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Contents

Introduction

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Chapter 1

Coupon coloring in planar graphs

In this chapter we will examine the so-called total domatic number of graphs, especially in the case of triangulated planar graphs.

Let $G = (V, E)$ be a graph without isolated vertices.

Definition 1.0.1. *$S \subseteq V$ is a total dominating set if every vertex has a neighbor in S. The total domatic number of G is the maximum number of disjoint total dominating sets.*

Sometimes it is more convenient to look at total dominating sets as color classes.

Definition 1.0.2. *A coloring of the vertices is called a k-coupon coloring if every vertex has a neighbor from each color class. The coupon coloring number of G is the maximum k for which a k-coupon coloring exists. The coupon coloring number is denoted by $\chi_c(G)$.*

1.1 Complexity for arbitrary graphs

It turns out that determining the total domatic number (or equivalently the coupon coloring number) of a graph is rather hard. Even determining if the total domatic number of a graph is at least 2 is NP-complete. We prove this by showing that a variant of 3SAT is reducible to this question in polynomial time.

Definition 1.1.1. *An instance of the not-all-equal 3-satisfiability (NAE-3SAT) problem consists of a set C of clauses on a finite set X of Boolean variables, where each clause contains three literals. The question is whether there is a truth assignment for X that satisfies all the clauses in C such that each clause contains a false literal.*

Theorem 1.1.1. *NAE-3SAT is NP-complete.*

Proof. It can be checked in polynomial time whether a given truth assignment meets the requirement, so NAE-3SAT is in NP.

To prove NP completeness, we show first a reduction from 3SAT to NAE-4SAT. Let C be the set of clauses and X be the set of variables in an instance of 3SAT. Let $X' = X \cup y$, and $C' = \{(x_1 \vee x_2 \vee x_3 \vee y) \mid (x_1 \vee x_2 \vee x_3) \in C\}$. We claim that the NAE-4SAT problem defined by (C', X') is satisfiable if and only if the 3SAT problem defined by (C, X) is satisfiable. If the 3SAT formula is satisfied by a truth assignment then the same assignment with assigning the value false to y satisfies the NAE-4SAT problem. Now suppose a truth assignment satisfies the NAE-4SAT problem. If y has value false, then the same assignment of X satisfies the 3SAT problem. If y has value true, then changing every truth value in X to its opposite gives a truth assignment satisfying the 3SAT formula.

Finally, the reduction from NAE-4SAT to NAE-3SAT is by adding clauses $(x_1 \vee x_2 \vee z)$ and $(\bar{z} \vee x_3 \vee y)$ instead of each clause $(x_1 \vee x_2 \vee x_3 \vee y) \in C'$. □

Theorem 1.1.2. *It is NP-complete to decide whether the total domatic number of a graph is at least* 2*.*

Proof. Given a partition of the vertices into 2 sets, it can be checked in polynomial time whether these sets are total dominating sets. So the problem is a member of NP.

For proving NP-completeness, we will show that NAE-3SAT is reducible to this problem in polynomial time. Let C be the set of clauses and X be the set of variables in an instance of NAE-3SAT. We can assume that every variable x appears in at least one clause. Otherwise we add a new clause containing x and $\bar{x}$ to the formula. Now we construct the corresponding graph G. For each variable x, introduce 3 vertices x_1, x_2, x_3, and 2 edges x_1x_2, x_2x_3. For each clause c, introduce a vertex c. If x is a literal in c, then add the edge cx_1 to the graph. If $\bar{x}$ is a literal in c, then add the edge cx_3.

Suppose G has a partition into 2 disjoint total dominating sets: T and F. Assign the value true for each variable x with $x_1 \in T$ and assign the value false otherwise. For any variable x, x_1 and x_3 are the only neighbors of x_2, so x_1 and x_3 must be in different sets of the partition. If c is a vertex corresponding to a clause, then it must have neighbors both in T and F, and so the literals in c cannot be all true nor false.

Suppose now that the variables have a truth assignment such that each clause contains both true and false literals. Define T and F as follows. Put all the vertices corresponding to clauses into T. For each variable x put x_2 into F. Furthermore, if true was assigned to x, then put x_1 into T, x_3 into F, and conversely otherwise. □

Let us note that the constructed graph in the proof is always a bipartite graph.

Corollary 1.1.1. *It is NP-complete to decide whether the total domatic number of a bipartite graph is at least* 2*.*

1.2 Degree restrictions for arbitrary graphs

(remove???)

