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About a conjecture of Goddard and Henning

Diploma Thesis

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Contents

Introduction

TODO

Chapter 1

Coupon coloring in arbitrary graphs

In this chapter we will examine the so-called total domatic number of graphs.

Let $G = (V, E)$ be a graph without isolated vertices.

Definition 1.0.1. *$S \subseteq V$ is a total dominating set if every vertex has a neighbor in S. The total domatic number of G is the maximum number of disjoint total dominating sets.*

Sometimes it is more convenient to look at total dominating sets as color classes.

Definition 1.0.2. *A coloring of the vertices is called a k-coupon coloring if every vertex has a neighbor from each color class. The coupon coloring number of G is the maximum k for which a k-coupon coloring exists. The coupon coloring number is denoted by $\chi_c(G)$.*

Remark 1.0.1. *We refer as proper colorings to colorings in the usual sense. I.e. colorings of the vertices such that for every vertex v, none of the neighbors of v has the same color as v.*

1.1 Complexity

It turns out that determining the total domatic number (or equivalently the coupon coloring number) of a graph is rather hard. Heggernes and Telle [4] showed that even determining whether the total domatic number of a graph is at least 2 is NP-complete. We prove this by showing that a variant of 3SAT is reducible to this question in polynomial time.

Definition 1.1.1. *An instance of the not-all-equal 3-satisfiability (NAE-3SAT) problem consists of a set C of clauses on a finite set X of Boolean variables, where each clause contains three literals. The question is whether there is a truth assignment for X that satisfies all the clauses in C such that each clause contains a false literal.*

Theorem 1.1.1. *NAE-3SAT is NP-complete.*

Proof. It can be checked in polynomial time whether a given truth assignment meets the requirement, so NAE-3SAT is in NP.

To prove NP completeness, we show first a reduction from 3SAT to NAE-4SAT. Let C be the set of clauses and X be the set of variables in an instance of 3SAT. Let $X' = X \cup y$, and $C' = \{(x_1 \vee x_2 \vee x_3 \vee y) \mid (x_1 \vee x_2 \vee x_3) \in C\}$. We claim that the NAE-4SAT problem defined by (C', X') is satisfiable if and only if the 3SAT problem defined by (C, X) is satisfiable. If the 3SAT formula is satisfied by a truth assignment then the same assignment with assigning the value false to y satisfies the NAE-4SAT problem. Now suppose a truth assignment satisfies the NAE-4SAT problem. If y has value false, then the same assignment of X satisfies the 3SAT problem. If y has value true, then changing every truth value in X to its opposite gives a truth assignment satisfying the 3SAT formula.

Finally, the reduction from NAE-4SAT to NAE-3SAT is by adding clauses $(x_1 \vee x_2 \vee z)$ and $(\bar{z} \vee x_3 \vee y)$ instead of each clause $(x_1 \vee x_2 \vee x_3 \vee y) \in C'$. $\square$

Theorem 1.1.2. *It is NP-complete to decide whether the total domatic number of a graph is at least* 2.

Proof. Given a partition of the vertices into 2 sets, it can be checked in polynomial time whether these sets are total dominating sets. So the problem is a member of NP.

For proving NP-completeness, we will show that NAE-3SAT is reducible to this problem in polynomial time. Let C be the set of clauses and X be the set of variables in an instance of NAE-3SAT. We can assume that every variable x appears in at least one clause. Otherwise we add a new clause containing x and $\bar{x}$ to the formula. Now we construct the corresponding graph G. For each variable x, introduce 3 vertices x_1, x_2, x_3, and 2 edges x_1x_2, x_2x_3. For each clause c, introduce a vertex c. If x is a literal in c, then add the edge cx_1 to the graph. If $\bar{x}$ is a literal in c, then add the edge cx_3.

Suppose G has a partition into 2 disjoint total dominating sets: T and F. Assign the value true for each variable x with $x_1 \in T$ and assign the value false otherwise. For any variable x, x_1 and x_3 are the only neighbors of x_2, so x_1 and x_3 must be in different sets of the partition. If c is a vertex corresponding to a clause, then it must have neighbors both in T and F, and so the literals in c cannot be all true nor false.

Suppose now that the variables have a truth assignment such that each clause contains both true and false literals. Define T and F as follows. Put all the vertices corresponding to clauses into T. For each variable x put x_2 into F. Furthermore, if true was assigned to x, then put x_1 into T, x_3 into F, and conversely otherwise. $\square$

Let us note that the constructed graph in the proof is always a bipartite graph.

Corollary 1.1.1. *It is NP-complete to decide whether the total domatic number of a bipartite graph is at least* 2*.*

1.2 Degree restrictions

A natural question is whether graphs with an appropriately big minimum degree always have a total domatic number of at least 2. Zelinka [3] showed that this is not the case.

Theorem 1.2.1. *For every $\delta \in \mathbb{Z}^+$ there exists a graph without* 2 *disjoint total dominating sets with minimum degree δ.*

Proof. We define a bipartite graph $G = (U, V; E)$ for arbitrary δ as follows. Let U be a set of cardinality $2\delta - 1$, and V be the set of all subsets of U consisting of δ elements. For all $u \in U, v \in V$ vertices $uv \in E$ if and only if v contains u. Clearly, each vertex of U has at least δ neighbors, and each vertex of V has exactly δ neighbors. Suppose that there exists a 2-coupon coloring. Then there exists a monochromatic $U_0 \in U$ containing δ vertices. Thus, there exists a $v \in V$ corresponding to U_0. That is a contradiction, since the neighborhood of v is U_0, and that is monochromatic. ☐

A natural next question might arise. The graph constructed above has a large number of vertices compared to the minimum degree. The other extreme is the case of complete graphs, where the minimum degree δ is $n - 1$. This is not an interesting case: if $n \geq 4$, then every subset U of the vertices is a total dominating set, if $|U| \geq 2$. We show a less strict but sufficient condition for the existence of 2 disjoint dominating sets: a lower bound for the minimum degree defined by the order of the graph. The next theorem is also from Zelinka [3]. We show here an easier proof than the original.

