14 December 2021 16:36

Bridgeland Stability

§0: Π-stability for D-branes in string theory

AKA: The handwavey part.

[Douglas 2001]

N=2 Superconformal field theory
SCFT

SCFT + topological twist (TCFT)

$\Updownarrow$

A_∞-category

SCFT + $\mathcal{D}$

$\mathcal{L} \rightsquigarrow$ same $\mathcal{D}$

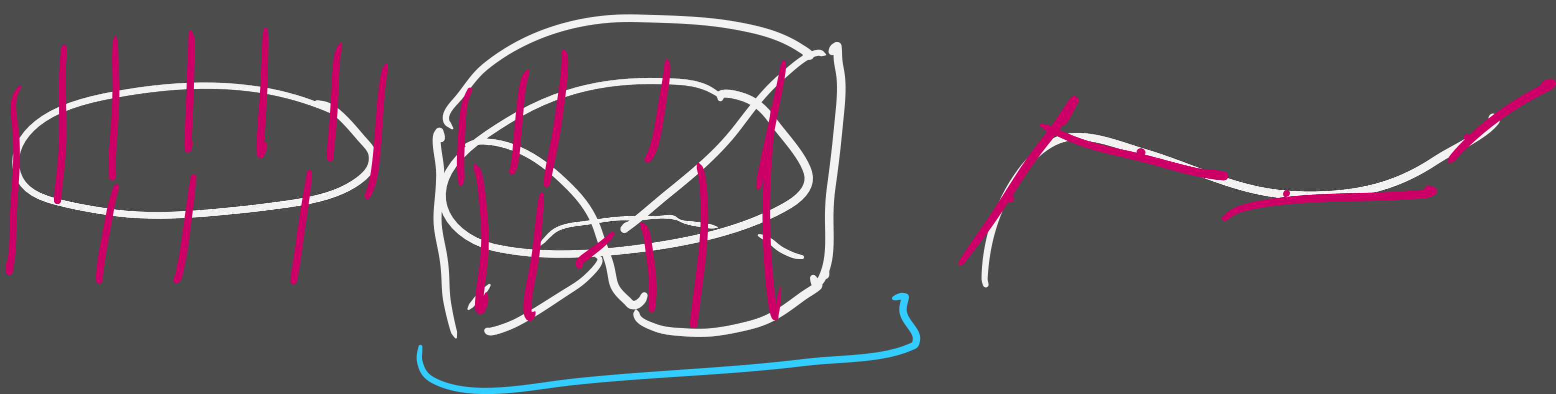
π-stability $\longleftrightarrow$ Q: why is $\mathcal{D}$ special

$\rightsquigarrow$ $\underset{E \in \mathcal{D}}{Z(E)}$, $\Phi(E)$

§1 Stability

Mumford '62: slope stability for vector bundles on curves.

E: vector bundle on a smooth curve C



Slope: $\mu(E) = \dfrac{\deg(E)}{\operatorname{rank}(E)}$

E is μ-stable if $0 \neq F \subsetneq E$,
$\neq 0$ semistable

$$\mu(F) < \mu(E)$$
$$\leq$$

e.g. $\mathcal{L}$ line bundle (rank = 1)

$$\mu(\mathcal{L}) = \deg(\mathcal{L})$$

$0 \neq F \subseteq \mathcal{L}$ must have rank = 1

but $\deg(F) < \deg(L)$

$\therefore$ $\mathcal{L}$ is stable

IN FACT: stable bundles are the building blocks for all vector bundles over C

M : moduli space of vector bdls on C

$M_{r,d}$: E : vector bundle, rank $= r$, degree $= d$

$\hookrightarrow M^{st}_{r,d}$ is "nice"

<u>Also:</u> $\mathrm{Coh}(C)$ similar thing:

e.g. $\mathbb{P}1$: $\mathrm{Coh}(\mathbb{P}1)$ generated by line bundles and skyscraper sheaves. $\mathcal{O}(n)$ x_0^n

