This text is adapted from *Computer Architecture and Digital Design* by Harris and Harris.

1 From Zero to One

1.3 THE DIGITAL ABSTRACTION

Most physical variables are continuous. For example, the voltage on a wire, the frequency of an oscillation, or the position of a mass are all continuous quantities. Digital systems, on the other hand, represent information with *discrete-valued variables*—that is, variables with a finite number of distinct values.

An early digital system using variables with ten discrete values was Charles Babbage's Analytical Engine. Babbage labored from 1834 to 1871, designing and attempting to build this mechanical computer. The Analytical Engine used gears with ten positions labeled 0 through 9, much like a mechanical odometer in a car. Figure 1.3 shows a prototype of the Analytical Engine, in which each row processes one digit. Babbage chose 25 rows of gears, so the machine has 25-digit precision.

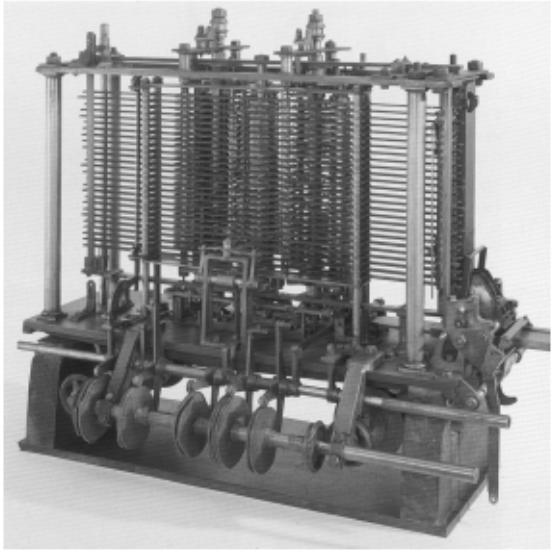


Figure 1.3 Babbage's Analytical Engine, under construction at the time of his death in 1871
(image courtesy of Science Museum/Science and Society Picture Library).

Unlike Babbage's machine, most electronic computers use a binary (two-valued) representation in which a high voltage indicates a '1' and a low voltage indicates a '0,' because it is easier to distinguish between two voltages than ten. The *amount of information* D in a discrete valued variable with N distinct states is measured in units of *bits* as

$$D = \log_2 N \text{ bits} \qquad (1.1)$$

A binary variable conveys $\log_2 2 = 1$ bit of information. Indeed, the word bit is short for *b*inary dig*it*. Each of Babbage's gears carried $\log_2 10 = 3.322$ bits of information because it could be in one of $2^{3.322} = 10$ unique positions. A continuous signal theoretically contains an

infinite amount of information because it can take on an infinite number of values. In practice, noise and measurement error limit the information to only 10 to 16 bits for most continuous signals. If the measurement must be made rapidly, the information content is lower (e.g., 8 bits).

This book focuses on digital circuits using binary variables: 1's and 0's. George Boole developed a system of logic operating on binary variables that is now known as *Boolean logic*. Each of Boole's variables could be TRUE or FALSE. Electronic computers commonly use a positive voltage to represent '1' and zero volts to represent '0'. In this book, we will use the terms '1,' TRUE, and HIGH synonymously. Similarly, we will use '0,' FALSE, and LOW interchangeably.

The beauty of the *digital abstraction* is that digital designers can focus on 1's and 0's, ignoring whether the Boolean variables are physically represented with specific voltages, rotating gears, or even hydraulic fluid levels. A computer programmer can work without needing to know the intimate details of the computer hardware. On the other hand, understanding the details of the hardware allows the programmer to optimize the software better for that specific computer.

An individual bit doesn't carry much information. In the next section, we examine how groups of bits can be used to represent numbers. In later chapters, we will also use groups of bits to represent letters and programs.

