Stability conditions and group actions

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Where the work was done in collaboration with others, I have made a significant contribution. In particular, Chapters 7 and 8 are based on joint work with Edmund Heng and Anthony M. Licata.

This work has not been submitted for any other award or professional qualification.

Hannah Dell

To the women who taught, mentored, and inspired me, especially Deborah Evans, Liana Heuberger, and Vicky Neale.

Abstract

We study slope-stable vector bundles and Bridgeland stability conditions on varieties which are a quotient of a smooth projective variety by a finite group G acting freely. We show that there is an analytic isomorphism between $\mathrm{rep}(G)$-invariant geometric stability conditions on the quotient and G-invariant geometric stability conditions on the cover. We use this to describe a connected component inside the stability manifolds of free quotients when the cover has finite Albanese morphism. This applies to varieties with non-finite Albanese morphism which are free quotients of varieties with finite Albanese morphism, such as Beauville-type and bielliptic surfaces. This gives a partial answer to a question raised by Lie Fu, Chunyi Li, and Xiaolei Zhao: If a variety X has non-finite Albanese morphism, does there always exist a non-geometric stability condition on X? We also give counterexamples to a conjecture of Fu–Li–Zhao concerning the Le Potier function, which characterises Chern classes of slope-semistable sheaves. As a result of independent interest, we give a description of the set of geometric stability conditions on an arbitrary surface in terms of a refinement of the Le Potier function. This generalises a result of Fu–Li–Zhao from Picard rank one to arbitrary Picard rank.

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Contents

Notation and Conventions

k	an algebraically closed field
$\mathcal{D}$	a k-linear essentially small Ext-finite triangulated category with a Serre functor
G	a finite group such that $(\mathrm{char}(k), \lvert G \rvert) = 1$
$\mathrm{rep}(G)$	the category of finite dimensional representations of G over k
$\mathcal{D}_G$	the category of G-equivariant objects
X	a smooth connected projective variety over k
$\mathrm{D^b}(X)$	the bounded derived category of coherent sheaves on X
$\mathrm{D^b_G}(X)$	the bounded derived category of G-equivariant coherent sheaves
$\mathrm{K}(\mathcal{D}), \mathrm{K}(X)$	the Grothendieck group of $\mathcal{D}$, resp. $\mathrm{D^b}(X)$
$\mathrm{K_{num}}(\mathcal{D}), \mathrm{K_{num}}(X)$	the numerical Grothendieck group of $\mathcal{D}$, resp. $\mathrm{D^b}(X)$
$\mathrm{Stab}(\mathcal{D}), \mathrm{Stab}(X)$	the space of numerical Bridgeland stability conditions on $\mathcal{D}$, resp. $\mathrm{D^b}(X)$
$\mathrm{Stab^{Geo}}(X)$	the space of geometric numerical stability conditions on $\mathrm{D^b}(X)$
$\mathrm{Stab_{lf}}(\mathcal{D})$	the space of locally-finite Bridgeland stability conditions on $\mathcal{D}$
$\mathrm{ch}(E)$	the Chern character of an object $E \in \mathrm{D^b}(X)$
$\mathrm{NS}(X)$	$\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, the Néron–Severi group of X
$\mathrm{NS}_{\mathbf{R}}(X)$	$\mathrm{NS}(X) \otimes \mathbf{R}$
$\mathrm{Amp}_{\mathbf{R}}(X)$	the ample cone inside $\mathrm{NS}_{\mathbf{R}}(X)$
$\mathrm{Eff}_{\mathbf{R}}(X)$	the effective cone inside $\mathrm{NS}_{\mathbf{R}}(X)$
$\mathrm{Chow}(X)$	the Chow group of X, i.e. algebraic cycles on X modulo rational equivalence
$\mathrm{Chow_{num}}(X)$	the numerical Chow group of X

Chapter 1

Introduction

Given a smooth projective complex variety X, one can construct an invariant, the complex manifold of Bridgeland stability conditions $\mathrm{Stab}(X)$. This invariant plays many important roles in algebraic geometry. In this thesis, we study how it can be used to detect geometric properties when X admits a free action by a finite group.

1.1 Motivation: Geometric stability conditions

A Bridgeland stability condition $\sigma \in \mathrm{Stab}(X)$ picks out certain objects in $\mathrm{Coh}(X)$ – and more generally, in $\mathrm{D^b}(X)$ – and calls them *σ-stable*. Let us first see an example when $\dim X = 1$.

Example 1.1.1. Let X be a curve. The pair $\sigma_\mu = (\mathrm{Coh}(X), -\deg(E) + i\mathrm{rank}(E))$ is a Bridgeland stability condition. Let E be a non-zero coherent sheaf on X. We define the slope with respect to σ_μ as $\mu(E) = +\infty$ if $\mathrm{rank}(E) = 0$, and $\mu(E) = \deg(E)/\mathrm{rank}(E)$ otherwise. Then E is σ_μ-stable if and only if any non-zero non-trivial subsheaf F satisfies $\mu(F) < \mu(E)$. This is the original definition of slope-stability which was used to classify vector bundles [Mum63]. Examples of σ_μ-stable sheaves are line bundles, $\mathcal{O}(n)$, and all skyscraper sheaves of points, $\mathcal{O}_x$ for $x \in X$.

That was an example of a so-called geometric stability condition. Such stability conditions always pick out the points of your variety.

Definition 1.1.2. A stability condition $\sigma \in \mathrm{Stab}(X)$ is *geometric* if for all points $x \in X$, $\mathcal{O}_x$ is σ-stable. We denote by $\mathrm{Stab}^{\mathrm{Geo}}(X)$ the set of all geometric stability conditions.

Question 1.1.3. *Do there exist non-geometric stability conditions?*

The short answer is "sometimes". We summarise what was known in the literature (before [FLZ22]) in the table below. See Section 1.3 for further details and references.

$\dim X$	$\mathrm{Stab}^{\mathrm{Geo}}(X)$	$\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$	$\mathrm{Stab}(X) \neq \mathrm{Stab}^{\mathrm{Geo}}(X)$
1	$= \widetilde{\mathrm{GL}}_2^+(\mathbf{R}) \cdot \sigma_\mu \cong \mathbf{C} \times \mathbf{H}$	$g(X) > 0$	$\mathrm{Stab}(\mathbf{P}^1) \cong \mathbf{C}^2$
2	controlled by invariants of slope-stable sheaves on X	abelian surfaces	$\mathbf{P}^2$, K3 surfaces, rational surfaces, $X \supset C \cong \mathbf{P}^1$ s.t. $C^2 < 0$
≥ 3	$\neq \emptyset$ for some 3folds and $\mathbf{P}^n$	???	$\mathbf{P}^n$

This raises the question: which geometric properties lead to geometric and non-geometric stability conditions? The first general answer was given by [FLZ22, Theorem 1.1]: Lie Fu, Chunyi Li, and Xiaolei

Zhao showed that if X has finite Albanese morphism alb_X (i.e. if X admits a finite map to an abelian variety), then $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$. This led them to pose the following question.

Question 1.1.4 ([FLZ22, Q. 4.11])**.** *If* alb_X *is not finite, then is* $\mathrm{Stab}(X) \neq \mathrm{Stab}^{\mathrm{Geo}}(X)$*?*

In all known examples (i.e. the rightmost column in the table above) the answer to Question 1.1.4 was positive. The goal of this thesis is to study a different flavour of examples: *free quotients*, i.e. quotients $Y = X/G$ of varieties by the free action of a finite group G. In particular, suppose alb_X is finite while alb_Y is not. This occurs in several examples including Beauville-type and bielliptic surfaces (see Examples 1.2.1 and 1.2.2). To study Question 1.1.4 for Y, we will compare $\mathrm{Stab}(Y)$ with $\mathrm{Stab}(X)$ in two different ways. One crucial consequence will be the following.

Theorem 8.3.1. *Let X be a surface with finite Albanese morphism. Let G be a finite group acting freely on X. Let $S = X/G$. Then* $\mathrm{Stab}^{\mathrm{Geo}}(S)$ *is a contractible connected component of* $\mathrm{Stab}(S)$.

This tells us that if S is a Beauville-type or bielliptic surface, then either Question 1.1.4 is false, or $\mathrm{Stab}(S)$ is disconnected. Both of these cases would be surprising.

1.2 Summary of results

We first study the Le Potier function introduced by Fu–Li–Zhao in [FLZ22, §3.1]. In Theorem 2.4.7 we give counterexamples to the conjecture stated in [FLZ22, §4.4], and explain how a refinement of the Le Potier function controls the set of geometric Bridgeland stability conditions on any surface in Theorem 4.3.1.

Our second approach is via group actions on triangulated categories. In Theorem 5.3.6 we sharpen the correspondence between G-invariant stability conditions on $\mathcal{D}$ and stability conditions on the G-equivariant category $\mathcal{D}_G$ introduced by Macrì–Mehrotra–Stellari in [MMS09, Theorem 1.1]. This is used to control the set of geometric stability conditions on any free quotient by a finite group in Theorem 6.3.2. We first consider abelian groups, and later generalise this to any finite group via $\mathrm{rep}(G)$ in Theorem 8.2.1 and Theorem 8.3.1.

Chapters 2-6 are based on [D23] and Chapters 7-8 are based on [DHL24].

1.2.1 Slope-stability on free quotients via the Le Potier function

A fundamental problem in the study of stable sheaves on a variety X is to understand the set of Chern characters of stable sheaves. This can be used to control wall-crossing and hence indirectly control Brill–Noether phenomena as in [Bay18, Theorem 1.1] and [Fey20]. Later in this thesis, we will use the set of Chern characters of stable sheaves to describe $\mathrm{Stab}^{\mathrm{Geo}}(X)$ for surfaces (see Theorem 4.3.1).

When studying slope-stable sheaves on a variety X, a natural question is for which topological invariants (i.e. Chern character) slope-stable sheaves exist. For $X = \mathbf{P}^2$, Drézet and Le Potier gave a complete solution in [DL85, Theorem B] in terms of a function of the slope, $\delta\colon \mathbf{R} \to \mathbf{R}$. In [FLZ22, §3.1], the authors define the Le Potier function $\Phi_{X,H}$ which gives a generalisation of Drézet and Le Potier's function to any polarised surface (X, H). They use this to control geometric Bridgeland stability conditions with respect to a sublattice of the numerical K-group of X, $\mathrm{K}_{\mathrm{num}}(X)$, coming from the polarisation.

Let $\mathrm{NS}_{\mathbf{R}}(X) \coloneqq \mathrm{NS}(X) \otimes \mathbf{R}$, where $\mathrm{NS}(X)$ is the Néron–Severi group of X, and let $\mathrm{Amp}_{\mathbf{R}}(X)$ denote the ample cone inside $\mathrm{NS}_{\mathbf{R}}(X)$. In Chapter 2 we introduce a generalisation of the Le Potier function. We state the version for surfaces below to ease notation.

Definition 4.3.3. *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We define the Le Potier function twisted by B, $\Phi_{X,H,B} : \mathbf{R} \to \mathbf{R}$, by*

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{\mathrm{ch}_2(F) - B \,.\, \mathrm{ch}_1(F)}{H^2 \mathrm{ch}_0(F)} : F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable with } \frac{H \,.\, \mathrm{ch}_1(F)}{H^2 \mathrm{ch}_0(F)} = \mu \right\}.$$

The Bogomolov–Gieseker inequality gives an upper bound for $\Phi_{X,H,B}$ (see Lemma 2.2.6). If $B = 0$, this is the same as [FLZ22, Definition 3.1], i.e. $\Phi_{X,H,0} = \Phi_{X,H}$, and the upper bound is $\frac{x^2}{2}$. The function $\Phi_{X,H,B}$ naturally generalises to higher dimensions, see Definition 2.2.4.

The Le Potier function partially determines the non-emptiness of moduli spaces of H-semistable sheaves of a fixed Chern character, which in turn controls wall crossing, along with the birational geometry of these moduli spaces. Examples of this include for $\mathbf{P}^2$ [LZ19, Theorem 0.2, Theorem 0.4], K3 surfaces [BM14, Theorem 5.7], and abelian surfaces [MYY12, Theorem 4.4.1].

The Le Potier function is known for abelian surfaces [Muk84, Corollary 0.2][Yos01], K3 surfaces [Huy16, Chapter 10, Theorem 2.7], del Pezzo surfaces of degrees $9 - m$ for $m \leq 6$ (for $H = -K_X$, $B = 0$) [LZ23, Theorem 7.15], Hirzebruch surfaces [CH21, Theorem 9.7], and for surfaces with finite Albanese morphism [LR21, Example 2.12(2)].

In this thesis, we relate the Le Potier function of X to the Le Potier function of any free quotient of X by a finite group. We state these results for surfaces with $B = 0$ below.

Proposition 2.3.4. *Let X be a surface, and let G be a finite group acting freely on X. Consider the quotient map $\pi\colon X \to X/G =: S$ and let $H_S \in \mathrm{Amp}_{\mathbf{R}}(S)$. Then $\Phi_{S,H_S} = \Phi_{X,\pi^* H_S}$.*

Recall that any variety X has a map alb_X, the Albanese morphism, to $\mathrm{Alb}(X) := \mathrm{Pic}^0(\mathrm{Pic}^0(X))$, the Albanese variety. This map is algebraic, unique up to translation, and every morphism $f\colon X \to A$ to another abelian variety A factors via alb_X. The dimension of $\mathrm{Alb}(X)$ is $h^1(X, \mathcal{O}_X)$. When X is a curve, $\mathrm{alb}(X)$ is the Abel–Jacobi map to the Jacobian of the curve. Over $\mathbf{C}$, $\mathrm{Alb}(X) = \frac{H^0(X,\Omega^1)^\vee}{H_1(X,\mathbf{Z})}$. For more details, see [Bea96, Chapter V] and [GH94, Chapter 2.6]. In Proposition 2.4.6 we compute the Le Potier function for varieties with finite Albanese morphism. Together with Proposition 2.3.4, this allows us to compute the Le Potier function for varieties that are finite free quotients of varieties with finite Albanese morphism.

Theorem 2.4.7. *Let X be a surface with finite Albanese morphism. Let G be a finite group acting freely on X. Let $\pi\colon X \to X/G =: S$ denote the quotient map. Let $H_X = \mathrm{alb}_X^* H = \pi^* H_S \in \mathrm{Amp}_{\mathbf{R}}(X)$ be an ample class pulled back from $\mathrm{Alb}(X)$ and S. Then $\Phi_{S,H_S}(x) = \frac{x^2}{2}$.*

In Example 2.4.8 we explain how to choose appropriate ample classes such that Theorem 2.4.7 applies to the following two classes of minimal surfaces.

Example 1.2.1 (Beauville-type surfaces, $q = 0$)**.** Let $X = C_1 \times C_2$, where $C_i \subset \mathbf{P}^2$ are curves of genus $g(C_i) \geq 2$. Each curve has finite Albanese morphism, and hence so does X. Suppose there is a free action of a finite group G on X, such that $S = X/G$ has $q(S) := h^1(S, \mathcal{O}_S) = 0$ and $p_g(S) := h^2(S, \mathcal{O}_S) = 0$. This generalises a construction due to Beauville in [Bea96, Chapter X, Exercise 4], and we call S a *Beauville-type surface*. These were classified in [BCG08, Theorem 0.1]. There are 17 families, 5 of which involve an abelian group. In the abelian cases, G is one of the following groups: $(\mathbf{Z}/2\mathbf{Z})^3$, $(\mathbf{Z}/2\mathbf{Z})^4$, $(\mathbf{Z}/3\mathbf{Z})^2$, $(\mathbf{Z}/5\mathbf{Z})^2$. Since $\dim(\mathrm{Alb}(S)) = q(S) = 0$, alb_S is trivial for any Beauville-type surface.

Example 1.2.2 (Bielliptic surfaces, $q = 1$)**.** Let $S \cong (E \times F)/G$, where E, F are elliptic curves, and G is a finite group of translations of E acting on F such that $F/G \cong \mathbf{P}^1$. Then $q(S) = 1$ and $\mathrm{Alb}(S) \cong E/G$, so alb_S is an elliptic fibration. Such surfaces are called *bielliptic*. There are 7 families, see [Bea96, List VI.20].

In particular, Beauville-type surfaces provide counterexamples to the following conjecture.

Conjecture 1.2.3 ([FLZ22, §4.4]). *Let (S, H) be a polarised surface with $q = 0$, then the Le Potier function $\Phi_{S,H}$ is not continuous at 0.*

This conjecture was motivated by Question 1.1.4 and the expectation that discontinuities of $\Phi_{S,H}$ could be used to show the existence of a wall of the geometric chamber for regular surfaces, as in the cases of rational and K3 surfaces.

1.2.2 The Le Potier Function and Geometric Stability Conditions

Let X be a surface and fix $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. In [FLZ22, Theorem 3.4, Proposition 3.6], the authors showed that $\Phi_{X,H}$ gives precise control over $\mathrm{Stab}_H^{\mathrm{Geo}}(X)$, the set of geometric numerical Bridgeland stability conditions with respect to a sublattice $\Lambda_H \subset \mathrm{K}_{\mathrm{num}}(X)$ (see Remark 6.3.6). When X has Picard rank 1, $\mathrm{Stab}_H^{\mathrm{Geo}}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$.

In Chapter 4, we generalise this to the set of all geometric numerical Bridgeland stability conditions.

Theorem 4.3.1. *Let X be a surface. There is a homeomorphism of topological spaces*

$$\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \left\{ (H, B, \alpha, \beta) \in (\mathrm{NS}_{\mathbf{R}}(X))^2 \times \mathbf{R}^2 : H \textit{ is ample, } \alpha > \Phi_{X,H,B}(\beta) \right\}.$$

In particular, $\mathrm{Stab}^{\mathrm{Geo}}(X)$ is connected. We discuss in Section 4.4 how Theorem 4.3.1 could be used to describe the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(X)$. This emphasises how $\Phi_{X,H,B}$ is a crucial tool for understanding the existence of non-geometric stability conditions on surfaces. In particular, if one can compute the Le Potier function, one should be able to tell whether the boundary of the set of geometric stability conditions has a wall.

1.2.3 Geometric Stability Conditions and Abelian Group Actions

Let k be an algebraically closed field, and let G be a finite abelian group such that $(\mathrm{char}(k), |G|) = 1$. Let $\mathcal{D}$ be a k-linear idempotent complete triangulated category with an action of G in the sense of [Del97] (see Definition 5.1.1). This induces an action on $\mathrm{Stab}(\mathcal{D})$, the space of all numerical Bridgeland stability conditions on $\mathcal{D}$. Let $\mathcal{D}_G$ denote the corresponding category of G-equivariant objects (see Definition 5.1.4). There is a residual action by $\widehat{G} = \mathrm{Hom}(G, k^*)$ on $\mathcal{D}_G$ (see Proposition 5.1.6), and $(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}$ by [Ela15, Theorem 4.2]. Theorem 5.3.6 describes an analytic isomorphism between G-invariant stability conditions on $\mathcal{D}$ and $\widehat{G}$-invariant stability conditions on $\mathcal{D}_G$. This builds on [Pol07, Proposition 2.2.3] and [MMS09, Theorem 1.1].

In this thesis, we focus on the case where $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ for X a variety, and the action of G on $\mathcal{D}$ is induced by a free action by G on X. Then $(\mathrm{D}^{\mathrm{b}}(X))_G \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) := \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_G(X))$, the bounded derived category of G-equivariant coherent sheaves on X. We call $Y := X/G$ a *free abelian quotient*. Recall that $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$. Let $\pi \colon X \to Y$ denote the quotient morphism. There is a decomposition of $\pi_* \mathcal{O}_X$ into line bundles $\mathcal{L}_\chi$ according to the 1-dimensional representations $\chi \in \widehat{G}$. Then $- \otimes \mathcal{L}_\chi \colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(Y)$ describes the residual action of $\widehat{G}$.

In all known examples, the stability manifold contains an open set of *geometric* stability conditions (see Definition 1.1.2). We prove that geometric stability conditions are preserved under the analytic isomorphism of Theorem 5.3.6.

Theorem 6.1.1. *Suppose G is a finite abelian group acting freely on a variety X. Let $\pi \colon X \to Y := X/G$ denote the quotient map. Consider the action of $\widehat{G}$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ as in Proposition 5.1.6. Then the* *functors π^*, π_* induce analytic isomorphisms between G-invariant stability conditions on $\mathrm{D^b}(X)$ and $\widehat{G}$-invariant stability conditions on $\mathrm{D^b}(Y)$ which preserve geometric stability conditions:*

$$(\pi^*)^{-1} \colon (\mathrm{Stab}(X))^G \overset{\cong}{\rightleftarrows} (\mathrm{Stab}(Y))^{\widehat{G}} : (\pi_*)^{-1}.$$

The compositions $(\pi_*)^{-1} \circ (\pi^*)^{-1}$ and $(\pi^*)^{-1} \circ (\pi_*)^{-1}$ fix slicings and rescale central charges by $|G|$.

In particular, suppose $\sigma = (\mathcal{P}_\sigma, Z_\sigma) \in (\mathrm{Stab}(X))^G$ satisfies the support property with respect to (Λ, λ). Then $(\pi^*)^{-1}(\sigma) =: \sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in (\mathrm{Stab}(Y))^{\widehat{G}}$ is defined by

$$\begin{aligned} \mathcal{P}_{\sigma_Y}(\phi) &= \{\mathcal{E} \in \mathrm{D^b}(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\}, \\ Z_{\sigma_Y} &= Z_\sigma \circ \pi^*, \end{aligned}$$

where π^* is the natural induced map on $\mathrm{K}(\mathrm{D^b}(Y))$, and σ_Y satisfies the support property with respect to $(\Lambda, \lambda \circ \pi^*)$.

As discussed in Section 1.1, very little is known about how the geometry of a variety X relates to the geometry of $\mathrm{Stab}(X)$. In [FLZ22, Theorem 1.1], the authors showed that if X has finite Albanese morphism, then all stability conditions on $\mathrm{D^b}(X)$ are geometric. In this setup, we obtain a union of connected components of geometric stability conditions on any free abelian quotient of X.

Theorem 6.3.2. *Let X be a variety with finite Albanese morphism. Let G be a finite abelian group acting freely on X and let $Y = X/G$. Then $\mathrm{Stab}^\ddagger(Y) := (\mathrm{Stab}(Y))^{\widehat{G}} \cong \mathrm{Stab}(X)^G$ is a union of connected components consisting only of geometric stability conditions.*

When X is a surface, we have the following stronger result.

Theorem 6.3.3. *Let X be a surface with finite Albanese morphism. Let G be a finite abelian group acting freely on X. Let $S = X/G$. Then $\mathrm{Stab}^\ddagger(S) = \mathrm{Stab}^{\mathrm{Geo}}(S)$. In particular, $\mathrm{Stab}^{\mathrm{Geo}}(S)$ is a contractible connected component of $\mathrm{Stab}(S)$.*

Note that Theorem 4.3.1 gives us an explicit description of $\mathrm{Stab}^\ddagger(S)$. Moreover, let $S = X/G$ be a Beauville-type or bielliptic surface with G abelian. As discussed in Examples 1.2.1 and 1.2.2, S has non-finite Albanese morphism. By Theorem 6.3.3, $\mathrm{Stab}^{\mathrm{Geo}}(S) \subset \mathrm{Stab}(S)$ is a connected component. In particular, if $\mathrm{Stab}(S)$ is connected, then Question 1.1.4 would have a negative answer.

1.2.4 Geometric stability conditions and fusion actions

In the rest of this thesis, we generalise the results of Section 1.2.3 to actions of non-abelian groups. Let X be a variety with a free action of a finite group G, and consider the quotient $\pi \colon X \to Y := X/G$. When G is abelian, we saw in Theorem 6.1.1 that the action of the group of 1-dimensional representations $\widehat{G}$ on $\mathrm{D^b}(Y)$ can be used to relate $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$. When G is non-abelian, the action of $\widehat{G}$ is too small to capture this relation. Instead, we need to use all of the (irreducible) representations of G. In particular, $\pi_*\mathcal{O}_X$ has a decomposition into vector bundles E_ρ corresponding to the irreducible representations $\rho \in \mathrm{Irr}(G)$ (see (8.1)). There are endofunctors $- \otimes E_\rho \colon \mathrm{D^b}(Y) \to \mathrm{D^b}(Y)$, whose composition corresponds to taking the tensor product in $\mathrm{rep}(G)$. This gives $\mathrm{D^b}(Y)$ the structure of a *(right) module category over* $\mathrm{rep}(G)$. We also call this an *action* of $\mathrm{rep}(G)$. In Section 7.1, we study such structures in greater generality. In particular, we work with *fusion categories* – monoidal categories with strong finiteness properties – acting on categories (see Definitions 7.1.7 and 7.1.14).

Before we can generalise Theorem 6.1.1, we need to understand stability conditions on the free quotient $Y = X/G$ that *behave well* with respect to the $\mathrm{rep}(G)$-action, that is

$$\mathrm{Stab}_{\mathrm{rep}(G)}(Y) := \left\{ \sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(Y) : \begin{array}{l} \mathcal{P}(\phi) \otimes E_\rho \subset \mathcal{P}(\phi), \\ Z(F \otimes E_\rho) = \dim \rho \cdot Z(F), \end{array} \forall \rho \in \mathrm{Irr}(G) \right\}.$$

We call such stability conditions $\mathrm{rep}(G)$-*equivariant*. When G is abelian, an action of $\mathrm{rep}(G)$ is equivalent to an action of $\widehat{G}$. Thus $\mathrm{Stab}_{\mathrm{rep}(G)}(Y) = (\mathrm{Stab}(Y))^{\widehat{G}}$ is the set of fixed points of a group action on the manifold $\mathrm{Stab}(Y)$. Therefore, it is a closed submanifold. In the non-abelian case, we instead build on techniques from [Hen22] to study *fusion-equivariant* stability conditions in Section 7.2. We prove that such stability conditions always form a closed submanifold in Theorem 7.2.15, which in particular applies to $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$.

In Section 7.3, we consider again the setting of a finite group G acting on a triangulated category $\mathcal{D}$. Theorem 7.3.10 describes an analytic isomorphism between G-invariant stability conditions on $\mathcal{D}$ and $\mathrm{rep}(G)$-equivariant stability conditions on $\mathcal{D}_G$. This generalises Theorem 5.3.6. When $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(Y)$ with $Y = X/G$ a free quotient, this analytic isomorphism again preserves geometric stability conditions.

Theorem 8.2.1. *Suppose G is a finite group acting freely on a variety X. Let $\pi\colon X \to Y := X/G$ denote the quotient map. Consider the action of $\mathrm{rep}(G)$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$ as in (7.2). Then the functors π^*, π_* induce an analytic isomorphism between the closed submanifolds of stability conditions $(\mathrm{Stab}(X))^G$ and $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$,*

$$(\pi^*)^{-1} \colon (\mathrm{Stab}(X))^G \overset{\cong}{\rightleftarrows} \mathrm{Stab}_{\mathrm{rep}(G)}(Y) : (\pi_*)^{-1},$$

which are mutual inverses up to rescaling the central charge by $|G|$. Moreover, this preserves geometric stability conditions.

This generalises Theorem 6.1.1. In Lemma 8.1.1, we prove that $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ is still open. This allows us to generalise the remaining results from Chapter 6.

Theorem 8.3.1. *Let X be a variety with finite Albanese morphism. Let G be a finite group acting freely on X and let $Y = X/G$. Then*

(1) $\mathrm{Stab}^\ddagger(Y) := \mathrm{Stab}_{\mathrm{rep}(G)}(Y) \cong \mathrm{Stab}_{\mathrm{vec}_G}(X)$ is a union of connected components consisting only of geometric stability conditions.
(2) if X is a surface, then $\mathrm{Stab}^\ddagger(Y) = \mathrm{Stab}^{\mathrm{Geo}}(Y) \cong (\mathrm{Stab}(X))^G$. In particular, $\mathrm{Stab}^\ddagger(Y)$ is a contractible connected component of $\mathrm{Stab}(Y)$.

This produces new examples of varieties with non-finite Albanese morphism whose stability manifold has a connected component of only geometric stability conditions (see Examples 8.3.2, 8.3.3, and 8.3.4). Since the stability manifold of any variety is expected to be connected, this provides further evidence that Question 1.1.4 has a negative answer.

Motivation from representation theory

The study of actions of fusion categories in Chapter 7 was also motivated by questions in representation theory. For example, can a group G always be realised as a symmetry of a (triangulated) category $\mathcal{D}$? In [Hen22, Chapter 2], Edmund Heng used fusion categories to construct new (faithful) categorical actions coming from certain non-simply laced braid groups. In the upcoming work [HL], Heng and Licata study the case where $G = B_W$ is the braid group associated to any (non-simply laced) Coxeter system W. They construct a triangulated category $\mathcal{D}^W$ which has an action of a fusion category $\mathcal{C}^W$ associated to W. Moreover, they conjecture that $\mathrm{Stab}_{\mathcal{C}^W}(\mathcal{D}^W)$ is a universal covering of the hyperplane complement associated to W, whose contractibility would prove the $K(\pi, 1)$ conjecture (see [DHL24, §1.4]).

In [DHL24], we also studied the representation theory of so-called *(separated) McKay quivers* Q. This includes many examples of wild quivers, whose stability manifolds are not well understood. In [DHL24, Corollary 5.2] we used Theorem 7.3.10 to describe a submanifold of $\mathrm{Stab}(\mathrm{D^b}(\mathrm{Rep}(Q)))$ explicitly.

1.3 Survey: Geometric stability conditions

To give context for the results in this thesis, we survey the cases where a connected component of the stability manifold is known, and where geometric and non-geometric stability conditions have been described. Throughout, X is assumed to be a smooth projective variety over $\mathbf{C}$.

There are the following general results:

- Varieties with alb_X finite: $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$ [FLZ22, Theorem 1.1]
- Quotients of varieties with alb_X finite: Let $Y = X/G$ be a free quotient of X, and assume alb_X is finite. If G-invariant stability conditions exist on X, then $\mathrm{Stab}^{\mathrm{Geo}}(Y) \cong (\mathrm{Stab}(X))^G$ is a union of connected components consisting only of geometric stability conditions, see Theorem 8.3.1.

The results for specific examples are summarised in the following table:

$\dim X$	$\mathrm{Stab}^{\mathrm{Geo}}(X)$	$\mathrm{Stab}(X) \neq \mathrm{Stab}^{\mathrm{Geo}}(X)$?
1	$\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$	$\mathrm{Stab}(\mathbf{P}^1) \cong \mathbf{C}^2$
2	controlled by $\Phi_{X,H,B}$	$\mathbf{P}^2$, K3 surfaces, rational surfaces, $X \supset C$ rational curve s.t. $C^2 < 0$
≥ 3	$\neq \emptyset$ for some 3folds and $\mathbf{P}^n$	any variety with a complete exceptional collection

Note that the examples in the rightmost column have non-finite Albanese morphism. This gives a positive answer to Question 1.1.4 in those cases.

Curves: For any curve C, $\mathrm{Stab}^{\mathrm{Geo}}(C) \cong \widetilde{\mathrm{GL}}_2^+(\mathbf{R})$. Up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$, this corresponds to slope-stability for $\mathrm{Coh}(C)$ [Mac07b, Theorem 2.7]. We saw this already in Example 1.1.1

- $\mathrm{Stab}(\mathbf{P}^1) \cong \mathbf{C}^2$ [Oka06, Theorem 1.1]. Okada's construction uses the identification $\mathrm{D^b}(\mathbf{P}^1) \cong \mathrm{D^b}(\mathrm{Rep}(K_2))$ where K_2 is the Kroneker quiver with two arrows. In particular, these are not all geometric.
- Let C be a curve of genus $g(C) \geq 1$, then $\mathrm{Stab}(C) = \mathrm{Stab}^{\mathrm{Geo}}(C) \cong \widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ [Bri07, Theorem 9.1], [Mac07b, Theorem 2.7].

Surfaces: There is a construction called *tilting* which gives an open set of geometric stability conditions on any surface, see for example [AB13; MS17] and Chapter 4.

A connected component is known in the following cases:

- Surfaces with finite Albanese morphism: This connected component is precisely the set of geometric stability conditions which come from tilting. This follows from [FLZ22, Theorem 1.1] together with Theorem 4.4.1.
- K3 surfaces: There is a distinguished connected component $\mathrm{Stab}^{\dagger}(X)$ described by taking the closure and translates under autoequivalences of the open set of geometric stability conditions [Bri08, Theorem 1.1]. By [Bri08, Theorem 12.1], at general points of the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(X)$, either
  - all skyscraper sheaves have a spherical vector bundle as a stable factor, or
  - $\mathcal{O}_x$ is strictly semistable if and only if $x \in C$, a smooth rational curve in X.
- $\mathbf{P}^2$: $\mathrm{Stab}(\mathbf{P}^2)$ has a simply-connected component, $\mathrm{Stab}^{\dagger}(\mathbf{P}^2)$, which is a union of geometric and algebraic stability conditions [Li17, Theorem 0.1].
- Enriques surfaces: Suppose Y is an Enriques surface with K3 cover X, and let $\mathrm{Stab}^{\dagger}(X)$ be the connected component of $\mathrm{Stab}(X)$ described above. Then there exists a connected component $\mathrm{Stab}^{\dagger}(Y)$ which embeds into $\mathrm{Stab}^{\dagger}(X)$ as a closed submanifold. Moreover, when Y is very

general, $\mathrm{Stab}^{\dagger}(Y) \cong \mathrm{Stab}^{\dagger}(X)$ [MMS09, Theorem 1.2]. $\mathrm{Stab}^{\dagger}(X)$ has non-geometric stability conditions, hence by Theorem 6.1.1 so does $\mathrm{Stab}^{\dagger}(Y)$.
- Beauville-type and bielliptic surfaces: Let $S = X/G$. By Theorem 8.3.1(2) there is a connected component $\mathrm{Stab}^{\ddagger}(S) = \mathrm{Stab}^{\mathrm{Geo}}(S) \cong (\mathrm{Stab}(X))^G$. If $\mathrm{Stab}(S)$ is connected, this would give a negative answer to Question 1.1.4, in contrast to all previous examples.

Non-geometric stability conditions are known to exist in the following cases.

- Rational surfaces: the boundary of the geometric chamber contains points where skyscrapers sheaves are destabilised by exceptional bundles. This is explained for $\mathrm{Tot}(\mathcal{O}_{\mathbf{P}^2}(-3))$ in [BM11, §5], and the arguments generalise to any rational surface.
- Surfaces which contain a smooth rational curve C with negative self intersection: these have a wall of the geometric chamber such that $\mathcal{O}_x$ is stable if $x \notin C$, and strictly semistable if $x \in C$ [TX22, Lemma 7.2], [LR22, Proposition 5.3].

Threefolds: Fix $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. Denote by $\mathrm{Stab}_H(X)$ the space of stability conditions such that the central charge factors via a certain lattice $\Lambda_H \subset \mathrm{K}_{\mathrm{num}}(X)$. If $\rho(X) = 1$, this gives rise to elements of $\mathrm{Stab}(X)$. A strategy for constructing stability conditions in $\mathrm{Stab}_H(X)$ for threefolds was first introduced in [BMT13, §3, §4]. This uses so-called tilt stability conditions to construct geometric stability conditions if a stronger Bogomolov–Gieseker-type inequality is satisfied.

Geometric stability conditions in $\mathrm{Stab}_H(X)$ exist for some threefolds, see [BMS16, Theorem 1.4], [Ber+17, Theorem 1.1], [Piy17, Theorem 1.3], [Kos18, Theorem 1.2], [Li19, Theorem 1.3], [Kos20, Theorem 1.2], [Kos22, Theorem 1.3], [Liu22, Theorem 1.2].

Below we describe the only threefolds where $\mathrm{Stab}(X)$ is known to be non-empty. These are also the only cases where a connected component of $\mathrm{Stab}_H(X)$ was previously known.

- Abelian threefolds: There is a distinguished connected component $\mathrm{Stab}_H^{\dagger}(X)$ of $\mathrm{Stab}_H(X)$. This was completely described in [BMS16, Theorem 1.4]. These have been shown to satisfy the full support property, in particular, they lie in a connected component $\mathrm{Stab}^{\dagger}(X) \subset \mathrm{Stab}(X)$ [OPT22, Theorem 3.21]. Abelian threefolds are also a case of [FLZ22, Theorem 1.1].
- Calabi–Yau threefolds of abelian type: Let Y be a Calabi–Yau threefold admitting an abelian threefold X as a finite étale cover. Then $Y = X/G$, where G is $(\mathbf{Z}/2)^{\oplus 2}$ or D_4 (the dihedral group of order 8) [OS01, Theorem 0.1]. There is a distinguished connected component $\mathfrak{P}$ of $\mathrm{Stab}_H(Y)$ induced from $\mathrm{Stab}_H^{\dagger}(X)$ which contains only geometric stability conditions [BMS16, Corollary 10.3]. By the previous paragraph together with Theorem 8.3.1, $\mathfrak{P}$ lies in a connected component of $\mathrm{Stab}(Y)$, and this connected component consists only of geometric stability conditions.

The only examples where non-geometric stability conditions are known to exist on threefolds are those with complete exceptional collections. We explain this in greater generality below.

Exceptional collections: There are stability conditions on any triangulated category with a complete exceptional collection $\{E_1, \ldots, E_n\}$ called *algebraic stability conditions* [Mac07b, §3]. More precisely, choose $p_i \in \mathbf{Z}$ such that the exceptional collection $\{E_1[p_1], \ldots, E_n[p_n]\}$ is Ext, i.e. for all $i \neq j$, $\mathrm{Hom}^{\leq 0}(E_i[p_i], E_j[p_j]) = 0$. Then the extension closed subcategory, $\mathcal{A}$, generated by $E_i[p_i]$ is the heart of a bounded t-structure on $\mathrm{D^b}(X)$. A stability condition can then be constructed on $\mathcal{A}$ by fixing $Z(E_i[p_i]) = z_i \in \mathbf{H}$. On $\mathbf{P}^n$, this construction has been used to show the existence of geometric stability conditions [Mu21, Proposition 3.5] [Pet22, §3.3]. If X is a variety with a complete exceptional collection, non-geometric stability conditions can be constructed from hearts that do not contain skyscraper sheaves [Mac07a, §4.2].