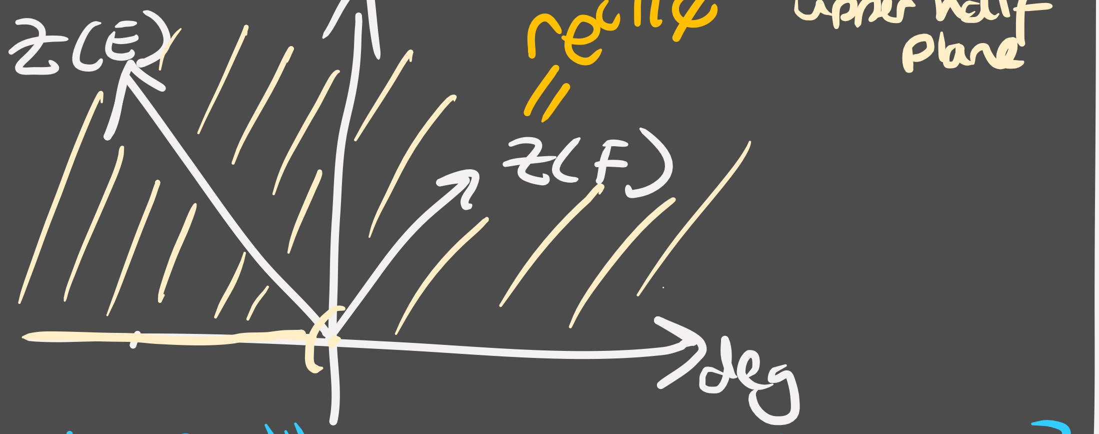
$x \in X \rightsquigarrow \mathcal{O}_x$ $\mathcal{O}_x$

$Z(E) = -\deg(E) + i\,\mathrm{rank}(E) \in \mathbb{H}$

E stable

$F \subset E$

$\mathbb{C}$:



[we will see later in "$P(\phi)$" def. why we use ϕ not $\mathrm{Arg}(Z)$]

Stability for Quivers

· Q finite quiver, $Q_0 = \{0, \dots, n\}$

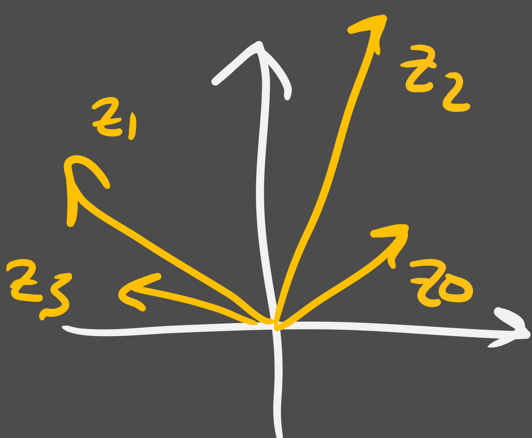
· Consider $\mathrm{Rep}_{\mathbb{C}} Q$

EXAMPLE P_2: $0 \overset{x}{\underset{y}{\rightrightarrows}} 1$

$\underline{V} \in \mathrm{Rep}_k Q : (V_0, V_1, \phi_x, \phi_y : V_0 \to V_1)$

e.g.: $\mathbb{C} \overset{\lambda \in \mathbb{C}}{\underset{\mu \in \mathbb{C}}{\rightrightarrows}} \mathbb{C}$ and more generally: $\underline{V} : \mathbb{C}^n \overset{\phi_x}{\underset{\phi_y}{\rightrightarrows}} \mathbb{C}^m$