Claim 1.2.1. *Let $G = (V, E)$ be a graph of order n with minimum degree δ. Then every subset of V with at least $n - \delta + 1$ vertices is a total dominating set.*

Proof. Let $S \subseteq V$ be a set of cardinality at least $n - \delta + 1$ and $v \in V$ be an arbitrary vertex. v has at least δ neighbors, and $|(V - v) \cap S| \geq n - \delta$, so there must exist a vertex $u \in S$ adjacent to v. ☐

Corollary 1.2.1. *Let $G = (V, E)$ be a graph of order n with minimum degree δ. If $\delta \geq 1 + n/2$, then there exist* 2 *disjoint dominating sets in G.*

Proof. If $\delta \geq 1 + n/2$, then $2(n - \delta + 1) \leq n$, and therefore the total domatic number of G is at least 2. ☐

Remark 1.2.1. *Aram, Sheikholeslami and Volkmann [5] proved a lower bound for the total domatic number in terms of the order, the minimal and the maximal degree of the graph. Using this lower bound they proved that the domatic number of an* r*-regular graph is at least* $r/3ln(r)$*. By an easy calculation follows from this theorem that if a graph is* r*-regulat with* $r \geq 5$*, then the total domatic number is at least* 2*.*

Chapter 2

Introducing the Goddard-Henning conjecture

2.1 Formulate the conjecture

From now on we will focus on 2-coupon colorings and planar graphs. A conjecture of Goddard and Henning [2] is the following.

Conjecture 2.1.1. *If G is a simple triangulated planar graph of order at least 4, then the total domatic number of G is at least 2.*

Remark 2.1.1. *The simplicity of the graph is necessary. Suppose the graph on figure 2.1 has a 2-coupon coloring. Then A and C must have different colors, because they are the only neighbors of B. Similarly, C and E must have different colors, as well as E and A. That is a contradiction, since A, C and E form a triangular.*

Remark 2.1.2. *Allowing triangulated disks (i.e. planar graphs with at most one face greater than 3), the conjecture does not hold. For example, the graph on figure 2.2 does not have a 2-coupon coloring from similar reasons as the previous one. We will show later that this graph is a member of a bigger graph family without 2 disjoint dominating sets.*

2.2 Fast proofs for some special cases

There are some sufficient conditions known for having a total domatic number of at least 2. We will cover most of the known cases along the way. In this section we take a look at special cases with relatively easy proofs.

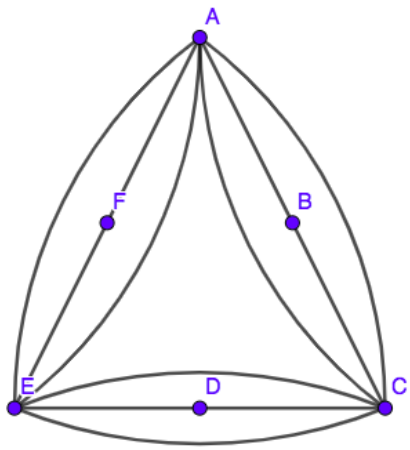

Figure 2.1: Simplicity is necessary

The first example is a graph family for which an easy induction shows that they are 2-coupon colorable.

Definition 2.2.1. *A graph is called a stacked graph if it can be constructed from a triangle by repeatedly putting a new vertex in a face and connecting it with the vertices on the boundary of that face.*

Remark 2.2.1. *Stacked graphs are triangulated.*

Claim 2.2.1. *Stacked graphs with at least* 4 *vertices are* 2*-coupon colorable.*

Proof. We can determine the colors of the vertices as the graph is constructed. The current coloring will maintain two following two properties.

1. It is a 2-coupon coloring of the current graph.

2. Every face has vertices from both color classes.

The construction of the graph starts with a simple triangle. Color two vertices of the triangle to black, and the remaining vertex to white. Color the vertex added to the graph in the first step to white. This coloring has the desired properties. When a vertex is inserted into a face, color the new vertex to white, if there is only one white vertex on the face's boundary, and black otherwise. This trivially maintains the desired properties. □

Goddard and Henning [2] established the conjecture for some cases. We show now three of them.

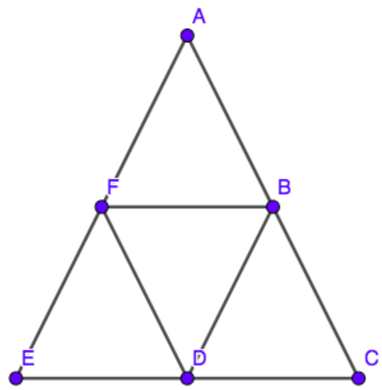

Figure 2.2: The conjecture does not hold for triangulated disks

Claim 2.2.2. *Let G be a a simple triangulated graph. If all the vertices of G have an odd degree, then there exists a coupon coloring with 2 colors.*

Proof. There exists a proper 4-coloring of the vertices. As every vertex v has an odd degree, there exists an odd circle in the neighborhood of v. Hence, in a proper 4-coloring v has neighbors from at least 3 color classes. This means that the union of any two color classes forms a total dominating set. □

Claim 2.2.3. *If G can be obtained from a triangulated graph H by putting a new vertex on every face and connecting them with the vertices of that face, then G it 2-coupon colorable.*

Proof. Take a proper 4-coloring on the vertices of H and define a 2-coloring by taking the union of $2-2$ color classes. The obtained coloring has the property that none of the faces is monochromatic. Color the added vertices to black if the face has only one black vertex, and color it to white otherwise. □

Claim 2.2.4. *Let G be a a simple triangulated graph of order at least 4. If the dual of G is Hamiltonian, then G admits two disjoint total dominating sets.*

Proof. Let C be the Hamiltonian graph in the dual graph. Color the vertices of G inside C to black, and the vertices outside of C to white. □

2.3 Variations on the conjecture

As a reminder: the Goddard-Henning conjecture states that every simple triangulated planar graph of order at least 4 has total domatic number at least 2. In this chapter we try to find equivalent statements to the conjecture, as well as (hopefully) slightly stronger statements. The motivation for this chapter is that even if the stronger statements are not true, they can be useful for proving the conjecture in special cases.

Definition 2.3.1. *Let G be a triangulated planar graph. For a vertex v, each triangle containing v has an edge not containing v. We call the circle consisting of these edges a wheel.*

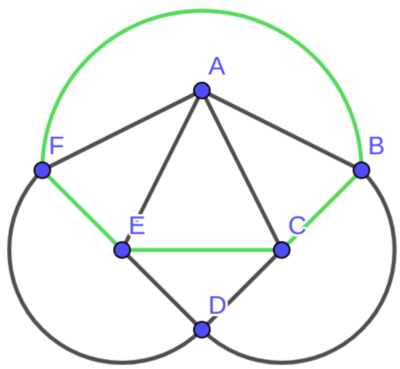

Figure 2.3: The green edges form the wheel defined by A

Statement 2.3.1. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a bipartite $H = (V, F)$ subgraph of G, such that F contains at least one edge from each wheel of G.*

Claim 2.3.1. *The Goddard-Henning conjecture holds if and only if Statement 2.3.1 holds.*

Proof. Let G be a triangulated graph. Suppose first that it has a 2-coupon coloring.

$$F = \{uv \in E \mid u \text{ and } v \text{ are in different color classes}\}$$

defines a bipartite subgraph of G that contains at least one edge from each wheel.

