STABILITY CONDITIONS ON FREE ABELIAN QUOTIENTS

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ABSTRACT. We study slope-stable vector bundles and Bridgeland stability conditions on varieties which are a quotient of a smooth projective variety by a finite abelian group G acting freely. We show there is a one-to-one correspondence between \widehat{G} -invariant geometric stability conditions on the quotient and G-invariant geometric stability conditions on the cover. We apply our results to describe a connected component inside the stability manifolds of free abelian quotients when the cover has finite Albanese morphism. This applies to varieties with non-finite Albanese morphism which are free abelian quotients of varieties with finite Albanese morphism, such as Beauville-type and bielliptic surfaces. This gives a partial answer to a question raised by Lie Fu, Chunyi Li, and Xiaolei Zhao: If a variety X has non-finite Albanese morphism, does there always exist a non-geometric stability condition on X? We also give counterexamples to a conjecture of Fu-Li-Zhao concerning the Le Potier function, which characterises Chern classes of slope-semistable sheaves. As a result of independent interest, we give a description of the set of geometric stability conditions on an arbitrary surface in terms of a refinement of the Le Potier function. This generalises a result of Fu-Li-Zhao from Picard rank one to arbitrary Picard rank.

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1. Introduction

In this article, we study stability conditions on varieties that are free quotients by finite abelian groups, especially quotients of varieties with finite Albanese morphism such as bielliptic and Beauville-type surfaces.

One approach is via group actions on triangulated categories. We sharpen the correspondence between G-invariant stability conditions on $\mathcal D$ and stability conditions on the G-equivariant category $\mathcal D_G$ introduced by Macrì, Mehrotra, and Stellari in [MMS09, Proposition 2.17]. This is used to control the set of geometric stability conditions on any free quotient by a finite abelian group.

We also study the Le Potier function introduced by Fu, Li, and Zhao in [FLZ22, §3.1]. We give counter examples to the conjecture stated in [FLZ22, §4], and explain how a refinement of the Le Potier function controls the set of geometric Bridgeland stability conditions on any surface.

1.1. **Geometric Stability Conditions and Group Actions.** Let k be an algebraically closed field, and let G be a finite abelian group such that $(\operatorname{char}(k), |G|) = 1$. Let \mathcal{D} be a k-linear additive idempotent complete triangulated category with an action of G in the sense of [Del97]. This induces an action on $\operatorname{Stab}(\mathcal{D})$, the space of all numerical Bridgeland stability conditions on \mathcal{D} . Let \mathcal{D}_G denote the corresponding category of G-equivariant objects. There is a residual action by $\widehat{G} = \operatorname{Hom}(G, k^*)$

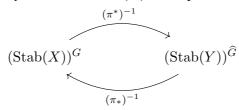
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on \mathcal{D}_G (see Proposition 2.5), and $(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}$ by [Ela15, Theorem 4.2]. Lemma 2.23 describes a one-to-one correspondence between G-invariant stability conditions on \mathcal{D} and \widehat{G} -invariant stability conditions on \mathcal{D}_G . This builds on the abelian case of [Pol07, Proposition 2.2.3] and [MMS09, Theorem 1.1], and was independently obtained in [PPZ23, Lemma 4.1].

In this paper, we focus on the case where $\mathcal{D}=\mathrm{D}^{\mathrm{b}}(X)$ for X a smooth projective variety over \mathbf{C} , and the action of G on \mathcal{D} is induced by a free action by G on X. Then $(\mathrm{D}^{\mathrm{b}}(X))_G\cong\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X):=\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_G(X))$, the bounded derived category of G-equivariant coherent sheaves on X. Let $\pi\colon X\to Y:=X/G$. We call Y a free abelian quotient. Then $\mathrm{D}^{\mathrm{b}}(Y)\cong\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$. There is a decomposition of $\pi_*\mathcal{O}_X$ into line bundles \mathcal{L}_χ according to the 1-dimensional representations $\chi\in\widehat{G}$. Then $-\otimes\mathcal{L}_\chi\colon\mathrm{D}^{\mathrm{b}}(Y)\to\mathrm{D}^{\mathrm{b}}(Y)$ describes the residual action of \widehat{G} .

A stability condition $\sigma \in \operatorname{Stab}(X) := \operatorname{Stab}(\operatorname{D^b}(X))$ is called *geometric* if all skyscraper sheaves of points \mathcal{O}_x are σ -stable and of the same phase. In all known examples, the stability manifold contains an open set of geometric stability conditions. We prove that geometric stability conditions are preserved under the correspondence of Lemma 2.23:

Theorem 3.3. Suppose G is a finite abelian group acting freely on a smooth projective variety X. Let $\pi\colon X\to Y:=X/G$ denote the quotient map. Consider the action of \widehat{G} on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)\cong\mathrm{D}^{\mathrm{b}}(Y)$ as in Proposition 2.5. Then there is a one-to-one correspondence between G-invariant stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ and \widehat{G} -invariant stability conditions on $\mathrm{D}^{\mathrm{b}}(Y)$ which preserves geometric stability conditions:



The compositions $(\pi_*)^{-1} \circ (\pi^*)^{-1}$ and $(\pi^*)^{-1} \circ (\pi_*)^{-1}$ fix slicings and rescale central charges by |G|. In particular, suppose $\sigma = (\mathcal{P}_{\sigma}, Z_{\sigma}) \in (\operatorname{Stab}(X))^G$ satisfies the support property with respect to (Λ, λ) . Then $(\pi^*)^{-1}(\sigma) =: \sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in (\operatorname{Stab}(Y))^{\widehat{G}}$ is defined by:

$$\mathcal{P}_{\sigma_Y}(\phi) = \{ \mathcal{E} \in D^b(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \},$$
$$Z_{\sigma_Y} = Z_{\sigma} \circ \pi^*,$$

where π^* is the natural induced map on $K(D^b(Y))$, and σ_Y satisfies the support property with respect to $(\Lambda, \lambda \circ \pi^*)$.

Very little is known about how the geometry of a variety X relates to the geometry of $\mathrm{Stab}(X)$. Recall that every algebraic variety X has a map alb_X , the Albanese morphism, to $\mathrm{Alb}(X) := \mathrm{Pic}^0(\mathrm{Pic}^0(X))$, the Albanese variety. It is algebraic, and every morphism $f\colon X\to A$ to another abelian variety A factors via alb_X . In [FLZ22, Theorem 1.1], the authors showed that if X has finite Albanese morphism, then all stability conditions on $\mathrm{D^b}(X)$ are geometric. In this set-up, we obtain a union of connected components of geometric stability conditions on any free abelian quotient of X.

Theorem 3.9. Let X be a smooth projective variety with finite Albanese morphism. Let G be a finite abelian group acting freely on X and let Y = X/G. Then $\operatorname{Stab}^{\dagger}(Y) := (\operatorname{Stab}(Y))^{\widehat{G}}$ is a union of connected components consisting only of geometric stability conditions.

When X is a surface, we have the following stronger result.

Corollary 3.10. Let X be a smooth projective surface with finite Albanese morphism. Let G be an abelian group acting freely on X. Let S = X/G. Then $\operatorname{Stab}^{\dagger}(S) = (\operatorname{Stab}(X))^G = \operatorname{Stab}^{\operatorname{Geo}}(S)$. In particular, $\operatorname{Stab}^{\operatorname{Geo}}(S)$ is a connected component of $\operatorname{Stab}(S)$.

We explain in §1.3 how to describe $\operatorname{Stab}^{\operatorname{Geo}}(S)$ explicitly for any surface S. Moreover, Corollary 3.10 applies to the following 2 classes of minimal surfaces.

Example 1.1 (Beauville-type surfaces, q=0). Let $X=C_1\times C_2$, where $C_i\subset {\bf P}^2$ are smooth projective curves of genus $g(C_i)\geq 2$. Each curve has finite Albanese morphism, and hence so does X. Suppose there is a free action of a finite group G on X, such that S=X/G has $q(S):=h^1(S,\mathcal{O}_S)=0$ and $p_g(S):=h^2(S,\mathcal{O}_S)=0$. Then alb_S is trivial. This generalises a construction due to Beauville in [Bea96, Chapter X, Exercise 4], and we call S a Beauville-type surface. These are classified in [BCG08, Theorem 0.1]. There are 17 families, 5 of which involve an abelian group. In the abelian cases, G is one of the following groups: $({\bf Z}/2{\bf Z})^3$, $({\bf Z}/2{\bf Z})^4$, $({\bf Z}/3{\bf Z})^2$, $({\bf Z}/5{\bf Z})^2$.

Example 1.2 (Bielliptic surfaces, q=1). Let $S\cong (E\times F)/G$, where E,F are elliptic curves, and G is a finite group of translations of E acting on F such that $F/G\cong \mathbf{P}^1$. Then q(S)=1 and $\mathrm{Alb}(S)\cong E/G$, so alb_S is an elliptic fibration. Such surfaces are called *bielliptic* and were first classified in [BD07]. There are 7 families, see [Bea96, List VI.20].

Let S be a Beauville-type or bielliptic surface. As discussed above, S has non-finite Albanese morphism. By Corollary 3.10, $\operatorname{Stab}^{\operatorname{Geo}}(S) \subset \operatorname{Stab}(S)$ is a connected component. In particular, if $\operatorname{Stab}(S)$ is connected, then the following question would have a negative answer.

Question 1.3 ([FLZ22, Question 4.11]). Let X be a smooth projective variety whose Albanese morphism is not finite. Are there always non-geometric stability conditions on $D^b(X)$?

This is the converse of [FLZ22, Theorem 1.1]. In all other known examples, the answer to Question 1.3 is positive (see §1.4).

1.2. **The Le Potier Function.** A fundamental problem in the study of stable sheaves on a smooth projective variety X is to understand the set of Chern characters of stable sheaves. This can be used to describe $\operatorname{Stab}^{\operatorname{Geo}}(X)$ for surfaces (see Theorem 5.10) and to control wall-crossing and hence indirectly control Brill-Noether phenomena as in [Bay18, Theorem 1.1] and [Fey20].

When studying slope-stable sheaves on a smooth projective complex variety X, a natural question is for which topological invariants (i.e. Chern character) slope-stable sheaves exist. For $X = \mathbf{P}^2$, Drézet and Le Potier gave a complete solution in [DL85, Theorem B] in terms of a function of the slope, $\delta \colon \mathbf{R} \to \mathbf{R}$. In [FLZ22, §3.1], the authors define a Le Potier function $\Phi_{X,H}$ which gives a generalisation of Drézet and Le Potier's function to any smooth projective polarised surface (X,H). They use this to control geometric Bridgeland stability conditions with respect to a sublattice of the numerical K-group of X, $K_{\text{num}}(X)$, coming from the polarisation.

Let $\mathrm{NS}_{\mathbf{R}}(X) := \mathrm{NS}(X) \otimes \mathbf{R}$, where $\mathrm{NS}(X)$ is the Néron-Severi group of X, and let $\mathrm{Amp}_{\mathbf{R}}(X)$ denote the ample cone inside $\mathrm{NS}_{\mathbf{R}}(X)$. In §4 we introduce a generalisation of the Le Potier function which will be used to control the set of all geometric stability conditions in §5. We state the version for surfaces below to ease notation.

Definition 5.8. Let X be a smooth projective surface. Let $(H, B) \in Amp_{\mathbf{R}}(X) \times NS_{\mathbf{R}}(X)$. We define the Le Potier function twisted by B, $\Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R}$, as

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} \, : \begin{matrix} F \in \operatorname{Coh}(X) \text{ is H-semistable with} \\ \mu_H(F) = \mu \end{matrix} \right\}.$$

The Bogomolov-Gieseker inequality gives an upper bound for $\Phi_{X,H,B}$ (see Lemma 4.6). If B=0, this is the same as [FLZ22, Definition 3.1], i.e. $\Phi_{X,H,0}=\Phi_{X,H}$, and the upper bound is $\frac{x^2}{2}$. $\Phi_{X,H,B}$ naturally generalises to higher dimensions, see Definition 4.4.

The Le Potier function partially determines the non-emptiness of moduli spaces of H-semistable sheaves of a fixed Chern character, which in turn controls wall crossing, along with the birational geometry of these moduli spaces, for example for \mathbf{P}^2 [LZ19, Theorem 0.2, Theorem 0.4], K3 surfaces [BM14, Theorem 5.7], and abelian surfaces [MYY12, Theorem 4.4.1].

The Le Potier function is known for abelian surfaces [Muk84, Corollary 0.2][Yos01], K3 surfaces [Huy16, Chapter 10, Theorem 2.7], del Pezzo surfaces of degrees 9-m for $m \le 6$ [LZ23, Theorem 7.19], Hirzebruch surfaces [CH21, Theorem 9.7], and for surfaces with finite Albanese morphism [LR21, Example 2.12(2)].

In this paper, we relate the Le Potier function of X to the Le Potier function of any free quotient of X by a finite group. We state these results for surfaces with B=0 below.

Proposition 4.10. Let X be a smooth projective surface, and let G be a finite group acting freely on X. Let $\pi \colon X \to X/G =: S$ denote the quotient map, and let $H_S \in \mathrm{Amp}_{\mathbf{R}}(S)$. Then $\Phi_{S,H_S} = \Phi_{X,\pi^*H_S}$.

Proposition 4.10 gives us a way to compute the Le Potier function of varieties that are finite free quotients of varieties with finite Albanese morphism.

Corollary 4.15. Let X be a smooth projective surface with finite Albanese morphism alb_X . Let G be a finite group acting freely on X. Let $\pi\colon X\to X/G=:S$ denote the quotient map. Let $H_X=\mathrm{alb}_X^*H=\pi^*H_S\in\mathrm{Amp}_{\mathbf{R}}(X)$ be an ample class pulled back from $\mathrm{Alb}(X)$ and S. Then $\Phi_{S,H_S}(x)=\frac{x^2}{2}$.

In Example 4.16 we explain how to choose appropriate ample classes such that Corollary 4.15 applies to bielliptic and Beauville-type surfaces. In particular, Beauville-type surfaces provide counterexamples to the following conjecture:

Conjecture 1.4 ([FLZ22, §4]). Let (S, H) be a smooth polarised surface with q = 0, then the Le Potier function $\Phi_{S,H}$ is not continuous at 0.

This conjecture was motivated by Question 1.3 and the expectation that discontinuities of $\Phi_{S,H}$ could be used to show the existence of a wall of the geometric chamber for regular surfaces, as in the cases of rational and K3 surfaces.

1.3. The Le Potier Function and Geometric Stability Conditions. Let X be a surface and fix $H \in \operatorname{Amp}_{\mathbf{R}}(X)$. In [FLZ22, Theorem 3.4, Proposition 3.6], Fu, Li, and Zhao show that $\Phi_{X,H}$ gives precise control over $\operatorname{Stab}_{H}^{\operatorname{Geo}}(X)$, the set of geometric numerical Bridgeland stability conditions with respect to a specific lattice, Λ_H . When X has Picard rank 1, $\operatorname{Stab}_{H}^{\operatorname{Geo}}(X) = \operatorname{Stab}_{H}^{\operatorname{Geo}}(X)$.

We generalise this to the set of all geometric numerical Bridgeland stability conditions.

Theorem 5.10. Let X be a smooth projective surface. Then

$$\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \left\{ (H, B, \alpha, \beta) \in (\mathrm{NS}_{\mathbf{R}}(X))^2 \times \mathbf{R}^2 : H \text{ is ample, } \alpha > \Phi_{X,H,B}(\beta) \right\}.$$

In particular, $\operatorname{Stab}^{\operatorname{Geo}}(X)$ is connected. We discuss in Remark 5.40 how Theorem 5.10 could be used to describe the boundary of $\operatorname{Stab}^{\operatorname{Geo}}(X)$. This emphasises how $\Phi_{X,H,B}$ is a crucial tool for understanding the existence of non-geometric stability conditions on surfaces. In particular, if one can compute the Le Potier function, one should be able to tell whether the boundary of the set of geometric stability conditions has a wall.

1.4. **Survey: Geometric Stability Conditions.** To give context for the results in this paper, we survey the cases where a connected component of the stability manifold is known, and where geometric and non-geometric stability conditions have been described.

There are the following general results:

- Varieties with alb_X finite: $Stab(X) = Stab^{Geo}(X)$ [FLZ22, Theorem 1.1]
- Quotients of varieties with alb_X finite: Let Y = X/G be a free abelian quotient of X, and assume alb_X is finite. If G-invariant stability conditions exist on X, then $\mathrm{Stab}^{\mathrm{Geo}}(Y) \cong (\mathrm{Stab}(X))^G$ is a union of connected components consisting only of geometric stability conditions, see Theorem 3.9.

The results for specific examples are summarised in the following table:

$\dim X$	$\operatorname{Stab}^{\operatorname{Geo}}(X)$	$\operatorname{Stab}(X) \neq \operatorname{Stab}^{\operatorname{Geo}}(X)$?
1	$\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$	$\operatorname{Stab}(\mathbf{P}^1) \cong \mathbf{C}^2$
2	controlled by $\Phi_{X,H,B}$	${f P}^2$, K3 surfaces, rational surfaces, $X\supset C$ rational
		curve s.t. $C^2 < 0$
3	$\neq \emptyset$ for some 3folds	\mathbf{P}^3
≥ 4	$ \neq \emptyset \text{ for } \mathbf{P}^n $	\mathbf{P}^n

Note that the examples in the rightmost column have non-finite Albanese morphism. This gives a positive answer to Question 1.3 in those cases.

Curves: For any curve C, $\operatorname{Stab}^{\operatorname{Geo}}(C) \cong \widetilde{\operatorname{GL}}_2^+(\mathbf{R})$. Up to the action of $\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$, this corresponds to Mukai's slope-stability for $\operatorname{Coh}(C)$ [Mac07b, Theorem 2.7].

- Stab(\mathbf{P}^1) $\cong \mathbf{C}^2$ [Oka06, Theorem 1.1]. Okada's construction uses the identification $\mathrm{D}^\mathrm{b}(\mathbf{P}^1) \cong \mathrm{D}^\mathrm{b}(\mathrm{Rep}(K_2))$ where K_2 is the Kroneker quiver. In particular, these are not all geometric.
- Let C be a curve of genus $g(C) \geq 1$, then $\operatorname{Stab}(C) = \operatorname{Stab}^{\operatorname{Geo}}(C) \cong \widetilde{\operatorname{GL}}_2^+(\mathbf{R})$ [Bri07, Theorem 9.1], [Mac07b, Theorem 2.7].

Surfaces: There is a construction called *tilting* which gives an open set of geometric stability conditions on any smooth projective surface, see for example [AB13],[MS17].

A connected component is known in the following cases:

- Surfaces with finite Albanese morphism: This connected component is precisely the set of geometric stability conditions which come from tilting. This follows from [FLZ22, Theorem 1.1] together with Corollary 5.39.
- K3 surfaces: There is a distinguished connected component $\operatorname{Stab}^{\dagger}(X)$ described by taking the closure and translates under autoequivalences of the open set of geometric stability conditions [Bri08, Theorem 1.1]. By [Bri08, Theorem 12.1], at general points of the boundary of $\operatorname{Stab}^{\operatorname{Geo}}(X)$, either
 - all skyscraper sheaves have a spherical vector bundle as a stable factor, or
 - \mathcal{O}_x is strictly semistable if and only if $x \in C$, a smooth rational curve in X.
- \mathbf{P}^2 : $\operatorname{Stab}(\mathbf{P}^2)$ has a simply-connected component, $\operatorname{Stab}^{\dagger}(\mathbf{P}^2)$, which is a union of geometric and algebraic stability conditions [Li17, Theorem 0.1].
- Enriques surfaces: Suppose Y is an Enriques surface with K3 cover X, and let $\operatorname{Stab}^{\dagger}(X)$ be the connected component of $\operatorname{Stab}(X)$ described above. Then there exists a connected component $\operatorname{Stab}^{\dagger}(Y)$ which embeds into $\operatorname{Stab}^{\dagger}(X)$ as a closed submanifold. Moreover, when Y is very general, $\operatorname{Stab}^{\dagger}(Y) \cong \operatorname{Stab}^{\dagger}(X)$ [MMS09, Theorem 1.2]. $\operatorname{Stab}^{\dagger}(X)$ has non-geometric stability conditions, hence by Theorem 3.3 so does $\operatorname{Stab}^{\dagger}(Y)$.
- Beauville-type and bielliptic surfaces: Let S = X/G. There is a connected component $\operatorname{Stab}^{\dagger}(S) = \operatorname{Stab}^{\operatorname{Geo}}(X) = (\operatorname{Stab}(X))^G$, see Corollary 3.10. If $\operatorname{Stab}(S)$ is connected, this would give a negative answer to Question 1.3, in contrast to all previous examples.

Non-geometric stability conditions are known to exist in the following cases:

- Rational surfaces: the boundary of the geometric chamber contains points where skyscrapers sheaves are destabilised by exceptional bundles. This is explained for $Tot(\mathcal{O}_{\mathbf{P}^2}(-3))$ in [BM11, §5], and the arguments generalise to any rational surface.
- Surfaces which contain a smooth rational curve C with negative self intersection: these have a wall of the geometric chamber such that \mathcal{O}_x is stable if $x \notin C$, and strictly semistable if $x \in C$ [TX22, Lemma 7.2].