1.4 Related works

Theorem 5.3.6 was independently obtained in [PPZ23, Lemma 4.11]. Theorem 4.3.1 was used to prove that $\mathrm{Stab}^{\mathrm{Geo}}(X)$ is contractible for any surface X in [Rek23, Theorem A]. Theorem 7.2.11 was obtained independently in [QZ23, Theorem 4.9], where the authors considered fusion-equivariant stability conditions in relation to cluster theory.

Chapter 2

Slope-stability on free quotients

We compute the Le Potier function of free abelian quotients and varieties with finite Albanese morphism. We apply this to Beauville-type surfaces which provides counterexamples to Conjecture 1.2.3. Throughout, X will be a smooth projective variety over $\mathbf{C}$.

2.1 H-stability

Notation 2.1.1. Let $A \cdot B$ denote the intersection product of elements of $\mathrm{Chow}_{\mathrm{num}}(X) \otimes \mathbf{R}$. If $A \cdot B$ is 0-dimensional, we define $A \,.\, B := \deg(A \cdot B)$.

Definition 2.1.2. Let $\dim X = n$. Fix an ample class $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. Given $0 \neq F \in \mathrm{Coh}(X)$, we define the *H-slope* of F as follows:

$$\mu_H(F) := \begin{cases} \frac{H^{n-1}.\mathrm{ch}_1(F)}{H^n \mathrm{ch}_0(F)}, & \text{if } \mathrm{ch}_0(F) > 0; \\ +\infty, & \text{if } \mathrm{ch}_0(F) = 0. \end{cases}$$

We say that F is *H-(semi)stable* if for every non-zero subobject $E \subsetneq F$,

$$\mu_H(E) < (\leq) \mu_H(F).$$

2.2 The Le Potier function

In the study of H-stability, a natural question to ask is whether there are necessary and sufficient conditions on a cohomology class $\gamma \in H^*(X, \mathbf{Q})$ for there to exist a H-semistable sheaf F with $\mathrm{ch}(F) = \gamma$.

The Bogomolov–Gieseker inequality (see [Bog79, §10], or [HL10, Theorem 3.4.1]) gives the following necessary condition for H-semistable sheaves on surfaces:

$$2\mathrm{ch}_0(F)\mathrm{ch}_2(F) \leq \mathrm{ch}_1(F)^2.$$

This generalises to the following statement for any variety X of dimension $n \geq 2$ via the Mumford–Mehta–Ramanathan restriction theorem.

Theorem 2.2.1 ([Lan04, Theorem 3.2], [HL10, Theorem 7.3.1])**.** *Assume* $\dim X = n \geq 2$. *Fix* $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. *If* F *is a torsion-free* H*-semistable sheaf, then*

$$2\mathrm{ch}_0(F)(H^{n-2} \,.\, \mathrm{ch}_2(F)) \leq H^{n-2} \,.\, \mathrm{ch}_1(F)^2.$$

Remark 2.2.2. Let $B \in \mathrm{NS}_{\mathbf{R}}(X)$. The *twisted Chern character* is defined by $\mathrm{ch}^B \coloneqq \mathrm{ch} \cdot e^{-B}$. Then

$$2\mathrm{ch}_0^B(F)(H^{n-2} \,.\, \mathrm{ch}_2^B(F)) - H^{n-2} \,.\, (\mathrm{ch}_1^B(F))^2 = 2\mathrm{ch}_0(F)(H^{n-2} \,.\, \mathrm{ch}_2(F)) - H^{n-2} \,.\, \mathrm{ch}_1(F)^2,$$

hence Theorem 2.2.1 also holds for twisted Chern characters.

Lemma 2.2.3 (Consequence of the Hodge Index Theorem)**.** *Assume* $\dim X = n \geq 2$. *Let* $(H, D) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. *Then* $H^n \left(H^{n-2} \,.\, D^2\right) \leq (H^{n-1} \,.\, D)^2$.

Proof. Let $E \coloneqq D - \frac{H^{n-1} . D}{H^n} H$. Then $H^{n-1} \,.\, E = 0$. Hence, by the Hodge Index Theorem [Luo90, Theorem 1], we have $H^{n-2} \,.\, E^2 \leq 0$. Since H is ample, $H^n > 0$. Therefore,

$$0 \geq H^n \left(H^{n-2} \,.\, E^2\right) = H^n \left(H^{n-2} \,.\, \left(D - \frac{H^{n-1} \,.\, D}{H^n} H\right)^2\right) = H^n \left(H^{n-2} \,.\, D^2\right) - \left(H^{n-1} \,.\, D\right)^2.$$

□

Now assume $\dim X = n \geq 2$ and fix $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Let F be any H-semistable torsion-free sheaf. By the twisted version of Theorem 2.2.1 and the fact that $H^n > 0$, we have

$$2H^n \mathrm{ch}_0(F)(H^{n-2} \,.\, \mathrm{ch}_2^B(F)) \leq H^n(H^{n-2} \,.\, \mathrm{ch}_1^B(F)^2) \leq (H^{n-1} \,.\, \mathrm{ch}_1^B(F))^2,$$

where the final inequality is by Lemma 2.2.3. Since F is torsion-free, $\mathrm{ch}_0(F) > 0$, and hence

$$\frac{H^{n-2} \,.\, \mathrm{ch}_2^B(F)}{H^n \mathrm{ch}_0(F)} \leq \frac{1}{2} \left(\frac{H^{n-1} \,.\, \mathrm{ch}_1^B(F)}{H^n \mathrm{ch}_0(F)}\right)^2.$$

Now we expand the expressions for $\mathrm{ch}_2^B(F)$ and $\mathrm{ch}_1^B(F)$:

$$\begin{aligned} &\frac{H^{n-2} \,.\, \mathrm{ch}_2(F) - H^{n-2} \,.\, B \,.\, \mathrm{ch}_1(F) + \frac{1}{2} H^{n-2} \,.\, B^2 \,.\, \mathrm{ch}_0(F)}{H^n \mathrm{ch}_0(F)} \\ &\leq \frac{1}{2} \left(\frac{H^{n-1} \,.\, \mathrm{ch}_1(F) - H^{n-1} \,.\, B \mathrm{ch}_0(F)}{H^n \mathrm{ch}_0(F)}\right)^2 \\ &= \frac{1}{2} \left(\mu_H(F) - \frac{H^{n-1} \,.\, B}{H^n}\right)^2. \end{aligned}$$

Therefore,

$$\nu_{H,B}(F) \coloneqq \frac{H^{n-2} \,.\, \mathrm{ch}_2(F) - H^{n-2} \,.\, B \,.\, \mathrm{ch}_1(F)}{H^n \mathrm{ch}_0(F)} \leq \frac{1}{2} \left(\mu_H(F) - \frac{H^{n-1} \,.\, B}{H^n}\right)^2 - \frac{1}{2} \frac{H^{n-2} \,.\, B^2}{H^n}. \tag{2.1}$$

For a given $\mu \in \mathbf{R}$, if $\mu_H(F) = \mu$, we can therefore ask how large $\nu_{H,B}(F)$ can be. These leads us to make the following definition.

Definition 2.2.4. Assume $\dim X = n \geq 2$. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We define the *Le Potier function twisted by* B, $\Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$, by

$$\Phi_{X,H,B}(x) \coloneqq \limsup_{\mu \to x} \left\{\nu_{H,B}(F) \colon F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable with } \mu_H(F) = \mu\right\}. \tag{2.2}$$

Remark 2.2.5. (1) If $n = 2$ then for every rational number $\mu \in \mathbf{Q}$ there exists an H-stable sheaf F with $\mu_H(F) = \mu$ [HL10, Theorem 5.2.5]. Since $\nu_{H,B}$ is bounded above by (2.1) and $\Phi_{X,H,B}$ is a limit supremum over values of $\nu_{H,B}$, it follows that the value of $\Phi_{X,H,B}$ at every point is in $\mathbf{R}$.

(2) For $n > 2$, it is an open question whether for a given $\mu \in \mathbf{Q}$ there exists an H-semistable sheaf F with $\mu_H(F) = \mu$. Note that $\sup(\emptyset) = -\infty$. This explains why we define $\Phi_{X,H,B}$ to be valued in $\mathbf{R} \cup \{-\infty\}$.

(3) If $B = 0$, we will write $\Phi_{X,H} \coloneqq \Phi_{X,H,0}$. If $n = 2$, then $\Phi_{X,H}$ is exactly [FLZ22, Definition 3.1].

The following result generalises the case for surfaces in [FLZ22, Proposition 3.2].

Lemma 2.2.6. *Assume* $\dim X = n \geq 2$. *Let* $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. *Then* $\Phi_{X,H,B}$ *satisfies*

$$\Phi_{X,H,B}(x) \leq \frac{1}{2}\left[\left(x - \frac{H^{n-1} \,.\, B}{H^n}\right)^2 - \frac{H^{n-2} \,.\, B^2}{H^n}\right]. \tag{2.3}$$

It is the smallest upper semi-continuous function such that

$$\nu_{H,B}(F) \leq \Phi_{X,H,B}\left(\mu_H(F)\right) \tag{2.4}$$

for every torsion-free H*-semistable sheaf* F.

Proof. The upper bound in (2.3) follows from (2.1). Moreover, $\Phi_{X,H,B}(x)$ is by definition the least upper bound for $\nu_{H,B}(F)$ for all torsion-free H-semistable sheaves F with $\mu_H(F) = x$. Hence the inequality in (2.4) holds.

Fix $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We now unpack Definition 2.2.4 to see why $\Phi_{X,H,B}$ is upper semi-continuous. For any $\delta > 0$ and $x \in \mathbf{R}$, let $\mathcal{B}(x, \delta) = (x - \delta, x + \delta) \subset \mathbf{R}$ and define

$$g_x(\delta) \coloneqq \sup_{\mu \in \mathcal{B}(x,\delta)} \left\{\nu_{H,B}(F) \colon F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable with } \mu_H(F) = \mu\right\}.$$

Then $\Phi_{X,H,B}(x) = \inf_{\delta > 0} g_x(\delta)$.

Now fix $x_0 \in \mathbf{R}$. To prove $\Phi_{X,H,B}$ is upper semi-continuous, we need to show that for every real number $y > \Phi_{X,H,B}(x_0)$ there exists an open neighbourhood U of x_0 such that $y > \Phi_{X,H,B}(x)$ for all $x \in U$. Suppose $y > \Phi_{X,H,B}(x_0)$, then there exists $\delta_0 > 0$ such that $y > g_{x_0}(\delta_0)$. Moreover, for all $x \in \mathcal{B}(x_0, \delta)$, $g_{x_0}(\delta_0) \geq \Phi_{X,H,B}(x)$. Thus $y > \Phi_{X,H,B}(x)$ for all $x \in U \coloneqq \mathcal{B}(x_0, \delta)$. □

Example 2.2.7. Let X be a K3 surface and let $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. One can use [Bri08, Theorem 5.3] to compute that

$$\Phi_{X,H,0}(x) = \begin{cases} \frac{1}{2}x^2 - \frac{1}{H^2}\left(1 - \frac{1}{q_H(x)^2}\right) & \text{if } x \in \mathbf{Q}, \text{ where } q_H(x) = \min\{r \in \mathbf{Z}_{>0} : rH^2x \in \mathbf{Z}\} \\ \frac{1}{2}x^2 - \frac{1}{H^2} & \text{otherwise.} \end{cases}$$

Note that the Le Potier function is indeed upper semicontinuous, but not continuous in this case.

2.3 The Le Potier function for free quotients

Let G be a finite group acting freely on X. There is an étale covering, $\pi \colon X \to X/G =: Y$. Then $\mathrm{Pic}(Y) \cong \mathrm{Pic}_G(X)$, the group of isomorphism classes of G-equivariant line bundles on X [MFK94, Chapter 1.3]. Fix $H_S \in \mathrm{Amp}_{\mathbf{R}}(Y)$. Then $\pi^* H_S \in \mathrm{Amp}_{\mathbf{R}}(X)$ is G-invariant. Beauville-type and bielliptic surfaces provide examples of such quotients.

Example 2.3.1 (Ample classes on Beauville-type surfaces)**.** Let $S = X/G$ be a Beauville-type surface, as introduced in Example 1.2.1. Then $X = C_1 \times C_2$ is a product of curves of genus $g(C_i) > 1$, $q(S) := h^1(S, \mathcal{O}_S) = 0$, $p_g(S) := h^2(S, \mathcal{O}_S) = 0$ so $\chi(\mathcal{O}_S) = 1$, and $K_S^2 = 8$ where K_S is the canonical divisor of S.

Assume that there are actions of G on each curve C_i such that the action of G on $C_1 \times C_2$ is the diagonal action. This is called the *unmixed case* in [BCG08, Theorem 0.1] and excludes 3 families of dimension 0. To classify ample classes on S, we follow similar arguments to [GS13, §2.2]. See also [Sha14]. Consider the projections to each curve, $p_1 \colon X \to C_1$ and $p_2 \colon X \to C_2$. For $i, j \in \mathbf{Z}$ and points $c_i \in C_i$, consider the G-invariant divisor class

$$[\mathcal{O}(i,j)] := p_1^*([\mathcal{O}_{C_1}(ic_i)]) \otimes p_2^*([\mathcal{O}_{C_2}(jc_2)]) \in \mathrm{NS}_G(X).$$

Moreover,

$$\chi_{\mathrm{top}}(S) = \frac{\chi_{\mathrm{top}}(C_1) \cdot \chi_{\mathrm{top}}(C_2)}{|G|} = 4\frac{(1-g(C_1))(1-g(C_2))}{|G|} = 4\chi(\mathcal{O}_S) = 4.$$

Therefore, $\operatorname{rank} \mathrm{NS}(S) = b_2(S) = 2$ and

$$\mathrm{NS}_{\mathbf{Q}}(S) \cong \mathbf{Q} \cdot [\mathcal{O}(1,0)] \oplus \mathbf{Q} \cdot [\mathcal{O}(0,1)].$$

In particular, $\mathrm{Amp}_{\mathbf{R}}(S) \cong \mathbf{R}_{>0} \cdot [\mathcal{O}(1,0)] \oplus \mathbf{R}_{>0} \cdot [\mathcal{O}(0,1)]$.

Lemma 2.3.2 ([HL10, Lemma 3.2.2])**.** *Let $f \colon X \to Y$ be a finite morphism of varieties of dimension $n \geq 2$ and let $E \in \mathrm{Coh}(Y)$. Let $(H_Y, B_Y) \in \mathrm{Amp}_{\mathbf{R}}(Y) \times \mathrm{NS}_{\mathbf{R}}(Y)$. Then E is H_Y-semistable if and only if f^*E is f^*H_Y-semistable. Moreover, if $\mathrm{ch}_0(E) \neq 0$, then $\mu_{H_Y}(E) = \mu_{f^*H_Y}(f^*E)$ and $\nu_{H_Y,B_Y}(E) = \nu_{f^*H_Y,f^*B_Y}(f^*E)$. In particular, $\Phi_{X,f^*H_Y,f^*B_Y} \geq \Phi_{Y,H_Y,B_Y}$.*

Proof. The claim that E is H_Y-semistable if and only if f^*E is f^*H_Y-semistable follows from the same arguments as in the proof of [HL10, Lemma 3.2.2]. If $\mathrm{ch}_0(E) \neq 0$, then

$$\begin{aligned}
\mu_{f^*H_Y}(f^*E) &= \frac{\deg((f^*H_Y)^{n-1} \cdot f^*(\mathrm{ch}_1(E)))}{\deg((f^*H_Y)^n \cdot f^*(\mathrm{ch}_0(E)))} \\
&= \frac{\deg(f^*(H_Y^{n-1} \cdot \mathrm{ch}_1(E)))}{\deg(f^*(H_Y^n \cdot \mathrm{ch}_0(E)))} \quad (f \text{ is flat, so } f^* \text{ is a ring morphism}) \\
&= \frac{\deg(f)\deg(H_Y^{n-1} \cdot \mathrm{ch}_1(E))}{\deg(f)\deg(H_Y^n \cdot \mathrm{ch}_0(E))} \quad \text{(projection formula)} \\
&= \mu_{H_Y}(E)
\end{aligned}$$

By the same arguments, $\nu_{f^*H_Y,f^*B_Y}(f^*E) = \nu_{H_Y,B_Y}(E)$. ☐

Lemma 2.3.3 ([D23, Lemma 4.9])**.** *Suppose a finite group G acts freely on X. Let $\pi \colon X \to Y := X/G$ denote the quotient map. If $F \in \mathrm{Coh}(X)$ is π^*H_Y-semistable, then π_*F is H_Y-semistable. Moreover, if $\mathrm{ch}_0(F) \neq 0$, then $\mu_{H_Y}(\pi_*F) = \mu_{\pi^*H_Y}(F)$ and $\nu_{H_Y,B_Y}(\pi_*F) = \nu_{\pi^*H_Y,\pi^*B_Y}(F)$. In particular, $\Phi_{X,\pi^*H_Y,\pi^*B_Y} \leq \Phi_{Y,H_Y,B_Y}$.*

Proof. Suppose that $F \in \mathrm{Coh}(X)$ is π^*H_Y-semistable. We will explain in Section 6.1 that there is an equivalence to the category of G-equivariant coherent sheaves, $\mathrm{Coh}(Y) \cong \mathrm{Coh}_G(X)$. There are also functors $\mathrm{Forg}_G \colon \mathrm{Coh}_G(X) \to \mathrm{Coh}(X)$ and $\mathrm{Inf}_G \colon \mathrm{Coh}(X) \to \mathrm{Coh}_G(X)$ such that, under this equivalence,

$$\pi^*(\pi_*(F)) \cong \mathrm{Forg}_G \circ \mathrm{Inf}_G(F) = \bigoplus_{g \in G} g^*F.$$

Since $\pi^* H_Y$ is G-invariant, it follows that $g^* F$ is $\pi^* H_Y$-semistable for every $g \in G$. In particular, $\bigoplus_{g \in G} g^* F$ is $\pi^* H_Y$-semistable. By Lemma 2.3.2, $\pi_* F$ is H_Y-semistable.

Now suppose $\mathrm{ch}_0(F) \neq 0$. Since the Chern character is additive, $\mu_{\pi^* H_Y}(\pi^* \pi_* F) = \mu_{\pi^* H_Y}(F)$. By Lemma 2.3.2, $\mu_{H_Y}(\pi_* F) = \mu_{\pi^* H_Y}(\pi^* \pi_* F) = \mu_{\pi^* H_Y}(F)$, as required.

By the same arguments, $\nu_{H_Y, B_Y}(\pi_* F) = \nu_{\pi^* H_Y, \pi^* B_Y}(F)$. □

Proposition 2.3.4 ([D23, Proposition 4.10]). *Let X be a variety, and let G be a finite group acting freely on X. Let $\pi\colon X \to X/G =: Y$ denote the quotient. Let $(H_Y, B_Y) \in \mathrm{Amp}_{\mathbf{R}}(Y) \times \mathrm{NS}_{\mathbf{R}}(Y)$. Then $\Phi_{Y, H_Y, B_Y} = \Phi_{X, \pi^* H_Y, \pi^* B_Y}$.*

Proof. This follows from Lemma 2.3.2 and Lemma 2.3.3. □

2.4 The Le Potier function for varieties with finite Albanese morphism

The Le Potier function for surfaces with finite Albanese morphism was known previously [LR21, Example 2.12(2)]. Below, we give a different proof which works for $\Phi_{X,H,B}$ in any dimension. We first need the following definition.

Definition 2.4.1 ([Muk78, Definitions 4.4, 5.2]). A vector bundle E on an abelian variety A is *homogeneous* if it is invariant under translations, i.e. for every $x \in A$, $T_x^*(E) \cong E$, where T_x is translation on A by x. E is called *semi-homogeneous* if for every $x \in A$, there exists a line bundle L on A such that $T_x^*(E) \cong E \otimes L$.

See [Muk78, Proposition 5.1] for some equivalent characterisations for when a vector bundle is semi-homogeneous. We will need the following properties of homogeneous and semi-homogeneous vector bundles.

Theorem 2.4.2 ([Muk78, Theorem 4.17, Lemma 6.11]). *Let E be a vector bundle with $ch_0(E) = r$ on an abelian variety A.*

(1) E is homogeneous if and only if $E \cong \oplus_{i=1}^k (P_i \otimes U_i)$, where each P_i is a numerically trivial line bundle, i.e. $c_1(P_i) = 0$, and each U_i is a unipotent line bundle, i.e. an iterated self-extension of $\mathcal{O}_A$.

(2) Suppose E is semihomogeneous and consider the multiplication by r map, $r_A\colon A \to A$. Then $r_A^* E \cong \det(E)^{\otimes r} \otimes V$, where V is a homogeneous vector bundle with $\mathrm{ch}_0(V) = \mathrm{ch}_0(r_A^* E)$ and $c_1(V) = 0$.

There are many H-semistable semi-homogeneous vector bundles on any abelian variety.

Proposition 2.4.3 ([Muk78, Theorem 7.11]). *Let A be an abelian variety and fix $H \in \mathrm{Amp}_{\mathbf{R}}(A)$. For every divisor class $C \in \mathrm{NS}_{\mathbf{Q}}(A)$ there exists a H-semistable semi-homogeneous vector bundle E_C on A with $C = \frac{\mathrm{ch}_1(E_C)}{\mathrm{ch}_0(E_C)}$ and $\mathrm{ch}(E_C) = \mathrm{ch}_0(E_C) \cdot e^C$.*

Proof. These vector bundles are constructed as follows: for any $C \in \mathrm{NS}_{\mathbf{Q}}(A)$, write $C = \frac{[L]}{l}$, where $[L]$ is the equivalence class of $L \in \mathrm{NS}(A)$ and $l \in \mathbf{Z}_{>0}$. Let $l_A\colon A \to A$ denote the multiplication by l map, and define $F = (l_A)_*((L)^{\otimes l})$. By [Muk78, Proposition 6.22], F is a semi-homogeneous vector bundle with $C = \delta(F) := \frac{\det(F)}{\mathrm{ch}_0(F)}$. Moreover, F has a filtration by semi-homogeneous vector bundles $E_1, \ldots, E_m$. By [Muk78, Proposition 6.15], each E_i is μ_H-semistable for any $H \in \mathrm{Amp}_{\mathbf{R}}(A)$ and satisfies $C = \delta(E_i)$.

Let $E_C := E_1$ and let $r = \mathrm{ch}_0(E_C)$. We claim that $\mathrm{ch}(E_C) = re^C$. Consider the multiplication by r map $r_A\colon X \to X$. By Theorem 2.4.2(2), $r_A^* E_C \cong \det(E_C)^{\otimes r} \otimes V$, where V is a homogeneous vector bundle, and

$$\mathrm{ch}(r_A^* E_C) = \mathrm{ch}(V) \cdot \mathrm{ch}\left(\det(E_C)^{\otimes r}\right) = \mathrm{ch}_0(r_A^* E_C) e^{r \det(E_C)} = \mathrm{ch}_0(r_A^* E_C) e^{r^2 C}. \tag{2.5}$$

Now recall that $H^{2i}(A, \mathbf{C}) = \bigwedge^{2i} H^1(A, \mathbf{C})$. On $H^1(A, \mathbf{C})$, r_A^* is multiplication by r. It follows that on $\mathrm{Chow}_{\mathrm{num}}{}^i(X)$, r_A^* is multiplication by r^{2i}. In particular, $\delta(r_A^* E_C) = r^2\delta(E)$. Hence (2.5) becomes

$$\mathrm{ch}(r_A^* E_C) = \mathrm{ch}_0(r_A^* E_C) e^{\delta(r_A^* E_C)} = r_A^* \left(\mathrm{ch}_0(E) e^{\delta(E)}\right) = r_A^* \left(re^C\right).$$

Since r_A is flat, the claim follows. Hence $\frac{\mathrm{ch}_1(E_C)}{\mathrm{ch}_0(E_C)} = C$. ☐

Proposition 2.4.4. *Let A be an abelian variety of dimension $n \geq 2$. Fix $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(A) \times \mathrm{NS}_R(A)$. Then*

$$\Phi_{A,H,B}(x) = \frac{1}{2}\left[\left(x - \frac{H^{n-1} \,.\, B}{H^n}\right)^2 - \frac{H^{n-2} \,.\, B^2}{H^2}\right].$$

Proof. For any $k \in \mathbf{Q}$ define $C_k := kH + B$. Then by Proposition 2.4.3, there exists a μ_H-semistable vector bundle E_{C_k} with $C_k = \frac{\mathrm{ch}_1(E_{C_k})}{\mathrm{ch}_0(E_{C_k})}$ and $\mathrm{ch}(E_{C_k}) = \mathrm{ch}_0(E_{C_k}) \cdot e^{C_k}$. Let $r = \mathrm{ch}_0(E_{C_k})$. Hence

$$\mu_H(E_{C_k}) = \frac{H^{n-1} \,.\, rC_k}{H^n r} = k + \frac{H^{n-1} \,.\, B}{H^n},$$

and

$$\begin{aligned}
\nu_{H,B}(E_{C_k}) &= \frac{H^{n-2} \,.\, \frac{1}{2} r C_k^2 - H^{n-2} \,.\, B \,.\, rC_k}{H^n r} \\
&= \frac{1}{2}\frac{H^{n-2} \,.\, \left(k^2 H^2 + 2kH \,.\, B + B^2\right) - H^{n-2} \,.\, (2kH \,.\, B + 2B^2)}{H^n} \\
&= \frac{1}{2}\left[k^2 - \frac{H^{n-2} \,.\, B^2}{H^n}\right] \\
&= \frac{1}{2}\left[\left(\mu_H(E_{C_k}) - \frac{H^{n-1} \,.\, B}{H^n}\right)^2 - \frac{H^{n-2} \,.\, B^2}{H^2}\right].
\end{aligned}$$

This gives a lower bound for $\Phi_{A,H,B}\left(\mu_H(E_{C_k})\right)$, which is the same as the upper bound in Lemma 2.2.6. Now note that for any $x \in \mathbf{Q}$, we can choose k so that $\mu_H(E_{C_k}) = x$. Hence $\Phi_{A,H,B}(x)$ attains its upper bound for all $x \in \mathbf{Q}$. Finally, by definition of the Le Potier function, it must attain this upper bound for all $x \in \mathbf{R}$. ☐

We next use this to compute the Le Potier function for varieties with finite Albanese morphism.

Proposition 2.4.5 ([LR21, Example 2.12(2)], [D23, Proposition 4.13])**.** *Let X be a variety with finite Albanese morphism $a\colon X \to \mathrm{Alb}(X)$ and $n := \dim X \geq 2$. Let $H_X \in \mathrm{Amp}_{\mathbf{R}}(X)$. Then $a^* E_C$ is H_X-semistable for every $C \in \mathrm{NS}_{\mathbf{Q}}(\mathrm{Alb}(X))$.*

Proof. Fix $C \in \mathrm{NS}_{\mathbf{Q}}(\mathrm{Alb}(X))$ and $H_A \in \mathrm{Amp}_{\mathbf{R}}(\mathrm{Alb}(X))$. Let E_C be the corresponding H_A-semistable semi-homogeneous vector bundle on $\mathrm{Alb}(X)$ from Proposition 2.4.3. Let $r := \mathrm{ch}_0(E_C)$ and consider the multiplication by r map $r_{\mathrm{Alb}(X)}\colon \mathrm{Alb}(X) \to \mathrm{Alb}(X)$. By Theorem 2.4.2,

$$r_{\mathrm{Alb}(X)}^*(E_C) = L^{-1} \bigoplus_{i=1}^{k} P_i \otimes U_i,$$

where L is a line bundle, and for all i, P_i is a numerically trivial line bundle and U_i is an iterated self-extension of $\mathcal{O}_{\mathrm{Alb}(X)}$. Therefore, $L \otimes r_{\mathrm{Alb}(X)}^*(E_C)$ is an iterated extension of numerically trivial line bundles.

Now consider the fibre square:

$$\begin{tikzcd}
\mathcal{Z} \coloneqq X \times_{\mathrm{Alb}(X)} \mathrm{Alb}(X) \arrow[r, "p_A"] \arrow[d, "p_X"] & \mathrm{Alb}(X) \arrow[d, "r_A"] \\
X \arrow[r, "a"] & \mathrm{Alb}(X)
\end{tikzcd}$$

Without loss of generality, fix a connected component Z of $\mathcal{Z}$. Then on Z,

$$(p_X|_Z)^*a^*(E_C) = (p_A|_Z)^*r_A^*(E_C).$$

The property of being an extension of numerically trivial line bundles is preserved by taking pullback. Hence $p_A^*(L) \otimes (p_X|_Z)^*a^*(E_C)$ is an iterated extension of numerically trivial line bundles. Recall that line bundles are stable with respect to any ample class. Thus $p_A^*(L) \otimes (p_X|_Z)^*a^*(E_C)$ is $(p_X|_Z)^*H_X$-semistable, hence so is $(p_X|_Z)^*a^*(E_C)$. Finally, by Lemma 2.3.2, $a^*(E_C)$ is H_X-semistable. $\square$

Proposition 2.4.6 ([LR21, Example 2.12(2)], [D23, Proposition 4.14]). *Let X be a variety with finite Albanese morphism $a\colon X \to \mathrm{Alb}(X)$. Fix $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(\mathrm{Alb}(X)) \times \mathrm{NS}_{\mathbf{R}}(\mathrm{Alb}(X))$ and assume $n \coloneqq \dim X \geq 2$. Then*

$$\Phi_{\mathrm{Alb}(X),H,B}(x) = \Phi_{X,a^*H,a^*B}(x) = \frac{1}{2}\left[\left(x - \frac{(a^*H)^{n-1} \,.\, a^*B}{(a^*H)^n}\right)^2 - \frac{(a^*H)^{n-2} \,.\, (a^*B)^2}{(a^*H)^n}\right].$$

Proof. First note that, by the projection formula, the upper bounds of $\Phi_{\mathrm{Alb}(X),H,B}$ and Φ_{X,a^*H,a^*B} are the same. By Lemma 2.3.2, $\Phi_{\mathrm{Alb}(X),H,B} \leq \Phi_{X,a^*H,a^*B}$. Hence it suffices to show that $\Phi_{\mathrm{Alb}(X),H,B}$ attains this upper bound. This follows from Proposition 2.4.4. $\square$

We now combine this with Proposition 2.3.4.

Theorem 2.4.7 ([D23, Corollary 4.15]). *Let X be a variety with finite Albanese morphism $a\colon X \to \mathrm{Alb}(X)$, and let G be a finite group acting freely on X. Let $\pi\colon X \to X/G =: Y$ denote the quotient map. Suppose we have:*
- *$H_X = a^*H = \pi^*H_Y$: a class in $\mathrm{Amp}_{\mathbf{R}}(X)$ pulled back from $\mathrm{Alb}(X)$ and Y, and*
- *$B_X = a^*B = \pi^*B_Y$: a class in $\mathrm{NS}_{\mathbf{R}}(X)$ pulled back from $\mathrm{Alb}(X)$ and Y.*

Then
$$\Phi_{Y,H_Y,B_Y}(x) = \frac{1}{2}\left[\left(x - \frac{H_Y^{n-1} \,.\, B_Y}{H_Y^n}\right)^2 - \frac{H_Y^{n-2} \,.\, B_Y^2}{H_Y^n}\right].$$

Proof. By Proposition 2.3.4 and Proposition 2.4.6, it follows that:

$$\Phi_{Y,H_Y,B_Y}(x) = \Phi_{X,\pi^*H_Y,\pi^*B_Y}(x) = \frac{1}{2}\left[\left(x - \frac{(\pi^*H_Y)^{n-1} \,.\, \pi^*B_Y}{(\pi^*H_Y)^n}\right)^2 - \frac{(\pi^*H_Y)^{n-2} \,.\, (\pi^*B_Y)^2}{(\pi^*H_Y)^n}\right].$$

The result follows again by the projection formula. $\square$

Example 2.4.8. Suppose X has finite Albanese morphism $a\colon X \to \mathrm{Alb}(X)$, and let G be a finite group acting freely on X. Then for each $g \in G$, $\mathrm{alb}_X \circ g : X \to \mathrm{Alb}(X)$. By the universal property of alb_X, there is a morphism $g\colon \operatorname{Alb}(X) \to \operatorname{Alb}(X)$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\mathrm{alb}_X"] & X \arrow[d, "\mathrm{alb}_X"] \\ \operatorname{Alb}(X) \arrow[r, "g"] & \operatorname{Alb}(X) \end{tikzcd}.$$

In particular, this induces an action of G on $\operatorname{NS}(\operatorname{Alb}(X))$. Fix $L \in \operatorname{Amp}(\operatorname{Alb}(X))$ and $B = 0$. Then $H := \otimes_{g\in G} g^* L \in \operatorname{Amp}_{\mathbf{R}}(X)$ satisfies the hypotheses of Theorem 2.4.7. In particular, this applies to bielliptic surfaces ($q = 1$) and Beauville-type surfaces ($q = 0$). The latter provides a counterexample to Conjecture 1.2.3, since $\Phi_{Y,H_Y,0}(x) = \frac{x^2}{2}$ is continuous.

Chapter 3

Bridgeland stability conditions

In this chapter we introduce Bridgeland stability conditions together with some of their properties which will be used later in this thesis. We assume throughout that $\mathcal{D}$ is a k-linear essentially small Ext-finite triangulated category with a Serre functor.

3.1 Stability conditions via slicings

Definition 3.1.1 ([Bri07, Definition 3.3]). A *slicing* $\mathcal{P}$ on $\mathcal{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbf{R}$ such that

1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$;
2. if $F_1 \in \mathcal{P}(\phi_1), F_2 \in \mathcal{P}(\phi_2)$, then $\phi_1 > \phi_2 \implies \mathrm{Hom}_{\mathcal{D}}(F_1, F_2) = 0$;
3. every $E \in \mathcal{D}$ has a Harder–Narasimhan (HN) filtration, i.e. there exist objects $E_1, \dots E_m \in \mathcal{D}$, real numbers $\phi_1 > \phi_2 > \dots > \phi_m$, and a collection of distinguished triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_m \arrow[ul, dotted] &
\end{tikzcd}$$

where $A_i \in \mathcal{P}(\phi_i)$ for $1 \leq i \leq m$. We call the A_i the *HN factors* of E.

Notation 3.1.2. (1) If $0 \neq E \in \mathcal{P}(\phi)$, we call $\phi(E) = \phi$ the *phase* of E.

(2) Given an interval $I \subset \mathbf{R}$, we denote by $\mathcal{P}(I)$ the smallest additive subcategory of $\mathcal{D}$ containing all objects E whose HN factors all have phases lying in I, i.e. $\phi_i \in I$.

The next piece of data needed to define a stability condition will be a way to assign a complex number to every object of $\mathcal{D}$ which is additive on exact triangles. To ensure this we will work with the Grothendieck group.

Definition 3.1.3. Let $\mathcal{A}$ be an abelian category. Its *Grothendieck group* $\mathrm{K}(\mathcal{A})$ is the abelian group freely generated by isomorphism classes $[A]$ of objects $A \in \mathcal{A}$ modulo the relation

$$[B] = [A] + [C] \text{ if } 0 \to A \to B \to C \to 0 \text{ is exact.}$$

Similarly, for $\mathcal{D}$ a triangulated category, its *Grothendieck group* $\mathrm{K}(\mathcal{D})$ is the abelian group freely generated by isomorphism classes $[X]$ of objects $X \in \mathcal{D}$ modulo the relation

$$[Y] = [X] + [Z] \text{ if } X \to Y \to Z \to X[1] \text{ is a distinguished triangle.}$$

The *numerical Grothendieck group*, $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$, is the quotient of $\mathrm{K}(\mathcal{D})$ by the null-space of the Euler form $\chi(E,F) = \sum_i (-1)^i \dim(\mathrm{Hom}(E, F[i]))$. The Euler form is finite since we assumed $\mathcal{D}$ is Ext-finite. Moreover, since we assumed $\mathcal{D}$ has a Serre-functor, the left and right null spaces of $\chi(-,-)$ coincide.

Definition 3.1.4 ([Bri07, Definition 5.1]). A *Bridgeland pre-stability condition* on $\mathcal{D}$ is a pair $\sigma = (\mathcal{P}, Z)$ such that

(1) $\mathcal{P}$ is a slicing, and
(2) $Z\colon \mathrm{K}(\mathcal{D}) \to \mathbf{C}$ is a group homomorphism such that, if $0 \neq E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbf{R}$, then $Z([E]) = m(E)e^{i\pi\phi}$, where $m(E) \in \mathbf{R}_{>0}$.

We call Z the *central charge*.

Remark 3.1.5. (1) To ease notation, we write $Z(E) := Z([E])$.

(2) The HN filtration in Definition 3.1.1(3) is unique up to isomorphism. We will denote by $\phi^+_\sigma(E) := \phi_1$, $\phi^-_\sigma(E) := \phi_m$, and $m_\sigma(E) := \sum_i |Z(A_i)|$.
(3) Each $\mathcal{P}(\phi)$ is an abelian category [Bri07, Lemma 5.2]. Non-zero objects of $\mathcal{P}(\phi)$ are called σ-*semistable* of phase ϕ, and non-zero simple objects of $\mathcal{P}(\phi)$ are called σ-*stable* of phase ϕ.

