Eötvös Loránd University
Faculty of Science

Hanna Gábor

About a conjecture of Goddard and Henning

Diploma Thesis

Supervisor:

Kristóf Bérczi
Department of Operations Research

Budapest, 2018

Contents

Introduction

TODO

Chapter 1

Coupon coloring in arbitrary graphs

In this chapter we will examine the so-called total domatic number of graphs. Let $G = (V, E)$ be a graph without isolated vertices.

Definition 1. *$S \subseteq V$ is a total dominating set if every vertex has a neighbor in S. The total domatic number of G is the maximum number of disjoint total dominating sets.*

Sometimes it is more convenient to look at total dominating sets as color classes.

Definition 2. *A coloring of the vertices is called a k-coupon coloring if every vertex has a neighbor from each color class. The coupon coloring number of G is the maximum k for which a k-coupon coloring exists. The coupon coloring number is denoted by $\chi_c(G)$.*

Remark 3. *We refer as proper colorings to colorings in the usual sense. I.e. colorings of the vertices such that for every vertex v, none of the neighbors of v has the same color as v.*

1.1 Complexity

It turns out that determining the total domatic number (or equivalently, the coupon coloring number) of a graph is rather difficult. Heggernes and Telle [4] showed that even determining whether the total domatic number of a graph is at least 2 is NP-complete. We prove this by showing that a variant of 3SAT is reducible to this question in polynomial time.

Definition 4. *An instance of the not-all-equal 3-satisfiability (NAE-3SAT) problem consists of a set C of clauses on a finite set X of Boolean variables, where each clause contains three literals. The question is whether there is a truth assignment for X that satisfies all the clauses in C such that each clause contains a false literal.*

Theorem 5. *NAE-3SAT is NP-complete.*

Proof. It can be checked in polynomial time whether a given truth assignment meets the requirement, so NAE-3SAT is in NP.

To prove NP completeness, we show first a reduction from 3SAT to NAE-4SAT. Let C be the set of clauses and X be the set of variables in an instance of 3SAT. Let $X' = X \cup y$, and $C' = \{(x_1 \vee x_2 \vee x_3 \vee y) \mid (x_1 \vee x_2 \vee x_3) \in C\}$. We claim that the NAE-4SAT problem defined by (C', X') is satisfiable if and only if the 3SAT problem defined by (C, X) is satisfiable. If the 3SAT formula is satisfied by a truth assignment then the same assignment with assigning the value false to y satisfies the NAE-4SAT problem. Now suppose a truth assignment satisfies the NAE-4SAT problem. If y has value false, then the same assignment of X satisfies the 3SAT problem. If y has value true, then changing every truth value in X to its opposite gives a truth assignment satisfying the 3SAT formula.

Finally, the reduction from NAE-4SAT to NAE-3SAT is by adding clauses $(x_1 \vee x_2 \vee z)$ and $(\bar{z} \vee x_3 \vee y)$ instead of each clause $(x_1 \vee x_2 \vee x_3 \vee y) \in C'$. $\square$

Theorem 6. *It is NP-complete to decide whether the total domatic number of a graph is at least 2.*

Proof. Given a partition of the vertices into 2 sets, it can be checked in polynomial time whether these sets are total dominating sets. So the problem is a member of NP.

For proving NP-completeness, we will show that NAE-3SAT is reducible to this problem in polynomial time. Let C be the set of clauses and X be the set of variables in an instance of NAE-3SAT. We can assume that every variable x appears in at least one clause. Otherwise we add a new clause containing x and $\bar{x}$ to the formula. Now we construct the corresponding graph G. For each variable x, introduce 3 vertices x_1, x_2, x_3, and 2 edges x_1x_2, x_2x_3. For each clause c, introduce a vertex c. If x is a literal in c, then add the edge cx_1 to the graph. If $\bar{x}$ is a literal in c, then add the edge cx_3.

Suppose G has a partition into 2 disjoint total dominating sets: T and F. Assign the value true for each variable x with $x_1 \in T$ and assign the value false otherwise. For any variable x, x_1 and x_3 are the only neighbors of x_2, so x_1 and x_3 must be in different sets of the partition. If c is a vertex corresponding to a clause, then it must have neighbors both in T and F, and so the literals in c cannot be all true nor false.

Suppose now that the variables have a truth assignment such that each clause contains both true and false literals. Define T and F as follows. Put all the vertices corresponding to clauses into T. For each variable x put x_2 into F. Furthermore, if true was assigned to x, then put x_1 into T, x_3 into F, and conversely otherwise. $\square$

Let us note that the constructed graph in the proof is always a bipartite graph.

Corollary 7. *It is NP-complete to decide whether the total domatic number of a bipartite graph is at least* 2*.*

1.2 Degree restrictions

A natural question is whether graphs with a sufficiently large minimum degree always have a total domatic number of at least 2. Zelinka [3] showed that this is not the case.

Theorem 8. *For every $\delta \in \mathbb{Z}^+$ there exists a graph of minimum degree δ without* 2 *disjoint total dominating sets.*

Proof. We define a bipartite graph $G = (U, V; E)$ for arbitrary δ as follows. Let U be a set of cardinality $2\delta - 1$, and V be the set of all subsets of U consisting of δ elements. For all $u \in U, v \in V$, $uv \in E$ if and only if v contains u. Clearly, each vertex of U has at least δ neighbors, and each vertex of V has exactly δ neighbors. Suppose that there exists a 2-coupon coloring. Then there exists a monochromatic $U_0 \in U$ containing δ vertices. Thus, there exists a $v \in V$ corresponding to U_0. That is a contradiction, since the neighborhood of v is U_0, and that is monochromatic. □

A natural next question arises. The graph constructed above has a large number of vertices compared to the minimum degree. The other extreme case is formed by complete graphs, where the minimum degree is $n - 1$. This is not an interesting case: if $n \geq 4$, then every subset U of the vertices of size at least 2 is a total dominating set. We show a less strict but sufficient condition for the existence of 2 disjoint dominating sets: a lower bound for the minimum degree defined by the order of the graph. The next theorem is also from Zelinka [3]. We show here an easier proof than the original.

Claim 9. *Let $G = (V, E)$ be a graph of order n with minimum degree δ. Then every subset of V with at least $n - \delta + 1$ vertices is a total dominating set.*

Proof. Let $S \subseteq V$ be a set of cardinality at least $n - \delta + 1$ and $v \in V$ be an arbitrary vertex. v has at least δ neighbors, and $|(V - v) \cap S| \geq n - \delta$, so there must exist a vertex $u \in S$ adjacent to v. □

Corollary 10. *Let $G = (V, E)$ be a graph of order n with minimum degree δ. If $\delta \geq 1 + n/2$, then there exist* 2 *disjoint dominating sets in G.*

Proof. If $\delta \geq 1 + n/2$, then $2(n - \delta + 1) \leq n$, and therefore the total domatic number of G is at least 2. □

Remark 11. *Aram, Sheikholeslami and Volkmann [5] proved a lower bound for the total domatic number in terms of the order, the minimal and the maximal degree of the graph. Using this lower bound, they proved that the domatic number of an* r*-regular graph is at least* $r/3ln(r)$. *By an easy calculation, it follows from this theorem that if a graph is* r*-regular with* $r \geq 5$, *then the total domatic number is at least* 2.

Chapter 2

Introducing the Goddard-Henning conjecture

2.1 Formulating the conjecture

From now on we will focus on 2-coupon colorings and planar graphs. A conjecture of Goddard and Henning [2] is the following.

Conjecture 12. *If G is a simple triangulated planar graph of order at least 4, then the total domatic number of G is at least 2.*

Remark 13. *The simplicity of the graph is necessary. Suppose the graph on figure 2.1 has a 2-coupon coloring. Then A and C must have different colors, because they are the only neighbors of B. Similarly, C and E must have different colors, as well as E and A. That is a contradiction, since A, C and E form a triangle.*

Remark 14. *Allowing triangulated disks (i.e. planar graphs with at most one face greater than 3), the conjecture does not hold. For example, the graph on figure 2.2 does not have a 2-coupon coloring for similar reasons as the previous one. We will show later that this graph is a member of a bigger graph family without 2 disjoint dominating sets.*

2.2 Easy proofs for some special cases

There are some sufficient conditions known for having a total domatic number of at least 2. We will cover most of the known cases along the way. In this section we take a look at special cases with relatively easy proofs.