1.4 NUMBER SYSTEMS

You are accustomed to working with decimal numbers. In digital systems consisting of 1's and 0's, binary or hexadecimal numbers are often more convenient. This section introduces the various number systems that will be used throughout the rest of the book.

1.4.1 Decimal Numbers

In elementary school, you learned to count and do arithmetic in *decimal*. Just as you (probably) have ten fingers, there are ten decimal digits, 0, 1, 2, ..., 9. Decimal digits are joined together to form longer decimal numbers. Each column of a decimal number has ten times the weight of the previous column. From right to left, the column weights are 1, 10, 100, 1000, and so on. Decimal numbers are referred to as *base 10*. The base is indicated by a subscript after the number to prevent confusion when working in more than one base. For example, Figure 1.4 shows how the decimal number 9742_{10} is written as the sum of each of its digits multiplied by the weight of the corresponding column.

An N-digit decimal number represents one of 10^N possibilities: 0, 1, 2, 3, ..., 10^{N-1}. This is called the *range* of the number. For example, a three-digit decimal number represents one of 1000 possibilities in the range of 0 to 999.

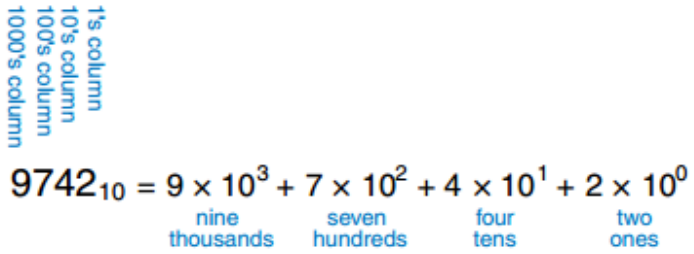


Figure 1.4 Representation of a decimal number

1.4.2 Binary Numbers

Bits represent one of two values, 0 or 1, and are joined together to form binary numbers. Each column of a binary number has twice the weight of the previous column, so binary numbers are *base 2*. In binary, the column weights (again from right to left) are 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, and so on. If you work with binary numbers often, you'll save time if you remember these powers of two up to 2^{16}.

An *N*-bit binary number represents one of 2^N possibilities: 0, 1, 2, 3, ..., 2^{N-1}. Table 1.1 shows 1, 2, 3, and 4-bit binary numbers and their decimal equivalents.

Table 1.1 Binary numbers and their decimal equivalent

1-Bit Binary Numbers	2-Bit Binnary Numbers	3-Bit Binary Numbers	4-Bit Binary Numbers	Decimal Equivalents
0	00	000	0000	0
1	01	001	0001	1
	10	010	0010	2
	11	011	0011	3
		100	0100	4
		101	0101	5
		110	0110	6
		111	0111	7
			1000	8
			1001	9
			1010	10
			1011	11
			1100	12
			1101	13
			1110	14
			1111	15

Example 1.1 BINARY TO DECIMAL CONVERSION

Convert the binary number 10110_2 to decimal.

Solution: Figure 1.5 shows the conversion.

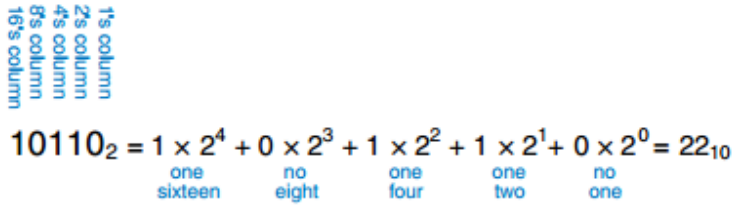


Figure 1.5 Conversion of a binary number to decimal

Example 1.2 DECIMAL TO BINARY CONVERSION

Convert the decimal number 84_{10} to binary.

Solution: Determine whether each column of the binary result has a 1 or a 0. We can do this starting at either the left or the right column.