Pick $z_0, \dots, z_n \in \mathbb{H}$



$$Z(\underline{V}) = \sum_{i=0}^{n} \dim V_i \cdot z_i$$
$$= re^{i\pi\phi}$$

$\underset{\neq \underline{0}}{\underline{V}}$ is $\underset{\text{s.s.}}{\text{stable}}$ if $\underline{W} \subsetneq \underline{V} \Rightarrow \phi(\underline{W}) \underset{\leq}{<} \phi(\underline{V})$

e.g. P_2: Pick $z_0, z_1 \in \mathbb{H}$

RECALL: $\underline{W} = (W_0, W_1, \psi_x, \psi_y)$ is a subrep. of $\underline{V} = (V_0, V_1, \phi_x, \phi_y)$ if $W_0 \subset V_0$ and $W_1 \subset V_1$ as subvector spaces, and $\exists\, \alpha, \beta$ linear maps s.t.

$$\begin{tikzcd}
W_0 \arrow[r, shift left, "\psi_x"] \arrow[r, shift right, "\psi_y"'] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\copyright"] & W_1 \arrow[d, "\beta"] \\
V_0 \arrow[r, shift left, "\phi_x"] \arrow[r, shift right, "\phi_y"'] & V_1
\end{tikzcd}$$

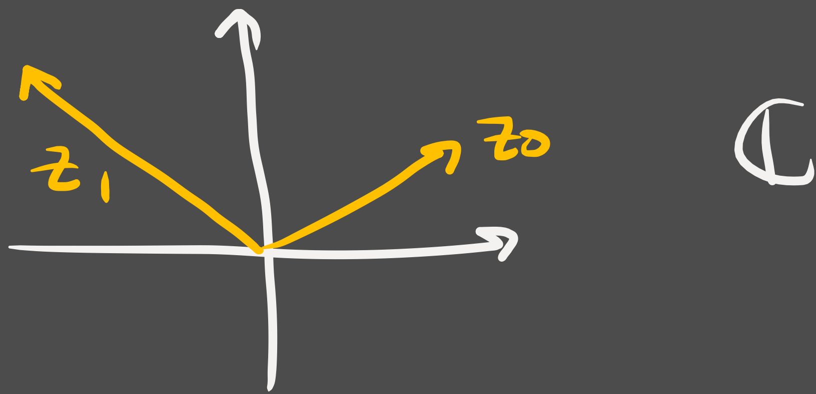
HENCE If $\underline{V}$ is a subrep of P_2 then:

$$\begin{tikzcd}
S_0: & 0 \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "\alpha"] \\
\underline{V}: & \mathbb{C}^n \arrow[r, shift left, "\phi_x"] \arrow[r, shift right, "\phi_y"'] & \mathbb{C}^m
\end{tikzcd}$$

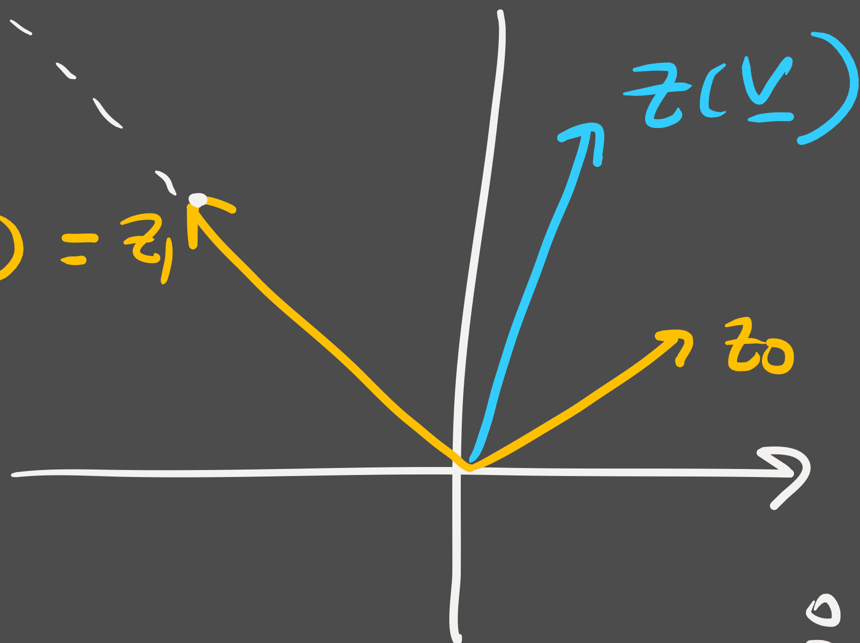
(simple rep.)