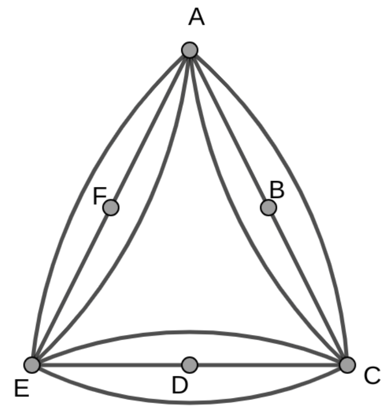


Figure 2.1: Simplicity is necessary

The first example is a graph family for which an easy induction shows that they are 2-coupon colorable.

Definition 15. *A graph is called a stacked graph if it can be constructed from a triangle by repeatedly putting a new vertex in a face and connecting it with the vertices on the boundary of that face.*

Remark 16. *Stacked graphs are triangulated.*

Claim 17. *Stacked graphs with at least* 4 *vertices are* 2*-coupon colorable.*

Proof. We can determine the colors of the vertices as the graph is constructed. The current coloring will maintain two following two properties.

1. It is a 2-coupon coloring of the current graph.

2. Every face has vertices from both color classes.

The construction of the graph starts with a simple triangle. Color two vertices of the triangle to black, and the remaining vertex to white. Color the vertex added to the graph in the first step to white. This coloring has the desired properties. When a vertex is inserted into a face, color the new vertex to white if there is only one white vertex on the face's boundary, and black otherwise. This trivially maintains the desired properties. □

Goddard and Henning [2] verified the conjecture for several cases. We show now three of them.

Claim 18. *Let G be a simple triangulated graph. If all the vertices of G have an odd degree, then there exists a coupon coloring with* 2 *colors.*

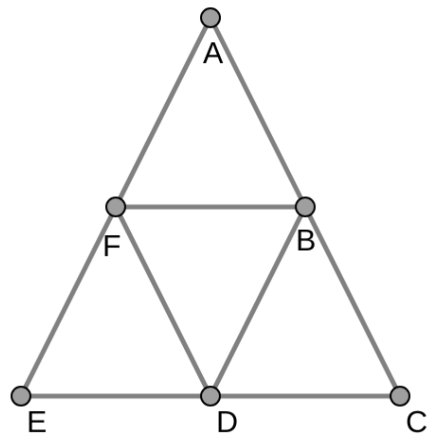


Figure 2.2: The conjecture does not hold for triangulated disks

Proof. There exists a proper 4-coloring of the vertices. As every vertex v has an odd degree, there exists an odd cycle in the neighborhood of v. Hence, in a proper 4-coloring v has neighbors from at least 3 color classes. This means that the union of any two color classes forms a total dominating set. $\square$

Claim 19. *If G can be obtained from a triangulated graph H by putting a new vertex on every face and connecting them with the vertices of that face, then G is 2-coupon colorable.*

Proof. Take a proper 4-coloring on the vertices of H and define a 2-coloring by taking the union of $2-2$ color classes. The obtained coloring has the property that none of the faces is monochromatic. Color the additional vertices to black if the face has only one black vertex, and color it to white otherwise. $\square$

Claim 20. *Let G be a a simple triangulated graph of order at least 4. If the dual of G is Hamiltonian, then G admits two disjoint total dominating sets.*

Proof. Let C be the Hamiltonian cycle in the dual graph G^*. Consider C as a curve on the plane. Color the vertices of G inside C to black, and the vertices outside of C to white. We claim that this way we defined a 2-coupon coloring of G. To prove this, we need to show that for every face f of G^*, there exist a face f' inside C and a face f'' outside of C (inclusively) such that f has a common edge with both f' and f''.

Let f be a face inside C. We claim that f has an edge on C. Otherwise the vertices of f would be of degree at least 4, which is a contradiction, as G^* is 3-regular. Let f'' be the face on the other side of this edge.

If f has a neighboring face inside C, then choose f' to be that face. If f does not have any neighboring faces inside C, then f must be a face defined by C. As G does not have parallel edges, all the edges of C must belong to different faces outside of C. Index the vertices along C from c_1 to c_{n^*}. If there is an edge c_ic_j going between two non-consecutive vertices of C, then by the 3-regularity of G^*, c_ic_{i+1} and $c_{j-1}c_j$ belong to the same face outside of C. It also follows from the 3-regularity, that if there are two parallel c_ic_{i+1} edges in G^*, then there is only one $c_{i+1}c_{i+2}$ edge. Thus, the only case in which all the edges of C belong to different faces outside of C, is when C is of length 2. This means, that G is a triangle, and that is a contradiction.

The same proof works for faces outside of C. □

2.3 Variations of the conjecture

As a reminder: the Goddard-Henning conjecture states that every simple triangulated planar graph of order at least 4 has total domatic number at least 2. In this chapter we try to find equivalent statements to the conjecture, as well as (hopefully) slightly stronger statements. The motivation for this chapter is that even if the stronger statements are not true, they can be useful for proving the conjecture in special cases.

Definition 21. *Let G be a triangulated planar graph. For a vertex v, each triangle containing v has an edge not containing v. We call the cycle consisting of these edges a wheel.*

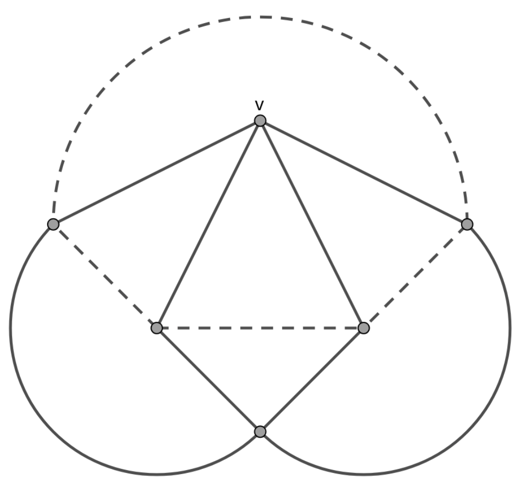


Figure 2.3: The dashed edges form the wheel defined by v

Statement 22. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a bipartite $H = (V, F)$ subgraph of G such that F contains at least one edge from each wheel of G.*

Claim 23. *The Goddard-Henning conjecture holds if and only if Statement 22 holds.*

Proof. Let G be a triangulated graph. Suppose first that it has a 2-coupon coloring.

$$F = \{uv \in E \mid u \text{ and } v \text{ are in different color classes}\}$$

defines a bipartite subgraph of G that contains at least one edge from each wheel.

Now suppose that there exists bipartite subgraph that meets our requirement. Color the vertices in one of the classes to black, and the vertices in the other class to white. This is a 2-coupon coloring of the original graph. □

Statement 24. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a forest in G containing at least one edge from each wheel.*

Remark 25. *If Statement 24 holds, then Statement 22 also holds.*

Statement 26. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a subgraph $H' = (V, F')$ having the following two properties.*

1. *F' contains exactly 1 edge from each face of G.*
2. *There are no isolated vertices in H'.*

Lemma 27. *A connected planar graph is bipartite if and only if each of its faces have an even number of edges.*

Proof. Suppose that the graph is not bipartite and thus there exists a cycle C of odd length. We show that there exists an odd face. The proof goes by induction on the number of faces in the inner side of C. If C is a face, then we are done. If C is not a face, then there exists a face f in the inner side of C having at least one common edge with C. f does not contain every edge of C, since G is connected. Let C' be the symmetric difference of the edge sets of C and f. By the parity of C either f is an odd face or C' is an odd cycle containing less faces in its inner side than C.

The other direction is trivial. □

Claim 28. *Statement 26 holds if and only if Statement 22 (and thus the Goddard-Henning conjecture) also holds.*

Proof. Let $H' = (V, F')$ be the subgraph required by Statement 26. We show that $H = (V, E - F')$ is a subgraph required by Statement 22. H is a bipartite graph by Lemma 27, as each of its faces have 4 edges. Take a wheel $v_1 v_2 \dots v_k$ defined by a vertex v. v is not an isolated vertex in H', so there exists a vertex v_i such that $vv_i \in F'$. As F' contains exactly one edge from each face, $v_i v_{i+1} \in F$.

Now let $H = (V, F)$ be the subgraph required by Statement 22. Clearly, F cannot contain all three edges of a face. We show that there exists an edge set F_0, such that $(V, F \cup F_0)$ is a bipartite subgraph of G that contains exactly two edges of each face. Let $F_0 = \emptyset$. We will add edges to F_0 maintaining that $(V, F \cup F_0)$ is a bipartite graph.

Suppose that there exists a face uvw in G with $uv, vw \notin F \cup F_0$, $wu \in F \cup F_0$. If $F \cup F_0 \cup \{uv\}$ or $F \cup F_0 \cup \{vw\}$ is bipartite, then add the appropriate edge to F_0. If adding either of these edges to F_0 creates an odd cycle in $(V, F \cup F_0)$, then there exists a path P_{uv} of odd length from u to v and a path P_{vw} of odd length from v to w. Thus $Puv + Pvw + wu$ is a closed walk of odd length. But that is a contradiction as $(V, F \cup F_0)$ is a bipartite graph.