Working from the left, start with the largest power of 2 less than the number (in this case, 64). $84 \geq 64$, so there is a 1 in the 64's column, leaving $84 - 64 = 20$. $20 < 32$, so there is a 0 in the 32's column. $20 \geq 16$, so there is a 1 in the 16's column, leaving $20 - 16 = 4$. $4 < 8$, so there is a 0 in the 8's column. $4 \geq 4$, so there is a 1 in the 4's column, leaving $4 - 4 = 0$. Thus there must be 0's in the 2's and 1's column. Putting this all together, $84_{10} = 1010100_2$.

Working from the right, repeatedly divide the number by 2. The remainder goes in each column. $84/2 = 42$, so 0 goes in the 1's column. $42/2 = 21$, so 0 goes in the 2's column. $21/2 = 10$ with a remainder of 1 going in the 4's column. $10/2 = 5$, so 0 goes in the 8's column. $5/2 = 2$ with a remainder of 1 going in the 16's column. $2/2 = 1$, so 0 goes in the 32's column. Finally $1/2 = 0$ with a remainder of 1 going in the 64's column. Again, $84_{10} = 1010100_2$

1.5 LOGIC GATES

Now that we know how to use binary variables to represent information, we explore digital systems that perform operations on these binary variables. *Logic gates* are simple digital circuits that take one or more binary inputs and produce a binary output. Logic gates are drawn with a symbol showing the input (or inputs) and the output. Inputs are usually drawn on the left (or top) and outputs on the right (or bottom). Digital designers typically use letters near the beginning of the alphabet for gate inputs and the letter *Y* for the gate output. The relationship between the inputs and the output can be described with a *truth table* or a Boolean equation. A truth table lists inputs on the left and the corresponding output on the right. It has one row for each possible combination of inputs. A *Boolean equation* is a mathematical expression using binary variables.

1.5.1 NOT Gate

A *NOT gate* has one input, *A*, and one output, *Y*, as shown in Figure 1.12. The NOT gate's output is the inverse of its input. If *A* is FALSE, then *Y* is TRUE. If *A* is TRUE, then *Y* is FALSE. This relationship is summarized by the truth table and Boolean equation in the figure. The line over A in the Boolean equation is pronounced *NOT*, so $Y = \bar{A}$ is read "*Y* equals NOT *A*." The NOT gate is also called an *inverter*.

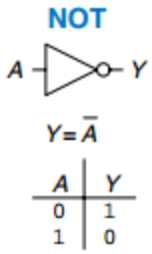


Figure 1.12 NOT gate

1.5.2 Buffer

The other one-input logic gate is called a *buffer* and is shown in Figure 1.13. It simply copies the input to the output.

From the logical point of view, a buffer is no different from a wire, so it might seem useless. However, from the analog point of view, the buffer might have desirable characteristics such as the ability to deliver large amounts of current to a motor or the ability to quickly send its output to many gates. This is an example of why we need to consider multiple levels of abstraction to fully understand a system; the digital abstraction hides the real purpose of a buffer.

The triangle symbol indicates a buffer. A circle on the output is called a *bubble* and indicates inversion, as was seen in the NOT gate symbol of Figure 1.12.

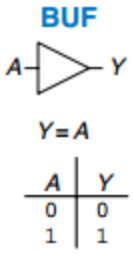
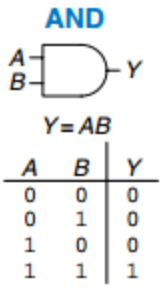


Figure 1.13 Buffer

1.5.3 AND Gate

Two-input logic gates are more interesting. The *AND gate* shown in Figure 1.14 produces a TRUE output, *Y*, if and only if both *A and B* are TRUE. Otherwise, the output is FALSE. By convention, the inputs are listed in the order 00, 01, 10, 11, as if you were counting in binary. The Boolean equation for an AND gate can be written in several ways: $Y = A \cdot B$, $Y = AB$, or $Y = A \cap B$. The $\cap$ symbol is pronounced "intersection" and is preferred by logicians. We prefer $Y = AB$, read "*Y* equals *A* and *B*," because we are lazy.