Threefolds: Fix $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. Denote by $\mathrm{Stab}_H(X)$ the space of stability conditions such that the central charge factors via a certain lattice $\Lambda_H \subset \mathrm{K}_{\mathrm{num}}(X)$. If $\rho(X) = 1$, this gives rise to elements of $\mathrm{Stab}(X)$. A strategy for constructing stability conditions in $\mathrm{Stab}_H(X)$ for threefolds was first introduced in [BMT13, §3, §4]. This uses so-called tilt stability conditions to construct geometric stability conditions if a stronger BG-type inequality is satisfied.

Geometric stability conditions in $\operatorname{Stab}_H(X)$ exist for some threefolds, see [BMS16, Theorem 1.4], [Ber+17, Theorem 1.1], [Piy17, Theorem 1.3], [Kos18, Theorem 1.2], [Li19, Theorem 1.3], [Kos20, Theorem 1.2], [Kos22, Theorem 1.3], [Liu22, Theorem 1.2].

Below we describe the only threefolds where $\operatorname{Stab}(X)$ is known to be non-empty. These are also the only cases where a connected component of $\operatorname{Stab}_H(X)$ was previously known.

• Abelian threefolds: There is a distinguished connected component $\operatorname{Stab}_H^{\dagger}(X)$ of $\operatorname{Stab}_H(X)$ which has been completely described [BMS16, Theorem 1.4]. These have been shown to satisfy the full support property, in particular, they lie in a connected component $\operatorname{Stab}^{\dagger}(X) \subset \operatorname{Stab}(X)$ [OPT22, Theorem 3.21]. Abelian threefolds are also a case of [FLZ22, Theorem 1.1].

• Calabi-Yau threefolds of abelian type: Let Y be a Calabi-Yau threefold admitting an abelian threefold X as a finite étale cover. There is a distinguished connected component $\operatorname{Stab}_H^{\dagger}(Y)$ of $\operatorname{Stab}_H(Y)$ induced from $\operatorname{Stab}_H^{\dagger}(X)$ [BMS16, Corollary 10.3]. By the previous paragraph together with Theorem 3.3, $\operatorname{Stab}(Y)$ is also non-empty.

The only examples where non-geometric stability conditions are known to exist are those with complete exceptional collections. We explain this in greater generality below.

Exceptional collections: There are stability conditions on any triangulated category with a complete exceptional collections called *algebraic stability conditions* [Mac07b, §3]. On \mathbf{P}^n , this has been used to show the existence of geometric stability conditions [Mu21, Proposition 3.5] [Pet22, §3.3]. If X is a smooth projective variety with a complete exceptional collection, non-geometric stability conditions can be constructed from hearts that do not contain skyscraper sheaves [Mac07a, §4.2].

1.5. Notation.

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\mathcal{D} a triangulated category
                        G a finite group such that (\operatorname{char}(k), |G|) = 1
                      \mathcal{D}_G the category of G-equivariant objects
                        X a smooth projective variety over an algebraically closed field
                 \mathrm{D^b}(X) the bounded derived category of coherent sheaves on X
                 D_{\mathbf{G}}^{\mathbf{b}}(X)
                             the bounded derived category of G-equivariant coherent sheaves
         K(\mathcal{D}), K(X)
                             the Grothendieck group of \mathcal{D}, resp. D^{b}(X)
                             the numerical Grothendieck group of \mathcal{D}, resp. D^{b}(X)
K_{\text{num}}(\mathcal{D}), K_{\text{num}}(X)
                             the space of numerical Bridgeland stability conditions on \mathcal{D}, D^{b}(X)
  \operatorname{Stab}(\mathcal{D}), \operatorname{Stab}(X)
           \operatorname{Stab}^{\operatorname{Geo}}(X)
                             the space of geometric numerical stability conditions on D^{b}(X)
                             the Chern character of an object E \in D^b(X)
                  ch(E)
                             \operatorname{Pic}(X)/\operatorname{Pic}^0(X), the Néron-Severi group of X
                 NS(X)
              NS_{\mathbf{R}}(X)
                             NS(X) \otimes \mathbf{R}
            \mathrm{Amp}_{\mathbf{R}}(X)
                             the ample cone inside NS_{\mathbf{R}}(X)
              \mathrm{Eff}_{\mathbf{R}}(X)
                             the effective cone inside NS_{\mathbf{R}}(X)
             \operatorname{Chow}(X)
                             the Chow group of X
        \operatorname{Chow}_{\operatorname{num}}(X)
                             the numerical Chow group up of X
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2. G- and \widehat{G} -invariant stability conditions

We review the notions of equivariant triangulated categories in §2.1 and Bridgeland stability conditions in §2.2. In §2.3 we describe a correspondence between stability conditions on a triangulated category with an action of a finite abelian group and stability conditions on the corresponding equivariant category.

2.1. Review: G-equivariant triangulated categories. Let \mathcal{C} be a pre-additive category, linear over a ring k. Let G be a finite group with $(\operatorname{char}(k), |G|) = 1$. The definition of a group action on a category and the corresponding equivariant category are due to Deligne [Del97]. We will follow the treatment by Elagin from [Ela15] in our presentation below.

Definition 2.1 ([Ela15, Definition 3.1]). A (right) action of G on C is defined by the following data:

- a functor $\phi_q \colon \mathcal{C} \to \mathcal{C}$, for every $g \in G$;
- a natural isomorphism $\varepsilon_{g,h} \colon \phi_g \phi_h \to \phi_{hg}$ for every $g,h \in G$, for which all diagrams

$$\begin{array}{ccc} \phi_f \phi_g \phi_h & \xrightarrow{\varepsilon_{g,h}} \phi_f \phi_{hg} \\ \downarrow^{\varepsilon_{f,g}} & \downarrow^{\varepsilon_{f,gh}} \\ \phi_{gf} \phi_h & \xrightarrow{\varepsilon_{gf,h}} \phi_{hgf} \end{array}$$

are commutative.

Example 2.2 ([Ela15, Example 3.4]). Let G be a group acting on a scheme X. For each $g \in G$, let $\phi_g := g^* \colon \operatorname{Coh}(X) \to \operatorname{Coh}(X)$. Then for all $g, h \in G$ there are canonical isomorphisms:

$$\phi_g \phi_h = g^* h^* \xrightarrow{\sim} (hg)^* = \phi_{hg}.$$

Together these define an action of G on the category Coh(X).

Definition 2.3 ([Ela15, Definition 3.5]). Suppose G acts on a category C. A G-equivariant object in C is a pair $(F, (\theta_g)_{g \in G})$ where $F \in Ob \mathcal{C}$ and $(\theta_g)_{g \in G}$ is a family of isomorphisms

$$\theta_q \colon F \to \phi_q(F),$$

such that all diagrams

$$F \xrightarrow{\theta_g} \phi_g(F)$$

$$\downarrow^{\theta_{hg}} \qquad \qquad \downarrow^{\phi_g(\theta_h)}$$

$$\phi_{hg}(F) \xleftarrow{\varepsilon_{g,h}} \phi_g(\phi_h(F))$$

are commutative. We call the family of isomorphisms a G-linearisation. A morphism of G-equivariant objects from $(F_1,(\theta_g^1))$ to $(F_2,(\theta_g^2))$ is a morphism $f\colon F_1\to F_2$ compatible with θ_g , i.e. such that the below diagrams commute for all $g\in G$

$$F_{1} \xrightarrow{\theta_{g}^{1}} \phi_{g}(F_{1})$$

$$\downarrow^{f} \qquad \qquad \downarrow^{\phi_{g}(f)}$$

$$F_{2} \xrightarrow{\theta_{g}^{2}} \phi_{g}(F_{2}).$$

The category of G-equivariant objects in \mathcal{C} is denoted \mathcal{C}_G

Example 2.4. Let G be a group acting on a scheme X with ϕ_g and $\varepsilon_{g,h}$ defined as in Example 2.2. G-equivariant objects are G-equivariant coherent sheaves. Let $\mathrm{Coh}_G(X) := (\mathrm{Coh}(X))_G$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_G(X))$. Suppose $k = \overline{k}$ and G acts freely on a smooth projective variety X over k. Let $\pi \colon X \to X/G$ be the quotient map. Then $\mathrm{Coh}(X/G) \cong \mathrm{Coh}_G(X)$ via $\mathcal{E} \mapsto (\pi^*\mathcal{E}, (\theta_g))$, where $\theta_g \colon \pi^*\mathcal{E} = \bigoplus_{h \in G} h^*\mathcal{E} \stackrel{\sim}{\to} \bigoplus_{h \in G} g^*(h^*\mathcal{E})$, and $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong \mathrm{D}^{\mathrm{b}}(X/G)$.

Proposition 2.5. [Ela15, p. 12] Suppose G is an abelian group and k is algebraically closed. Let $\widehat{G} = \operatorname{Hom}(G, k^*)$ be the group of 1-dimensional representations of G. Then there is an action of \widehat{G} on C_G . For every $\chi \in \widehat{G}$, ϕ_{χ} is given by

$$\phi_{\chi}((F,(\theta_h))) := (F,(\theta_h)) \otimes \chi := (F,(\theta_h \cdot \chi(h)))$$

For $\chi, \psi \in \widehat{G}$ the equivariant objects $\phi_{\chi}(\phi_{\psi}((F), (\theta_h)))$ and $\phi_{\psi\chi}((F, (\theta_h)))$ are the same, hence we set the isomorphisms $\varepsilon_{\chi,\psi}$ to be the identities.

Definition 2.6. Suppose G acts on a category \mathcal{C} . Then we denote by $\operatorname{Forg}_G \colon \mathcal{C}_G \to \mathcal{C}$ the forgetful functor $\operatorname{Forg}_G(F,(\theta_g)) = F$. Also let $\operatorname{Inf}_G \colon \mathcal{C} \to \mathcal{C}_G$ be the inflation functor which is defined by

$$\operatorname{Inf}_G(F) := \left(\bigoplus_{g \in G} \phi_g(F), (\xi_g) \right),$$

where

$$\xi_g \colon \bigoplus_{h \in G} \phi_h(F) \xrightarrow{\sim} \bigoplus_{h \in G} \phi_g \phi_h(F)$$

is the collection of isomorphisms

$$\varepsilon_{g,h}^{-1} \colon \phi_{hg}(F) \to \phi_g \phi_h(F).$$

Lemma 2.7. The forgetful functor $Forg_G$ is faithful, and it is left and right adjoint to Inf_G .

Proof. Faithfulness follows immediately from the definition of morphisms between G-equivariant objects. For the fact that $Forg_G$ is left and right adjoint to Inf_G see [Ela15, Lemma 3.8]

The following proposition builds on a result of Balmer in [Bal11, Theorem 5.17].

Proposition 2.8 ([Ela15, Corollary 6.10]). Suppose G acts on a triangulated category C which has a DG-enhancement, then C_G is triangulated in such a way that $Forg_G$ is exact.

The proof of the following theorem will use comonads. The full definitions can be found in [Ela15, §2] but for the proof we will only need to know the following: Given a comonad T on a category \mathcal{C} , a comodule over T is a pair (F,h) where $F\in \mathrm{Ob}\,\mathcal{C}$ and $h\colon F\to TF$ is a morphism, called the comonad structure, satisfying certain conditions (see [Ela15, Definition 2.5]). All comodules over a given comonad T on \mathcal{C} form a category which is denoted \mathcal{C}_T . There is a forgetful functor $\mathrm{Forg}_T\colon \mathcal{C}_T\to \mathcal{C}$ which forgets the comonad structure, i.e. $(F,h)\mapsto F$.

Theorem 2.9 ([Ela15, Theorem 4.2]). Suppose k is an algebraically closed field and let \mathcal{D} be a k-linear additive idempotent complete category. Let G be a finite abelian group with $(\operatorname{char}(k), |G|) = 1$. Suppose \mathcal{D} is a k-linear additive idempotent complete category and G acts on \mathcal{D} . Then

$$(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}.$$

In particular, under this equivalence $\operatorname{Forg}_{\widehat{G}} \colon (\mathcal{D}_G)_{\widehat{G}} \to \mathcal{D}_G$ is identified with $\operatorname{Inf}_G \colon \mathcal{D} \to \mathcal{D}_G$ and their adjoints $\operatorname{Inf}_{\widehat{G}} \colon \mathcal{D}_G \to (\mathcal{D}_G)_{\widehat{G}}$ and $\operatorname{Forg}_G \colon \mathcal{D}_G \to \mathcal{D}$ are also identified.

Proof. Elagin's proof that $(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}$ uses the following chain of equivalences:

$$(\mathcal{D}_G)_{\widehat{G}} \overset{(1)}{\cong} (\mathcal{D}_G)_{T(\operatorname{Forg}_{\widehat{G}},\operatorname{Inf}_{\widehat{G}})} \overset{(2)}{\cong} (\mathcal{D}_G)_{\mathcal{R}} \overset{(3)}{\cong} (\mathcal{D}_G)_{T(\operatorname{Inf}_G,\operatorname{Forg}_G)} \overset{(4)}{\cong} \mathcal{D},$$

where $T(\operatorname{Forg}_{\widehat{G}},\operatorname{Inf}_{\widehat{G}}),\mathcal{R},T(\operatorname{Inf}_{G},\operatorname{Forg}_{G})$ are comonads on the corresponding categories. The equivalences in (1) and (4) are the comparison functors from [Ela15, Proposition 2.6]. In particular, under (1), $\operatorname{Forg}_{\widehat{G}}\cong\operatorname{Forg}_{T(\operatorname{Forg}_{\widehat{G}},\operatorname{Inf}_{\widehat{G}})}$ and under (4), $\operatorname{Forg}_{T(\operatorname{Inf}_{G},\operatorname{Forg}_{G})}\cong\operatorname{Inf}_{G}$. Moreover the equivalences (2) and (3) only change the comonad structure, hence the images of the forgetful functors for each category of comodules are the same. Therefore under the equivalence $(\mathcal{D}_{G})_{\widehat{G}}$, $\operatorname{Forg}_{\widehat{G}}\cong\operatorname{Inf}_{G}$. Finally recall that $\operatorname{Forg}_{\widehat{G}}$ and $\operatorname{Inf}_{\widehat{G}}$ are left and right adjoint, and as are Forg_{G} and Inf_{G} . Hence $\operatorname{Inf}_{\widehat{G}}\cong\operatorname{Forg}_{G}$ follows immediately. \square

2.2. Review: Bridgeland stability conditions. Let \mathcal{D} be a triangulated category.

Definition 2.10 ([Bri07, Definition 3.3]). A *slicing* \mathcal{P} on \mathcal{D} is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbf{R}$ such that:

- (1) $P(\phi)[1] = P(\phi + 1)$
- (2) If $F_1 \in \mathcal{P}(\phi_1), F_2 \in \mathcal{P}(\phi_2)$, then $\phi_1 > \phi_2 \implies \operatorname{Hom}(F_1, F_2) = 0$
- (3) Every $E \in \mathcal{D}$ has a Harder–Narasimhan (HN) filtration, i.e. there exist real numbers $\phi_1 > \phi_2 > \cdots > \phi_m$, objects $E_i \in \mathcal{D}$, and a collection of distinguished triangles:

$$0 = E_0 \xrightarrow{r} E_1 \xrightarrow{r} E_2 \xrightarrow{r} E_m = E$$

$$A_1 \qquad A_2 \qquad A_m$$

where $A_i \in P(\phi_i)$ for all $1 \le i \le m$.

Remark 2.11. We will denote by $\phi_{\mathcal{P}}^+(E) := \phi_1$, $\phi_{\mathcal{P}}^-(E) := \phi_m$, and $m_{\sigma}(E) := \sum_i |Z(A_i)|$. Moreover, non-zero objects of $\mathcal{P}(\phi)$ are called *semistable* of phase ϕ , and non-zero simple objects of $\mathcal{P}(\phi)$ are called *stable* of phase ϕ .

Definition 2.12 ([Bri07, Definition 5.1]). A *Bridgeland pre-stability condition* on \mathcal{D} is a pair $\sigma = (\mathcal{P}, Z)$ such that:

- (1) \mathcal{P} is a slicing
- (2) $Z \colon K(\mathcal{D}) \to \mathbf{C}$ is a homomorphism such that: for any $E \neq 0$, if $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbf{R}$, then $Z([E]) = m(E)e^{i\pi\phi}$, where $m(E) \in \mathbf{R}_{>0}$.

We call Z the *central charge*.

Definition 2.13. A Bridgeland pre-stability condition $\sigma = (\mathcal{P}, Z)$ on \mathcal{D} satisfies the *support property* (with respect to (Λ, λ)) if

- (1) Z factors via a finite rank lattice Λ , i.e. $Z : K(\mathcal{D}) \xrightarrow{\lambda} \Lambda \to \mathbb{C}$, and
- (2) there exists a quadratic form Q on $K_{num}(\mathcal{D}) \otimes \mathbf{R}$ such that
 - (a) Ker Z is negative definite with respect to Q, and
 - (b) every σ -semistable object $E \in D^b(X)$ satisfies $Q(\lambda(E)) \geq 0$.

A Bridgeland pre-stability condition that satisfies the support property is called a *Bridgeland stability* condition. If λ factors via $K_{num}(X)$, we call σ a numerical Bridgeland stability condition

The set of stability conditions with respect to (Λ, λ) will be denoted $\operatorname{Stab}_{\Lambda}(\mathcal{D})$. Unless stated otherwise, we will assume that all Bridgeland stability conditions are numerical. The set of numerical stability conditions on \mathcal{D} will be denoted by $\operatorname{Stab}(\mathcal{D})$.

As described in [Bri08, §2], $\operatorname{Stab}(\mathcal{D})$ has a natural topology induced by the generalised metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ \left| \phi_{\sigma_2}^-(E) - \phi_{\sigma_1}^-(E) \right|, \left| \phi_{\sigma_2}^+(E) - \phi_{\sigma_1}^+(E) \right|, \left| \log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)} \right| \right\}.$$

Theorem 2.14 ([Bri07, Theorem 7.1]). The space of stability conditions $Stab(\mathcal{D})$ has the natural structure of a complex manifold of dimension $rank(K_{num}(\mathcal{D}))$, such that the map:

$$\mathcal{Z} \colon \operatorname{Stab}(\mathcal{D}) \to \operatorname{Hom}_{\mathbf{Z}}(K_{\operatorname{num}}(\mathcal{D}), \mathbf{C})$$

$$\sigma = (\mathcal{P}, Z) \longmapsto Z$$

is a local homeomorphism at every point of $Stab(\mathcal{D})$.

In other words, the central charge gives a local system of coordinates for the stability manifold.

Remark 2.15 ([MS17, Remark 5.14]). There is a right action on $\operatorname{Stab}(\mathcal{D})$ by the universal cover $\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$ of $\operatorname{GL}^+(\mathbf{R})$, see [MS17, Remark 5.14] for details. If we consider \mathbf{C}^* as a subgroup of $\operatorname{GL}^+(\mathbf{R})$, then this induces an action of $\widetilde{\mathbf{C}^*} = \mathbf{C}$ on $\operatorname{Stab}(\mathcal{D})$.

There is an equivalent characterisation of Bridgeland stability conditions, which uses the notion of a t-structure on a triangulated category. For the general theory of t-structures we refer the reader to [Bei+82, §1.3]. We first need the following definitions:

Definition 2.16. The *heart of a bounded t-structure* in a triangulated category \mathcal{D} is a full additive subcategory \mathcal{A} such that:

- (1) If $k_1 > k_2$ then $\text{Hom}(A[k_1], A[k_2]) = 0$.
- (2) For any object E in \mathcal{D} there are integers $k_1 > k_2 > \cdots > k_n$, and a sequence of exact triangles:

$$0 = E_0 \xrightarrow{F} E_1 \xrightarrow{F} E_2 \xrightarrow{F} \cdots \xrightarrow{F} E_{n-1} \xrightarrow{F} E_n = E$$

$$A_1 \qquad A_2 \qquad A_n$$

such that $A_i \in \mathcal{A}[k_i]$ for 1 < i < n.

Definition 2.17 ([Bri07, Definitions 2.1, 2.2]). Let \mathcal{A} be an abelian category on a triangulated category \mathcal{D} . A *stability function* for \mathcal{A} is a group homomorphism $Z \colon K(\mathcal{A}) \to \mathbf{C}$ such that for every non-zero object E of \mathcal{A} ,

$$Z([E]) \in \mathbf{H} := \{ m \cdot e^{i\pi\phi} \mid m \in \mathbf{R}_{>0}, \phi \in (0,1] \} \subset \mathbf{C}.$$

For every non-zero object E, we define the *phase* by $\phi(E) = \frac{1}{\pi} \arg(Z([E])) \in (0,1]$. We say an object E is Z-stable (resp. semistable) if $E \neq 0$ and for every proper non-zero subobject A we have $\phi(A) < \phi(E)$ (resp. $\phi(A) \leq \phi(E)$).