Now suppose that there exists a face uvw in G such that none of its edges is contained in $F \cup F_0$. If either of its edges can be added to F_0 maintaining a bipartite graph, then put those edges in F_0. Otherwise there exist odd paths P_{uv}, P_{vw}, P_{wu} as above. Concatenating these paths gives a closed walk of odd length and that yields a contradiction.

$(V, F + F_0)$ clearly contains an edge from each wheel and contains two edges of each face of G. So $H' = (V, E - (F \cup F_0))$ contains exactly one edge from each face, and has no isolated vertices. □

One can rephrase the Goddard-Henning conjecture in the dual graph as well.

Claim 29. *$G^* = (V^*, E^*)$ is the dual of a simple triangulated graph of order at least 4 if and only if G^* is a 3-regular 2-edge-connected planar graph of order at least 4.*

Proof. It is trivial that G^* is 3-regular if and only if its dual is triangulated.

It is also easy to see that a cut consisting of one edge corresponds to a loop edge in the dual, and a cut consisting of two edges corresponds to a pair of parallel edges.

Finally, by 3-regularity and using Euler's formula

$$f^* = m^* - n^* + 2 = 3n^*/2 - n + 2 = n^*/2 + 2,$$

where f^*, m^*, and n^* denote the number of faces, edges and vertices of G^*. Thus the dual of G^* has at least 4 vertices if and only if $4 \leq n^*/2 + 2$, i.e. G^* has at least 4 vertices. □

Statement 30. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph $H^* = (V^*, F^*)$ in G^* that has the following 2 properties.*

1. *H^* does not contain any odd cut of G^*.*

2. *For every face f of G^*, H^* contains an edge e not on f that has at least one endpoint on f. We say that e leaves the face f.*

Claim 31. *Statement 30 is equivalent with Statement 22.*

Proof. We show that given a subgraph $H = (V, F)$ that meets the requirements of 22, the edges corresponding to F in the dual of G form a subgraph H^* required by 30, and vice versa. It may be worth noting that the defined H^* is not necessarily the same as the dual graph of H.

It follows from the fact that cycles of a planar graph correspond to minimal cutsets in the dual graph, that H is bipartite if and only if H^* does not contain any odd cut of G^*.

Moreover, an edge from a wheel defined by v in G, corresponds to an edge that leaves the face that corresponds to v in the dual graph of G. Hence H contains at least one edge from each wheel if and only if for every face of G^*, H^* contains at least one edge that leaves that face. □

Statement 32. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph that has the following 2 properties.*

1. *It intersects every odd cut of G^*.*

2. *For every face f of G^*, it does not contain all the edges leaving f.*

Claim 33. *Statement 30 holds if and only if Statement 32 holds.*

Proof. If H^* meets the requirements of either of the statements, the complementer subgraph in G^* meets the requirements of the other. □

A 2-factor of a graph $G = (V, E)$ consists of disjoint cycles covering V. We can formulate a sufficient condition for the Goddard-Henning conjecture with the help of 2-factors. The motivation for such a reformulation is the fact that the existence of 2-factors in which some cycles are not allowed is a well-studied area in graph theory.

Statement 34. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a 2-factor not containing any of the faces.*

Claim 35. *If Statement 34 holds, then 30 holds.*

Proof. Let $H^* = (V^*, F^*)$ be the 2-factor containing none of the faces of G^*.

Every cut of G^* has an even number of common edges with every cycle in H^*. Therefore H^* does not contain any odd cuts of G^*.

Let $f = v_1 v_2 \dots v_l$ be a face of G^*. As F^* does not contain f, there must exist a v_i such that $v_i v_{i+1} \notin F^*$. Moreover, every vertex has degree 2 in H^*, so there is an edge starting from v_i that leaves f. □

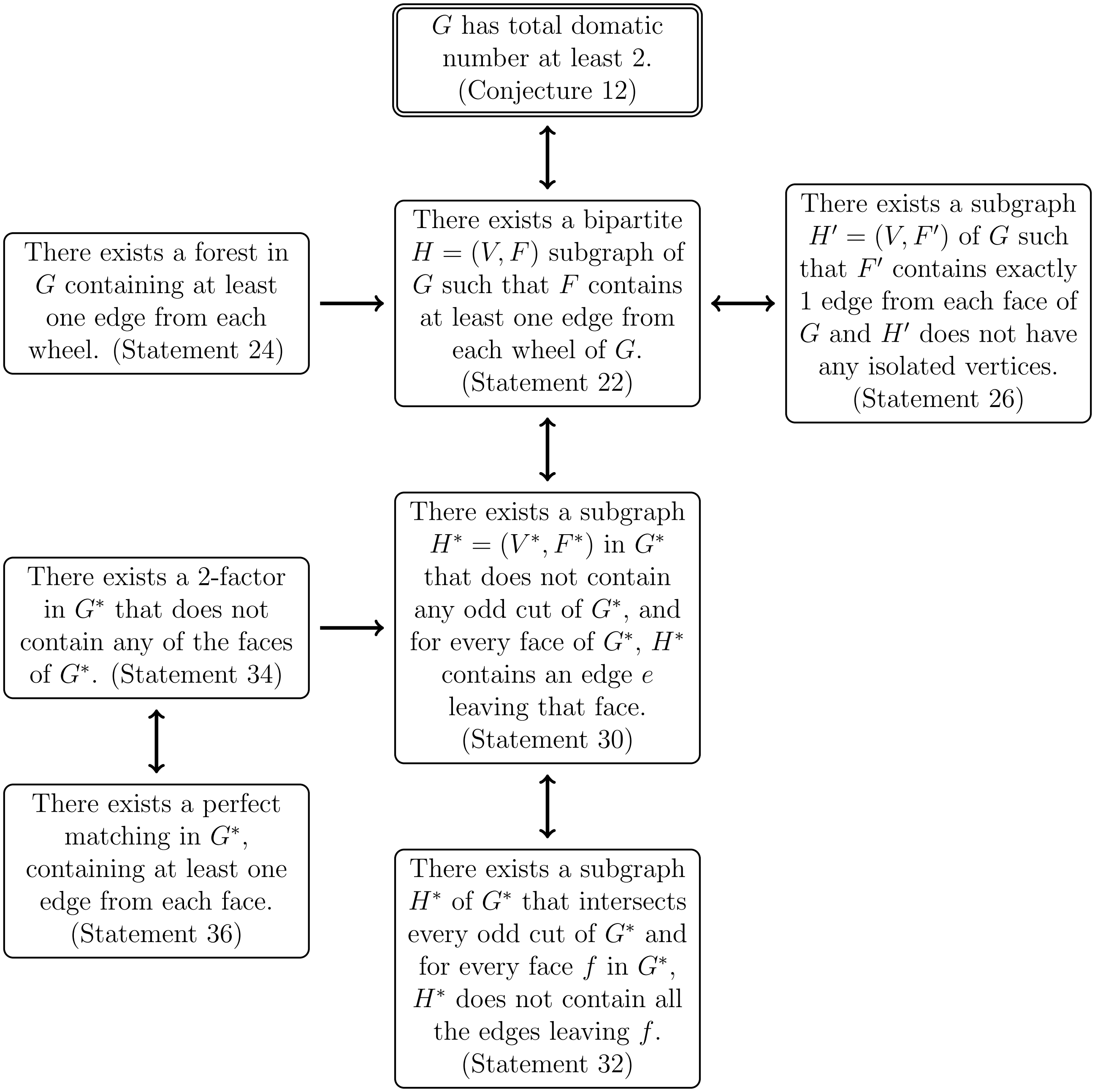
Statement 34 can easily be converted into a statement about perfect matchings.

Statement 36. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a perfect matching containing at least one edge from each face.*

Claim 37. *Statement 34 holds if and if Statement 36 holds.*

Proof. As G^* is 3-regular, a subgraph is a 2-factor if and only if the complementer subgraph is a perfect matching. Clearly, a subgraph contains none of the faces if and only if the complementer subgraph does contain at least one edge from each face. $\square$

The following figure summarizes the statements of this section. For this, let G be a simple triangulated planar graph of order at least 4 and let G^* denote its dual. Note, that by Claim 29, G^* is a 3-regular 2-edge-connected planar graph of order at least 4.