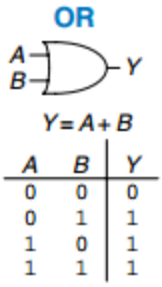


A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

Figure 1.14 AND gate

1.5.4 OR Gate

The *OR gate* shown in Figure 1.15 produces a TRUE output, *Y*, if either *A* or *B* (or both) are TRUE. The Boolean equation for an OR gate is written as $Y = A + B$ or $Y = A \cup B$. The $\cup$ symbol is pronounced union and is preferred by logicians. Digital designers normally use the + notation, $Y = A + B$ is pronounced "*Y* equals *A* or *B*".

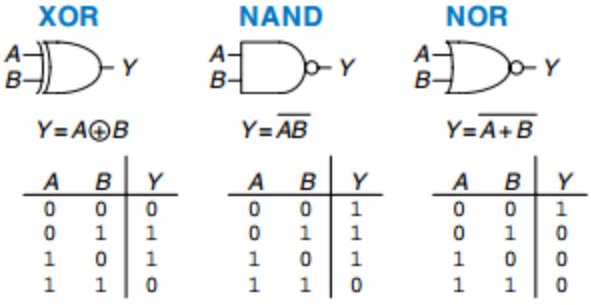


A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

Figure 1.15 OR gate

1.5.5 Other Two-Input Gates

Figure 1.16 shows other common two-input logic gates. *XOR* (exclusive OR, pronounced "ex-OR") is TRUE if *A* or *B*, but not both, are TRUE. Any gate can be followed by a bubble to invert its operation. The *NAND gate* performs NOT AND. Its output is TRUE unless both inputs are TRUE. The *NOR gate* performs NOT OR. Its output is TRUE if neither *A* nor *B* is TRUE.



A	B	Y
0	0	0
0	1	1
1	0	1
1	1	0

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

Figure 1.16 More two-input logic gates

2 Combinational Logic Design

(Note: though the "something to hold the new state" component of our circuits is the part that made them sequential, i.e. state-holding, the rest of the circuit was combinational logic so we will read about this to help us understand our circuits and Boolean equations/operations.)

2.1 INTRODUCTION

In digital electronics, a *circuit* is a network that processes discrete-valued variables. A circuit can be viewed as a black box, shown in Figure 2.1, with

- one or more discrete-valued input terminals
- one or more discrete-valued output terminals
- a functional specification describing the relationship between inputs and outputs

Peering inside the black box, circuits are composed of nodes and elements. An *element* is itself a circuit with inputs, outputs, and a specification. A *node* is a wire, whose voltage conveys a discrete-valued variable. Nodes are classified as *input*, *output*, or *internal*. Inputs receive values from the external world. Outputs deliver values to the external world. Wires that are not inputs or outputs are called internal nodes. Figure 2.2 illustrates a circuit with three elements, E1, E2, and E3, and six nodes. Nodes *A*, *B*, and *C* are inputs. *Y* and *Z* are outputs. n1 is an internal node between E1 and E3.

Digital circuits are classified as *combinational* or *sequential*. A combinational circuit's outputs depend only on the current values of the inputs; in other words, it combines the current input values to compute the output. For example, a logic gate is a combinational circuit. A sequential circuit's outputs depend on both current and previous values of the inputs; in other words, it depends on the input sequence. A combinational circuit is *memoryless*, but a sequential circuit has *memory*.

The functional specification of a combinational circuit expresses the output values in terms of the current input values. The functional specification of a combinational circuit is usually expressed as a truth table or a Boolean equation.

2.2 BOOLEAN EQUATIONS

Boolean equations deal with variables that are either TRUE or FALSE, so they are perfect for describing digital logic. This section defines some terminology commonly used in Boolean equations, then shows how to write a Boolean equation for any logic function given its truth table.