Now suppose that there exists bipartite subgraph that meets our requirement. Color the vertices in one of the classes to black, and the vertices in the other class to white. This is a 2-coupon coloring of the original graph. $\square$

Statement 2.3.2. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a forest in G containing at least one edge from each wheel.*

Remark 2.3.1. *If Statement 2.3.2 holds, then Statement 2.3.1 also holds.*

Statement 2.3.3. *Let $G = (V, E)$ be a simple triangulated graph of order at least 4. Then there exists a subgraph of $H' = (V, F')$ having the following two properties.*

1. *F' contains exactly 1 edge from each face of G.*
2. *There are no isolated vertices in H'.*

Lemma 2.3.1. *A connected planar graph is bipartite if end only if each of its faces have an even number of edges.*

Proof. Suppose that the graph is not bipartite and thus there exists a circle C of odd length. We show that then exists an odd face. The proof goes by induction on the number of faces in the inner side of C. If C is a face, then we are done. If C is not a face, then there exists a face f in the inner side of C having at least one common edge with C. f does not contain every edge of C, since G is connected. Let C' be the symmetric difference of the edge sets of C and f. By the parity of C either f is an odd face or C' is an odd circle containing less faces in its inner side than C.

The other direction is trivial. □

Claim 2.3.2. *Statement 2.3.3 holds if and only if Statement 2.3.1 (and thus the Goddard-Henning conjecture) also holds.*

Proof. Let $H' = (V, F')$ be the subgraph required by Statement 2.3.3. We show that $H = (V, E - F')$ is a subgraph required by Statement 2.3.1. H is a bipartite graph by Lemma 2.3.1, as each of its faces have 4 edges. Take a wheel $v_1 v_2 \dots v_k$ defined by a vertex v. v is not an isolated vertex in H', so there exists a vertex v_i such that $vv_i \in F'$. As F' contains exactly one edge from each face, $v_i v_{i+1} \in F$.

Now let $H = (V, F)$ be the subgraph required by Statement 2.3.1. Clearly, F cannot contain all three edges of a face. We show that there exists an edge set F_0, such that $(V, F \cup F_0)$ is a bipartite subgraph of G that contains exactly two edges of each face. Let $F_0 = \emptyset$. We will add edges to F_0 maintaining that $(V, F \cup F_0)$ is a bipartite graph.

Suppose that there exists a face uvw in G with $uv, vw \notin F \cup F_0$, $wu \in F \cup F_0$. If $F \cup F_0 \cup \{uv\}$ or $F \cup F_0 \cup \{vw\}$ is bipartite, then add the appropriate edge to F_0. If adding either of these edges to F_0 creates an odd circle in $(V, F \cup F_0)$, then there exists a path P_{uv} of odd length from u to v and a path P_{vw} of odd length from v to w. Thus $Puv + Pvw + wu$ is a closed walk of odd length. But that is a contradiction as $(V, F \cup F_0)$ is a bipartite graph.

Now suppose that there exists a face uvw in G such that none of its edges is contained in $F \cup F_0$. If either of its edges can be added to F_0 maintaining a bipartite graph, then put those edges in F_0. Otherwise there exist odd paths P_{uv}, P_{vw}, P_{wu} as above. Concatenating these paths gives a closed walk of odd length and that yields a contradiction.

$(V, F + F_0)$ clearly contains an edge from each wheel and contains two edges of each face of G. So $H' = (V, E - (F \cup F_0))$ contains exactly one edge from each face, and has no isolated vertices. □

One can phrase the Goddard-Henning conjecture in the dual graph as well.

Claim 2.3.3. *$G^* = (V^*, E^*)$ is the dual of a simple triangulated graph of order at least 4 if and only if G^* is a 3-regular 2-edge-connected planar graph of order at least 4.*

Proof. It is trivial that G^* is 3-regular if and only if its dual is triangulated.

It is also easy to see that a cut consisting of one edge corresponds to a loop edge in the dual, and a cut consisting of two edges corresponds to a pair of parallel edges.

Finally, by 3-regularity and using Euler's formula

$$f^* = m^* - n^* + 2 = 3n^*/2 - n + 2 = n^*/2 + 2,$$

where f^*, m^*, and n^* denote the number of faces, edges and vertices of G^*. Thus the dual of G^* has at least 4 vertices if and only if $4 \leq n^*/2 + 2$, i.e. G^* has at least 4 vertices. □

Statement 2.3.4. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph $H^* = (V^*, F^*)$ in G^* that has the following 2 properties.*

1. *It does not contain any odd cut of G^*.*
2. *For every face f of G^*, H^* contains an edge e not on f that has at least one endpoint on f. We say that e leaves the face f.*

Claim 2.3.4. *2.3.4 is equivalent with 2.3.1.*

Proof. We show that given a subgraph $H = (V, F)$ that meets the requirements of 2.3.1, the edges corresponding to F in the dual of G form a subgraph H^* required by 2.3.4, and vice versa. It may be worth noting that the defined H^* is not necessarily the same as the dual graph of H.

It follows from the fact that circles of a planar graph correspond to minimal cutsets in the dual graph, that H is bipartite if and only if H^* does not contain any odd cut of G^*.

Moreover, an edge from a wheel defined by v in G, corresponds to an edge that leaves the face that corresponds to v in the dual graph of G. Hence H contains at least one edge from each wheel if and only if for every face of G^*, H^* contains at least one edge that leaves that face. □

Statement 2.3.5. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a subgraph that has the following 2 properties.*

1. *It intersects every odd cut of G^*.*

2. *For every face f of G^* it does not contain all the edges leaving f.*

Claim 2.3.5. *2.3.4 holds if and only if 2.3.5 holds.*

Proof. If H^* meets the requirements of either of the statements, the complementer subgraph in G^* meets the requirements of the other. □

A 2-factor of a graph $G = (V, E)$ consists of disjoint circles covering V. We can formulate a sufficient condition for the Goddard-Henning conjecture with the help of 2-factors. The motivation for such a reformulation is the fact that the existence of 2-factors in which some cycles are not allowed, is a well-studied part of graph theory.

Statement 2.3.6. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a 2-factor not containing any of the faces.*

Claim 2.3.6. *If Statement 2.3.6 holds, then 2.3.4 holds.*

Proof. Let $H^* = (V^*, F^*)$ be the 2-factor containing none of the faces of G^*.