G has total domatic number at least 2. (Conjecture 12)
There exists a forest in G containing at least one edge from each wheel. (Statement 24)
There exists a bipartite $H = (V, F)$ subgraph of G such that F contains at least one edge from each wheel of G. (Statement 22)
There exists a subgraph $H' = (V, F')$ of G such that F' contains exactly 1 edge from each face of G and H' does not have any isolated vertices. (Statement 26)
There exists a 2-factor in G^* that does not contain any of the faces of G^*. (Statement 34)
There exists a subgraph $H^* = (V^*, F^*)$ in G^* that does not contain any odd cut of G^*, and for every face of G^*, H^* contains an edge e leaving that face. (Statement 30)
There exists a perfect matching in G^*, containing at least one edge from each face. (Statement 36)
There exists a subgraph H^* of G^* that intersects every odd cut of G^* and for every face f in G^*, H^* does not contain all the edges leaving f. (Statement 32)

Chapter 3

Proofs for special cases of the Goddard-Henning conjecture

3.1 Outerplanar and Hamiltonian graphs

Nagy [1] showed that the conjecture of Goddard and Henning holds for Hamiltonian graphs. For this, he characterized the 2-coupon colorable maximal outerplanar graphs first.

Definition 38. *A graph is outerplanar if it has a planar drawing for which all vertices belong to the outer face. A maximal outerplanar graph is a simple outerplanar graph such that adding any edge results in a non-outerplanar graph.*

Remark 39. *The outer face of a maximal outerplanar graph is a Hamiltonian cycle.*

In order to provide the mentioned characterization, we need to introduce a few notions first.

Definition 40. *Let G be a maximal outerplanar graph of order $n \geq 3$. The sun graph $M(G)$ of G is obtained by gluing a triangle to each edge of the outer face.*

Remark 41. *$M(G)$ is a maximal outerplanar graph with $2n$ vertices, from which n has degree 2.*

Remark 42. *If G has an odd number of vertices, then $M(G)$ does not have two disjoint total dominating sets, as in a 2-coupon coloring of $M(G)$ the vertices of G must have alternating colors. The graph on figure 2.2 is the sun graph of the BDE triangle.*

Definition 43. *A vertex v of a maximal outerplanar graph is called a central vertex if the following conditions hold.*

1. *$deg(v) \geq 3$*

2. *Every neighbor of v has degree at least 3.*

3. *For every u, w neighbors of v the length of the uw path on the outer face not containing v is divisible by 4.*

Claim 44. *The outer face of a maximal outerplanar graph does not contain two consecutive central vertices.*

Proof. Suppose there exists an uv edge on the outer face such that u and v are central vertices. Because of the maximality of the graph, there exists a uvw triangle. Index the vertices along the outer face form $v = v_1$ to $u = v_n$. Suppose $w = v_i$. From the centrality of v it follows that $i \equiv 2 \pmod 4$. On the other hand, u is also a central vertex, hence $i \equiv 1 \pmod 4$, a contradiction. $\square$

Definition 45. *A generalized sun graph is a maximal outerplanar graph of order $n \equiv 2 \pmod 4$ such that the number of degree 2 vertices plus the number of central vertices is $n/2$.*

Remark 46. *Every second vertex of the outer face in a generalized sun graph is either central or has degree 2.*

The key theorem is the following.

Theorem 47. *Let G be a maximal outerplanar graph of order at least 4. G admits 2 disjoint total dominating sets if and only if G is not a generalized sun graph.*

For proving this theorem, we need some observations about generalized sun graphs.

Definition 48. *Let G be a maximal outerplanar graph and $i \geq 2$. We say that a uv edge is a chord of length i if there is a uv path of length i on the outer face. (This means that if uv is a chord of length i, then it is also a chord of length $n - i$.)*

Lemma 49. *A maximal outerplanar graph of order $n \geq 3$ has a chord of length 2.*

Proof. The statement is trivial for $n = 3$. Suppose that $n \geq 4$ and let uv be a chord of minimal length. By the maximality of the graph there exists a uvw face, where w is on the shorter uv path of the outer face. If uv is not a chord of length 2, then uw or vw is a chord of length less than the length of the uv chord. $\square$

Lemma 50. *A maximal outerplanar graph of order $n \geq 5$ has a chord of length 3 or 4.*

Proof. Let uv be a chord of minimal length among chords of length at least 3. By the maximality of the graph there exists a uvw face, where w is on the uv path on the outer face that defines the length of the chord. If on this path w has a distance bigger than 2 from either u or v, than uw or vw is a chord contradicting the minimality of uv. Thus, the length of the uv path is at most 4. $\square$

Lemma 51. *If G is a maximal planar graph of order $n \geq 7$, then there exists a bounded face, such that the deletion of this face divides G into three graphs with the following properties.*

1. *At most one of the three graphs has more than 3 bounded faces.*

2. *At least one of the three graphs has 2 or 3 faces.*

Proof. For $n \leq 11$, the statement is easy to verify.

For $n > 11$ delete the faces sharing 2 common edges with the unbounded face. Then in the remaining graph G_1 delete the faces that now have 2 common edges with the unbounded face.

We claim that the remaining graph G_2 is not empty. G has $m = \frac{3(f-1)+n}{2}$ edges, where f denotes the number of faces. Then, by Euler's formula, $n = f + 1$, hence G has at least 11 faces. In the first step at most $n/2$ faces are deleted, and in the second step at most $|G_1|/2$ faces are deleted. Thus at most $3(f+1)/4$ faces are deleted. $3(f+1)/4 \leq f - 2$ if $f \geq 11$.

Finally, choose a face f_0 from the remaining graph G_2, that has 2 common edges with the unbounded face of G_2. We claim that f_0 has the desired properties. f_0 has at most one neighboring face in G_2, and one or two neighboring faces f_1 and maybe f_2 outside of G_2. Both of f_1 and f_2 has at most two neighboring faces outside of G_1, and at least one of f_1 and f_2 has at least one neighboring face outside of G_1. □

Proof of Theorem 47. First we show that generalized sun graphs do not have 2 disjoint total dominating sets. The proof goes by induction on the number of vertices $n = 4k + 2$.

For $k = 1$ there is only one generalized sun graph and it does not admit 2 disjoint total dominating sets. (Shown on figure 2.2.)

Suppose $k \geq 2$ and G is a generalized sun graph of order $4k + 2$. Index the vertices along the outer face from v_1 to v_{4k+2}, such that every vertex with an odd index is central or has degree 2. Let c be a 2-coloring of the graph. We show that c cannot be a 2-coupon coloring. The cardinality of the vertices implies that there must be two consecutive vertices v_{2i} and v_{2i+2} with the same color (say white). If v_{2i+1} has only white neighbors, then this coloring is not a 2-coupon coloring. So suppose v_{2i+1} has a black neighbor v_j. In this case v_{2i+1} is a central vertex. The $v_{2i+1}v_j$ edge divides the graph into two parts ($v_{2i+1}v_j$ is an edge in both graphs). Both of these graphs are generalized sun graphs, as v_{2i+1} either remains a central vertex or becomes a vertex of degree 2 in these smaller graphs, whereas other central vertices remain central vertices. By induction, the restriction of c is not a 2-coupon coloring in either of the smaller graphs. If there is a vertex v_l with a monochromatic neighborhood in one of the smaller graphs and $l \neq 2i+1$, $l \neq j$, then v_l has the same neighborhood in G, hence all its neighbors are from the same color class. v_{2i+1} cannot violate the condition, as it was chosen in a way that it has both a black and a white neighbor in both graphs. Thus the only remaining case is when v_j has a monochromatic neighborhood in both graphs. But in this case all of its neighbors are from the same color class as v_{2i+1}, so it has a monochromatic neighborhood also in G.

Now we show that if a graph G of order n is not a generalized sun graph then it does have two disjoint total dominating sets.

If $n \equiv 0 \pmod 4$, then it is easy to find a 2-coupon coloring: color the vertices along the boundary of the outer face by repeating the pattern $BBWW$.

If $n \equiv 1 \pmod 4$, then the same coloring method works, if you start the coloring from the right vertex. By lemma 49 there exists a chord uv of length 2. Alternating colors in pairs starting from v does the job. (See Figure 3.1.)

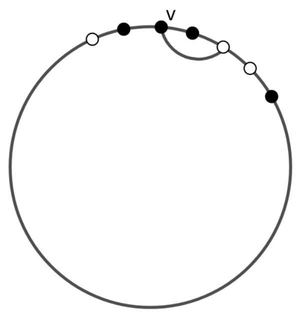


Figure 3.1: Coloring an outerplanar graph of order $4k + 1$

If $n \equiv 3 \pmod 4$, then start the coloring from a vertex next to v. (See Figure 3.2.)