Definition 2.18 ([Bri07, Definition 2.3]). Let \mathcal{A} be an abelian category and let $Z \colon K(\mathcal{A}) \to \mathbf{C}$ be a stability function on \mathcal{A} . A *Harder-Narasimhan (HN) filtration* of a non zero object E of \mathcal{A} is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots E_{n-1} \subset E_n = E$$

such that each factor $F_i = E_i/E_{i-1}$ (called a Harder-Narasimhan factor) is a Z-semistable object of A, and $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$. Moreover, we say that Z has the Harder-Narasimhan property if every non-zero object of A has a Harder-Narasimhan filtration.

Proposition 2.19 ([Bri07, Proposition 5.3]). To give a Bridgeland pre-stability condition (\mathcal{P}, Z) on a triangulated category \mathcal{D} is equivalent to giving a pair $(Z_{\mathcal{A}}, \mathcal{A})$, where \mathcal{A} is the heart of a bounded t-structure \mathcal{A} on $D^b(X)$ and $Z_{\mathcal{A}}$ is a stability function for \mathcal{A} which has the Harder-Narasimhan property. Moreover, (\mathcal{P}, Z) is a numerical Bridgeland stability condition if and only if $Z_{\mathcal{A}}$ factors via $K_{num}(\mathcal{D})$ and satisfies the support property (Definition 2.13(2)) for $Z_{\mathcal{A}}$ -semistable objects.

2.3. **Inducing stability conditions.** Suppose a finite group G acts on a triangulated category $\mathcal D$ by exact autoequivalences, $\{\Phi_g \mid g \in G\}$. This induces an action on the stability manifold via $\Phi_g \cdot (\mathcal P, Z) = (\Phi_g(\mathcal P), Z \circ (\Phi_g)_*^{-1})$, where $(\Phi_g)_* \colon \mathrm{K}(\mathcal D) \otimes \mathbf C \to \mathrm{K}(\mathcal D) \otimes \mathbf C$ is the natural morphism induced by Φ_g . We say that a stability condition σ is G-invariant if $\Phi_g \cdot \sigma = \sigma$.

The results in [MMS09, §2.2] are stated for *locally finite* Bridgeland stability conditions: If $\sigma = (\mathcal{P}, Z)$ is a pre-stability condition and there exists $\varepsilon > 0$ such that, for all $\phi \in \mathbf{R}$, $\mathcal{P}((\phi - \varepsilon, \phi + \varepsilon))$ is of finite length, then we call σ *locally-finite*. We will write $\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D})$ for the space of all locally-finite stability conditions on \mathcal{D} , and $(\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$ for the G-invariant ones.

Let $\sigma \in (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}))^G$. By Lemma 2.7 and Proposition 2.8, $\operatorname{Forg}_G \colon \mathcal{D}_G \to \mathcal{D}$ is exact and faithful, so we can apply the construction of [MMS09, §2.2]: Define $\operatorname{Forg}_G^{-1}(\sigma) := \sigma_G = (\mathcal{P}_{\sigma_G}, Z_{\sigma_G})$, where

$$\begin{split} \mathcal{P}_{\sigma_G}(\phi) &:= \{\mathcal{E} \in \mathcal{D}_G : \mathrm{Forg}_G(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \}, \\ Z_{\sigma_G} &:= Z_{\sigma} \circ (\mathrm{Forg}_G)_*. \end{split}$$

Here $(\operatorname{Forg}_G)_* \colon \mathrm{K}(\mathcal{D}_G) \otimes \mathbf{C} \to \mathrm{K}(\mathcal{D}) \otimes \mathbf{C}$ is the natural morphism induced by Forg_G .

Proposition 2.20 ([MMS09, Theorem 2.14]). Suppose k is an algebraically closed field. Let \mathcal{D} be an essentially small k-linear additive idempotent complete triangulated category with a DG-enhancement. Let G be a finite abelian group such that $(\operatorname{char}(k), |G|) = 1$. Suppose G acts on \mathcal{D} by exact autoequivalences Φ_g for every $g \in G$. Suppose $\sigma = (\mathcal{P}, Z) \in (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$ is a G-invariant pre-stability condition on \mathcal{D} . Then $\operatorname{Forg}_G^{-1}(\sigma) \in \operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G)$.

Proof. By [Ela15, Corollary 6.10] and our assumptions on \mathcal{D} , it follows that \mathcal{D}_G is a triangulated category.

Suppose $\mathcal{E} \in \mathcal{P}(\phi)$. Then $\operatorname{Forg}_G(\operatorname{Inf}_G(\mathcal{E})) = \bigoplus_{g \in G} \Phi_g(\mathcal{E})$. Since σ is G-invariant, $\Phi_g(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi)$ for all $g \in G$. Moreover, $\mathcal{P}_{\sigma}(\phi)$ is extension closed, hence $\bigoplus_{g \in G} \Phi_g(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi)$. The result then follows from [MMS09, Theorem 2.14].

Lemma 2.21. Assume the hypotheses of Proposition 2.20 and let \widehat{G} act on \mathcal{D}_G by twisting as in Proposition 2.5. Then $\operatorname{Forg}_G^{-1}(\sigma)$ is \widehat{G} -invariant.

Proof. First note that, for every class $[\mathcal{E}] = [(E,(\theta_g))] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D}_G) \otimes \mathbf{C}$, $(\mathrm{Forg}_G)_*([(E,(\theta_g))]) = [E]$. Hence $Z_{\sigma_G}([\mathcal{E}]) = Z_{\sigma} \circ (\mathrm{Forg}_G)_*([(E,(\theta_g))]) = Z_{\sigma}([E])$, where $[E] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D}) \otimes \mathbf{C}$. Moreover, from the definition of \mathcal{P}_{σ_G} , we have:

$$\mathcal{P}_{\sigma_G}(\phi) = \{ \mathcal{E} \in \mathcal{D}_G : \operatorname{Forg}_G(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \}$$
$$= \{ (E, (\theta_g)) \in \mathcal{D}_G : E \in \mathcal{P}_{\sigma}(\phi) \}$$

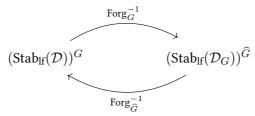
In particular, since the action of \widehat{G} on $(E,(\theta_g)) \in \mathcal{D}_G$ does not change E, it follows that the central charge Z_{σ_G} and slicing \mathcal{P}_{σ_G} are \widehat{G} -invariant, and hence $\sigma_G \in (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$.

Proposition 2.22 ([MMS09, Proposition 2.17]). Under the hypotheses of Proposition 2.20, the morphism $\operatorname{Forg}_G^{-1} : (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}))^G \to (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}_G))^{\widehat{G}}$ is continuous, and the image of $\operatorname{Forg}_G^{-1}$ is a closed embedded submanifold.

Proof. The proof of [MMS09, Proposition 2.17] is for the action of a finite group G on $D^b(X)$, induced by the action of G on X, a smooth projective variety over \mathbb{C} (i.e. $\Phi_g = g^*$). The result follows in our setting by replacing this with the action of exact autoequivalences Φ_g on \mathcal{D} in the proof. \square

In the case where G is abelian, we have the following description of the image of $\operatorname{Forg}_C^{-1}$:

Lemma 2.23. Suppose k is an algebraically closed field. Let $\mathcal D$ be an essentially small k-linear additive idempotent complete triangulated category with a DG-enhancement. Let G be a finite abelian group such that $(\operatorname{char}(k), |G|) = 1$. Suppose G acts on $\mathcal D$ by exact autoequivalences Φ_g for every $g \in G$, and consider the action of $\widehat G$ on $\mathcal D_G$ as in Proposition 2.5. Then there is a one-to-one correspondence between G-invariant stability conditions on $\mathcal D_G$:



In particular, the compositions $\operatorname{Forg}_{\widehat{G}}^{-1} \circ \operatorname{Forg}_{G}^{-1}$ and $\operatorname{Forg}_{G}^{-1} \circ \operatorname{Forg}_{\widehat{G}}^{-1}$ fix slicings and rescale central charges by |G|.

Proof. First note that by [Ela15, Corollary 6.10] and our assumptions on \mathcal{D} it follows that \mathcal{D}_G is a triangulated category.

Let $\sigma \in (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}))^G$. Then by Proposition 2.20 and Lemma 2.21, $\sigma_G := \operatorname{Forg}_G^{-1}(\sigma) \in (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}_G))^{\widehat{G}}$. We now apply the construction of [MMS09, §2.2] again, with \widehat{G} . In particular let $\sigma_{\widehat{G}} := \operatorname{Forg}_{\widehat{G}}^{-1}(\sigma_G)$, where

$$\mathcal{P}_{\sigma_{\widehat{G}}}(\phi) = \{ \mathcal{E} \in (\mathcal{D}_G)_{\widehat{G}} : \operatorname{Forg}_{\widehat{G}}(\mathcal{E}) \in \mathcal{P}_{\sigma_G}(\phi) \}$$
$$= \{ \mathcal{E} \in (\mathcal{D}_G)_{\widehat{G}} : \operatorname{Forg}_G(\operatorname{Forg}_{\widehat{G}}(\mathcal{E})) \in \mathcal{P}_{\sigma}(\phi) \}.$$

By Proposition 2.20, $\operatorname{Forg}_{\widehat{G}}^{-1}(\sigma_G) \in \operatorname{Stab}_{\operatorname{lf}}((\mathcal{D}_G)_{\widehat{G}})$. To complete the proof, we need to show that, under the equivalence $(\mathcal{D}_{\widehat{G}})_G \cong \mathcal{D}$, $\sigma_{\widehat{G}} = \sigma$ up to rescaling the central charge by |G|. From Theorem 2.9 we know that under this equivalence, $\operatorname{Forg}_{\widehat{G}} \cong \operatorname{Inf}_G$. Hence we can apply the same argument as in the proof of [MMS09, Proposition 2.17]. In particular:

$$\begin{split} \mathcal{P}_{\sigma_{\widehat{G}}}(\phi) &= \{\mathcal{E} \in \mathcal{D} : \mathrm{Forg}_G(\mathrm{Inf}_G(\mathcal{E})) \in \mathcal{P}_{\sigma}(\phi)\} \\ &= \left\{\mathcal{E} \in \mathcal{D} : \bigoplus_{g \in G} \Phi_g(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \right\}. \end{split}$$

Suppose $\mathcal{E} \in \mathcal{P}_{\sigma_{\widehat{G}}}(\phi)$. Since taking cohomology commutes with direct sums, $\Phi_g(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi)$ for all $g \in G$. In particular $\mathcal{E} \in \mathcal{P}_{\sigma}(\phi)$ and hence $\mathcal{P}_{\sigma_{\widehat{G}}}(\phi) \subseteq \mathcal{P}_{\sigma}(\phi)$ for all $\phi \in \mathbf{R}$. Now suppose $\mathcal{E} \in \mathcal{P}_{\sigma}(\phi)$, then by the proof of Proposition 2.20 it follows that $\operatorname{Forg}_G(\operatorname{Inf}_G(\mathcal{E})) \in \mathcal{P}_{\sigma}$, and hence $\mathcal{E} \in \mathcal{P}_{\sigma_{\widehat{G}}}(\phi)$. In particular, $\mathcal{P}_{\sigma_{\widehat{G}}}(\phi) \supseteq \mathcal{P}_{\sigma}(\phi)$ for all $\phi \in \mathbf{R}$, so $\mathcal{P}_{\sigma_{\widehat{G}}} = \mathcal{P}_{\sigma}$. Now let $[\mathcal{E}] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D}) \otimes \mathbf{C}$ and consider the central charge:

$$Z_{\sigma_{\widehat{G}}}([\mathcal{E}]) = Z_{\sigma_{G}} \circ (\operatorname{Forg}_{\widehat{G}})_{*}([\mathcal{E}]) = Z_{\sigma} \circ (\operatorname{Forg}_{G})_{*} \circ (\operatorname{Inf}_{G})_{*}([\mathcal{E}]) = Z_{\sigma} \left(\sum_{g \in G} ([\Phi_{g}(\mathcal{E})]) \right).$$

 $Z_{\sigma} \text{ is G-invariant, hence } Z_{\sigma}([\mathcal{E}]) = (\Phi_g)_* Z_{\sigma}([\mathcal{E}]) = Z_{\sigma}([\Phi_g(\mathcal{E})]) \text{ for all } g \in G. \text{ Finally, since } Z_{\sigma} \text{ is a homomorphism, it follows that } Z_{\sigma_{\widehat{G}}}([\mathcal{E}]) = |G| \cdot Z_{\sigma}([\mathcal{E}]).$

Note that if we start instead with a stability condition $\sigma_G \in (\operatorname{Stab}_{\operatorname{lf}}(\mathcal{D}_G))^{\widehat{G}}$, then by a symmetric argument it follows that $\sigma_G = \operatorname{Forg}_G^{-1} \circ \operatorname{Forg}_{\widehat{G}}^{-1}(\sigma_G)$, up to rescaling the central charge by $|\widehat{G}| = |G|$.

Remark 2.24. If $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ where X is a scheme, and if the action of G on \mathcal{D} is induced by an action of G on X, i.e. $\Phi_g = g^*$, then the one-to-one correspondence above follows from the abelian case of [Pol07, Proposition 2.2.3].

3. Geometric stability conditions on abelian quotients

We apply the methods of §2 to describe geometric stability conditions on free abelian quotients. In particular, we show that geometric stability conditions are preserved under the correspondence in Lemma 2.23, and use this to describe a union of connected components of geometric stability conditions on free abelian quotients of varieties with finite Albanese morphism. In the case of surfaces, we obtain a stronger result using a description of the set of geometric stability conditions from §5.

3.1. Inducing geometric stability conditions. Let X be a smooth projective variety over $\mathbb C$. Let G be a finite group acting freely on X. Let Y=X/G and denote by $\pi\colon X\to Y$ the quotient map. Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$ denote the derived category of G-equivariant coherent sheaves on X as in Example 2.4. Recall that $\mathrm{D}^{\mathrm{b}}(Y)\cong\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$, where the equivalence is given by:

$$\Psi \colon \mathrm{D}^{\mathrm{b}}(Y) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$$
$$\mathcal{E} \longmapsto (\pi^{*}(\mathcal{E}), \lambda_{\mathrm{nat}}),$$

and $\lambda_{\mathrm{nat}} = \{\lambda_g\}_{g \in G}$ is the G-linearisation given by:

$$\lambda_g \colon \pi^* \mathcal{E} \xrightarrow{\sim} g^* \pi^* \mathcal{E} = (\pi \circ g)^* \mathcal{E} \cong \pi^* \mathcal{E}.$$

Combining this with §2, we have the following commutative diagram:

The residual action of \widehat{G} on $\mathrm{D^b}(Y)$ is given by tensoring with degree 0 line bundles \mathcal{L}_{χ} for each $\chi \in \widehat{G}$.

Definition 3.1. A Bridgeland stability condition σ on $D^b(X)$ is called *geometric* if for every point $x \in X$, the skyscraper sheaf \mathcal{O}_x is σ -stable.

Proposition 3.2 ([FLZ22, Proposition 2.9]). Let X be a smooth projective variety. Let σ be a geometric numerical stability condition on $D^b(X)$. Then all skyscraper sheaves are of the same phase.

In this context, the correspondence from Lemma 2.23 preserves geometric stability.

Theorem 3.3. Suppose G is a finite abelian group acting freely on a smooth projective variety X. Let $\pi\colon X\to Y:=X/G$ denote the quotient map. Consider the action of \widehat{G} on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)\cong\mathrm{D}^{\mathrm{b}}(Y)$ as in Proposition 2.5. Then there is a one-to-one correspondence between G-invariant stability conditions on $\mathrm{D}^{\mathrm{b}}(X)$ and \widehat{G} -invariant stability conditions on $\mathrm{D}^{\mathrm{b}}(Y)$ which preserves geometric stability conditions:

$$(\operatorname{Stab}(X))^{G} \qquad (\operatorname{Stab}(Y))^{\widehat{G}}$$

$$(\pi_{*})^{-1}$$

The compositions $(\pi_*)^{-1} \circ (\pi^*)^{-1}$ and $(\pi^*)^{-1} \circ (\pi_*)^{-1}$ fix slicings and rescale central charges by |G|. In particular, suppose $\sigma = (\mathcal{P}_\sigma, Z_\sigma) \in (\operatorname{Stab}(X))^G$ satisfies the support property with respect to (Λ, λ) . Then $(\pi^*)^{-1}(\sigma) =: \sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in (\operatorname{Stab}(Y))^{\widehat{G}}$ is defined by:

$$\mathcal{P}_{\sigma_Y}(\phi) = \{ \mathcal{E} \in D^b(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \},$$
$$Z_{\sigma_Y} = Z_{\sigma} \circ \pi^*,$$

where π^* is the natural induced map on $K(D^b(Y))$, and σ_Y satisfies the support property with respect to $(\Lambda, \lambda \circ \pi^*)$.

Proof. First note that $\pi_* \circ \pi^* \colon \mathrm{K}_{\mathrm{num}}(Y) \to \mathrm{K}_{\mathrm{num}}(Y)$ is just multiplication by |G|, as it sends [E] to $\left[E \otimes \bigoplus_{\chi \in \widehat{G}} \mathcal{L}_{\chi}\right]$. Therefore, $\pi^* \colon \mathrm{K}_{\mathrm{num}}(Y) \to \mathrm{K}_{\mathrm{num}}(X)$ is injective.

Together with Lemma 2.23, it follows that $(\pi^*)^{-1}$ and $(\pi_*)^{-1}$ give a one-to-one correspondence

Together with Lemma 2.23, it follows that $(\pi^*)^{-1}$ and $(\pi_*)^{-1}$ give a one-to-one correspondence between numerical Bridgeland stability conditions as described above. It remains to show that $\sigma \in (\operatorname{Stab}(X))^G$ is geometric if and only if $\sigma_Y = (\pi^*)^{-1}(\sigma)$ is.

Step 1. Suppose $\sigma = (\mathcal{P}_{\sigma}, Z_{\sigma}) \in (\operatorname{Stab}(X))^G$ is geometric. Let $y \in Y$, this corresponds to the orbit Gx for some $x \in X$ (so x is unique up to the action of G). We need to show $\pi^*(\mathcal{O}_y)$ is σ_Y -stable. Recall,

$$\mathcal{P}_{\sigma_Y}(\phi) = \{ \mathcal{E} \in D^b(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \}$$

for every $\phi \in \mathbf{R}$. Now consider:

$$\pi^*(\mathcal{O}_y) = \bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} \in \mathrm{D}^\mathrm{b}(X).$$

By our assumption on σ and Proposition 3.2, all skyscraper sheaves of points of X are σ -stable and of the same phase which we denote by ϕ_{sky} . In particular, $\mathcal{O}_{g^{-1}x} \in \mathcal{P}_{\sigma}(\phi_{\text{sky}})$ for all $g \in G$. Moreover, $\mathcal{P}_{\sigma}(\phi_{\text{sky}})$ is extension closed, hence $\bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} \in \mathcal{P}_{\sigma}(\phi_{\text{sky}})$, and thus $\mathcal{O}_y \in \mathcal{P}_{\sigma_Y}(\phi_{\text{sky}})$. Now suppose that \mathcal{O}_y is strictly semistable, then there exist $\mathcal{E}, \mathcal{F} \in \mathcal{P}_{\sigma_Y}(\phi_{\text{sky}})$ such that:

$$\mathcal{E} \hookrightarrow \mathcal{O}_y \twoheadrightarrow \mathcal{F}$$

is non-trivial, i.e. \mathcal{E} is not isomorphic to 0 or \mathcal{O}_y . By definition of $\mathcal{P}_{\sigma_Y}(\phi_{\text{sky}})$, $\pi^*(\mathcal{E})$, $\pi^*(\mathcal{F}) \in \mathcal{P}_{\sigma}(\phi_{\text{sky}})$. Moreover, π^* is exact, hence there is an exact sequence:

$$\pi^*(\mathcal{E}) \hookrightarrow \pi^*(\mathcal{O}_y) = \bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} \twoheadrightarrow \pi^*(\mathcal{F})$$

in $\mathcal{P}_{\sigma}(\phi_{\text{sky}}) \subset D^{\text{b}}(X)$. Since $\pi^*(\mathcal{E})$ is a subobject of $\pi^*(\mathcal{O}_y)$, we must have $\pi^*(\mathcal{E}) = \bigoplus_{a \in A} \mathcal{O}_{a^{-1}x}$, where $A \subset G$ is a subset. Hence

$$supp(\pi^*(\mathcal{E})) = \{a^{-1}x : a \in A\} \subset \{g^{-1}x : g \in G\} = supp(\pi^*(\mathcal{O}_y)).$$

Note that $\pi^*(\mathcal{E})$ is a G-invariant sheaf. But $\operatorname{supp}(\pi^*(\mathcal{E}))$ is G-invariant if and only if $A=\emptyset$ or A=G. Hence $\mathcal{E}=0$ or $\mathcal{E}=\mathcal{O}_v$, which is a contradiction.