Every cut of G^* has an even number of common edges with every circle in H^*. Therefore H^* does not contain any odd cuts of G^*.

Let $f = v_1 v_2 \dots v_l$ be a face of G^*. As F^* does not contain f, there must exist a v_i such that $v_i v_{i+1} \notin F^*$. Moreover, every vertex has degree 2 in H^*, so there is an edge starting from v_i that leaves f. □

Statement 2.3.6 can easily be converted into a statement about perfect matchings.

Statement 2.3.7. *Let $G^* = (V^*, E^*)$ be a 3-regular 2-edge-connected planar graph of order at least 4. Then there exists a perfect matching containing at least one edge from each face.*

Claim 2.3.7. *Statement 2.3.6 holds if and if Statement 2.3.7 holds.*

Proof. As G^* is 3-regular, a subgraph is a 2-factor if and only if the complementer subgraph is a perfect matching. Clearly, a subgraph contains none of the faces if and only if the complementer subgraph does contain at least one edge from each face. □

TODO: Add a figure about these statements.

Chapter 3

Proofs for special cases of the Goddard-Henning conjecture

3.1 Outerplanar and Hamiltonian graphs

Zoltán Lóránt Nagy [1] showed that the conjecture of Goddard and Henning holds for Hamiltonian graphs. For this, he characterized the 2-coupon colorable maximal outerplanar graphs first.

Definition 3.1.1. *A graph is outerplanar if it has a planar drawing for which all vertices belong to the outer face. A maximal outerplanar graph is a simple outerplanar graph such that adding any edge results in a non-outerplanar graph.*

Remark 3.1.1. *The outer face of a maximal outerplanar graph is a Hamiltonian cycle.*

In order to provide the mentioned characterization we need to introduce a few notions first.

Definition 3.1.2. *Let G be a maximal outerplanar graph of order $n \geq 3$. The $M(G)$ sun graph of G is obtained by gluing a triangle to each edge of the outer face.*

Remark 3.1.2. *$M(G)$ is a maximal outerplanar graph with $2n$ vertices, from which n has degree 2.*

Remark 3.1.3. *If G has an odd number of vertices, then $M(G)$ does not have two disjoint total dominating sets, as in a 2-coupon coloring of $M(G)$ the vertices of G must have alternating colors. The graph on figure 2.2 is the sun graph of the BDE triangle.*

Definition 3.1.3. *A vertex v of a maximal outerplanar graph is called a central vertex if the following 3 conditions hold.*

1. *$deg(v) \geq 3$*
2. *Every neighbor of v has degree at least 3.*
3. *For every u, w neighbors of v the length of the uw path on the outer face not containing v is divisible by 4.*

Claim 3.1.1. *The outer face of a maximal outerplanar graph does not contain two consecutive central vertices.*

Proof. Suppose there exists an uv edge on the outer face such that u and v are central vertices. Because of the maximality of the graph there exists a uvw triangle. Index the vertices along the outer face form $v = v_1$ to $u = v_n$. Suppose $w = v_i$. From the centrality of v follows that $i \equiv 2 \pmod 4$. On the other hand, u is also a central vertex, hence $i \equiv 1 \pmod 4$. □

Definition 3.1.4. *A generalized sun graph is a maximal outerplanar graph of order $n \equiv 2 \pmod 4$ such that the number of degree 2 vertices plus the number of central vertices is $n/2$.*

Remark 3.1.4. *Every second vertex of the outer face in a generalized sun graph is either central or has degree 2.*

The key characterization theorem is the following.

Theorem 3.1.1. *Let G be a maximal outerplanar graph of order at least 4. G admits 2 disjoint total dominating sets if and only if G is not a generalized sun graph.*

For proving this theorem we need some observations about generalized sun graphs.

Definition 3.1.5. *Let G be a maximal outerplanar graph and $i \geq 2$. We say that a uv edge is a chord of length i, if there is a uv path of length i on the outer face. (This means that if uv is a chord of length i, then it is also a chord of length $n - i$.)*

Lemma 3.1.1. *A maximal outerplanar graph of order $n \geq 3$ has a chord of length 2.*

Proof. It is trivial for $n = 3$. Suppose $n \geq 4$ and let uv be a chord of minimal length. By the maximality of the graph there exists a uvw face, where w is on the shorter uv path of the outer face. If uv is not a chord of length 2, then uw or vw is a chord of length less than the length of the uv chord. □

Lemma 3.1.2. *A maximal outerplanar graph of order $n \geq 5$ has a chord of length 3 or 4.*

Proof. Let uv be a chord of minimal length among chords of length at least 3. By the maximality of the graph there exists a uvw face, where w is on the uv path on the outer face that defines the length of the chord. If on this path w has a distance bigger than 2 from either u or v, than uw or vw is a chord contradicting the minimality of uv. Thus, the length of the uv path is at most 4. $\square$

Lemma 3.1.3. *If G is a maximal planar graph of order $n \geq 7$, then there exists a bounded face, such that the deletion of this face divides G into three graphs with the following properties.*

1. *At most one of the three graphs has more than 3 bounded faces.*
2. *At least one of the three graphs has 2 or 3 faces.*

Proof. For $n \leq 11$, the statement is easy to verify.

For $n > 11$ delete the faces with 2 common edges with the unbounded face. Then in the remaining graph G_1 delete the faces that now have 2 common edges with the unbounded face.

We claim that the remaining graph G_2 is not empty. G has $m = \frac{3(f-1)+n}{2}$ edges, where f denotes the number of faces. Then by Euler's formula $n = f + 1$, hence G has at least 11 faces. In the first deleting step at most $n/2$ faces are deleted, and in the second step at most $|G_1|/2$ faces are deleted. Thus at most $3(f+1)/4$ faces are deleted. $3(f+1)/4 \leq f - 2$ if $f \geq 11$.

Finally, choose a face f_0 from the remaining graph G_2, that has 2 common edges with the unbounded face of G_2. We claim that f_0 has the desired properties. f_0 has at most one neighboring face in G_2, and one or two neighboring faces f_1 and maybe f_2 outside of G_2. Both of f_1 and f_2 has at most two neighboring faces outside of G_1, and at least one of f_1 and f_2 has at least one neighboring face outside of G_1. $\square$

Proof of Theorem 3.1.1. First we show that generalized sun graphs do not have 2 disjoint total dominating sets. The proof goes by induction on the $n = 4k + 2$ number of vertices.

For $k = 1$ there is only one generalized sun graph and it does not admit 2 disjoint total dominating sets. (Shown on figure 2.2.)