2.2.1 Terminology

The complement of a variable, *A*, is its inverse, $\bar{A}$. The variable or its complement is called a *literal*. For example, *A*, $\bar{A}$, *B*, and are literals. We call *A* the *true form* of the variable and the complementary form; "true form" does not mean that A is TRUE, but merely that A does not have a line over it.

The AND of one or more literals is called a *product* or an *implicant*. $\bar{A}B$, $A\overline{B}\,\overline{C}$, and B are all implicants for a function of three variables. A *minterm* is a product involving all of the inputs to the function. $A\overline{B}\,\overline{C}$ is a minterm for a function of the three variables *A, B,* and *C*, but $\bar{A}B$ is not, because it does not involve *C*. Similarly, the OR of one or more literals is called a *sum*. A *maxterm* is a sum involving all of the inputs to the function. $A + \overline{B} + C$ is a maxterm for a function of the three variables *A*, *B*, and *C*.

The *order of operations* is important when interpreting Boolean equations. Does $Y = A + BC$ mean Y = (*A* OR *B*) AND *C* or $Y = A$ OR (*B* AND *C*)? In Boolean equations, NOT has the highest *precedence*, followed by AND, then OR. Just as in ordinary equations, products are performed before sums. Therefore, the equation is read as $Y = A$ OR (*B* AND *C*). Equation 2.1 gives another example of order of operations.

$$\overline{A}B + BC\overline{D} = ((\overline{A})B) + (BC(\overline{D})) \qquad (2.1)$$

2.2.2 Sum-of-Products Form

A truth table of *N* inputs contains 2^N rows, one for each possible value of the inputs. Each row in a truth table is associated with a minterm that is TRUE for that row. Figure 2.8 shows a truth table of two inputs, *A* and *B*. Each row shows its corresponding minterm. For example, the minterm for the first row is $\overline{A}\,\overline{B}$ because $\overline{A}\,\overline{B}$ is TRUE when $A = 0$, $B = 0$.

We can write a Boolean equation for any truth table by summing each of the minterms for which the output, *Y*, is TRUE. For example, in Figure 2.8, there is only one row (or minterm) for which the output *Y* is TRUE, shown circled in blue. Thus, $Y = \bar{A}B$. Figure 2.9 shows a truth table with more than one row in which the output is TRUE. Taking the sum of each of the circled minterms gives $Y = \bar{A}B + AB$.

This is called the sum-of-products canonical form of a function because it is the sum (OR) of products (ANDs forming minterms). Although there are many ways to write the same function, such as $Y = B\bar{A} + BA$, we will sort the minterms in the same order that they appear in the truth table, so that we always write the same Boolean expression for the same truth table.

A	B	Y	minterm
0	0	0	$\overline{A}\,\overline{B}$
0	1	1	$\overline{A}\,B$
1	0	0	$A\,\overline{B}$
1	1	0	$A\,B$

Figure 2.8 Truth table and minterms

Canonical form is just a fancy word for standard form. You can use the term to impress your friends and scare your enemies.

A	B	Y	minterm
0	0	0	$\overline{A}\,\overline{B}$
0	1	1	$\overline{A}\,B$
1	0	0	$A\,\overline{B}$
1	1	1	$A\,B$

Figure 2.9 Truth table with multiple TRUE minterms

The sum-of-products form provides a Boolean equation for any truth table with any number of variables. Figure 2.12 shows a random three-input truth table.

A	B	C	Y
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Figure 2.12 Random three-input truth table

The sum-of-products form of the logic function is

$$Y = \overline{A}B\overline{C} + A\overline{B}\overline{C} + A\overline{B}C \qquad (2.2)$$

Unfortunately, sum-of-products form does not necessarily generate the simplest equation. In Section 2.3 we show how to write the same function using fewer terms.