Step 2. Suppose that $\sigma_Y=(\mathcal{P}_{\sigma_Y},Z_{\sigma_Y})\in (\mathrm{Stab}(Y))^{\widehat{G}}$ is geometric. Recall

$$\mathcal{P}_{\sigma_Y}(\phi) = \{ \mathcal{E} \in D^b(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_{\sigma}(\phi) \},$$

for all $\phi \in \mathbf{R}$. Fix $x \in X$, and let $y \in Y$ be the point corresponding to the orbit Gx. By assumption, \mathcal{O}_y is σ_Y -stable. Let ϕ_{sky} denote its phase. Then $\pi^*(\mathcal{O}_y) = \bigoplus_{g \in G} g^*\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$. Moreover, since taking cohomology commutes with direct summands, $g^*\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ for all $g \in G$. In particular, $\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$. Now suppose that \mathcal{O}_x is strictly semistable, then there exist $\mathcal{A}, \mathcal{B} \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ such that

$$\mathcal{A} \hookrightarrow \mathcal{O}_x \twoheadrightarrow \mathcal{B}$$

is non-trivial, i.e. \mathcal{A} is not isomorphic to 0 or \mathcal{O}_x . By Theorem 3.3, $(\pi_*)^{-1}$ sends $\mathcal{P}_{\sigma}(\phi_{\text{sky}})$ to $\mathcal{P}_{\sigma_Y}(\phi_{\text{sky}})$. Hence we have a short exact sequence in $\mathcal{P}_{\sigma_Y}(\phi_{\text{sky}})$:

$$\pi_*(\mathcal{A}) \hookrightarrow \pi_*(\mathcal{O}_x) = \mathcal{O}_y \twoheadrightarrow \pi_*(\mathcal{B}).$$

However, \mathcal{O}_y is stable, hence $\pi_*(A) = 0$ or $\pi_*(B) = 0$. But π is finite, hence π_* is conservative. Therefore A = 0 or B = 0, which is a contradiction.

3.2. Group actions and the classification of geometric stability conditions on surfaces. We denote by $\operatorname{Stab}^{\operatorname{Geo}}(X)$ the set of all geometric stability conditions on X. The following result is proved in §5:

Theorem 5.5 ([Bri08, Proposition 10.3]). Let X be a smooth projective surface, and let $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}^{\operatorname{Geo}}(X)$. Then σ is determined by its central charge up to shifting the slicing by [2n] for any $n \in \mathbf{Z}$.

Moreover, if σ is normalised using the action of $\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$ such that $Z(\mathcal{O}_x) = -1$ and $\phi(\mathcal{O}_x) = 1$ for any $x \in X$. Then

(1) the central charge can be written as follows:

$$Z([E]) = (\alpha - i\beta)\omega^2 \operatorname{rank}([E]) + (B + i\omega) \cdot \operatorname{ch}_1([E]) - \operatorname{ch}_2([E])$$

where $\alpha, \beta \in \mathbf{R}$, $B, \omega \in \mathrm{NS}(X) \otimes \mathbf{R}$, and ω is ample. Moreover,

(2) the heart is of the form $\mathcal{P}((0,1]) = \langle \mathcal{T}, \mathcal{F}[1] \rangle$, where $(\mathcal{T}, \mathcal{F})$ is the torsion pair on Coh(X) given by:

$$\mathcal{T}:=\left\{E\in\operatorname{Coh}(X): \begin{array}{l} \operatorname{Any}\,\omega\text{-semistable Harder-Narasimhan factor }F\text{ of the tor-}\\ \operatorname{sion free part of }E\text{ satisfies }\operatorname{Im}Z(F)>0. \end{array}\right\},$$

$$\mathcal{F}:=\left\{E\in\operatorname{Coh}(X): \begin{array}{l} E\text{ is torsion free, and any }\omega\text{-semistable Harder-}\\ \operatorname{Narasimhan factor }F\text{ of }E\text{ satisfies }\operatorname{Im}Z(F)\leq0. \end{array}\right\},$$

Corollary 3.4. Let G be a group acting on a smooth projective surface X. Then $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}^{\operatorname{Geo}}(X)$ is G-invariant if and only if Z is G-invariant.

Proof. If $\sigma=(\mathcal{P},Z)\in \operatorname{Stab}^{\operatorname{Geo}}(X)$ is G-invariant, then so is Z. Suppose $\sigma=(\mathcal{P},Z)\in \operatorname{Stab}^{\operatorname{Geo}}(X)$ and Z is G-invariant. Fix $g\in G$. Then $g^*\sigma=(g^*(\mathcal{P}),Z\circ g^*)$ and σ are both geometric, and skyscraper sheaves have the same phase. By Theorem 5.5, $\sigma=g^*\sigma$.

Lemma 3.5. Let X be a smooth projective variety, and let $G \subseteq \operatorname{Pic}^0(X)$ be a finite subgroup. Then the induced action of G on $K_{num}(X)$ is trivial.

Proof. Let $\mathcal{L} \in G$ and $[E] \in \mathrm{K}_{\mathrm{num}}(X)$. The induced action of G on $\mathrm{K}_{\mathrm{num}}(X)$ is given by $\mathcal{L} \cdot [E] := [E \otimes \mathcal{L}]$. Since \mathcal{L} is a degree 0 line bundle, $\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})}$ and $c_1(\mathcal{L}) = 0$ in $\mathrm{Chow}_{\mathrm{num}}(X)$. Therefore,

$$\operatorname{ch}\mid_{\mathrm{K_{num}}} ([E\otimes \mathcal{L}]) = \operatorname{ch}\mid_{\mathrm{K_{num}}} ([E]) \cdot \operatorname{ch}\mid_{\mathrm{K_{num}}} (\mathcal{L}) = \operatorname{ch}\mid_{\mathrm{K_{num}}} ([E]).$$

By Hirzebruch-Riemann-Roch, ch: $\mathrm{K}(X) \to \mathrm{Chow}(X)$ induces an injective map ch: $\mathrm{K}_{\mathrm{num}}(X) \to \mathrm{Chow}_{\mathrm{num}}(X)$. Therefore, $\mathcal{L} \cdot [E] = [E \otimes \mathcal{L}] = [E]$ in $\mathrm{K}_{\mathrm{num}}(X)$.

Corollary 3.6. Let S be a smooth projective surface and let $G \subseteq \operatorname{Pic}^0(S)$ be a finite subgroup. Then every geometric stability condition on S is G-invariant.

Proof. Let $\sigma=(\mathcal{P},Z)\in \operatorname{Stab}^{\operatorname{Geo}}(S)$. By Corollary 3.4 it is enough to show that Z is G invariant. By Lemma 3.5, G acts trivially on $\operatorname{K}_{\operatorname{num}}(S)$. Since σ is numerical, $Z\colon \operatorname{K}(S)\to \mathbf{C}$ factors via $\operatorname{K}_{\operatorname{num}}(S)$, hence Z is G invariant.

Example 3.7. Suppose G is a finite abelian group acting freely on a smooth projective variety X, and let Y:=X/G. Then by Proposition 2.5 there is also an action of $\widehat{G}=\operatorname{Hom}(G,\mathbf{C})$ on $\operatorname{D}^{\operatorname{b}}_{\operatorname{G}}(X)\cong\operatorname{D}^{\operatorname{b}}(Y)$. As discussed in §3.1. The corresponding action on $\operatorname{D}^{\operatorname{b}}(Y)$ is given by tensoring with degree 0 line bundles \mathcal{L}_{χ} for each $\chi\in\widehat{G}$. Corollary 3.6 tells us that every geometric stability condition on $\operatorname{D}^{\operatorname{b}}(Y)$ is \widehat{G} invariant.

Let X be a smooth projective variety. Note that, by [Bri08, Proposition 9.4], $\operatorname{Stab}^{\operatorname{Geo}}(X)$ is open. Moreover, in §5 we prove the following result for surfaces:

Proposition 5.39. Let X be a smooth projective surface. Then $Stab^{Geo}(X)$ is connected.

3.3. Applications to varieties with finite Albanese morphism.

Lemma 3.8. Suppose that a finite group G acts on a triangulated category \mathcal{D} and that the induced action on $K_{num}(\mathcal{D})$ is trivial. Then $(\operatorname{Stab}(\mathcal{D}))^G$ is a union of connected components inside $\operatorname{Stab}(\mathcal{D})$.

Proof. By Bridgeland's deformation theorem [Bri07, Theorem 1.2], there is a local homeomorphism

$$\mathcal{Z} \colon \operatorname{Stab}(\mathcal{D}) \to \operatorname{Hom}(K_{num}(\mathcal{D}), \mathbf{C}).$$

Let $g \in G$, and denote by g_* the induced action of g on $K(\mathcal{D})$ and $K_{num}(\mathcal{D})$. Recall that the action of G on $Stab(\mathcal{D})$ is given by $g \cdot \sigma = (g(\mathcal{P}), Z \circ g_*)$. The induced action of G on $K_{num}(\mathcal{D})$ is trivial, hence $\mathcal{Z}(\sigma)$ is G-invariant and $\mathcal{Z}(g \cdot \sigma) = \mathcal{Z}(\sigma)$. Since \mathcal{Z} commutes with the action of G on $Stab(\mathcal{D})$, the properties of being G-invariant and not being G-invariant are open in $Stab(\mathcal{D})$, so the result follows.

We now combine this with the results of §3.1 and §3.2.

Theorem 3.9. Let X be a smooth projective variety with finite Albanese morphism. Let G be a finite abelian group acting freely on X and let Y = X/G. Then $\operatorname{Stab}^{\dagger}(Y) := (\operatorname{Stab}(Y))^{\widehat{G}}$ is a union of connected components consisting only of geometric stability conditions.

Proof. X has finite Albanese morphism, so it follows from [FLZ22, Theorem 2.13] that all stability conditions on X are geometric. In particular, all G-invariant stability conditions on X are geometric, so from the correspondence of Theorem 3.3 it follows that all \widehat{G} -invariant stability conditions on Y are geometric. Hence $(\operatorname{Stab}(Y))^{\widehat{G}} \subset \operatorname{Stab}^{\operatorname{Geo}}(S)$.

Recall from Example 3.7 that \widehat{G} acts on $\mathrm{D^b}(Y)$ by tensoring with degree 0 line bundles. Now we may apply Lemma 3.5, so it follows that \widehat{G} acts trivially on $\mathrm{K}_{\mathrm{num}}(Y)$. Hence, by Lemma 3.8, $(\mathrm{Stab}(Y))^{\widehat{G}}$ is a union of connected components.

When X is a surface, we can apply the results of §3.2 to describe all of $\operatorname{Stab}^{\operatorname{Geo}}(X)$:

Corollary 3.10. Let X be a smooth projective surface with finite Albanese morphism. Let G be an abelian group acting freely on X. Let S = X/G. Then $\operatorname{Stab}^{\dagger}(S) = (\operatorname{Stab}(X))^G = \operatorname{Stab}^{\operatorname{Geo}}(S)$. In particular, $\operatorname{Stab}^{\dagger}(S)$ is a connected component of $\operatorname{Stab}(S)$.

Proof. By Theorem 3.9, $(\operatorname{Stab}(S))^{\widehat{G}} \subset \operatorname{Stab}^{\operatorname{Geo}}(S)$ is a union of connected components. By Corollary 3.6, $\operatorname{Stab}^{\operatorname{Geo}}(S) \subset (\operatorname{Stab}(S))^{\widehat{G}}$. Hence $\operatorname{Stab}^{\operatorname{Geo}}(S) = (\operatorname{Stab}(S))^{\widehat{G}}$. By Proposition 5.39, $\operatorname{Stab}^{\dagger}(S) \cong \operatorname{Stab}^{\operatorname{Geo}}(S)$ is connected. In particular, $\operatorname{Stab}^{\dagger}(S)$ is a connected component of $\operatorname{Stab}(S)$.

See Example 5.42 for an explicit description of $\operatorname{Stab}^{\dagger}(S) = \operatorname{Stab}^{\operatorname{Geo}}(S)$.

Example 3.11. Let $S=(C_1\times C_2)/G$ be the quotient of a product of smooth curves such that $g(C_1)$, $g(C_2)\geq 1$, and G is a finite abelian group acting freely on S. Then $C_1\times C_2$ has finite Albanese morphism. By Corollary 3.10, $\operatorname{Stab}(S)$ has a connected component consisting only of all geometric stability conditions. In particular, we could take S to be a Beauville-type or bielliptic surface (see Example 1.1 and Example 1.2).

Remark 3.12. Let X be a smooth projective surface with finite Albanese morphism, and let G be an abelian group acting freely on X. Let S=X/G, and denote by $\pi\colon X\to S$ the quotient map. Moreover, let H_X be a G-invariant polarization of X and let H_S be the corresponding polarization on S such that $\pi^*H_S=H_X$. Then all stability conditions in $\operatorname{Stab}_H(X)$ are G-invariant.

Therefore, under the correspondence of Theorem 3.3, $\operatorname{Stab}_{H_S}^{\dagger}(S) = (\operatorname{Stab}_{H_S}(S))^{\widehat{G}} \cong \operatorname{Stab}_{H_X}(X)$. From Corollary 3.10 it follows that $\operatorname{Stab}_{H_S}^{\operatorname{Geo}}(S) \cong \operatorname{Stab}_{H_X}(X)$. $\operatorname{Stab}_{H_X}(X)$ is the same as the connected component described in [FLZ22, Corollary 3.7]. In particular, $\operatorname{Stab}_{H_S}^{\dagger}(S) \cong \operatorname{Stab}_{H_S}^{\operatorname{Geo}}(S)$ is connected and contractible.

Example 3.13. A Calabi–Yau threefold of abelian type is an étale quotient Y = X/G of an abelian threefold X by a finite group G acting freely on X such that the canonical line bundle of Y is

trivial and $H^1(Y, \mathbf{C}) = 0$. As discussed in [BMS16, Example 10.4(i)], these are classified in [OS01, Theorem 0.1]. In particular, G can be chosen to be $(\mathbf{Z}/2)^{\oplus 2}$ or D_8 , and the Picard rank of Y is 3 or 2 respectively.

Fix a polarization (Y, H), and consider stability conditions that factor via Λ_H , i.e. $\operatorname{Stab}_H(Y)$. This has a connected component $\mathfrak P$ of geometric stability conditions induced from $\operatorname{Stab}_H(X)$ [BMS16, Corollary 10.3] which is described explicitly in [BMS16, Lemma 8.3]. Moreover, by [OPT22, Theorem 3.21] together with Theorem 3.9, $\sigma \in \mathfrak P$ satisfies the full support property. Hence $\mathfrak P$ lies in a a connected component $\operatorname{Stab}^\dagger(Y) \subset \operatorname{Stab}(X)$ consisting only of geometric stability conditions.

4. The Le Potier function

We compute the Le Potier function of free abelian quotients and varieties with finite Albanese morphism. We apply this to Beauville-type surfaces which provides counter examples to Conjecture 1.4.

4.1. H **stability.** Let X be a smooth projective variety over \mathbb{C} .

Definition 4.1. Let X be a smooth projective variety over \mathbb{C} . Fix an ample class $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. Given $F \in \mathrm{Coh}(X)$ we define the H-slope of F as follows:

$$\mu_H(F) := \begin{cases} \frac{H^{n-1}\operatorname{ch}_1(F)}{H^n\operatorname{ch}_0(F)}, & \text{if } \operatorname{ch}_0(F) > 0; \\ +\infty, & \text{if } \operatorname{ch}_0(F) = 0. \end{cases}$$

We say that *F* is *H*-(*semi*)*stable* if for every $0 \neq E \subsetneq F$

$$\mu_H(E) < (\leq)\mu_H(F/E).$$

4.2. **The Le Potier Function.** When studying H-stability, a natural question that arises is whether there are necessary and sufficient conditions on a cohomology class $\gamma \in H^*(X, \mathbf{Q})$ for there to exist a H-semistable sheaf F with $\mathrm{ch}(F) = \gamma$.

The Bogomolov–Gieseker inequality (see [Bog79, $\S10$], or [HL10, Theorem 3.4.1]) gives the following necessary condition for H-semistable sheaves on surfaces:

$$2\operatorname{ch}_{0}(F)\operatorname{ch}_{2}(F) < \operatorname{ch}_{1}(F)^{2}$$
.

This generalises to the following statement for any smooth projective variety X of dimension $n \geq 2$ via the Mumford–Mehta–Ramanathan restriction theorem.

Theorem 4.2 ([Lan04, Theorem 3.2], [HL10, Theorem 7.3.1]). Let X be a smooth projective variety of dimension $n \ge 2$. Fix $H \in \operatorname{Amp}_{\mathbf{R}}(X)$. If F is a torsion-free H-semistable sheaf, then

$$2\mathsf{ch}_0(F)(H^{n-2}\,.\,\mathsf{ch}_2(F)) \leq H^{n-2}\,.\,\mathsf{ch}_1(F)^2.$$

Remark 4.3. (1) Let $A \cdot B$ denote the intersection product of elements of $\operatorname{Chow}_{\operatorname{num}}(X) \otimes \mathbf{R}$. If $A \cdot B$ is 0-dimensional, we define $A \cdot B := \deg(A \cdot B)$.

(2) Let $B \in \mathrm{NS}_{\mathbf{R}}(X)$. The *twisted Chern character* is defined by $\mathrm{ch}^B := \mathrm{ch} \cdot e^{-B}$. Then

$$2\mathsf{ch}_0(F)^B(H^{n-2} \cdot \mathsf{ch}_2(F)^B) - H^{n-2} \cdot (\mathsf{ch}_1(F)^B)^2 = 2\mathsf{ch}_0(F)(H^{n-2} \cdot \mathsf{ch}_2(F)) - H^{n-2} \cdot \mathsf{ch}_1(F)^2,$$

hence Theorem 4.2 holds for twisted Chern characters.

Now fix $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Then $H^n > 0$. Let F be any H-semistable sheaf. By the twisted version of Theorem 4.2,

$$2H^n\mathrm{ch}_0(F)(H^{n-2}\,.\,\mathrm{ch}_2^B(F)) \leq H^n(H^{n-2}\,.\,\mathrm{ch}_1^B(F)^2) \leq (H^{n-1}\,.\,\mathrm{ch}_1^B(F))^2,$$

where the final inequality is by the Hodge Index Theorem. Since F is torsion free,

$$\frac{H^{n-2} \cdot \operatorname{ch}_2^B(F)}{H^n \operatorname{ch}_0(F)} \le \frac{1}{2} \left(\frac{H^{n-1} \cdot \operatorname{ch}_1^B(F)}{H^n \operatorname{ch}_0(F)} \right)^2.$$

Now we expand the expressions for $\operatorname{ch}_2^B(F)$ and $\operatorname{ch}_1^B(F)$:

$$\begin{split} &\frac{H^{n-2} \cdot \operatorname{ch}_2(F) - H^{n-2} \cdot B \cdot \operatorname{ch}_1(F) + \frac{1}{2}H^{n-2} \cdot B^2 \cdot \operatorname{ch}_0(F)}{H^n \operatorname{ch}_0(F)} \\ & \leq \frac{1}{2} \left(\frac{H^{n-1} \cdot \operatorname{ch}_1(F) - H^{n-1} \cdot B \operatorname{ch}_0(F)}{H^n \operatorname{ch}_0(F)} \right)^2 \\ & = \frac{1}{2} \left(\mu_H(F) - \frac{H^{n-1} \cdot B}{H^n} \right)^2. \end{split}$$

Therefore,

$$\frac{H^{n-2} \cdot \operatorname{ch}_2(F) - H^{n-2} \cdot B \cdot \operatorname{ch}_1(F)}{H^n \operatorname{ch}_0(F)} \leq \frac{1}{2} \left(\mu - \frac{H^{n-1} \cdot B}{H^n} \right)^2 - \frac{1}{2} \frac{H^{n-2} \cdot B^2}{H^n}$$

This motivates the following Definition.

Definition 4.4. Let X be a smooth projective variety of dimension $n \geq 2$. Let $(H, B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. We define the *Le Potier function twisted by* B, $\Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$, by

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{H^{n-2} \cdot \operatorname{ch}_2(F) - H^{n-2} \cdot B \cdot \operatorname{ch}_1(F)}{H^n \operatorname{ch}_0(F)} \colon \underset{\mu_H(F) = \mu}{F \in \operatorname{Coh}(X) \text{ is H-semistable with}} \right\}$$

if the limit exists, and $\Phi_{X,H,B}(x) := -\infty$ otherwise.

Remark 4.5. If B=0, we will write $\Phi_{X,H}:=\Phi_{X,H,0}$. If n=2, then $\Phi_{X,H}$ is exactly [FLZ22, Definition 3.1].

