Stability Conditions on F.A.Q.s

§1: Geometric Stability on Surfaces

- X: smooth projective surface $/\mathbb{C}$
- $H \in \mathrm{Amp}_{\mathbb{R}}(X)$

$E \in \mathrm{Coh}(X) \rightsquigarrow \mu_H(E) := \begin{cases} +\infty & \mathrm{rk}(E)=0 \\ \dfrac{H\cdot ch_1(E)}{H^2\cdot \mathrm{rk}(E)} & \text{otherwise} \end{cases}$

"H-stable = all subsheaves have $<\mu$"

- $\beta \in \mathbb{R}$

1st ingredient:

$$\mathcal{T}_{H,\beta} := \{E \in \mathrm{Coh}(X) : \forall E \twoheadrightarrow Q \neq 0,\ \mu_H(Q) > \beta\}$$

$$\mathcal{F}_{H,\beta} := \{E \in \mathrm{Coh}(X) : \forall 0 \neq F \subset E,\ \mu_H(F) \leq \beta\}$$

torsion pair $\rightsquigarrow$ tilt:

$$\mathrm{Coh}^{H,\beta}(X) := \left\{ E \in D^b(X) : \begin{array}{l} H^0(E) \in \mathcal{T}_{H,\beta} \\ H^{-1}(E) \in \mathcal{F}_{H,\beta} \\ H^i(E) = 0 \ \text{otherwise} \end{array} \right\}$$

2nd ingredient: $B \in NS_{\mathbb{R}}(X)$, $\alpha \in \mathbb{R}$

$$Z_{H,B,\alpha,\beta}(E) = (\alpha - i\beta) H^2 \,\mathrm{rk}(E) + (B + iH)\cdot ch_1(E) - ch_2(E)$$

$\rightsquigarrow$ new slope $= \dfrac{-\mathrm{Re}}{\mathrm{Im}}$

"Arcara–Bertram, Macrì–Schmidt,..."

Th^m [Bridgeland '08, ...] For $\alpha \gg 0$, $\{\sigma_{H,B,\alpha,\beta}\}$ is a continuous family of Bridgeland stability conditions.

$\frac{1}{2}\left[\left(\beta_0 - \frac{H\cdot B}{H^2}\right)^2 - \frac{B^2}{H^2}\right]$

⚠ $\mathcal{O}_x$ is simple in $\mathrm{Coh}^{H,0}(X) \Rightarrow \mathcal{O}_x$ is $\sigma_{H,\beta}$-stable

Def^n $\sigma \in \mathrm{Stab}(X)$ is **geometric** if $\mathcal{O}_x$ is σ-stable $\forall x \in X$

all stability conditions

write $\mathrm{Stab}^{Geo}(X)$ for all geometric stability conditions

"is this everything"

(Q1): $\exists$ nongeometric stability conditions?

"originally hope: all stabilities this nice, and Stab(X) connected"
"Curves: no, except for $\mathbb{P}^1$ (from quivers)"
"dim ≥ 3: little is known, so here's what we know in dim 2"

Surfaces:

① abelian surfaces: no

yes: ② K3 surfaces (spherical) ③ rational surfaces (exceptional)] $\mathcal{O}_x$ destabilized by rigid sheaf

④ $X \supset C$ rational curve s.t. $C^2 < 0$
$\forall x \in C$, $\mathcal{O}_x$ destab. by $\mathcal{O}_C(k)$

[Bridgeland, Huybrechts–Macrì–Stellari]
K3: Bridgeland
Rat. surface: Bayer–Macrì (on $\mathbb{P}^2$)
Tramel–Xia (Bingyu)

Th^m [Lie Fu – Chunyi Li – Xiaolei Zhao '22] "in any dim"
X finite Albanese morphism $alb_X \Rightarrow \mathrm{Stab}(X) = \mathrm{Stab}^{Geo}(X)$
$\Updownarrow$
has finite map to abelian variety

(Q2) [FLZ, Q4.11] non finite Albanese morphism
$\overset{??}{\Rightarrow}$ $\exists$ nongeometric stability conditions?

example/expectation: yes!

§2 The Le Potier function

"capture what was observed in examples"

Conj [FLZ] X minimal surface with $h^1(\mathcal{O}_X) = 0$ $\Leftrightarrow$ alb_X trivial
"rigid"
$\Rightarrow$ $\Phi_{X,H}$ discontinuous at 0.

hope: capture existence of destabilizing sheaves as for K3 & rational

• $B = 0$

Def^n X surface, (H, B) as before. The Le Potier function twisted by B, $\Phi_{X,H,B} : \mathbb{R} \to \mathbb{R}$, is:

$$\Phi_{H,B}(x) := \underset{\mu \to x}{\limsup}\ \cancel{\max} \left\{ \frac{ch_2(E) + B \cdot ch_1(E)}{H^2\, rk(E)} : \begin{array}{l} E \in Coh(X) \\ H\text{-semistable} \\ \text{with } \mu_H(E) = \cancel{x}\ \mu \end{array} \right\}$$

SLOGAN: fix $\mu_H(F) = \mu$, how big can this get?