If $m > 0$ then S_0 is always a subrep. of $\underline{V}$.

Suppose $\underline{\phi(z_1) > \phi(z_0)}$:



- $Z(S_0) = z_1$
- If $n > 0$, $\phi(S_0) > \phi(\underline{V})$, so $\underline{V}$ is not stable.



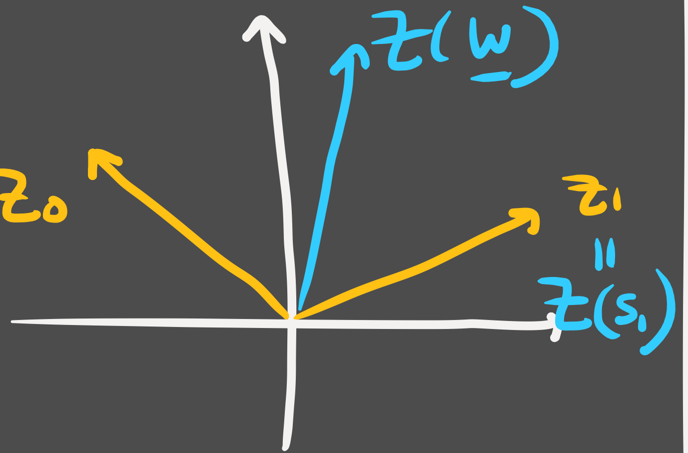
Similar argument for $S_1 : \mathbb{C} \overset{0}{\underset{0}{\rightrightarrows}} 0$

shows that S_0, S_1 are the <u>only</u> stable reps

Suppose instead $\phi(z_1) < \phi(z_0)$:

Consider:

$$S_1: \quad \begin{tikzcd} \mathcal{O} \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d, "0"'] & \mathbb{C} \arrow[d, "\alpha"] \\ \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd} \quad :\underline{W}$$



$\rightsquigarrow \phi(S_1) < \phi(\underline{W})$

Hence S_1 does not destabilise $\underline{W}$.

Moreover:

$$S_0: \quad \begin{tikzcd} \mathbb{C} \arrow[r, shift left, "0"] \arrow[r, shift right, "0"'] \arrow[d] & \mathcal{O} \arrow[d, "0"] \\ \mathbb{C} \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mu"'] & \mathbb{C} \end{tikzcd} \quad :\underline{W}$$

commutes $\iff (\lambda, \mu) = (0,0)$

IN PARTICULAR:

$(\lambda, \mu) \neq (0,0) \Rightarrow \underline{W}$ stable

Remark: stable objects

$\mathbb{C} \overset{\lambda}{\underset{\mu}{\rightrightarrows}} \mathbb{C}$ up to iso. $\overset{1:1}{\longleftrightarrow}$ points in $\mathbb{P}^1$

This hints at: $\rightsquigarrow$ $D^b(\mathrm{Rep}_{\mathbb{C}}(P_2)) \simeq D^b(\mathbb{P}^1_{\mathbb{C}})$

Slope stability

- $\mathcal{E}, \mathcal{F}$ vector bundles or $Coh(X)$
- $\mathcal{E}$ semistable of phase ϕ
  $\mathcal{F} \hookrightarrow \mathcal{E} \Rightarrow \phi(\mathcal{F}) \leq \phi(\mathcal{E})$
  If $\mathcal{E}$ is s.s., $\nexists$ no map $\mathcal{F} \to \mathcal{E}$
  $\phi(\mathcal{F}) > \phi(\mathcal{E})$
- $\mathcal{E}$ stable bundle of phase ϕ
  $\mathcal{F} \hookrightarrow \mathcal{E} \Rightarrow \phi(\mathcal{F}) < \phi(\mathcal{E})$
  $\mathcal{E}$ has no nontrivial subobjects of same phase