Suppose $k \geq 2$ and G is a generalized sun graph of order $4k + 2$. Index the vertices along the outer face from v_1 to v_{4k+2}, such that every vertex with an odd index is central or has degree 2. Let c be a 2-coloring of the graph. We show that c cannot be a 2-coupon coloring. The cardinality of the vertices implies that there must be two consecutive vertices v_{2i} and v_{2i+2} with the same color (say white). If v_{2i+1} has only white neighbors, then this coloring is not a 2-coupon coloring. So suppose v_{2i+1} has a black neighbor v_j. In this case v_{2i+1} is a central vertex. The $v_{2i+1}v_j$ edge divides the graph into two parts ($v_{2i+1}v_j$ is an edge in both graphs). Both of these graphs are generalized sun graphs, as v_{2i+1} either remains a central vertex or becomes a vertex of degree 2 in these smaller graphs, whereas other central vertices remain central vertices. By induction, the restriction of c is not a 2-coupon coloring in either of the smaller graphs. If there is a vertex v_l

with a monochromatic neighborhood in one of the smaller graphs and $l \neq 2i+1$, $l \neq j$, then v_l has the same neighborhood in G, hence all its neighbors are from the same color class. v_{2i+1} cannot violate the condition, as it was chosen in a way that it has both a black and a white neighbor in both graphs. Thus the only remaining case is when v_j has a monochromatic neighborhood in both graphs. But in this case all of its neighbors are from the same color class as v_{2i+1}, so it has a monochromatic neighborhood also in G.

Now we show that if a graph G of order n is not a generalized sun graph then it does have two disjoint total dominating sets.

If $n \equiv 0 \pmod 4$, then it is easy to find a 2-coupon coloring: color the vertices along the boundary of the outer face by repeating the pattern $BBWW$.

If $n \equiv 1 \pmod 4$, then the same coloring method works, if you start the coloring from the right vertex. By lemma 3.1.1 there exists a chord uv of length 2. Alternating colors in pairs starting from v does the job. (See Figure 3.1.)

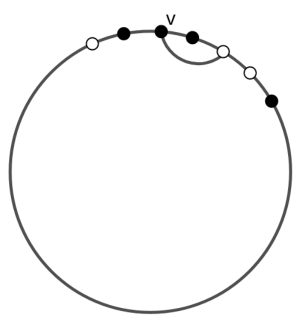

Figure 3.1: Coloring an outerplanar graph of order $4k+1$

If $n \equiv 3 \pmod 4$, then start the coloring from a vertex next to v. (See Figure 3.2.)

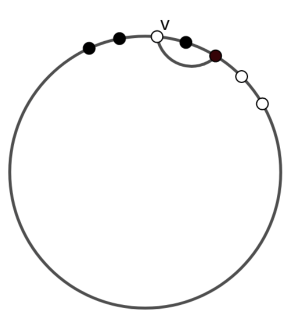

Figure 3.2: Coloring an outerplanar graph of order $4k+3$

Suppose $n \equiv 2 \pmod 4$. We show by induction that if G does not have 2 disjoint total dominating sets, then it is a generalized sun graph.
The case $k = 1$ is easy to check.
If G has a chord uv of length 3, then uv divides the graph into two parts: G_1 of order 4 and G_2 of order $4k$. By alternating colors in pairs one can obtain 2-coupon colorings of G_1 and G_2, where both u and v are colored to black in both graphs.
If G does not have a chord of length 3, then by Lemma 3.1.2 there exists a chord uv of length 4. Then uv divides G into two graphs. One of them must be the maximal outerplanar graph G_5 of order 5. u and v must be the degree 3 vertices of G_5, as otherwise there would be a chord of length 3 in G. Note that in a 2-coupon coloring u and v must have the same colors in order to create a proper neighborhood for the degree 2 vertices of G_5. Consider the face uvw, where w is not in G_5. The deletion of this face divides G into 3 graphs: G', G'' and G_5. (It might be that G' or G'' is degenerated in the sense that it consists only of one edge.) Let G' be the graph containing u and w. Without loss of generality we may assume that $|G''| \leq |G'|$.
Choose the uv chord in a way, such that $|G'|$ is minimal. We may assume that G' has at most 3 faces by Lemma 3.1.3, and thus $|G'| \leq 5$. Thus, there are 4 cases depending on the size of G'.

- Case 1: $|G'| = 2$. In this case $G''| = 4k - 2$. If G'' has a 2-coupon coloring, then it can easily be expanded to a 2-coupon coloring of G. If G'' does not have a 2-coupon coloring, then it is a generalized sun graph by induction. If v is a degree 2 or central vertex in G'', then color the vertices of G'' by alternating colors in pairs, starting with white from w, but color v to black. Let x denote the vertex before v. (See Figure 3.3.) This way, only v can

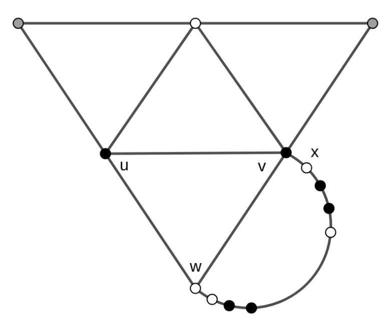

Figure 3.3: Case 1

have a monochromatic neighborhood in G'': if v has degree 2, then wx is an edge, and if v is central, then v and x have a common neighbor in G'' and v has only white neighbors. G_5 can be colored to provide v the missing color.

If v is neither a degree 2 vertex nor a central vertex, then w is. In this case w is a central vertex in G and thus G is a generalized sun graph.

- Case 2: $|G'| = 3$. In a 2-coupon coloring u and v must have the same color, whereas u and w must have different colors. Let H be the graph obtained from G by deleting G_5 and identifying uw with vw. (See 3.4) G is 2-coupon colorable if and only if H is 2-coupon colorable. On the other hand, G is a generalized sun graph if and only if H is a generalized sun graph.