Definition 3.1.6. Let Λ be a finite rank lattice with a surjective group homomorphism $\lambda\colon \mathrm{K}(\mathcal{D}) \twoheadrightarrow \Lambda$.

(1) A Bridgeland pre-stability condition $\sigma = (\mathcal{P}, Z)$ on $\mathcal{D}$ satisfies the *support property with respect to* (Λ, λ) if
    (a) Z factors via Λ, i.e. $Z\colon \mathrm{K}(\mathcal{D}) \overset{\lambda}{\twoheadrightarrow} \Lambda \to \mathbf{C}$, and
    (b) there exists a quadratic form Q on $\Lambda \otimes \mathbf{R}$ such that
        (i) $\operatorname{Ker} Z \otimes \mathbf{R}$ is negative definite with respect to Q, and
        (ii) every σ-semistable object $E \in \mathcal{D}$ satisfies $Q(\lambda(E)) \geq 0$.
(2) A Bridgeland pre-stability condition σ on $\mathcal{D}$ that satisfies the support property with respect to (Λ, λ) is called a *Bridgeland stability condition* (with respect to (Λ, λ)). If λ also factors via $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$, we call σ a *numerical Bridgeland stability condition.*

Remark 3.1.7. The support property was first introduced in [KS08, Section 2.1], and reformulated in [BMS16, Definition A.3].

Remark 3.1.8. Suppose $\sigma = (\mathcal{P}, Z)$ is a stability condition on $\mathcal{D}$. Let $E \in \mathcal{D}$ and let A_i be its HN factors from Definition 3.1.1(3). The support property implies that the abelian subcategories $\mathcal{P}(\phi)$ are finite length (i.e. noetherian and artinian) [MS17, Exercise 5.9]. Hence every HN factor $A_i \in \mathcal{P}(\phi_i)$ admits finite Jordan–Hölder filtrations,

$$0 = A_{i,0} \subsetneq A_{i,1} \subsetneq \cdots A_{i,m_i-1} \subsetneq A_{i,m_i} = A_i,$$

such that the quotients $B_{i,j} = A_{i,j+1}/A_{i,j}$ are simple in $\mathcal{P}(\phi_i)$. Such filtrations may not be unique, however the factors $B_{i,j}$ are unique up to reordering – we call these the *JH factors* of E (with respect to σ).

Example 3.1.9. Let X be a curve. The slope of a non-zero sheaf $E \in \mathrm{Coh}(X)$ is defined by $\mu(E) = +\infty$ if $\operatorname{rank}(E) = 0$ and $\mu(E) = \deg(E)/\operatorname{rank}(E)$ otherwise. Then E is called μ-semistable if any non-zero non-trivial subsheaf F satisfies $\mu(F) \leq \mu(E)$.

For $[E] \in \mathrm{K}_{\mathrm{num}}(X)$, define $Z(E) = -\deg([E]) + i \operatorname{rank}([E]) \in \mathbf{C}$. For $\phi \in (0,1]$, also define

$$\mathcal{P}(\phi) := \{0\} \cup \left\{ 0 \neq E \in \mathrm{Coh}(X) : E \text{ is } \mu\text{-semistable with } \frac{\arg(Z(E))}{\pi} = \phi \right\}.$$

Then for any $\phi \in \mathbf{R}$, we can uniquely write $\phi = n_\phi + \psi_\phi$ with $n_\phi \in \mathbf{Z}$ and $\psi_\phi \in (0,1]$. Hence define $\mathcal{P}(\phi) = \mathcal{P}(\psi_\phi)[n_\phi]$. One can use the Harder–Narasimhan filtrations for μ-stability to show that $\mathcal{P}$ defines a slicing. Therefore, $\sigma = (\mathcal{P}, Z)$ is a Bridgeland pre-stability condition on $\mathrm{D}^{\mathrm{b}}(X)$.

Now let $\Lambda = H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \cong \mathbf{Z}^2$, and consider the isomorphism

$$\lambda = (\mathrm{rank}(-), \deg(-)) : \mathrm{K}_{\mathrm{num}}(X) \to \Lambda.$$

Note that there are no non-zero objects in $\mathrm{K}_{\mathrm{num}}(X)$ with $Z(E) = 0$, hence we can take $Q = 0$ for Definition 3.1.6(1)(b). It follows that $\sigma = (\mathcal{P}, Z)$ is a numerical Bridgeland stability condition on $\mathrm{D}^{\mathrm{b}}(X)$ with respect to (Λ, λ).

3.2 The stability manifold

The set of stability conditions with respect to (Λ, λ) will be denoted $\mathrm{Stab}_\Lambda(\mathcal{D})$. Unless stated otherwise, we will assume that all Bridgeland stability conditions are numerical. The set of numerical stability conditions on $\mathcal{D}$ will be denoted by $\mathrm{Stab}(\mathcal{D})$.

As described in [Bri07, Proposition 8.1], $\mathrm{Stab}_\Lambda(\mathcal{D})$ has a natural topology induced by the generalised metric

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ \left|\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)\right|, \left|\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)\right|, \left|\log \frac{m_{\sigma_2}(E)}{m_{\sigma_1}(E)}\right| \right\}.$$

Bridgeland's main result tells us that the set of all stability conditions is a complex manifold, hence it is often called the stability manifold.

Theorem 3.2.1 ([Bri07, Theorem 1.2])**.** *The space of stability conditions* $\mathrm{Stab}_\Lambda(\mathcal{D})$ *has the natural structure of a complex manifold of dimension* $\mathrm{rank}(\Lambda)$*. The forgetful map* $\mathcal{Z}$ *defines the local homeomorphism*

$$\begin{aligned} \mathcal{Z}\colon \mathrm{Stab}_\Lambda(\mathcal{D}) &\to \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{C}) \\ \sigma = (\mathcal{P}, Z) &\longmapsto Z. \end{aligned}$$

In other words, the central charge gives a local system of coordinates for the stability manifold.

Remark 3.2.2. The above theorem was originally stated for locally-finite stability conditions: Suppose $\sigma = (\mathcal{P}, Z)$ is a pre-stability condition and there exists $\varepsilon > 0$ such that $\mathcal{P}(\phi - \varepsilon, \phi + \varepsilon)$ is a quasi-abelian category of finite length for all $\phi \in \mathbf{R}$, then σ is called *locally-finite*, see [Bri07, Definition 5.7]. Definition 3.1.6, which instead assumes the support property, is now the standard one. The support property implies locally-finiteness, see [BMS16, Appendix A] for details. Denote by $\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D})$ the space of all locally-finite stability conditions on $\mathcal{D}$.

Remark 3.2.3 ([MS17, Remark 5.14])**.** The stability manifold comes with two commuting continuous actions.

(1) There is a right action by the universal cover $\widetilde{\mathrm{GL}}^+_2(\mathbf{R})$ of $\mathrm{GL}^+_2(\mathbf{R})$, the 2×2 matrices with real entries and positive determinant. Consider the presentation

$$\widetilde{\mathrm{GL}}^+_2(\mathbf{R}) = \left\{ (T, f) : \begin{array}{l} f\colon \mathbf{R} \to \mathbf{R} \text{ increasing}, f(\phi+1) = f(\phi) + 1 \\ T \in \mathrm{GL}^+_2(\mathbf{R}) \\ f|_{\mathbf{R}/2\mathbf{Z}} = T|_{\mathbf{R}^2 \setminus \{0\}/\mathbf{R}_{>0}} \end{array} \right\}.$$

Each $(T, F) \in \widetilde{\mathrm{GL}}^+_2(\mathbf{R})$ acts on $\mathrm{Stab}_\Lambda(\mathcal{D})$ via

$$(T, f) \cdot (\mathcal{P}, Z) = (\mathcal{P}', Z'), \quad \mathcal{P}'(\phi) \coloneqq \mathcal{P}(f(\phi)), \quad Z'(E) \coloneqq T^{-1}(Z(E)).$$

If we consider $\mathbf{C}^*$ as a subgroup of $\mathrm{GL}_2^+(\mathbf{R})$, there is an induced action of the group $\widetilde{\mathbf{C}^*} = \mathbf{C}$ (under addition) on $\mathrm{Stab}_\Lambda(\mathcal{D})$. Each $a + ib \in \mathbf{C}$ acts via

$$(a+ib)\cdot(\mathcal{P}, Z) = (\mathcal{P}', Z'), \quad \mathcal{P}'(\phi) \coloneqq \mathcal{P}(\phi + a), \quad Z'(E) \coloneqq e^{-(a+ib)i\pi} Z(E).$$

(2) There is a left action by the group of exact autoequivalences $\mathrm{Aut}(\mathcal{D})$ via

$$\Phi \cdot (\mathcal{P}, Z) = (\mathcal{P}', Z'), \quad \mathcal{P}'(\phi) \coloneqq \Phi(\mathcal{P}(\phi)), \quad Z'(E) \coloneqq Z(\Phi_*^{-1}(E)),$$

where $\Phi \in \mathrm{Aut}(\mathcal{D})$ and Φ_* is the natural isomorphism induced by Φ on $\mathrm{K}(\mathcal{D})$.

3.3 Stability conditions via hearts

There is an equivalent characterisation of Bridgeland stability conditions, which uses the notion of a t-structure on a triangulated category. The theory of t-structures was first introduced in [Bei+82, §1.3]. We first need the following definitions.

Definition 3.3.1. A *heart of a bounded t-structure* in $\mathcal{D}$ is a full additive subcategory $\mathcal{A}$ such that

(1) if $k_1 > k_2$ then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$;

(2) for any object E in $\mathcal{D}$ there are integers $k_1 > k_2 > \cdots > k_n$, and a sequence of exact triangles

$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dotted] & & A_2 \arrow[ul, dotted] & & & & A_n \arrow[ul, dotted] &
\end{tikzcd}$$

such that $A_i \in \mathcal{A}[k_i]$ for $1 \leq i \leq n$.

Remark 3.3.2. Given a slicing $\mathcal{P}$ on $\mathcal{D}$, $\mathcal{P}(\phi, \phi+1]$ is the heart of a bounded t-structure for any $\phi \in \mathbf{R}$ [Bri07, §3]. We call $\mathcal{P}(0,1]$ the *standard heart* of the slicing $\mathcal{P}$.

Definition 3.3.3 ([Bri07, Definitions 2.1, 2.2])**.** Let $\mathcal{A}$ be an abelian category or a triangulated category $\mathcal{D}$. A *stability function* for $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbf{C}$ such that for every non-zero object E of $\mathcal{A}$,

$$Z([E]) \in \mathbf{H} \coloneqq \{m \cdot e^{i\pi\phi} \mid m \in \mathbf{R}_{>0}, \phi \in (0,1]\} \subset \mathbf{C}.$$

For every non-zero object E, we define the *phase* by $\phi(E) = \frac{1}{\pi}\arg(Z([E])) \in (0,1]$. We say an object E is *Z-stable* (resp. *Z-semistable*) if $E \neq 0$ and for every proper non-zero subobject A we have $\phi(A) < \phi(E)$ (resp. $\phi(A) \leq \phi(E)$).

Definition 3.3.4 ([Bri07, Definition 2.3])**.** Let $\mathcal{A}$ be an abelian category and let $Z\colon K(\mathcal{A}) \to \mathbf{C}$ be a stability function on $\mathcal{A}$. A *Harder–Narasimhan (HN) filtration* of a non-zero object E of $\mathcal{A}$ is a finite chain of subobjects in $\mathcal{A}$,

$$0 = E_0 \subset E_1 \subset \cdots E_{n-1} \subset E_n = E,$$

such that each factor $F_i = E_i/E_{i-1}$ (called a *Harder–Narasimhan factor*) is a Z-semistable object of $\mathcal{A}$, and $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$. Moreover, we say that Z has the *Harder–Narasimhan property* if every non-zero object of $\mathcal{A}$ has a Harder–Narasimhan filtration.

Remark 3.3.5. The HN filtration is again unique if it exists.

Proposition 3.3.6 ([Bri07, Proposition 5.3])**.** *To give a Bridgeland pre-stability condition $(\mathcal{P}, Z)$ on $\mathcal{D}$ is equivalent to giving a pair $(Z_{\mathcal{A}}, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ and $Z_{\mathcal{A}}$ is a stability function for $\mathcal{A}$ which has the Harder–Narasimhan property.*

Moreover, $(\mathcal{P}, Z)$ is a numerical Bridgeland stability condition if and only if $Z_{\mathcal{A}}$ factors via $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$ and satisfies the support property (Definition 3.1.6(1)) for $Z_{\mathcal{A}}$-semistable objects.

Remark 3.3.7. Given $(Z_{\mathcal{A}}, \mathcal{A})$ as in the statement of Proposition 3.3.6, the corresponding slicing $\mathcal{P}$ gives an object $E \in \mathcal{D}$ a HN filtration by combining the filtration from $\mathcal{A}$ in Definition 3.3.1(2) with the HN filtration in Definition 3.3.4. In particular, the HN filtration from the slicing is a refinement of the filtration in Definition 3.3.1(2).

To construct stability conditions in Chapter 4, we will also need the following definition.

Definition 3.3.8 ([HRS96, Chapter I.2])**.** Let $\mathcal{A}$ be an abelian category. A *torsion pair* in $\mathcal{A}$ is a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ such that
(1) for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$, $\mathrm{Hom}(T, F) = 0$, and
(2) for any $E \in \mathcal{A}$ there are $T \in \mathcal{T}$, $F \in \mathcal{F}$, and an exact sequence

$$0 \to T \to E \to F \to 0.$$

Proposition 3.3.9 ([HRS96, Proposition 2.1])**.** *Suppose $(\mathcal{T}, \mathcal{F})$ is a torsion pair in an abelian category $\mathcal{A}$. Then*

$$\mathcal{A}^\sharp := \left\{E \in \mathrm{D}^b(\mathcal{A}) \mid \mathcal{H}^0_{\mathcal{A}}(E) \in \mathcal{T},\ \mathcal{H}^{-1}_{A}(E) \in \mathcal{F},\ \mathcal{H}^i_{\mathcal{A}}(E) = 0 \text{ for all } i \neq 0, -1\right\}$$

*is the heart of a bounded t-structure on $\mathrm{D}^b(\mathcal{A})$. We call $\mathcal{A}^\sharp$ the **tilt** of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$.*

3.4 The deformation property

In Section 4.3 and Section 7.2.2, we will need the following refinement of Theorem 3.2.1.

Proposition 3.4.1 ([BMS16, Proposition A.5], [Bay19, Theorem 1.2])**.** *Assume $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_\Lambda(\mathcal{D})$ satisfies the support property with respect to (Λ, λ) and a quadratic form Q on $\Lambda \otimes \mathbf{R}$. Consider the open subset of $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{C})$ consisting of central charges whose kernel is negative definite with respect to Q, and let U be the connected component containing Z. Let $\mathcal{Z}$ denote the local homeomorphism from Theorem 3.2.1, and let $\mathcal{U} \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ be the connected component of the preimage $\mathcal{Z}^{-1}(U)$ containing σ. Then*
(1) the restriction $\mathcal{Z}|_{\mathcal{U}} \colon \mathcal{U} \to U$ is a covering map, and
(2) any stability condition $\sigma' \in \mathcal{U}$ satisfies the support property with respect to Q.

Theorem 3.4.2 ([D23, Corollary 5.2])**.** *Assume $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_\Lambda(\mathcal{D})$ satisfies the support property with respect to (Λ, λ) and a quadratic form Q on $\Lambda \otimes \mathbf{R}$. Let $U \subset \mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{C})$, and $\mathcal{U} \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ be the connected components from Proposition 3.4.1. Suppose there is a path Z_t in U parametrised by $t \in [0,1]$ such that $\mathrm{Im}\, Z_t$ is constant and $Z_{t_0} = Z$ for some $t_0 \in [0,1]$. Then this lifts to a unique path in $\mathcal{U}$ given by $\sigma_t = (\mathcal{Q}_t, Z_t)$ for $t \in [0,1]$ such that $\sigma_{t_0} = \sigma$, $\mathcal{Q}_t(0,1] = \mathcal{P}(0,1]$, and σ_t satisfies the support property with respect to Q.*

Proof. Let $\mathcal{Z}$ denote the local homeomorphism from Theorem 3.2.1. By Proposition 3.4.1(1), the restriction $\mathcal{Z}|_{\mathcal{U}} \colon \mathcal{U} \to U$ is a covering map. By the path lifting property, there is a unique path in $\mathcal{U}$ given by $\sigma_t = (\mathcal{Q}_t, Z_t)$ for $t \in [0,1]$ with $\sigma_{t_0} = \sigma$. By Proposition 3.4.1(2), σ_t satisfies the support property with respect to Q for all t. It remains to show that $\mathcal{Q}_t(0,1] = \mathcal{P}(0,1]$.

Fix a non-zero object $E \in \mathcal{D}$, and define

$$\Sigma_E := \{t \in [0,1] : E \in \mathcal{Q}_t(0,1]\}.$$

We first claim that Σ_E is open in $[0,1]$. Suppose $T \in \Sigma_E$. Then the JH factors of E with respect to σ_T, B_i, are all in $\mathcal{Q}_T(0,1]$ and satisfy $\mathrm{Im}\, Z_T(B_i) \geq 0$. The property for an object to be stable is open in

$\mathrm{Stab}_\Lambda(\mathcal{D})$ (see [BM11, Proposition 3.3]). Moreover, $0 < \phi_{\mathcal{Q}_T}(B_i)$ which is also an open property. Thus there is an open neighbourhood $V \subset [0,1]$ with $T \in V$ such that for all $t \in V$, all B_i are σ_t-stable with $0 < \phi_{\mathcal{Q}_t}(B_i)$. Now assume for a contradiction that Σ_E is not open. Therefore $1 < \phi_{\mathcal{Q}_t}(B_i)$ for some $t \in V$ and some i. However, for V sufficiently small this would mean that $\mathrm{Im}\, Z_t(B_i) < 0$ which contradicts the assumption that $\mathrm{Im}\, Z_t$ is constant. Hence Σ_E is open.

We next claim that the complement, $(\Sigma_E)^\mathsf{c} = [0,1] \setminus \Sigma_E$ is open in $[0,1]$. Assume for a contradiction that $T \in (\Sigma_E)^\mathsf{c}$ is in the closure and not the interior of Σ_E. Recall that $\phi^+(E)$ and $\phi^-(E)$ are continuous functions on $\mathrm{Stab}_\Lambda(\mathcal{D})$. Therefore $\phi^+_{\mathcal{Q}_T}(E) \le 1$ and $\phi^-_{\mathcal{Q}_T}(E) \ge 0$. However, if $\phi^-_{\mathcal{Q}_T}(E) > 0$, then $E \in \mathcal{Q}_T(0,1]$, so T would be in the interior of Σ_E which is a contradiction. Hence $\phi^-_{\mathcal{Q}_T}(E) = 0$. In particular, there is a JH factor of E with respect to σ_E, B, such that $B \in \mathcal{Q}_T(0)$. Again, since the property for an object to be stable is open, there is an open neighbourhood $W \subset [0,1]$ with $T \in W$ such that B is σ_t-stable for all $t \in W$. So for all $t \in W$, E has a morphism to a σ_t-stable object $B \in \mathcal{Q}_t(0)$. Therefore, $E \notin \mathcal{Q}_t(0,1]$ for all $t \in W$. But T is in the closure of Σ_E, hence $W \cap \Sigma_E \neq \emptyset$, which gives a contradiction. Therefore, $(\Sigma_E)^\mathsf{c}$ is open.

It follows that Σ_E is open and closed in $[0,1]$, therefore $\Sigma_E = [0,1]$. In particular, $\mathcal{Q}_t(0,1]$ is constant for all $t \in T$. Since $\mathcal{Q}_{t_0}(0,1] = \mathcal{P}(0,1]$, the result follows. $\square$

Chapter 4

Geometric stability conditions on surfaces

We use the Le Potier function to describe the set of geometric stability conditions on any surface. This was previously known for surfaces with Picard rank 1 [FLZ22, Theorem 3.4, Proposition 3.6].

4.1 Geometric stability conditions

Let X be a variety. Recall from Definition 1.1.2 that $\sigma \in \mathrm{Stab}(X)$ is called geometric if all skyscraper sheaves $\mathcal{O}_x$ are σ-stable. Denote by $\mathrm{Stab}^{\mathrm{Geo}}(X)$ the set of all (numerical) geometric stability conditions. We now collect some useful properties.

Proposition 4.1.1 ([FLZ22, Proposition 2.9])**.** *Let $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(X)$. Then all skyscraper sheaves are of the same phase.*

Proposition 4.1.2 ([Bri08, Proposition 9.4])**.** $\mathrm{Stab}^{\mathrm{Geo}}(X)$ *is open in* $\mathrm{Stab}(X)$.

Definition 4.1.3. A geometric stability condition $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ is called *normalised* if $Z(\mathcal{O}_x) = -1$ and $\phi(\mathcal{O}_x) = 1$. We write $\mathrm{Stab}^{\mathrm{Geo}}_N(X)$ for the set of all normalised geometric stability conditions.

Proposition 4.1.4. *There is an isomorphism* $\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \mathrm{Stab}^{\mathrm{Geo}}_N(X)$.

Proof. Let $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(X)$. Then $\mathcal{O}_x$ is σ-stable and of the same phase for every point $x \in X$ by Proposition 4.1.1. As discussed in Remark 3.2.3, $\mathbf{C}$ acts on $\mathrm{Stab}(X)$. In particular, there is a unique element $g \in \mathbf{C}$ such that $g \cdot \sigma = (\mathcal{P}', Z')$ satisfies $Z'([\mathcal{O}_x]) = -1$ and $\mathcal{O}_x \in \mathcal{P}'(1)$ for all $x \in X$. ☐

4.2 The central charge of a geometric stability condition

For the rest of this chapter, X will be a surface. The following result tells us that normalised geometric stability conditions on X are determined by their central charge.

Theorem 4.2.1 ([Bri08, Proposition 10.3], [D23, Theorem 5.5])**.** *Let X be a surface, and consider a geometric stability condition $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$. Then σ is determined by its central charge up to shifting the slicing by $[2n]$ for any $n \in \mathbf{Z}$.*

Moreover, if $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$, then

(1) *the central charge can be uniquely written in the following form*

$$Z([E]) = (\alpha - i\beta)H^2\mathrm{ch}_0([E]) + (B + iH) \,.\, \mathrm{ch}_1([E]) - \mathrm{ch}_2([E]),$$

where $\alpha, \beta \in \mathbf{R}$, $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$.

(2) *the heart,* $\mathcal{P}(0, 1]$, *is the tilt of* $\mathrm{Coh}(X)$ *with respect to the torsion pair* $(\mathcal{T}, \mathcal{F})$, *where*

$$\mathcal{T} := \left\{ E \in \mathrm{Coh}(X) : \begin{array}{l} \text{Any } H\text{-semistable HN factor } F \text{ of the torsion free} \\ \text{part of } E \text{ satisfies } \operatorname{Im} Z([F]) > 0. \end{array} \right\},$$

$$\mathcal{F} := \left\{ E \in \mathrm{Coh}(X) : \begin{array}{l} E \text{ is torsion free, and any } H\text{-semistable HN factor} \\ F \text{ of } E \text{ satisfies } \operatorname{Im} Z([F]) \le 0. \end{array} \right\}.$$

Notation 4.2.2. We will use $Z_{H,B,\alpha,\beta} = Z$ to denote central charges of the above form. Since the imaginary part $\operatorname{Im} Z_{H,B,\alpha,\beta}$ only depends on H and β, we will write $(\mathcal{T}_{H,\beta}, \mathcal{F}_{H,\beta})$ for the torsion pair, and $\mathrm{Coh}^{H,\beta}(X)$ for the corresponding tilted heart. Then $\sigma_{H,B,\alpha,\beta} := (Z_{H,B,\alpha,\beta}, \mathrm{Coh}^{H,\beta}(X))$.

The proof of Theorem 4.2.1 is similar to the case of K3 surfaces proved in [Bri08, §10]. We first need the following result, the proof of which is also written for K3 surfaces but immediately generalises to any surface.

Lemma 4.2.3 ([Bri08, Lemma 10.1])**.** *Suppose* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_N^{\mathrm{Geo}}(X)$. *Let* $E \in \mathrm{D^b}(X)$. *Then*

a) *if* $E \in \mathcal{P}(0, 1]$ *then* $H^i(E) = 0$ *unless* $i \in \{-1, 0\}$, *and moreover* $H^{-1}(E)$ *is torsion free,*
b) *if* $E \in \mathcal{P}(1)$ *is stable, then either* $E = \mathcal{O}_x$ *for some* $x \in X$, *or* $E[-1]$ *is a locally-free sheaf,*
c) *if* $E \in \mathrm{Coh}(X)$ *is a sheaf then* $E \in \mathcal{P}(-1, 1]$; *if* E *is a torsion sheaf then* $E \in \mathcal{P}(0, 1]$,
d) *the pair of subcategories*

$$\mathcal{T} = \mathrm{Coh}(X) \cap \mathcal{P}(0, 1] \text{ and } \mathcal{F} = \mathrm{Coh}(X) \cap \mathcal{P}(-1, 0]$$

defines a torsion pair on $\mathrm{Coh}(X)$ *and* $\mathcal{P}(0, 1]$ *is the corresponding tilt.*

Proof. (of Theorem 4.2.1) It is enough to show that normalised geometric stability conditions are determined by their central charge, since by Proposition 4.1.4 the central charge of $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ is unique up to shifting the slicing by $[2n]$ for $n \in \mathbf{Z}$.

Now fix $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_N^{\mathrm{Geo}}(X)$. We will prove that σ is the unique normalised geometric stability condition with this central charge in two steps.

Step 1 Recall that over $\mathbf{Q}$ the Chern character gives an isomorphism $\mathrm{K_{num}}(X) \cong \mathbf{Q} \oplus \mathrm{NS}(X) \oplus \mathbf{Q}$. Since σ is numerical, the central charge factors via $\mathrm{K_{num}}(X)$, so it can be written as follows.

$$Z([E]) = a\,\mathrm{ch}_0([E]) + B \,.\, \mathrm{ch}_1([E]) + c\,\mathrm{ch}_2([E]) + i(d\,\mathrm{ch}_0([E]) + H \,.\, \mathrm{ch}_1([E]) + e\,\mathrm{ch}_2([E])),$$

where $a, c, d, e \in \mathbf{R}$ and $B, H \in \mathrm{NS}_{\mathbf{R}}(X)$.

Since $\sigma \in \mathrm{Stab}_N^{\mathrm{Geo}}(X)$, $Z([\mathcal{O}_x]) = -1$. Hence $-1 = c$ and $e = 0$. Let $C \subset X$ be a curve. By Lemma 4.2.3(c), $\mathcal{O}_C \in \mathcal{P}(0, 1]$. Since $\mathrm{ch}_0(\mathcal{O}_C) = 0$ and $\mathrm{ch}_1(\mathcal{O}_C) = C$,

$$\operatorname{Im} Z([\mathcal{O}_C]) = H \,.\, C \ge 0.$$

This holds for any curve $C \subset X$, so $H \in \mathrm{NS}_{\mathbf{R}}(X)$ is nef. By Proposition 4.1.2, $\mathrm{Stab}^{\mathrm{Geo}}(X)$ is open. Moreover, by Theorem 3.2.1, a small deformation from σ to σ' in $\mathrm{Stab}^{\mathrm{Geo}}(X)$ corresponds to a small deformation of the central charges Z to Z', and in turn a small deformation of H to H' inside $\mathrm{NS}_{\mathbf{R}}(X)$. In particular, $H' \,.\, C \ge 0$ for any curve $C \subset X$. Therefore, H lies in the interior of the nef cone, hence H is ample.

Now let $\alpha \coloneqq \frac{a}{H^2}$ and $\beta \coloneqq \frac{-d}{H^2}$. Then the central charge is of the form

$$Z([E]) = (\alpha - i\beta)H^2\mathrm{ch}_0([E]) + (B + iH) \,.\, \mathrm{ch}_1([E]) - \mathrm{ch}_2([E]).$$

Step 2 Consider the torsion pair $(\mathcal{T}, \mathcal{F})$ of Lemma 4.2.3(d), so $\mathcal{P}(0,1]$ is the tilt of $\mathrm{Coh}(X)$ at $(\mathcal{T}, \mathcal{F})$. By Lemma 4.2.3(c), all torsion sheaves lie in $\mathcal{T}$. To complete the proof, we need the following claim.

$$E \in \mathrm{Coh}(X) \text{ is } H\text{-stable and torsion-free} \implies \begin{cases} E \in \mathcal{T} \text{ if } \operatorname{Im} Z([E]) > 0, \\ E \in \mathcal{F} \text{ if } \operatorname{Im} Z([E]) \leq 0. \end{cases} \tag{$*$}$$

This is Step 2 of the proof of [Bri08, Lemma 10.3]. Bridgeland first shows that E must lie in $\mathcal{T}$ or $\mathcal{F}$. We explain why it then follows that $\operatorname{Im} Z([E]) = 0$ implies $E \in \mathcal{F}$. Assume E is non-zero and $E \in \mathcal{T}$. Since $Z([E]) \in \mathbf{R}$, it follows that $E \in \mathcal{P}(1)$. For any $x \in \mathrm{Supp}(E)$, E has a non-zero map $f\colon E \to \mathcal{O}_x$. Let E_1 be its kernel in $\mathrm{Coh}(X)$. Since $\mathcal{O}_x$ is stable, f is a surjection in $\mathcal{P}(1)$. Thus E_1 also lies in $\mathcal{P}(1)$ and hence in $\mathcal{T}$. Moreover, $Z([E_1]) = Z([E]) - Z([\mathcal{O}_x]) = Z([E]) - 1$. Repeating this by replacing E with E_1 and so on creates a chain of strict subobjects in $\mathcal{P}(1)$, $E \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots$, such that $Z([E_n]) = Z([E]) - n$. If this process does not terminate, then $Z([E_k]) \in \mathbf{R}_{>0}$ for some $k \in \mathbf{N}$, contradicting the fact that $E_n \in \mathcal{P}(1)$. Otherwise, $E_i \cong \mathcal{O}_x$ for some i, contradicting the fact that E is torsion-free. □

Proposition 4.2.4. *Let X be a surface. Then there is an injective local homeomorphism*

$$\begin{aligned} \Pi\colon \mathrm{Stab}_N^{\mathrm{Geo}}(X) &\longrightarrow \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 \\ \sigma = \sigma_{H,B,\alpha,\beta} &\longmapsto (H, B, \alpha, \beta). \end{aligned}$$

Proof. Let $\mathcal{Z}\colon \mathrm{Stab}(X) \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$ denote the local homeomorphism from Theorem 3.2.1. Also define $\mathcal{N} \coloneqq \{(\mathcal{P}, Z) \in \mathrm{Stab}(X) : Z(\mathcal{O}_x) = -1\}$ and consider the following diagram

$$\begin{tikzcd} & & \mathrm{Stab}(X) \arrow[dll, "\mathcal{Z}"'] & \supset & \mathcal{N} \arrow[d, "\mathcal{Z}|_{\mathcal{N}}"'] & \supset & \mathrm{Stab}_N^{\mathrm{Geo}}(X) \arrow[d, "\mathcal{Z}|_{\mathrm{Stab}_N^{\mathrm{Geo}}(X))}"'] \\ \mathrm{Hom}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C}) & \supset & \{Z : Z(\mathcal{O}_x) = -1\} & \supset & \mathcal{Z}(\mathcal{N}) & \supset & \mathcal{Z}(\mathrm{Stab}_N^{\mathrm{Geo}}(X)) \end{tikzcd}$$

Since $\mathcal{Z}$ is a local homeomorphism and restriction is injective, $\mathcal{Z}|_{\mathcal{N}}$ and $\mathcal{Z}|_{\mathrm{Stab}_N^{\mathrm{Geo}}(X)}$ are also local homeomorphisms. Moreover, by the same argument as Step 1 of the proof of Theorem 4.2.1,

$$\begin{aligned} \{Z : Z(\mathcal{O}_x) = -1\} &\cong (\mathrm{NS}_{\mathbf{R}}(X))^2 \times \mathbf{R}^2 \\ Z = Z_{H,B,\alpha,\beta} &\mapsto (H, B, \alpha, \beta). \end{aligned}$$

Define Π to be the composition of $\mathcal{Z}|_{\mathrm{Stab}_N^{\mathrm{Geo}}(X)}$ with this isomorphism. Together with Theorem 4.2.1, it follows that Π is an injective local homeomorphism with

$$\operatorname{im} \Pi \subseteq \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2.$$

□

4.3 The set of all geometric stability conditions on surfaces

In this section we prove the following theorem, which is the main result of this Chapter.

Theorem 4.3.1 ([D23, Theorem 5.10])**.** *Let X be surface. There is a homeomorphism of topological spaces*

$$\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \left\{(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X,H,B}(\beta)\right\}.$$

Remark 4.3.2. We saw in Section 2.2 that any H-semistable sheaf F satisfies the Bogomolov–Gieseker (BG) inequality,

$$2\mathrm{ch}_0(F)\mathrm{ch}_2(F) \leq \mathrm{ch}_1(F)^2.$$

In [MS17, Theorem 6.10], the authors use the BG inequality to help construct a subset of $\mathrm{Stab}^{\mathrm{Geo}}(X)$ parametrised by $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We will see that this corresponds to the range where $\alpha > \frac{1}{2}\left[\left(\beta - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]$ in the above Theorem, see Proposition 4.3.12 for details.

In Section 4.2, we saw that any normalised geometric stability condition on a surface is determined by parameters $(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2$. To characterise geometric stability conditions on surfaces, we will find necessary and sufficient conditions for when these parameters define a geometric stability condition. To do this, we will use the Le Potier function twisted by B which was introduced in Definition 2.2.4. Since X is a surface, the value of $\Phi_{X,H,B}$ at every point is in $\mathbf{R}$ by Remark 2.2.5(1). Hence the definition specialises as follows.

Definition 4.3.3. Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We define the *Le Potier function twisted by* B, $\Phi_{X,H,B} \colon \mathbf{R} \to \mathbf{R}$, by

$$\Phi_{X,H,B}(x) := \limsup_{\mu \to x} \left\{ \frac{\mathrm{ch}_2(F) - B\,.\,\mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)} : \begin{array}{l} F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable with} \\ \mu_H(F) = \mu \end{array} \right\}.$$

Notation 4.3.4. To ease notation, we define

$$\mathcal{U} := \left\{(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X,H,B}(\beta)\right\}$$

Idea of the proof. By Proposition 4.1.4, $\mathrm{Stab}^{\mathrm{Geo}}(X) \cong \mathbf{C} \times \mathrm{Stab}^{\mathrm{Geo}}_N(X)$. Hence to prove Theorem 4.3.1 it is enough to show that there is a homeomorphism $\mathrm{Stab}^{\mathrm{Geo}}_N(X) \cong \mathcal{U}$. We will do this in the following two steps.

Step 1 *Construct an injective, local homeomorphism,* $\Pi \colon \mathrm{Stab}^{\mathrm{Geo}}_N(X) \to \mathcal{U}$. By Proposition 4.2.4, there is an injective local homeomorphism

$$\begin{aligned} \Pi \colon \mathrm{Stab}^{\mathrm{Geo}}_N(X) &\longrightarrow \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 \\ \sigma = \sigma_{H,B,\alpha,\beta} &\longmapsto (H, B, \alpha, \beta). \end{aligned}$$

So it remains to show that the image is contained in $\mathcal{U}$, which we do in Proposition 4.3.6.

Step 2 *Construct a pointwise inverse,* $\Sigma \colon \mathcal{U} \to \mathrm{Stab}^{\mathrm{Geo}}_N(X)$, i.e. show that $\sigma_{H,B,\alpha,\beta}$ is a stability condition if $\alpha > \Phi_{X,H,B}(\beta)$, since it will then follow by construction that $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$. Our strategy is to first show these $\sigma_{H,B,\alpha,\beta}$ is a stability condition for $\alpha \gg 0$. We then use the deformation property to show that, as α decreases, $\sigma_{H,B,\alpha,\beta}$ is still a stability condition if $\alpha > \Phi_{X,H,B}(\beta)$. More precisely:

(1) Fix $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$, and $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Let $\sigma_0 \coloneqq \sigma_{H,B,\alpha_0,\beta_0}$. To construct Σ it is enough to show that σ_0 is a stability condition.

(2) Fix $\alpha_1 > \max\left\{\alpha_0, \frac{1}{2}\left[\left(\beta_0 - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]\right\}$. Let $\sigma_1 \coloneqq \sigma_{H,B,\alpha_1,\beta_0}$.

(3) In [MS17, §6], the authors construct a subset of geometric stability conditions on X (Theorem 4.3.10). In Proposition 4.3.12 we show that σ_1 lies in this subset – up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ on $\mathrm{Stab}(X)$ – so in particular it is a stability condition.

(4) In Proposition 4.3.26, we show σ_0 is a stability condition by decreasing α and applying Theorem 3.4.2 as follows.
   (a) For $t \in [0,1]$, let $\alpha_t \coloneqq \alpha_0 + t(\alpha_1 - \alpha_0)$ and $Z_t \coloneqq Z_{H,B,\alpha_t,\beta_0}$. Then $\{Z_t\}_{t\in[0,1]}$ is a path in the space of central charges $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$.
   (b) $\mathrm{Im}\, Z_t$ does not depend on α_t, hence it is constant for all $t \in [0,1]$.
   (c) There is a quadratic form $Q = Q^{\delta,\epsilon}_{H,B,\alpha_0,\beta_0}$ (see Proposition 4.3.19) such that
      (i) Q is negative definite on $\mathrm{Ker}\, Z_t \otimes \mathbf{R}$ (Lemma 4.3.20),
      (ii) σ_1 satisfies the support property with respect to Q (Lemma 4.3.25).
   (d) By (a)-(c), the hypotheses of Theorem 3.4.2 are satisfied. Therefore, for all $t \in [0,1]$ Z_t lifts to $\sigma_t = (\mathrm{Coh}^{H,\beta}(X), Z_{H,B,\alpha_t,\beta_0}) \in \mathrm{Stab}(X)$. In particular, σ_0 is a stability condition.