Thm A [$\rho(X)=1$ FLZ, $\rho>1$ D.] $\Phi_{X,H,B}$ controls $\mathrm{Stab}^{geo}(X)$, i.e.

$$\mathrm{Stab}^{geo}(X) \simeq \mathbb{C} \times \left\{ (H,B,\alpha,\beta) \in (NS_{\mathbb{R}}(X))^2 \times \mathbb{R}^2 : \begin{array}{l} H \text{ ample} \\ \alpha > \Phi_{X,H,B}(\beta) \end{array} \right\}$$

Rmk can use this to answer Q1

· wall ↔ line bdy Φ edges ↔

· really happens for K3, not.

⚠ $\mathrm{Stab}(X)$ connected ⇒ enough to look at today

§3 Free Abelian Quotients (FAQs)

"If I lost you, then all you need for the rest of the talk is"

#Q2:

KEY IDEA: finite abelian $G \overset{\text{free}}{\curvearrowright} X$ — alby finite

$$\pi \downarrow$$

F.A.Q. $Y := X/G$ — alby not finite

GOAL: Compare $\mathrm{Stab}(X)$ & $\mathrm{Stab}(X/G)$

"need a way to pushforward / pullback stability"

G: finite group, $\mathcal{D}$: 

Def[n] [Deligne '97] $G \curvearrowright \mathcal{D}$ if $\forall g, h \in G$

· $\phi_g : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$ equiv. — autoequiv.

· $\varepsilon_{g,h} : \phi_g \phi_h \xrightarrow{\sim} \phi_{hg}$ — natural iso.

& compatible with triples of grp elements — assoc.

more than $\leq \mathrm{Aut}(\mathcal{D})$

EX: $G \curvearrowright X$, $\phi_g = g^* : D^b(X) \to D^b(X)$,

$$g^* h^* \xrightarrow{\sim} (hg)^*$$

Recall $\mathrm{Coh}_G(X)$: G-equivariant sheaves,

objects: $(E, \{\lambda_g\}_G)$ E G-invariant,

$\lambda_g : E \xrightarrow{\sim} g^* E$ choice of iso.

This generalizes!

$G \curvearrowright \mathcal{D} \rightsquigarrow \mathcal{D}_G$: category of G-equivariant objects

EX ① $G \curvearrowright X$, $(D^b(X))_G = D^b(\mathrm{Coh}_G(X)) \simeq D^b([X/G])$

if G acts freely, $\simeq D^b(X/G)$

② G abelian $\curvearrowright \mathcal{D}$

"ex of an action not coming from $G \curvearrowright X$"

$\hat{G} := \mathrm{Hom}(G, \mathbb{C}^*) \curvearrowright \mathcal{D}_G$

Thm [Elagin '15] G abelian $\curvearrowright \mathcal{D}$. Then $(\mathcal{D}_G)_{\hat G} \simeq \mathcal{D}$

LEMMA [Polishchuk '07, Macrì–Mehrotra–Stellari '09, Perry–Pertusi–Zhao '23, D. '23] G abelian (+...)

$$(\mathrm{Stab}(\mathcal{D}))^{G} \xleftrightarrow[1:1]{\circledast} (\mathrm{Stab}(\mathcal{D}_G))^{\hat G}$$

G-inv.

"mostly due to" "MMS: $\longrightarrow$ closed embed, me/PPZ (use Elagin to) describe image"

𝒟: ess. small, additive, ℂ-lin., idempotent complete, DG-enhanced

G finite abelian, acts by exact auto equivalences

Now $X \xrightarrow{\pi} Y = X/G$ f.A.G.