2.3 BOOLEAN ALGEBRA

In the previous section, we learned how to write a Boolean expression given a truth table. However, that expression does not necessarily lead to the simplest set of logic gates. Just as you use algebra to simplify mathematical equations, you can use *Boolean algebra* to simplify Boolean equations. The rules of Boolean algebra are much like those of ordinary algebra but are in some cases simpler, because variables have only two possible values: 0 or 1.

Boolean algebra is based on a set of axioms that we assume are correct. Axioms are unprovable in the sense that a definition cannot be proved. From these axioms, we prove all the theorems of Boolean algebra. These theorems have great practical significance, because they teach us how to simplify logic to produce smaller and less costly circuits.

Axioms and theorems of Boolean algebra obey the principle of *duality*. If the symbols 0 and 1 and the operators • (AND) and + (OR) are interchanged, the statement will still be correct. We use the prime (') symbol to denote the *dual* of a statement.

2.3.1 Axioms

Table 2.1 states the axioms of Boolean algebra. These five axioms and their duals define Boolean variables and the meanings of NOT, AND, and OR. Axiom A1 states that a Boolean variable B is 0 if it is not 1. The axiom's dual, A1', states that the variable is 1 if it is not 0. Together, A1 and A1' tell us that we are working in a Boolean or binary field of 0's and 1's. Axioms A2 and A2' define the NOT operation. Axioms A3 to A5 define AND; their duals, A3' to A5' define OR.

Table 2.1 Axioms of Boolean algebra

	Axiom		Dual	Name
A1	$B = 0$ if $B \neq 1$	A1′	$B = 1$ if $B \neq 0$	Binary field
A2	$\overline{0} = 1$	A2′	$\overline{1} = 0$	NOT
A3	$0 \bullet 0 = 0$	A3′	$1 + 1 = 1$	AND/OR
A4	$1 \bullet 1 = 1$	A4′	$0 + 0 = 0$	AND/OR
A5	$0 \bullet 1 = 1 \bullet 0 = 0$	A5′	$1 + 0 = 0 + 1 = 1$	AND/OR

2.3.2 Theorems of One Variable

Theorems T1 to T5 in Table 2.2 describe how to simplify equations involving one variable.

Table 2.2 Boolean theorems of one variable

	Theorem		Dual	Name
T1	$B \bullet 1 = B$	T1′	$B + 0 = B$	Identity
T2	$B \bullet 0 = 0$	T2′	$B + 1 = 1$	Null Element
T3	$B \bullet B = B$	T3′	$B + B = B$	Idempotency
T4		$\overline{\overline{B}} = B$		Involution
T5	$B \bullet \overline{B} = 0$	T5′	$B + \overline{B} = 1$	Complements

The *identity* theorem, T1, states that for any Boolean variable B, B AND 1 = B. Its dual states that B OR 0 = B. In hardware, as shown in Figure 2.14, T1 means that if one input of a two-input AND gate is always 1, we can remove the AND gate and replace it with a wire connected to the variable input (B). Likewise, T1' means that if one input of a two-input OR gate is always 0, we can replace the OR gate with a wire connected to B. In general, gates cost money, power, and delay, so replacing a gate with a wire is beneficial.

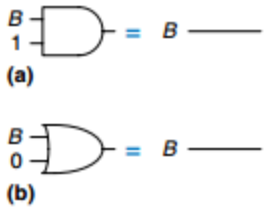


Figure 2.14 **Identity theorem in hardware: (a) T1, (b) T1′**

The *null element theorem*, T2, says that *B* AND 0 is always equal to 0. Therefore, 0 is called the *null* element for the AND operation, because it nullifies the effect of any other input. The dual states that *B* OR 1 is always equal to 1. Hence, 1 is the null element for the OR operation. In hardware, as shown in Figure 2.15, if one input of an AND gate is 0, we can replace the AND gate with a wire that is tied LOW (to 0). Likewise, if one input of an OR gate is 1, we can replace the OR gate with a wire that is tied HIGH (to 1).