4.3.1 STEP 1: Construction of the map $\mathrm{Stab}^{\mathrm{Geo}}_N(X) \to \mathcal{U}$.

Before we can prove that $\Pi(\mathrm{Stab}^{\mathrm{Geo}}_N(X)) \subseteq \mathcal{U}$, we need the following lemma which will help us to vary the parameters $(\alpha,\beta) \in \mathbf{R}^2$.

Lemma 4.3.5 ([D23, Lemma 5.16]). *Suppose $\sigma = \sigma_{H,B,\alpha,\beta} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$ is geometric. There there is an open neighbourhood $W \subset \mathbf{R}^2$ of (α,β), such that for every $(\alpha',\beta') \in W$, $\sigma_{H,B,\alpha',\beta'} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$.*

Proof. By Proposition 4.1.2, there is an open neighbourhood V of σ in $\mathrm{Stab}(X)$ where skyscraper sheaves are all stable. Let $\mathcal{Z}\colon \mathrm{Stab}(X) \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$ denote the local homeomorphism from Theorem 3.2.1. By Proposition 4.2.4(3), $\Pi(V)$ is open in $\mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2$. Therefore, $W \coloneqq \Pi(V)|_{\mathbf{R}^2}$ is open in $\mathbf{R}^2$. □

Proposition 4.3.6 ([FLZ22, Proposition 3.6], [D23, Lemma 5.13]). *Let $\sigma = \sigma_{H,B,\alpha,\beta} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$. Then $\alpha > \Phi_{X,H,B}(\beta)$, i.e. $\Pi(\mathrm{Stab}^{\mathrm{Geo}}_N(X)) \subseteq \mathcal{U}$, where Π is the injective local homeomorphism from Proposition 4.2.4.*

Proof. Suppose for a contradiction that $\alpha \leq \Phi_{X,H,B}(\beta)$. Let $W \subset \mathbf{R}^2$ be the open neighbourhood of (α,β) from Lemma 4.3.5. Recall that

$$\Phi_{X,H,B}(\beta) \coloneqq \limsup_{\mu\to\beta} \left\{\nu_{H,B}(F) : F \in \mathrm{Coh}(X) \text{ is } H\text{-semistable with } \mu_H(F) = \mu\right\}.$$

Therefore, there exist H-semistable sheaves with slopes arbitrarily close to β, and $\nu_{H,B}$ arbitrarily close to $\Phi_{X,H,B}(\beta)$. In particular, there exists $(\alpha_0,\beta_0) \in W$ and a torsion-free H-semistable sheaf F with

$$\beta_0 = \mu_H(F) = \frac{H \,.\, \mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}, \quad \text{and} \quad \alpha_0 \leq \nu_{H,B}(F) = \frac{\mathrm{ch}_2(F) - B \,.\, \mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}. \tag{4.1}$$

Since $(\alpha_0,\beta_0) \in W$, $\sigma_{H,B,\alpha_0,\beta_0} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$. Also $\mathrm{ch}_0(F) > 0$, hence

$$\mathrm{Im}(Z_{H,B,\alpha_0,\beta_0}([F])) = H \,.\, \mathrm{ch}_1([F]) - \beta_0 H^2\mathrm{ch}_0([F]) = 0.$$

By definition of the torsion pair $(\mathcal{T}_{H,\beta_0}, \mathcal{F}_{H,\beta_0})$ in Theorem 4.2.1, it follows that $F \in \mathcal{F}_{H,\beta_0}$. This implies that $Z_{H,B,\alpha_0,\beta_0}([F]) \in \mathbf{R}_{>0}$. However, by (4.1),

$$\mathrm{Re}(Z_{H,B,\alpha_0,\beta_0}([F])) = \alpha_0 H^2 \mathrm{ch}_0([F]) + B \,.\, \mathrm{ch}_1([F]) - \mathrm{ch}_2([F]) \leq 0,$$

Hence $Z_{H,B,\alpha_0,\beta_0}([F]) \in \mathbf{R}_{\leq 0}$ which is a contradiction. □

4.3.2 STEP 2: Construction of the pointwise inverse $\mathcal{U} \to \mathrm{Stab}_N^{\mathrm{Geo}}(X)$.

We will start by showing that $\sigma_{H,B,\alpha,\beta}$ is a stability condition for $\alpha \gg 0$ by relating these to the stability conditions constructed in [MS17, §6]. Let us first recall the construction of the latter.

Definition 4.3.7. Let X be a surface, and fix classes $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Define the pair $\sigma_{H,B} \coloneqq (\mathrm{Coh}^{H,B}(X), Z_{H,B})$, where

$$\begin{aligned} Z_{H,B}([E]) &= \left(-\mathrm{ch}_2^B([E]) + \frac{H^2}{2} \,.\, \mathrm{ch}_0^B([E])\right) + iH \,.\, \mathrm{ch}_1^B([E]) \\ &= \left[\frac{1}{2}\left(1 - \frac{B^2}{H^2}\right) - i\frac{H \,.\, B}{H^2}\right] H^2 \mathrm{ch}_0([E]) + (B + iH) \,.\, \mathrm{ch}_1([E]) - \mathrm{ch}_2([E]), \\ \mathcal{T}_{H,B} &= \left\{E \in \mathrm{Coh}(X) : \begin{array}{l}\text{Any } H\text{-semistable Harder–Narasimhan factor } F \text{ of the} \\ \text{torsion free part of } E \text{ satisfies } \mathrm{Im}\, Z_{H,B}([F]) > 0.\end{array}\right\}, \\ \mathcal{F}_{H,B} &= \left\{E \in \mathrm{Coh}(X) : \begin{array}{l}E \text{ is torsion free, and any } H\text{-semistable Harder–} \\ \text{Narasimhan factor } F \text{ of } E \text{ satisfies } \mathrm{Im}\, Z_{H,B}([F]) \leq 0.\end{array}\right\}, \end{aligned}$$

and $\mathrm{Coh}^{H,B}(X)$ is the tilt of $\mathrm{Coh}(X)$ at the torsion pair $(\mathcal{T}_{H,B}, \mathcal{F}_{H,B})$.

For $\sigma_{H,B}$ to be a stability condition, it needs to satisfy the support property (see Definition 3.1.6(1)). In particular, there should exist a quadratic form on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ which is positive semi-definite on $\sigma_{H,B}$ semistable objects. The follow result will be used to construct such a quadratic form.

Lemma 4.3.8 ([MS17, Exercise 6.11], [D23, Lemma 5.20])**.** *Let X be a surface. Then there exists a continuous function $C_{(-)} \colon \mathrm{Amp}_{\mathbf{R}}(X) \to \mathbf{R}_{\geq 0}$ such that, for every $D \in \mathrm{Eff}_{\mathbf{R}}(X)$,*

$$C_H (H \,.\, D)^2 + D^2 \geq 0.$$

Proof. $C_H(H \,.\, D)^2 + D^2 \geq 0$ is invariant under rescaling. If we consider $\mathrm{Eff}_{\mathbf{R}}(X) \subset \mathrm{NS}_{\mathbf{R}}(X)$ as normed vector spaces, it is therefore enough to look at the subspace of unit vectors $U \subset \mathrm{Eff}_{\mathbf{R}}(X)$.

Since $D \in U$ is effective and non-zero, $H \,.\, D > 0$. Hence there exists $C \in \mathbf{R}_{\geq 0}$ such that $C(H \,.\, D)^2 + D^2 \geq 0$. Define:

$$C_{H,D} \coloneqq \inf\{C \in \mathbf{R}_{\geq 0} : C(H \,.\, D)^2 + D^2 \geq 0\}.$$

Since $\mathrm{Amp}_{\mathbf{R}}(X)$ is open, $H' \,.\, D > 0$ for a small deformation H' of H. It follows that $\overline{U}$ is strictly contained in the subspace $\{E \in \mathrm{NS}_{\mathbf{R}}(X) : E \,.\, H > 0\}$. Moreover, $C_{H,D}$ is a continuous function on $\overline{U}$, and $\overline{U}$ is compact as it is a closed subset of the unit sphere in $\mathrm{NS}_{\mathbf{R}}(X)$. Therefore, $C_{H,D}$ has a maximum, which we call C_H. By construction, this is a continuous function on $\mathrm{Amp}_{\mathbf{R}}(X)$. □

Definition 4.3.9. Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. We define the following quadratic forms on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$:

$$
\begin{aligned}
Q_{BG} &\coloneqq \mathrm{ch}_1^2 - 2\mathrm{ch}_2\mathrm{ch}_0 \\
\Delta_{H,B}^{C_H} &\coloneqq Q_{BG} + C_H(H \,.\, \mathrm{ch}_1^B)^2,
\end{aligned}
$$

where $C_H \in \mathbf{R}_{\geq 0}$ is the constant from Lemma 4.3.8.

Theorem 4.3.10 ([MS17, Theorem 6.10]). *Let X be a surface. Fix a pair of divisor classes $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Then $\sigma_{H,B} \in \mathrm{Stab}_N^{\mathrm{Geo}}(X)$, and $\sigma_{H,B}$ satisfies the support property with respect to $\Delta_{H',B'}^{C_{H'}}$, where $(H', B') \in \mathrm{Amp}_{\mathbf{Q}}(X) \times \mathrm{NS}_{\mathbf{Q}}(X)$ are nearby rational classes.*

Remark 4.3.11. Theorem 4.3.10 was first proved for K3 surfaces in [Bri08], along with the fact that this gives rise to a continuous family. In [MS17, Theorem 6.10], the authors first prove the result holds for rational classes (H, B) and sketch how to extend this to arbitrary classes. In particular, $\sigma_{H,B}$ can be obtained as a deformation of $\sigma_{H',B'}$ for nearby rational classes (H', B'), and both $\sigma_{H',B'}$ and $\sigma_{H,B}$ satisfy the same support property with respect to $\Delta_{H',B'}^{C_{H'}}$. This uses the fact that $\Delta_{H,B}^{C_H}$ varies continuously with (H, B), together with similar arguments to Proposition 3.4.1.

We are now ready to prove that $\sigma_{H,B,\alpha,\beta}$ is a stability condition for $\alpha \gg 0$.

Proposition 4.3.12 ([D23, Lemma 5.24]). *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$ and fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Suppose $\alpha > \frac{1}{2}\left[\left(\beta_0 - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]$. Define $b \coloneqq \beta_0 - \frac{H.B}{H^2} \in \mathbf{R}$ and $a \coloneqq \sqrt{2\alpha - b^2 + \frac{B^2}{H^2}} \in \mathbf{R}_{>0}$. Then $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{aH,B+bH}$ are the same up to the action of $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$. Moreover, for $\alpha > \frac{1}{2}\left[\left(\beta - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]$, this is a continuous family in $\mathrm{Stab}_N^{\mathrm{Geo}}(X)$.*

Proof. Throughout this proof we abuse notation by considering central charges as homomorphisms $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R} \to \mathbf{C}$. We first claim that $\operatorname{Ker} Z_{H,B,\alpha,\beta_0} = \operatorname{Ker} Z_{aH,B+bH}$ as sub-vector spaces of $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. Fix $u \in \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$, then $\operatorname{Im} Z_{aH,B+bH}(u) = 0$ if and only if

$$
\begin{aligned}
0 &= aH \,.\, B\mathrm{ch}_0(u) + abH^2\mathrm{ch}_0(u) - aH \,.\, \mathrm{ch}_1(u) \\
&= a\left(H \,.\, B\mathrm{ch}_0(u) + \left(\beta_0 - \frac{H \,.\, B}{H^2}\right) H^2\mathrm{ch}_0(u) - H \,.\, \mathrm{ch}_1(u)\right) \\
&= a\left(\beta_0 H^2\mathrm{ch}_0(u) - H \,.\, \mathrm{ch}_1(u)\right) \\
&= -a \operatorname{Im} Z_{H,B,\alpha,\beta_0}(u).
\end{aligned}
$$

Since $a > 0$, $\operatorname{Im} Z_{aH,B+bH}(u) = 0$ if and only if $\operatorname{Im} Z_{H,B,\alpha,\beta_0}(u) = 0$. Suppose $\operatorname{Im} Z_{aH,B+bH}(u) = 0$, so $H \,.\, \mathrm{ch}_1(u) = \beta_0 H^2\mathrm{ch}_0(u)$. Then $\operatorname{Re} Z_{aH,B+bH}(u) = 0$ if and only if

$$
\begin{aligned}
0 &= \frac{1}{2}\left((aH)^2 - (B + bH)^2\right)\mathrm{ch}_0(u) + B \,.\, \mathrm{ch}_1(u) + bH \,.\, \mathrm{ch}_1(u) - \mathrm{ch}_2(u) \\
&= \frac{1}{2}\left(a^2 - \frac{(B + bH)^2}{H^2} + 2b\beta_0\right) H^2\mathrm{ch}_0(u) + B \,.\, \mathrm{ch}_1(u) - \mathrm{ch}_2(u).
\end{aligned}
$$

Moreover,

$$\begin{aligned}\frac{1}{2}\left(a^2-\frac{(B+bH)^2}{H^2}+2b\beta_0\right)&=\frac{1}{2}\left(a^2-\frac{B^2}{H^2}+2b\left(\beta_0-\frac{B\,.\,H}{H^2}\right)-b^2\right)\\&=\frac{1}{2}\left(2\alpha-b^2+\frac{B^2}{H^2}-\frac{B^2}{H^2}+b^2\right)\\&=\alpha.\end{aligned}$$

It follows that $u\in\operatorname{Ker}Z_{aH,B+bH}$ if and only if $u\in\operatorname{Ker}Z_{H,B,\alpha,\beta_0}$. Therefore, $Z_{aH,B+bH}$ and Z_{H,B,α,β_0} are the same up to the action of $\mathrm{GL}_2^+(\mathbf{R})$.

Moreover, by Theorem 4.3.10, $\sigma_{aH,B+bH}\in\operatorname{Stab}_N^{\mathrm{Geo}}(X)$. Together with Theorem 4.2.1, it follows that $\sigma_{H,B,\alpha,\beta_0}=g\cdot\sigma_{aH,B+bH}\in\operatorname{Stab}(X)$ for some $g\in\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$. Then, by definition, $\sigma_{H,B,\alpha,\beta_0}\in\operatorname{Stab}_N^{\mathrm{Geo}}(X)$. It remains to show this gives rise to a continuous family. By Proposition 4.2.4 and Proposition 4.3.6,

$$\Pi\colon\operatorname{Stab}_N^{\mathrm{Geo}}(X)\to\mathcal{U},\quad\sigma_{H,B,\alpha,\beta}\mapsto(H,B,\alpha,\beta).$$

is an injective local homeomorphism. Let $V:=\left\{(H,B,\alpha,\beta):\alpha>\frac{1}{2}\left[\left(\beta-\frac{H.B}{H^2}\right)^2-\frac{B^2}{H^2}\right]\right\}$. The restriction $\Pi|_{\Pi^{-1}(V)}$ is still an injective local homeomorphism. Moreover, By the arguments above, $\Pi|_{\Pi^{-1}(V)}$ is surjective. Hence $\sigma_{H,B,\alpha,\beta}$ is a continuous family. $\square$

Remark 4.3.13. Let $\mathrm{Sh}_2^+(\mathbf{R})\subset\mathrm{GL}_2^+(\mathbf{R})$ denote the subgroup of shearings, i.e. transformations that preserves the real line in $\mathbf{C}\cong\mathbf{R}^2$. It is simply connected, hence it embeds as a subgroup into $\widetilde{\mathrm{GL}}_2^+(\mathbf{R})$ and acts on $\operatorname{Stab}(X)$. In the above proof, $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{aH,B+bH}$ have the same hearts, so they are the same up to the action of $\mathrm{Sh}_2^+(\mathbf{R})$.

The next result follows from the proof of Theorem 4.3.10. We explain this part of the argument explicitly, as it will be essential for extending the support property to any $\sigma_{H,B,\alpha,\beta}$ with $\alpha>\Phi_{X,H,B}(\beta)$ in Lemma 4.3.25.

Lemma 4.3.14 ([D23, Lemma 5.26]). *Let X be a surface. Let $(H,B)\in\operatorname{Amp}_{\mathbf{R}}(X)\times\operatorname{NS}_{\mathbf{R}}(X)$. There exists $(H',B')\in\operatorname{Amp}_{\mathbf{Q}}(X)\times\operatorname{NS}_{\mathbf{Q}}(X)$ such that, for $a\geq1$, $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $\operatorname{Ker}Z_{aH,B}\otimes\mathbf{R}$. In particular, $\Delta_{H',B'}^{C'_H}$ gives the support property for $\sigma_{aH,B}$.*

Proof. By Theorem 4.3.10, we have $\sigma_{aH,B}\in\operatorname{Stab}_N^{\mathrm{Geo}}(X)$ for $a\geq1$, and nearby to (H,B) there exist rational classes $(H',B')\in\operatorname{Amp}_{\mathbf{Q}}(X)\times\operatorname{NS}_{\mathbf{Q}}(X)$ such that $\Delta_{H',B'}^{C_{H'}}$ gives the support property for $\sigma_{H,B}\in\operatorname{Stab}_N^{\mathrm{Geo}}(X)$. In particular, $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $\mathrm{K}_1:=\operatorname{Ker}Z_{H,B}\otimes\mathbf{R}$. By Proposition 3.4.1, it is enough to show $\Delta_{H',B'}^{C_{H'}}$ is negative definite on $K_a:=\operatorname{Ker}Z_{aH,B}\otimes\mathbf{R}$ for $a\geq1$.

Recall that $u=(\mathrm{ch}_0^B(u),\mathrm{ch}_1^B(u),\mathrm{ch}_2^B(u))\in\mathrm{K}_a$ if and only if

$$a^2\frac{H^2}{2}\mathrm{ch}_0^B(u)=\mathrm{ch}_2^B(u)\quad\text{ and }\quad H\,.\,\mathrm{ch}_1^B(u)=0.$$

Let $\Psi_a\colon\mathrm{K}_1\to\mathrm{K}_a$ be the isomorphism of sub-vector spaces of $\mathrm{K}_{\mathrm{num}}(X)\otimes\mathbf{R}$ given by

$$\Psi_a\colon v=(\mathrm{ch}_0^B(v),\mathrm{ch}_1^B(v),\mathrm{ch}_2^B(v))\mapsto\left(\mathrm{ch}_0^B(v),\mathrm{ch}_1^B(v),\mathrm{ch}_2^B(v)+\left(a^2-1\right)\frac{H^2}{2}\mathrm{ch}_0^B(v)\right).$$

Let $u \in \mathrm{K}_a$. Then $u = \Psi_a(v)$ for some $v \in \mathrm{K}_1$. Clearly $\Delta^{C_{H'}}_{H',B'}(0) = 0$, so we may assume $u \neq 0$. Hence $v \neq 0$, and it is enough to show that $\Delta^{C_{H'}}_{H',B'}(\Psi_a(v)) < 0$. Recall that $\mathrm{ch}_1^{B'} = \mathrm{ch}_1 - B' \,.\, \mathrm{ch}_0$, hence $\mathrm{ch}_1^{B'}(\Psi_a(v)) = \mathrm{ch}_1^{B'}(v)$. Therefore,

$$\begin{aligned}\Delta^{C_{H'}}_{H',B'}(\Psi_a(v)) &= \left(\mathrm{ch}_1^B(v)\right)^2 - 2\mathrm{ch}_0^B(v)\mathrm{ch}_2^B(v) - 2\left(a^2-1\right)\frac{H^2}{2}\left(\mathrm{ch}_0^B(v)\right)^2 + C_{H'}\left(H' \,.\, \mathrm{ch}_1^{B'}(v)\right)^2 \\ &= \Delta^{C_{H'}}_{H',B'}(v) - 2\left(a^2-1\right)\frac{H^2}{2}\left(\mathrm{ch}_0^B(v)\right)^2 \\ &\leq \Delta^{C_{H'}}_{H',B'}(v).\end{aligned}$$

The last inequality uses the fact that $a^2 - 1 \geq 0$. Since $\Delta^{C_{H'}}_{H',B'}$ is negative definite on K_1, it follows that $\Delta^{C_{H'}}_{H',B'}(\Psi_a(v)) < 0$. ☐

We next introduce a new quadratic form on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. We will combine this with Q_{BG} in Proposition 4.3.19 to define the quadratic form that will later give the support property for $\sigma_{H,B,\alpha,\beta}$ for any $\alpha > \Phi_{X,H,B}(\beta)$.

Definition 4.3.15. Let X be a surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Let $\alpha > \Phi_{X,H,B}(\beta)$, and let $\delta > 0$. We define the following quadratic form on $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$.

$$Q_{H,B,\alpha,\beta,\delta} \coloneqq \delta^{-1}(H \,.\, \mathrm{ch}_1 - \beta_0 H^2\mathrm{ch}_0)^2 - (H^2\mathrm{ch}_0)\left(\mathrm{ch}_2 - B \,.\, \mathrm{ch}_1 - (\alpha_0 - \delta)H^2\mathrm{ch}_0\right).$$

The upper semi-continuity of the Le Potier function can now be used to prove that this new quadratic form is positive semi-definite on H-semistable sheaves.

Lemma 4.3.16 ([D23, Lemma 5.28])**.** *Let X be a surface. Let $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Then there exists $\delta > 0$ such that, for every H-semistable torsion-free sheaf F, we have $Q_{H,B,\alpha_0,\beta_0,\delta}([F]) \geq 0$.*

Proof. Since $\Phi_{X,H,B}$ is upper semi-continuous and bounded above by a quadratic polynomial in x, the same argument as in [FLZ22, Remark 3.5] applies. In particular, there exists a sufficiently small $\delta > 0$ such that

$$\frac{(x-\beta_0)^2}{\delta} + \alpha_0 - \delta \geq \Phi_{X,H,B}(x).$$

Suppose F is an H-semistable torsion-free sheaf. Let $x = \mu_H(F) = \frac{H.\mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}$, then

$$\delta^{-1}(H \,.\, \mathrm{ch}_1(F) - \beta_0 H^2\mathrm{ch}_0(F))^2 + (\alpha_0 - \delta)(H^2\mathrm{ch}_0(F))^2 \geq (H^2\mathrm{ch}_0(F))^2\Phi_{X,H,B}\left(\frac{H \,.\, \mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}\right).$$

From Lemma 2.2.6 it follows that

$$\delta^{-1}(H \,.\, \mathrm{ch}_1(F) - \beta_0 H^2\mathrm{ch}_0(F))^2 + (\alpha_0 - \delta)(H^2\mathrm{ch}_0(F))^2 \geq (H^2\mathrm{ch}_0(F))^2\frac{\mathrm{ch}_2(F) - B \,.\, \mathrm{ch}_1(F)}{H^2\mathrm{ch}_0(F)}.$$

In particular,

$$\delta^{-1}(H \,.\, \mathrm{ch}_1(F) - \beta_0 H^2\mathrm{ch}_0(F))^2 - (H^2\mathrm{ch}_0(F))\left(\mathrm{ch}_2(F) - B \,.\, \mathrm{ch}_1(F) - (\alpha_0 - \delta)H^2\mathrm{ch}_0(F)\right) \geq 0.$$

☐

Remark 4.3.17. Let $u \in \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. We now consider Z_{H,B,α_0,β_0} again as a homomorphism $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R} \to \mathbf{C}$. Recall that $u \in \mathrm{K}_{\alpha_0} \coloneqq \operatorname{Ker} Z_{H,B,\alpha_0,\beta_0} \subseteq \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ if and only if

$$\alpha_0 H^2 \mathrm{ch}_0(u) + B \,.\, \mathrm{ch}_1(u) - \mathrm{ch}_2(u) = 0 \quad \text{and} \quad H \,.\, \mathrm{ch}_1(u) - \beta_0 H^2 \mathrm{ch}_0(u) = 0.$$

Then

$$Q_{H,B,\alpha_0,\beta_0,\delta}(u) = -\delta \left(H^2 \mathrm{ch}_0(u)\right)^2 \leq 0,$$

for all $u \in \mathrm{K}_{\alpha_0}$. In particular, $Q_{H,B,\alpha_0,\beta_0,\delta}$ is negative semi-definite on K_{α_0}. Hence $Q_{H,B,\alpha_0,\beta_0,\delta}$ does not fulfil the support property.

To construct a quadratic form which is negative definite on $K_{\alpha_0} = \operatorname{Ker} Z_{H,B,\alpha_0,\beta_0}$, we will next combine $Q_{H,B,\alpha_0,\beta_0,\delta}$ with Q_{BG}, the quadratic form coming from the Bogomolov–Gieseker inequality introduced in Definition 4.3.9. We first need the following result.

Lemma 4.3.18 ([Bog79, §10], [HL10, Theorem 3.4.1])**.** *Let X be a surface. Let $H \in \mathrm{Amp}_{\mathbf{R}}(X)$. Then $Q_{BG}([F]) \geq 0$ for every H-semistable torsion-free sheaf F.*

Proposition 4.3.19 ([D23, Proposition 5.31])**.** *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta > 0$ as in Lemma 4.3.16. Let $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0} \coloneqq Q_{H,B,\alpha_0,\beta_0,\delta} + \varepsilon Q_{BG}$. Then there exists $\varepsilon > 0$ such that*

(1) $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}([F]) \geq 0$ for every H-semistable torsion-free sheaf F,

(2) $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}([T]) \geq 0$ for every torsion sheaf T, and

(3) $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$ is negative definite on $K_{\alpha_0} \coloneqq \operatorname{Ker} Z_{H,B,\alpha_0,\beta_0} \subseteq \mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$.

Proof. (1) follows immediately for any $\varepsilon > 0$ from Lemma 4.3.16 and Lemma 4.3.18. For (2), let C_H be the constant from Lemma 4.3.8. Choose $\varepsilon_1 > 0$ such that $\varepsilon_1 < \frac{\delta^{-1}}{C_H}$. Let T be a torsion sheaf, then

$$\begin{aligned} Q^{\delta,\varepsilon_1}_{H,B,\alpha_0,\beta_0}([T]) &= \delta^{-1}(H \,.\, \mathrm{ch}_1([T]))^2 + \varepsilon_1 \mathrm{ch}_1([T])^2 \\ &= \varepsilon_1 \left(\frac{\delta^{-1}}{\varepsilon_1}(H \,.\, \mathrm{ch}_1([T]))^2 + \mathrm{ch}_1([T])^2 \right) \\ &> \varepsilon_1 \left(C_H (H \,.\, \mathrm{ch}_1([T]))^2 + \mathrm{ch}_1([T])^2 \right) \\ &\geq 0, \end{aligned}$$

For (3), fix a norm on $\mathrm{K}_{\mathrm{num}}(X)$ and let U denote the set of unit vectors in K_{α_0} with respect to this norm. It will be enough to show there exists $\varepsilon_2 > 0$ such that $Q^{\delta,\varepsilon_2}_{H,B,\alpha_0,\beta_0}|_U < 0$.

Let $A \coloneqq \{u \in U \mid Q_{H,B,\alpha_0,\beta_0,\delta}(u) = 0\}$. We first claim that $Q_{BG} < 0$ on A. For any $a \in A$, $\mathrm{ch}_0(a) = 0$. The condition that $Z_{H,B,\alpha_0,\beta_0}(a) = 0$ then becomes

$$B \,.\, \mathrm{ch}_1(a) = \mathrm{ch}_2(a) \quad \text{and} \quad H \,.\, \mathrm{ch}_1(a) = 0.$$

H is ample, so $\mathrm{ch}_1(a)^2 \leq 0$ by the Hodge Index Theorem. If $\mathrm{ch}_1^2(a) = 0$, then $\mathrm{ch}_1(a) = 0$, and hence $0 = B \,.\, \mathrm{ch}_1(a) = \mathrm{ch}_2(a)$. So $a = 0$, which contradicts the fact that $a \in U$. Therefore, for all $a \in A$,

$$Q_{BG}(a) = \mathrm{ch}_1(a)^2 < 0.$$

We next claim that there exists a sufficiently small $\varepsilon_2 > 0$ such that $Q^{\delta,\varepsilon_2}_{H,B,\alpha_0,\beta_0} < 0$ on U. Note that $Q^{\delta,\varepsilon_2}_{H,B,\alpha_0,\beta_0}\Big|_A = \varepsilon_2 Q_{BG}\Big|_A < 0$, so we only need to check the claim on $U \setminus A$. Now suppose the converse, so for every $\varepsilon > 0$, there exists $u \in U \setminus A$ such that

$$Q_{BG}(u) \geq -\frac{1}{\varepsilon} Q_{H,B,\alpha_0,\beta_0,\delta}(u).$$

Thus $Q_{H,B,\alpha_0,\beta_0,\delta}(u) < 0$ since $Q_{H,B,\alpha_0,\beta_0,\delta}$ is negative semi-definite on U, and $u \in U \setminus A$. Therefore,

$$P(u) \coloneqq \frac{Q_{BG}(u)}{-Q_{H,B,\alpha_0,\beta_0,\delta}(u)} \geq \frac{1}{\varepsilon}.$$

Thus P is not bounded above on $U \setminus A$. Moreover, A is closed and $Q_{BG}\big|_A < 0$ on A. Hence Q_{BG} is negative definite on some open neighbourhood V of A, so $P\big|_V < 0$. Finally, $U \setminus V$ is compact, so P must be bounded above on $U \setminus V$. In particular, P is bounded above on $U \setminus A$ which gives a contradiction. It follows that there exists $\varepsilon_2 > 0$ such that $Q^{\delta,\varepsilon_2}_{H,B,\alpha_0,\beta_0}$ is negative definite on K_{α_0}. Finally, let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. $\square$

We will use Proposition 4.3.19 soon to prove that $\sigma_{H,B,\alpha_0,\beta_0}$ satisfies the support property with respect to $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$. The following result will then help us to extend the support property to any $\sigma_{H,B,\alpha,\beta_0}$ with $\alpha \geq \alpha_0$.

Lemma 4.3.20 ([D23, Lemma 5.32])**.** *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$, and fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta, \varepsilon > 0$ as in Proposition 4.3.19. Then $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$ is negative definite on $K_\alpha \coloneqq \mathrm{Ker}\, Z_{H,B,\alpha,\beta_0} \otimes \mathbf{R}$ for all $\alpha \geq \alpha_0$.*

Proof. Recall that $u = (\mathrm{ch}_0(u), \mathrm{ch}_1(u), \mathrm{ch}_2(u)) \in \mathrm{K}_\alpha = \mathrm{Ker}\, Z_{H,B,\alpha,\beta_0} \otimes \mathbf{R}$ if and only if

$$\alpha H^2 \mathrm{ch}_0(u) + B\,.\,\mathrm{ch}_1(u) - \mathrm{ch}_2(u) = 0, \quad H\,.\,\mathrm{ch}_1(u) - \beta_0 H^2 \mathrm{ch}_0(u) = 0.$$

Let $\Psi_\alpha \colon \mathrm{K}_{\alpha_0} \to \mathrm{K}_\alpha$ be the isomorphism of sub-vector spaces of $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$ given by

$$\Psi_\alpha \colon v = (\mathrm{ch}_0(v), \mathrm{ch}_1(v), \mathrm{ch}_2(v)) \mapsto (\mathrm{ch}_0(v), \mathrm{ch}_1(v), \mathrm{ch}_2(v) + (\alpha - \alpha_0) H^2 \mathrm{ch}_0(v)).$$

Let $u \in \mathrm{K}_\alpha$, then $u = \Psi_\alpha(v)$ for some $v \in \mathrm{K}_{\alpha_0}$. Clearly $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(0) = 0$, so we may assume $u \neq 0$. Hence $v \neq 0$, and it is enough to show that $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(\Psi_\alpha(v)) < 0$. Moreover,

$$\begin{aligned}
Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(\Psi_\alpha(v)) &= Q_{H,B,\alpha_0,\beta_0,\delta}(\Psi_\alpha(v)) + \varepsilon Q_{BG}(\Psi_\alpha(v)) \\
&= Q_{H,B,\alpha_0,\beta_0,\delta}(v) - (\alpha - \alpha_0)(H^2 \mathrm{ch}_0(v))^2 + \varepsilon Q_{BG}(v) - 2\varepsilon(\alpha - \alpha_0) H^2 \mathrm{ch}_0(v)^2 \\
&= Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(v) - (\alpha - \alpha_0) H^2 \mathrm{ch}_0(v)^2 (H^2 + 2\varepsilon) \\
&\leq Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(v).
\end{aligned}$$

Finally, by Proposition 4.3.19(3), $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}(v) < 0$. $\square$

We next return to studying stability conditions of the form $\sigma_{H,B}$ (see Definition 4.3.7). The following result characterises objects that are $\sigma_{aH,B}$-semistable for all $a \gg 0$. This is sometimes called the *large volume limit*.

Lemma 4.3.21 ([MS17, Lemma 6.18])**.** *Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. If $E \in \mathrm{Coh}^{H,B}(X)$ is $\sigma_{aH,B}$-semistable for all $a \gg 0$, then it satisfies one of the following conditions:*

(1) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a H-semistable torsion-free sheaf.

(2) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a torsion sheaf.

(3) $\mathcal{H}^{-1}(E)$ *is a* H*-semistable torsion-free sheaf and* $\mathcal{H}^0(E)$ *is either 0 or a torsion sheaf supported in dimension zero.*

We now combine this with Proposition 4.3.19(1) and (2) to check that $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$ is positive semi-definite on objects that are semistable at the large volume limit.

Proposition 4.3.22 ([D23, Corollary 5.34]). *Let* $(H,B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$*. Fix* $\alpha_0, \beta_0 \in \mathbf{R}$ *such that* $\alpha_0 > \Phi_{X,H,B}(\beta_0)$*. Choose* $\delta, \varepsilon > 0$ *as in Proposition 4.3.19. If* $E \in \mathrm{Coh}^{H,B}(X)$ *is* $\sigma_{aH,B}$*-semistable for all* $a \gg 0$*, then* $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}([E]) \geq 0$.

Proof. Let $Q \coloneqq Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$. By our hypotheses, E satisfies one of the three conditions in Lemma 4.3.21. If E satisfies (1), then $Q([E]) = Q([\mathcal{H}^0(E)])$, where $\mathcal{H}^0(E)$ is a H-semistable torsion-free sheaf, and the result follows from Proposition 4.3.19(1). Similarly, if E satisfies (2), then by Proposition 4.3.19(2), $Q([E]) = Q([\mathcal{H}^0(E)]) \geq 0$. Now assume E satisfies (3). Then

$$\mathrm{ch}([E]) = -\mathrm{ch}(\mathcal{H}^{-1}(E)) + \mathrm{length}(\mathcal{H}^0(E))$$

Hence

$$Q_{BG}([E]) = Q_{BG}([\mathcal{H}^{-1}(E)]) - 2(-\mathrm{ch}_0(\mathcal{H}^{-1}(E)))\,\mathrm{length}(\mathcal{H}^0(E)) \geq Q_{BG}(\mathcal{H}^{-1}(E)).$$

The same argument applies to $Q_{H,B,\alpha_0,\beta_0,\delta}$. Hence $Q([E]) \geq Q([\mathcal{H}^{-1}(E)])$. The result follows by Proposition 4.3.19(1). □

To prove the support property, we first want to extend Proposition 4.3.22 to $\sigma_{H,B,\alpha_1,\beta_0}$-semistable objects (for some $\alpha_1 > \alpha_0$). We will prove this in Lemma 4.3.25 by contradiction using the JH factors (see Remark 3.1.8 for the definition). The following two lemmas will be used to control how the quadratic form behaves on JH factors.

Lemma 4.3.23 ([D23, Lemma 5.35]). *Let* $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$ *with support property given by a quadratic form* Q *on* $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$*. Suppose* $E \in \mathrm{D^b}(X)$ *is strictly* σ*-semistable and satisfies* $Q(E) \neq 0$*. Let* $A_1, \dots, A_m$ *be the JH factors of* E*. Then* $Q(A_i) < Q(E)$ *for all* $1 \leq i \leq m$.

Proof. It is enough to prove that $Q(A_1) < Q(E)$. Since E is σ-semistable, $E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbf{R}$. By definition, $A_1 \in \mathcal{P}(\phi)$, and hence $E/A_1 \in \mathcal{P}(\phi)$ also. Therefore, by the support property, $Q(A_1) \geq 0$ and $Q(E/A_1) \geq 0$. Moreover, since A_1 and E/A_1 have the same phase, there exists $\lambda \in \mathbf{R}_{>0}$ such that $Z(A_1) - \lambda Z(E/A_1) = 0$. Hence $[A_1] - \lambda[E/A_1] \in \mathrm{Ker}\, Z \otimes \mathbf{R}$.

