a Hamiltonian cycle.*

In order to provide the mentioned characterization we need to introduce a few notions first.

Definition 2.2.2. *Let G be a maximal outerplanar graph of order $n \geq 3$. The $M(G)$ sun graph of G is obtained by gluing a triangle to each edge of the outer face.*

Remark 2.2.2. *$M(G)$ is a maximal outerplanar graph with $2n$ vertices, from which n has degree 2. The graph on figure 2.2 is the sun graph of the BDE triangle.*

Definition 2.2.3. *A vertex v of a maximal outerplanar graph is called a central vertex if the following 3 conditions hold.*

1. *$deg(v) \geq 3$*
2. *Every neighbor of v has degree at least 3*
3. *For every u, w neighbors of v the length of the uw path on the outer face not containing v is divisible by 4.*

Claim 2.2.1. *The outer face of a maximal outerplanar graph does not contain two consecutive central vertices.*

Proof. Suppose there exists an uv edge on the outer face such that u and v are central vertices. Because of the maximality of the graph there exists a uvw triangle. Index the vertices along the outer face form $v = v_1$ to $u = v_n$. Suppose $w = v_i$. From the centrality of v follows that $i \equiv 2 \pmod 4$. On the other hand u is also a central vertex, hence $i \equiv 1 \pmod 4$. □

Definition 2.2.4. *A generalized sun graph is a maximal outerplanar graph of order* $n \equiv 2 \ (mod\ 4)$ *such that the number of degree* 2 *vertices plus the number of central vertices is* $n/2$.

Remark 2.2.3. *Every second vertex of the outer face in a generalized sun graph is either central or has degree* 2.

The key characterization theorem is the following.

Theorem 2.2.1. *Let* G *be a maximal outerplanar graph.* G *admits* 2 *disjoint total dominating sets if and only if* G *is not a generalized sun graph.*

For proving this theorem we need some observations about generalized sun graphs. TODO

Lemma 2.2.1. *The outer face of a maximal outerplanar graph has a chord TODO*

Proof of Theorem 2.2.1. First we show that generalized sun graphs do not have 2 disjoint total dominating sets. The proof goes by induction on the $n = 4k + 2$ number of vertices.

For $k = 1$ there is only one generalized sun graph and it does not admit 2 disjoint total dominating sets. (Shown on figure 2.2.)

Suppose $k \geq 2$ and G is a generalized sun graph of order $4k + 2$. Index the vertices along the outer face from v_1 to v_{4k+2}, such that every vertex with an odd index is central or has degree 2. Let c be a 2-coloring of the graph. We show that c cannot be a 2-coupon coloring. The cardinality of the vertices implies that there must be two consecutive vertices v_{2i} and v_{2i+2} with the same color (say white). If v_{2i} has only white neighbors, then this coloring is not a 2-coupon coloring. So suppose v_{2i} has a black neighbor v_j. In this case, v_{2i} is a central vertex. The $v_{2i}v_j$ edge cuts the graph into two parts ($v_{2i}v_j$ is an edge in both graphs). Both of these graphs are generalized sun graphs, as v_{2i} either remains a central vertex or become a vertex of degree 2 in these smaller graphs, whereas other central vertices remain central vertices. By induction, the restriction of c is not a 2-coupon coloring in either of the smaller graphs. If there is a vertex v_l with a monochromatic neighborhood in one of the smaller graphs and $l \neq 2i, l \neq j$, then v_l has the same neighborhood in G, hence all its neighbors are from the same color class. v_{2i} cannot violate the condition, as it was chosen in a way that it has both a black and a white neighbor in both graphs. Thus the only remaining case is when v_j has a monochromatic neighborhood in both graphs. But in this case, all of its neighbors are from the same color class as v_{2i}, so it has a monochromatic neighborhood also in G.

Now we show that if a graph G of order n is not a generalized sun graph then it does have 2 disjoint total dominating sets.

If $n \equiv 0 \pmod 4$, then it is easy to find a 2-coupon coloring: color the vertices along the boundary of the outer face by repeating the pattern $BBWW$.