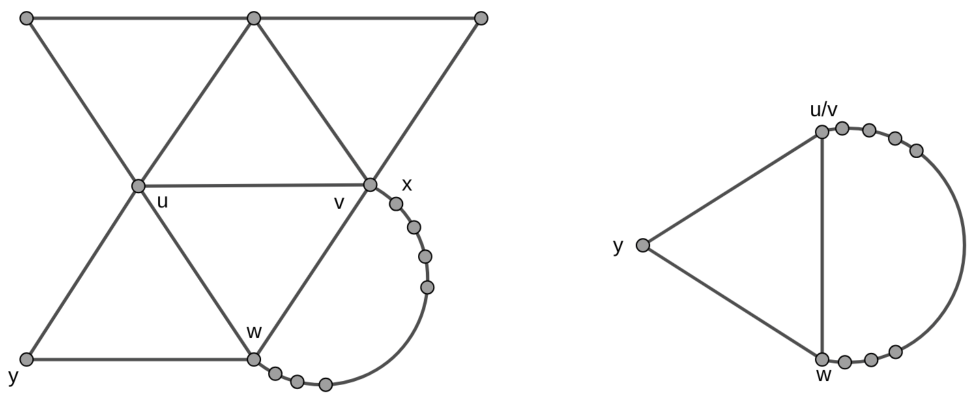

Figure 3.4: Case 2: G (left) and H (right)

- Case 3: $|G'| = 4$. If $|G'| = 4$, then uw is a chord of length 3, and that is a contradiction.

- Case 4: $|G'| = 5$. u and w must be the degree 3 vertices in G', otherwise we could find a chord of length 3. In a 2-coupon coloring u and w must have the same color. Let H be the graph obtained from G by deleting G_5 and identifying uw with vw. (See 3.5) G is 2-coupon colorable if and only if H is 2-coupon colorable. On the other hand, G is a generalized sun graph if and only if H is a generalized sun graph.

□

Remark 3.1.5. *With a slight modification of the proof it can be shown that the vertices of a generalized sun graph cannot be colored in a way that every degree* 2 *or central vertex has neighbors from both color classes.*

As mentioned earlier, based on this theorem Zoltán Lóránt Nagy [1] also showed that the total dominating number of Hamiltonian triangulated graphs is at least two. We still need a Lemma for proceeding with the proof.

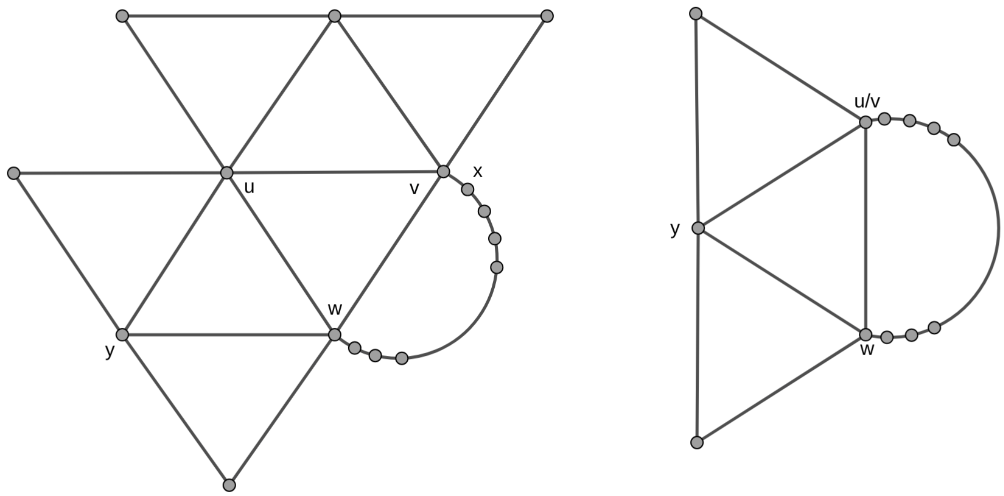

Figure 3.5: Case 4: G (left) and H (right)

Lemma 3.1.4. *If G is a generalized sun graph of order $4k+2$, then there exist at most $k-1$ chords incident to a central vertex.*

Proof. The proof goes by induction. If $k=1$, then there exists only one generalized sun graph of order $4k+2$, and it does not have any central vertices. Suppose $k>1$. We are done, if there is no chord incident to a central vertex. Otherwise, let uv be such a chord of minimal length, where v is central. uv divides the graph into two generalized sun graphs. In both of these graphs, v is either a degree 2 vertex or a central vertex. Hence all the chords incident to a central vertex in G must be either uv or a chord with the same property in one of the smaller graphs. By the minimality of uv, one of these graphs does not have chords incident to a central vertex. The other graph is of order at most $4(k-1)+2$, thus by induction it contains at most $k-2$ chords incident to a central vertex. $\square$

Corollary 3.1.1. *In a generalized sun graph the number of central vertices is less than the number of degree 2 vertices.*

Proof. Let G be a generalized sun graph of order $4k+2$. The number of central vertices is at most $k-1$ by the previous lemma. The number of degree 2 or central vertices is $2k+1$. $\square$

Theorem 3.1.2. *Every triangulated graph with a Hamiltonian circle admits 2 disjoint dominating sets.*

Proof. A Hamiltonian triangulated graph G can be obtained by identifying the Hamiltonian cycle of two maximal outerplanar graphs G_1 and G_2. If at least one of these outerplanar graphs is

2-coupon colorable, then the same coloring is a 2-coupon coloring of G. By Theorem 3.1.1 we are done if at least one of these graphs is not a generalized sun graph.

Suppose that G_1 and G_2 are generalized sun graphs. If the union of degree 2 vertices and central vertices is the same in G_1 and G_2, then by Corollary 3.1.1 there exists a vertex with degree 2 in both graphs. Then this vertex is a degree 2 vertex in G and that is a contradiction, as in a triangulated planar graph every degree must be at least 3.

Assume that each vertex is a degree 2 or central vertex in G_1 or G_2. Index the vertices along the Hamiltonian cycle from v_1 to v_{4k+2}. We claim that there exists an index i such that v_i is a degree 2 vertex in G_1 and v_{i+3} is a degree 2 vertex in G_2. (If $i+3$ is bigger than $4k+2$, we take $v_{i+3-(4k+2)}$.) Let $I_1 = \{i|deg_{G_1}(v_i) = 2\}$, $I_2 = \{i|deg_{G_2}(v_i) = 2\}$ and $J = \{i+3|i \in I_1\}$. By Corollary 3.1.1 the cardinality of all of these sets is at least $k+2$, hence there exists an index i such that $i+3 \in I_2 \cap J$, proving the claim.