Let Q also denote the associated symmetric bilinear form. Now assume $[A_1] - \lambda[E/A_1] \neq 0$ in $\mathrm{K}_{\mathrm{num}}(X) \otimes \mathbf{R}$. By the support property, Q is negative definite on $\mathrm{Ker}(Z) \otimes \mathbf{R}$, hence

$$0 > Q([A_1] - \lambda[E/A_1]) = Q(A_1) - 2\lambda Q(A_1, E/A_1) + \lambda^2 Q(E/A_1).$$

Moreover, $\lambda > 0$ and $Q(A_1), Q(E/A_1) \geq 0$. It follows that $Q(A_1, E/A_1) > 0$. Therefore,

$$Q(E) = Q(A_1) + Q(E/A_1) + 2Q(A_1, E/A_1) > Q(A_1).$$

Otherwise, if $[A_1] = \lambda[E/A_1]$, then $\mu \coloneqq 1/\lambda > 0$ and

$$Q(E) = Q(A_1) + \mu(\mu + 2)Q(A_1).$$

If $Q(A_1) = 0$, then $Q(E) = 0$ which is a contradiction. Hence $Q(A_1) > 0$, so $Q(E) > Q(A_1)$. □

Lemma 4.3.24 ([Bay19, Lemma 6.1]). *Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(X)$, and let Q be a quadratic form which is negative definite on* $\mathrm{Ker}\, Z \otimes \mathbf{R}$. *Suppose $E \in \mathrm{D^b}(X)$ is strictly σ-semistable and let $A_1, \ldots, A_m$ be the JH factors of E. If $Q(E) < 0$, then for some $1 \leq j \leq m$, $Q(A_j) < 0$.*

Proof. We will prove this result by induction on the length m of the JH filtration of E. If $m = 1$, then $E = A_1$ so $Q(A_1) < 0$. Now let $m > 1$. Assume for a contradiction that $Q(A_1), Q(E/A_1) \geq 0$. Let Q also denote the associated symmetric bilinear form. By the same argument as in the proof of Lemma 4.3.23, it follows that $Q(A, E/A_1) > 0$. Therefore,

$$Q(E) = Q(A_1) + Q(E/A_1) + 2Q(A_1, E/A_1) > 0,$$

which is a contradiction. Hence either $Q(A_1) < 0$ and we are done, or $Q(E/A_1) < 0$. The JH factors of E are $A_2, \ldots A_m$, hence by induction $Q(A_i) < 0$ for some $2 \leq i \leq m$. □

We can now prove that there is a stability condition $\sigma_1 = \sigma_{H,B,\alpha_1,\beta_0}$ which satisfies the support property with respect to our new quadratic form $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$.

Lemma 4.3.25 ([D23, Lemma 5.37]). *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Choose $\delta, \varepsilon > 0$ as in Proposition 4.3.19. Fix $\alpha_1 \in \mathbf{R}$ such that $\alpha_1 > \max\left\{\alpha_0, \frac{1}{2}\left[\left(\beta_0 - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]\right\}$. Assume $E \in \mathrm{D^b}(X)$ is $\sigma_{H,B,\alpha_1,\beta_0}$-semistable. Then $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}([E]) \geq 0$. In particular, $\sigma_{H,B,\alpha_1,\beta_0}$ satisfies the support property with respect to $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$.*

Proof. To ease notation, let $Q \coloneqq Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$. From Proposition 4.3.12, we know that for every $\alpha \geq \alpha_1$, $\sigma_{H,B,\alpha,\beta_0}$ and $\sigma_{a_\alpha H, B+bH}$, have the same heart when $b = \beta_0 - \frac{H.B}{H^2}$ and $a_\alpha = \sqrt{2\alpha - b^2 + \frac{B^2}{H^2}}$.

Moreover, by Lemma 4.3.14, there exists $(H', B') \in \mathrm{Amp}_{\mathbf{Q}}(X) \times \mathrm{NS}_{\mathbf{Q}}(X)$ such that $\Delta^{C_{H'}}_{H',B'}$ gives the support property for $\sigma_{aH,B+bH}$ if $a \geq a_{\alpha_1}$. We may assume $\Delta^{C_{H'}}_{H',B'} \in \mathbf{Z}$, since it is true after rescaling by some integer. Furthermore, since E is $\sigma_{a_{\alpha_1}H,B+bH}$-semistable, $\Delta^{C_{H'}}_{H',B'}([E]) \in \mathbf{Z}_{\geq 0}$.

If E is $\sigma_{H,B,\alpha,\beta_0}$-stable for $\alpha \gg 0$, then by definition of a_α, E is $\sigma_{aH,B}$-stable for $a \gg 0$. It then follows by Proposition 4.3.22 that $Q([E]) \geq 0$. Otherwise, there exists some $\alpha_2 \geq \alpha_1$ such that E is strictly $\sigma_{H,B,\alpha_2,\beta_0}$-semistable. Let $A_1, \ldots A_m$ denote the JH factors of E. Then by Lemma 4.3.23, $\Delta^{C_{H'}}_{H',B'}([A_i]) < \Delta^{C_{H'}}_{H',B'}([E])$ for all $1 \leq i \leq m$. Each A_i is $\sigma_{H,B,\alpha_2,\beta_0}$-stable, so $\Delta^{C_{H'}}_{H',B'}([A_i]) \geq 0$ for all $1 \leq i \leq m$.

Assume for a contradiction that $Q([E]) < 0$. Then it follows from Lemma 4.3.24 that $Q([A_j]) < 0$ for some $1 \leq j \leq m$. Let $E_2 \coloneqq A_j$. We can now repeat this process for E_2 in place of $E_1 \coloneqq E$, and so on. This gives a sequence $E_1, E_2, E_3, \ldots, E_k, \ldots$ and $\alpha_1 \leq \alpha_2 < \alpha_3 \ldots < \alpha_k \ldots$ such that $E_k \in \mathrm{D^b}(X)$ is $\sigma_{H,B,\alpha_k,\beta_0}$-semistable, $Q(E_k) < 0$, and $0 \leq \Delta^{C_{H'}}_{H',B'}([E_{k+1}]) < \Delta^{C_{H'}}_{H',B'}([E_k])$ for all $k \geq 1$. But $\Delta^{C_{H'}}_{H',B'}([E_k]) \in \mathbf{Z}_{\geq 0}$ for all k, so no such sequence can exist. This gives a contradiction.

Finally, by Lemma 4.3.20, $Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$ is negative definite on $\mathrm{Ker}\, Z_{H,B,\alpha_1,\beta_0} \otimes \mathbf{R}$. □

We are finally ready to apply Theorem 3.4.2, which will allow us to extend the support property from $\sigma_1 = \sigma_{H,B,\alpha_1,\beta_0}$ to any $\sigma_{H,B\alpha,\beta_0}$ with $\alpha \geq \alpha_0$.

Proposition 4.3.26 ([D23, Proposition 5.38]). *Let X be a surface. Let $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$. Fix $\alpha_0, \beta_0 \in \mathbf{R}$ such that $\alpha_0 > \Phi_{X,H,B}(\beta_0)$. Then $\sigma_{H,B,\alpha,\beta_0} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$ for all $\alpha \geq \alpha_0$.*

Proof. Fix $\alpha_1 \in \mathbf{R}$ such that $\alpha_1 > \max\left\{\alpha_0, \frac{1}{2}\left[\left(\beta_0 - \frac{H.B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]\right\}$. By Theorem 4.3.10 and Proposition 4.3.12, it follows that $\sigma_1 \coloneqq \sigma_{H,B,\alpha_1,\beta_0} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$. The same argument applies for any $\alpha > \alpha_1$, hence it remains to prove that $\sigma_{H,B,\alpha,\beta_0} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$ for $\alpha_1 > \alpha \geq \alpha_0$.

Choose $\delta, \varepsilon > 0$ as in Proposition 4.3.19. Then by Lemma 4.3.25, σ_1 satisfies the support property with respect to $Q := Q^{\delta,\varepsilon}_{H,B,\alpha_0,\beta_0}$. For $t \in [0,1]$, let $\alpha_t := \alpha_0 + t(\alpha_1 - \alpha_0)$ and $Z_t := Z_{H,B,\alpha_t,\beta_0}$. Consider the open subset of $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$ consisting of central charges whose kernel is negative definite with respect to Q. Let U be the connected component containing Z_1. Then for all $t \in [0,1]$, $Z_t \in U$ by Lemma 4.3.20. Moreover, $\operatorname{Im} Z_{H,B,\alpha,\beta_0}$ remains constant as α varies. In particular, $\operatorname{Im} Z_t$ is constant. Therefore, the hypotheses of Theorem 3.4.2 are satisfied. It follows that for all $t \in [0,1]$ Z_t lifts to $\sigma_t = (\mathrm{Coh}^{H,\beta}(X), Z_{H,B,\alpha_t,\beta_0}) \in \mathrm{Stab}(X)$. In particular, $\sigma_{H,B,\alpha,\beta_0} \in \mathrm{Stab}^{\mathrm{Geo}}_N(X)$ for all $\alpha_1 \geq \alpha \geq \alpha_0$. □

Proof. (of Theorem 4.3.1) By Proposition 4.1.4, it is enough to show that $\mathrm{Stab}^{\mathrm{Geo}}_N(X) \cong \mathcal{U}$, where

$$\mathcal{U} = \left\{(H, B, \alpha, \beta) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X) \times \mathbf{R}^2 : \alpha > \Phi_{X,H,B}(\beta)\right\}.$$

This follows from Proposition 4.2.4, Proposition 4.3.6, and Proposition 4.3.26. □

4.4 Applications of Theorem 4.3.1

Theorem 4.4.1 ([D23, Corollary 5.39])**.** *Let X be a surface. Then $\mathrm{Stab}^{\mathrm{Geo}}(X)$ is connected.*

Theorem 4.3.1 can be used to study the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(X)$ as follows. If non-geometric stability conditions exist in the same connected component of $\mathrm{Stab}(X)$ as $\mathrm{Stab}^{\mathrm{Geo}}(X)$, then there exists $\tau \in \mathrm{Stab}(X)$ on the boundary of $\mathrm{Stab}^{\mathrm{Geo}}(X)$. This means that for some $x \in X$, $\mathcal{O}_x$ is strictly τ-semistable. To test whether this happens, we can look for walls: Let $v_0, w \in \mathrm{K}_{\mathrm{num}}(X) \setminus \{0\}$ be two non-parallel vectors. A *numerical wall* $W_w(v_0)$ for v_0 with respect to w is a non-empty subset of $\mathrm{Stab}(X)$ given by

$$W_w(v_0) := \{\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(X) : \operatorname{Re} Z(v_0) \cdot \operatorname{Im} Z(w) = \operatorname{Re}(w) \cdot (v_0)\}.$$

This gives us candidate stability conditions for which an object of class v_0 could be strictly semistable, if this happens, we call $W_w(v_0)$ an *(actual) wall.* See [MS17, S 5.5] for details on the wall and chamber structure in general. Now consider the slice $\mathcal{N} := \{\sigma : Z(\mathcal{O}_x) = -1\}$. Then

$$W_w([\mathcal{O}_x]) \cap \mathcal{N} = \{\sigma = (\mathcal{P}, Z) \in \mathcal{N} : \operatorname{Im} Z(w) = 0\}.$$

Note that $\operatorname{Im} Z(w) = 0$ is a linear condition in $\mathrm{Hom}(\mathrm{K}_{\mathrm{num}}(X), \mathbf{C})$. By the proof of Proposition 4.2.4,

$$\begin{gathered}\{Z : Z(\mathcal{O}_x) = -1\} \cong (\mathrm{NS}_{\mathbf{R}}(X))^2 \times \mathbf{R}^2 \\ Z = Z_{H,B,\alpha,\beta} \mapsto (H, B, \alpha, \beta).\end{gathered}$$

If non-geometric stability conditions exist, there will be (actual) walls which intersect the boundary of $\mathrm{Stab}^{\mathrm{Geo}}_N(X)$. In particular, this can only happen where the boundary of $\mathrm{Stab}^{\mathrm{Geo}}_N(X)$ is locally linear.

There are precisely two types of (actual) walls of the geometric chamber for K3 surfaces and rational surfaces. They either correspond to walls of the nef cone (see [TX22, Lemma 7.2] for a construction) or to discontinuities of the Le Potier function. For K3 surfaces, the second case comes from the existence of spherical bundles which is explained in [Yos09, Proposition 2.7]. For rational surfaces, the discontinuities correspond to exceptional bundles. This is explained for $\mathrm{Tot}(\mathcal{O}_{\mathbf{P}^2}(-3))$ in [BM11, §5], and the arguments generalise to any rational surface.

It seems reasonable to expect this to hold for all surfaces. The description of the geometric chamber given by Theorem 4.3.1 also supports this. Indeed, by the above discussion, a wall where $\mathcal{O}_x$ is destabilised corresponds locally to the boundary of $\mathcal{U}$ being linear. This boundary is exactly where

(1) H becomes nef and not ample. We expect that this only gives rise to walls in the following cases:
   - H is big and nef. Then H induces a contraction of rational curves. This can be used to construct non-geometric stability conditions [TX22, Lemma 7.2].
   - H is nef and induces a contraction to a curve whose fibres are rational curves. In this case, we expect a wall. For example, let $f\colon S \to C$ be a $\mathbf{P}^1$-bundle over a curve. We expect the existence of stability conditions on S such that all skyscraper sheaves are strictly semistable, and they are destabilised by

$$\mathcal{O}_{f^{-1}(x)} \to \mathcal{O}_x \to \mathcal{O}_{f^{-1}(x)}(-1)[1] \to \mathcal{O}_{f^{-1}(x)}[1].$$

(2) If $\Phi_{X,H,B}$ is discontinuous at β, then $\mathrm{Stab}_N^{\mathrm{Geo}}(X)$ locally has a linear boundary. We expect this to give rise to non-geometric stability conditions.
(3) $\alpha = \Phi_{X,H,B}(\beta)$. We expect no boundary in this case.

By the above discussion, we have the following result about walls of $\mathrm{Stab}^{\mathrm{Geo}}(X)$.

Theorem 4.4.2 ([D23, Corollary 5.41])**.** *Let X be a surface. Suppose that for all $(H, B) \in \mathrm{Amp}_{\mathbf{R}}(X) \times \mathrm{NS}_{\mathbf{R}}(X)$, $\Phi_{X,H,B}$ has no discontinuities and there exists no open $U \in \mathbf{R}$ such that $\Phi_{X,H,B}(U)$ is linear. Then any wall of $\mathrm{Stab}^{\mathrm{Geo}}(X)$ where $\mathcal{O}_x$ is destabilised corresponds to a class $H' \in \mathrm{NS}_{\mathbf{R}}(X)$ which is nef and not ample.*

Chapter 5

Group actions on categories and induced stability conditions

In this chapter we study triangulated categories $\mathcal{D}$ with actions of finite groups G. In Section 5.1 we introduce Deligne's notion of equivariant categories $\mathcal{D}_G$ which generalises the notion of equivariant sheaves on a variety with a group action. In Section 5.3 we use [MMS09] to compare stability conditions on $\mathcal{D}$ and $\mathcal{D}_G$.

5.1 Equivariant categories

Let $\mathcal{C}$ be a pre-additive category, linear over a ring k. Let G be a finite group with $(\mathrm{char}(k), |G|) = 1$. The definition of a group action on a category and the corresponding equivariant category are due to Deligne [Del97]. In this section we will follow the treatment by Elagin from [Ela15].

Definition 5.1.1 ([Ela15, Definition 3.1])**.** A *(right) action* of G on $\mathcal{C}$ is defined by the following data:

- a functor $\phi_g \colon \mathcal{C} \to \mathcal{C}$, for every $g \in G$;
- a natural isomorphism $\varepsilon_{g,h} \colon \phi_g\phi_h \to \phi_{hg}$ for every $g, h \in G$, for which all diagrams

$$\begin{array}{ccc} \phi_f\phi_g\phi_h & \xrightarrow{\varepsilon_{g,h}} & \phi_f\phi_{hg} \\ \downarrow{\scriptstyle \varepsilon_{f,g}} & & \downarrow{\scriptstyle \varepsilon_{f,gh}} \\ \phi_{gf}\phi_h & \xrightarrow{\varepsilon_{gf,h}} & \phi_{hgf} \end{array}$$

  are commutative.

Remark 5.1.2. Note that this definition of a G-action (first introduced in [Del97]) is more than a group homomorphism $G \to \mathrm{Aut}(\mathcal{C})$ as there is a fixed isomorphism $\phi_g\phi_h \xrightarrow{\sim} \phi_{hg}$ for each $g, h \in G$. This finer notion is required to define the category of G-equivariant objects in Definition 5.1.4. See [BO23, Section 2.2] for details on obstructions to lifting a group homomorphism $G \to \mathrm{Aut}(\mathcal{C})$ to a G-action.

Example 5.1.3 ([Ela15, Example 3.4])**.** Let G be a group acting on a scheme X. For each $g \in G$, let $\phi_g \coloneqq g^* \colon \mathrm{Coh}(X) \to \mathrm{Coh}(X)$. Then for all $g, h \in G$ there are canonical isomorphisms:

$$\phi_g\phi_h = g^*h^* \xrightarrow{\sim} (hg)^* = \phi_{hg}.$$

Together these define an action of G on the category $\mathrm{Coh}(X)$.

Definition 5.1.4 ([Ela15, Definition 3.5]). Suppose G acts on a category $\mathcal{C}$. A *G-equivariant object* in $\mathcal{C}$ is a pair $(F, (\theta_g)_{g\in G})$ where $F \in \operatorname{Ob}\mathcal{C}$ and $(\theta_g)_{g\in G}$ is a family of isomorphisms

$$\theta_g \colon F \to \phi_g(F),$$

such that all diagrams

$$\begin{tikzcd} F \arrow[r, "\theta_g"] \arrow[d, "\theta_{hg}"] & \phi_g(F) \arrow[d, "\phi_g(\theta_h)"] \\ \phi_{hg}(F) & \phi_g(\phi_h(F)) \arrow[l, "\varepsilon_{g,h}"'] \end{tikzcd}$$

are commutative. We call the family of isomorphisms a *G-linearisation.* A *morphism of G-equivariant objects* from $(F_1, (\theta_g^1))$ to $(F_2, (\theta_g^2))$ is a morphism $f \colon F_1 \to F_2$ compatible with θ_g, i.e. such that the below diagrams commute for all $g \in G$

$$\begin{tikzcd} F_1 \arrow[r, "\theta_g^1"] \arrow[d, "f"] & \phi_g(F_1) \arrow[d, "\phi_g(f)"] \\ F_2 \arrow[r, "\theta_g^2"] & \phi_g(F_2). \end{tikzcd}$$

The category of G-equivariant objects of $\mathcal{C}$ is denoted by $\mathcal{C}_G$. We also call $\mathcal{C}_G$ the *G-equivariantisation* of $\mathcal{C}$.

Example 5.1.5. Let G be a group acting on a scheme X with ϕ_g and $\varepsilon_{g,h}$ defined as in Example 5.1.3. G-equivariant objects in $\operatorname{Coh}(X)$ are G-equivariant coherent sheaves. Let $\operatorname{Coh}_G(X) := (\operatorname{Coh}(X))_G$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) := \mathrm{D}^{\mathrm{b}}(\operatorname{Coh}_G(X))$. Suppose $k = \overline{k}$ and G acts freely on a variety X over k. Let $\pi \colon X \to X/G$ be the quotient map. Then $\operatorname{Coh}(X/G) \cong \operatorname{Coh}_G(X)$ via $\mathcal{E} \mapsto (\pi^*\mathcal{E}, (\theta_g))$, where the linearisation is given by $\theta_g \colon \pi^*\mathcal{E} \xrightarrow{\sim} (\pi \circ g)^*\mathcal{E} = g^*\pi^*\mathcal{E}$. Thus $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong \mathrm{D}^{\mathrm{b}}(X/G)$.

There are few examples of group actions on categories which do not arise from a group action on a variety. The following result gives one such example. It will also be key in proving that $\mathcal{C}$ can be recovered from $\mathcal{C}_G$ when G is abelian, see Theorem 5.1.10.

Proposition 5.1.6 ([Ela15, p. 12]). *Suppose G is an abelian group acting on $\mathcal{C}$ and k is algebraically closed. Let $\widehat{G} = \operatorname{Hom}(G, k^*)$ be the group of 1-dimensional representations of G. Then there is an action of $\widehat{G}$ on $\mathcal{C}_G$. For every $\chi \in \widehat{G}$, on objects ϕ_χ is given by*

$$\phi_\chi((F, (\theta_h))) := (F, (\theta_h)) \otimes \chi := (F, (\theta_h \cdot \chi(h))),$$

and on morphisms ϕ_χ is the identity. For $\chi, \psi \in \widehat{G}$ the equivariant objects $\phi_\chi(\phi_\psi((F), (\theta_h)))$ and $\phi_{\psi\chi}((F, (\theta_h)))$ are the same, hence we set the isomorphisms $\varepsilon_{\chi,\psi}$ to be the identities.

There are two natural functors going between $\mathcal{C}$ and $\mathcal{C}_G$.

Definition 5.1.7. Suppose G acts on a category $\mathcal{C}$. Then we denote by $\operatorname{Forg}_G \colon \mathcal{C}_G \to \mathcal{C}$ the *forgetful functor* $\operatorname{Forg}_G(F, (\theta_g)) = F$. Also let $\operatorname{Inf}_G \colon \mathcal{C} \to \mathcal{C}_G$ be the *inflation functor* which is defined by

$$\operatorname{Inf}_G(F) := \left(\bigoplus_{g\in G} \phi_g(F), (\xi_g) \right),$$

where
$$\xi_g\colon \bigoplus_{h\in G}\phi_h(F) \xrightarrow{\sim} \bigoplus_{h\in G}\phi_g\phi_h(F)$$
is the collection of isomorphisms
$$\varepsilon_{g,h}^{-1}\colon \phi_{hg}(F) \to \phi_g\phi_h(F).$$

Lemma 5.1.8. *The forgetful functor* Forg_G *is faithful, and it is left and right adjoint to* Inf_G.

Proof. Faithfulness follows immediately from the definition of morphisms between G-equivariant objects. For the fact that Forg_G is left and right adjoint to Inf_G see [Ela15, Lemma 3.8] ☐

The following proposition builds on a result of Balmer in [Bal11, Theorem 5.17]. We will need it later to construct Bridgeland stability conditions on equivariant categories.

Proposition 5.1.9 ([Ela15, Corollary 6.10])**.** *Suppose G acts on a triangulated category $\mathcal{D}$ which has a DG-enhancement, then $\mathcal{D}_G$ is triangulated in such a way that Forg_G is exact.*

When G is abelian, the following result tells us that $\mathcal{C}$ can be recovered from $\mathcal{C}_G$ using the residual action of $\widehat{G}$. This will be an important ingredient in the proof of the main result of this chapter, Theorem 5.3.6.

Theorem 5.1.10 ([Ela15, Theorem 4.2])**.** *Suppose k is an algebraically closed field and let $\mathcal{C}$ be a k-linear idempotent complete category. Let G be a finite abelian group with $(\mathrm{char}(k), |G|) = 1$. Suppose G acts on $\mathcal{C}$. Then*
$$(\mathcal{C}_G)_{\widehat{G}} \cong \mathcal{C}.$$
In particular, under this equivalence $\mathrm{Forg}_{\widehat{G}}\colon (\mathcal{C}_G)_{\widehat{G}} \to \mathcal{C}_G$ is identified with $\mathrm{Inf}_G\colon \mathcal{C} \to \mathcal{C}_G$, and their adjoints $\mathrm{Inf}_{\widehat{G}}\colon \mathcal{C}_G \to (\mathcal{C}_G)_{\widehat{G}}$ and $\mathrm{Forg}_G\colon \mathcal{C}_G \to \mathcal{C}$ are also identified.

The proof of Theorem 5.1.10 uses comonads. In the next section we introduce the necessary background and explain why $\mathrm{Forg}_{\widehat{G}} \cong \mathrm{Inf}_G$ and $\mathrm{Inf}_{\widehat{G}} \cong \mathrm{Forg}_G$ under the equivalence.

5.2 Comonads and Elagin's reversion theorem

The following definitions are taken from [Ela15, §2]. For further details, see [BW05, Chapter 3] and [Mac71, Chapter 6]. We continue to assume that $\mathcal{C}$ is a pre-additive category, linear over a ring k.

Definition 5.2.1 ([Ela15, Definitions 2.1, 2.2])**.** A *comonad* $\mathbf{T} = (T, \varepsilon, \delta)$ on $\mathcal{C}$ consists of
(1) a functor $T\colon \mathcal{C} \to \mathcal{C}$, and
(2) natural transformations $\varepsilon\colon T \to \mathrm{Id}_{\mathcal{C}}$ and $\delta\colon T \to T^2 = TT$
such that the following diagrams commute

$$\begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"'] & T^2 \arrow[d, "T\varepsilon"] \\ T^2 \arrow[r, "\varepsilon T"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\delta"] \arrow[d, "\delta"] & T^2 \arrow[d, "T\delta"] \\ T^2 \arrow[r, "\delta T"] & T^3 \end{tikzcd}.$$

Two comonads $\mathbf{T}_1 = (T_1, \varepsilon_1, \delta_1)$ and $\mathbf{T}_2 = (T_2, \varepsilon_2, \delta_2)$ on $\mathcal{C}$ are *isomorphic* if there exists an isomorphism of functors $T \to T'$ compatible with the natural transformations ε_i and δ_i.

Definition 5.2.2 ([Ela15, Definition 2.5])**.** Let $\mathbf{T} = (T, \varepsilon, \delta)$ be a comonad on $\mathcal{C}$. A *comodule* over $\mathbf{T}$, also called a $\mathbf{T}$-coalgebra, is a pair (F, h) consisting of

(1) an object $F \in \operatorname{Ob} \mathcal{C}$,
(2) a morphism $h\colon F \to T(F)$ called the *comonad structure* such that
   (a) the composition $\varepsilon(F) \circ h\colon F \to T(F) \to F$ is the identity, and
   (b) the following diagram commutes

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[d, "h"'] & T(F) \arrow[d, "T(h)"] \\ T(F) \arrow[r, "\delta(F)"'] & T^2(F) \end{tikzcd} .$$

A morphism between two comodules over $\mathbf{T}$, $(F_1, h_1) \to (F_2, h_2)$, is a morphism $F_1 \to F_2$ in $\mathcal{C}$ compatible with the morphisms h_i (and their properties (a) and (b)).

All comodules over a given comonad $\mathbf{T}$ on $\mathcal{C}$ form a category which is denoted $\mathcal{C}_{\mathbf{T}}$.

Definition 5.2.3. There is a forgetful functor $\operatorname{Forg}_{\mathbf{T}}\colon \mathcal{C}_{\mathbf{T}} \to \mathcal{C}$ which forgets the comonad structure, i.e. $(F, h) \mapsto F$. It has a right adjoint $\operatorname{Inf}_{\mathbf{T}}\colon \mathcal{C} \to \mathcal{C}_{\mathbf{T}}$ which is defined by

$$F \mapsto (T(F), \delta(F)), \quad f \mapsto T(f)$$

for every object F and morphism f in $\mathcal{C}$.

Example 5.2.4 ([Ela15, Example 2.3])**.** Consider a pair of adjoint functors, $P^*\colon \mathcal{B} \to \mathcal{C}$ (left) and $P_*\colon \mathcal{C} \to \mathcal{B}$ (right). Let $\eta\colon \operatorname{Id}_{\mathcal{B}} \to P_* \circ P^*$ and $\varepsilon\colon P^*P_* \to \operatorname{Id}_{\mathcal{C}}$ be the unit and co-unit of the adjunction. Let $T = P^* \circ P_*$ and $\delta = P^* \circ \eta \circ P_*\colon P^* \circ P_* \to P^* \circ P_* \circ P^* \circ P_*$. Then $\mathbf{T}(P^*, P_*) := (T, \varepsilon, \delta)$ is a comonad on $\mathcal{C}$.

Now suppose $\mathcal{C}$ is additive and G is a finite group acting on $\mathcal{C}$. Then by Lemma 5.1.8, the functors $\operatorname{Forg}_G\colon \mathcal{C}_G \to \mathcal{C}$ and $\operatorname{Inf}_G\colon \mathcal{C} \to \mathcal{C}_G$ are both left and right adjoints. Hence $\mathbf{T}(\operatorname{Forg}_G, \operatorname{Inf}_G)$ defines a comonad on $\mathcal{C}$, and $\mathbf{T}(\operatorname{Inf}_G, \operatorname{Forg}_G)$ defines a comonad on $\mathcal{C}_G$.

From now on, we assume $\mathcal{C}$ is additive and G is a finite group acting on $\mathcal{C}$. The following result allows us to go between equivariant objects and comodules over the comonads associated to Inf_G and Forg_G.

Proposition 5.2.5 ([Ela15, Proposition 3.11])**.** *(1) There is an equivalence, $\mathcal{C}_G \cong \mathcal{C}_{\mathbf{T}(\operatorname{Forg}_G, \operatorname{Inf}_G)}$. Under this equivalence, $\operatorname{Forg}_G \cong \operatorname{Forg}_{\mathbf{T}(\operatorname{Forg}_G, \operatorname{Inf}_G)}$ and $\operatorname{Inf}_G \cong \operatorname{Inf}_{\mathbf{T}(\operatorname{Forg}_G, \operatorname{Inf}_G)}$.*
(2) Suppose moreover that $\mathcal{C}$ is idempotent complete. Then there is an equivalence $\mathcal{C} \cong (\mathcal{C}_G)_{\mathbf{T}(\operatorname{Inf}_G, \operatorname{Forg}_G)}$ under which $\operatorname{Inf}_G \cong \operatorname{Forg}_{\mathbf{T}(\operatorname{Inf}_G, \operatorname{Forg}_G)}$ and $\operatorname{Forg}_G \cong \operatorname{Inf}_{\mathbf{T}(\operatorname{Inf}_G, \operatorname{Forg}_G)}$.

Example 5.2.6 ([Ela15, §4])**.** Let $k[G]$ denote the regular representation of G with basis $\{e_g\}_{g \in G}$. Define $R\colon \mathcal{C}_G \to \mathcal{C}_G$ by

$$R\colon (F, (\theta_g)_{g \in G}) \mapsto k[G] \otimes (F, (\theta_g)_{g \in G}).$$

The morphism of representations $k[G] \to k$ given by $e_g \mapsto 1$ induces $\varepsilon_R\colon R \to \operatorname{id}$, a morphism of functors. The morphism of representations $k[G] \to k[G] \otimes k[G]$ given by $e_g \mapsto e_g \otimes e_g$ also induces a morphism of funtors $\delta_R\colon R \to R^2$. Altogether, $\mathcal{R} := (R, \varepsilon_R, \delta_R)$ defines a comonad on $\mathcal{C}_G$.

Proof. (of Theorem 5.1.10) We explain why $\operatorname{Forg}_{\widehat{G}} \cong \operatorname{Inf}_G$ and $\operatorname{Inf}_{\widehat{G}} \cong \operatorname{Forg}_G$. Elagin's proof that $(\mathcal{C}_G)_{\widehat{G}} \cong \mathcal{C}$ uses the following chain of equivalences.

$$(\mathcal{C}_G)_{\widehat{G}} \overset{(1)}{\cong} (\mathcal{C}_G)_{\mathbf{T}(\operatorname{Forg}_{\widehat{G}}, \operatorname{Inf}_{\widehat{G}})} \overset{(2)}{\cong} (\mathcal{C}_G)_{\mathcal{R}} \overset{(3)}{\cong} (\mathcal{C}_G)_{\mathbf{T}(\operatorname{Inf}_G, \operatorname{Forg}_G)} \overset{(4)}{\cong} \mathcal{C},$$

where $\mathbf{T}(\mathrm{Forg}_{\widehat{G}}, \mathrm{Inf}_{\widehat{G}})$, $\mathbf{T}(\mathrm{Inf}_G, \mathrm{Forg}_G)$, and $\mathcal{R}$ are the comonads on the corresponding categories defined in Example 5.2.4 and Example 5.2.6. The equivalence in (1) is from Proposition 5.2.5(1) for the action of $\widehat{G}$ on $\mathcal{C}_G$. The equivalence in (4) is from Proposition 5.2.5(2). In particular, under (1), $\mathrm{Forg}_{\widehat{G}} \cong \mathrm{Forg}_{\mathbf{T}(\mathrm{Forg}_{\widehat{G}},\mathrm{Inf}_{\widehat{G}})}$ and under (4), $\mathrm{Forg}_{\mathbf{T}(\mathrm{Inf}_G,\mathrm{Forg}_G)} \cong \mathrm{Inf}_G$. Moreover, the equivalences (2) and (3) only change the comonad structure, hence the images of the forgetful functors for each category of comodules are the same. Therefore, under the equivalence $(\mathcal{C}_G)_{\widehat{G}} \cong \mathcal{D}$, $\mathrm{Forg}_{\widehat{G}} \cong \mathrm{Inf}_G$. Finally, by Lemma 5.1.8, $\mathrm{Forg}_{\widehat{G}}$ and $\mathrm{Inf}_{\widehat{G}}$ are left and right adjoint, and as are Forg_G and Inf_G. Hence $\mathrm{Inf}_{\widehat{G}} \cong \mathrm{Forg}_G$. □

5.3 Stability conditions on equivariant categories

The goal of this section is to compare $\mathrm{Stab}(\mathcal{D})$ and $\mathrm{Stab}(\mathcal{D}_G)$, when G is a finite group acting on a triangulated category $\mathcal{D}$. Throughout, let k be an algebraically closed field, and let $\mathcal{D}$ be a k-linear essentially small Ext-finite triangulated category with a Serre functor and a DG-enhancement.

Suppose a finite group G with $(\mathrm{char}(k), |G|) = 1$ acts on $\mathcal{D}$ by exact autoequivalences, Φ_g. By Remark 3.2.3, this induces an action on the stability manifold via

$$\Phi_g \cdot (\mathcal{P}, Z) = (\Phi_g(\mathcal{P}), Z \circ (\Phi_g)_*^{-1})$$

where $(\Phi_g)_* \colon \mathrm{K}(\mathcal{D}) \to \mathrm{K}(\mathcal{D})$ is the natural morphism induced by Φ_g. We say that a stability condition σ is *G-invariant* if $\Phi_g \cdot \sigma = \sigma$. Write $(\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$ for the space of all G-invariant locally-finite stability conditions, and $(\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C}))^G$ for the space of G-invariant central charges, i.e. $Z = Z \circ (\Phi_g)_*^{-1}$ for all $g \in G$.

Let $\sigma \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$. By Lemma 5.1.8 and Proposition 5.1.9, $\mathrm{Forg}_G \colon \mathcal{D}_G \to \mathcal{D}$ is exact and faithful. This means we can apply the construction from [MMS09, §2.2], which induces a (locally-finite) stability condition on $\mathcal{D}_G$ as follows.

Define $\mathrm{Forg}_G^{-1}(\sigma) := \sigma_G = (\mathcal{P}_{\sigma_G}, Z_{\sigma_G})$, where

$$\begin{aligned} \mathcal{P}_{\sigma_G}(\phi) &:= \{\mathcal{E} \in \mathcal{D}_G : \mathrm{Forg}_G(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\}, \\ Z_{\sigma_G} &:= Z_\sigma \circ (\mathrm{Forg}_G)_*. \end{aligned}$$

Here $(\mathrm{Forg}_G)_* \colon \mathrm{K}(\mathcal{D}_G) \to \mathrm{K}(\mathcal{D})$ is the natural morphism induced by Forg_G.

Proposition 5.3.1 ([MMS09, Theorem 2.14])**.** *Suppose G acts on $\mathcal{D}$ and $\sigma = (\mathcal{P}, Z) \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$ is G-invariant. Then $\mathrm{Forg}_G^{-1}(\sigma) \in \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}_G)$.*

Proof. By Proposition 5.1.9 and our assumptions on $\mathcal{D}$, it follows that $\mathcal{D}_G$ is a triangulated category and that the assumptions stated before [MMS09, Theorem 2.14] are satisfied.

Suppose $\mathcal{E} \in \mathcal{P}(\phi)$. Then $\mathrm{Forg}_G(\mathrm{Inf}_G(\mathcal{E})) = \bigoplus_{g \in G} \Phi_g(\mathcal{E})$. Since σ is G-invariant, $\Phi_g(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)$ for all $g \in G$. Moreover, $\mathcal{P}_\sigma(\phi)$ is extension closed, hence $\bigoplus_{g \in G} \Phi_g(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)$. The result then follows from [MMS09, Theorem 2.14]. □

Remark 5.3.2. The above result is stated in greater generality in [MMS09, Theorem 2.14]. In particular, they use a functor $F \colon \mathcal{D} \to \mathcal{D}'$ to induce stability conditions from $\mathcal{D}'$ to $\mathcal{D}$ under some assumptions. We will see this later in Section 7.2.3.

The following result is an analog of Lemma 2.3.2.