The above discussion and definition generalises [FLZ22, Proposition 3.2]:

Lemma 4.6 ([FLZ22, Proposition 3.2]). Let X be a smooth projective variety of dimension $n \geq 2$. Let $(H, B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Then $\Phi_{X,H,B}$ is well defined and satisfies

$$\Phi_{X,H,B}(x) \le \frac{1}{2} \left[\left(x - \frac{H^{n-1} \cdot B}{H^n} \right)^2 - \frac{H^{n-2} \cdot B^2}{H^n} \right].$$

It is the smallest upper semi-continuous function such that

$$\frac{H^{n-2} \cdot \operatorname{ch}_{2}(F) - H^{n-2} \cdot B \cdot \operatorname{ch}_{1}(F)}{H^{n} \operatorname{ch}_{0}(F)} \leq \Phi_{X,H,B} \left(\frac{H^{n-1} \cdot \operatorname{ch}_{1}(F)}{H^{n} \operatorname{ch}_{0}(F)} \right)$$

for every torsion-free H-stable (or H-semistable) sheaf F.

4.3. The Le Potier function for free quotients. Let X be a smooth projective variety, and let G be a finite group acting freely on X. There is an étale covering of smooth projective varieties, $\pi\colon X\to X/G=:Y$. Then $\mathrm{Pic}(Y)\cong \mathrm{Pic}_G(X)$, the group of isomorphism classes of G-equivariant line bundles on X. Fix $H_S\in \mathrm{Amp}_{\mathbf{R}}(Y)$. Then $\pi^*H_S\in \mathrm{Amp}_{\mathbf{R}}(X)$ is G-invariant. Beauville-type and bielliptic surfaces provide examples of such quotients.

Example 4.7 (Ample classes on Beauville-type surfaces). Let S = X/G be a Beauville-type surface, as introduced in Example 1.1. Then $X = C_1 \times C_2$ is a product of curves of genus $g(C_i) > 1$, $q(S) := h^1(S, \mathcal{O}_S) = 0$, $p_g(S) := h^2(S, \mathcal{O}_S) = 0$ so $\chi(\mathcal{O}_S) = 1$, and $K_S^2 = 8$ where K_S is the canonical divisor of S.

Assume that there are actions of G on each curve C_i such that the action of G on $C_1 \times C_2$ is the diagonal action. This is called the *unmixed case* in [BCG08, Theorem 0.1] and excludes 3 families of dimension 0. To classify ample classes on S, we follow similar arguments to [GS13, §2.2]. Let $p_1 \colon X \to C_1$ and $p_2 \colon X \to C_2$ denote the projections to each curve. For $i, j \in \mathbf{Z}$, define

$$\mathcal{O}(i,j) := p_1^*(\mathcal{O}_{C_1}(i)) \otimes p_2^*(\mathcal{O}_{C_2}(j)) \in \operatorname{Pic}_G(X).$$

Moreover,

$$\chi_{\text{top}}(S) = \frac{\chi_{\text{top}}(C_1) \cdot \chi_{\text{top}}(C_2)}{|G|} = 4 \frac{(1 - g(C_1))(1 - g(C_2))}{|G|} = 4\chi(\mathcal{O}_S) = 4$$

Therefore, rank $Pic(S) = b_2 = 2$ and

$$\operatorname{Pic}_{\mathbf{Q}}(S) \cong \mathbf{Q} \cdot [\mathcal{O}(1,0)] \oplus \mathbf{Q} \cdot [\mathcal{O}(0,1)].$$

In particular, $\operatorname{Amp}_{\mathbf{R}}(S) \cong \mathbf{R}_{>0} \cdot [\mathcal{O}(1,0)] \oplus \mathbf{R}_{>0} \cdot [\mathcal{O}(0,1)].$

Lemma 4.8. Let $\mathcal{E} \in \operatorname{Coh}(Y)$. Then \mathcal{E} is H_Y -semistable if and only if $\pi^*\mathcal{E}$ is π^*H_Y -semistable.

Proof. This follows from the same arguments as in the proof of [HL10, Lemma 3.2.2]. \Box

Lemma 4.9. If $\mathcal{F} \in \operatorname{Coh}(X)$ is π^*H_Y -semistable, then $\pi_*\mathcal{F}$ is H_Y -semistable.

Proof. Suppose that $\mathcal{F} \in Coh(X)$ is π^*H_Y -semistable. Note that

$$\pi^*(\pi_*(\mathcal{F})) = \bigoplus_{g \in G} g^* \mathcal{F}.$$

Since π^*H_Y is G-invariant, it follows that $g^*\mathcal{F}$ is π^*H_Y -semistable for every $g\in G$. In particular, $\bigoplus_{g\in G}g^*\mathcal{F}$ is π^*H_Y -semistable. By Lemma 4.8, $\pi_*\mathcal{F}$ is H_Y -semistable.

Proposition 4.10. Let X be a smooth projective variety, and let G be a finite group acting freely on X. Let $\pi \colon X \to X/G =: Y$ denote the quotient. Let $(H_YB_Y) \in \mathrm{Amp}_{\mathbf{R}}(Y) \times \mathrm{NS}_{\mathbf{R}}(Y)$. Then $\Phi_{Y,H_Y,B_Y} = \Phi_{X,\pi^*H_Y,\pi^*B_Y}$.

Proof. To determine the Le Potier function, it is enough to consider torsion-free sheaves, since torsion sheaves have slope $+\infty$.

Step 1. Let $\mathcal{E} \in \operatorname{Coh}(Y)$ be torsion-free and H_Y -semistable. By Lemma 4.8, $\pi^*\mathcal{E}$ is π^*H_Y -semistable. Moreover,

$$\begin{split} \mu_{\pi^*H_Y}(\pi^*\mathcal{E}) &= \frac{\deg((\pi^*H_Y)^{n-1} \cdot \pi^*(\operatorname{ch}_1(\mathcal{E})))}{\deg((\pi^*H_Y)^n \cdot \pi^*(\operatorname{ch}_0(\mathcal{E})))} \\ &= \frac{\deg(\pi^*(H_Y^{n-1} \cdot \operatorname{ch}_1(\mathcal{E})))}{\deg(\pi^*(H_Y^n \cdot \operatorname{ch}_0(\mathcal{E})))} \quad (\pi \text{ is flat, so } \pi^* \text{ is a ring morphism}) \\ &= \frac{\deg(\pi) \deg(H_Y^{n-1} \cdot \operatorname{ch}_1(\mathcal{E}))}{\deg(\pi) \deg(H_Y^{n-1} \cdot \operatorname{ch}_0(\mathcal{E}))} \\ &= \mu_{H_Y}(\mathcal{E}) \end{split}$$

By the same arguments,

$$\frac{(\pi^* H_Y)^{n-2} \cdot \operatorname{ch}_2(\pi^* \mathcal{E}) - (\pi^* H_Y)^{n-1} \cdot (\pi^* B_Y) \cdot \operatorname{ch}_1(\pi^* \mathcal{E})}{(\pi^* H_Y)^n \cdot \operatorname{ch}_0(\pi^* \mathcal{E})} = \frac{H_Y^{n-2} \cdot \operatorname{ch}_2(\mathcal{E}) - H_Y^{n-1} \cdot B_Y \cdot \operatorname{ch}_1(\mathcal{E})}{H_V^n \cdot \operatorname{ch}_0(\mathcal{E})}.$$

Hence the contribution of $\pi^*\mathcal{E}$ to $\Phi_{X,\pi^*H_Y,\pi^*B_Y}$ is the same as the contribution of \mathcal{E} to Φ_{Y,H_Y,B_Y} . Step 2. Suppose $\mathcal{F} \in \operatorname{Coh}(X)$ is torsion-free and π^*H_Y -semistable. Note that

$$\pi^*(\pi_*(\mathcal{F})) = \bigoplus_{g \in G} g^* \mathcal{F}.$$

 π^*H_Y is G-invariant, hence $g^*\mathcal{F}$ is π^*H_Y -semistable for every $g\in G$. In particular, $\bigoplus_{g\in G}g^*\mathcal{F}$ is π^*H_Y -semistable. Since the Chern character is additive, $\pi^*\pi_*\mathcal{F}$ makes the same contribution to $\Phi_{X,\pi^*H_Y,\pi^*B_Y}$ as \mathcal{F} . By Lemma 4.8, $\pi_*\mathcal{F}$ is H_Y -semistable. The result follows from Step 1.

4.4. The Le Potier function for varieties with finite Albanese morphism. The Le Potier function for surfaces with finite Albanese morphism was known previously [LR21, Example 2.12(2)]. Below, we give a different proof which works for $\Phi_{X,H,B}$ in any dimension. We first need the following definition.

Definition 4.11 ([Muk78]). A vector bundle E on an abelian variety X is semi-homogeneous if for every $x \in X$, there exists a line bundle E on X such that $T_x^*(E) \cong E \otimes E$, where T_x is translation on X by X.

See [Muk78, Proposition 5.1] for some equivalent characterisations. There are many H-semistable semi-homogeneous vector bundles on any abelian variety:

Proposition 4.12 ([Muk78, Theorem 7.11]). Let A be an abelian variety and fix $H \in \mathrm{Amp}_{\mathbf{R}}(A)$. Then for every $\frac{p}{q} \in \mathbf{Q}$ there exists a H-semistable semi-homogeneous vector bundle $E_{p,q}$ on A with $\mu(E_{p,q}) = \frac{p}{q}$ and $\mathrm{ch}(E_{p,q}) = \mathrm{ch}_0(E_{p,q}) \cdot e^{\frac{pH}{q}}$.

We use this to compute the Le Potier function for varieties with finite Albanese morphism.

Proposition 4.13 ([LR21, Example 2.12(2)]). Let X be a smooth projective variety with finite Albanese morphism $a: X \to \mathrm{Alb}(X)$ and $n:=\dim X \geq 2$. Let $H_X \in \mathrm{Amp}_{\mathbf{R}}(X)$. Then $a^*E_{p,q}$ is H_X -semistable for every $\frac{p}{q} \in \mathbf{Q}$.

Proof. Fix $\frac{p}{q} \in \mathbf{Q}$ and $H_A \in \mathrm{Amp}_{\mathbf{R}}(\mathrm{Alb}(X))$. Let $E_{p,q}$ be the corresponding H_A -semistable semi-homogeneous vector bundle on $\mathrm{Alb}(X)$ from Proposition 4.12. Let $r := \mathrm{ch}_0(E_{p,q})$ and let $r_{\mathrm{Alb}(X)} \colon \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ denote the multiplication by r map. By [Muk78, Lemma 6.11], $r_{\mathrm{Alb}(X)}^*(E_{p,q})$ is a homogeneous vector bundle up to tensoring with a fixed line bundle L.

Moreover, by [Muk78, Theorem 4.17], any homogeneous vector bundle is a direct sum of vector bundles of the form $P_i \oplus U_i$, where P_i is a degree 0 line bundle and U_i is a unipotent line bundle (i.e. U_i is an iterated self-extension of $\mathcal{O}_{\mathrm{Alb}(X)}$). Therefore, $r^*_{\mathrm{Alb}(X)}(E_{p,q})$ is an iterated extension of degree 0 line bundles.

Now consider the fibre square:

$$Z := X \times_{\text{Alb}(X)} \text{Alb}(X) \xrightarrow{p_A} \text{Alb}(X)$$

$$\downarrow^{p_X} \qquad \qquad \downarrow^{r_A}$$

$$X \xrightarrow{a} \text{Alb}(X)$$

Then $p_X^*a^*(E_{p,q})=p_A^*r_A^*(E_{p,q})$ on Z. The property of being an extension of degree 0 line bundles is preserved by taking pullback. Hence $p_X^*a^*(E_{p,q})$ is an iterated extension of degree 0 line bundles. Recall that line bundles are stable with respect to any ample class. Thus $p_X^*a^*(E_{p,q})$ is $p_X^*H_X$ -semistable. Finally, by Lemma 4.8, $a^*(E_{p,q})$ is H_X -semistable.

Proposition 4.14 ([LR21, Example 2.12(2)]). Let X be a smooth projective variety with finite Albanese morphism $a: X \to \mathrm{Alb}(X)$ and $n:=\dim X \geq 2$. Fix $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(\mathrm{Alb}(X)) \times \mathrm{NS}_{\mathbf{R}}(\mathrm{Alb}(X))$. Then

$$\Phi_{X,a^*H,a^*B}(x) = \frac{1}{2} \left[\left(x - \frac{(a^*H)^{n-1} \cdot a^*B}{(a^*H)^n} \right)^2 - \frac{(a^*H)^{n-2} \cdot (a^*B)^2}{(a^*H)^n} \right].$$

Proof. $H_X := a^*H$ is ample, since it is the pullback of an ample class by a finite morphism. Let $\frac{p}{q} \in \mathbf{Q}$, and assume without loss of generality that $\gcd(p,q) = 1$. Then consider:

$$\mu_{H_X}(a^*E_{p,q}) = \frac{\deg\left(H_X^{n-1} \cdot a^*\operatorname{ch}_1(E_{p,q})\right)}{\deg\left(H_X^n \cdot a^*\operatorname{ch}_0(E_{p,q})\right)}$$

$$= \frac{\deg\left(H_X^{n-1} \cdot \operatorname{ch}_0(E_{p,q})\frac{p}{q}a^*H\right)}{\deg\left(H_X^n \cdot \operatorname{ch}_0(E_{p,q})\right)}$$

$$= \frac{p}{q}\frac{\deg\left(H_X^{n-1} \cdot H_X\right)}{\deg\left(H_X^n\right)}$$

$$= \mu_H(E_{p,q})$$

Similarly,

$$\begin{split} &\frac{H_X^{n-2} \cdot \operatorname{ch}_2(a^* E_{p,q}) - H_X^{n-2} \cdot B_X \cdot \operatorname{ch}_1(a^* E_{p,q})}{H_X^n \cdot \operatorname{ch}_0(a^* E_{p,q})} \\ &= \frac{\operatorname{deg}\left[H_X^{n-2} \cdot a^* \left(\operatorname{ch}_0(E_{p,q}) \frac{1}{2} \left(\frac{p}{q} H\right)^2\right) - H_X^{n-2} \cdot B_X \cdot a^* \left(\operatorname{ch}_0(E_{p,q}) \frac{p}{q} H\right)\right]}{\operatorname{deg}\left(H_X^n \cdot \operatorname{ch}_0(E_{p,q})\right)} \\ &= \frac{1}{2} \left(\frac{p}{q}\right)^2 \frac{\operatorname{deg}\left(H_X^{n-2} \cdot H_X^2\right)}{\operatorname{deg}\left(H_X^n\right)} - \frac{p}{q} \frac{\operatorname{deg}\left(H_X^{n-2} \cdot B_X \cdot H_X\right)}{\operatorname{deg}\left(H_X^n\right)} \\ &= \frac{1}{2} \left[\left(\mu_{H_X}(a^* E_{p,q}) - \frac{H_X^{n-1} \cdot B_X}{H_X^n}\right)^2 - \frac{H_X^{n-2} \cdot B_X^2}{H_X^n}\right] \end{split}$$

Moreover, by Proposition 4.13, $E_{p,q}$ is H_X -semistable. Hence this gives a lower bound for $\Phi_{X,H_X,B_X}\left(\frac{p}{q}\right)$. which is the same as the upper bound in Lemma 4.6. Since Φ_{X,H_X,B_X} is upper semicontinuous, the result follows.

We now combine this with Proposition 4.10.

Corollary 4.15. Let X be a smooth projective variety with finite Albanese morphism $a: X \to \text{Alb}(X)$, and let G be a finite group acting freely on X. Let $\pi\colon X\to X/G=:Y$ denote the quotient map. Suppose we have:

- $H_X = a^*H = \pi^*H_Y$: a class in $\mathrm{Amp}_{\mathbf{R}}(X)$ pulled back from $\mathrm{Alb}(X)$ and Y, and $B_X = a^*B = \pi^*B_Y$: a class in $\mathrm{NS}_{\mathbf{R}}(X)$ pulled back from $\mathrm{Alb}(X)$ and Y.

Then
$$\Phi_{Y,H_Y,B_Y}(x) = \frac{1}{2} \left[\left(x - \frac{H_Y^{n-1}.B_Y}{H_Y^n} \right)^2 - \frac{H_Y^{n-2}.B_Y^2}{H_Y^n} \right].$$

Proof. By Proposition 4.10 and Proposition 4.14, it follows that:

$$\Phi_{Y,H_Y,B_Y}(x) = \Phi_{X,\pi^*H_Y,\pi^*B_Y}(x) = \frac{1}{2} \left[\left(x - \frac{(\pi^*H_Y)^{n-1} \cdot \pi^*B_Y}{(\pi^*H_Y)^n} \right)^2 - \frac{(\pi^*H_Y)^{n-2} \cdot (\pi^*B_Y)^2}{(\pi^*H_Y)^n} \right].$$

The result follows by the projection formula.

Example 4.16. Suppose X has finite Albanese morphism $a: X \to Alb(X)$, and let G be a finite group acting freely on X. This induces an action of G on NS(Alb(X)). Fix $L \in Amp(Alb(X))$. Then $H_X := \bigotimes_{g \in G} g^*L \in \operatorname{Amp}_{\mathbf{R}}(X)$ satisfies the hypotheses of Corollary 4.15. In particular, this applies to bielliptic surfaces (q = 1) and Beauville-type surfaces (q = 0). The latter provides a counter example to Conjecture 1.4.

5. Geometric stability conditions and the Le Potier function

We use the Le Potier function to describe the set of geometric stability conditions on any surface. This was previously known for surfaces with Picard rank 1 [FLZ22, Theorem 3.4, Proposition 3.6].

5.1. The deformation property and tilting. To prove existence of stability conditions later in this section, we will need the following refinement of Theorem 2.14:

Proposition 5.1 ([BMS16, Proposition A.5], [Bay19, Theorem 1.2]). Let \mathcal{D} be a triangulated category. Assume $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{D})$ satisfies the support property with respect to a quadratic form Q on $K_{num}(\mathcal{D}) \otimes \mathbf{R}$. Consider the open subset of $Hom_{\mathbf{Z}}(K_{num}(\mathcal{D}), \mathbf{C})$ consisting of central charges whose kernel is negative definite with respect to Q, and let U be the connected component containing Z. Let $\mathcal Z$ denote the local homeomorphism from Theorem 2.14, and let $\mathcal{U} \subset \operatorname{Stab}(\mathcal{D})$ be the connected component of the preimage $\mathcal{Z}^{-1}(U)$ containing σ . Then

(1) the restriction $\mathcal{Z}|_{\mathcal{U}} \colon \mathcal{U} \to U$ is a covering map, and

(2) any stability condition $\sigma' \in \mathcal{U}$ satisfies the support property with respect to the same quadratic form Q.

Corollary 5.2. Let \mathcal{D} be a triangulated category. Assume $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{D})$ satisfies the support property with respect to a quadratic form Q on $K_{\operatorname{num}}(\mathcal{D}) \otimes \mathbf{R}$. Let $U \subset \operatorname{Hom}_{\mathbf{Z}}(K_{\operatorname{num}}(\mathcal{D}), \mathbf{C})$, and $\mathcal{U} \subset \operatorname{Stab}(\mathcal{D})$ be the connected components from Proposition 5.1. Suppose there is a path Z_t in U parametrised by $t \in [0,1]$, such that $\operatorname{Im} Z_t$ is constant and $Z_{t_0} = Z$ for some $t_0 \in [0,1]$. Then this lifts to a path $\sigma_t = (\mathcal{Q}_t, Z_t)$ in \mathcal{U} passing through σ along which $\mathcal{Q}_t(0,1] = \mathcal{P}(0,1]$ and σ_t satisfies the support property with respect to Q.

Proof. Let \mathcal{Z} denote the local homeomorphism from Theorem 2.14. By Proposition 5.1(1), $\mathcal{Z}|_{\mathcal{U}} \colon \mathcal{U} \to \mathcal{U}$ is a covering map. By the path lifting property, there is a unique path $\sigma_t = (\mathcal{Q}_t, Z_t)$ in \mathcal{U} with $\sigma = \sigma_{t_0}$. By Proposition 5.1(2), σ_t satisfies the support property with respect to Q for all t. It remains to show that $\mathcal{Q}_t(0,1] = \mathcal{P}(0,1]$.

Fix a non-zero object $E \in \mathrm{D}^{\mathrm{b}}(X)$. We claim that the set of points in the path σ_t where $E \in \mathcal{Q}_t(0,1]$ is open and closed. Suppose $E \in \mathcal{Q}_T(0,1]$ for some $T \in [0,1]$. Then all Jordan-Hölder (JH) factors of E with respect to σ_T , E_i , are in $\mathcal{Q}_T(0,1]$ and satisfy $\mathrm{Im}\, Z_T(E_i) \geq 0$. The property for an object to be stable is open in $\mathrm{Stab}(X)$ (see [BM11, Proposition 3.3]). Moreover, $0 < \phi_{\mathcal{Q}_t}(E_i)$ is an open property. Since $\mathrm{Im}\, Z_t$ is constant, $\mathrm{Im}\, Z_t(E_i) \geq 0$ for all t. Hence, for all sufficiently close σ_t , $\phi_{\mathcal{Q}_t}(E_i) \leq 1$ and $E \in \mathcal{Q}_t(0,1]$.