Finally, we define a 2-coupon coloring of G as follows. Let v_i be a vertex as above. Color the vertices by alternating colors in pairs starting from v_{i+2}. (See Figure 3.6.) It is clear that all the vertices apart from v_{i+1} and v_{i+2} have neighbors in both color classes. However, v_i is a degree 2 vertex in G_1, hence $v_{i-1}v_{i+1}$ is an edge in G_1. Similarly, $v_{i+2}v_{i+4}$ is an edge in G_2. These two edges ensure that v_{i+1} and v_{i+2} also have neighbors in both color classes. $\square$

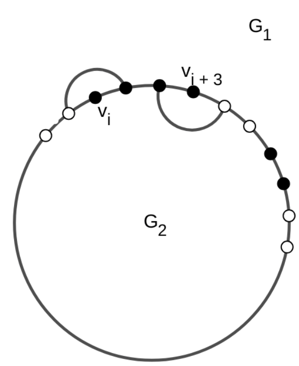

Figure 3.6: Coloring a Hamiltonian graph

Remark 3.1.6. *Whitney [8] proved that each triangulated planar graph without separating triangles is Hamiltonian. Helden [9] strengthened this statement by proving that each triangulated planar graph with at most five separating triangles is Hamiltonian.*

With the help of these theorems, we can also say something about graphs with another kind of 2-factors.

Claim 3.1.2. *Let G be a simple triangulated graph of order $n \geq 4$. If G has a 2-factor with none of its cycles having length congruent to 2 modulo 4, then the total dominating number of G is at least 2.*

Proof. If the 2-factor consists only of one cycle, then Theorem 3.1.2 proves the claim. Suppose that there are at least two cycles in the 2-factor. First note that by alternating colors in pairs, any cycle of length not congruent to 2 modulo 4 can be colored in a way, such that there is at most one vertex with a monochromatic neighborhood. (See Figures 3.1, 3.2.)

Contract each cycle to a single vertex. G is connected, hence there exists a tree T in the contracted graph. Choose T to have a minimal number of degree 1 vertices. Let $E(T)$ denote the edges of the original graph that were mapped to T. We show a 2-coupon coloring in the subgraph defined by the union of the 2-factor and $E(T)$. Choose a root node r from the degree 1 vertices in T.

We color the cycle C_0 corresponding to r first. Choose a vertex v in C_0, such that there exists a uv edge in $E(T)$. We color C_0 such that only v may have a monochromatic neighborhood and assign u the missing color.

After this, we iteratively color a child of a cycle that is already colored. We start by coloring cycles that do not correspond to leaves in T. Suppose that C_1 is a cycle like that, let C_2 be a child of C_1, and v_1v_2 be the edge in $E(T)$ such that $v_1 \in C_1$ and $v_2 \in C_2$. Let c_1 be a coloring of C_1 such that only v_1 may have a monochromatic neighborhood. There might be a vertex (but only one) in C_1 that already has a color, so flip the colors of c_1 if necessary. Color v_2 in a way that provides v_1 the missing color.

Now we color cycles that correspond to leaves, but have at least 4 vertices. Let C_l be a cycle like that. By Theorem 3.1.1 there exists a 2-coupon coloring c_l of C_l. Again, if there is a vertex in C_l that already has a color, then we might need to flip the colors of c_l.

Finally, we need to color cycles corresponding to leaves of T and having only 3 vertices. Let $u_lv_lw_l$ be a cycle like that, where v_l is the only vertex that may already have a color. Suppose it is colored to black. There exists a face $v_lw_lx_l$ where $x_l \neq u_l$. If x_l is colored to black, then color w_l and u_l to white. If x_l is colored to white, then color w_l to white, and u_l to black. The only remaining case is when x_l does not have a color yet. In this case, there must be a face $x_ly_lz_l$ corresponding to a leaf of T. These two leaves have a closest common ancestor t. t is of degree at least 2 in T, hence $t \neq r$, so t must have degree at least 3. By adding the edge corresponding to v_lx_l to T and removing the first edge of the tv_l path, we would get a tree T', where T' would have fewer degree 1 vertices then T has. This contradicts to the choice of T. □

3.2 Graphs without low-degree vertices

TODO

Definition 3.2.1. *Let H be a hypergraph. The incidence graph of H is a bipartite graph with one of its classes corresponding to the vertices of H, and the other class corresponding to the hyperedges of H. ve is an edge in the incidence graph if and only if the hyperedge e contains vertex v in H.*

Definition 3.2.2. *A hypergraph is called planar, if its incidence graph is planar.*

Definition 3.2.3. *A vertex coloring of a hypergraph is proper if all its hyperedges contain vertices from both color classes.*

Theorem 3.2.1. *Let $H = (V, E)$ be a planar hypergraph with at most 2 hyperedges of size 2. Then H has a proper vertex coloring with two colors.*

Proof. We may assume that H has exactly 2 hyperedges of size 2. Otherwise add one or two new hyperedges of size 2. A proper coloring of the resulting hypergraph is also a proper coloring of the original one.

We define another planar graph based on the incidence graph of H as follows. For each hyperedge e, delete the corresponding vertex from the graph, and add edges between the vertices contained in e in a way that it results in a circle. The bounded faces of the resulting graph $G_H = (V, F)$ correspond to the hyperedges of H. (See Figure 3.7.) After this, triangulate the faces that have more than 3 vertices.

Let c_4 be a proper 4-coloring of the resulting graph with colors $\{1, 2, 3, 4\}$. By permuting the color classes if necessary, we may assume the followings.

1. If the two hyperedges of size 2 are not disjoint, i.e. they are $\{u, v\}$ and $\{u, w\}$, then $c_4(u)1$, and $c_4(v), c_4(w) \in \{3, 4\}$.

2. If the two hyperedges of size 2 are disjoint, i.e. they are $\{u, v\}$ and $\{x, y\}$, then $c_4(u), c_4(x) \in \{1, 2\}$ and $c_4(v), c_4(y) \in \{3, 4\}$.

We define a 2-coloring of G_H as follows.

$$c_2(v) = \begin{cases} 1, & \text{if } c_4(v) = 1 \text{ or } c_4(v) = 2 \\ 2, & \text{if } c_4(v) = 3 \text{ or } c_4(v) = 4 \end{cases}$$

We claim that c_2 is a proper 2-coloring of H. The hyperedges of size 2 have vertices in both color classes due to our assumptions on c_4. For the other hyperedges there exists a face in G containing the vertices of the hyperedge. As c_4 was a proper 4-coloring of the triangulated graph, each face of G of size at least 3 has vertices from 3 or 4 color classes. □

Claim 3.2.1. *Let G be a simple triangulated graph. If there are at most two vertices of degree at most 4, then G has a 2-coupon coloring.*

