Measurement and Time Evolution

Observables and Measurement in QM

Consider an observable represented by operator $\hat{Q}$, and a state $|\psi\rangle$, which is normalized. $\hat{Q}$ has eigenvalues $\lambda_1, \lambda_2, \ldots$

$\hat{Q}$ has eigenvectors $|v_1\rangle, |v_2\rangle, \ldots$

By property ③ of Hermitian operators,

$$|\psi\rangle = c_1 |v_1\rangle + c_2 |v_2\rangle + \ldots$$

(eigenvectors span the space).

Postulate III of QM

- If Q is measured, the only possible results of such a measurement are one of the eigenvalues $\lambda_1, \lambda_2, \lambda_3$, etc....
- The probability of getting λ_1 is $P(\lambda_1) = |\langle v_1|\psi\rangle|^2 = |c_1|^2$ where $|v_1\rangle$ is a normalized eigenfunction.
- After the measurement, the state of the system has collapsed to the eigenvector $|v_i\rangle$ corresponding to the measured value λ_i.

$|\psi\rangle$ (initial state)

$\downarrow$

measure Q

possible outcome :	λ_1	λ_2	λ_3
probability :	$\lvert\langle v_1\vert\psi\rangle\rvert^2$	$\lvert\langle v_2\vert\psi\rangle\rvert^2$	$\lvert\langle v_3\vert\psi\rangle\rvert^2$
State after measurement :	$\vert v_1\rangle$	$\vert v_2\rangle$	$\vert v_3\rangle$

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Time evolution in QM

This will also be review from previous courses. In the absence of measurement, the ket describing a quantum system obeys the Schrödinger Equation.

$$i\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle$$

Postulate IV of QM

H is an observable (the energy), so $\hat{H}$ is Hermitian ... any vector can be expressed as a combination of its eigenstates. Let's work in the "energy basis" with states labeled by $|n\rangle$. $\hat{H}|n\rangle = E_n|n\rangle$

(I could keep calling them $|e_n\rangle$ ... but this is simpler).

$\hat{H}$ is Hermitian so any vector can be expressed as a combination of eigenstates.

$|\psi\rangle = \sum_n c_n |n\rangle$ so the Schrödinger equation is

$$i\hbar \frac{d}{dt}\sum_n c_n|n\rangle = \hat{H}\sum_n c_n|n\rangle = \sum_n c_n \hat{H}|n\rangle$$

Now, $\hat{H}$ can be represented by a matrix and the energy eigenstates are found from that matrix.

If $\hat{H}$ doesn't depend on time, then the matrix doesn't depend on time, so its eigenvalues don't either. But c_n could depend on t.

$$i\hbar \sum_n |n\rangle \frac{d}{dt}c_n(t) = \sum_n E_n c_n(t)\cdot|n\rangle$$

$$i\hbar \sum_n |n\rangle \frac{d}{dt} c_n(t) = \sum_n E_n c_n(t) \cdot |n\rangle$$

Now multiply both sides on the left by $\langle m|$, one particular eigenstate.

$$i\hbar \sum_n \underbrace{\langle m|n\rangle} \frac{d}{dt} c_n(t) = \sum_n E_n c_n(t) \langle m|n\rangle$$

$= \delta_{mn}$ only nonzero if $m=n$.
Since $\hat{H}$ is Hermitian, these eigenstates are orthogonal and can be normalized. Only on term in the sum survives.

$$i\hbar \cdot \dot{c}_m(t) = E_m c_m(t)$$

$$\frac{dc_m}{dt} = -\frac{iE_m t}{\hbar} c_m(t)$$, solution is

$$c_m(t) = c_m(0) e^{-\frac{iE_m t}{\hbar}}$$

which is familiar from Griffiths Ch. 2.

$$\therefore \quad |\psi\rangle = \sum_n c_n(t)|n\rangle$$

$$|\psi\rangle = \sum_n c_n(0) e^{-\frac{iE_n t}{\hbar}} |n\rangle$$

Each eigenstate carries its own time dependence

e.g. For a 2-state system:

$$|\psi\rangle = \underbrace{c_1(0)} e^{-\frac{iE_1 t}{\hbar}} |1\rangle + \underbrace{c_2(0)} e^{-\frac{iE_2 t}{\hbar}} |2\rangle$$

I'll probably go back to writing these as just "c_1", "c_2" to save writing.

Measurement and Time Evolution

I'd like to get some practice now with an example, but first lets' warm up:

Poll Q: A basis has 2 (orthonormal) basis vectors: $|e_1\rangle$ and $|e_2\rangle$. An operator $\hat{M} = |e_2\rangle\langle e_1|$. What is the matrix M in this basis?

A.) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ B.) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

C.) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ D.) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Example: Based on Griffiths 3.25

A certain two-level system (2 energy levels) is

$$\hat{H} = \varepsilon\left[\,|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2| + \sqrt{3}\,|e_1\rangle\langle e_2| + \sqrt{3}\,|e_2\rangle\langle e_1|\,\right]$$

where ε is a real # w/dimensions of energy.

with respect to an orthonormal basis. $|e_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|e_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

a.) Find the matrix H in this basis.

b.) Determine the eigenvectors and eigenvalues of H in this basis.

c.) At $t=0$, the system is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|e_1\rangle + |e_2\rangle\right)$. if an energy measurement is made, at time $t=0$, what's the probability you'll get the higher of the two energies?

d.) If a measurement isn't made at $t=0$, what is $|\psi(t)\rangle$ in terms of the basis kets $|e_1\rangle$, $|e_2\rangle$?