Now suppose σ_T is in the closure and not the interior of $\{\sigma_t : E \in \mathcal{Q}_t(0,1]\}$ inside $\{\sigma_t : t \in [0,1]\}$. Recall that $\phi^+(E)$ and $\phi^-(E)$ are continuous. Hence $\phi_{\mathcal{Q}_T}^-(E) = 0$, and E has a morphism to a stable object in $\mathcal{Q}_T(0)$ which is also stable nearby. In particular, $\{\sigma_t : E \notin \mathcal{Q}_t(0,1]\}$ is open, which proves the claim. Hence $\mathcal{Q}_t(0,1]$ is constant. Since $\mathcal{Q}_{t_0} = \mathcal{P}(0,1]$, the result follows.

To construct stability conditions, we will also need the following definition.

Definition 5.3 ([HRS96, Chapter I.2]). Let \mathcal{A} be an abelian category. A *torsion pair* on \mathcal{A} is a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ of \mathcal{A} such that

- (1) for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, $\operatorname{Hom}(T, F) = 0$, and
- (2) for any $E \in \mathcal{A}$ there are $T \in \mathcal{T}$, $F \in \mathcal{F}$, and an exact sequence

$$0 \to T \to E \to F \to 0.$$

Proposition 5.4 ([HRS96, Proposition 2.1]). Let X be a smooth projective variety. Let A be the heart of a bounded t-structure on $D^b(X)$. Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair on A. Then

$$\mathcal{A}^{\sharp}:=\left\{E\in \mathrm{D}^{b}(\mathcal{A})\mid \mathcal{H}^{0}_{\mathcal{A}}(E)\in \mathcal{T},\ \mathcal{H}^{-1}_{A}(E)\in \mathcal{F},\ \mathcal{H}^{i}_{\mathcal{A}}(E)=0\ \textit{for all}\ i\neq 0,-1\right\}$$

is the heart of a bounded t-structure on $D^b(A)$. We call A^{\sharp} the **tilt** of A with respect to $(\mathcal{T}, \mathcal{F})$.

5.2. The central charge of a geometric stability condition. For the rest of this section, let X be a smooth projective surface over \mathbf{C} . We are particularly interested in geometric Bridgeland stability conditions, i.e. $\sigma \in \operatorname{Stab}(X)$ such that the skyscraper sheaf \mathcal{O}_x is σ -stable for every point $x \in X$. Denote by $\operatorname{Stab}^{\operatorname{Geo}}(X)$ the set of all geometric stability conditions.

Theorem 5.5 ([Bri08, Proposition 10.3]). Let X be a smooth projective surface, and let $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}^{\operatorname{Geo}}(X)$. Then σ is determined by its central charge up to shifting the slicing by [2n] for any $n \in \mathbf{Z}$. Moreover, if σ is normalised using the action of \mathbf{C} such that $Z(\mathcal{O}_x) = -1$ and $\phi(\mathcal{O}_x) = 1$ for all $x \in X$. Then

(1) the central charge can be uniquely written in the following form:

$$Z([E]) = (\alpha - i\beta)H^2 \operatorname{ch}_0([E]) + (B + iH) \cdot \operatorname{ch}_1([E]) - \operatorname{ch}_2([E]),$$
where $\alpha, \beta \in \mathbf{R}$, $(H, B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Moreover,

(2) the heart, $\mathcal{P}((0,1])$, is the tilt of Coh(X) at the torsion pair $(\mathcal{T},\mathcal{F})$, where

$$\mathcal{T} := \left\{ E \in \operatorname{Coh}(X) : \begin{array}{l} \operatorname{Any} \operatorname{H-semistable} \operatorname{Harder-Narasimhan} \operatorname{factor} F \ \text{of the tor-} \\ \operatorname{sion} \operatorname{free} \ \operatorname{part} \ \text{of} \ E \ \operatorname{satisfies} \ \operatorname{Im} Z([F]) > 0. \end{array} \right\},$$

$$\mathcal{F} := \left\{ E \in \operatorname{Coh}(X) : \begin{array}{l} E \ \text{is} \ \operatorname{torsion} \ \operatorname{free}, \ \operatorname{and} \ \operatorname{any} \ \operatorname{H-semistable} \ \operatorname{Harder-} \\ \operatorname{Narasimhan} \operatorname{factor} F \ \operatorname{of} E \ \operatorname{satisfies} \ \operatorname{Im} Z([F]) \leq 0. \end{array} \right\},$$

Notation 5.6. We will use $Z_{H,B,\alpha,\beta} = Z$ to denote central charges of the above form. Since $\operatorname{Im} Z_{H,B,\alpha,\beta}$ only depends on H and β , we will write $(\mathcal{T}_{H,\beta},\mathcal{F}_{H,\beta})$ for the torsion pair, and $\operatorname{Coh}^{H,\beta}(X)$ for the corresponding tilted heart. Then $\sigma_{H,B,\alpha,\beta} := (Z_{H,B,\alpha,\beta},\operatorname{Coh}^{H,\beta}(X))$.

The proof is similar to the case of K3 surfaces proved in [Bri08, §10]. We first need the following result which immediately generalises to any smooth projective surface:

Lemma 5.7 ([Bri08, Lemma 10.1]). Suppose $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(X)$ is a stability condition on a smooth projective surface X such that for each point $x \in X$ the sheaf \mathcal{O}_x is σ -stable of phase one. Let E be an object of $\operatorname{D}^{\operatorname{b}}(X)$. Then

- a) if $E \in \mathcal{P}((0,1])$ then $H^i(E) = 0$ unless $i \in \{-1,0\}$, and moreover $H^{-1}(E)$ is torsion free,
- b) if $E \in \mathcal{P}(1)$ is stable, then either $E = \mathcal{O}_x$ for some $x \in X$, or E[-1] is a locally-free sheaf,
- c) if $E \in Coh(X)$ is a sheaf the $E \in \mathcal{P}((-1,1])$; if E is a torsion sheaf then $E \in \mathcal{P}((0,1])$,
- d) the pair of subcategories

$$\mathcal{T} = \operatorname{Coh}(X) \cap \mathcal{P}((0,1])$$
 and $\mathcal{F} = \operatorname{Coh}(X) \cap \mathcal{P}((-1,0])$

defines a torsion pair on Coh(X) and $\mathcal{P}((0,1])$ is the corresponding tilt.

Proof. (of Theorem 5.5)

Step 1. Since σ is numerical, the central charge can be written as follows:

$$Z([E]) = a \operatorname{ch}_0([E]) + B \cdot \operatorname{ch}_1([E]) + c \operatorname{ch}_2([E]) + i(d \operatorname{ch}_0([E]) + H \cdot \operatorname{ch}_1([E]) + e \operatorname{ch}_2([E])),$$
 where $a, c, d, e \in \mathbf{R}$ and $B, H \in \operatorname{NS}_{\mathbf{R}}(X)$.

Since σ is geometric, \mathcal{O}_x is σ -stable and of the same phase for every point $x \in X$ by Proposition 3.2. As discussed in Remark 2.15, \mathbf{C} acts on $\mathrm{Stab}(X)$. In particular, there is a unique element $g \in \mathbf{C}$ such that $g^*\sigma = (\mathcal{P}', Z')$ satisfies $Z'([\mathcal{O}_x]) = -1$ and $\mathcal{O}_x \in \mathcal{P}'(1)$ for all $x \in X$. Now we may assume that $Z([\mathcal{O}_x]) = -1$ and $\mathcal{O}_x \in \mathcal{P}(1)$ for all $x \in X$. Hence -1 = c and e = 0. Let $C \subset X$ be a curve. By Lemma 5.7(c), $\mathcal{O}_C \in \mathcal{P}((0,1])$. Since $\mathrm{ch}_0(\mathcal{O}_C) = 0$ and $\mathrm{ch}_1(\mathcal{O}_C) = C$,

$$\operatorname{Im} Z([\mathcal{O}_C]) = H \cdot C > 0.$$

This holds for any curve $C \subset X$, so $H \in \mathrm{NS}_{\mathbf{R}}(X)$ is nef. By [BM02, Proposition 9.4], $\mathrm{Stab}^{\mathrm{Geo}}(X)$ is open. Moreover, by Theorem 2.14, a small deformation from σ to σ' in $\mathrm{Stab}^{\mathrm{Geo}}(X)$ corresponds to a small deformation of the central charges Z to Z', and in turn a small deformation of H to H' inside $\mathrm{NS}_{\mathbf{R}}(X)$. In particular, $H' \cdot C \geq 0$ for any curve $C \subset X$. Therefore, H lies in the interior of the nef cone, hence H is ample.

Now let $\alpha := \frac{a}{H^2}$ and $\beta := \frac{-d}{H^2}$. Then the central charge is of the form:

$$Z([E]) = (\alpha - i\beta)H^2 \operatorname{ch}_0([E]) + (B + iH) \cdot \operatorname{ch}_1([E]) - \operatorname{ch}_2([E]).$$

Step 2. Consider the torsion pair $(\mathcal{T}, \mathcal{F})$ of Lemma 5.7(d), so $\mathcal{P}((0, 1])$ is the tilt of $\operatorname{Coh}(X)$ at $(\mathcal{T}, \mathcal{F})$. By Lemma 5.7(c), all torsion sheaves lie in \mathcal{T} . To complete the proof, we need the following claim:

$$E \in \operatorname{Coh}(X) \text{ is H-stable and torsion-free } \Longrightarrow \begin{cases} E \in \mathcal{T} \text{ if } \operatorname{Im} Z([E]) > 0, \\ E \in \mathcal{F} \text{ if } \operatorname{Im} Z([E]) \leq 0. \end{cases} \tag{*}$$

This is Step 2 of the proof of [Bri08, Lemma 10.3]. Bridgeland first shows that E must lie in \mathcal{T} or \mathcal{F} . We explain why it then follows that $\operatorname{Im} Z([E]) = 0$ implies $E \in \mathcal{F}$. Assume E is non-zero and $E \in \mathcal{T}$. Since $Z([E]) \in \mathbf{R}$, it follows that $E \in \mathcal{P}(1)$. For any $x \in \operatorname{Supp}(E)$, E has a non-zero map $f: E \to \mathcal{O}_x$. Let E_1 be its kernel in $\operatorname{Coh}(X)$. Since \mathcal{O}_x is stable, f is a surjection in $\mathcal{P}(1)$. Thus E_1 also lies in $\mathcal{P}(1)$ and hence in \mathcal{T} . Moreover, $Z([E_1]) = Z([E]) - Z([\mathcal{O}_x]) = Z([E]) - 1$. Repeating

this by replacing E with E_1 and so on creates a chain of strict subobjects in $\mathcal{P}(1)$, $E \supsetneq E_1 \supsetneq E_2 \supsetneq$ \cdots , such that $Z([E_n]) = Z([E]) - n$. If this process does not terminate, then $Z([E_k]) \in \mathbf{R}_{>0}$ for some $k \in \mathbb{N}$, contradicting the fact that $E_n \in \mathcal{P}((0,1])$. Otherwise, $E_i \cong \mathcal{O}_x$ for some i, contradicting the fact that E is torsion-free.

5.3. The set of all geometric stability conditions on surfaces. In the previous section, we showed that any geometric stability condition on a surface which satisfies $Z(\mathcal{O}_x) = -1$, $\phi(\mathcal{O}_x) = 1$ is determined by its central charge. In particular, it depends on parameters $(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathbb{R}$ $NS_{\mathbf{R}}(X) \times \mathbf{R}^2$. To characterise geometric stability conditions on surfaces, we will find necessary and sufficient conditions for when these parameters define a geometric stability condition. In Definition 4.4 we introduced the Le Potier function twisted by B. We restate the version for surfaces below.

Definition 5.8. Let X be a smooth projective surface. Let $(H, B) \in Amp_{\mathbf{R}}(X) \times NS_{\mathbf{R}}(X)$. We define the Le Potier function twisted by $B, \Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R}$, by

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} \, : \begin{matrix} F \in \operatorname{Coh}(X) \text{ is H-semistable with} \\ \mu_H(F) = \mu \end{matrix} \right\}.$$

Remark 5.9. By [HL10, Theorem 5.2.5], for every rational number $\mu \in \mathbf{Q}$ there exists an H-stable sheaf F with $\mu_H(F) = \mu$. Together with the fact that $\Phi_{X,H,B}$ is bounded above, it follows that the value of $\Phi_{X,H,B}$ at every point is in **R**.

Theorem 5.10. Let X be a smooth projective surface. Then

$$\operatorname{Stab}^{\operatorname{Geo}}(X) \cong \mathbf{C} \times \left\{ (H, B, \alpha, \beta) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X, H, B}(\beta) \right\}.$$

Remark 5.11. In [MS17, Theorem 6.10], the authors describe a subset of $\operatorname{Stab}^{\operatorname{Geo}}(X)$ parametrised by $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. This corresponds to where $\alpha > \frac{1}{2} \left[\left(\beta - \frac{H.B}{H^2} \right)^2 - \frac{B^2}{H^2} \right]$ in the above Theorem (see Lemma 5.24 for details). We will call this the BG range.

Notation 5.12. To ease notation, we make the following definitions.

$$\mathcal{U} := \left\{ (H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X, H, B}(\beta) \right\}$$

$$\mathrm{Stab}_N^{\mathsf{Geo}}(X) := \left\{ \sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathsf{Geo}}(X) : Z(\mathcal{O}_x) = -1, \mathcal{O}_x \in \mathcal{P}(1) \ \forall x \in X \right\}$$

Idea of the proof. By Theorem 5.5, for every $\sigma \in \operatorname{Stab}^{\operatorname{Geo}}(X)$, there exists a unique $g \in \mathbf{C}$ such that $g^*\sigma \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. To prove Theorem 5.10 it is enough to show that $\operatorname{Stab}_N^{\operatorname{Geo}}(X) \cong \mathcal{U}$. We do this in two steps:

Step 1. Construct a continuous, injective, local homeomorphism Π : Stab $_N^{\text{Geo}}(X) \to \mathcal{U}$: Theorem 5.5 shows that, for every $\sigma \in \operatorname{Stab}_{N}^{\operatorname{Geo}}(X)$, there are unique $(H, B, \alpha, \beta) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X) \times \mathbf{R}^{2}$ such that $\sigma = \sigma_{H,B,\alpha,\beta}$. This gives an injective map:

$$\Pi \colon \mathrm{Stab}_{N}^{\mathrm{Geo}}(X) \longrightarrow \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^{2}$$
$$\sigma = \sigma_{H,B,\alpha,\beta} \longmapsto (H,B,\alpha,\beta)$$

We will show that the image is contained in \mathcal{U} (Lemma 5.13), and that Π is a local homeomorphism (Proposition 5.18).

Step 2. Construct a pointwise inverse $\Sigma \colon \mathcal{U} \to \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. We will first show this is possible for (H, B, α, β) in the BG range (Lemma 5.24). In Proposition 5.38, we extend this to any $\alpha > \Phi_{X,H,B}(\beta)$ by applying Corollary 5.2 as follows.

- Fix $(H, B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$, and $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Fix $\alpha_1 > \frac{1}{2} \left[\left(\beta_0 \frac{H.B}{H^2} \right)^2 \frac{B^2}{H^2} \right]$.

If only α varies, then $\operatorname{Im} Z_{H,B,\alpha,\beta_0}$ is constant. We construct a quadratic form (Proposition 5.31) and shows that it gives the support property for $\sigma_{H,B,\alpha_1,\beta_0}$ (Lemma 5.37) and is negative definite on $\operatorname{Ker} Z_{H,B,\alpha,\beta_0}$ for all $\alpha > \Phi_{X,H,B}(\beta_0)$ (Lemma 5.32).

5.3.1. STEP 1: Construction of the map $\operatorname{Stab}_N^{\operatorname{Geo}}(X) \to \mathcal{U}$.

Lemma 5.13 ([FLZ22, Proposition 3.6]). Let $\sigma = \sigma_{H,B,\alpha,\beta} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. Then $\alpha > \Phi_{X,H,B}(\beta)$. In particular, $\Pi(\operatorname{Stab}_N^{\operatorname{Geo}}(X)) \subseteq \mathcal{U}$.

To prove this statement, we first need the following lemmas.

Lemma 5.14. Let $\sigma_{H,B,\alpha,\beta}=(Z_{H,B,\alpha,\beta},\operatorname{Coh}^{H,\beta}(X))\in\operatorname{Stab}_N^{\operatorname{Geo}}(X)$. Then there is no torsion-free H-semistable sheaf F such that $Z([F])\in\mathbf{R}_{\leq 0}$.

Proof. Suppose such an F exists. Then $\operatorname{Im} Z([F]) = 0$. From the definition of the torsion pair $(\mathcal{T}_{H,\beta}, \mathcal{F}_{H,\beta})$ in Theorem 5.5, it follows that $F \in \mathcal{F}_{H,\beta}$. But this implies that $Z([F]) \in \mathbf{R}_{>0}$.

Lemma 5.15. Let X be a smooth projective surface. Let $(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2$. Suppose

$$\alpha \leq \max \left\{ \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} : F \in \operatorname{Coh}(X) \text{ is H-semistable with } \mu_H(F) = \beta \right\}.$$

Then there exists an H-semistable sheaf F with $\operatorname{ch}_0(F)>0$ and $Z_{H,B,\alpha,\beta}(F)\in\mathbf{R}_{\leq 0}$.

Proof. By our hypotheses, there exists an H-semistable sheaf F with $\beta=\mu_H(F)=\frac{H.\mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}$, and $\alpha\leq\frac{\mathrm{ch}_2(F)-B.\mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}$. Since $\mu_H(F)\neq+\infty$, $\mathrm{ch}_0(F)>0$. Moreover,

$$\operatorname{Re}(Z_{H,B,\alpha,\beta}([F])) = \alpha H^2 \operatorname{ch}_0([F]) + B \cdot \operatorname{ch}_1([F]) - \operatorname{ch}_2([F]) \le 0,$$

 $\operatorname{Im}(Z_{H,B,\alpha,\beta}([F])) = H \cdot \operatorname{ch}_1([F]) - \beta H^2 \operatorname{ch}_0([F]) = 0.$

Hence $Z_{H,B,\alpha,\beta}([F]) \in \mathbf{R}_{\leq 0}$, as required.

Lemma 5.16. Suppose $\sigma = \sigma_{H,B,\alpha,\beta} \in \operatorname{Stab}_{N}^{\operatorname{Geo}}(X)$ is geometric. There there is an open neighbourhood $W \subset \mathbf{R}^2$ of (α,β) , such that for every $(\alpha',\beta') \in W$, $\sigma_{H,B,\alpha',\beta'} \in \operatorname{Stab}_{N}^{\operatorname{Geo}}(X)$.

Proof. By [Bri08, Proposition 9.4], there is an open neighbourhood U of σ in $\mathrm{Stab}(X)$ where all skyscraper sheaves are stable. Together with Theorem 2.14, it follows that there is an open neighbourhood $V\subseteq\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(X),\mathbf{C})$ of $Z_{H,B,\alpha,\beta}$ such that for any $Z'\in V$, the associated stability condition $\sigma'=(Z',\mathcal{A}')\in\mathrm{Stab}_N^{\mathrm{Geo}}(X)$. By Theorem 5.5, V can be identified with a subset of $\mathrm{Amp}_{\mathbf{R}}(X)\times\mathrm{NS}_{\mathbf{R}}(X)\times\mathbf{R}^2$. Let W be the intersection of V with $\{H\}\times\{B\}\times\mathbf{R}^2$. Then W has the required properties. \square

Lemma 5.17. Suppose $\sigma = \sigma_{H,B,\alpha,\beta} = (Z_{H,B,\alpha,\beta}, \operatorname{Coh}^{H,\beta}(X))$ is geometric. Suppose $\alpha \leq \Phi_{X,H,B}(\beta)$. Then there exists $\sigma_0 = (Z_0, \mathcal{A}_0) \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$, and a torsion-free H-semistable sheaf F such that $Z_0([F]) \in \mathbf{R}_{\leq 0}$.

Proof. Let $W \subset \mathbf{R}^2$ be the open neighbourhood of (α, β) from Lemma 5.16. Recall that

$$\Phi_{X,H,B}(\beta) := \limsup_{\mu \to \beta} \left\{ \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} \, : \begin{matrix} F \in \operatorname{Coh}(X) \text{ is H-semistable with} \\ \mu_H(F) = \mu \end{matrix} \right\}.$$

Therefore, there exist H-semistable sheaves with slopes arbitrarily close to β , and $\frac{\operatorname{ch}_2 - B \cdot \operatorname{ch}_1}{H^2 \operatorname{ch}_0}$ arbitrarily close to $\Phi_{X,H,B}(\beta)$. Hence there exists $(\alpha_0,\beta_0) \in W$ and an H-semistable sheaf F with

$$\mu_H(F) = \beta_0 \text{ and } \alpha_0 < \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2\operatorname{ch}_0(F)}.$$

In particular,

$$\alpha_0 \leq \max \left\{ \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} : F \in \operatorname{Coh}(X) \text{ is H-semistable with } \mu_H(F) = \beta_0 \right\}.$$

By Lemma 5.15, there exists an H-semistable sheaf F' with $\operatorname{ch}_0(F')>0$ and $Z_{H,B,\alpha_0,\beta_0}([F'])\in\mathbf{R}_{\leq 0}$. By Lemma 5.16, $\sigma_0:=\sigma_{H,B,\alpha_0,\beta_0}\in\operatorname{Stab}_N^{\operatorname{Geo}}(X)$.