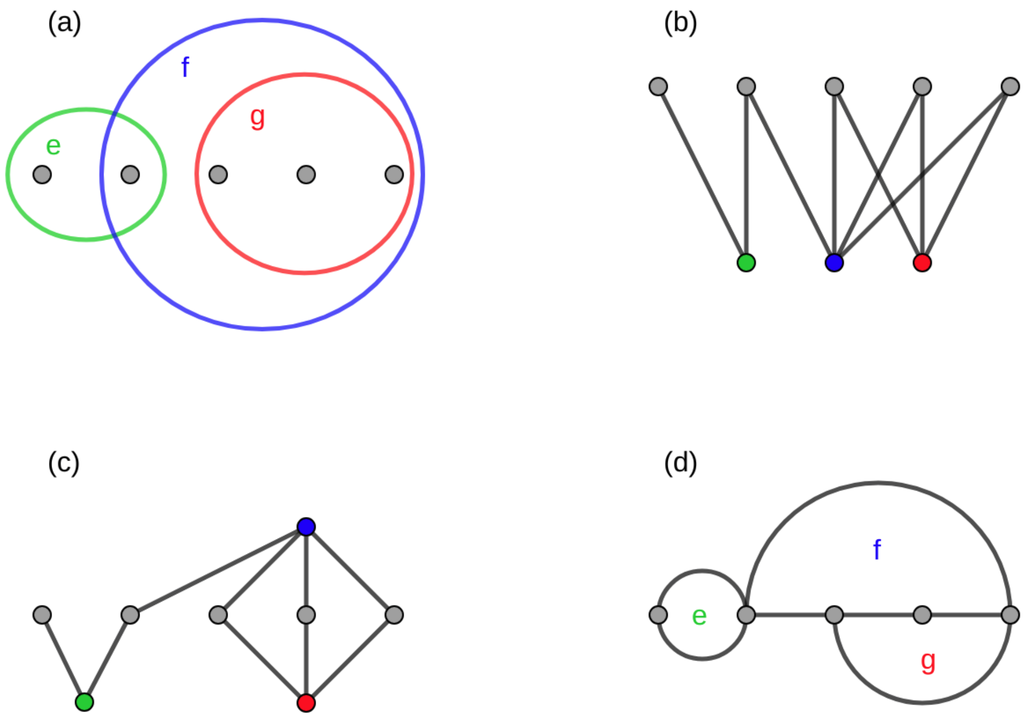

Figure 3.7: A hypergraph (a), its incidence graph (b), a planar embedding of the incidence graph (c) and the graph G_H (d)

Proof. By Claim 2.3.3 the dual G^* is a 3-regular 2-edge-connected planar graph. Thus, by Petersen's theorem there exists a perfect matching M in G^*. By deleting the edges corresponding to M from G, we get a graph G' such that all its faces contains 4 vertices. We call such graphs quadrangulated. Note, that for each vertex v, we deleted at most the half of the vertices starting from v, thus there are at most two vertices in G' of degree 2 and none of the vertices has less than 2 neighbors.

By Lemma 2.3.1, G' is a bipartite graph. Let U and V be the two classes of G'. G' is the incidence graph of two hypergraphs: let H_1 be the hypergraph defined on the vertex set U with hyperedges V, and H_2 be the hypergraph defined on the vertex set V with hyperedges U. By Theorem 3.2.1 H_1 and H_2 has proper 2-colorings. Take the union of these colorings c. I.e. on the vertices of U, c is defined by a proper coloring of H_1, whereas on the vertices of V, c is defined by a proper coloring of H_2. c is a 2-coupon coloring of G'. $\square$

Remark 3.2.1. *It is not true that every simple quadrangulated graph has a* 2*-coupon coloring. See*

Figure 3.8 for an example.

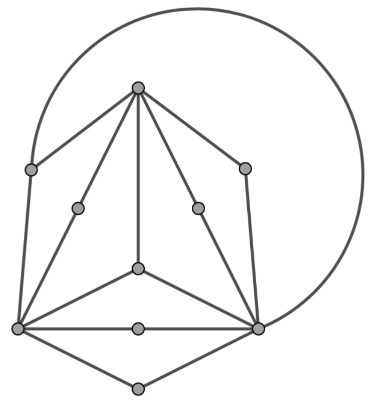

Figure 3.8: A quadrangulated graph without two disjoint dominating sets

Conjecture 3.2.1. *If G is a simple triangulated graph with all the vertices of degree at least 4, then G admits three disjoint total dominating sets.*

It is also worth noting that having total dominating number at least 2 can be translated into a statement about the so-called neighborhood hypergraph. The neighborhood hypergraph of G is a hypergraph defined on the vertex set of G, where for each vertex v, there is a hyperedge containing the neighboring vertices of v. G has total dominating number at least 2 if and only if the neighborhood hypergraph has a coloring such that every hyperedge contains vertices from both color classes. In other words, it is equivalent to the statement that the hyperedges possess property B. Its relevance is that property B has been studied extensively.

3.3 Barnette's conjecture

A conjecture of Barnette [6] is the following.

Conjecture 3.3.1. *Every 3-connected cubic planar bipartite graph is Hamiltonian.*

By 2.2.4 if the Barnette-conjecture holds, then the Goddard-Henning conjecture also holds for Eulerian triangulations.

Alt, Payne, Schmidt and Wood [7] proved that the conjecture holds for graphs, where the dual is an Eulerian planar triangulation and has a special 3-coloring.

TODO

Chapter 4

Algorithm for generating all planar triangulations

(TODO (or delete))

Bibliography

[1] Zoltán Lóránt Nagy *Coupon-coloring and total domination in Hamiltonian planar triangulations* arXiv preprint arXiv:1708.01725

[2] Wayne Goddard, Michael A. Henning *Thoroughly distributed colorings* arXiv preprint arXiv:1609.09684

[3] Bohdan Zelinka *Total domatic number and degree of vertices of a graph.* Mathematica Slovaca, Vol. 39 (1989), No. 1, 7–11

[4] Pinar Heggernes, Jan Arne Telle *Partitioning graphs into generalized dominating sets* Nord. J. Comput. 5 (2), 128-142Nord. J. Comput. 5 (2), 128-142

[5] H. Aram, S. M. Sheikholeslami, L. Volkmann *On the total domatic number of regular graphs* Transactions on Combinatorics 1, 45-51

[6] David W. Barnette *Conjecture 5* Recent progress in combinatorics (ed. W. T. Tutte), Academic Press, New York (1969) 343

[7] Helmut Alt, Michael S. Payne, Jens M. Schmidt, David R. Wood *Thoughts on Barnette's conjecture* arXiv preprint arXiv:1312.3783

[8] H. Whitney *A theorem on graphs* Ann. Math. 32 (1931), 378-390

[9] Guido Helden *Hamiltonicity of maximal planar graphs and planar triangulations* These de doctorat, RWTH Aachen University

[10] Zdenek Dvorák, Daniel Král *On planar mixed hypergraphs* Journal of Combinatorics 8 (1), 35-35