Reference: Griffiths 8.1

The Variational Principle

Here is another technique to address a quantum system which you don't know how to solve exactly.

The idea is (almost embarrasingly) simple. Suppose you're trying to solve the time-independent Schrödinger Equation

$\hat{H}\psi = E\psi$, and you'd like to know the ground state energy E_{gs}.

Now, if you can't solve the Schrödinger equation, you don't know ψ_{GS} or E_{gs}. <u>But</u>, you can determine an <u>upper bound</u> on E_{GS} as follows.

1. Make up a normalized wave function ϕ. This function will probably not be an energy eigenstate of $\hat{H}$ (unless you made a very lucky guess.)
2. Calculate $\langle \phi | \hat{H} | \phi \rangle$, the expectation value of H in this state.
3. I claim that $\langle \phi | \hat{H} | \phi \rangle \geq E_{GS}$ for <u>any</u> wave function ϕ.

<u>proof</u>: Because ϕ is a function, it can be expressed as a combination of the (unknown) true energy eigenstates. $\psi_n(x)$

$$\phi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

$$\hat{H}\phi(x) = \sum_{n=1}^{\infty} c_n \hat{H}\psi_n(x) = \sum_{n=1}^{\infty} c_n E_n \psi_n(x)$$

$$\phi^*(x)\hat{H}\phi(x) = \left[\sum_{m=1}^{\infty} c_m^* \psi_m^*(x)\right]\left[\sum_{n=1}^{\infty} c_n E_n \psi_n(x)\right]$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_n c_m^* E_n \psi_m^*(x)\psi_n(x)$$

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Now calculate $\langle \phi | \hat{H} | \phi \rangle = \int_{-\infty}^{\infty} \phi^*(x) \hat{H} \phi(x)\, dx$

$$\langle \phi | \hat{H} | \phi \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n c_m^* E_n \underbrace{\int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x)\, dx}_{= \delta_{m,n}}$$

By orthonormality:

$$\langle \phi | \hat{H} | \phi \rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n c_m^* E_n \delta_{m,n}$$

In the double sum, only terms where $m = n$ are non-zero.

$$= \sum_{n=1}^{\infty} c_n c_n^* E_n$$

$$= \sum_{n=1}^{\infty} |c_n|^2 E_n \geq \sum_{n=1}^{\infty} |c_n|^2 E_{gs}$$

since $E_n \geq E_{gs}$

$$\langle \phi | \hat{H} | \phi \rangle \geq E_{gs} \sum_{n=1} |c_n|^2$$

Probability that system is found in state ψ_n if a measurement is made.

$$\langle \phi | \hat{H} | \phi \rangle \geq E_{gs} \qquad \text{since } \sum_n |c_n|^2 = 1$$

which is what we wanted to show. ∎

Put another way, $\langle \phi | \hat{H} | \phi \rangle = \langle E \rangle$ in the state $\phi(x)$. It's the average of a bunch of energy measurements. This average will be equal to E_{gs} if $\phi = \psi_{GS}$. But if it's any other state, there will be some probability that measuring E will collapse to $\psi_2, \psi_3, \ldots$ etc ... and therefore $\langle E \rangle > E_{GS}$.

This technique is most useful if your guess $\phi_a(x)$ depends on some parameter (call it "a"). Then, by minimizing $\langle E \rangle$ as a function of a, you have an upper bound on E_{gs}.

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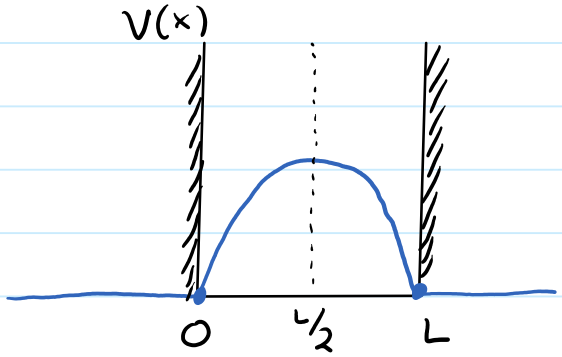
<u>Example</u>: (Griffiths 8.16). Suppose you didn't know the solution to the infinite square well. Find an upper bound on E_{gs} using



a.) $\psi(x) = A\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)$

b.) $\psi(x) = A\left[\left(\frac{x}{L}\right)\left(1-\frac{x}{L}\right)\right]^p$

and find value of p which minimizes the ground state energy.