Idempotency, T3, says that a variable AND itself is equal to just itself. Likewise, a variable OR itself is equal to itself. The theorem gets its name from the Latin roots: *idem* (same) and *potent* (power). The operations return the same thing you put into them. Figure 2.16 shows that idempotency again permits replacing a gate with a wire.

Involution, T4, is a fancy way of saying that complementing a variable twice results in the original variable. In digital electronics, two wrongs make a right. Two inverters in series logically cancel each other out and are logically equivalent to a wire, as shown in Figure 2.17. The dual of T4 is itself.

The *complement theorem*, T5 (Figure 2.18), states that a variable AND its complement is 0 (because one of them has to be 0). And, by duality, a variable OR its complement is 1 (because one of them has to be 1).

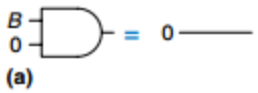


Figure 2.15 Null element theorem in hardware: (a) T2, (b) T2′

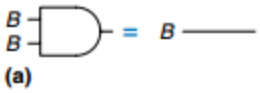


Figure 2.16 Idempotency theorem in hardware: (a) T3, (b) T3′

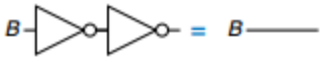


Figure 2.17 Involution theorem in hardware: T4

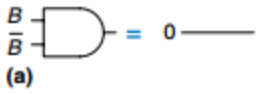


Figure 2.18 Complement theorem in hardware: (a) T5, (b) T5′

2.3.3 Theorems of Several Variables

Theorems T6 to T12 in Table 2.3 describe how to simplify equations involving more than one Boolean variable.

Commutativity and *associativity*, T6 and T7, work the same as in traditional algebra. By commutativity, the *order* of inputs for an AND or OR function does not affect the value of the output. By associativity, the specific groupings of inputs do not affect the value of the output.

The *distributivity theorem*, T8, is the same as in traditional algebra, but its dual, T8', is not. By T8, AND distributes over OR, and by T8', OR distributes over AND. In traditional algebra, multiplication distributes over addition but addition does not distribute over multiplication.

The *covering, combining,* and *consensus* theorems, T9 to T11, permit us to eliminate redundant variables. With some thought, you should be able to convince yourself that these theorems are correct.

De Morgan's Theorem, T12, is a particularly powerful tool in digital design. The theorem explains that the complement of the product of all the terms is equal to the sum of the complement of each term. Likewise, the complement of the sum of all the terms is equal to the product of the complement of each term.

Table 2.3 Boolean theorems of several variables

	Theorem		Dual	Name
T6	$B \bullet C = C \bullet B$	T6′	$B + C = C + B$	Commutativity
T7	$(B \bullet C) \bullet D = B \bullet (C \bullet D)$	T7′	$(B + C) + D = B + (C + D)$	Associativity
T8	$(B \bullet C) + (B \bullet D) = B \bullet (C + D)$	T8′	$(B + C) \bullet (B + D) = B + (C \bullet D)$	Distributivity
T9	$B \bullet (B + C) = B$	T9′	$B + (B \bullet C) = B$	Covering
T10	$(B \bullet C) + (B \bullet \overline{C}) = B$	T10′	$(B + C) \bullet (B + \overline{C}) = B$	Combining
T11	$(B \bullet C) + (\overline{B} \bullet D) + (C \bullet D)$ $= B \bullet C + \overline{B} \bullet D$	T11′	$(B + C) \bullet (\overline{B} + D) \bullet (C + D)$ $= (B + C) \bullet (\overline{B} + D)$	Consensus
T12	$\overline{B_0 \bullet B_1 \bullet B_2 \ldots}$ $= (\overline{B_0} + \overline{B_1} + \overline{B_2} \ldots)$	T12′	$\overline{B_0 + B_1 + B_2 \ldots}$ $= (\overline{B_0} \bullet \overline{B_1} \bullet \overline{B_2})$	De Morgan's Theorem