- slope-stable bundles/sheaves are building blocks of $Coh(X)$/vectbdls
- $Z(\mathcal{E}) = -\deg(\mathcal{E}) + i\,\mathrm{rank}(\mathcal{E}) \in \mathbb{H}$

If $\mathcal{E} \neq 0$, $Z(\mathcal{E}) = \underbrace{m(\mathcal{E})}_{\in \mathbb{R}_{>0}} \cdot e^{i\pi\phi}$

Bridgeland stability $\sigma = (P, Z)$

- $\mathcal{E}, \mathcal{F} \in \mathcal{D}$: triangulated category
- $\mathcal{E} \in \underbrace{P(\phi)}$ subcategory of $\mathcal{D}$ $\forall \phi \in \mathbb{R}$

$\phi_1 > \phi_2$, $\mathcal{F} \in P(\phi_1)$, $\mathcal{E} \in P(\phi_2)$
$\mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$

- $\mathcal{E} \in P(\phi)$ is stable if $\mathcal{E}$ is a simple object of $P(\phi$
- $P(\phi)[1] = P(\phi + 1)$

(Recall: $A \to B \to C \to A[1]$ exact triangle)

- stable objects: building blocks of $\mathcal{D}$
- $Z : \underbrace{K(\mathcal{D})} \to \mathbb{C}$ group homomorphism

[Grothendieck group, $A \to B \to C \to A[1]$ $\rightsquigarrow$ $[B] = [A] + [C]$

If $\mathcal{E} \neq 0$, $\mathcal{E} \in P(\phi)$:

$$Z(\mathcal{E}) := \underset{K(\mathcal{D})}{Z([\mathcal{E}])} = \underbrace{m(\mathcal{E})}_{\in \mathbb{R}_{>0}} \cdot e^{i\pi\phi}$$

PLUS

- Z factor via Λ fin. dim lattice
- "support property"

If Λ factors via $K_{num}(\mathcal{D})$, σ is called "numerical"

Rmk: can equivalently define:

$$\sigma = (\mathcal{A}, Z_{\mathcal{A}})$$

heart of a bdd t-structure on $\mathcal{D}$

stability function "$Z_{\mathcal{A}} : \mathcal{A} \to \mathbb{H}$" + H.N.

e.g. Rep Q is a ♡ for $D^b(\text{Rep } Q)$. Hence $\sigma = (\text{Rep } Q, Z)$ is a Bridgeland stability condition.

(from earlier

Let X: sm. proj. variety / $\mathbb{C}$, $D^b(X) := D^b(\text{Coh}(X))$.

Def^n A Bridgeland stability condition is geometric if all skyscraper sheaves of points are stable and of the same phase.

$$\mathcal{O}_x \leftrightsquigarrow x \in X$$

$\mathrm{Stab}_\Lambda(X) := \{\text{all Bridgeland stability conditions on } X\}$

↪ Bri'07: This is a complex manifold

$(P, Z) \to Z$ local coordinate

What we know:

- $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$
  $\quad \| $
  $\mathrm{Stab}(\mathrm{Rep}\, P_2)$
- $\mathrm{Stab}(C) \cong \widetilde{GL_2^+(\mathbb{C})}$
  C: sm. proj. curve of $g \geq 1$
- $\mathrm{Stab}(X)$ known when X: abelian surface

Thm [Lie Fu, Chunyi Li, Xiaolei Zhao]

X has finite Albanese morphism $\Rightarrow$ all stability conditions are geometric

?
$\Leftarrow$

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d] & \mathrm{Alb}(X) \arrow[dl, dashed, "\exists!"] \\ B & \end{tikzcd}$$

$\mathrm{Alb}(X) = \mathrm{Pic}^0(\mathrm{Pic}^0(X))$

B abelian