Grassmannians correct date

17 September 2021 14:42

Grassman...
correct date

Grassmannians

Hannah Dell

University of Edinburgh

GLaMS Example Seminar

Projective Space

Let K be a field. Then we define:

- **Projective n-space over K:**

$$\mathbb{P}^n_K = (K^{n+1} \setminus \{0\})/\sim \; = \{\text{lines through } \underline{0} \text{ in } K^{n+1}\}$$

$$(x_0, \dots, x_n) \sim \lambda(x_0, \dots, x_n) \quad \lambda \in K^*$$

we write $[x_0 : \dots : x_n] \in \mathbb{P}^n_K$

- **Projective variety:** $S \subset K[x_0, \dots, x_n]$ homogeneous polynomials

$$V(S) = \{\underline{x} \in \mathbb{P}^n_K : f(\underline{x}) = 0 \;\; \forall f \in S\}$$

We call sets of this form projective varieties

The Grassmannian

Let K be a field, then we define:

Definition

Let $n \in \mathbb{N}_{>0}$, and $k \in \mathbb{N}$ with $0 \leq k \leq n$. The **Grassmannian** of k-planes in K^n is the set:

$$Gr(k,n) = \{k\text{-dimensional linear subspaces of } K^n\}$$

Gr(k,n) is a MODULI SPACE

i.e. a geometric space whose points represent geometric objects of some fixed kind

First Examples

- $\mathrm{Gr}(1, n) = \{$1-dim. linear subspaces of $K^n\}$
  $= \{$lines through 0 in $K^{n}\} = \mathbb{P}^{n-1}_K$
- $\mathrm{Gr}(2, n) = \{$ 2-dim. lin. subspaces of $K^n\}$
  $= \{$1-dim lin. subspaces of $(K^n \setminus \{0\})/\sim\}$
  $=$ lines in $\mathbb{P}^{n-1}$

$$\mathbb{P}^n = \left(K^{n+1} \setminus \{0\}\right)/\sim$$

$\{x_0, \dots, x_n\}$

$[1 : \dots]$

$Gr(2,3) = \{$2-dim. lin subspaces of $\mathbb{C}^3\}$

let e_1, e_2, e_3 basis

$w \in G(2,3)$ is of the form: $w = \mathrm{Lin}(\underbrace{a_1e_1 + a_2e_2 + a_3e_3}_{=v_1}, \underbrace{b_1e_1 + b_2e_2 + b_3e_3}_{=v_2})$
$= \mathrm{Lin}(v_1, v_2)$

e.g. $w = \mathrm{Lin}(e_1 + e_2, e_1 + e_3)$

The Plücker Embedding

WTS $Gr(n,k)$ is a projective variety.

$\hookrightarrow \exists N$ s.t. $Gr(n,k) \hookrightarrow \mathbb{P}^N$

$\hookrightarrow \exists$ homogeneous polys S s.t.

$Gr(n,k) = V(S) \subseteq \mathbb{P}^N$

Alternating Tensor Product

Definition

Let V be a vector space over K, and let $k \in \mathbb{N}$. Then the **k-fold alternating tensor product**, denoted $\Lambda^k V$ is the quotient:

$$\Lambda^k V = {}^{\left(V^{\otimes k}\right)}\!/_{L}$$

where $L = \{v_1 \otimes \cdots \otimes v_k \mid v_1, \ldots, v_k \in V, \ v_i = v_j \text{ for some } i \neq j\}$.

For $x \in \Lambda^k V$ we write: $x = \sum_{i=1}^{N} a_i v_{i_1} \wedge \cdots \wedge v_{i_k}$, where $a_i \in k$.

wedge

$\Lambda^k V$ Key Properties and First Example

What we need to know: $\dim V = n$, $V \cong K^n$, $v_1, \ldots, v_k \in K^n$: $k \leq n$

- Alternating:
  $v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_k = -1 \cdot v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_k$
- Linear dependence: $v_1 \wedge \cdots \wedge v_k = 0 \iff v_1, \ldots, v_k$ linearly dependent
- Basis: let $e_1, \ldots, e_n$ basis of V.

$\Lambda^k V$ basis: $e_{i_1} \wedge \cdots \wedge e_{i_k}$: $i_1 < \cdots < i_k$ in $\{1, \ldots, n\}$ $\therefore \dim \Lambda^k V = \binom{n}{k}$

Consider $\Lambda^2 \mathbb{C}^3$, with the standard basis e_1, e_2, e_3 of $\mathbb{C}^3$.