A natural question is whether graphs with an appropriately big minimum degree always have a total domatic number of at least 2.

Theorem 1.2.1. *For every d there exists a graph with minimum degree d and without 2 disjoint total dominating sets.*

Proof. ... □

...Other degree stuff (k-regular, maxdeg-mindeg small enough)...

1.3 Observations for planar graphs

From now on we will focus on 2-coupon colorings and planar graphs. A conjecture of Goddard and Henning is the following.

Conjecture 1.3.1. *If G is a simple triangulated planar graph of order at least 4, then the total domatic number of G is at least 2.*

Remark 1.3.1. *The simplicity of the graph is necessary. Suppose the graph on figure 1.1 has a 2-coupon coloring. Then A and C must have different colors, because they are the only neighbors of B. Similarly, C and E must have different colors, as well as E and A. That is a contradiction, since A, C and E form a triangular.*

Remark 1.3.2. *Allowing triangulated disks (i.e. planar graphs with at most one face greater than 3), the conjecture does not hold. For example, the graph on figure 1.2 does not have a 2-coupon coloring from similar reasons as the previous one. We will show later that this graph is a member of a bigger graph family without 2 disjoint dominating sets.*

There are some sufficient conditions known for having a total domatic number of at least 2. We will cover some of them along the way. The first example is a graph family for which an easy induction shows that they are 2-coupon colorable.

Definition 1.3.1. *A graph is called a stacked graph if it can be constructed from a triangle by repeatedly putting a new vertex in a face and connecting it with the vertices on the boundary of that face.*

Remark 1.3.3. *Stacked graphs are triangulated.*

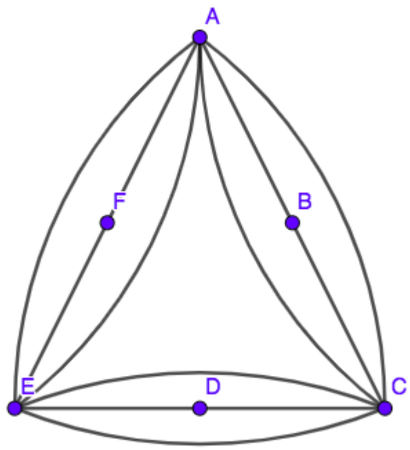

Figure 1.1: Simplicity is necessary

Claim 1.3.1. *Stacked graphs with at least* 4 *vertices are* 2*-coupon colorable.*

Proof. We can determine the colors of the vertices as the graph is constructed. The current coloring will maintain two following two properties.

1. It is a 2-coupon coloring of the current graph.
2. Every face has vertices from both color classes.

The construction of the graph starts with a simple triangle. Color two vertices of the triangle to blue, and the remaining vertex to red. Color the vertex added to the graph in the first step to red. This coloring has the desired properties. When a vertex is inserted into a face, color the new vertex to red, if there is only one red vertex on the face's boundary, and blue otherwise. This trivially maintains the desired properties. □

1.4 Outerplanar and triangulated Hamiltonian graphs

Zoltán Lóránt Nagy showed that the conjecture of Goddard and Henning holds for Hamiltonian graphs. For this, he characterized the 2-coupon colorable maximal outerplanar graphs.

Definition 1.4.1. *A graph is outerplanar if it has a planar drawing for which all vertices belong to the outer face. A maximal outerplanar graph is an outerplanar graph such that adding any edge results in a not outerplanar graphs.*

Remark 1.4.1. *Te outer face of a maximal outerplanar graph is a Hamiltonian cycle.*

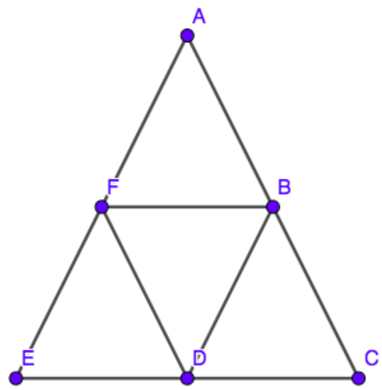

Figure 1.2: The conjecture does not hold for triangulated disks

In order to provide the mentioned characterization we need to introduce a few notions first.