We need the matrix elements

$$H = \begin{pmatrix} \langle e_1|\hat{H}|e_1\rangle & \langle e_1|\hat{H}|e_2\rangle \\ \langle e_2|\hat{H}|e_1\rangle & \langle e_2|\hat{H}|e_2\rangle \end{pmatrix}$$

$\langle e_1|\hat{H}|e_1\rangle =$

$$\varepsilon\,\langle e_1|\left[\,|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2| + \sqrt{3}\,|e_1\rangle\langle e_2| + \sqrt{3}\,|e_2\rangle\langle e_1|\,\right]|e_1\rangle$$

$$= \varepsilon\Big[\langle e_1|e_1\rangle\langle e_1|e_1\rangle - \langle e_1|e_2\rangle\langle e_2|e_1\rangle + \sqrt{3}\,\langle e_1|e_1\rangle\langle e_2|e_1\rangle + \sqrt{3}\,\langle e_1|e_2\rangle\langle e_1|e_1\rangle\Big]$$

$$= \varepsilon\left[1\cdot 1 - 0\cdot 0 + \sqrt{3}\cdot 0 + \sqrt{3}\,0\cdot 1\right] = \varepsilon$$

Similar:

$$\langle e_1|\hat{H}|e_2\rangle = \sqrt{3}\,\varepsilon$$
$$\langle e_2|\hat{H}|e_1\rangle = \sqrt{3}\,\varepsilon \quad \Rightarrow \quad \hat{H} \to \begin{pmatrix} \varepsilon & \sqrt{3}\,\varepsilon \\ \sqrt{3}\,\varepsilon & -\varepsilon \end{pmatrix}$$
$$\langle e_2|\hat{H}|e_2\rangle = -\varepsilon$$

Note: $\hat{H}^\dagger = \hat{H}$, so the matrix is Hermitian, good.

use E instead of λ. Eigenvalues are energies!

b.) Eigenvalues:

$$\begin{vmatrix} \varepsilon - E & \sqrt{3}\,\varepsilon \\ \sqrt{3}\,\varepsilon & -(\varepsilon+E) \end{vmatrix} = 0$$

$$-(\varepsilon - E)(\varepsilon + E) - 3\varepsilon^2 = 0$$
$$-(\varepsilon^2 - E^2) - 3\varepsilon^2 = 0$$
$$E^2 - 4\varepsilon^2 = 0$$
$$(E + 2\varepsilon)(E - 2\varepsilon) = 0 \quad \text{so} \quad \boxed{E = \pm 2\varepsilon}$$

Eigenvectors:

#1: $$\begin{pmatrix} \varepsilon & \sqrt{3}\,\varepsilon \\ \sqrt{3}\,\varepsilon & -\varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\varepsilon a + \sqrt{3}\,\varepsilon b = 2\varepsilon a$$
$$\sqrt{3}\, b = a$$
$$b = \frac{1}{\sqrt{3}} a$$

$$\Rightarrow \boxed{E_2 = 2\varepsilon \qquad |2\rangle = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}}$$

#2: $$\begin{pmatrix} \varepsilon & \sqrt{3}\,\varepsilon \\ \sqrt{3}\,\varepsilon & -\varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -2\varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\varepsilon a + \sqrt{3}\,\varepsilon b = -2\varepsilon a$$
$$-3a = \sqrt{3}\, b \Rightarrow a = -\frac{1}{\sqrt{3}} b$$

$$\boxed{E_1 = -2\varepsilon \qquad |1\rangle = \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}}$$

Note $|1\rangle$ & $|2\rangle$ are orthogonal as expected.

c.) Generally, we want to work with an orthonormal set of eigenvectors, so let's normalize them:

$$|2\rangle \rightarrow \frac{1}{2}\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix} \qquad |1\rangle \rightarrow \frac{1}{2}\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix}$$

so that $\langle 1|1\rangle = 1$

$\langle 2|2\rangle = 1$

If $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, we can get the coefficients c_1, c_2:

$$c_1 = \langle 1|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$c_2 = \langle 2|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix}\sqrt{3} & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

So $P(E_2) = \left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2 = 93.3\%$

$P(E_1) = \left|\frac{\sqrt{3}-1}{2\sqrt{2}}\right| = 6.70\%$

They add to 1 as they should!

d.) If we don't measure the energy at $t=0$, we must evolve the ket $|\psi(t)\rangle$ in time. This is done in the energy basis.

$$|\psi(t)\rangle = c_1 e^{\frac{-iE_1 t}{\hbar}}|1\rangle + c_2 e^{\frac{-iE_2 t}{\hbar}}|2\rangle$$