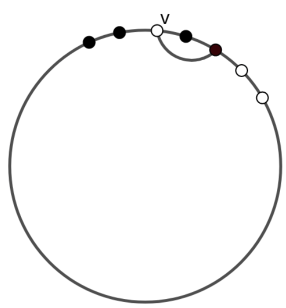


Figure 3.2: Coloring an outerplanar graph of order $4k + 3$

Suppose $n \equiv 2 \pmod 4$. We show by induction that if G does not have 2 disjoint total dominating sets, then it is a generalized sun graph. The case $k = 1$ is easy to check.

If G has a chord uv of length 3, then uv divides the graph into two parts: G_1 of order 4 and G_2 of order $4k$. By alternating colors in pairs one can obtain 2-coupon colorings of G_1 and G_2, where both u and v are colored to black in both graphs.

If G does not have a chord of length 3, then by Lemma 50 there exists a chord uv of length 4. Then uv divides G into two graphs. One of them must be the maximal outerplanar graph G_5 of order 5. u and v must be the degree 3 vertices of G_5, as otherwise there would be a chord of length 3 in G. Note that in a 2-coupon coloring u and v must have the same colors in order to create a proper neighborhood for the degree 2 vertices of G_5. Consider the face uvw, where w is not in G_5. The deletion of this face divides G into 3 graphs: G', G'' and G_5. (It might be that G' or G'' is degenerated in the sense that it consists only of one edge.) Let G' be the graph containing u and w. Without loss of generality we may assume that $|G''| \leq |G'|$.

Choose the uv chord in a way, such that $|G'|$ is minimal. We may assume that G' has at most 3 faces by Lemma 51, and thus $|G'| \leq 5$. Thus, there are 4 cases depending on the size of G'.

- Case 1: $|G'| = 2$. In this case $G''| = 4k - 2$. If G'' has a 2-coupon coloring, then it can easily be expanded to a 2-coupon coloring of G. If G'' does not have a 2-coupon coloring, then it is a generalized sun graph by induction. If v is a degree 2 or central vertex in G'', then color the vertices of G'' by alternating colors in pairs, starting with white from w, but color v to black. Let x denote the vertex before v. (See Figure 3.3.) This way, only v can

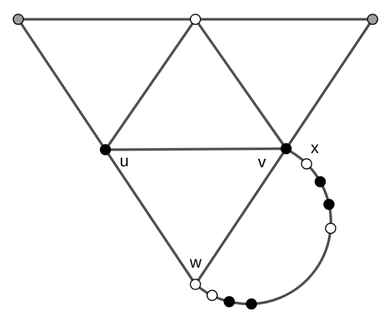


Figure 3.3: Case 1

have a monochromatic neighborhood in G'': if v has degree 2, then wx is an edge, and if v is central, then v and x have a common neighbor in G'' and v has only white neighbors. G_5 can be colored to provide v the missing color.

If v is neither a degree 2 vertex nor a central vertex, then w is. In this case w is a central vertex in G and thus G is a generalized sun graph.

- Case 2: $|G'| = 3$. In a 2-coupon coloring u and v must have the same color, whereas u and w must have different colors. Let H be the graph obtained from G by deleting G_5 and identifying uw with vw. (See 3.4) G is 2-coupon colorable if and only if H is 2-coupon colorable. On the other hand, G is a generalized sun graph if and only if H is a generalized sun graph.

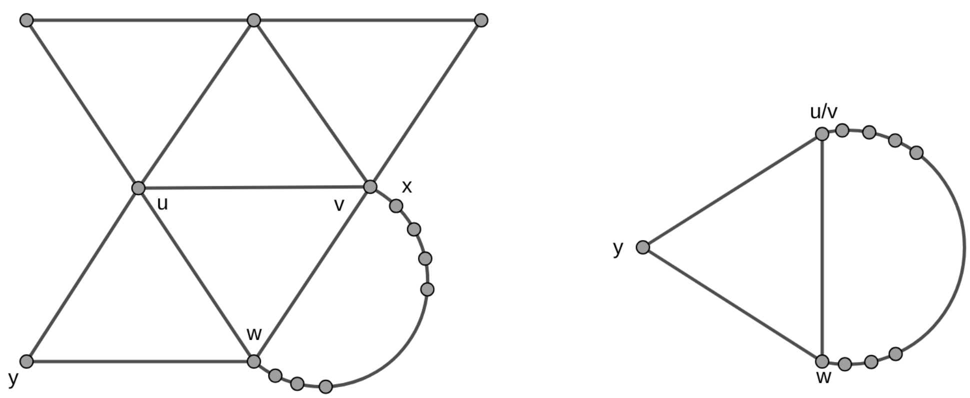


Figure 3.4: Case 2: G (left) and H (right)

- Case 3: $|G'| = 4$. If $|G'| = 4$, then uw is a chord of length 3, and that is a contradiction.
- Case 4: $|G'| = 5$. u and w must be the degree 3 vertices in G', otherwise we could find a chord of length 3. In a 2-coupon coloring u and w must have the same color. Let H be the graph obtained from G by deleting G_5 and identifying uw with vw. (See 3.5) G is 2-coupon colorable if and only if H is 2-coupon colorable. On the other hand, G is a generalized sun graph if and only if H is a generalized sun graph.

$\square$

Remark 52. *With a slight modification of the proof it can be shown that the vertices of a generalized sun graph cannot be colored in a way that every degree* 2 *or central vertex has neighbors from both color classes.*

As mentioned earlier, based on this theorem Zoltán Lóránt Nagy [1] also showed that the total dominating number of Hamiltonian triangulated graphs is at least two. We still need a Lemma for proceeding with the proof.

Lemma 53. *If G is a generalized sun graph of order $4k+2$, then there exist at most $k-1$ chords incident to a central vertex.*

Proof. The proof goes by induction. If $k = 1$, then there exists only one generalized sun graph of order $4k+2$, and it does not have any central vertices. Suppose $k > 1$. We are done, if there is no chord incident to a central vertex. Otherwise, let uv be such a chord of minimal length, where v is central. uv divides the graph into two generalized sun graphs. In both of these graphs, v is either

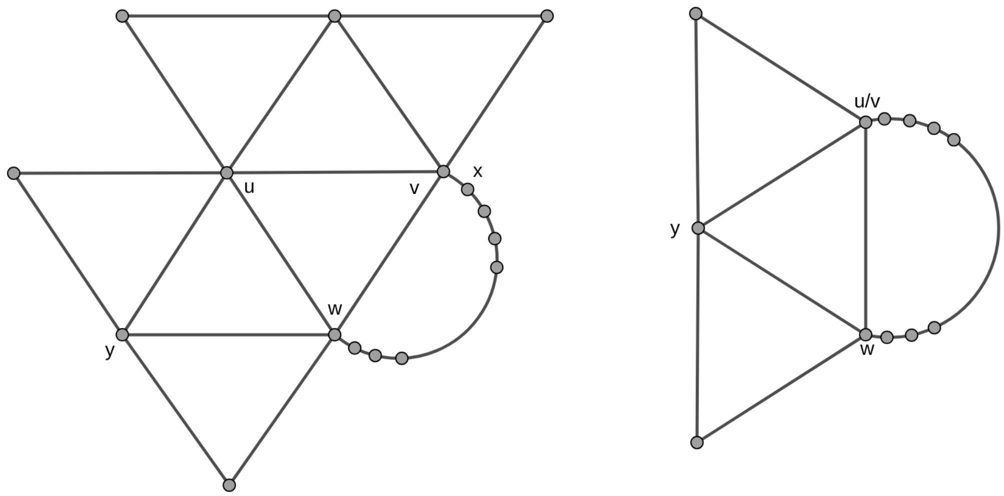


Figure 3.5: Case 4: G (left) and H (right)

a degree 2 vertex or a central vertex. Hence all the chords incident to a central vertex in G must be either uv or a chord with the same property in one of the smaller graphs. By the minimality of uv, one of these graphs does not have chords incident to a central vertex. The other graph is of order at most $4(k-1)+2$, thus by induction it contains at most $k-2$ chords incident to a central vertex. □

Corollary 54. *In a generalized sun graph the number of central vertices is less than the number of degree* 2 *vertices.*

Proof. Let G be a generalized sun graph of order $4k+2$. The number of central vertices is at most $k-1$ by the previous lemma. The number of degree 2 or central vertices is $2k+1$. □

Theorem 55. *Every triangulated graph with a Hamiltonian circle admits* 2 *disjoint dominating sets.*

Proof. A Hamiltonian triangulated graph G can be obtained by identifying the Hamiltonian cycle of two maximal outerplanar graphs G_1 and G_2. If at least one of these outerplanar graphs is 2-coupon colorable, then the same coloring is a 2-coupon coloring of G. By Theorem 47 we are done if at least one of these graphs is not a generalized sun graph.