Lemma 5.3.3 ([BO20, Lemma 4.3])**.** *Suppose $\sigma \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$, and let $\sigma_G = \mathrm{Forg}_G^{-1}(\sigma) \in \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}_G)$. Let $\mathcal{E} = (E, (\theta_g)) \in \mathcal{D}_G$.*

(1) $\mathcal{E}$ is σ_G-semistable if and only if $\mathrm{Forg}(\mathcal{E})$ is σ-semistable

(2) *If* $\operatorname{Forg}(\mathcal{E})$ *is* σ*-stable, then* $\mathcal{E}$ *is* σ_G*-stable.*
(3) *If* $\mathcal{E}$ *is* σ_G*-stable, then* $\operatorname{Forg}(\mathcal{E}) = \oplus_{g \in G/H} \theta_g^{-1}(\Phi_g(F))$ *for some subgroup* $H \subset G$ *and* σ*-stable object* F*. Hence* $\operatorname{Forg}(\mathcal{E})$ *is* σ*-polystable, and it is* σ*-stable if and only if it is simple.*

Proof. (1) is by definition of the construction of σ_G. The rest follows from [BO20, Lemma 4.3]. $\square$

Lemma 5.3.4 ([D23, Lemma 2.21]). *Suppose* G *is abelian and acts on* $\mathcal{D}$*. Consider the action of* $\widehat{G}$ *on* $\mathcal{D}_G$ *by tensoring as in Proposition 5.1.6. Then* $\operatorname{Forg}_G^{-1}(\sigma)$ *is* $\widehat{G}$*-invariant.*

Proof. First note that, for every class $[\mathcal{E}] = [(E, (\theta_g))] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D}_G)$, $(\operatorname{Forg}_G)_*([(E, (\theta_g))]) = [E]$. Hence $Z_{\sigma_G}([\mathcal{E}]) = Z_\sigma \circ (\operatorname{Forg}_G)_*([(E, (\theta_g))]) = Z_\sigma([E])$, where $[E] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D})$. Moreover, from the definition of $\mathcal{P}_{\sigma_G}$, we have

$$\begin{aligned}\mathcal{P}_{\sigma_G}(\phi) &= \{\mathcal{E} \in \mathcal{D}_G : \operatorname{Forg}_G(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\} \\ &= \{(E, (\theta_g)) \in \mathcal{D}_G : E \in \mathcal{P}_\sigma(\phi)\}.\end{aligned}$$

In particular, since the action of $\widehat{G}$ on $(E, (\theta_g)) \in \mathcal{D}_G$ does not change E, it follows that the central charge Z_{σ_G} and slicing $\mathcal{P}_{\sigma_G}$ are $\widehat{G}$-invariant, and hence $\sigma_G \in (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$. $\square$

Proposition 5.3.5 ([MMS09, Proposition 2.17]). *Under the hypotheses of Proposition 5.3.1, the morphism* $\operatorname{Forg}_G^{-1} \colon (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}))^G \to (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$ *is continuous, and the image of* $\operatorname{Forg}_G^{-1}$ *is a closed embedded submanifold.*

Proof. The proof of [MMS09, Proposition 2.17] is for the action of a finite group G on $\mathrm{D^b}(X)$, induced by the action of G on X, a variety over $\mathbf{C}$ (i.e. $\Phi_g = g^*$). The result follows in our setting by replacing this with the action of exact autoequivalences Φ_g on $\mathcal{D}$ in the proof. $\square$

In the case where G is abelian, we now describe the image of $\operatorname{Forg}_G^{-1}$. This leads to the following analytic isomorphism between submanifolds of $\operatorname{Stab}(\mathcal{D})$ and of $\operatorname{Stab}(\mathcal{D}_G)$.

Theorem 5.3.6 ([D23, Lemma 2.23]). *Suppose* $\mathcal{D}$ *has a DG-enhancement. Let* G *be a finite abelian group such that* $(\operatorname{char}(k), |G|) = 1$*. Suppose* G *acts on* $\mathcal{D}$ *by exact autoequivalences* Φ_g *for every* $g \in G$*, and consider the action of* $\widehat{G}$ *on* $\mathcal{D}_G$ *as in Proposition 5.1.6. Then the functors* Forg_G *and* Inf_G *induce an analytic isomorphism between* G*-invariant stability conditions on* $\mathcal{D}$ *and* $\widehat{G}$*-invariant stability conditions on* $\mathcal{D}_G$*,*

$$\operatorname{Forg}_G^{-1} \colon (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}))^G \overset{\cong}{\rightleftarrows} (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}} : \operatorname{Forg}_{\widehat{G}}^{-1}.$$

More precisely, the compositions $\operatorname{Forg}_{\widehat{G}}^{-1} \circ \operatorname{Forg}_G^{-1}$ *and* $\operatorname{Forg}_G^{-1} \circ \operatorname{Forg}_{\widehat{G}}^{-1}$ *fix slicings and rescale central charges by* $|G|$*.*

Proof. Let $\sigma \in (\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$. Therefore, by Proposition 5.3.1 and Lemma 5.3.4, $\sigma_G := \operatorname{Forg}_G^{-1}(\sigma)$ is in $(\operatorname{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$. We now apply Proposition 5.3.1 again but with $\operatorname{Forg}_{\widehat{G}}$. In particular, let $\sigma_{\widehat{G}} := \operatorname{Forg}_{\widehat{G}}^{-1}(\sigma_G)$, where

$$\begin{aligned}\mathcal{P}_{\sigma_{\widehat{G}}}(\phi) &= \{\mathcal{E} \in (\mathcal{D}_G)_{\widehat{G}} : \operatorname{Forg}_{\widehat{G}}(\mathcal{E}) \in \mathcal{P}_{\sigma_G}(\phi)\} \\ &= \{\mathcal{E} \in (\mathcal{D}_G)_{\widehat{G}} : \operatorname{Forg}_G(\operatorname{Forg}_{\widehat{G}}(\mathcal{E})) \in \mathcal{P}_\sigma(\phi)\}.\end{aligned}$$

By Proposition 5.3.1, $\operatorname{Forg}_{\widehat{G}}^{-1}(\sigma_G) \in \operatorname{Stab}_{\mathrm{lf}}((\mathcal{D}_G)_{\widehat{G}})$. To complete the proof, we need to show that, under the equivalence $(\mathcal{D}_G)_{\widehat{G}} \cong \mathcal{D}$, $\sigma_{\widehat{G}} = \sigma$ up to rescaling the central charge by $|G|$. From Theorem

5.1.10 we know that $\mathrm{Forg}_{\widehat{G}} \cong \mathrm{Inf}_G$ under this equivalence. Hence we can apply the same argument as in the proof of [MMS09, Proposition 2.17]. In particular,

$$\begin{aligned}\mathcal{P}_{\sigma_{\widehat{G}}}(\phi) &= \{\mathcal{E} \in \mathcal{D} : \mathrm{Forg}_G(\mathrm{Inf}_G(\mathcal{E})) \in \mathcal{P}_\sigma(\phi)\} \\ &= \left\{\mathcal{E} \in \mathcal{D} : \bigoplus_{g \in G} \Phi_g(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\right\}.\end{aligned}$$

Suppose $\mathcal{E} \in \mathcal{P}_{\sigma_{\widehat{G}}}(\phi)$. Since $\mathcal{P}(\phi)$ is closed under direct summands, $\Phi_g(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)$ for all $g \in G$. Thus $\mathcal{E} \in \mathcal{P}_\sigma(\phi)$. Now suppose $\mathcal{E} \in \mathcal{P}_\sigma(\phi)$, then by the proof of Proposition 5.3.1 it follows that $\mathrm{Forg}_G(\mathrm{Inf}_G(\mathcal{E})) = \oplus_{g \in G} \phi_g(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)$. Therefore, $\mathcal{E} \in \mathcal{P}_{\sigma_{\widehat{G}}}(\phi)$. In particular, $\mathcal{P}_{\sigma_{\widehat{G}}} = \mathcal{P}_\sigma$. Now let $[\mathcal{E}] \in \mathrm{K}_{\mathrm{num}}(\mathcal{D}) \otimes \mathbf{C}$ and consider the central charge

$$Z_{\sigma_{\widehat{G}}}([\mathcal{E}]) = Z_\sigma \circ (\mathrm{Forg}_G)_* \circ (\mathrm{Inf}_G)_*([\mathcal{E}]) = Z_\sigma\left(\sum_{g \in G}([\Phi_g(\mathcal{E})])\right).$$

Z_σ is G-invariant, hence $Z_\sigma([\mathcal{E}]) = (\Phi_g)_* Z_\sigma([\mathcal{E}]) = Z_\sigma([\Phi_g(\mathcal{E})])$ for all $g \in G$. Finally, since Z_σ is a homomorphism, it follows that $Z_{\sigma_{\widehat{G}}}([\mathcal{E}]) = |G| \cdot Z_\sigma([\mathcal{E}])$.

Note that if we start instead with a stability condition $\sigma_G \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$, then by a symmetric argument it follows that $\sigma_G = \mathrm{Forg}_G^{-1} \circ \mathrm{Forg}_{\widehat{G}}^{-1}(\sigma_G)$, up to rescaling the central charge by $|\widehat{G}| = |G|$. Therefore, Forg_G^{-1} and $\mathrm{Forg}_{\widehat{G}}^{-1}$ are homeomorphisms since they are continuous by Proposition 5.3.5, and rescaling the central charge is itself a homeomorphism. In fact, rescaling the central charge by $|G|$ is a linear isomorphism on $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C})$ and $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}_G), \mathbf{C})$. Hence Forg_G^{-1} and $\mathrm{Forg}_{\widehat{G}}^{-1}$ are analytic isomorphisms, since they are isomorphisms on the level of tangent spaces, i.e.

$$\begin{aligned}(\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C}))^G &\overset{\cong}{\rightleftarrows} (\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}_G), \mathbf{C}))^{\widehat{G}} \\ Z &\mapsto Z \circ \mathrm{Forg}_{\widehat{G}} \\ Z' \circ \mathrm{Forg}_G &\leftarrow\!\shortmid Z'.\end{aligned}$$

□

Remark 5.3.7. If $\mathcal{D} = \mathrm{D^b}(X)$ where X is a scheme, and if the action of G on $\mathcal{D}$ is induced by an action of G on X, i.e. $\Phi_g = g^*$, then the analytic isomorphism above gives the bijection in the abelian case of [Pol07, Proposition 2.2.3].

Remark 5.3.8. As in [BMS16, Theorem 10.1], Theorem 5.3.6 also goes through with the support property. In particular, a stability condition $\sigma \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}))^G$ satisfies support property with respect to (Λ, λ) if and only if the induced stability condition $\sigma_G \in (\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}_G))^{\widehat{G}}$ satisfies support property with respect to $(\Lambda, \lambda \circ (\mathrm{Forg}_G)_*)$.

Chapter 6

Bridgeland stability on free abelian quotients

We apply the methods of Section 5.3 to describe geometric stability conditions on free abelian quotients. In particular, we show that geometric stability conditions are preserved under the analytic isomorphism in Theorem 5.3.6, and use this to describe a union of connected components of geometric stability conditions on free abelian quotients of varieties with finite Albanese morphism. In the case of surfaces, we obtain a stronger result using a description of the set of geometric stability conditions from Chapter 4.

6.1 Inducing geometric stability conditions

Suppose G acts on a variety X. Let $Y = X/G$ and denote by $\pi\colon X \to Y$ the quotient map. In this section, we explain how Theorem 5.3.6 gives us a way to compare $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$.

Let $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$ denote the derived category of G-equivariant coherent sheaves on X as in Example 5.1.5. Recall that $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$, where the equivalence is given by

$$\begin{aligned} \Psi\colon \mathrm{D}^{\mathrm{b}}(Y) &\longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \\ \mathcal{E} &\longmapsto (\pi^*(\mathcal{E}), \lambda_{\mathrm{nat}}), \end{aligned}$$

and $\lambda_{\mathrm{nat}} = \{\lambda_g\}_{g\in G}$ is the G-linearisation given by

$$\lambda_g\colon \pi^*\mathcal{E} \xrightarrow{\sim} g^*\pi^*\mathcal{E} = (\pi\circ g)^*\mathcal{E} \cong \pi^*\mathcal{E}.$$

Now assume G is abelian. By Theorem 5.1.10, there is an equivalence $\Omega\colon \mathrm{D}^{\mathrm{b}}(X) \xrightarrow{\sim} (\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X))_{\widehat{G}}$. This fits into the following diagram of functors

$$\begin{tikzcd} \mathrm{D}^{\mathrm{b}}(Y) \arrow[r, "\Psi", "\sim"'] \arrow[d, shift left, "\pi^*"] & \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \arrow[d, shift left, "\mathrm{Inf}_{\widehat{G}}"] \arrow[dl, shift left, "\mathrm{Forg}_G"] \\ \mathrm{D}^{\mathrm{b}}(X) \arrow[r, "\Omega", "\sim"'] \arrow[u, shift left, "\pi_*"] \arrow[ur, shift left, "\mathrm{Inf}_G"] & (\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X))_{\widehat{G}} \arrow[u, shift left, "\mathrm{Forg}_{\widehat{G}}"] \end{tikzcd} \tag{6.1}$$

where

$$\pi^* \overset{\Psi}{\cong} \mathrm{Forg}_G \overset{\Omega}{\cong} \mathrm{Inf}_{\widehat{G}}\ ,\ \pi_* \overset{\Psi}{\cong} \mathrm{Inf}_G \overset{\Omega}{\cong} \mathrm{Forg}_{\widehat{G}}\ ,\ \pi_* \circ \pi^* \overset{\Psi}{\cong} \mathrm{Inf}_G \circ \mathrm{Forg}_G\ ,\ \mathrm{Forg}_G \circ \mathrm{Inf}_G \overset{\Omega}{\cong} \mathrm{Inf}_{\widehat{G}} \circ \mathrm{Forg}_{\widehat{G}}.$$

Recall that the pushforward of the structure sheaf decomposes as a direct sum of numerically trivial line bundles $\pi_*\mathcal{O}_X = \oplus_{\chi \in \widehat{G}} \mathcal{L}_\chi$, i.e. $c_1(\mathcal{L}_\chi) = 0$. The residual action of $\widehat{G}$ on $\mathrm{D^b}(Y)$ is given by $- \otimes \mathcal{L}_\chi$.

Now recall that a stability condition is called *geometric* if all skyscraper sheaves of points are stable (Definition 1.1.2). The following result shows that the analytic isomorphism from Theorem 5.3.6 preserves geometric stability.

Theorem 6.1.1 ([D23, Theorem 3.3]). *Suppose G is a finite abelian group acting freely on a variety X. Let $\pi\colon X \to Y := X/G$ denote the quotient map. Consider the action of $\widehat{G}$ on $\mathrm{D^b_G}(X) \cong \mathrm{D^b}(Y)$ as in Proposition 5.1.6. Then the functors π^*, π_* induce an analytic isomorphism between G-invariant stability conditions on $\mathrm{D^b}(X)$ and $\widehat{G}$-invariant stability conditions on $\mathrm{D^b}(Y)$ which preserve geometric stability conditions,*

$$(\pi^*)^{-1}\colon (\mathrm{Stab}(X))^G \overset{\cong}{\rightleftarrows} (\mathrm{Stab}(Y))^{\widehat{G}} : (\pi_*)^{-1}.$$

The compositions $(\pi_*)^{-1} \circ (\pi^*)^{-1}$ and $(\pi^*)^{-1} \circ (\pi_*)^{-1}$ fix slicings and rescale central charges by $|G|$.

In particular, suppose $\sigma = (\mathcal{P}_\sigma, Z_\sigma) \in (\mathrm{Stab}(X))^G$ satisfies the support property with respect to (Λ, λ). Then $(\pi^*)^{-1}(\sigma) =: \sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in (\mathrm{Stab}(Y))^{\widehat{G}}$ is defined by

$$\begin{aligned} \mathcal{P}_{\sigma_Y}(\phi) &= \{\mathcal{E} \in \mathrm{D^b}(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\}, \\ Z_{\sigma_Y} &= Z_\sigma \circ \pi^*, \end{aligned}$$

where π^* is the natural induced map on $\mathrm{K}(\mathrm{D^b}(Y))$, and σ_Y satisfies the support property with respect to $(\Lambda, \lambda \circ \pi^*)$.

Proof. Step 1 First note that $\pi_* \circ \pi^*\colon \mathrm{K_{num}}(Y) \to \mathrm{K_{num}}(Y)$ is multiplication by $|G|$, as it sends $[\mathcal{E}]$ to $\left[\mathcal{E} \otimes \bigoplus_{\chi \in \widehat{G}} \mathcal{L}_\chi\right]$ and all $\mathcal{L}_\chi$ are numerically trivial. Therefore, $\pi^*\colon \mathrm{K_{num}}(Y) \to \mathrm{K_{num}}(X)$ is injective.Now recall that under $\mathrm{D^b_G}(X) \cong \mathrm{D^b}(Y)$, $\mathrm{Forg}_G \cong \pi^*$ and $\mathrm{Inf}_G \cong \pi_*$. Together with Theorem 5.3.6 and Remark 5.3.8, it follows that $(\pi^*)^{-1}$ and $(\pi_*)^{-1}$ give an analytic isomorphism between numerical Bridgeland stability conditions as described above. It remains to show that $\sigma \in (\mathrm{Stab}(X))^G$ is geometric if and only if $\sigma_Y = (\pi^*)^{-1}(\sigma)$ is.

Step 2 Suppose $\sigma = (\mathcal{P}_\sigma, Z_\sigma) \in (\mathrm{Stab}(X))^G$ is geometric. Let $y \in Y$. This corresponds to the orbit Gx for some $x \in X$ (so x is unique up to the action of G). We need to show $\mathcal{O}_y$ is σ_Y-stable. Recall,

$$\mathcal{P}_{\sigma_Y}(\phi) = \{\mathcal{E} \in \mathrm{D^b}(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\}$$

for every $\phi \in \mathbf{R}$. Note that

$$\pi^*\mathcal{O}_y = \bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} \in \mathrm{D^b}(X).$$

By our assumption on σ and Proposition 4.1.1, all skyscraper sheaves of points of X are σ-stable and of the same phase which we denote by ϕ_{sky}. In particular, $\mathcal{O}_{g^{-1}x} \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ for all $g \in G$. Moreover, $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ is extension closed, hence $\bigoplus_{g \in G} \mathcal{O}_{g^{-1}x} \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$, and thus $\mathcal{O}_y \in \mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$. Now suppose that $\mathcal{O}_y$ is strictly semistable, then there exist objects $\mathcal{E}, \mathcal{F} \in \mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$ and the following exact sequence in $\mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$

$$\mathcal{E} \hookrightarrow \mathcal{O}_y \twoheadrightarrow \mathcal{F}$$

is non-trivial, i.e. $\mathcal{E}$ is not isomorphic to 0 or $\mathcal{O}_y$. By definition of $\mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$, the pullbacks $\pi^*(\mathcal{E}), \pi^*(\mathcal{F})$ are objects in $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$. Hence we have the following exact sequence in $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$.

$$\pi^*(\mathcal{E}) \hookrightarrow \pi^*(\mathcal{O}_y) = \bigoplus_{g\in G} \mathcal{O}_{g^{-1}x} \twoheadrightarrow \pi^*(\mathcal{F})$$

Then $\pi^*(\mathcal{E})$ is a subobject of $\pi^*(\mathcal{O}_y)$ which is a direct sum of simple objects. There can be no nontrivial morphism from $\pi^*(\mathcal{E})$ to any of these simples, hence $\pi^*(\mathcal{E}) = \bigoplus_{a\in A} \mathcal{O}_{a^{-1}x}$, where $A \subset G$ is a subset. Therefore,

$$\mathrm{supp}(\pi^*(\mathcal{E})) = \{a^{-1}x : a \in A\} \subset \{g^{-1}x : g \in G\} = \mathrm{supp}(\pi^*(\mathcal{O}_y)).$$

Note that $\pi^*(\mathcal{E})$ is a G-invariant sheaf. But $\mathrm{supp}(\pi^*(\mathcal{E}))$ is G-invariant if and only if $A = \emptyset$ or $A = G$. Hence $\mathcal{E} = 0$ or $\mathcal{E} = \mathcal{O}_y$, which is a contradiction.

Step 3 Suppose that $\sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in (\mathrm{Stab}(Y))^{\widehat{G}}$ is geometric. Recall

$$\mathcal{P}_{\sigma_Y}(\phi) = \{\mathcal{E} \in \mathrm{D^b}(Y) : \pi^*(\mathcal{E}) \in \mathcal{P}_\sigma(\phi)\},$$

for all $\phi \in \mathbf{R}$. Fix $x \in X$, and let $y \in Y$ be the point corresponding to the orbit Gx. By assumption, $\mathcal{O}_y$ is σ_Y-stable. Let ϕ_{sky} denote its phase. Then $\pi^*(\mathcal{O}_y) = \bigoplus_{g\in G} g^*\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$. Moreover, since $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ is closed under direct summands, $g^*\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ for all $g \in G$. In particular, $\mathcal{O}_x \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$. Now suppose that $\mathcal{O}_x$ is strictly semistable, then there exist $A, B \in \mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ such that

$$A \hookrightarrow \mathcal{O}_x \twoheadrightarrow B$$

is a non-trivial exact sequence in $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$, i.e. A is not isomorphic to 0 or $\mathcal{O}_x$. By Step 1, $(\pi_*)^{-1}$ sends $\mathcal{P}_\sigma(\phi_{\mathrm{sky}})$ to $\mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$. Hence we have a short exact sequence in $\mathcal{P}_{\sigma_Y}(\phi_{\mathrm{sky}})$,

$$\pi_*(A) \hookrightarrow \pi_*(\mathcal{O}_x) = \mathcal{O}_y \twoheadrightarrow \pi_*(B).$$

However, $\mathcal{O}_y$ is stable, hence $\pi_*(A) = 0$ or $\pi_*(B) = 0$. But π is finite, hence π_* is conservative. Therefore $A = 0$ or $B = 0$, which is a contradiction. □

6.2 Group actions and geometric stability conditions on surfaces

We next want to study the action of $\widehat{G}$ on $\mathrm{Stab}^{\mathrm{Geo}}(X/G)$. In this section, we will see that when S is a surface, this action is trivial (Proposition 6.2.4). We first consider the more general setting of finite subgroups of $\mathrm{Pic}^0(X)$ acting on $\mathrm{D^b}(X)$.

Lemma 6.2.1 ([D23, Lemma 3.5])**.** *Let X be a variety. Let $G \subseteq \mathrm{Pic}^0(X)$ be a finite subgroup, so this acts on $\mathrm{D^b}(X)$ by tensor product. Then the induced action of G on $\mathrm{K_{num}}(X)$ is trivial.*

Proof. We first recall the relation between $\mathrm{K_{num}}(X)$ and the numerical Chow group $\mathrm{Chow_{num}}(X)$. Consider the Chern character map $\mathrm{ch}\colon \mathrm{K}(X) \to \mathrm{Chow}(X)$, which is an isomorphism over $\mathbf{Q}$. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{K}(X) & \xrightarrow{\mathrm{ch}} & \mathrm{Chow}(X) \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle q_2} \\ \mathrm{K_{num}}(X) & \xrightarrow{\mathrm{ch}|_{\mathrm{K_{num}}}} & \mathrm{Chow_{num}}(X) \end{array},$$

where q_1 is the quotient by the null-space of the Euler form $\chi(-,-)$ on $\mathrm{K}(X)$ (see Definition 3.1.3), and q_2 is the quotient of $\mathrm{Chow}(X)$ by numerical equivalence. If $\mathrm{ch}|_{\mathrm{K}_{\mathrm{num}}(X)}([E]) = 0$, then by Hirzebruch–Riemann–Roch $\chi([E], F) = 0$ for all $F \in \mathrm{K}(X)$. Hence $[E] = 0$ in $\mathrm{K}_{\mathrm{num}}(X)$, so $\mathrm{ch}|_{\mathrm{K}_{\mathrm{num}}(X)}$ is injective.

Now let $\mathcal{L} \in G$ and $[E] \in \mathrm{K}_{\mathrm{num}}(X)$. The induced action of G on $\mathrm{K}_{\mathrm{num}}(X)$ is given by

$$[E] \mapsto \mathcal{L} \cdot [E] \coloneqq [E \otimes \mathcal{L}].$$

Since $\mathcal{L}$ is a line bundle, $\mathrm{ch}(\mathcal{L}) = e^{c_1(\mathcal{L})}$. Moreover, in the numerical Chow group $\mathrm{Chow}_{\mathrm{num}}(X)$, $c_1(\mathcal{L}) = 0$. Therefore,

$$\mathrm{ch}\,|_{\mathrm{K}_{\mathrm{num}}}([E \otimes \mathcal{L}]) = \mathrm{ch}\,|_{\mathrm{K}_{\mathrm{num}}}([E]) \cdot \mathrm{ch}\,|_{\mathrm{K}_{\mathrm{num}}}(\mathcal{L}) = \mathrm{ch}\,|_{\mathrm{K}_{\mathrm{num}}}([E]).$$

Since $\mathrm{ch}|_{\mathrm{K}_{\mathrm{num}}(X)}$ is injective, $\mathcal{L} \cdot [E] = [E \otimes \mathcal{L}] = [E]$ in $\mathrm{K}_{\mathrm{num}}(X)$. $\square$

Since $\widehat{G} \subseteq \mathrm{Pic}^0(X/G)$, acts trivially on $\mathrm{K}_{\mathrm{num}}(X/G)$. The next result tells us that if a group acts trivially on $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$, then the invariant stability conditions form a union of connected components.

Lemma 6.2.2 ([D23, Lemma 3.8]). *Suppose that a finite group G acts on a triangulated category $\mathcal{D}$ by exact autoequivalences such that the induced action on $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$ is trivial. Then $(\mathrm{Stab}(\mathcal{D}))^G$ is a union of connected components inside $\mathrm{Stab}(\mathcal{D})$.*

Proof. By Theorem 3.2.1, there is a local homeomorphism

$$\mathcal{Z} \colon \mathrm{Stab}(\mathcal{D}) \to \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C}).$$

Let $g \in G$, and denote by $(\Phi_g)_*$ the induced action of g on $\mathrm{K}(\mathcal{D})$ and $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$. Recall that the action of G on $\mathrm{Stab}(\mathcal{D})$ is given by $(\Phi_g)_* \cdot \sigma = (\Phi_g(\mathcal{P}), Z \circ (\Phi_g)_*^{-1})$. The induced action of G on $\mathrm{K}_{\mathrm{num}}(\mathcal{D})$ is trivial, hence $\mathcal{Z}(\sigma)$ is G-invariant and $\mathcal{Z}(g \cdot \sigma) = \mathcal{Z}(\sigma)$. Furthermore, G acts continuously on $\mathrm{Stab}(\mathcal{D})$, and the local homeomorphism $\mathcal{Z}$ commutes with this action. Hence the properties of being G-invariant and not being G-invariant are open in $\mathrm{Stab}(\mathcal{D})$, so the result follows. $\square$

Now let X be a surface. We saw in Theorem 4.2.1 that $\sigma \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ is determined by its central charge up to shifting by $[2n]$. This means that, to test if σ is G-invariant, we only have to check the central charge:

Lemma 6.2.3 ([D23, Corollary 3.4]). *Supposes G acts on a surface X. Then $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ is G-invariant if and only if Z is G-invariant.*

Proof. If $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ is G-invariant, then so is Z. Suppose $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$ and Z is G-invariant. Fix $g \in G$. Then $g^*\sigma = (g^*(\mathcal{P}), Z \circ (g^*)^{-1}) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$. Moreover, $g^*\mathcal{O}_x = \mathcal{O}_{g^{-1}x}$, hence skyscraper sheaves have the same phase with respect to σ and $g^*\sigma$. By Theorem 4.2.1, $\sigma = g^*\sigma$. $\square$

Proposition 6.2.4 ([D23, Corollary 3.6]). *Let X be a surface and let $G \subseteq \mathrm{Pic}^0(X)$ be a finite subgroup. Then every geometric stability condition on X is G-invariant.*

Proof. Let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}^{\mathrm{Geo}}(X)$. By Lemma 6.2.3 it is enough to show that Z is G invariant. By Lemma 6.2.1, G acts trivially on $\mathrm{K}_{\mathrm{num}}(X)$. Since σ is numerical, $Z \colon \mathrm{K}(X) \to \mathbf{C}$ factors via $\mathrm{K}_{\mathrm{num}}(X)$, hence Z is G invariant. $\square$

Example 6.2.5. Suppose G is a finite abelian group acting freely on a surface X, and let $Y \coloneqq X/G$. Then by Proposition 5.1.6 there is also an action of $\widehat{G} = \mathrm{Hom}(G, \mathbf{C})$ on $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong \mathrm{D}^{\mathrm{b}}(Y)$. As discussed in Section 6.1, the corresponding action on $\mathrm{D^b}(Y)$ is given by tensoring with a numerically trivial line bundle $\mathcal{L}_\chi$ for each $\chi \in \widehat{G}$. Proposition 6.2.4 tells us that every geometric stability condition on $\mathrm{D^b}(Y)$ is $\widehat{G}$ invariant.

6.3 Applications to varieties with finite Albanese morphism

We now apply our results to varieties with finite Albanese morphism. Such varieties only have geometric stability conditions.

Theorem 6.3.1 ([FLZ22, Theorem 1.1])**.** *If X is a variety with finite Albanese morphism, then* $\mathrm{Stab}(X) = \mathrm{Stab}^{\mathrm{Geo}}(X)$.

We now combine this with the results of Section 6.1 and Section 6.2 to study quotients of varieties with finite Albanese morphism.

Theorem 6.3.2 ([D23, Theorem 3.9])**.** *Let X be a variety with finite Albanese morphism. Let G be a finite abelian group acting freely on X and let $Y = X/G$. Then $\mathrm{Stab}^\ddagger(Y) := (\mathrm{Stab}(Y))^{\widehat{G}} \cong \mathrm{Stab}(X)^G$ is a union of connected components consisting only of geometric stability conditions.*

Proof. X has finite Albanese morphism, so it follows from Theorem 6.3.1 that all stability conditions on X are geometric. In particular, all G-invariant stability conditions on X are geometric, so from Theorem 6.1.1 it follows that all $\widehat{G}$-invariant stability conditions on Y are geometric. Hence $(\mathrm{Stab}(Y))^{\widehat{G}} \subset \mathrm{Stab}^{\mathrm{Geo}}(Y)$.

Recall from Example 6.2.5 that $\widehat{G}$ acts on $\mathrm{D^b}(Y)$ by tensoring with numerically trivial line bundles. Now we may apply Lemma 6.2.1, so it follows that $\widehat{G}$ acts trivially on $\mathrm{K_{num}}(Y)$. Hence, by Lemma 6.2.2, $(\mathrm{Stab}(Y))^{\widehat{G}}$ is a union of connected components. $\square$

When X is a surface, we have the following stronger result.

Theorem 6.3.3 ([D23, Corollary 3.10])**.** *Let X be a surface with finite Albanese morphism. Let G be a finite abelian group acting freely on X. Let $S = X/G$. Then $\mathrm{Stab}^\ddagger(S) = \mathrm{Stab}^{\mathrm{Geo}}(S) \cong (\mathrm{Stab}(X))^G$. In particular, $\mathrm{Stab}^\ddagger(S)$ is a contractible connected component of $\mathrm{Stab}(S)$.*

Proof. By Theorem 6.3.2, $(\mathrm{Stab}(S))^\ddagger \subset \mathrm{Stab}^{\mathrm{Geo}}(S)$ is a union of connected components. Moreover, $\mathrm{Stab}^{\mathrm{Geo}}(S)$ is connected by Theorem 4.4.1and contractible by [Rek23, Theorem A]. Hence $\mathrm{Stab}^\ddagger(S) = \mathrm{Stab}^{\mathrm{Geo}}(S)$, and the result follows. $\square$

Remark 6.3.4.
(1) The fact that $\mathrm{Stab}^{\mathrm{Geo}}(S) = (\mathrm{Stab}(S))^{\widehat{G}}$ also follows from Theorem 6.3.2 together with Proposition 6.2.4.
(2) $\mathrm{Stab}^\ddagger(S) = \mathrm{Stab}^{\mathrm{Geo}}(S)$ is explicitly described in Theorem 4.3.1.

Example 6.3.5. Let $S = (C_1 \times C_2)/G$ be the quotient of a product of smooth curves such that $g(C_1)$, $g(C_2) \geq 1$ and G is a finite abelian group acting freely on S. Then $C_1 \times C_2$ has finite Albanese morphism. By Theorem 6.3.3, $\mathrm{Stab}^{\mathrm{Geo}}(S)$ is a connected component. In particular, we could take S to be any bielliptic surface (see Example 1.2.2) or a Beauville-type surface with G abelian (see Example 1.2.1).

Remark 6.3.6. For an ample class H on a variety of dimension n, consider the following surjection from $K(X)$,

$$[E] \mapsto (H^n \mathrm{ch}_0(E), H^{n-1} \,.\, \mathrm{ch}_1(E), \dots, \mathrm{ch}_n(E)) \subseteq \mathbf{R}^n.$$

Let Λ_H denote the image. The submanifold $\mathrm{Stab}_H(X) := \mathrm{Stab}_{\Lambda_H}(X) \subseteq \mathrm{Stab}(X)$ is often studied. Note that these are the same when X has Picard rank 1.

Now let X be a surface with finite Albanese morphism, and let G be an abelian group acting freely on X. Let $S = X/G$, and denote by $\pi\colon X \to S$ the quotient map. Moreover, let H_X be a G-invariant polarization of X and let H_S be the corresponding polarization on S such that $\pi^* H_S = H_X$. Then if a homomorphism $Z\colon K(X) \to \mathbf{C}$ factors via Λ_{H_X}, it is G-invariant. Hence by Lemma 6.2.3, all stability conditions in $\mathrm{Stab}_{H_X}(X)$ are G-invariant.

From Theorem 6.3.3 it follows that $\mathrm{Stab}^{\ddagger}_{H_S}(S) \cong (\mathrm{Stab}_{H_X}(X))^G = \mathrm{Stab}_{H_X}(X)$. $\mathrm{Stab}_{H_X}(X)$ is the same as the component described in [FLZ22, Corollary 3.7]. This gives another proof that $\mathrm{Stab}^{\ddagger}_{H_S}(S) = \mathrm{Stab}^{\mathrm{Geo}}_{H_S}(S)$ is connected and contractible.

Example 6.3.7. A *Calabi–Yau threefold of abelian type* is an étale quotient $Y = X/G$ of an abelian threefold X by a finite group G acting freely on X such that the canonical line bundle of Y is trivial and $H^1(Y, \mathbf{C}) = 0$. As discussed in [BMS16, Example 10.4(i)], these are classified in [OS01, Theorem 0.1]. In particular, G can be chosen to be $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ or D_4 (the dihedral group of order 8), and the Picard rank of Y is 3 or 2 respectively.

Fix a polarization (Y, H), and consider $\mathrm{Stab}_H(Y)$ as in Remark 6.3.6. This has a connected component $\mathfrak{P}$ of geometric stability conditions induced from $\mathrm{Stab}_H(X)$ [BMS16, Corollary 10.3] which is described explicitly in [BMS16, Lemma 8.3]. When $G = (\mathbf{Z}/2)^{\oplus 2}$, by [OPT22, Theorem 3.21], the stability conditions constructed by Bayer–Macrì–Stellari in $\mathrm{Stab}_H(X)$ satisfy the full support property, i.e. they actually lie in $\mathrm{Stab}(X)$. Together with Theorem 6.3.2, it follows that $\sigma \in \mathfrak{P}$ also satisfies the full support property. In particular, $\mathfrak{P}$ lies in a connected component of $\mathrm{Stab}^{\ddagger}(Y)$.

Chapter 7

Fusion actions on categories and equivariant stability conditions

In this chapter, we study categories with actions of non-abelian groups, and more generally module categories over *fusion categories* – monoidal categories with strong finiteness properties. The main result of this chapter is Theorem 7.3.10, which generalises Theorem 5.3.6 to non-abelian groups.

We first introduce fusion categories and module categories over them in Section 7.1. In Section 7.2 we define *fusion-equivariant* stability conditions, and prove that they form a closed submanifold of the stability manifold. This generalises the notion of G-invariant stability conditions on a triangulated category $\mathcal{D}$ that we saw in Section 5.3. Finally, in Section 7.3 we prove Theorem 7.3.10, which describes an analytic isomorphism between G-invariant stability conditions on $\mathcal{D}$ and $\mathrm{rep}(G)$-invariant stability conditions on the equivariant category $\mathcal{D}_G$.

7.1 Preliminaries on fusion categories and module categories

7.1.1 Monoidal categories

We start by recalling the necessary definitions from the theory of monoidal categories. The following definitions are taken from [Eti+15, Chapter 2].

Definition 7.1.1 ([Eti+15, Definition 2.1.1])**.** A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$ where

(1) $\mathcal{C}$ is a category,
(2) $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a\colon (-\otimes-)\otimes- \xrightarrow{\sim} -\otimes(-\otimes-)$ is a natural isomorphism:

$$a_{X,Y,Z}\colon (X\otimes Y)\otimes Z \xrightarrow{\sim} X\otimes(Y\otimes Z), \qquad X, Y, Z \in \mathcal{C}$$

called the *associativity constraint*,
(3) $\mathbb{1} \in \mathcal{C}$ is an object of $\mathcal{C}$, and
(4) $\iota\colon \mathbb{1}\otimes\mathbb{1} \xrightarrow{\sim} \mathbb{1}$ is an isomorphism,

subject to the following two axioms.

(1) (The pentagon axiom.) The diagram

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "a_{W \otimes X,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X \otimes Y,Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y \otimes Z}"'] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

(2) (The unit axiom.) The functors

$$L_1 \colon X \mapsto \mathbb{1} \otimes X,$$
$$R_1 \colon X \mapsto X \otimes \mathbb{1}$$

of left and right multiplication by $\mathbb{1}$ are autoequivalences of $\mathcal{C}$. These functors are called the *unit constraints*.

We equivalently say that $\mathcal{C}$ has a monoidal structure $(\otimes, a., \iota)$.

Definition 7.1.2 ([Eti+15, Definition 2.4.1])**.** Let $(\mathcal{C}, \otimes, \mathbb{1}, a, i)$ and $(\mathcal{C}', \otimes', \mathbb{1}', a', i')$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a pair (F, J) where $F \colon \mathcal{C} \to \mathcal{C}'$ is a functor, and

$$J_{X,Y} \colon F(X) \otimes' F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

is a natural isomorphism such that $F(\mathbb{1})$ is isomorphic to $\mathbb{1}'$ and the diagram

$$\begin{tikzcd}
(F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "a'_{F(X),F(Y),F(Z)}"] \arrow[d, "J_{X,Y} \otimes' \mathrm{id}_{F(Z)}"'] & F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[d, "\mathrm{id}_{F(X)} \otimes' J_{Y,Z}"] \\
F(X \otimes Y) \otimes' F(Z) \arrow[d, "J_{X \otimes Y,Z}"'] & F(X) \otimes' F(Y \otimes Z) \arrow[d, "J_{X,Y \otimes Z}"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a_{X,Y,Z})"] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

is commutative for all X, Y, Z in $\mathcal{C}$.

A monoidal functor is said to be an *equivalence of normal categories* if it is an equivalence of ordinary categories.