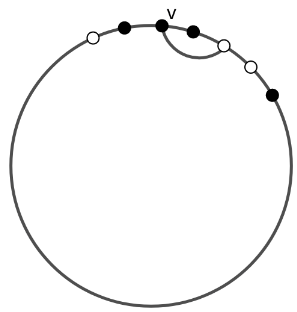

Figure 2.3: Coloring an outerplanar graph of order $4k + 1$

If $n \equiv 1 \pmod 4$, then the same coloring method works, if you start the coloring from the right vertex. By lemma **??** there exists a chord uv of length 2. Alternating colors in pairs starting from v does the job.

If $n \equiv 3 \pmod 4$, then start the coloring from a vertex next to v.

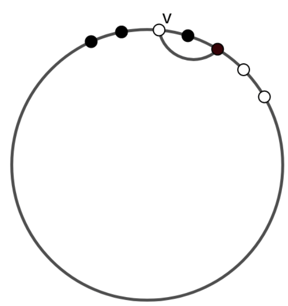

Figure 2.4: Coloring an outerplanar graph of order $4k + 3$

Suppose $n \equiv 2 \pmod 4$. We show that if G does not have 2 disjoint total dominating sets, then it is a generalized sun graph.

TODO □

Remark 2.2.4. *With a slight modification of the proof it can be shown that the vertices of a generalized sun graph cannot be colored in a way that every degree* 2 *or central vertex has neighbors from both color classes.*

Theorem 2.2.2. *Every triangulated graph with a Hamiltonian circle admits* 2 *disjoint dominating sets.*

Proof. Let G be a triangulated Hamiltonian graph and let n denote the number of vertices in G.

TODO □

2.3 Graphs without low-degree vertices

TODO (Find the related article)

2.4 A result using hypergraphs

TODO

Chapter 3

Variations on the Goddard-Henning conjecture

As a reminder: the Goddard-Henning conjecture states that every simple triangulated planar graph of order at least 4 has total domatic number at least 2. In this chapter we try to find equivalent statements to the conjecture, as well as (hopefully) slightly stronger statements. Even if the stronger statements are not true, they can be useful for proving the conjecture in special cases. (e.g. Theorem 4.2.1)

Definition 3.0.1. *Let G be a triangulated planar graph. For a vertex v, each triangle containing v has an edge not containing v. We call the circle consisting of these edges a wheel.*

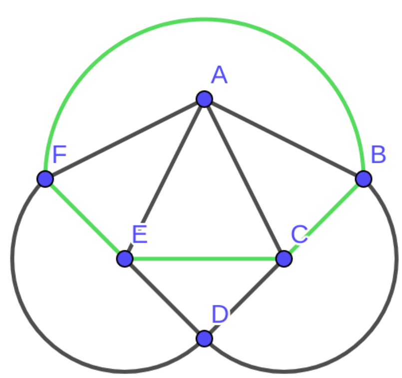

Figure 3.1: The green edges form the wheel defined by A

Statement 3.0.1. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a bipartite $H = (V, F)$ subgraph of G, such that F contains at least one edge from each wheel of G.*

Claim 3.0.1. *The Goddard-Henning conjecture holds if and only if Statement 3.0.1 holds.*

Proof. Let G be a triangulated graph. Suppose first that it has a 2-coupon coloring.

$$F = \{uv \in E \mid u \text{ and } v \text{ are in different color classes}\}$$

defines a bipartite subgraph of G that contains at least one edge from each wheel.

Now suppose that there exists bipartite subgraph that meets our requirement. Color the vertices in one of the classes to black, and the vertices in the other class to white. This is a 2-coupon coloring of the original graph. □

Statement 3.0.2. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a forest in G containing at least one edge from each wheel.*

Remark 3.0.1. *If Statement 3.0.2 holds, then Statement 3.0.1 also holds.*

Statement 3.0.3. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a subgraph of $H' = (V, F')$ having the following two properties.*

1. *F' contains exactly 1 edge from each face of G.*
2. *There are no isolated vertices in H'.*

Lemma 3.0.1. *A connected planar graph is bipartite if end only if each of its faces have an even number of edges.*

Proof. Suppose that the graph is not bipartite and thus there exists a circle C of odd length. We show that then exists an odd face. The proof goes by induction on the number of faces in the inner side of C. If C is a face, then we are done. If C is not a face, then there exists a face f in the inner side of C having at least one common edge with C. f does not contain every edge of C, since G is connected. Let C' be the symmetric difference of the edge sets of C and f. By the parity of C either f is an odd face or C' is an odd circle containing less faces in its inner side than C.