Proof of Lemma 5.13. From Theorem 5.5, we know that $\sigma = g^*\sigma_{H,B,\alpha,\beta}$ for some $g \in \mathbf{C}$. Suppose $\alpha \leq \Phi_{X,H,B}(\beta)$. By Lemma 5.17, there exists a geometric stability condition $\sigma_0 = (Z_0, \mathcal{A}_0)$, and an H-semistable sheaf F with $\mathrm{ch}_0(F) > 0$ such that $Z_0([F]) \in \mathbf{R}_{\leq 0}$. However, σ_0 is geometric, so this contradicts Lemma 5.14.

Proposition 5.18. Let X be a smooth projective surface. Then the following map is an injective local homeomorphism onto its image

Π:
$$\operatorname{Stab}_{N}^{\operatorname{Geo}}(X) \longrightarrow \mathcal{U} = \{(H, B, \alpha, \beta) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X) \times \mathbf{R}^{2} : \alpha > \Phi_{X,H,B}(\beta)\}$$

$$\sigma = \sigma_{H,B,\alpha,\beta} \longmapsto (H, B, \alpha, \beta)$$

Proof. Let $\mathcal{Z} \colon \operatorname{Stab}(X) \to \operatorname{Hom}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$ denote the local homeomorphism from Theorem 2.14. Let

$$\mathcal{N} := \{ \sigma \in \operatorname{Stab}(X) : Z(\mathcal{O}_x) = -1 \ \forall x \in X \}.$$

Then $\mathcal{Z}|_{\mathcal{N}}$ is a continuous local homeomorphism onto its image, hence so is $\mathcal{Z}|_{\operatorname{Stab}^{\operatorname{Geo}}_{\mathcal{N}}(X)}$.

By Theorem 5.5, any $\sigma \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$ is determined by its central charge, hence $\mathcal{Z}|_{\operatorname{Stab}_N^{\operatorname{Geo}}(X)}$ is injective, and Π factors via $\mathcal{Z}|_{\operatorname{Stab}_N^{\operatorname{Geo}}(X)}$. By the same argument as Step 1 of the proof of Theorem 5.5,

$$\mathcal{Z}(\mathcal{N}) \cong \left\{ (H, B, \alpha, \beta) \in (\mathrm{NS}_{\mathbf{R}}(X))^2 \times \mathbf{R}^2 \right\}.$$

 Π is exactly the above isomorphism composed with $\mathcal{Z}|_{\operatorname{Stab}_N^{\operatorname{Geo}}(X)}$. Hence it is an injective local homeomorphism onto its image. \square

5.3.2. STEP 2: Construction of the pointwise inverse $\mathcal{U} \to \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. We first recall the construction of stability conditions in [MS17, Theorem 6.10].

Definition 5.19. Let X be a smooth projective surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Define $\sigma_{H,B} := (\mathrm{Coh}^{H,B}(X), Z_{H,B})$, where

$$\begin{split} Z_{H,B}([E]) &= \left(-\operatorname{ch}_2^B([E]) + \frac{H^2}{2} \cdot \operatorname{ch}_0^B([E])\right) + iH \cdot \operatorname{ch}_1^B([E]) \\ &= \left[\frac{1}{2}\left(1 - \frac{B^2}{H^2}\right) - i\frac{H \cdot B}{H^2}\right] H^2 \operatorname{ch}_0([E]) + (B + iH) \cdot \operatorname{ch}_1([E]) - \operatorname{ch}_2([E]), \\ \mathcal{T}_{H,B} &= \left\{E \in \operatorname{Coh}(X) : \underset{\text{torsion free part of E satisfies $\operatorname{Im} Z_{H,B}([F]) > 0$.} \right\}, \\ \mathcal{F}_{H,B} &= \left\{E \in \operatorname{Coh}(X) : \underset{\text{Narasimhan factor F of E satisfies $\operatorname{Im} Z_{H,B}([F]) \leq 0$.} \right\}, \end{split}$$

and $\operatorname{Coh}^{H,B}(X)$ is the tilt of $\operatorname{Coh}(X)$ at the torsion pair $(\mathcal{T}_{H,B}, \mathcal{F}_{H,B})$.

Lemma 5.20 ([MS17, Exercise 6.11]). Let X be a smooth projective surface. Then there exists a continuous function $C_{(-)}$: $\operatorname{Amp}_{\mathbf{R}}(X) \to \mathbf{R}_{\geq 0}$ such that, for every $D \in \operatorname{Eff}_{\mathbf{R}}(X)$,

$$C_H(H \cdot D)^2 + D^2 \ge 0.$$

Proof. $C_H(H \cdot D)^2 + D^2 \geq 0$ is invariant under rescaling. If we consider $\mathrm{Eff}_{\mathbf{R}}(X) \subset \mathrm{NS}_{\mathbf{R}}(X)$ as normed vector spaces, it is therefore enough to look at the subspace of unit vectors $U \subset \mathrm{Eff}_{\mathbf{R}}(X)$. Since $D \in U$ is effective and $D \neq 0$, $H \cdot D > 0$. Hence there exists $C \in \mathbf{R}_{\geq 0}$ such that $C(H \cdot D)^2 + D^2 \geq 0$. Define:

$$C_{H,D} := \inf\{C \in \mathbf{R}_{\geq 0} : C_H(H \cdot D)^2 + D^2 \geq 0\}.$$

Since $\operatorname{Amp}_{\mathbf{R}}(X)$ is open, H'. D>0 for a small deformation H' of H. It follows that \overline{U} is strictly contained in the subspace $\{E\in\operatorname{NS}_{\mathbf{R}}(X):E:H>0\}$. Moreover, $C_{H,D}$ is a continuous function on \overline{U} , and \overline{U} is compact as it is a closed subset of the unit sphere in $\operatorname{NS}_{\mathbf{R}}(X)$. Therefore, $C_{H,D}$ has a maximum, which we call C_H . By construction, this is a continuous function on $\operatorname{Amp}_{\mathbf{R}}(X)$. \square

Definition 5.21. Let X be a smooth projective surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We define the following quadratic forms on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$:

$$egin{aligned} Q_{BG} &:= \operatorname{ch}_1^2 - 2 \operatorname{ch}_2 \operatorname{ch}_0 \ \Delta^{C_H}_{H,B} &:= Q_{BG} + C_H(H \cdot \operatorname{ch}_1^B)^2, \end{aligned}$$

where $C_H \in \mathbf{R}_{>0}$ is the constant from Lemma 5.20.

Theorem 5.22 ([MS17, Theorem 6.10]). Let X be a smooth projective surface. Let $(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Then $\sigma_{H,B} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. In particular, $\sigma_{H,B}$ satisfies the support property with respect to $\Delta_{H',B'}^{C_{H'}}$, where $(H',B') \in \operatorname{Amp}_{\mathbf{Q}}(X) \times \operatorname{NS}_{\mathbf{Q}}(X)$ are nearby rational classes.

Remark 5.23. Theorem 5.22 was first proved for K3 surfaces in [Bri08], along with the fact that this gives rise to a continuous family. In [MS17, Theorem 6.10], the authors first prove the result holds for rational classes (H,B) and sketch how to extend this to arbitrary classes. In particular, $\sigma_{H,B}$ can be obtained as a deformation of $\sigma_{H',B'}$ for nearby rational classes (H',B'), and $\sigma_{H,B}$ satisfies the same support property, $\Delta_{H',B'}^{C_{H'}}$. This uses the fact that $\Delta_{H,B}^{C_{H}}$ varies continuously with (H,B), together with similar arguments to Proposition 5.1.

Lemma 5.24. Let X be a smooth projective surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$ and fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Suppose $\alpha > \frac{1}{2} \left[\left(\beta_0 - \frac{H \cdot B}{H^2} \right)^2 - \frac{B^2}{H^2} \right]$. Define $b := \beta_0 - \frac{H \cdot B}{H^2} \in \mathbf{R}$ and $a := \sqrt{2\alpha - b^2 + \frac{B^2}{H^2}} \in \mathbf{R}_{>0}$. Then $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{aH,B+bH}$ are the same up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$. Moreover, this is a continuous family for $\alpha > \frac{1}{2} \left[\left(\beta_0 - \frac{H \cdot B}{H^2} \right)^2 - \frac{B^2}{H^2} \right]$.

Proof. We abuse notation and consider the central charges as homomorphisms $K_{num}(X) \otimes \mathbf{R} \to \mathbf{C}$. By Theorem 5.5, it is enough to show $\operatorname{Ker} Z_{H,B,\alpha,\beta_0} = \operatorname{Ker} Z_{aH,B+bH}$ as sub-vector spaces of $K_{num}(X) \otimes \mathbf{R}$. Fix $u \in K_{num}(X) \otimes \mathbf{R}$. Since a > 0, $\operatorname{Im} Z_{aH,B+bH}(u) = 0$ if and only if

$$0 = aH \cdot B\operatorname{ch}_{0}(u) + abH^{2}\operatorname{ch}_{0}(u) - aH \cdot \operatorname{ch}_{1}(u)$$

$$= a\left(H \cdot B\operatorname{ch}_{0}(u) + \left(\beta_{0} - \frac{H \cdot B}{H^{2}}\right)H^{2}\operatorname{ch}_{0}(u) - H \cdot \operatorname{ch}_{1}(u)\right)$$

$$= a\left(\beta_{0}H^{2}\operatorname{ch}_{0}(u) - H \cdot \operatorname{ch}_{1}(u)\right)$$

$$= -a\operatorname{Im} Z_{H,B,\alpha,\beta_{0}}(u).$$

Therefore, $\operatorname{Im} Z_{aH,B+bH}(u)=0$ if and only if $\operatorname{Im} Z_{H,B,\alpha,\beta_0}(u)=0$. Now assume $\operatorname{Im} Z_{aH,B+bH}(u)=0$, so H . $\operatorname{ch}_1(u)=\beta_0H^2\operatorname{ch}_0(u)$. Then $\operatorname{Re} Z_{aH,B+bH}(u)=0$ if and only if

$$0 = \frac{1}{2} ((aH)^2 - (B+bH)^2) \operatorname{ch}_0 + B \cdot \operatorname{ch}_1 + bH \cdot \operatorname{ch}_1(u) - \operatorname{ch}_2(u)t$$

= $\frac{1}{2} \left(a^2 - \frac{(B+bH)^2}{H^2} + 2b\beta_0 \right) H^2 \operatorname{ch}_0(u) + B \cdot \operatorname{ch}_1(u) - \operatorname{ch}_2(u).$

Moreover,

$$\frac{1}{2} \left(a^2 - \frac{(B+bH)^2}{H^2} + 2b\beta_0 \right) = \frac{1}{2} \left(a^2 - \frac{B^2}{H^2} + 2b \left(\beta_0 - \frac{B \cdot H}{H^2} \right) - b^2 \right)
= \frac{1}{2} \left(2\alpha - b^2 + \frac{B^2}{H^2} - \frac{B^2}{H^2} + b^2 \right)
= \alpha.$$

It follows that $u \in \operatorname{Ker} Z_{aH,B+bH}$ if and only if $u \in Z_{H,B,\alpha,\beta_0}$. Therefore, by Theorem 5.22, $\sigma_{aH,B+bH} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. Moreover, $\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$ acts on $\operatorname{Stab}(X)$ by autoequivalences, hence $\sigma_{H,B,\alpha,\beta_0} \in \operatorname{Stab}(X)$. Then, by definition, $\sigma_{H,B,\alpha,\beta_0} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. It remains to show this gives rise to a continuous family. By Proposition 5.18,

$$\Pi : \operatorname{Stab}_{N}^{\operatorname{Geo}}(X) \to \mathcal{U}, \quad \sigma_{H,B,\alpha,\beta} \mapsto (H,B,\alpha,\beta).$$

is an injective local homeomorphism. Let $V:=\left\{(H,B,\alpha,\beta):\alpha>\frac{1}{2}\left[\left(\beta-\frac{H.B}{H^2}\right)^2-\frac{B^2}{H^2}\right]\right\}$. The restriction $\Pi|_{\Pi^{-1}(V)}$ is still an injective local homeomorphism. Moreover, By the arguments above, $\Pi|_{\Pi^{-1}(V)}$ is surjective, hence it is continuous.

Remark 5.25. Let $\operatorname{Sh}_2^+(\mathbf{R}) \subset \operatorname{GL}_2^+(\mathbf{R})$ denote the subgroup of shearings, i.e. transformations that preserves the real line. It is simply connected, hence it embeds as a subgroup into $\widetilde{\operatorname{GL}}_2^+(\mathbf{R})$ and acts on $\operatorname{Stab}(X)$. In the above proof, $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{aH,B+bH}$ have the same hearts, so they are the same up to the action of $\operatorname{Sh}_2^+(\mathbf{R})$.

The next result follows from the proof of Theorem 5.22. We explain this part of the argument explicitly, as it will be essential for extending the support property in Lemma 5.37.

Lemma 5.26. Let X be a smooth projective surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. There exists $(H',B') \in \mathrm{Amp}_{\mathbf{Q}}(X) \times \mathrm{NS}_{\mathbf{Q}}(X)$ such that, for $a \geq 1$, $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $\mathrm{Ker}\ Z_{aH,B} \otimes \mathbf{R}$. In particular, $\Delta_{H',B'}^{C'_{H}}$ gives the support property for $\sigma_{aH,B}$.

Proof. By Theorem 5.22, $\sigma_{aH,B} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$ for $a \geq 1$, and there exists $(H',B') \in \operatorname{Amp}_{\mathbf{Q}}(X) \times \operatorname{NS}_{\mathbf{Q}}(X)$ nearby to (H,B) such that $\Delta_{H',B'}^{C_{H'}}$ gives the support property for $\sigma_{H,B} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. In particular, $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $K_1 := \operatorname{Ker} Z_{H,B} \otimes \mathbf{R}$. By Proposition 5.1, it is enough to show $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $K_a := \operatorname{Ker} Z_{aH,B} \otimes \mathbf{R}$ for $a \geq 1$.

Recall that $u=(\mathop{\mathrm{ch}}\nolimits_0^B(u), \mathop{\mathrm{ch}}\nolimits_1^B(u), \mathop{\mathrm{ch}}\nolimits_2^B(u)) \in \mathcal{K}_a$ if and only if

$$a^2 \frac{H^2}{2} \operatorname{ch}_0^B(u) = \operatorname{ch}_2^B(u), \quad H \cdot \operatorname{ch}_1^B(u) = 0.$$

Let $\Psi_a \colon \mathrm{K}_1 \to \mathrm{K}_a$ be the isomorphism of sub-vector spaces of $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ given by

$$\Psi_a\colon v=(\mathrm{ch}_0^B(v),\mathrm{ch}_1^B(v),\mathrm{ch}_2^B(v))\mapsto \left(\mathrm{ch}_0^B(v),\mathrm{ch}_1^B(v),\mathrm{ch}_2^B(v)+(a^2-1)\frac{H^2}{2}\mathrm{ch}_0^B(v)\right).$$

Let $u \in K_a$. Then $u = \Psi_a(v)$ for some $v \in K_1$. Clearly $\Delta_{H',B'}^{C_{H'}}(0) = 0$, so we may assume $u \neq 0$. Hence $v \neq 0$, and it is enough to show that $\Delta_{H',B'}^{C_{H'}}(\Psi_a(v)) < 0$. Recall that $\operatorname{ch}_1^{B'} = \operatorname{ch}_1 - B'$. ch_0 , hence $\operatorname{ch}_1^{B'}(\Psi_a(v)) = \operatorname{ch}_1^{B'}(v)$. Therefore,

$$\begin{split} \Delta^{C_{H'}}_{H',B'}(\Psi_a(v)) &= (\operatorname{ch}_1^B(v))^2 - 2\operatorname{ch}_0^B(v)\operatorname{ch}_2^B(v) - 2(a^2 - 1)\frac{H^2}{2}(\operatorname{ch}_0^B(v))^2 + C_{H'}(H' \cdot \operatorname{ch}_1^{B'}(v))^2 \\ &= \Delta^{C_{H'}}_{H',B'}(v) - 2(a^2 - 1)\frac{H^2}{2}(\operatorname{ch}_0^B(v))^2 \\ &\leq \Delta^{C_{H'}}_{H'B'}(v). \end{split}$$

Since $\Delta_{H',B'}^{C_{H'}}$ is negative definite on K_1 , it follows that $\Delta_{H',B'}^{C_{H'}}(\Psi_a(v)) < 0$.

Definition 5.27. Let X be a smooth projective surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Let $\alpha > \Phi_{X,H,B}(\beta)$, and let $\delta > 0$. We define the following quadratic form on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$:

$$Q_{H,B,\alpha,\beta,\delta} := \delta^{-1}(H \cdot \mathsf{ch}_1 - \beta_0 H^2 \mathsf{ch}_0)^2 - (H^2 \mathsf{ch}_0) \left(\mathsf{ch}_2 - B \cdot \mathsf{ch}_1 - (\alpha_0 - \delta) H^2 \mathsf{ch}_0 \right).$$

Lemma 5.28. Let X be a smooth projective surface. Let $(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Then there exists $\delta > 0$ such that, for every H-semistable torsion-free sheaf F, we have $Q_{H,B,\alpha_0,\beta_0,\delta}([F]) \geq 0$.

Proof. Since $\Phi_{X,H,B}$ is upper semi-continuous and bounded above by a quadratic polynomial in x, the same argument as in [FLZ22, Remark 3.5] applies. In particular, there exists a sufficiently small $\delta>0$ such that

$$\frac{(x-\beta_0)^2}{\delta} + \alpha_0 - \delta \ge \Phi_{X,H,B}(x).$$

Suppose F is an H-semistable torsion-free sheaf. Let $x = \mu_H(F) = \frac{H.\operatorname{ch}_1(F)}{H^2\operatorname{ch}_0(F)}$, then

$$\delta^{-1}(H \cdot \operatorname{ch}_1(F) - \beta_0 H^2 \operatorname{ch}_0(F))^2 + (\alpha_0 - \delta)(H^2 \operatorname{ch}_0(F))^2 \ge (H^2 \operatorname{ch}_0(F))^2 \Phi_{X,H,B} \left(\frac{H \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)} \right).$$

From Lemma 4.6 it follows that

$$\delta^{-1}(H \cdot \operatorname{ch}_1(F) - \beta_0 H^2 \operatorname{ch}_0(F))^2 + (\alpha_0 - \delta)(H^2 \operatorname{ch}_0(F))^2 \ge (H^2 \operatorname{ch}_0(F))^2 \frac{\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F)}{H^2 \operatorname{ch}_0(F)}.$$

In particular,

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$$\delta^{-1}(H.\operatorname{ch}_1(F) - \beta_0 H^2 \operatorname{ch}_0(F))^2 - (H^2 \operatorname{ch}_0(F)) \left(\operatorname{ch}_2(F) - B \cdot \operatorname{ch}_1(F) - (\alpha_0 - \delta) H^2 \operatorname{ch}_0(F) \right) \ge 0.$$

Remark 5.29. Let $u \in \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. Now consider Z_{H,B,α_0,β_0} as a homomorphism $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. $\mathbf{R} \to \mathbf{C}$. Recall that $[E] \in \mathrm{K}_{\alpha_0} := \mathrm{Ker}\, Z_{H,B,\alpha_0,\beta_0} \subseteq \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ if and only if

$$\alpha_0 H^2 \operatorname{ch}_0(u) + B \cdot \operatorname{ch}_1(u) - \operatorname{ch}_2(u) = 0,$$
 $H \cdot \operatorname{ch}_1(u) - \beta_0 H^2 \operatorname{ch}_0(u) = 0.$

Then

$$Q_{H,B,\alpha_0,\beta_0,\delta}(u) = -\delta \left(H^2 \operatorname{ch}_0(u)\right)^2 \le 0,$$

for all $u \in K_{\alpha_0}$. In particular, $Q_{H,B,\alpha_0,\beta_0,\delta}$ is negative semi-definite on K_{α_0} . Hence $Q_{H,B,\alpha_0,\beta_0,\delta}$ does not fulfil the support property.

To construct a quadratic form which is negative definite on $K_{\alpha_0} = \text{Ker } Z_{H,B,\alpha_0,\beta_0}$, we will combine $Q_{H,B,\alpha_0,\beta_0,\delta}$ with Q_{BG} , the quadratic form coming from the Bogomolov-Gieseker inequality introduced in Definition 5.21.

Lemma 5.30 ([Bog79, §10], [HL10, Theorem 3.4.1]). Let X be a smooth projective surface. Let $H \in$ $\operatorname{Amp}_{\mathbf{R}}(X)$. Then $Q_{BG}([F]) \geq 0$ for every H-semistable torsion-free sheaf F.