Definition 7.1.3 ([Eti+15, Definition 2.8.1])**.** A monoidal category $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$ is called strict if

(1) $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$,

(2) $X \otimes \mathbb{1} = X = \mathbb{1} \otimes X$,

and the associativity and unit constraints are the identity maps.

Example 7.1.4. Let $\mathcal{C}$ be a category. Consider the category $\mathrm{End}(\mathcal{C})$ of all endofunctors $\mathcal{C} \to \mathcal{C}$ with tensor product given by composition. This is a strict monoidal category.

In the following definition, we suppress the unit constraints to ease notation.

Definition 7.1.5 ([Eti+15, Definition 2.10.1])**.** Let $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$ be a monoidal category and let $X \in \mathcal{C}$. An object ${}^{\vee}X$ in $\mathcal{C}$ is said to be a *left dual* of X if there exists morphisms $\mathrm{ev} \colon {}^{\vee}X \otimes X \to \mathbb{1}$ and

$\mathrm{coev}\colon \mathbb{1} \to X \otimes {}^\vee X$, called *evaluation* and *coevaluation*, such that the compositions

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes {}^\vee X) \otimes X \xrightarrow{a_{X,{}^\vee X,X}} X \otimes ({}^\vee X \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X,$$
$${}^\vee X \xrightarrow{\mathrm{id}_{{}^\vee X} \otimes \mathrm{coev}_X} {}^\vee X \otimes (X \otimes {}^\vee X) \xrightarrow{a_{{}^\vee X,X,{}^\vee X}} ({}^\vee X \otimes X) \otimes {}^\vee X \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{{}^\vee X}} X,$$

are the identity morphisms.

An object $X^\vee$ in $\mathcal{C}$ is said to be a *right dual* of X if there exists morphisms $\mathrm{ev}'\colon X \otimes X^\vee \to \mathbb{1}$ and $\mathrm{coev}'\colon \mathbb{1} \to X^\vee \otimes X$, such that the corresponding compositions to those for left duals are also the identities.

Example 7.1.6. In $\mathrm{End}(\mathcal{C})$, a left (resp. right) dual is the same as a left (resp. right) adjoint.

7.1.2 Fusion categories

We next review the notion of a fusion category and some of its properties. We refer the reader to [ENO05; Eti+15] for further details. Throughout this chapter, k will be an algebraically closed field.

Definition 7.1.7. A (strict) *fusion category* $\mathcal{C}$ is a k-linear (i.e. hom spaces are finite-dimensional k-vector spaces), abelian category with a (strict) monoidal structure $(- \otimes -, \mathbb{1})$ such that

(1) the monoidal unit $\mathbb{1}$ is simple;
(2) $\mathcal{C}$ is semisimple (i.e. any object is a direct sum of finitely many simples);
(3) $\mathcal{C}$ has finitely many isomorphism classes of simple objects; and
(4) every object $C \in \mathcal{C}$ has a left dual ${}^\vee C$ and a right dual $C^\vee$ with respect to the tensor product $\otimes$. This is often called *rigidity*.

A (finite) set of representatives of isomorphism classes of simple objects in $\mathcal{C}$ will be denoted by $\mathrm{Irr}(\mathcal{C})$.

Example 7.1.8. (1) The category of finite dimensional k-vector spaces, vec, is a fusion category. More generally, let G be a finite group. Then the category of G-graded vector spaces, vec_G, is fusion. The simples are given by $g \in G$ representing the one-dimensional vector space with grading concentrated in g.

(2) Let G be a finite group. If $(\mathrm{char}(k), |G|) = 1$, then the category of representations of G, $\mathrm{rep}(G)$, is fusion. If $(\mathrm{char}(k), |G|) \neq 1$ and $|G| > 1$, then $\mathrm{rep}(G)$ is not a fusion category since it is not semisimple [Eti+11, Proposition 3.2].

(3) The Fibonacci fusion category Fib is the fusion category with two simple objects $\mathbb{1}, \Pi$ and fusion rules
$$\Pi \otimes \Pi \cong \mathbb{1} \oplus \Pi. \tag{7.1}$$
As a fusion category, Fib is unique up to equivalence. It also called the *golden ratio* fusion category.

Remark 7.1.9. Let $\mathcal{C}$ be a fusion category. Its Grothendieck group $\mathrm{K}(\mathcal{C})$ (see Definition 3.1.3) is also a fusion category, with tensor product given by $[X] \otimes [Y] := [X \otimes Y]$. $\mathrm{K}(\mathcal{C})$ has a basis given by the (finite) isomorphism classes of simples $\{[S_j]\}_{j=0}^n$. The tensor product defines multiplication on $\mathrm{K}(\mathcal{C})$ by $[S_k] \cdot [S_i] := [S_k \otimes S_i]$. This satisfies

$$[S_k] \cdot [S_i] = \sum_{i=0}^{n} {}_k r_{i,j} [S_j], \quad {}_k r_{i,j} \in \mathbf{Z}_{\geq 0},$$

where ${}_k r_{i,j}$ is the multiplicity of the simple S_j in the direct sum decomposition of $S_k \otimes S_i$. $[S_0] = [\mathbb{1}]$ is the multiplicative unit. Hence we call $\mathrm{K}(\mathcal{C})$ the *fusion ring* of $\mathcal{C}$.

There is a well-defined notion of dimension for non-zero objects in a fusion category.

Proposition 7.1.10 ([Eti+15, Proposition 3.3.6(3)]). *Let $\mathcal{C}$ be a fusion category. Any ring homomorphism $\varphi\colon \mathrm{K}(\mathcal{C}) \to \mathbf{R}$ satisfying the following properties is unique:*

(1) $\varphi([X]) \geq 0$ for all $X \in \mathcal{C}$; and

(2) $\varphi([X]) = 0$ if and only if $X \cong 0 \in \mathcal{C}$.

The above proposition makes the following definition of dimension unique.

Definition 7.1.11. Let $\mathcal{C}$ be a fusion category with $n+1$ simple objects $\{S_k\}_{k=0}^{n}$. For each simple S_k, we define $\mathrm{FPdim}([S_k]) \in \mathbf{R}$ to be the Frobenius–Perron eigenvalue (the unique largest real eigenvalue) of the non-negative matrix $({}_k r_{i,j})_{i,j\in\{0,1,\ldots,n\}}$. This extends to a ring homomorphism [Eti+15, Proposition 3.3.6(1)] which we denote by

$$\mathrm{FPdim}\colon \mathrm{K}(\mathcal{C}) \to \mathbf{R}.$$

The *Frobenius–Perron dimension*, $\mathrm{FPdim}(X)$, of an object $X \in \mathcal{C}$ is defined to be $\mathrm{FPdim}([X])$.

Note that FPdim is not necessarily integral. It is unique in the following sense.

Remark 7.1.12. If $X \in \mathcal{C}$ is non-zero, then $\mathrm{FPdim}(X) \geq 1$ [Eti+15, Proposition 3.3.4].

Example 7.1.13. (1) In both Example 7.1.8(1) and (2), $\mathrm{FPdim}(X)$ agrees with $\dim(X)$, the dimension of the underlying vector space. Indeed, $\dim$ defines a ring homomorphism $\dim\colon \mathrm{K}(\mathcal{C}) \to \mathbf{R}$ satisfying the conditions in Proposition 7.1.10. More generally, any fusion category with a fibre functor (i.e. an exact k-linear functor that commutes with tensor products) to the category of vector spaces has $\mathrm{FPdim} = \dim$.

(2) For the fusion category Fib from Example 7.1.8(3), the relation in (7.1) dictates that Π must have $\mathrm{FPdim}(\Pi) = \frac{1}{2}(1+\sqrt{5})$, i.e. the golden ratio.

7.1.3 Module categories over fusion categories

Definition 7.1.14. A left (resp. right) *module category* over a fusion category $\mathcal{C}$ is an additive category $\mathcal{M}$ together with an additive monoidal functor

$$\mathcal{C} \ \left(\text{resp. } \mathcal{C}^{\otimes^{\mathrm{op}}}\right) \ \to \mathrm{End}(\mathcal{M})$$

into the additive, monoidal category of additive endofunctors $\mathrm{End}(\mathcal{M})$. We will also say that $\mathcal{C}$ *acts* on $\mathcal{M}$. If $\mathcal{M}$ is moreover abelian or triangulated, then we insist that $\mathcal{C}$ acts on $\mathcal{M}$ by exact endofunctors.

Notation 7.1.15. Suppose $\mathcal{C}$ acts on $\mathcal{M}$. Given an object $C \in \mathcal{C}$, we denote the associated endofunctor by $C \otimes -$.

Remark 7.1.16. If $\mathcal{M}$ is a left (resp. right) module category over $\mathcal{C}$, then by definition the monoidal structures of $\mathrm{End}(\mathcal{M})$ and $\mathcal{C}$ have to be compatible. This means that, for any $C, D \in \mathcal{C}$, there is a fixed isomorphism of functors $\varepsilon_{C,D}\colon C \otimes (D \otimes -) \xrightarrow{\sim} (C \otimes D) \otimes -$ (resp. $(D \otimes C) \otimes -$), and these morphisms are compatible with the coherence conditions for the natural transformations that are part of the monoidal structures.

Suppose $\mathcal{M}$ is an abelian or triangulated category, and $\mathcal{M}$ is a left (resp. right) module category over a fusion category $\mathcal{C}$. Then $\mathrm{K}(\mathcal{M})$ is a left (resp. right) module category over the fusion ring $\mathrm{K}(\mathcal{C})$, with action given by

$$[C] \cdot [A] \coloneqq [C \otimes A] \in \mathrm{K}(\mathcal{M}).$$

Example 7.1.17. (1) Every additive, k-linear category $\mathcal{A}$ is naturally a module category over the category of finite dimensional k-vector spaces vec. The action of $k^m \in \mathrm{vec}$ is defined on $A \in \mathcal{A}$

by $k^m \boxtimes A := A^{\oplus m}$, and every morphism $A \xrightarrow{f} A'$ in $\mathcal{A}$ is sent to $k^m \boxtimes f \colon A^{\oplus m} \xrightarrow{\bigoplus_{i=1}^m f} A'^{\oplus m}$. Each $n \times m$ matrix $M \in \mathrm{Hom}_{\mathrm{vec}}(k^m, k^n)$ can be viewed as a morphism from $A^{\oplus m}$ to $A^{\oplus n}$ for each $A \in \mathcal{A}$ (using k-linearity of $\mathcal{A}$), so it defines a natural transformation. The whole category vec acts on $\mathcal{A}$ by a choice of equivalence to its skeleton generated by k.

(2) Any fusion category is always a left (and right) module category over itself. Each object C gets sent to the functor $C \otimes -$ (resp. $- \otimes C$).

The examples of module categories that we will study in more detail in Section 7.3 come from actions of finite groups on categories (and, as we will see, actions of $\mathrm{rep}(G)$ on G-equivariant categories).

Definition 7.1.18. Let G be a finite group. Consider the monoidal category $\mathrm{Cat}(G)$ defined as follows

- objects are elements of G,
- morphisms are identities,
- tensor product is multiplication in G (so the identity of G is the monoidal unit).

A left (resp. right) *action of* G on a category $\mathcal{M}$ is an additive monoidal functor

$$\Phi \colon \mathrm{Cat}(G) \text{ (resp. } \mathrm{Cat}(G)^{\otimes \mathrm{op}}) \to \mathrm{End}(\mathcal{M}).$$

If $\mathcal{M}$ is moreover abelian or triangulated, we require that the functors $\Phi_g := \Phi(g)$ are exact.

Remark 7.1.19. (1) Definition 7.1.18 agrees with Definition 5.1.1: Recall that a (left) action of G consists of an assignment of an (invertible) endofunctor $\Phi_g \in \mathrm{End}(\mathcal{M})$ for each $g \in G$, together with an isomorphism of functors $\varepsilon_{g,h} \colon \Phi_g \circ \Phi_h \xrightarrow{\sim} \Phi_{gh}$ for each pair $g, h \in G$, which satisfy $\varepsilon_{gh,k} \circ \varepsilon_{g,h} = \varepsilon_{g,hk} \circ \varepsilon_{h,k}$. The compatibility data is encapsulated by the term "monoidal functor" in Definition 7.1.18.

(2) An action of G determines an action of vec_G: We need to define an additive monoidal functor $\Psi \colon \mathrm{vec}_G \to \mathrm{End}(\mathcal{M})$. Since vec_G is semisimple, it is enough to specify what happens to the simples $g \in G$. Indeed, let $\Psi(g) := \Phi(g)$, then this extends to any G-graded vector space as follows

$$\Psi \colon \oplus_{g \in G} V_g \mapsto \sum_{g \in G} \dim V_g \Phi(g) \in \mathrm{End}(\mathcal{M}).$$

The data of the natural isomorphisms $\Phi(g) \otimes \Phi(h) \xrightarrow{\sim} \Phi(gh)$ then also determines natural isomorphisms $\Psi(X) \otimes \Psi(Y) \xrightarrow{\sim} \Psi(X \otimes Y)$ for any $X, Y \in \mathrm{vec}_G$.

(3) An action of vec_G determines an action of G: the action of vec_G specifies an endofunctor $\Phi(g)$ for each simple $g \in G$. The rest of the data of a G-action can again be determined by the natural isomorphisms of the action of vec_G.

7.2 The closed submanifold of fusion-equivariant stability conditions

Throughout this section, let $\mathcal{D}$ be a k-linear essentially small Ext-finite triangulated category with a Serre functor. Fix a surjective group homomorphism $\lambda \colon \mathrm{K}(\mathcal{D}) \to \Lambda$, and let $\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}_\Lambda(\mathcal{D})$. In Section 5.3 we saw that, if a finite group G acts on a triangulated category $\mathcal{D}$ by exact autoequivalences, then there is an induced action on $\mathrm{Stab}(\mathcal{D})$. Moreover, $(\mathrm{Stab}(\mathcal{D}))^G$ is a *closed* submanifold:

Theorem 7.2.1 ([MMS09, Theorem 1.1])**.** *The subset of G-invariant stability conditions $(\mathrm{Stab}(\mathcal{D}))^G$ is a closed submanifold of $\mathrm{Stab}(\mathcal{D})$.*

In this section we will generalise this to module categories over fusion categories. Without loss of generality, we only work with *left* module categories.

7.2.1 Fusion-equivariant stability conditions

Recall from Section 5.3 that $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{D})$ is called G-invariant if for all $g \in G$,

$$(\Phi_g(\mathcal{P}), Z \circ (\Phi_g)_*^{-1}) = (\mathcal{P}, Z).$$

This definition can be reformulated as follows.

Definition 7.2.2. Suppose $\mathcal{D}$ is equipped with an action of a finite group G (see Definition 7.1.18). A stability condition $(\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{D})$ is *G-invariant* if:

(1) $\Phi_g(\mathcal{P}(\phi)) = \mathcal{P}(\phi)$ for all $g \in G$ and for all $\phi \in \mathbf{R}$; and

(2) $Z \in \operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})^G \subseteq \operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})$,

where $\operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})^G$ denotes the $\mathbf{C}$-linear subspace of G-invariant homomorphisms Z, i.e. those satisfying $Z(\Phi_g(X)) = Z(X)$ for all $X \in \mathcal{D}$ and $g \in G$. We use $(\operatorname{Stab}(\mathcal{D}))^G \subseteq \operatorname{Stab}(\mathcal{D})$ to denote the subset of all G-invariant stability conditions.

We now generalise this to the case that $\mathcal{D}$ is equipped with the structure of a module category over a fusion category $\mathcal{C}$. We already saw that the triangulated $\mathcal{C}$-module category structure on $\mathcal{D}$ induces a $\mathrm{K}(\mathcal{C})$-module structure on its Grothendieck group $\mathrm{K}(\mathcal{D})$. On the other hand, the ring homomorphism induced by the Frobenius–Perron dimension (see Definition 7.1.11),

$$\operatorname{FPdim}\colon \mathrm{K}(\mathcal{C}) \to \mathbf{R} \subset \mathbf{C},$$

provides $\mathbf{C}$ with a $\mathrm{K}(\mathcal{C})$-module structure as well. It therefore makes sense to consider the $\mathbf{C}$-linear subspace of $\mathrm{K}(\mathcal{C})$-module homomorphisms inside $\operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})$. These are exactly the homomorphisms $Z\colon \mathrm{K}(\mathcal{C}) \to \mathbf{C}$ such that

$$Z([C] \cdot [X]) = Z([C \otimes X]) = \operatorname{FPdim}(C) Z([X])$$

for all $C \in \mathcal{C}$ and $[X] \in \mathrm{K}(\mathcal{D})$. We denote the subspace of such homomorphisms by $\operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})$.

We use this to define fusion-equivariant stability conditions as follows.

Definition 7.2.3. Let $\mathcal{D}$ be a triangulated module category over $\mathcal{C}$ (see Definition 7.1.14). A stability condition $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}(\mathcal{D})$ is *fusion-equivariant* over $\mathcal{C}$, or *$\mathcal{C}$-equivariant*, if:

(1) $C \otimes \mathcal{P}(\phi) \subseteq \mathcal{P}(\phi)$ for all $C \in \mathcal{C}$ and all $\phi \in \mathbf{R}$; and

(2) $Z \in \operatorname{Hom}_{\mathrm{K}(\mathcal{C})}(\mathrm{K}(\mathcal{D}), \mathbf{C}) \subseteq \operatorname{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})$.

We use $\operatorname{Stab}_{\mathcal{C}}(\mathcal{D}) \subseteq \operatorname{Stab}(\mathcal{D})$ to denote the subset of all $\mathcal{C}$-equivariant stability conditions.

Example 7.2.4. (1) When $\mathcal{C} = \mathrm{vec}$ acts on $\mathcal{D}$, all stability conditions are vec-equivariant.

(2) Let G be a finite group acting on $\mathcal{D}$; equivalently $\mathcal{D}$ is a module category over vec_G. Then a stability condition $\sigma \in \operatorname{Stab}(\mathcal{D})$ is G-invariant if and only if it is vec_G-equivariant.

If $\sigma \in \operatorname{Stab}_{\mathcal{C}}(\mathcal{D})$, then each $\mathcal{P}(\phi)$ is itself an abelian module category over $\mathcal{C}$. It follows that the standard heart $\mathcal{A} \coloneqq \mathcal{P}(0, 1]$ is an abelian module category over $\mathcal{C}$; in fact any $\mathcal{P}(a, a+1]$ is. Hence the Grothendieck group $\mathrm{K}(\mathcal{A}) \cong \mathrm{K}(\mathcal{D})$ also has a module structure over $\mathrm{K}(\mathcal{C})$. With the $\mathrm{K}(\mathcal{C})$-module structure on $\mathbf{C}$, we see that the induced stability function $Z\colon \mathrm{K}(\mathcal{A}) \to \mathbf{C}$ on $\mathcal{A}$ is a $\mathrm{K}(\mathcal{C})$-module homomorphism. Motivated by this, we make the following definition.

Definition 7.2.5. Let $\mathcal{A}$ be an abelian module category over $\mathcal{C}$. A stability function $Z\colon \mathrm{K}(\mathcal{A}) \to \mathbf{C}$ is *$\mathcal{C}$-equivariant* if Z is a $\mathrm{K}(\mathcal{C})$-module homomorphism.

The following crucial result puts restrictions on the behaviour of semistable objects under the action of a fusion category $\mathcal{C}$ when a stability function is $\mathcal{C}$-equivariant and satisfies the HN property. We emphasise here that $C \otimes -$ need not be invertible.

Proposition 7.2.6 ([Hen22, Proposition 3.3.4], [DHL24, Proposition 3.13]). *Let $\mathcal{A}$ be an abelian module category over a fusion category $\mathcal{C}$. Suppose $Z \in \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\mathrm{K}(\mathcal{A}), \mathbf{C})$ is a $\mathcal{C}$-equivariant stability function on $\mathcal{A}$ that satisfies the Harder–Narasimhan property. Then for any object A in $\mathcal{A}$, the following are equivalent:*

(1) A is semistable of phase ϕ;
(2) $C \otimes A$ is semistable of phase ϕ for all non-zero $C \in \mathcal{C}$; and
(3) $C \otimes A$ is semistable of phase ϕ for some non-zero $C \in \mathcal{C}$.

Remark 7.2.7. The triangulated category analogue of Proposition 7.2.6 does not hold. Namely Definition 7.2.3(1) does not necessarily follow from Definition 7.2.3(2). This can already be seen in the case of $\mathcal{C} = \mathrm{vec}_G$, i.e. there are stability conditions whose central charge is G-invariant but has some $\mathcal{P}(\phi)$ not closed under the action of G. This stresses the importance of the hypothesis that the abelian category in Proposition 7.2.6 is a module category over $\mathcal{C}$.

We can now state the following variant of Proposition 3.3.6.

Theorem 7.2.8 ([Hen22, Theorem 3.3.5], [DHL24, Theorem 3.15]). *A $\mathcal{C}$-equivariant stability condition on $\mathcal{D}$ is equivalent to the data of a heart of a bounded t-structure $\mathcal{A}$ on $\mathcal{D}$ such that $\mathcal{C} \otimes \mathcal{A} \subseteq \mathcal{A}$, together with a $\mathcal{C}$-equivariant stability function on $\mathcal{A}$ that satisfies the Harder–Narasimhan property.*

Proof. The proof of ($\Rightarrow$) follows from the discussion before Definition 7.2.5. The proof for ($\Leftarrow$) follows from the usual construction of a stability condition from a heart of a bounded t-structure with a stability function that satisfies the Harder–Narasimhan property together with Proposition 7.2.6. $\square$

7.2.2 Submanifold property

The forgetful map $\mathcal{Z}$ in Bridgeland's deformation theorem (Theorem 3.2.1) maps $\sigma \in \mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$ into the $\mathbf{C}$-linear subspace $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\mathrm{K}(\mathcal{D}), \mathbf{C}) \subseteq \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C})$ of $\mathrm{K}(\mathcal{C})$-module morphisms. To show that $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$ is a submanifold, it will suffice to show that any local deformation of $Z = \mathcal{Z}(\sigma)$ within the linear subspace $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\mathrm{K}(\mathcal{D}), \mathbf{C})$ lifts to a stability condition ς that is still $\mathcal{C}$-equivariant.

Lemma 7.2.9 ([DHL24, Lemma A.3]). *Suppose $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}(\mathcal{D})$ in Proposition 3.4.1 is moreover $\mathcal{C}$-equivariant. Then there exists $V \subset U$ so that, if $W \in V$ is $\mathcal{C}$-equivariant, the lifted stability condition $\varsigma = (\mathcal{Q}, W) \in \mathcal{U}$, is also $\mathcal{C}$-equivariant.*

Proof. Since U is open, there exists $\varepsilon > 0$ such that the open ball $V := B_\varepsilon(Z)$ is contained in U. Therefore, there is a path W_t in V parametrised by $t \in [0,1]$ such that $W_0 = Z$, $W_1 = W$, and there exists $T \in [0,1]$ such that

(1) $\mathrm{Im}\, W_t = \mathrm{Im}\, Z$ for $t \in [0, T]$,
(2) $\mathrm{Re}\, W_t = \mathrm{Re}\, W$ for $t \in [T, 1]$.

Furthermore, $\mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\Lambda, \mathbf{C})$ is a $\mathbf{C}$-linear subspace of $\mathrm{Hom}_{\mathbf{Z}}(\Lambda, \mathbf{C})$. Hence we can choose W_t to be $\mathcal{C}$-equivariant for all $t \in [0,1]$. By Proposition 3.4.1, W_t lifts to a path $\varsigma_t = (\mathcal{Q}_t, W_t)$ in $\mathcal{U}$.

It is well known that, if there is a path in $\mathrm{Stab}(\mathcal{D})$ where the imaginary part of the central charge is constant, then the standard heart $\mathcal{P}(0,1]$ doesn't change along that path. For a proof of this see for example Theorem 3.4.2. Therefore, by the construction of W_t, we have that

$$\mathcal{Q}_t(0,1] = \mathcal{P}(0,1] \qquad \text{for } t \in [0,T].$$

For $t \in [T,1]$, we instead consider the rotated path of stability conditions, $\frac{1}{2} \cdot \varsigma_t = (\frac{1}{2} \cdot \mathcal{Q}_t, -iW_t)$. In this case $\mathrm{Im}\, -iW_t = \mathrm{Im}\, -iW_T$, which gives us

$$\mathcal{Q}_t(1/2, 3/2] = \mathcal{Q}_T(1/2, 3/2] \qquad \text{for } t \in [T, 1].$$

Note that $\mathcal{Q}_t(1/2, 3/2]$ is the standard heart of $\frac{1}{2} \cdot \varsigma_t$.

By Theorem 7.2.8, since W_t is $\mathcal{C}$-equivariant, $\varsigma_t = (W_t, \mathcal{P}(0,1])$ is $\mathcal{C}$-equivariant for all $t \in [0, T]$. In particular, ς_T is $\mathcal{C}$-equivariant. Hence $\mathcal{Q}_T(1/2, 3/2]$ is an abelian module category over $\mathcal{C}$ – in fact, any $\mathcal{Q}(a, a+1]$ is. Therefore, again by Theorem 7.2.8, we also know that $\frac{1}{2} \cdot \varsigma_t = (-iW_t, \mathcal{Q}_T(1/2, 3/2])$ is $\mathcal{C}$-equivariant for all $t \in [T, 1]$. Since the $\mathbf{C}$-action on $\mathrm{Stab}(\mathcal{D})$ preserves $\mathcal{C}$-equivariant stability conditions, $\varsigma_1 = (\mathcal{Q}, W)$ is $\mathcal{C}$-equivariant as required. □

Remark 7.2.10. In [DHL24, Lemma 3.17] we gave an alternate proof of Lemma 7.2.9 under the (weaker) assumption that σ is locally-finite.

Theorem 7.2.11 ([QZ23, Theorem 4.9], [DHL24, Theorem 3.18])**.** *The space of $\mathcal{C}$-equivariant stability conditions, $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$, is a complex submanifold of $\mathrm{Stab}(\mathcal{D})$ of dimension $\leq \mathrm{rank}(\Lambda)$. The forgetful map $\mathcal{Z}$ defines a local homeomorphism*

$$\begin{aligned} \mathcal{Z}\colon \mathrm{Stab}_{\mathcal{C}}(\mathcal{D}) &\to \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(\mathrm{K}(\mathcal{D}), \mathbf{C}) \subseteq \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}(\mathcal{D}), \mathbf{C}) \\ (\mathcal{P}, Z) &\mapsto Z. \end{aligned}$$

Proof. This is a direct consequence of Lemma 7.2.9 together with Theorem 3.2.1. □

7.2.3 Closed property

In this subsection, we show that $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$ is moreover a closed subset of $\mathrm{Stab}(\mathcal{D})$. We will use the construction of inducing stability conditions to do this. Recall that in Section 5.3 we saw how to use the functor $\mathrm{Forg}_G\colon \mathcal{D}_G \to \mathcal{D}$ to induce stability conditions from $\mathrm{Stab}(\mathcal{D})$ to $\mathrm{Stab}(\mathcal{D}_G)$. The construction from [MMS09, §2.2] works more generally as follows.

Let $\Phi\colon \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories which satisfies

$$\mathrm{Hom}_{\mathcal{D}'}(\Phi(X), \Phi(Y)) = 0 \implies \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \quad \text{for all } X, Y \in \mathcal{D}. \tag{P1}$$

Given a locally-finite stability condition $\sigma' = (\mathcal{P}', Z') \in \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}')$, we define an additive subcategory of $\mathcal{D}$ for each $\phi \in \mathbf{R}$ by

$$\Phi^{-1}\mathcal{P}'(\phi) := \{X \in \mathcal{D} \mid \Phi(X) \in \mathcal{P}(\phi)\}.$$

The functor Φ induces a natural morphism $\Phi_*\colon \mathrm{K}(\mathcal{D}) \to \mathrm{K}(\mathcal{D}')$. We use this to define a group homomorphism $\Phi_*^{-1}Z'\colon \mathrm{K}(\mathcal{D}) \to \mathbf{C}$ given by

$$[X] \mapsto Z'[\Phi(X)].$$

The pair $\Phi^{-1}\sigma' := (\Phi^{-1}\mathcal{P}', \Phi_*^{-1}Z')$ might not always define a locally-finite stability condition on $\mathcal{D}$, as one has to check if the Harder–Narasimhan property is satisfied. Nonetheless, if we restrict to those that do, they form a closed subset.

Proposition 7.2.12 ([MMS09, Lemma 2.8, 2.9])**.** *Suppose Φ satisfies property* (P1)*. Then the following subset of $\mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}')$,*

$$\mathrm{Dom}_{lf}(\Phi^{-1}) := \left\{\sigma' \in \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D}') \mid \Phi^{-1}\sigma' \in \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D})\right\},$$

is closed. Moreover, the map from this subset

$$\Phi^{-1}\colon \mathrm{Dom}_{lf}(\Phi^{-1}) \to \mathrm{Stab}_{\mathrm{lf}}(\mathcal{D})$$

is continuous.

Remark 7.2.13. The method of inducing stability conditions also works for stability conditions with support property as follows. Let $\Phi\colon \mathcal{D} \to \mathcal{D}'$ be an exact functor satisfying (P1). Given $\sigma' = (\mathcal{P}', Z') \in \operatorname{Stab}_{\Lambda'}(\mathcal{D}')$ satisfying the support property with respect to (Λ', λ') and quadratic form Q', we define for each $\phi \in \mathbf{R}$ the following additive subcategory of $\mathcal{D}$

$$\Phi^{-1}\mathcal{P}'(\phi) := \{E \in \mathcal{D} \mid \Phi(E) \in \mathcal{P}'(\phi)\},$$

and group homomorphisms $\Phi^{-1}\lambda' := \lambda' \circ \Phi\colon \mathrm{K}(\mathcal{D}) \to \Lambda'$ and $\Phi_*^{-1}Z' := Z' \circ \Phi^{-1}\lambda'$. As before, we consider the pair $\Phi^{-1}\sigma' := (\Phi^{-1}\mathcal{P}', \Phi_*^{-1}Z')$. If this is a pre-stability condition, then one can use the same quadratic form Q' to prove that $\Phi^{-1}\sigma'$ satisfies the support property with respect to $(\Lambda', \Phi^{-1}\lambda')$. Define $\operatorname{Dom}(\Phi^{-1}) := \{\sigma' \in \operatorname{Stab}_{\Lambda'}(\mathcal{D}') \mid \Phi^{-1}\sigma' \in \operatorname{Stab}_{\Lambda'}(\mathcal{D})\}$.

Now assume $\mathcal{D}$ is a triangulated module category over $\mathcal{C}$. We will next use the endofunctors $C \otimes -$ to induce stability conditions. This will mean we can use Proposition 7.2.12 to prove the closed property.

Lemma 7.2.14 ([DHL24, Lemma 3.21])**.** *Let $0 \neq C \in \mathcal{C}$. Then*
(1) the endofunctor $C \otimes -\colon \mathcal{D} \to \mathcal{D}$ satisfies (P1)*;*
(2) for all $\mathcal{C}$-equivariant stability conditions $\sigma = (\mathcal{P}, Z) \in \operatorname{Stab}_{\mathcal{C}}(\mathcal{D})$, we have

$$(C \otimes -)^{-1}\mathcal{P} = \mathcal{P} \quad \textit{and} \quad (C \otimes -)^{-1}Z = \operatorname{FPdim}(C) \cdot Z.$$

Hence $(C \otimes -)^{-1}\sigma$ is also a $\mathcal{C}$-equivariant stability condition.
In particular,

$$\operatorname{Stab}_{\mathcal{C}}(\mathcal{D}) \subseteq \operatorname{Dom}((C \otimes -)^{-1}) = \{\sigma \in \operatorname{Stab}(\mathcal{D}) \mid (C \otimes -)^{-1}\sigma \in \operatorname{Stab}(\mathcal{D})\},$$

and $(C \otimes -)^{-1}$ maps $\operatorname{Stab}_{\mathcal{C}}(\mathcal{D})$ to itself.

Proof. (1): By [Eti+15, Proposition 2.10.8(b)], the existence of duals for C gives the following natural isomorphism

$$\operatorname{Hom}_{\mathcal{D}}(C \otimes X, C \otimes Y) \cong \operatorname{Hom}_{\mathcal{D}}(X, C^\vee \otimes C \otimes Y).$$

By semisimplicity and duality, the monoidal unit $\mathbb{1}$ is a summand of $C^\vee \otimes C$, so $\operatorname{Hom}_{\mathcal{D}}(X, Y)$ appears as a summand of $\operatorname{Hom}_{\mathcal{D}}(X, C^\vee \otimes C \otimes Y)$. This proves that (P1) is satisfied by $C \otimes -$.

For (2), suppose $\sigma = (\mathcal{P}, Z)$ is $\mathcal{C}$-equivariant. Then for all $[X] \in \mathrm{K}(\mathcal{D})$,

$$(C \otimes -)^{-1}Z([X]) = Z[C \otimes X] = \operatorname{FPdim}(C) \cdot Z([X]).$$

It remains to show that $(C \otimes -)^{-1}\mathcal{P} = \mathcal{P}$.

Suppose $X \in \mathcal{P}(\phi)$. Since σ is $\mathcal{C}$-equivariant, $C \otimes X \in \mathcal{P}(\phi)$. Thus $X \in (C \otimes -)^{-1}\mathcal{P}(\phi)$. Conversely, let $X \in (C \otimes -)^{-1}\mathcal{P}(\phi)$. Then by definition, $C \otimes X \in \mathcal{P}(\phi)$. Applying Proposition 7.2.6 to any preferred choice of heart containing $\mathcal{P}(\phi)$ shows that X must also be semistable of the same phase, i.e. $X \in \mathcal{P}(\phi)$ as required.

Note that $(C \otimes -)^{-1}\sigma = (\mathcal{P}, \operatorname{FPdim}(C) \cdot Z)$ is indeed a stability condition, since $(\mathcal{P}, \operatorname{FPdim}(C) \cdot Z)$ is obtained from the (continuous) action of $\mathbf{C}$ on $\sigma \in \operatorname{Stab}(\mathcal{D})$, i.e. $i \log \operatorname{FPdim}(C)/\pi \in \mathbf{C}$ applied to σ. The fact that it is $\mathcal{C}$-equivariant is immediate.

The final statement in the lemma is a direct consequence of (1) and (2). $\square$

Now we are ready to prove that $\operatorname{Stab}_{\mathcal{C}}(\mathcal{D})$ is a closed submanifold. This generalises Theorem 7.2.1.

Theorem 7.2.15 ([DHL24, Theorem 3.22, Corollary 3.23])**.** *Let $\mathcal{D}$ be a triangulated module category over $\mathcal{C}$. The subset $\operatorname{Stab}_{\mathcal{C}}(\mathcal{D}) \subseteq \operatorname{Stab}(\mathcal{D})$ of $\mathcal{C}$-equivariant stability conditions is a closed submanifold.*

Proof. By Theorem 7.2.11, $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$ is a submanifold of $\mathrm{Stab}(\mathcal{D})$. It remains to prove that it is closed.

Consider the set

$$\Delta := \left\{((\mathcal{P}, \mathrm{FPdim}(S) \cdot Z))_{S \in \mathrm{Irr}(\mathcal{C})}\right\} \subseteq \prod_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Stab}(\mathcal{D}).$$

The diagonal is closed in $\prod_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Stab}(\mathcal{D})$ and Δ is obtained by the continuous action of $\mathbf{C}$ on the appropriate components of $\mathrm{Stab}(\mathcal{D})$ (acted on by $i \log \mathrm{FPdim}(S)/\pi \in \mathbf{C}$). Hence Δ is a closed subset. By Proposition 7.2.12, it follows that $U := \bigcap_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Dom}((S \otimes -)^{-1})$ is a closed subset of $\mathrm{Stab}(\mathcal{D})$ and the map

$$\Psi := \prod_{S \in \mathrm{Irr}(\mathcal{C})} (S \otimes -)^{-1}|_U \colon\ U \longrightarrow \prod_{S \in \mathrm{Irr}(\mathcal{C})} \mathrm{Stab}(\mathcal{D})$$

is continuous. We denote by $I := \Psi^{-1}(\Delta)$ the inverse image of Δ through Ψ. I is a closed subset of U. Moreover, since U itself is closed in $\mathrm{Stab}(\mathcal{D})$, I is also a closed subset of $\mathrm{Stab}(\mathcal{D})$.

We claim that I is exactly $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$, which will then complete the proof. Lemma 7.2.14 shows that $\mathrm{Stab}_{\mathcal{C}}(\mathcal{D})$ is contained in I, so it remains to show the converse. Suppose $\sigma = (\mathcal{P}, Z) \in I$, which means for all $S \in \mathrm{Irr}(\mathcal{C})$, we have $(S \otimes -)^{-1}\sigma = (\mathcal{P}, \mathrm{FPdim}(S) \cdot Z)$. Since FPdim is a ring homomorphism and $\mathcal{C}$ is semisimple, it follows that

$$(C \otimes -)^{-1}\sigma = (\mathcal{P}, \mathrm{FPdim}(C) \cdot Z) \quad \text{for all } C \in \mathcal{C}.$$

Therefore, $(C \otimes -)^{-1}\mathcal{P}(\phi) = \mathcal{P}(\phi)$ for all $C \in \mathcal{C}$ and all $\phi \in \mathbf{R}$. Thus $C \otimes \mathcal{P}(\phi) \subseteq \mathcal{P}(\phi)$ for all C and all ϕ. On the other hand, $(C \otimes -)^{-1}Z = \mathrm{FPdim}(C) \cdot Z$. Hence Z is a $\mathrm{K}(\mathcal{C})$-module homomorphism, and σ is indeed $\mathcal{C}$-equivariant. □

7.3 A duality for stability conditions

Throughout this section, we continue to assume that k is an algebraically closed field, and $\mathcal{D}$ is a k-linear essentially small Ext-finite triangulated category with a Serre functor. Moreover, let G be a finite group and suppose $\mathcal{D}$ is a (left) module category over vec_G. By Remark 7.1.19, this is equivalent to an action of G on $\mathcal{D}$. To ensure that $\mathrm{rep}(G)$ is a fusion category, we also assume that $(\mathrm{char}(k), |G|) = 1$.