Suppose that G_1 and G_2 are generalized sun graphs. If the union of degree 2 vertices and central vertices is the same in G_1 and G_2, then by Corollary 54 there exists a vertex with degree 2 in both graphs. Then this vertex is a degree 2 vertex in G and that is a contradiction, as in a triangulated planar graph every degree must be at least 3.

Assume that each vertex is a degree 2 or central vertex in G_1 or G_2. Index the vertices along the Hamiltonian cycle from v_1 to v_{4k+2}. We claim that there exists an index i such that v_i is a degree 2 vertex in G_1 and v_{i+3} is a degree 2 vertex in G_2. (If $i+3$ is bigger than $4k+2$, we take $v_{i+3-(4k+2)}$.) Let $I_1 = \{i|deg_{G_1}(v_i) = 2\}$, $I_2 = \{i|deg_{G_2}(v_i) = 2\}$ and $J = \{i+3|i \in I_1\}$. By Corollary 54 the cardinality of all of these sets is at least $k+2$, hence there exists an index i such that $i+3 \in I_2 \cap J$, proving the claim.

Finally, we define a 2-coupon coloring of G as follows. Let v_i be a vertex as above. Color the vertices by alternating colors in pairs starting from v_{i+2}. (See Figure 3.6.) It is clear that all the vertices apart from v_{i+1} and v_{i+2} have neighbors in both color classes. However, v_i is a degree 2 vertex in G_1, hence $v_{i-1}v_{i+1}$ is an edge in G_1. Similarly, $v_{i+2}v_{i+4}$ is an edge in G_2. These two edges ensure that v_{i+1} and v_{i+2} also have neighbors in both color classes. $\square$

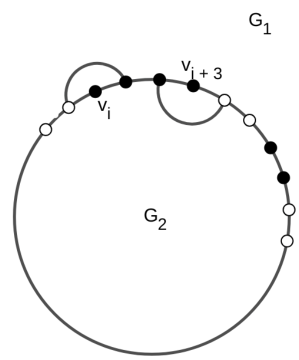


Figure 3.6: Coloring a Hamiltonian graph

Remark 56. *Whitney [12] proved that each triangulated planar graph without separating triangles is Hamiltonian. Helden [13] strengthened this statement by proving that each triangulated planar graph with at most five separating triangles is Hamiltonian.*

With the help of these theorems, we can also say something about graphs with another kind of 2-factors.

Claim 57. *Let G be a simple triangulated graph of order $n \geq 4$. If G has a 2-factor with none of its cycles having length congruent to 2 modulo 4, then the total dominating number of G is at least 2.*

Proof. If the 2-factor consists only of one cycle, then Theorem 55 proves the claim. Suppose that there are at least two cycles in the 2-factor. First note that by alternating colors in pairs, any cycle of length not congruent to 2 modulo 4 can be colored in a way, such that there is at most one vertex with a monochromatic neighborhood. (See Figures 3.1, 3.2.)

Contract each cycle to a single vertex. G is connected, hence there exists a tree T in the contracted graph. Choose T to have a minimal number of degree 1 vertices. Let $E(T)$ denote the edges of the original graph that were mapped to T. We show a 2-coupon coloring in the subgraph defined by the union of the 2-factor and $E(T)$. Choose a root node r from the degree 1 vertices in T.

We color the cycle C_0 corresponding to r first. Choose a vertex v in C_0, such that there exists a uv edge in $E(T)$. We color C_0 such that only v may have a monochromatic neighborhood and assign u the missing color.

After this, we iteratively color a child of a cycle that is already colored. We start by coloring cycles that do not correspond to leaves in T. Suppose that C_1 is a cycle like that, let C_2 be a child of C_1, and v_1v_2 be the edge in $E(T)$ such that $v_1 \in C_1$ and $v_2 \in C_2$. Let c_1 be a coloring of C_1 such that only v_1 may have a monochromatic neighborhood. There might be a vertex (but only one) in C_1 that already has a color, so flip the colors of c_1 if necessary. Color v_2 in a way that provides v_1 the missing color.

Now we color cycles that correspond to leaves, but have at least 4 vertices. Let C_l be a cycle like that. By Theorem 47 there exists a 2-coupon coloring c_l of C_l. Again, if there is a vertex in C_l that already has a color, then we might need to flip the colors of c_l.

Finally, we need to color cycles corresponding to leaves of T and having only 3 vertices. Let $u_lv_lw_l$ be a cycle like that, where v_l is the only vertex that may already have a color. Suppose it is colored to black. There exists a face $v_lw_lx_l$ where $x_l \neq u_l$. If x_l is colored to black, then color w_l and u_l to white. If x_l is colored to white, then color w_l to white, and u_l to black. The only remaining case is when x_l does not have a color yet. In this case, there must be a face $x_ly_lz_l$ corresponding to a leaf of T. These two leaves have a closest common ancestor t. t is of degree at least 2 in T, hence $t \neq r$, so t must have degree at least 3. By adding the edge corresponding to v_lx_l to T and removing the first edge of the tv_l path, we would get a tree T', where T' would have fewer degree 1 vertices then T has. This contradicts to the choice of T. □

3.2 Graphs without low-degree vertices

In this section we will show that the Goddard-Henning conjecture holds for graphs without too many low-degree vertices. For proving this, we will use a theorem about coloring hypergraphs. First, let us introduce the necessary definitions.

Definition 58. *Let H be a hypergraph. The incidence graph of H is a bipartite graph with one of its classes corresponding to the vertices of H, and the other class corresponding to the hyperedges of H. ve is an edge in the incidence graph if and only if the hyperedge e contains vertex v in H.*

Definition 59. *A hypergraph is called planar, if its incidence graph is planar.*

Definition 60. *A vertex coloring of a hypergraph is proper if all its hyperedges contain vertices from both color classes.*

The following theorem is from Zdenek Dvorák, Daniel Král [14].

Theorem 61. *Let $H = (V, E)$ be a planar hypergraph with at most 2 hyperedges of size 2. Then H has a proper vertex coloring with two colors.*

Proof. We may assume that H has exactly 2 hyperedges of size 2. Otherwise add one or two new hyperedges of size 2. A proper coloring of the resulting hypergraph is also a proper coloring of the original one.

We define another planar graph based on the incidence graph of H as follows. For each hyperedge e, delete the corresponding vertex from the graph, and add edges between the vertices contained in e in a way that it results in a circle. The bounded faces of the resulting graph $G_H = (V, F)$ correspond to the hyperedges of H. (See Figure 3.7.) After this, triangulate the faces that have more than 3 vertices.

Let c_4 be a proper 4-coloring of the resulting graph with colors $\{1, 2, 3, 4\}$. By permuting the color classes if necessary, we may assume the followings.

1. If the two hyperedges of size 2 are not disjoint, i.e. they are $\{u, v\}$ and $\{u, w\}$, then $c_4(u) = 1$, and $c_4(v), c_4(w) \in \{3, 4\}$.

2. If the two hyperedges of size 2 are disjoint, i.e. they are $\{u, v\}$ and $\{x, y\}$, then $c_4(u), c_4(x) \in \{1, 2\}$ and $c_4(v), c_4(y) \in \{3, 4\}$.

We define a 2-coloring of G_H as follows.

$$c_2(v) = \begin{cases} 1, & \text{if } c_4(v) = 1 \text{ or } c_4(v) = 2 \\ 2, & \text{if } c_4(v) = 3 \text{ or } c_4(v) = 4 \end{cases}$$

We claim that c_2 is a proper 2-coloring of H. The hyperedges of size 2 have vertices in both color classes due to our assumptions on c_4. For the other hyperedges there exists a face in G containing the vertices of the hyperedge. As c_4 was a proper 4-coloring of the triangulated graph, each face of G of size at least 3 has vertices from 3 or 4 color classes. □

Now we are ready to prove our claim about graphs without too many low-degree vertices.