$\circledast$: $(\mathrm{Stab}(X))^{G} \xleftrightarrow{1:1} (\mathrm{Stab}(Y))^{\hat G}$

$\sigma_X \longmapsto \sigma_Y$

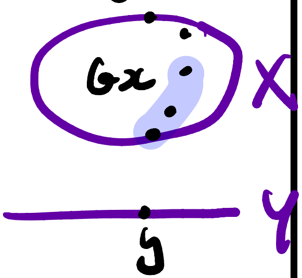
where: • $\hat G \curvearrowright D^b(Y)$ by $-\otimes \mathcal{L}$ — deg 0 line bundle

• $E \in D^b(Y)$ σ_Y-semistable

$\Leftrightarrow$ $\pi^*(E)$ σ_X-semistable

LEMMA [D.] $\circledast$ preserves geometric stability conditions

Sketch proof: $\Rightarrow$



• $y \in Y$, $\pi^*\mathcal{O}_y = \bigoplus_{g \in G} g^*\mathcal{O}_x$

• $\sigma_X \in (\mathrm{Stab}(X))^{G}$ geometric

$\Rightarrow$ $g^*\mathcal{O}_x = \mathcal{O}_{g^{-1}x}$ σ_X-stable $\forall g \in G, x \in X$

$\Rightarrow$ $\pi^*\mathcal{O}_y$ σ_X-semistable $\overset{\circledast}{\Leftrightarrow}$ $\mathcal{O}_y$ σ_Y-semistable

• only non.inv. subobjects of $\pi^*\mathcal{O}_y$ are

$\bigoplus_{h \in H} h^*\mathcal{O}_x$, $H \subsetneq G$

"stable = semistable + simple"

$\hookrightarrow$ cannot be of the form $\pi^*\mathcal{E}$, $\mathcal{E} \in D^b(Y)$

$\Rightarrow$ $\mathcal{O}_y$ is σ_Y-stable

⇐ Similar • $\sigma_Y \in (\mathrm{Stab}(Y))^{\hat G}$ geometric
⇒ $\pi^*\mathcal{O}_y$ σ_X-semistable
⇒ $\mathcal{O}_x$ σ_X-semistable
$E \hookrightarrow \mathcal{O}_x \twoheadrightarrow F$
$\pi_* E \hookrightarrow \mathcal{O}_y \twoheadrightarrow \pi_* F$

Thm B [D.] • X ~~variety~~ surface with finite Albanese morphism
• $Y = X/G$ F.A.Q.
Then $\underline{(\mathrm{Stab}(Y))^{\hat G}} \; \cancel{\subseteq} \; = \; \mathrm{Stab}^{geo}(Y)$
& this is a ~~union of~~ connected components in $\mathrm{Stab}(Y)$.

• (*) ⇒ $\subseteq$
• $\hat G \curvearrowright K_{num}$ trivial
• surface: $\supseteq$ & Thm A ⇒ connected

APPLICATIONS:

Ex 1 $Y = (C_1 \times C_2)/G$
• $g(C_i) = 1$, 7 families with $h^1(\mathcal{O}_Y) = 1$ called bielliptic surfaces.
"alb. Y elliptic fibration"
• $g(C_i) > 1$, 5 families with $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ called Beauville type surfaces
"alb. Y trivial"

$X = C_1 \times C_2$ has finite Albanese morphism,
but Y does not.

By Thm B either:
- $\mathrm{Stab}(Y) = \mathrm{Stab}^{geo}(Y)$ so Q2 false
- $\mathrm{Stab}(Y)$ disconnected   no such examples known

↳ disproves expectation that has a wall, like for K3, $\mathbb{P}^2$

Ex 2 "Calabi-Yau 3folds of abelian-type"
X abelian 3fold
$Y = X/G$ F.A.Q., $\omega_Y \cong \mathcal{O}_Y$ and $H^1(Y, \mathbb{C}) = 0$
$\mathfrak{B}$: connected comp. of geos in $\mathrm{Stab}(X)$ [Bayer-Macrì-Stellari + Oberdieck-Piyaratne-Toda]
Thm B ⇝ connected comp. of geos. in $\mathrm{Stab}(Y)$.

§4 LePotier on F.A.Q.s

X surface, $Y = X/G$ F.A.Q, $H_X \in \mathrm{Amp}_{\mathbb{R}}(X)$, $H_Y \in \mathrm{Amp}_{\mathbb{R}}(Y)$

Prop^{n}
- $\Phi_{Y,H_Y} = \Phi_{X,\pi^* H_Y}$ $(B=0)$
- X finite Alb case morphism, H_X pulled back from Alb(X) $\Rightarrow$ $\Phi_{X,H_X}(x) = \frac{x^2}{2}$

Corollary $Y = X/G$ Beauville type surface $(h^1(\mathcal{O}_X)=0)$
& $\pi^* H_Y$ pulled back from Alb(X)

$\Rightarrow$ $\Phi_{Y,H_Y} = \Phi_{X,\pi^* H_Y} = \frac{x^2}{2}$

i.e. FLZ-conjecture is false.

$\mathcal{O}_Y$ excep. but does not give discts or wall (Thm B)

$\rightsquigarrow$ need a sharper condition for $\Phi_{X,H}$ discts/existence of nongeos.