The main result of this section is to generalise Theorem 5.3.6 to non-abelian groups. In particular, we will show that, under mild assumptions on $\mathcal{D}$, the submanifold of vec_G-equivariant stability conditions has an analytic isomorphism to the submanifold of $\mathrm{rep}(G)$-equivariant stability conditions on the category of G-equivariant objects, $\mathcal{D}_G$, i.e.

$$\mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D}) \cong \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G).$$

7.3.1 Recap: G-equivariantisation

In Definition 5.1.4, we introduced the category of G-equivariant objects when $\mathcal{D}$ has a right action of G. There is an analagous definition for a left action of G. We reformulate this for left module categories over vec_G below.

Definition 7.3.1. Let $\mathcal{D}$ be a k-linear (additive) module category over vec_G (see Definition 7.1.14). We define the *G-equivariantisation* of $\mathcal{D}$, denoted by $\mathcal{D}_G$, as follows:

- the objects, called *G-equivariant objects*, are pairs $(X, (\theta_g)_{g \in G})$ with $X \in \mathcal{D}$ and isomorphisms

$\theta_g \colon g \otimes X \to X$ in $\mathcal{D}$ such that the following diagram commutes for all $g, h \in G$:

$$\begin{tikzcd} g \otimes h \otimes X \arrow[r, "g\otimes\theta_h"] \arrow[d, "\mathrm{mult} \otimes \mathrm{id}_X"'] & g \otimes X \arrow[d, "\theta_g"] \\ gh \otimes X \arrow[r, "\theta_{gh}"] & X \end{tikzcd} ;$$

- the morphisms are those in $\mathcal{D}$ that commute with the isomorphisms θ_g for all $g \in G$.

The category $\mathcal{D}_G$ is a k-linear (left) module category over $\mathrm{rep}(G)$. Recall that objects in $\mathrm{rep}(G)$ are pairs $(V, (\varphi_g)_{g\in G})$, where V is a k-vector space, and each $\varphi_g \colon V \to V$ is a linear map. Then each $(V, (\varphi_g)_{g\in G}) \in \mathrm{rep}(G)$ acts on $(X, (\theta_g)_{g\in G})$ by

$$(V, (\varphi_g)_{g\in G}) \otimes (X, (\theta_g)_{g\in G}) \coloneqq (V \boxtimes X, (\varphi_g \boxtimes \theta_g)_{g\in G}). \tag{7.2}$$

Note that the $\mathrm{rep}(G)$-action is determined by how the irreducible representations $\mathrm{Irr}(G)$ act. In particular, when G is abelian, $\mathrm{Irr}(G) = \widehat{G} \coloneqq \mathrm{Hom}(G, k^*)$ is a group, and (7.2) describes an action of $\widehat{G}$ on $\mathcal{D}_G$ in the sense of Definition 7.1.18. This is exactly the action we saw in Proposition 5.1.6.

In Section 5.3, we made the assumption that $\mathcal{D}$ has a DG-enhancement to ensure that $\mathcal{D}_G$ was triangulated in such a way that Forg_G was exact, and so that the assumptions stated before [MMS09, Theorem 2.14] were satisfied. In this chapter, to work in greater generality, we instead make the following assumption.

Assumption 7.3.2. Let $\mathrm{Forg}_G \colon \mathcal{D}_G \to \mathcal{D}$ denote the forgetful functor sending $(X, (\theta_g)_{g\in G}) \mapsto X$. We assume $\mathcal{D}_G$ is triangulated in such a way that Forg_G preserves and reflects exact triangles. In particular,

$$\begin{aligned} &(X, (\theta_g)_{g\in G}) \to (Y, (\theta'_g)_{g\in G}) \to (Z, (\theta''_g)_{g\in G}) \xrightarrow{[1]} (X, (\theta_g)_{g\in G})[1] \text{ is exact in } \mathcal{D}_G \\ \iff\ & X \to Y \to Z \xrightarrow{[1]} X[1] \text{ is exact in } \mathcal{D}. \end{aligned}$$

Example 7.3.3. Assumption 7.3.2 is satisfied in the following situations of interest:
- Let $\mathcal{A}$ be an abelian module category over vec_G. Then $\mathrm{D^b}(\mathcal{A})$ is a triangulated module category over vec_G. Moreover, $(\mathrm{D^b}(\mathcal{A}))_G \cong \mathrm{D^b}(\mathcal{A}_G)$ by [Ela15, Theorem 7.1] and it follows that $\mathrm{D^b}(\mathcal{A})$ satisfies Assumption 7.3.2.
- Let $\mathcal{D}$ be a triangulated module category over vec_G with a DG-enhancement. Then by Proposition 5.1.9, $\mathcal{D}$ satisfies Assumption 7.3.2.

Remark 7.3.4. Suppose G is abelian, so that $\mathrm{rep}(G) \cong \mathrm{vec}_{\widehat{G}}$. Then we saw in Theorem 5.1.10 that $(\mathcal{C}_G)_{\widehat{G}} \cong \mathcal{C}$. Note that this equivalence preserves the triangulated structure.

Now assume $\mathrm{char}(k) = 0$ and let G be any finite group. Let vec_G-MOD and $\mathrm{rep}(G)$-MOD denote the 2-category of k-linear, idempotent complete (additive) module categories over vec_G and $\mathrm{rep}(G)$ respectively. The 2-functor sending $\mathcal{C} \in \mathrm{vec}_G$-MOD to its G-equivariantisation $\mathcal{C}_G \in \mathrm{rep}(G)$-MOD is a 2-equivalence [Dri+10, Theorem 4.4]. This is known as the (1-)categorical Morita duality of vec_G and $\mathrm{rep}(G)$. This is an analog of the equivalence $(\mathcal{C}_G)_{\widehat{G}} \cong \mathcal{C}$ in the abelian case, however it is not clear whether this duality also preserves the triangulated structure. Hence we cannot use it to directly generalise Theorem 5.3.6.

In analogy with Definition 5.1.7, we define the following functor from $\mathcal{D}$ to $\mathcal{D}_G$.

Definition 7.3.5. Let $\mathcal{D}$ be a k-linear (additive) triangulated module category over vec_G that satisfies

Assumption 7.3.2. Define the *inflation functor*, $\mathrm{Inf}_G\colon \mathcal{D} \to \mathcal{D}_G$, by

$$\mathrm{Inf}_G(X) \coloneqq \left(\bigoplus_{g\in G} (g \otimes X), (\xi_h)_{h\in G} \right),$$

where ξ_h is defined on each summand by the isomorphism

$$h \otimes h^{-1}g \otimes X \xrightarrow{\mathrm{mult}\otimes \mathrm{id}_X} g \otimes X.$$

Lemma 7.3.6 ([DHL24, Lemma 4.5]). *Let $\mathcal{D}$ be a k-linear (additive) triangulated module category over* vec_G*.*

1. *The forgetful functor* $\mathrm{Forg}_G\colon \mathcal{D}_G \to \mathcal{D}$ *is faithful, and it is left and right adjoint to* Inf_G*.*
2. *The inflation functor* $\mathrm{Inf}_G\colon \mathcal{D} \to \mathcal{D}_G$ *is exact.*
3. *The composite functor* $\mathrm{Inf}_G \circ \mathrm{Forg}_G\colon \mathcal{D}_G \to \mathcal{D}_G$ *is equivalent to the functor* $k[G] \otimes -$ *given by the action of the regular representation on* $\mathcal{D}_G$*.*
4. *The composite functor* $\mathrm{Forg}_G \circ \mathrm{Inf}_G\colon \mathcal{D} \to \mathcal{D}$ *sends objects* $X \mapsto \bigoplus_{g\in G} g \otimes X$*.*

Proof. (1) follows from the same argument as Lemma 5.1.8. (2) follows from the exactness of the action of vec_G (which is part of the definition of a module structure) and Assumption 7.3.2 on $\mathcal{D}$. (3) and (4) follow from a standard calculation; see for example [Dem11; Dri+10] for the case of abelian categories. □

7.3.2 G-invariant and $\mathrm{rep}(G)$-equivariant stability conditions

Let $\mathcal{D}$ be a k-linear triangulated (left) module category over vec_G such that the G-equivariantisation $\mathcal{D}_G$ satisfies Assumption 7.3.2. In Section 5.3, we assumed G was abelian and used the functors Forg_G and $\mathrm{Forg}_{\widehat{G}} \cong \mathrm{Inf}_G$ to induce (locally-finite) stability conditions between $\mathcal{D}$ and $\mathcal{D}_G$. For the general case we will apply the construction from Remark 5.3.8 to the functors Forg_G and Inf_G.

Lemma 7.3.7 ([DHL24, Lemma 4.6]). *Both the functors* $\mathrm{Inf}_G\colon \mathcal{D} \to \mathcal{D}_G$ *and* $\mathrm{Forg}_G\colon \mathcal{D}_G \to \mathcal{D}$ *satisfy property* (P1).

Proof. By Lemma 7.3.6, it follows that

$$\mathrm{Hom}_{\mathcal{D}_G}\left(\mathrm{Inf}_G(X), \mathrm{Inf}_G(Y)\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(X, \left(\mathrm{Forg}_G \circ \mathrm{Inf}_G\right)(Y)\right) \cong \mathrm{Hom}_{\mathcal{D}}\left(X, \bigoplus_{g\in G} g \otimes Y\right).$$

The right-most hom space contains $\mathrm{Hom}_{\mathcal{D}}(X, Y)$ as a summand (pick g to be the identity element of G), so Inf_G satisfies property (P1). The same argument can be applied to Forg_G, or one could use the fact that Forg_G is faithful. □

Before we state our main theorem, we impose the following assumptions on $\mathcal{D}$.

Assumption 7.3.8.
1. $\mathcal{D}$ satisfies Assumption 7.3.2.
2. The vec_G action on $\mathcal{D}$ extends to an action on a triangulated category $\widetilde{\mathcal{D}}$ (containing $\mathcal{D}$), which contains all small coproducts and satisfies Assumption 7.3.2.

Example 7.3.9. Suppose $\mathcal{A}$ is an essentially small abelian category with a G-action that extends to a G-action on an abelian category $\widetilde{\mathcal{A}}$ containing all small coproducts. This includes:

- $\mathcal{A} = A\text{-mod}$ and $\widetilde{\mathcal{A}} = A\text{-Mod}$, where A is a finite-dimensional algebra over k equipped with a G-action and $A\text{-Mod}$ denotes the category of all modules; and

- $\mathcal{A} = \mathrm{Coh}(X)$ and $\widetilde{\mathcal{A}} = \mathrm{QCoh}(X)$, where X is a scheme over k equipped with a G-action.

Then the assumptions above hold for $\mathcal{D} = \mathrm{D^b}(\mathcal{A})$ (together with its induced G-action), with $\widetilde{\mathcal{D}} = \mathrm{D}(\widetilde{\mathcal{A}})$, the unbounded derived category on $\widetilde{\mathcal{A}}$.

We now generalise Theorem 5.3.6. In particular, we apply the construction from Remark 7.2.13 to Forg_G and Inf_G, and show that this gives rise to analytic isomorphisms between certain submanifolds of $\mathrm{Stab}_\Lambda(\mathcal{D})$ and $\mathrm{Stab}_\Lambda(\mathcal{D}_G)$. We continue to drop Λ from our notation.

Theorem 7.3.10 ([DHL24, Theorem 4.8])**.** *Suppose $\mathcal{D}$ satisfies Assumption 7.3.8. Then the functors Inf_G and Forg_G induce analytic isomorphisms between $\mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})$ and $\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$,*

$$\mathrm{Forg}_G^{-1} \colon \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D}) \overset{\cong}{\rightleftarrows} \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G) : \mathrm{Inf}_G^{-1},$$

which are mutual inverses up to rescaling the central charge by $|G|$.

Proof. We first show that $\mathrm{Dom}(\mathrm{Inf}_G^{-1})$ and $\mathrm{Dom}(\mathrm{Forg}_G^{-1})$ contain $\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$ and $\mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})$ respectively. To do this we will apply [MMS09, Theorem 2.14].

By Assumption 7.3.8, there are triangulated categories $\widetilde{\mathcal{D}} \supseteq \mathcal{D}$ and $\widetilde{\mathcal{D}}_G \supseteq \mathcal{D}_G$, such that $\widetilde{\mathcal{D}}$ satisfies Assumption 7.3.2. Hence Lemma 7.3.6 also applies to $\widetilde{\mathcal{D}}$. Moreover, the forgetful functor $\mathrm{Forg}_G^{\sim}$ and inflation functor $\mathrm{Inf}_G^{\sim}$ between $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}_G$ restrict to the usual forgetful functor Forg_G and inflation functor Inf_G between $\mathcal{D}$ and $\mathcal{D}_G$.

Furthermore, $\mathrm{Forg}_G^{\sim}(X) \in \mathcal{D}$ implies that $X \in \mathcal{D}_G$, since $(\mathrm{Inf}_G^{\sim} \circ \mathrm{Forg}_G^{\sim})(X) = k[G] \otimes X$ and $\mathcal{D}_G$ is closed under the action of $\mathrm{rep}(G)$. Similarly $\mathrm{Inf}_G^{\sim}(X') \in \mathcal{D}_G$ implies $X' \in \mathcal{D}$. Also, by Lemma 7.3.7, both Forg_G and Inf_G satisfy assumption (P1). Therefore, all the assumptions stated before [MMS09, Theorem 2.14] are satisfied. Moreover,

- for all $(\mathcal{P}', Z') \in \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$,

$$(\mathrm{Inf}_G \circ \mathrm{Forg}_G)(\mathcal{P}'(\phi)) = k[G] \otimes \mathcal{P}'(\phi) \subseteq \mathcal{P}'(\phi);$$

- for all $(\mathcal{P}, Z) \in \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})$,

$$(\mathrm{Forg}_G \circ \mathrm{Inf}_G)(\mathcal{P}(\phi)) = \bigoplus_{g \in G} g \otimes \mathcal{P}(\phi) \subseteq \mathcal{P}(\phi).$$

Hence it follows from [MMS09, Theorem 2.14] that

$$\mathrm{Dom}(\mathrm{Inf}_G^{-1}) \supseteq \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G); \quad \mathrm{Dom}(\mathrm{Forg}_G^{-1}) \supseteq \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D}).$$

We next claim that Inf_G^{-1} and Forg_G^{-1} map to the correct codomain, i.e.

$$\mathrm{Inf}_G^{-1}(\mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)) \subseteq \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D}); \quad \mathrm{Forg}_G^{-1}(\mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})) \subseteq \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G).$$

Indeed, let $\sigma' = (\mathcal{P}', Z') \in \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$. Since $\mathrm{Inf}_G(X) \cong \mathrm{Inf}_G(g \otimes X) \in \mathcal{D}_G$ for all $g \in G$, it follows that $\mathrm{Inf}_G^{-1}\sigma'$ is G-invariant. On the other hand, let $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})$. Suppose $X \in \mathrm{Forg}_G^{-1}\mathcal{P}(\phi)$. By construction, $\mathrm{Forg}_G(X) \in \mathcal{P}(\phi)$. Thus for any $V = (V, (\varphi_g)_{g \in G}) \in \mathrm{rep}(G)$, we have $\mathrm{Forg}_G(V \otimes X) = X^{\oplus \dim(V)}$. Since $\mathcal{P}(\phi)$ is an abelian category, it is closed under taking direct sums and hence $V \otimes X \in \mathrm{Forg}_G^{-1}\mathcal{P}(\phi)$. Moreover, $Z(\mathrm{Forg}_G(V \otimes X)) = \dim(V) \cdot Z(X)$. This shows that $\mathrm{Forg}_G^{-1}\sigma$ is $\mathrm{rep}(G)$-equivariant. Therefore the claim holds.

Finally, we will prove that Inf_G^{-1} and Forg_G^{-1} are mutual inverses up to rescaling the central charge by $|G| \neq 0$. The fact that Inf_G^{-1} and Forg_G^{-1} are analytic isomorphisms will then follow. Indeed, by Proposition 7.2.12 they are continuous, and rescaling the central charge by $|G|$ is a linear isomorphism on $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C})$ and $\mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}_G), \mathbf{C})$. Hence Inf_G^{-1} and Forg_G^{-1} give isomorphisms on the level of tangent spaces, i.e.

$$\begin{aligned} \mathrm{Hom}_{\mathrm{K}(\mathrm{vec}_G)}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}), \mathbf{C}) &\overset{\cong}{\rightleftarrows} \mathrm{Hom}_{\mathrm{K}_{\mathrm{num}}(\mathrm{rep}(G))}(\mathrm{K}_{\mathrm{num}}(\mathcal{D}_G), \mathbf{C}) \\ Z &\mapsto Z \circ \mathrm{Inf}_G \\ Z' \circ \mathrm{Forg}_G &\leftarrow\!\shortmid Z'. \end{aligned}$$

Now suppose $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_{\mathrm{vec}_G}(\mathcal{D})$. By Lemma 7.3.6(4) and the G-invariance of Z, we know that $Z((\mathrm{Forg}_G \circ \mathrm{Inf}_G)(X)) = |G| \cdot Z$. Furthermore, the G-invariance of σ also guarantees that, for all $g \in G$, $X \in \mathcal{P}(\phi)$ if and only if $g \otimes X \in \mathcal{P}(\phi)$. Together with the fact that $\mathcal{P}(\phi)$ is closed under taking direct sums and summands, we see that $(\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1})\mathcal{P}(\phi) = \mathcal{P}(\phi)$. Hence

$$(\mathrm{Inf}_G^{-1} \circ \mathrm{Forg}_G^{-1})\sigma = (\mathcal{P}, |G| \cdot Z).$$

On the other hand, suppose $\sigma' = (\mathcal{P}', Z') \in \mathrm{Stab}_{\mathrm{rep}(G)}(\mathcal{D}_G)$. By Lemma 7.3.6(3) together with Lemma 7.2.14(2), it follows that

$$(\mathrm{Forg}_G^{-1} \circ \mathrm{Inf}_G^{-1})\sigma' = (\mathcal{P}', \mathrm{FPdim}(k[G]) \cdot Z') = (\mathcal{P}', |G| \cdot Z').$$

Therefore, Forg_G^{-1} and Inf_G^{-1} are mutual inverses up to rescaling the central charge by $|G|$. □

Remark 7.3.11. If $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ where X is a scheme, and the action of G on $\mathcal{D}$ is induced by an action of G on X, i.e. $\Phi_g = g^*$, then the analytic isomorphism above gives the bijection in [Pol07, Proposition 2.2.3].

Chapter 8

Bridgeland stability on non abelian quotients

In this chapter, we study geometric stability conditions on free quotients of varieties by a finite group. In particular, we will use the results from Chapter 7 to generalise Chapter 6 to non-abelian group actions.

8.1 The action of $\operatorname{rep}(G)$

Let X be a variety. We first recall the setup from Section 6.1. Let G be a finite group acting freely on X. This induces a right action of G on $\operatorname{Coh}(X)$ by pullback. In this case, the G-equivariantisation $\operatorname{Coh}_G(X) \coloneqq \operatorname{Coh}(X)_G$ is the category of G-equivariant coherent sheaves. Let $\pi\colon X \to Y \coloneqq X/G$ denote the quotient morphism. Since G acts freely, Y is smooth. Moreover, $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \coloneqq \mathrm{D}^{\mathrm{b}}(\operatorname{Coh}_G(X))$. This equivalence is given by

$$\begin{aligned} \Psi\colon \mathrm{D}^{\mathrm{b}}(Y) &\longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \\ E &\longmapsto (\pi^*(E), \lambda_{\mathrm{nat}}), \end{aligned}$$

and $\lambda_{\mathrm{nat}} = \{\lambda_g\}_{g\in G}$ is the G-linearisation given by:

$$\lambda_g\colon \pi^*E \xrightarrow{\sim} g^*\pi^*E = (\pi \circ g)^*E \cong \pi^*E.$$

As before, $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X) \cong (\mathrm{D}^{\mathrm{b}}(X))_G$. Under $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$, there is an isomorphism of functors $\operatorname{Forg}_G \cong \pi^*$, $\operatorname{Inf}_G \cong \pi_*$.

We now describe the $\operatorname{rep}(G)$-action on $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$. Recall that $\pi_*\mathcal{O}_X$ splits as a direct sum of vector bundles corresponding to the irreducible representations of G,

$$\pi_*\mathcal{O}_X = \bigoplus_{\rho\in\operatorname{Irr}(\operatorname{rep}(G))} E_\rho^{\bigoplus \dim\rho}. \tag{8.1}$$

The action of $\operatorname{rep}(G)$ on $\mathrm{D}^{\mathrm{b}}(Y)$ is determined by $-\otimes E_\rho$. When G is abelian, $\operatorname{Irr}(\operatorname{rep}(G)) \cong \widehat{G}$, and this determines a (right) $\widehat{G}$-action on $\mathrm{D}^{\mathrm{b}}(Y)$. The E_ρ are then the line bundles $\mathcal{L}_\chi$ that we saw in Section 6.1

Under the equivalence $\Psi\colon \mathrm{D}^{\mathrm{b}}(Y) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{G}}(X)$,

$$\Psi(\pi_*\mathcal{O}_X) = (\pi^* \circ \pi_*\mathcal{O}_X, \lambda_{\mathrm{nat}}) = (\mathcal{O}_X, \mathrm{id}) \otimes \mathbf{C}[G].$$

The final equality can also be understood as an application of Lemma 7.3.6(iii). The decomposition in (8.1) corresponds to the decomposition of $\mathbf{C}[G]$ into irreducible representations,

$$(\mathcal{O}_X, \mathrm{id}) \otimes \mathbf{C}[G] = \bigoplus_{\rho \in \mathrm{Irr}(\mathrm{rep}(G))} ((\mathcal{O}_X, \mathrm{id}) \otimes \rho)^{\oplus \dim \rho}.$$

In particular, $\Psi(E_\rho) = (\mathcal{O}_X, \mathrm{id}) \otimes \rho$. Since pullbacks commute with tensor products and direct sums, the equivalence Ψ is equivariant with respect to the action of $\mathrm{rep}(G)$, i.e. $\Psi(F \otimes E_\rho) \cong \Psi(F) \otimes \rho$ for all $F \in \mathrm{D^b}(Y)$ and $\rho \in \mathrm{Irr}(\mathrm{rep}(G))$.

The following result generalises Lemma 6.2.1 and Lemma 6.2.2.

Lemma 8.1.1 ([DHL24, Lemma 5.6]). *Suppose a finite group G acts freely on X. Let $\pi\colon X \to Y := X/G$ denote the quotient. Then all central charges on $\mathrm{D^b}(Y)$ are $\mathrm{rep}(G)$-equivariant, more precisely*

$$\mathrm{Hom}_{\mathrm{K}(\mathrm{rep}(G))}(\mathrm{K}(Y), \mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}(Y), \mathbf{C}).$$

In particular, $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ is a union of connected components of $\mathrm{Stab}(Y)$.

Proof. To prove that $\mathrm{Hom}_{\mathrm{K}(\mathrm{rep}(G))}(\mathrm{K}(Y), \mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{K}(Y), \mathbf{C})$, it is enough to show that $\rho \cdot [F] = (\dim \rho)[F]$ for any $[F] \in \mathrm{K}(X)$ and $\rho \in \mathrm{Irr}(\mathrm{rep}(G))$.

By Hirzebruch–Riemann–Roch, the Chern character $\mathrm{ch}\colon \mathrm{K}(X) \to \mathrm{Chow}(X)$ is injective. We first claim that $\mathrm{ch}([E_\rho]) = (\dim \rho, 0, \cdots, 0)$, where $[E_\rho] \in \mathrm{K}(X)$. First note that

$$\Psi(\pi_* \mathcal{O}_X \otimes E_\rho) = ((\mathcal{O}_X, \mathrm{id}) \otimes \mathbf{C}[G]) \otimes \rho = (\mathcal{O}_X, \mathrm{id}) \otimes (\mathbf{C}[G] \otimes \rho).$$

Since $\mathbf{C}[G] \otimes \rho \cong \mathbf{C}[G]^{\oplus \dim \rho}$, we have

$$\Psi(\pi_* \mathcal{O}_X \otimes E_\rho) \cong ((\mathcal{O}_X, \mathrm{id}) \otimes \mathbf{C}[G])^{\oplus \dim \rho} = \Psi((\pi_* \mathcal{O}_X)^{\oplus \dim \rho}).$$

It follows that $\pi_* \mathcal{O}_X \otimes E_\rho \cong (\pi_* \mathcal{O}_X)^{\oplus \dim \rho}$.

Since π is finite and étale, by Grothendieck–Riemann–Roch, $\mathrm{ch}([\pi_* \mathcal{O}_X]) = (|G|, 0, \cdots, 0)$. Hence

$$\mathrm{ch}([\pi_* \mathcal{O}_X \otimes E_\rho]) = \mathrm{ch}([\pi_* \mathcal{O}_X]) \cdot \dim \rho = (|G| \dim \rho, 0, \cdots, 0). \tag{8.2}$$

On the other hand,

$$\mathrm{ch}([\pi_* \mathcal{O}_X \otimes E_\rho]) = \mathrm{ch}([\pi_* \mathcal{O}_X]) \cdot \mathrm{ch}([E_\rho]) = (|G|\mathrm{ch}_0(E_\rho), |G|\mathrm{ch}_1(E_\rho), \cdots, |G|\mathrm{ch}_n(E_\rho)). \tag{8.3}$$

The claim follows from comparing (8.2) and (8.3). Since ch is injective, it follows that, for any $[F] \in \mathrm{K}(X)$, $\rho \cdot [F] = [F \otimes E_\rho] = (\dim \rho)[F]$ as required.

Since all central charges are $\mathrm{rep}(G)$-equivariant, by Theorem 7.2.11 it follows that $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ is open in $\mathrm{Stab}(Y)$. Moreover, $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ is also closed by Theorem 7.2.15, hence it is a union of connected components of $\mathrm{Stab}(Y)$. □

8.2 Inducing geometric stability conditions

Recall that $\sigma \in \mathrm{Stab}(X)$ is called *geometric* if all skyscraper sheaves of points are σ-stable. As before, let $\mathrm{Stab}^{\mathrm{Geo}}(X)$ denote the set of all geometric stability conditions. We now use Theorem 7.3.10 to generalise Theorem 6.1.1.

Theorem 8.2.1 ([DHL24, Corollary 5.5]). *The functors π^*, π_* induce an analytic isomorphism between the closed submanifolds of stability conditions* $\mathrm{Stab}_{\mathrm{vec}_G}(X)$ *and* $\mathrm{Stab}_{\mathrm{rep}(G)}(Y)$,

$$(\pi^*)^{-1}\colon \mathrm{Stab}_{\mathrm{vec}_G}(X) \overset{\cong}{\rightleftarrows} \mathrm{Stab}_{\mathrm{rep}(G)}(Y) : (\pi_*)^{-1},$$

which are mutual inverses up to rescaling the central charge by $|G|$. Moreover, this preserves geometric stability conditions.

In particular, suppose $\sigma = (\mathcal{P}_\sigma, Z_\sigma) \in \mathrm{Stab}_{\mathrm{vec}_G}(X)$ satisfies the support property with respect to (Λ, λ). Then $(\pi^*)^{-1}(\sigma) =: \sigma_Y = (\mathcal{P}_{\sigma_Y}, Z_{\sigma_Y}) \in \mathrm{Stab}_{\mathrm{rep}(G)}(Y)$ is defined by:

$$\begin{aligned}\mathcal{P}_{\sigma_Y}(\phi) &= \{E \in \mathrm{D^b}(Y) : \pi^*(E) \in \mathcal{P}_\sigma(\phi)\},\\ Z_{\sigma_Y} &= Z_\sigma \circ \pi^*,\end{aligned}$$

where π^* is the natural induced map on $\mathrm{K_{num}}(Y)$, and σ_Y satisfies the support property with respect to $(\Lambda, \lambda \circ \pi^*)$.

Proof. First recall that being vec_G-equivariant is equivalent to being G-invariant. We first generalise Step 1 of the proof of Theorem 6.1.1.

Step 1 The composition $\pi_* \circ \pi^*\colon \mathrm{K_{num}}(Y) \to \mathrm{K_{num}}(Y)$ is still multiplication by $|G|$, as

$$\pi_* \circ \pi^*([E]) = [E \otimes \pi_* \mathcal{O}_X] = |G|[E],$$

where the last equality follows from the proof of Lemma 8.1.1. Therefore, $\pi^*\colon \mathrm{K_{num}}(Y) \to \mathrm{K_{num}}(X)$ is still injective. Now recall that under $\mathrm{D^b_G}(X) \cong \mathrm{D^b}(Y)$, $\mathrm{Forg}_G \cong \pi^*$ and $\mathrm{Inf}_G \cong \pi_*$. Together with Theorem 7.3.10, it follows that $(\pi^*)^{-1}$ and $(\pi_*)^{-1}$ give an analytic isomorphism between numerical Bridgeland stability conditions as described above.

Step 2 It remains to show that $\sigma \in (\mathrm{Stab}(X))^G$ is geometric if and only if $\sigma_Y = (\pi^*)^{-1}\sigma$ is. This follows from the same arguments as Step 2 and 3 of the proof Theorem 6.1.1. □

When G is abelian, being $\mathrm{rep}(G)$-equivariant is equivalent to being $\widehat{G}$-invariant. Hence, in the abelian case, Theorem 8.2.1 is exactly Theorem 6.1.1.

8.3 Applications to varieties with finite Albanese morphism

We can now generalise the arguments from the case that G is abelian in Theorem 6.3.2 and Theorem 6.3.3.

Theorem 8.3.1 ([DHL24, Theorem 5.7]). *Let X be a variety with finite Albanese morphism. Let G be a finite group acting freely on X and let $Y = X/G$. Then*

(1) $\mathrm{Stab}^\ddagger(Y) := \mathrm{Stab}_{\mathrm{rep}(G)}(Y) \cong \mathrm{Stab}_{\mathrm{vec}_G}(X)$ *is a union of connected components consisting only of geometric stability conditions.*

(2) if X is a surface, then $\mathrm{Stab}^\ddagger(Y) = \mathrm{Stab}^{\mathrm{Geo}}(Y) \cong \mathrm{Stab}_{\mathrm{vec}_G}(X)$. *In particular,* $\mathrm{Stab}^\ddagger(Y)$ *is a contractible connected component of* $\mathrm{Stab}(Y)$.

Proof. (1) X has finite Albanese morphism, so it follows from Theorem 6.3.1 that all stability conditions on X are geometric. In particular, $\mathrm{Stab}_{\mathrm{vec}(G)}(X) \subset \mathrm{Stab}^{\mathrm{Geo}}(X)$. By Theorem 8.2.1, $\mathrm{Stab}_{\mathrm{vec}_G}(X) \cong \mathrm{Stab}_{\mathrm{rep}(G)}(Y) \subset \mathrm{Stab}^{\mathrm{Geo}}(Y)$. The result now follows from Lemma 8.1.1.

(2) Since X is a surface, $\mathrm{Stab}^{\mathrm{Geo}}(Y)$ is connected by Theorem 4.4.1 and contractible by [Rek23, Theorem A]. Hence the result follows. $\square$

Example 8.3.2 (Calabi–Yau threefolds of abelian type)**.** A *Calabi–Yau threefold of abelian type* is an étale quotient $Y = X/G$ of an abelian threefold X by a finite group G acting freely on X such that the canonical line bundle of Y is trivial and $H^1(Y, \mathbf{C}) = 0$. These are classified in [OS01, Theorem 0.1]. In particular, G is $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$ or D_4, and the Picard rank of Y is 3 or 2 respectively.

By the same discussion in Example 6.3.7, Theorem 8.3.1 produces a non-empty union of connected components of $\mathrm{Stab}(Y)$, which is new in the case of $G = D_4$.

Example 8.3.3 (Generalised hyperelliptic varieties)**.** A *generalised hyperelliptic variety* $Y = A/G$ is a quotient of an abelian variety A by the free action of a finite group G which contains no translations. These varieties are Kähler and have Kodaira dimension zero. Theorem 8.3.1 applies to these varieties.

In dimension 3 with $G = D_4$, Y is a Calabi–Yau threefold of abelian type. Their construction in [CD20] has been generalised to produce explicit examples of generalised hyperelliptic varieties in dimension $2n+1$ with $G = D_{4n}$ [Agu21, Theorem 2.1]. In this case, $A = E^{2n} \times E'$, where E, E' are elliptic curves. One can use the explicit description of the D_{4n} action to prove that A has non-finite Albanese morphism.

Another way to produce examples with G non-abelian is to take semi-direct products of abelian groups acting freely on any abelian variety [AL22, §7].

Example 8.3.4 (Non-abelian Beauville-type surfaces)**.** Let $Y = (C_1 \times C_2)/G$ be the quotient of a product of curves such that $g(C_1), g(C_2) \geq 1$, and G is a finite abelian group acting freely on Y. Then $C_1 \times C_2$ has finite Albanese morphism. By Theorem 8.3.1(2), $\mathrm{Stab}(Y)$ has a contractible connected component consisting only of geometric stability conditions. In particular, we could take Y to be a Beauville-type or bielliptic surface (see Example 1.2.1 and Example 1.2.2).

In the case of Beauville-type surfaces, this provides a description of a connected component of $\mathrm{Stab}(Y)$ for *all* 17 families. The 5 families where G is abelian were studied in Example 6.3.5. Our result now includes the 12 other families where G is non-abelian. By [BCG08, Theorem 0.1], these are either A_5, S_4, $S_4 \times \mathbf{Z}/2\mathbf{Z}$, $D_4 \times \mathbf{Z}/2\mathbf{Z}$ (where D_4 is the dihedral group of order 8), or one of a list of certain higher order groups: $G(16)$, $G(32)$, $G(256, 1)$, $G(256, 2)$. Note that the final two are the only cases that have group elements which act by exchanging C_1 and C_2.

Chapter 9

Further Questions

Let X be a variety. There are no examples in the literature where $\mathrm{Stab}(X)$ is known to be disconnected. It would be interesting to investigate the following examples.

Question 9.0.1. *Let S be a Beauville-type or bielliptic surface. Is $\mathrm{Stab}(S)$ connected?*

S has non-finite Albanese morphism and $\mathrm{Stab}^{\mathrm{Geo}}(S) \subset \mathrm{Stab}(S)$ is a connected component by Theorem 8.3.1(2). If $\mathrm{Stab}(S)$ is connected, the following question would have a negative answer.

Question 1.1.4 ([FLZ22, Question 4.11])**.** *Let X be a variety whose Albanese morphism is not finite. Are there always non-geometric stability conditions on $\mathrm{D^b}(X)$?*

If this is not the case, then which geometric properties imply the existence of non-geometric stability conditions? Answering this could help us find new constructions of stability conditions.

It is still an open question whether geometric stability conditions exist on any variety X. As mentioned in Section 1.3, if X is $\mathbf{P}^n$, then the construction of stability conditions from a strong exceptional collection has been used to prove the existence of geometric stability conditions [Mu21, Proposition 3.5] [Pet22, §3.3]. It would be interesting to know whether this strategy works for any strong exceptional collection.

Question 9.0.2. *Suppose $\mathrm{D^b}(X)$ has a strong exceptional collection of vector bundles, and a corresponding heart $\mathcal{A}$ that can be used to construct stability conditions as in [Mac07a, §4.2]. If $\mathcal{O}_x \in \mathcal{A}$, then does $\mathcal{O}_x$ correspond to a stable quiver representation?*

Equivariant categories are a useful tool to study questions in algebraic geometry even when they arise from actions on varieties that are not free. For example, in [DJR23] we used equivariant categories together with techniques from Hodge theory and topological K-theory to study cyclic branched covers, and obtained new categorical Torelli theorems for the lowest degree prime Fano threefolds of index 1 and 2. It would be interesting to study what happens to stability conditions in this setting.

Question 9.0.3. *Suppose $\pi\colon X \to Y$ is a branched Galois cover. Can we relate $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$?*

Let $G = \mathrm{Aut}(X/Y)$, then since $\pi\colon X \to Y$ is Galois, $Y \cong X/G$, and π factors via the coarse moduli space map $c\colon [X/G] \to X/G$. By Theorem 7.3.10, there is an analytic isomorphism $\mathrm{Stab}_{\mathrm{vec}_G}(X) \cong \mathrm{Stab}_{\mathrm{rep}(G)}([X/G])$. Hence one strategy would be use these stability conditions on $[X/G]$ to induce stability conditions on Y, for example by applying Proposition 7.2.12 c^* or c_*. Note that the stability conditions constructed on $\mathrm{Stab}(X/G)$ from $\mathrm{Stab}(X)$ use the lattice $\mathrm{K_{num}}(X)$. But $\mathrm{K}([X/G])$ can be much larger than $\mathrm{K}(X)$.

An interesting test case for Question 9.0.3 would be when X and Y are curves and π is a double cover, since $\mathrm{Stab}([X/G])$ is contractible and has an explicit description [CP10, §7].

Another setting where Theorem 7.3.10 could be applied is the following.

Question 9.0.4. *Suppose a finite group G acts on X, and $Y \to [X/G]$ is a crepant resolution of the quotient such that $\mathrm{D}^{\mathrm{b}}(Y) \cong \mathrm{D}^{\mathrm{b}}([X/G])$. Can we relate $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$?*

This was recently done for the case of certain product varieties in [PS24].

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