Eötvös Loránd University
Faculty of Science

Hanna Gábor

Super-duper Thesis Title

Diploma Thesis

Supervisor:

Kristóf Bérczi
Department of Operations Research

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Contents

Introduction

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Chapter 1

Coupon coloring in planar graphs

1.1 ...

We will examine the so-called total domatic number of graphs. Let $G = (V, E)$ be a graph without isolated vertices.

Definition 1.1.1. *$S \subseteq V$ is a total dominating set if every vertex has a neighbor in S. The total domatic number of G is the maximum number of disjoint total dominating sets.*

Sometimes it's more convenient to look at total dominating sets as color classes.

Definition 1.1.2. *A coloring of the vertices is called a k-coupon coloring if every vertex has a neighbor from each color class. The coupon coloring number of G is the maximum k for which a k-coupon coloring exists. The coupon coloring number is denoted by $\chi_c(G)$.*

It turns out that determining the total domatic number (or equivalently the coupon coloring number) of a graph is rather hard.

Definition 1.1.3. *NAE-3SAT ...*

Theorem 1.1.1. *NAE-3SAT is NP-complete.*

Proof. ... $\square$

Theorem 1.1.2. *It's NP-complete to decide whether the total domatic number of a graph is at least* 2.

Proof. Given a partition of the vertices into 2 sets, it can be checked in polynomial time whether these sets are total dominating sets. So the problem is a member of NP.

For proving NP-completeness, we will show that NAE-3SAT is reducible to this problem in polynomial time. Let C be the set of clauses and X be the set of variables in an instance of NAE-3SAT. We can assume that every variable x appears in at least one clause. Otherwise we add a new clause containing x and $/barx$ to the formula. Now we construct the corresponding graph G. For each variable x, introduce 3 vertices x_1, x_2, x_3, and 2 edges x_1x_2, x_2x_3. For each clause c, introduce a vertex c. If x is a literal in c, then add the edge cx_1 to the graph. If $\bar{x}$ is a literal in c, then add the edge cx_3.

Suppose G has a partition into 2 disjoint total dominating sets: T and F. Assign the value true for each variable x with $x_1 \in T$ and assign the value false otherwise. For any variable x, x_1 and x_3 are the only neighbors of x_2, so x_1 and x_3 must be in different sets of the partition. If c is a vertex corresponding to a clause, then it must have neighbors both in T and F, and so the literals in c can't be all true nor false.

Suppose now that the variables have a truth assignment such that each clause contains both true and false literals. Define T and F as follows. Put all the vertices corresponding to clauses into T. For each variable x put x_2 into F. Furthermore, if true was assigned to x, then put x_1 into T, x_3 into F, and conversely otherwise.

(https://pdfs.semanticscholar.org/d44c/a18638eadeeb17b69f657cfa06c313646b8c.pdf or for bipartite: https://people.cs.clemson.edu/ goddard/papers/twoTotalDominationAugment.pdf) $\square$

Let us note that the construction of the graph in the proof is always a bipartite graph.

Corollary 1.1.1. *It's NP-complete to decide whether the total domatic number of a bipartite graph is at least* 2.

1.2 Degree restrictions

A natural question is whether graphs with an appropriately big minimum degree always have a total domatic number of at least 2.

Theorem 1.2.1. *For every d there exists a graph with minimum degree d and without* 2 *disjoint total dominating sets.*

Proof. ... $\square$

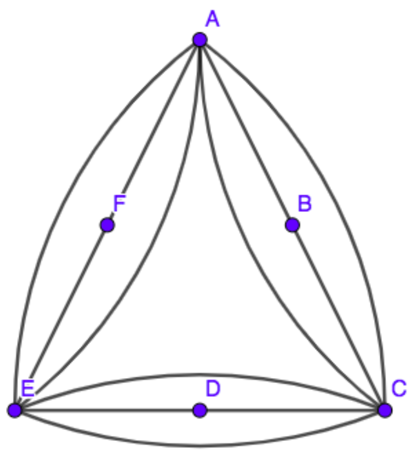
...Other degree stuff (k-regular, maxdeg-mindeg small enough)...

1.3 A conjecture of Goddard and Henning

From now on we will focus on 2-coupon colorings and planar graphs. A conjecture of Goddard and Henning is the following.

Conjecture 1.3.1. *If G is a simple triangulated planar graph of order at least 4, then the total domatic number of G is at least 2.*

Remark 1.3.1. *The simplicity of the graph is necessary. Suppose the graph below has a 2-coupon coloring. Then A and C must have different colors, because they are the only neighbors of B. Similarly, C and E must have different colors, as well as E and A. That's a contradiction, since A, C and E form a triangular.*



Remark 1.3.2. *Allowing triangulated disks (i.e. planar graphs with at most one face greater than 3), the conjecture doesn't hold. For example, the graph below doesn't have a 2-coupon coloring from similar reasons as the previous one. We will show later that this graph is a member of a bigger graph family without 2 disjoint dominating sets.*

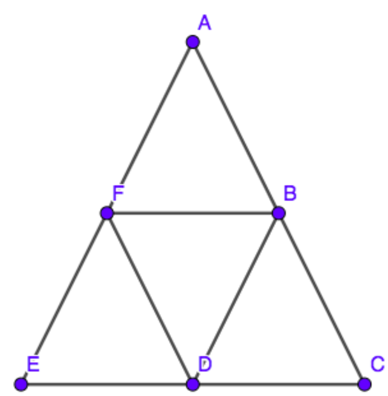
Theorem 1.3.1. *Let G be a triangulated planar graph. If all the vertices of G have an odd degree, then there exists a coupon coloring with 2 colors.*

Proof. ... □

Theorem 1.3.2. *Outerplanar graphs ...*

Theorem 1.3.3. *Every triangulated graph with a Hamiltonian circle admits 2 disjoint dominating sets.*

Hypergraph connection...


A
F
B
E
D
C

Chapter 2

Restricted 2-factors

2.1 Connection

2.2 Barnette's conjecture

Conjecture 2.2.1. *Every 3-connected cubic planar bipartite graph is Hamiltonian.*