Let $v = e_1 + e_2$, $w = e_1 + e_3 \in \mathbb{C}^3$

$$\begin{aligned} v \wedge w &= (e_1 + e_2) \wedge (e_1 + e_3) \\ &= e_1 \wedge e_1 + e_2 \wedge e_1 + e_1 \wedge e_3 + e_2 \wedge e_3 \\ &= 0 - e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3 \end{aligned}$$

Example continued

More generally, for any $v, w \in \mathbb{C}^3$:

$$v = a_1 e_1 + a_2 e_2 + a_3 e_3,$$
$$w = b_1 e_1 + b_2 e_2 + b_3 e_3.$$

So we have:

$$v \wedge w = (a_1 b_2 - a_2 b_1) e_1 \wedge e_2 + (a_1 b_3 - a_3 b_1) e_1 \wedge e_3 + (a_2 b_3 - a_3 b_2) e_2 \wedge e_3.$$

cross product

$\Lambda^k V$: (k vectors in $V \cong K^n$) $\rightsquigarrow$ (point in $\Lambda^k K^n$)

⚠ $\mathrm{Lin}(e_1, \dots, e_k) = \mathrm{Lin}(\lambda_1 e_1, \dots, \lambda_k e_k)$

$\lambda_1 e_1 \wedge \dots \wedge \lambda_k e_k = (\lambda_1 \cdots \lambda_k)\, e_1 \wedge \dots \wedge e_k \quad \lambda_j \in K^*$

$v_1 \wedge \dots \wedge v_k = \lambda\, w_1 \wedge \dots \wedge w_k \quad \lambda \in K^*$

$\Leftrightarrow \mathrm{Lin}(v_1, \dots, v_k) = \mathrm{Lin}(w_1, \dots, w_k)$

QUOTIENT BY SCALARS: $\mathrm{Gr}(n,k) \hookrightarrow \Lambda^k V \cong K^{\binom{n}{k}} \xrightarrow{\sim} (K^{\binom{n}{k}} - \{0\})/\sim = \mathbb{P}^{\binom{n}{k}-1}$

The Plücker Embedding

Definition

Consider the map:

$$f : \mathrm{Gr}(k, n) \to \mathbb{P}^{\binom{n}{k}-1} = \mathbb{P}^N$$
$$\mathrm{Lin}(v_1, \dots, v_k) \longmapsto [v_1 \wedge \dots \wedge v_k]$$

This is called the **Plücker embedding** of $\mathrm{Gr}(k, n)$. For $L \in \mathrm{Gr}(k, n)$, the homogeneous coordinates of $f(L)$ in $\mathbb{P}^{\binom{n}{k}-1}$ are called the **Plücker Coordinates** of L.

↳ well defined: basis ✓, $v_1 \wedge \dots \wedge v_k \neq 0$

↳ injective

Plücker Embedding Example

Consider $\mathrm{Gr}(2,3)$, where $V = \mathbb{C}^3$.

Then: $f : \mathrm{Gr}(2,3) \hookrightarrow \mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$

Using the standard basis, we denote the homogeneous coordinates of $\mathbb{P}^2$ by:

$$x_{1,2} = e_1 \wedge e_2 \qquad x_{1,3} = e_1 \wedge e_3 \qquad x_{2,3} = e_2 \wedge e_3 \qquad [x_{1,2} : x_{1,3} : x_{2,3}]$$

Let $L = \mathrm{Lin}(e_1 + e_2, e_1 + e_3) \in \mathrm{Gr}(2,3)$. We saw already that:

$$(e_1 + e_2) \wedge (e_1 + e_3) = -e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3$$

So $f(L) = [-1 : 1 : 1]$

Remark:

L rowspan of $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ minors: $-1, 1, 1$

The Grassmannian as a Projective Variety

Fix any non-zero $\omega \in \Lambda^k K^n$ with $k < n$. Then we can define a K-linear map:

$$f_\omega : K^n \to \Lambda^{k+1} K^n$$
$$v \longmapsto v \wedge \omega$$

Can show: $\omega \in \mathbb{P}^{\binom{n}{k}-1}$ lies in $\mathrm{Gr}(k,n)$

$\iff$ rank $f_\omega = n-k$

$\rightsquigarrow$ vanishing of $(n-k+1)\times(n-k+1)$ minors of matrix of

$\rightsquigarrow$ polynomials in matrix entries = coordinates of ω

$\Rightarrow$ $\mathrm{Gr}(n,k)$ is a projective variety.

Extended Example: Gr(2, 4)

Consider $\omega \in \mathrm{Gr}(2,4)$ and let e_1, e_2, e_3, e_4 be the standard basis for $\mathbb{C}^4$. Then ω corresponds to the row span of the matrix:

$$\begin{array}{c} \\ M_\omega := \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 \end{matrix} \\ \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix} \end{array}$$

$\mathbb{C}\mathbb{P}^N$

↳ minors give Plücker coordinates of ω

↳ look at a coordinate patch: $(x_{1,2} \neq 0) \subseteq \mathbb{P}^N$

$\omega \in (x_{1,2} \neq 0)$

ie. $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$ invertible

$A^{-1} M\omega$ same row span as $M\omega$

$U_0 = (x_{1,2} \neq 0) \cap \mathrm{Gr}(2,4)$

ie. $\omega \leftrightarrow \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ $\cong k^4$

$\omega = [1 : c : d : -a : -b : ad-bc]$

Explicit equations for Gr(2, 4)

We denote the homogeneous coordinates of $\mathbb{P}^{\binom{4}{2}-1} = \mathbb{P}^5$ by $x_{i,j}$: $1 \leq i < j \leq 4$.