Definition 1.4.2. *Let G be a maximal outerplanar graph of order $n \geq 3$. The $M(G)$ sun graph of G is obtained by gluing a triangle to each edge of the outer face.*

Remark 1.4.2. *$M(G)$ is a maximal outerplanar graph with $2n$ vertices, from which n has degree 2. The graph on figure 1.2 is the sun graph of the BDE triangle.*

Definition 1.4.3. *A vertex v of a maximal outerplanar graph is called a central vertex if the following 3 conditions hold.*

1. *$deg(v) \geq 3$*
2. *Every neighbor of v has degree at least 3*
3. *For every u, w neighbors of v the length of the uw path on the outer face not containing v is divisible by 4.*

Claim 1.4.1. *The outer face of a maximal outerplanar graph does not contain two consecutive central vertices.*

Proof. Suppose there exists an uv edge on the outer face such that u and v are central vertices. Because of the maximality of the graph there exists a uvw triangle. Index the vertices along the outer face form $v = v_1$ to $u = v_n$. Suppose $w = v_i$. From the centrality of v follows that $i \equiv 2 \pmod 4$. On the other hand u is also a central vertex, hence $i \equiv 1 \pmod 4$. □

Definition 1.4.4. *A generalized sun graph is a maximal outerplanar graph of order $n \equiv 2 \pmod 4$ such that the number of degree 2 vertices plus the number of central vertices is $n/2$.*

Remark 1.4.3. *Every second vertex of the outer face in a generalized sun graph is either central or has degree 2.*

The key characterization theorem is the following.

Theorem 1.4.1. *Let G be a maximal outerplanar graph. G admits 2 disjoint total dominating sets if and only if G is not a generalized sun graph.*

For proving this theorem we need some observations about generalized sun graphs.

Proof of Theorem 1.4.1. First we show that generalized sun graphs do not have 2 disjoint total dominating sets. The proof goes by induction on the $n = 4k + 2$ number of vertices. For $k = 1$ there is only one generalized sun graph and it does not admit 2 disjoint total dominating sets. (Shown on figure 1.2.) Suppose $k \geq 2$ and G is a generalized sun graph of order $4k + 2$. Index the vertices along the outer face from v_1 to v_{4k+2}, such that every vertex with an odd index is central or has degree 2. Let c be a 2-coloring of the graph. We show that c cannot be a 2-coupon coloring. The cardinality of the vertices implies that there must be two consecutive vertices v_{2i} and v_{2i+2} with the same color (say white). If v_{2i} has only white neighbors, then this coloring is not a 2-coupon coloring. So suppose v_{2i} has a black neighbor v_j. In this case, v_{2i} is a central vertex. The $v_{2i}v_j$ edge cuts the graph into two parts ($v_{2i}v_j$ is an edge in both graphs). Both of these graphs are generalized sun graphs, as v_{2i} either remains a central vertex or become a vertex of degree 2 in these smaller graphs, whereas other central vertices remain central vertices. By induction, the restriction of c is not a 2-coupon coloring in either of the smaller graphs. If there is a vertex v_l with a monochromatic neighborhood in one of the smaller graphs and $l \neq 2i, l \neq j$, then v_l has the same neighborhood in G, hence all its neighbors are from the same color class. v_{2i} cannot violate the condition, as it was chosen in a way that it has both a black and a white neighbor in both graphs. Thus the only remaining case is when v_j has a monochromatic neighborhood in both graphs. But in this case, all of its neighbors are from the same color class as v_{2i}, so it has a monochromatic neighborhood also in G. □

Remark 1.4.4. *With a slight modification of the proof it can be shown that the vertices of a generalized sun graph cannot be colored in a way that every degree 2 or central vertex has neighbors from both color classes.*

Lemma 1.4.1. *The outer face of a maximal outerplanar graph has a chord ...*

Theorem 1.4.2. *Every triangulated graph with a Hamiltonian circle admits 2 disjoint dominating sets.*

Proof. Let G be a triangulated Hamiltonian graph and let n denote the number of vertices in G.

If $n \equiv 0 \pmod 4$, then it is easy to find a 2-coupon coloring: color the vertices along the boundary of the outer face by repeating the pattern $BBWW$.

If $n \equiv 1 \pmod 4$, then the same coloring method almost works. There is only 1 vertex that has 2 black neighbors on the outer face. $\square$

(Hypergraph connection ???)

Chapter 2

Restricted 2-factors

2.1 Connection

Theorem 2.1.1. *Let G be a triangulated planar graph. If all the vertices of G have an odd degree, then there exists a coupon coloring with 2 colors.*

Proof. ... □

2.2 Barnette's conjecture

Conjecture 2.2.1. *Every 3-connected cubic planar bipartite graph is Hamiltonian.*