The other direction is trivial. □

Claim 3.0.2. *Statement 3.0.3 holds if and only if Statement 3.0.1 (and thus the Goddard-Henning conjecture) also holds.*

Proof. Let $H' = (V, F')$ be the subgraph required by Statement 3.0.3. We show that $H = (V, E - F')$ is a subgraph required by Statement 3.0.1. H is a bipartite graph by Lemma 3.0.1, as each of its faces have 4 edges. Take a wheel $v_1 v_2 \ldots v_k$ defined by a vertex v. v is not an isolated vertex in H', so there exists a vertex v_i such that $vv_i \in F'$. As F' contains exactly one edge from each face, $v_i v_{i+1} \in F$.

Now let $H = (V, F)$ be the subgraph required by Statement 3.0.1. Clearly, F cannot contain all three edges of a face. We show that there exists an edge set F_0, such that $(V, F \cup F_0)$ is a bipartite subgraph of G that contains exactly two edges of each face. Let $F_0 = \emptyset$. We will add edges to F_0 maintaining that $(V, F \cup F_0)$ is a bipartite graph.

Suppose that there exists a face uvw in G with $uv, vw \notin F \cup F_0$, $wu \in F \cup F_0$. If $F \cup F_0 \cup \{uv\}$ or $F \cup F_0 \cup \{vw\}$ is bipartite, then add the appropriate edge to F_0. If adding either of these edges to F_0 creates an odd circle in $(V, F \cup F_0)$, then there exists a path P_{uv} of odd length from u to v and a path P_{vw} of odd length from v to w. Thus $Puv + Pvw + wu$ is a closed walk of odd length. But that is a contradiction as $(V, F \cup F_0)$ is a bipartite graph.

Now suppose that there exists a face uvw in G such that none of its edges is contained in $F \cup F_0$. If either of its edges can be added to F_0 maintaining a bipartite graph, then put those edges in F_0. Otherwise there exist odd paths P_{uv}, P_{vw}, P_{wu} as above. Concatenating these paths gives a closed walk of odd length and that yields a contradiction.

$(V, F + F_0)$ clearly contains an edge from each wheel and contains two edges of each face of G. So $H' = (V, E - (F \cup F_0))$ contains exactly one edge from each face, and has no isolated vertices. $\square$

One can phrase the Goddard-Henning conjecture in the dual graph as well.

Claim 3.0.3. *$G^* = (V^*, E^*)$ is the dual of a simple triangulated graph of order at least 4 if and only if G^* is a 3-regular 2-edge-connected planar graph of order at least 4.*

Proof. It is trivial that G^* is 3-regular if and only if its dual is triangulated.

It is also easy to see that a cut consisting of one edge corresponds to a loop edge in the dual, and a cut consisting of two edges corresponds to a pair of parallel edges.

Finally, by 3-regularity and using Euler's formula

$$f^* = m^* - n^* + 2 = 3n^*/2 - n + 2 = n^*/2 + 2,$$

where f^*, m^*, and n^* denote the number of faces, edges and vertices of G^*. Thus the dual of G^* has at least 4 vertices if and only if $4 \leq n^*/2 + 2$, i.e. G^* has at least 4 vertices. $\square$

Statement 3.0.4. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph $H^* = (V^*, F^*)$ in G^* that has the following 2 properties.*

1. *It does not contain any odd cut of G^*.*

2. *For every face f of G^*, H^* contains an edge e not on f that has at least one endpoint on f. We say that e leaves the face f.*

Claim 3.0.4. *3.0.4 is equivalent with 3.0.1.*

Proof. We show that given a subgraph $H = (V, F)$ that meets the requirements of 3.0.1, the edges corresponding to F in the dual of G form a subgraph H^* required by 3.0.4, and vice versa. It may be worth noting that the defined H^* is not necessarily the same as the dual graph of H.

It follows from the fact that circles of a planar graph correspond to minimal cutsets in the dual graph, that H is bipartite if and only if H^* does not contain any odd cut of G^*.