Now substitute $|1\rangle$ and $|2\rangle$

$$|\psi(t)\rangle = c_1 e^{\frac{-i(-2\varepsilon)t}{\hbar}}\frac{1}{2}\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix} + \frac{1}{2}c_2 e^{\frac{-i(2\varepsilon)t}{\hbar}}\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}$$

Let $\omega = \frac{\varepsilon}{\hbar}$

$$|\psi(t)\rangle = c_1 e^{\frac{-i(-2\varepsilon)t}{\hbar}} \frac{1}{2}\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix} + \frac{1}{2} c_2 e^{\frac{-i(2\varepsilon)t}{\hbar}} \begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}$$

$$|\psi(t)\rangle = \frac{1}{4\sqrt{2}}\left\{(\sqrt{3}-1)e^{+2i\omega t}\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix} + (\sqrt{3}+1)e^{-2i\omega t}\begin{pmatrix}\sqrt{3}\\ 1\end{pmatrix}\right\}$$

$$|\psi(t)\rangle = \frac{1}{4\sqrt{2}}\begin{pmatrix}(1-\sqrt{3})e^{2i\omega t} + (3+\sqrt{3})e^{-2i\omega t}\\ (3-\sqrt{3})e^{2i\omega t} + (\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$ with $\omega = \varepsilon/\hbar$.

<u>Check:</u> $|\psi(0)\rangle = \frac{1}{4\sqrt{2}}\begin{pmatrix}1-\cancel{\sqrt{3}}+3+\cancel{\sqrt{3}}\\ 3-\cancel{\sqrt{3}}+\cancel{\sqrt{3}}+1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$

matches given initial conditions.

It is also possible to check $i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

$$i\hbar|\dot{\psi}\rangle = \frac{i\hbar}{4\sqrt{2}}\begin{pmatrix}2i\omega(1-\sqrt{3})e^{2i\omega t} - 2i\omega(3+\sqrt{3})e^{-2i\omega t}\\ 2i\omega(3-\sqrt{3})e^{2i\omega t} - 2i\omega(\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$

$$= -\frac{2\omega\hbar}{4\sqrt{2}}\begin{pmatrix}(1-\sqrt{3})e^{2i\omega t} - (3+\sqrt{3})e^{-2i\omega t}\\ (3-\sqrt{3})e^{2i\omega t} - (\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$

$$= \frac{-\varepsilon}{2\sqrt{2}}\begin{pmatrix}(1-\sqrt{3})e^{2i\omega t} - (3+\sqrt{3})e^{-2i\omega t}\\ (3-\sqrt{3})e^{2i\omega t} - (\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$

And, $$\hat{H}|\psi\rangle = \frac{1}{4\sqrt{2}}\begin{pmatrix}\varepsilon & \sqrt{3}\varepsilon\\ \sqrt{3}\varepsilon & -\varepsilon\end{pmatrix}\begin{pmatrix}(1-\sqrt{3})e^{2i\omega t} + (3+\sqrt{3})e^{-2i\omega t}\\ (3-\sqrt{3})e^{2i\omega t} + (\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$

$$\hat{H}|\psi\rangle = \frac{\varepsilon}{4\sqrt{2}}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}\begin{pmatrix}(1-\sqrt{3})e^{2i\omega t} + (3+\sqrt{3})e^{-2i\omega t}\\ (3-\sqrt{3})e^{2i\omega t} + (\sqrt{3}+1)e^{-2i\omega t}\end{pmatrix}$$

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$$\hat{H}|\psi\rangle = \frac{\varepsilon}{4\sqrt{2}}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} (1-\sqrt{3})e^{2i\omega t} + (3+\sqrt{3})e^{-2i\omega t} \\ (3-\sqrt{3})e^{2i\omega t} + (\sqrt{3}+1)\,e^{-2i\omega t} \end{pmatrix}$$

$$= \frac{\varepsilon}{4\sqrt{2}}\begin{pmatrix} e^{2i\omega t}\left[1-\sqrt{3}+3\sqrt{3}-3\right] + e^{-2i\omega t}\left[3+\sqrt{3}+3+\sqrt{3}\right] \\ e^{2i\omega t}\left[\sqrt{3}-3-3+\sqrt{3}\right] + e^{-2i\omega t}\left[3\sqrt{3}+3-\sqrt{3}-1\right] \end{pmatrix}$$

$$= \frac{\varepsilon}{4\sqrt{2}}\begin{pmatrix} e^{2i\omega t}(-2)(1-\sqrt{3}) & - e^{-2i\omega t}(-2)(3+\sqrt{3}) \\ e^{2i\omega t}\cdot(-2)(3-\sqrt{3}) & - e^{-2i\omega t}(-2)(\sqrt{3}+1) \end{pmatrix}$$

$$= \frac{\varepsilon(-2)}{4\sqrt{2}}\begin{pmatrix} e^{2i\omega t}(1-\sqrt{3}) - e^{-2i\omega t}(3+\sqrt{3}) \\ e^{2i\omega t}(3-\sqrt{3} - e^{-2i\omega t}(\sqrt{3}+1)) \end{pmatrix}$$

matches $i\hbar\,|\dot{\psi}(t)\rangle$ on previous page ✓