Claim 62. *Let G be a simple triangulated graph. If there are at most two vertices of degree at most 4, then G has a 2-coupon coloring.*

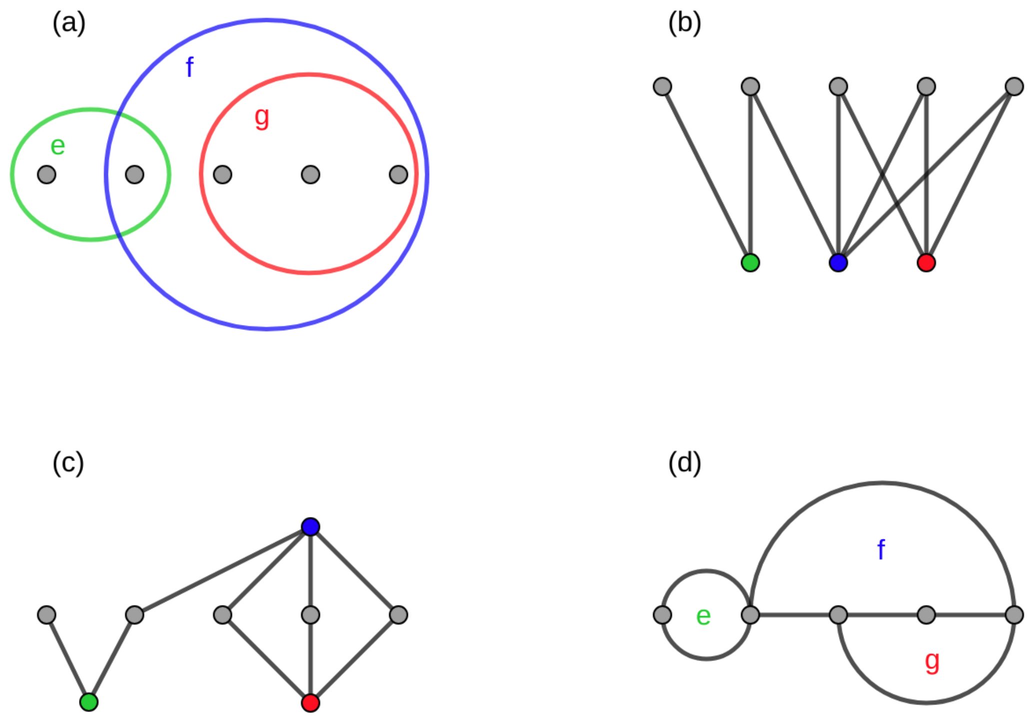


Figure 3.7: A hypergraph (a), its incidence graph (b), a planar embedding of the incidence graph (c) and the graph G_H (d)

Proof. By Claim 29 the dual G^* is a 3-regular 2-edge-connected planar graph. Thus, by Petersen's theorem there exists a perfect matching M in G^*. By deleting the edges corresponding to M from G, we get a graph G' such that all its faces contains 4 vertices. We call such graphs quadrangulated. Note, that for each vertex v, we deleted at most half of the vertices starting from v, thus there are at most two vertices in G' of degree 2 and none of the vertices has less than 2 neighbors.

By Lemma 27, G' is a bipartite graph. Let U and V be the two classes of G'. G' is the incidence graph of two hypergraphs: let H_1 be the hypergraph defined on the vertex set U with hyperedges V, and H_2 be the hypergraph defined on the vertex set V with hyperedges U. By Theorem 61, H_1 and H_2 has proper 2-colorings. Take the union of these colorings c. I.e. on the vertices of U, c is defined by a proper coloring of H_1, whereas on the vertices of V, c is defined by a proper coloring of H_2. c is a 2-coupon coloring of G'. □

Remark 63. *It is not true that every simple quadrangulated graph has a 2-coupon coloring. See Figure 3.8 for an example.*

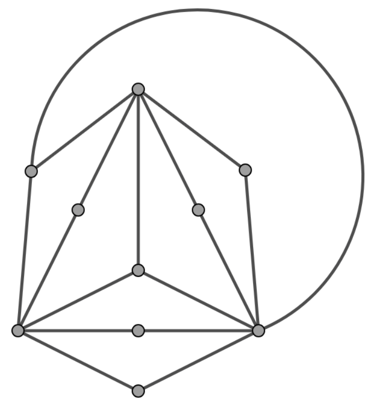


Figure 3.8: A quadrangulated graph without two disjoint dominating sets

Goddard and Henning [2] conjectured the following stronger statement about planar triangulations without degree 3 vertices.

Conjecture 64. *If G is a simple triangulated graph with all its vertices having degree at least 4, then G admits three disjoint total dominating sets.*

It is also worth noting that having total dominating number at least 2 can be translated into a statement about the so-called neighborhood hypergraph. The neighborhood hypergraph of G is a hypergraph defined on the vertex set of G, where for each vertex v, there is a hyperedge containing the neighboring vertices of v. G has total dominating number at least 2 if and only if the neighborhood hypergraph has a proper 2-coloring. In other words, it is equivalent to the statement that the hyperedges possess property B. Its relevance is that property B has been studied extensively.

3.3 Barnette's conjecture

A conjecture of Barnette [6] is the following.

Conjecture 65. *Every 3-connected 3-regular bipartite planar graph is Hamiltonian.*

By Claim 20 and Lemma 27, if Barnette's conjecture holds, then the Goddard-Henning conjecture also holds for Eulerian triangulations. In this section, we give a brief overview of its history and show two equivalent forms of the conjecture.

Barnette's conjecture originates from Tait's conjecture. Tait [8] conjectured that every 3-connected 3-regular planar graph is Hamiltonian. This is stronger than Barnette's conjecture, as it does not requires the graph to be bipartite. Tutte [9] constructed a counterexample for Tait's conjecture, and conjectured a similar statement. Namely, that every 3-connected 3-regular bipartite graph is Hamiltonian. Again, this is stronger than Barnette's conjecture, as it does not require the graph to be planar. Tutte's conjecture was also disproven: Horton [7] found a counterexample.

Kelmans [10] showed that Barnette's conjecture is equivalent to the statement that for every two edges on the same face of a 3-connected 3-regular bipartite planar graph, there exists a Hamiltonian cycle containing both edges. Note, that Barnette's conjecture trivially follows from this statement.

Florek [11] showed that Barnette's conjecture can be rephrased using the dual graph. Namely, the following statement is equivalent with Barnette's conjecture. The vertices of the dual of every 3-connected 3-regular bipartite planar graph can be partitioned into two subsets whose induced subgraphs are trees.

Chapter 4

Algorithm for generating all planar triangulations

TODO: write some stuff about the motivation for generating these triangulations.

Avis [15] gave an algorithm for generating all 3-connected triangulated disks without too many repetitions. For phrasing it more precisely, we call $(G, v_1, \dots v_r)$ an r-rooted triangulation, if $v_1, \dots, v_r$ are vertices of a face in G, and G can be embedded in the plane with the following properties.

1. The outer face of G is formed by $v_1, \dots, v_r$, and they appear in this clockwise order.

2. All the interior faces are triangles.

The algorithm generates all 3-connected r-rooted simple triangulations of order n exactly once. We suppose that $r < n$ for avoiding a trivial case.

Remark 66. *Every r-rooted triangulation has a unique embedding to the plane, in the sense that for every vertex v, the order of its neighbors is the same in all embeddings.*

Claim 67. *An r-rooted triangulation is 3-connected if and only if there does not exist an edge between two non-consecutive vertices of the outer face.*

Proof. TODO □

Corollary 68. *If $r = 3$, then every rooted triangulation is 3-connected.*

The idea of the algorithm is the following. First we construct an auxiliary graph $H = (T, F)$, where T is the set of 3-connected r-rooted planar triangulations of order n. Then we will show that H is connected. Finally, we perform a depth-first-search on H.

Definition 69. *We call an edge of a planar graph internal, if it is not on the outer face of the graph.*

Let t be an r-rooted triangulation and let uv be an internal edge of t. There exist two faces containing uv: uvx and uvy. We say that uv is a transformable edge if and only if xy is not an edge in t.

If uv is transformable, we refer to the following operation as flipping uv: delete the edge uv and add the edge xy to the graph. We denote the resulting graph with $flip(t, uv)$.

We define the edge set of H as follows. For $t_1, t_2 \in T$, $t_1t_2 \in F$ if and only if there exists a transformable edge e in t_1 such that it is not on the outer face, and the flipping of e results in t_2.

Remark 70. *H is a well-defined undirected graph.*

Proof. Suppose that t_2 can be obtained from t_1 by flipping uv by adding the edge xy. Note, that xy cannot be on the outer face of t_2. uv is not an edge in t_2, so xy is transformable, and flipping xy results in the graph t_1. □

For proving the connectivity of H, we need the following lemma.

Lemma 71. *Let v be a vertex in a 3-connected r-rooted triangulation G. Suppose that v is of degree at least 4 and has at least one neighbor on the outer face of G. If u_1, u_2, u_3, u_4 are consecutive neighbors of v such that u_1 is on the outer face an u_2 is not, then at least one of vu_3 and u_2u_3 is a transformable edge. Furthermore, the graph obtained by the transformation is also a 3-connected r-rooted triangulation.*

Proof. v_u3 in an internal edge bounding faces vu_2u_3 and vu_3u_4. if vu_3 is not transformable, then u_2u_4 is an edge. It is an internal edge, as u_2 does not lie on the outer face. Let xu_2u_4 and yu_2u_4 be the two faces bounded by u_2u_4. One of them must lie inside the triangle vu_2u_4, the other one must lie outside of it. (x or y might be v.) Thus, u_2u_4 is transformable. (See Figure 4.1.)