$x_{i,j} \leftrightarrow e_i \wedge e_j$

$\mathrm{Lin}(e_1, e_2) \leftrightarrow [1 : 0 : \cdots : 0]$

$\uparrow x_{1,2}$

Consider any $\omega \in \mathbb{P}^5$, then we can write:

$$\omega = [a_{1,2} : a_{1,3} : a_{1,4} : a_{2,3} : a_{2,4} : a_{3,4}]$$

ie.

$\omega = a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + a_{1,4}e_1 \wedge e_4 + a_{2,3}e_2 \wedge e_3 + a_{2,4}e_2 \wedge e_4 + a_{3,4}e_3 \wedge e_4$

TURNS OUT:

$\omega \in \mathrm{Gr}(2,4) \overset{\text{fw}}{\iff} \omega$ completely reducible

$\omega = v_1 \wedge v_2$

$\iff \omega \wedge \omega = 0$

Explicit equations for Gr(2, 4)

So consider:

$\omega = a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + a_{1,4}e_1 \wedge e_4 + a_{2,3}e_2 \wedge e_3 + a_{2,4}e_2 \wedge e_4 + a_{3,4}e_3 \wedge e_4$

$\omega = a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + a_{1,4}e_1 \wedge e_4 + a_{2,3}e_2 \wedge e_3 + a_{2,4}e_2 \wedge e_4 + a_{3,4}e_3 \wedge e_4$

$\ldots\; e_1 \wedge e_2 \wedge e_3 \wedge e_4$

Thus $\omega \wedge \omega =$ $(a_{1,2}a_{3,4}$

$= (a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3} + a_{2,3}a_{1,4} - a_{2,4}a_{1,3} + a_{3,4}a_{1,2})e_1 \wedge e_2 \wedge e_3 \wedge e_4$

$= 2(a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3})e_1 \wedge e_2 \wedge e_3 \wedge e_4$

$\omega \in Gr(2,4) \iff \omega \wedge \omega = 0 \iff \omega \in V(x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}) \subseteq \mathbb{P}^N$. $\mathbb{P}^5$

$\| \; Gr(2,4)$ is a quadric hypersurface

(!) dim 4

Counting lines in $\mathbb{P}^3$.

Q: Given 4 general lines in $\mathbb{P}^3$, how many other lines intersect all of them?

Fix $L \subset \mathbb{P}^3$: $\Sigma_L = \{ l \subset \mathbb{P}^3 \text{ line} : l \cap L \neq \emptyset \}$

After lin. change of coordinates, assume

$L \subset \mathbb{P}^3 \longleftrightarrow \mathrm{Lin}(e_1, e_2) \subseteq \mathbb{C}^4$

ie. rowspan $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

Then $\forall\, l \subseteq \mathbb{P}^3, l \neq L,\ l \leftrightarrow \mathrm{Lin}(v_1, v_2)$, $v_1, v_2 \in \mathbb{C}^4$

Then $l \cap L \neq \emptyset \iff e_1, e_2, v_1, v_2$ do not span $\mathbb{C}^4$

ie. they are linearly dependent

Q: Given 4 general lines in $\mathbb{P}^3$, how many other lines intersect all of them?

WORK ON: $x_{1,2} \neq 0$

$$l \longleftrightarrow \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

e_1, e_2, v_1, v_2 lindependent $\iff \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} = 0$

$\iff \underbrace{ad - bc}_{\text{coeff. of } x_{3,4}} = 0$

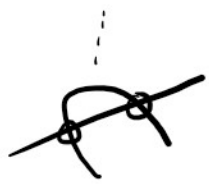
ie. $\boxed{x_{3,4} = 0}$

so: $l \cap L \neq \emptyset \iff l \in V(x_{3,4}) \subseteq \mathbb{P}^5$. Let $H_L := V(x_{3,4}) \subseteq \mathbb{P}^5$

$\Sigma_L = Gr(2,4) \cap H_L$

"$x_{3,4} = 0$"

Q: Given 4 general lines in $\mathbb{P}^3$, how many other lines intersect all of them?



coord. chase $\longrightarrow$ linear equation.

so given $L_1, \dots, L_4$: find $H_{L_1}, \dots, H_{L_4}$

$$\Sigma_{L_1, \dots, L_4} = \underbrace{Gr(2,4)}_{\dim 4} \cap \underbrace{H_{L_1}}_{-1} \cap \dots \cap \underbrace{H_{L_4}}_{-1}$$

Bézout's Theorem

Let $H_1, \dots, H_n$ be hypersurfaces in $\mathbb{P}^n$, with degrees $d_1, \dots, d_n$ respectively. Assume that the H_i have no common components, so that $H_1 \cap \ldots \cap H_n$ is a finite set. Then

$$\sum_{P \in H_1 \cap \ldots \cap H_n} m_P(H_1, \dots, H_n) = d_1 \cdots d_n$$

where $m_P(H_1, \dots, H_n)$ is the intersection multiplicity of $H_1, \dots, H_n$ at P.

$$\#(\Sigma_{L_1, \dots, L_4}) = 2 \cdot 1 \cdots 1 = 2$$

Thank you for listening!

Any questions?

GATHMANN: Algebraic Geometry, §8

HARRIS: Alg. Geom. First Course

Lecture 6+.