Proposition 5.31. Let X be a smooth projective surface. Let $(H, B) \in Amp_{\mathbf{R}}(X) \times NS_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta > 0$ as in Lemma 5.28. Let $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon} := Q_{H,B,\alpha_0,\beta_0,\delta} + \varepsilon Q_{BG}$. Then there exists $\varepsilon > 0$ such that

- (1) $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}([F]) \geq 0$ for every H-semistable torsion-free sheaf F,
- (2) $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}([T]) \geq 0$ for every torsion sheaf T, and
- (3) $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$ is negative definite on $K_{\alpha_0}:=\operatorname{Ker} Z_{H,B,\alpha_0,\beta_0}\subseteq \operatorname{K}_{\operatorname{num}}(X)\otimes \mathbf{R}$.

Proof. (1) follows immediately for any $\varepsilon>0$ from Lemma 5.28 and Lemma 5.30. For (2), let C_H be the constant from Lemma 5.20. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \frac{\delta^{-1}}{C_H}$. Let T be a torsion sheaf, then

$$\begin{split} Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon_1}([T]) &= \delta^{-1}(H \cdot \operatorname{ch}_1([T]))^2 + \varepsilon_1 \operatorname{ch}_1([T])^2 \\ &= \varepsilon_1 \left(\frac{\delta^{-1}}{\varepsilon_1} (H \cdot \operatorname{ch}_1([T]))^2 + \operatorname{ch}_1([T])^2 \right) \\ &> \varepsilon_1 \left(C_H(H \cdot \operatorname{ch}_1([T]))^2 + \operatorname{ch}_1([T])^2 \right) \\ &> 0. \end{split}$$

For (3), fix a norm on $K_{num}(X)$ and let U denote the set of unit vectors in K_{α_0} with respect to this norm. It will be enough to show there exists $\varepsilon_2>0$ such that $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon_2}|_{U}<0$. Let $A:=\{u\in U\mid Q_{H,B,\alpha_0,\beta_0,\delta}=0\}$. For any $a\in A$, $\operatorname{ch}_0(a)=0$. The condition that

 $Z_{H,B,\alpha_0,\beta_0}(a) = 0$ becomes

$$B \cdot \text{ch}_1(a) = \text{ch}_2(a),$$
 $H \cdot \text{ch}_1(a) = 0.$

H is ample, so $ch_1(a)^2 \le 0$ by the Hodge index theorem. If $ch_1^2(a) = 0$, then $ch_1(a) = 0$, and hence 0 = B. $\operatorname{ch}_1(a) = \operatorname{ch}_2(a)$. So a = 0, which contradicts the fact that $a \in U$. Therefore,

$$Q_{BG}|_{A}([E]) = \operatorname{ch}_{1}([E])^{2} < 0.$$

We now claim that there exists a sufficiently small $\varepsilon_2>0$ such that $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon_2}<0$ on U. Note that $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon_2}\Big|_A=\varepsilon_2Q_{BG}\Big|_A<0$, so we only need to check the claim on $U\setminus A$. Now suppose the converse, so for every $\varepsilon>0$, there exists $u\in U\setminus A$ such that

$$Q_{BG}(u) \ge -\frac{1}{\varepsilon} Q_{H,B,\alpha_0,\beta_0,\delta}(u)$$

 $Q_{H,B,\alpha_0,\beta_0,\delta}(u) < 0$ since $Q_{H,B,\alpha_0,\beta_0,\delta}$ is negative semi-definite on U, and $u \in U \setminus A$. Therefore,

$$P(u) := \frac{Q_{BG}(u)}{-Q_{H,B,\alpha_0,\beta_0,\delta}(u)} \ge \frac{1}{\varepsilon}.$$

Thus P is not bounded above on $U\setminus A$. Moreover, A is closed and $Q_{BG}\big|_A<0$ on A. Hence Q_{BG} is negative definite on some open neighbourhood V of A, so $P\big|_V<0$. Finally, $U\setminus V$ is compact, so P must be bounded above on $U\setminus V$. In particular, P is bounded above on $U\setminus A$ which gives a contradiction. It follows that there exists $\varepsilon_2>0$ such that $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon_2}$ is negative definite on K_{α_0} . Finally, let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$.

Lemma 5.32. Let X be a smooth projective surface. Let $(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$, and fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta, \varepsilon > 0$ as in Proposition 5.31. Then $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$ is negative definite on $K_{\alpha} := \operatorname{Ker} Z_{H,B,\alpha,\beta} \otimes \mathbf{R}$ for all $\alpha \geq \alpha_0$.

Proof. Recall that $u=(\operatorname{ch}_0(u),\operatorname{ch}_1(u),\operatorname{ch}_2(u))\in \operatorname{K}_{\alpha}=\operatorname{Ker} Z_{H,B,\alpha,\beta_0}\otimes \mathbf{R}$ if and only if

$$\alpha H^2 \operatorname{ch}_0(u) + B \cdot \operatorname{ch}_1(u) - \operatorname{ch}_2(u) = 0, \quad H \cdot \operatorname{ch}_1(u) - \beta_0 H^2 \operatorname{ch}_0(u) = 0.$$

Let $\Psi_{\alpha} \colon \mathrm{K}_{\alpha_0} \to \mathrm{K}_{\alpha}$ be the isomorphism of sub-vector spaces of $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ given by

$$\Psi_{\alpha}$$
: $v = (\operatorname{ch}_{0}(v), \operatorname{ch}_{1}(v), \operatorname{ch}_{2}(v)) \mapsto (\operatorname{ch}_{0}(v), \operatorname{ch}_{1}(v), \operatorname{ch}_{2}(v) + (\alpha - \alpha_{0})H^{2}\operatorname{ch}_{0}(v)).$

Let $u \in \mathcal{K}_{\alpha}$, then $u = \Psi_{\alpha}(v)$ for some $v \in \mathcal{K}_{\alpha_0}$. Clearly $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(0) = 0$, so we may assume $u \neq 0$. Hence $v \neq 0$, and it is enough to show that $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(\Psi_{\alpha}(v)) < 0$. Moreover,

$$\begin{split} Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(\Psi_\alpha(v)) &= Q_{H,B,\alpha_0,\beta_0,\delta}(\Psi_\alpha(v)) + \varepsilon Q_{BG}(\Psi_\alpha(v)) \\ &= Q_{H,B,\alpha_0,\beta_0,\delta}(v) - (\alpha - \alpha_0)(H^2\mathrm{ch}_0(v))^2 + \varepsilon Q_{BG}(v) - 2\varepsilon(\alpha - \alpha_0)H^2\mathrm{ch}_0(v)^2 \\ &= Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(v) - (\alpha - \alpha_0)H^2\mathrm{ch}_0(v)^2(H^2 + 2\varepsilon) \\ &\leq Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(v). \end{split}$$

Finally, by Proposition 5.31(3), $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}(v) < 0$.

Lemma 5.33 ([MS17, Lemma 6.18]). Let $(H, B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. If $E \in \operatorname{Coh}^{H,B}(X)$ is $\sigma_{aH,B}$ -semistable for all $a \gg 0$, then it satisfies one of the following conditions:

- (1) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^{0}(E)$ is a H-semistable torsion-free sheaf.
- (2) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^{0}(E)$ is a torsion sheaf.
- (3) $\mathcal{H}^{-1}(E)$ is a H-semistable torsion-free sheaf and $\mathcal{H}^{0}(E)$ is either 0 or a torsion sheaf supported in dimension zero.

Corollary 5.34. Let $(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta, \varepsilon > 0$ as in Proposition 5.31. If $E \in \operatorname{Coh}^{H,B}(X)$ is $\sigma_{aH,B}$ -semistable for all $a \gg 0$, then $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}([E]) \geq 0$.

Proof. Let $Q:=Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$. By our hypotheses, E satisfies one of the three conditions in Lemma 5.33. If E satisfies (1), then $Q([E])=Q([\mathcal{H}^0(E)])$, where $\mathcal{H}^0(E)$ is a H-semistable torsion-free sheaf, and the result follows from Proposition 5.31(1). Similarly, if E satisfies (2), then by Proposition 5.31(2), $Q([E])=Q([\mathcal{H}^0(E)])\geq 0$. Now assume E satisfies (3). Then

$$\operatorname{ch}([E]) = -\operatorname{ch}(\mathcal{H}^{-1}(E)) + \operatorname{length}(\mathcal{H}^{0}(E))$$

Hence

$$Q_{BG}([E]) = Q_{BG}([\mathcal{H}^{-1}(E)]) - (-\operatorname{ch}_0(\mathcal{H}^{-1}(E))) \operatorname{length}(E) \ge Q_{BG}(\mathcal{H}^{-1}(E)).$$

The same argument applies to $Q_{H,B,\alpha_0,\beta_0,\delta}$. Hence $Q([E]) \geq Q([\mathcal{H}^{-1}(E)])$. The result follows by Proposition 5.31(1).

Lemma 5.35. Let $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}(X)$ with support property given by a quadratic form Q on $\operatorname{K}_{\operatorname{num}}(X) \otimes \mathbf{R}$. Suppose $E \in \operatorname{D^b}(X)$ is strictly σ -semistable and let A_1, \ldots, A_m be the Jordan-Hölder factors of E. Then $Q(A_i) < Q(E)$ for all $1 \le i \le m$.

Proof. It is enough to prove that $Q(A_1) < Q(E)$. Since E is σ -semistable, $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbf{R}$. By definition, $A_1 \in \mathcal{P}(\phi)$, and hence $E/A_1 \in \mathcal{P}(\phi)$ also. Therefore, by the support property, $Q(A_1) \geq 0$ and $Q(E/A_1) \geq 0$. Moreover, since A_1 and E/A_1 have the same phase, there exists $\lambda > 0$ such that $Z(A_1) - \lambda Z(E/A_1) = 0$. Hence $[A_1] - \lambda [E/A_1] \in \operatorname{Ker} Z \otimes \mathbf{R}$ and is non-zero. Let Q also denote the associated symmetric bilinear form. By the support property,

$$0 > Q([A_1] - \lambda[E/A_1]) = Q(A_1) - 2\lambda Q(A, E/A_1) + \lambda^2 Q(E/A_1).$$

Moreover, $\lambda, Q(A_1), Q(E/A_1) > 0$. It follows that $Q(A_1, E/A_1) > 0$. Therefore,

$$Q(E) = Q(A_1) + Q(E/A_1) + 2Q(A_1, E/A_1) > Q(A_1).$$

Lemma 5.36. Let $\sigma = (Z, \mathcal{P}) \in \operatorname{Stab}(X)$, and let Q be a quadratic form which is negative definite on $\operatorname{Ker} Z \otimes \mathbf{R}$. Suppose $E \in \operatorname{D}^{\operatorname{b}}(X)$ is strictly σ -semistable and let A_1, \ldots, A_m be the Jordan-Hölder factors of E. If Q(E) < 0, then for some $1 \leq j \leq m$, $Q(A_j) < 0$.

Proof. Assume for a contradiction that $Q(A_1), Q(E/A_1) \geq 0$. Let Q also denote the associated symmetric bilinear form. By the same argument as in the proof of Lemma 5.35, it follows that $Q(A, E/A_1) > 0$. Therefore,

$$Q(E) = Q(A_1) + Q(E/A_1) + 2(A_1, E/A_1) > 0,$$

which is a contradiction. Hence either $Q(A_1) < 0$ and we are done, or $Q(E/A_1) < 0$. If $Q(E/A_1) < 0$, we can repeat the argument with E/A_1 and A_2 instead of E and A_1 . There are finitely many Jordan-Hölder factors, so this process terminates. Therefore, $Q(A_j) < 0$ for some $1 \le j \le n$.

Lemma 5.37. Let X be a smooth projective surface. Let $(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta, \varepsilon > 0$ as in Proposition 5.31. Fix $\alpha_1 \in \mathbf{R}$ such that $\alpha_1 > \max\left\{\alpha_0, \frac{1}{2}\left[\left(\beta_0 - \frac{H \cdot B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]\right\}$. Assume $E \in \operatorname{D}^{\mathrm{b}}(X)$ is $\sigma_{H,B,\alpha_1,\beta_0}$ -semistable. Then $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}([E]) \geq 0$. In particular, $\sigma_{H,B,\alpha_1,\beta_0}$ satisfies the support property with respect to $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$.

Proof. From Lemma 5.24, we know that for every $\alpha \geq \alpha_1$, $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{a_\alpha H,B+bH}$, have the same heart when $b=\beta_0-\frac{H.B}{H^2}$ and $a_\alpha=\sqrt{2\alpha-b^2+\frac{B^2}{H^2}}$.

Moreover, by Lemma 5.26, there exists $(H',B')\in \mathrm{Amp}_{\mathbf{Q}}(X)\times \mathrm{NS}_{\mathbf{Q}}(X)$ such that $\Delta_{H',B'}^{C_{H'}}$ gives the support property for $\sigma_{aH,B+bH}$ if $a\geq a_{\alpha_1}$. We may assume $\Delta_{H',B'}^{C_{H'}}\in\mathbf{Z}$, since it is true after rescaling by some integer. Furthermore, since E is $\sigma_{a_{\alpha_1}H,B+bH}$ -semistable, $\Delta_{H',B'}^{C_{H'}}([E])\in\mathbf{Z}_{\geq 0}$.

If E is $\sigma_{H,B,\alpha,\beta_0}$ -stable for $\alpha\gg 0$, then by definition of a_α , E is $\sigma_{aH,B}$ -stable for $a\gg 0$. It then follows by Corollary 5.34 that $Q([E])\geq 0$. Otherwise, there exists some $\alpha_2\geq \alpha_1$ such that E is strictly $\sigma_{H,B,\alpha_2,\beta_0}$ -semistable. Let $A_1,\ldots A_m$ denote the Jordan-Hölder factors of E. Then by Lemma 5.35, $\Delta_{H',B'}^{C_{H'}}([A_i])<\Delta_{H',B'}^{C_{H'}}([E])$ for all $1\leq i\leq m$. Each A_i is $\sigma_{H,B,\alpha_2,\beta_0}$ -stable, so $\Delta_{H',B'}^{C_{H'}}([A_i])\geq 0$ for all $1\leq i\leq m$.

Assume for a contradiction that Q([E]) < 0. From Lemma 5.36, $Q([A_j]) < 0$ for some $1 \le j \le m$. Let $E_2 := A_j$. We can now repeat this process for E_2 in place of $E_1 := E$, and so on. This gives a sequence $E_1, E_2, E_3, \ldots, E_k, \ldots$ and $\alpha_1 \le \alpha_2 < \alpha_3 \ldots < \alpha_k \ldots$ such that $E_k \in \mathrm{D^b}(X)$ is

 $\sigma_{H,B,\alpha_k,\beta_0}$ -semistable, $Q(E_k)<0$, and $0\leq \Delta_{H',B'}^{C_{H'}}([E_{k+1}])<\Delta_{H',B'}^{C_{H'}}([E_k])$ for all $k\geq 1$. But $\Delta_{H',B'}^{C_{H'}}([E_k])\in \mathbf{Z}_{\geq 0}$ for all k, so no such sequence can exist. This gives a contradiction.

Finally, by Lemma 5.32,
$$Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$$
 is negative definition on $\operatorname{Ker} Z_{H,B,\alpha_1,\beta_0} \otimes \mathbf{R}$.

We are finally ready to apply Corollary 5.2.

Proposition 5.38. Let X be a smooth projective surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Then $\sigma_{H,B,\alpha,\beta_0} \in \mathrm{Stab}_N^{\mathrm{Geo}}(X)$ for all $\alpha \geq \alpha_0$.

Proof. Fix $\alpha_1 \in \mathbf{R}$ such that $\alpha_1 > \max\left\{\alpha_0, \frac{1}{2}\left[\left(\beta_0 - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]\right\}$. By Lemma 5.24, it follows that $\sigma_{H,B,\alpha_1,\beta_0} \in \operatorname{Stab}_N^{\operatorname{Geo}}(X)$. Choose $\delta, \varepsilon > 0$ as in Proposition 5.31, then by Lemma 5.37, $\sigma_{H,B,\alpha_1,\beta_0}$ satisfies the support property with respect to $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$.

By Lemma 5.32, $Q_{H,B,\alpha_0,\beta_0}^{\delta,\varepsilon}$ is negative definite on $\operatorname{Ker} Z_{H,B,\alpha,\beta_0}$ for all $\alpha \geq \alpha_0$. Moreover, $\operatorname{Im} Z_{H,B,\alpha,\beta_0}$ remains constant as α varies. Therefore, the result follows by Corollary 5.2.

Proof. (of Theorem 5.10) By Theorem 5.5, for every $\sigma \in \operatorname{Stab}^{\operatorname{Geo}}(X)$ there exists a unique $g \in \mathbf{C}$ such that $g^*\sigma \in \operatorname{Stab}^{\operatorname{Geo}}_N(X)$. Hence it is enough to show that $\operatorname{Stab}^{\operatorname{Geo}}_N(X) \cong \mathcal{U}$, where

$$\mathcal{U} = \left\{ (H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X, H, B}(\beta) \right\}$$

This follows from Proposition 5.18 and Proposition 5.38.

5.4. Applications of Theorem 5.10.

Corollary 5.39. Let X be a smooth projective surface. Then $Stab^{Geo}(X)$ is connected.

Remark 5.40. There are precisely two types of walls of the geometric chamber for K3 surfaces and rational surfaces. They either correspond to walls of the nef cone (see [TX22, Lemma 7.2] for a construction) or to discontinuities of the Le Potier function. For K3 surfaces, the second case comes from the existence of spherical bundles which is explained in [Yos09, Proposition 2.7]. For rational surfaces, the discontinuities correspond to exceptional bundles. This is explained for $Tot(\mathcal{O}_{\mathbf{P}^2}(-3))$ in [BM11, §5], and the arguments generalise to any rational surface.

It seems reasonable to expect this to hold for all surfaces. The description of the geometric chamber given by Theorem 5.10 also supports this. Indeed, a wall where \mathcal{O}_x is destabilised corresponds locally to the boundary of \mathcal{U} being linear. This boundary is exactly where

- (1) H becomes nef and not ample. We expect that this only gives rise to walls in the following cases:
 - H is big and nef. Then H induces a contraction of rational curves. This can be used to construct non-geometric stability conditions [TX22, Lemma 7.2].
 - H is nef and induces a contraction to a curve whose fibres are rational curves. In this case, we expect a wall. For example, let $f \colon S \to C$ be a \mathbf{P}^1 -bundle over a curve. We expect the existence of stability conditions on S such that all skyscraper sheaves are strictly semistable, and they are destabilised by

$$\mathcal{O}_{f^{-1}(x)} \to \mathcal{O}_x \to \mathcal{O}_{f^{-1}(x)}(-1)[1] \to \mathcal{O}_{f^{-1}(x)}[1].$$

- (2) If $\Phi_{X,H,B}$ is discontinuous at β , then $\mathrm{Stab}_N^{\mathrm{Geo}}(X)$ locally has a linear boundary. We expect this to give rise to non-geometric stability conditions.
- (3) $\alpha = \Phi_{X,H,B}(\beta)$. We expect no boundary in this case.

Corollary 5.41. Let X be a smooth projective surface. If $\Phi_{X,H,B}$ has no discontinuities and no linear pieces for any $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$, then any wall of $\mathrm{Stab}^{\mathrm{Geo}}(X)$ where \mathcal{O}_x is destabilised corresponds to a class $H' \in \mathrm{NS}_{\mathbf{R}}(X)$ which is nef and not ample.

Example 5.42. Let X be a smooth projective surface. Suppose

$$\Phi_{X,H,B}(x) = \frac{1}{2} \left[\left(x - \frac{H \cdot B}{H^2} \right)^2 - \frac{B^2}{H^2} \right].$$

By Lemma 5.24, $\operatorname{Stab}^{\operatorname{Geo}}(S) \cong \{(H,B) \in \operatorname{Amp}_{\mathbf{R}}(X) \times \operatorname{NS}_{\mathbf{R}}(X)\}$. In particular, by Corollary 4.15, this holds for free abelian quotients of surfaces with finite Albanese morphism, such as Beauville-type and bielliptic surfaces. For these examples, we know by Corollary 3.10 that there are no walls of $\operatorname{Stab}^{\operatorname{Geo}}(X)$ where \mathcal{O}_x is destabilised.

6. Further Questions

Let X be a smooth projective variety. There are no examples in the literature where $\operatorname{Stab}(X)$ is known to be disconnected. It would be interesting to investigate the following examples.

Question 6.1. Let S be a Beauville-type or bielliptic surface. Is Stab(S) connected?

S has non-finite Albanese morphism and $\operatorname{Stab}^{\operatorname{Geo}}(S) \subset \operatorname{Stab}(S)$ is a connected component by Corollary 3.10. If $\operatorname{Stab}(S)$ is connected, the following question would have a negative answer.

Question 1.3 ([FLZ22, Question 4.11]). Let X be a smooth projective variety whose Albanese morphism is not finite. Are there always non-geometric stability conditions on $D^b(X)$?

Question 6.2. Suppose $D^b(X)$ has a strong exceptional collection of vector bundles, and a corresponding heart A that can be used to construct stability conditions as in [Mac07a, §4.2]. If $\mathcal{O}_x \in A$, then does \mathcal{O}_x correspond to a stable quiver representation?

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