By Claim 67, it is enough to show for 3-connectivity, that the new edge has an internal vertex. If u_2u_4 is the new edge, then u_2 is an internal vertex. Otherwise, the new edge is xy. One of them lies inside the vu_2u_4, and hence it is internal. □

Theorem 72. *H is connected.*

Proof. The proof goes as follows. First, we fix an r-rooted triangulation t^*. Then for every $t \in T$, we construct a path starting from t. Finally, we show that each of these paths ends in t^*.

Define t^* as follows. $v_1, v_2 \ldots, v_r$ form the outer face of t^*. Connect v_{r+1} to all the vertices of the outer face. For $i \geq r+2$, put v_i on the $v_{r-1}v_rv_{i-1}$ face and connect it with all of these three vertices. (See figure 4.2.)

Algorithm 73 Construct path

```
let p be an array of length n /* p[i] will be the i^th vertex of the path.
p[0] := t
while true do
   j := j + 1
   i := 1
   while deg(v_i) = 3 and i ≤ r − 2 do
      i := i + 1
   end while
   if i ≤ r − 2 then
      /* The outer face of t differs from the outer face of t*.
      let v_{i−1}, u_2, u_3, u_4 be consecutive neighbors of v_i in counterclockwise order
      if u_2u_4 ∉ E then
         p[j] := flip(t, v_iu_3)
      else
         p[j] := flip(t, u_2u_4)
      end if
   else
      /* The outer face looks good, identify the possible v_{r+1}.
      let w be the vertex connected to v_1, ... v_{r−2}.
      a = v_1
      while w has exactly one neighbor b ≠ a not on the outer face of t do
         /* If w was the possible v_k, then b is the possible v_{k+1}.
         a = w
         w = b
      end while
      if deg(w) = 3 then
         return p
      else
         let v_r, u_2, u_3, u_4 be consecutive neighbors of w in counterclockwise order
         if u_2u_4 ∉ E then
            p[j] := flip(t, wu_3)
         else
            p[j] := flip(t, u_2u_4)
         end if
      end if
   end if
   t := p[j]
end while
```

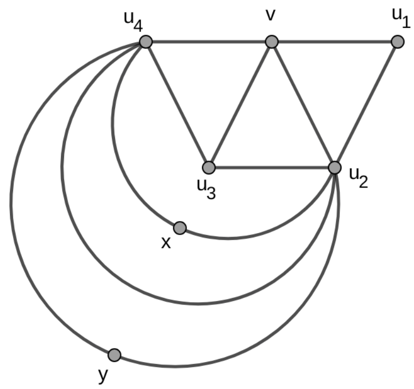


Figure 4.1: Lemma 71

Let $t = (G, v_1, \ldots v_r)$ be an r-rooted triangulation. We show now that Algorithm 73 stops in finitely many steps and constructs a path starting in t and ending in t^*. Note, that when the algorithm flips an edge, then it is really transformable.

We claim that after enough steps, $deg(v_1) = \cdots = deg(v_{r-2}) = 3$. Suppose that $deg(v_i) \neq 3$ for some $i \leq r-2$. Let i be the smallest such index. Then the algorithm takes v_{i-1}, u_2, u_3, u_4 such that they are consecutive neighbors of v_i in counterclockwise order. If $v_i u_3$ is transformable, then the degree of v_i decreases and for every $j < i, deg(v_j)$ does not change. Suppose that $v_i u_3$ is not transformable. Then let $x u_2 u_4$ and $y u_2 u_4$ be the two faces bounding $u_2 u_4$. Suppose $y = v_j$ for some $j < i$. From $deg(v_j) = 3$ it follows, that u_4 must be v_{j-1}. On the other hand, by 3-connectivity, if u_4 is an external vertex, then it is v_{i+1} and hence $i = r-1$, which is a contradiction. So adding xy to the graph does not change the degree of v_j for any $j < i$. In the next step of the algorithm, $v_i u_3$ is transformable. So we showed, that after enough steps, $deg(v_i) = 3$ for every $i \leq r-2$.

If $v_1, \ldots v_{r-2}$ are all degree 3 vertices, then there exists a vertex w such that w is connected to $v_1, \ldots v_r$. If w has more than one internal neighbor, than the algorithm chooses v_r, u_2, u_3, u_4 to be consecutive neighbors of w in counterclockwise order. If wu_3 is transformable, then the algorithm deletes wu_3 from the graph and adds $u_2 u_4$ to it. By executing this step, the degree of w decreases, and for $j \leq r-2$, $deg(v_j)$ does not change. The other case is when wu_3 is not transformable. Let x and y be the vertices of the two faces bounded by $u_2 u_4$, In this case the algorithm deletes $u_2 u_4$ from the graph and adds xy to the it. This does not change the degree of v_j for $j \leq r-2$. In the next step of the algorithm wu_3 will be transformable. Thus, after some steps, w will have only one internal neighbor b. It follows, that bv_r and bv_{r-1} are edges. With the same process, we achieve that b has at most 1 internal neighbor apart from v. It is easy to see, that the changes does not affect vertices outside of $bv_r v_{r-1}$. Thus, by iterating these steps we can achieve t^*. □

TODO: DFS,futasido

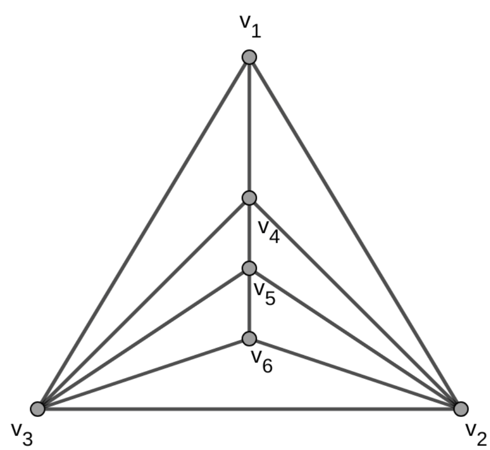


Figure 4.2: t^* for $r = 3$, $n = 6$

Bibliography

[1] Z. L. Nagy, *Coupon-coloring and total domination in Hamiltonian planar triangulations*, arXiv preprint arXiv:1708.01725.

[2] W. Goddard, M. A. Henning, *Thoroughly distributed colorings*, arXiv preprint arXiv:1609.09684.

[3] B. Zelinka, *Total domatic number and degree of vertices of a graph*, Mathematica Slovaca, Vol. 39 (1989), No. 1, 7–11.

[4] P. Heggernes, J. A. Telle, *Partitioning graphs into generalized dominating sets*, Nordic Journal of Computing. 5 (2), 128-142.

[5] H. Aram, S. M. Sheikholeslami, L. Volkmann, *On the total domatic number of regular graphs*, Transactions on Combinatorics 1, 45-51.

[6] D. W. Barnette, *Conjecture 5*, Recent progress in combinatorics (ed. W. T. Tutte), Academic Press, New York (1969) 343.

[7] J. A. Bondy, U. S. R. Murty, *Graph theory with applications*, New York: North Holland, pp. 61 and 242, 1976.

[8] P. G. Tait, *Listing's topologie*, Philosophical Magazine, 5th Series, 17: 30–46. Reprinted in Scientific Papers, Vol. II, pp. 85–98.

[9] W. T. Tutte, *On Hamiltonian circuits*, Journal of the London Mathematical Society, 21 (2): 98–101.

[10] A. K. Kelmans, *Constructions of cubic bipartite 3-connected graphs without Hamiltonian cycles*, Selected Topics in Discrete Mathematics: Proceedings of the Moscow Discrete Mathematics Seminar 1972–1990, American Mathematical Society Translations, Series 2, 158, pp. 127–140.

[11] J. Florek, *On Barnette's conjecture*, Discrete Mathematics, 310 (10-11): 1531–1535.

[12] H. Whitney, *A theorem on graphs*, Annals of Mathematics 32 (1931), 378-3 90.

[13] G. Helden, *Hamiltonicity of maximal planar graphs and planar triangulations*, These de doctorat, RWTH Aachen University.

[14] Z. Dvorák, D. Král, *On planar mixed hypergraphs*, Journal of Combinatorics 8 (1), 35-35.

[15] D.Avis, *Generating rooted triangulations without repetitions*, Algorithmica (1996) 16: 618-632