Moreover, an edge from a wheel defined by v in G, corresponds to an edge that leaves the face that corresponds to v in the dual graph of G. Hence H contains at least one edge from each wheel if and only if for every face of G^*, H^* contains at least one edge that leaves that face. □

Statement 3.0.5. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph that has the following 2 properties.*

1. *It intersects every odd cut of G^*.*

2. *For every face f of G^* it does not contain all the edges leaving f.*

Claim 3.0.5. *3.0.4 holds if and only if 3.0.5 holds.*

Proof. If H^* meets the requirements of either of the statements, the complementer subgraph in G^* meets the requirements of the other. □

A 2-factor of a graph $G = (V, E)$ consists of disjoint circles covering V. We can formulate a sufficient condition for the Goddard-Henning conjecture with the help of 2-factors.

Statement 3.0.6. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a 2-factor not containing any of the faces.*

Claim 3.0.6. *If Statement 3.0.6 holds, then 3.0.4 holds.*

Proof. Let $H^* = (V^*, F^*)$ be the 2-factor containing none of the faces of G^*.

Every cut of G^* has an even number of common edges with every circle in H^*. Therefore H^* does not contain any odd cuts of G^*.

Let $f = v_1 v_2 \dots v_l$ be a face of G^*. As F^* does not contain f, there must exist a v_i such that $v_i v_{i+1} \notin F^*$. Moreover, every vertex has degree 2 in H^*, so there is an edge starting from v_i that leaves f. □

The existence of 2-factors in which some cycles are not allowed, is a well-studied part of graph theory. We will cover some of these results in Chapter 4.

Statement 3.0.6 can easily be converted into a statement about perfect matchings.

Statement 3.0.7. *Let* $G^* = (V^*, E^*)$ *be a* 3*-regular* 2*-edge-connected planar graph of order at least* 4*. Then there exists a perfect matching containing at least one edge from each face.*

Claim 3.0.7. *Statement 3.0.6 holds if and if Statement 3.0.7 holds.*

Proof. As G^* is 3-regular, a subgraph is a 2-factor if and only if the complementer subgraph is a perfect matching. Clearly, a subgraph contains none of the faces if and only if the complementer subgraph does contain at least one edge from each face. □

TODO: Add a figure about these statements, mention how Barnette's conjecture fits here.

Chapter 4

Restricted 2-factors

4.1 Complexity results

TODO

4.2 Connection to the Goddard-Henning conjecture

Statement 3.0.6 gives a satisfactory condition for the conjecture of Goddard and Henning. With the help of this, we can show that the conjecture holds in the special case when there are only odd-degree vertices.

Theorem 4.2.1. *Let G be a a simple triangulated graph of order at least 4. If all the vertices of G have an odd degree, then there exists a coupon coloring with 2 colors.*

Proof. Let G^* be the dual of G. Take a 4-coloring c_{face} of the faces of G with the colors $\{1, 2, 3, 4\}$, such that faces with common edges belong to different color classes. Now we define a coloring on the edges. For an edge e, let f_e^1 and f_e^2 denote the faces containing e. Define the coloring as follows.

$$c_{edge}(e) = \begin{cases} 1, & \text{if } (c_{face}(f_e^1) = 1 \text{ and } c_{face}(f_e^2) = 2) \text{ or } (c_{face}(f_e^1) = 3 \text{ and } c_{face}(f_e^2) = 4) \\ 2, & \text{if } (c_{face}(f_e^1) = 2 \text{ and } c_{face}(f_e^2) = 3) \text{ or } (c_{face}(f_e^1) = 1 \text{ and } c_{face}(f_e^2) = 4) \\ 3, & \text{if } (c_{face}(f_e^1) = 1 \text{ and } c_{face}(f_e^2) = 3) \text{ or } (c_{face}(f_e^1) = 2 \text{ and } c_{face}(f_e^2) = 4) \end{cases}$$

This way c_{edge} is a coloring without edges of the same color having a common endpoint. Thus the union of any two color classes defines a 2-factor. Let H^* be such a 2-factor. From the degree restriction on G follows that every face in G^* has an odd number of edges. Hence H^* does not contain any faces. With this, we have proven that Statement 3.0.6 holds and then by Claim 3.0.6 G has a 2-coupon coloring. $\square$