a.) Normalize: $\int_0^L |\psi(x)|^2 dx = 1$

$$1 = |A|^2 \int_0^L \left(\frac{x}{L}\right)^2\left(1-\frac{x}{L}\right)^2 dx \qquad u = \frac{x}{L} \qquad du = \frac{dx}{L}$$

$$1 = |A|^2 \int_0^1 u^2(1-u)^2 L\,du$$

$$\frac{1}{L} = |A|^2 \int_0^1 u^2(1-u)^2 du \Rightarrow \frac{1}{L} = |A|^2\left[\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right]$$

$$|A| = \sqrt{30/L}$$

Now, calculate $\langle\psi|\hat{H}|\psi\rangle = \int_0^L \psi(x)^* \frac{\hat{p}^2}{2m}\psi(x)\,dx$

$$\hat{p} = (-i\hbar)\frac{\partial}{\partial x}$$

$$\langle\psi|\hat{H}|\psi\rangle = \frac{(-i\hbar)^2}{2m}\int_0^1 \psi(u)^* \frac{\partial^2}{\partial x^2}\psi(u)\,(L\cdot du)$$

Change derivative $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{1}{L}\frac{\partial\psi}{\partial u}$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{L^2}\frac{\partial^2\psi}{\partial u^2}$$

$$\langle\psi|\hat{H}|\psi\rangle = \frac{(-i\hbar)^2}{2m}\int_0^1 \psi^*(u)\cdot\frac{1}{L^2}\frac{\partial^2}{\partial u^2}\psi(u)\cdot L\cdot du$$

$$= \frac{-\hbar^2}{2m}\cdot\frac{1}{L}\,|A|^2\int_0^1 u(1-u)\cdot\underbrace{\frac{\partial^2}{\partial u^2}\left[u(1-u)\right]}_{(-2)}du$$

$$= \frac{\hbar^2}{mL}\cdot\left(\frac{30}{L}\right)\cdot\frac{1}{6} = \boxed{\frac{5\hbar^2}{mL^2}}$$

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Therefore, $E_{gs} \leq \frac{5\hbar^2}{mL^2}$

Note, the actual value is $E_{gs} = \frac{\pi^2\hbar^2}{2mL^2} = 4.93 \cdot \frac{\hbar^2}{mL^2}$

It works! $(4.93) \leq 5$ [within ~2% of the true value].

b.) Proceed as above. Normalize:

$$\phi(u) = A[u(1-u)]^p \qquad L|A|^2\int_0^1 |\phi(u)|^2 du = 1$$

$$L|A|^2 \int_0^1 [u(1-u)]^{2p} = 1$$

Leave as-is for now.

And, $$\langle\phi|\hat{H}|\phi\rangle = \frac{-\hbar^2}{2m}\cdot\frac{1}{L}\,|A|^2\int_0^1 [u(1-u)]^p \cdot \frac{\partial^2}{\partial u^2}[u(1-u)]^p du$$

integrate by parts: $$+\frac{\hbar^2|A|^2}{2mL}\int_0^1 \left\{\frac{\partial}{\partial u}[u(1-u)]^p\right\}^2$$

$$+\frac{\hbar^2|A|^2}{2mL}\cdot\int_0^1 \left\{p\,[u(1-u)]^{p-1}\cdot[1-2u]\right\}^2 du$$

$$\langle\phi|\hat{H}|\phi\rangle = +\frac{\hbar^2}{2mL^2}\cdot \underbrace{p^2\frac{\int_0^1 [u(1-u)]^{2p-2}[1-2u]^2 du}{\int_0^1 [u(1-u)]^{2p} du}}_{\text{Mathematica}}$$

```
FullSimplify[Integrate[(u (1 - u))^(2 p-2) (1 - 2 u)^2, {u, 0, 1}] / Integrate[u^(2 p) (1 - u)^(2 p), {u, 0, 1}]]
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$$-\frac{2+8p}{p-2p^2} \text{ if } \mathrm{Re}[p] > \frac{1}{2}$$

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$$\langle \phi | \hat{H} | \phi \rangle = \frac{\hbar^2}{mL^2} \underbrace{\left[\frac{p}{2} \frac{(2+8p)}{(2p-1)} \right]}$$

minimize this!

```
Plot[p/2 (2 + 8 p)/(2 p - 1), {p, 1/2, 3},
 Frame → True,
 FrameLabel → {"p", "<ϕ|H|ϕ>/(h²/mL²)"},
 LabelStyle → {16, Bold}]
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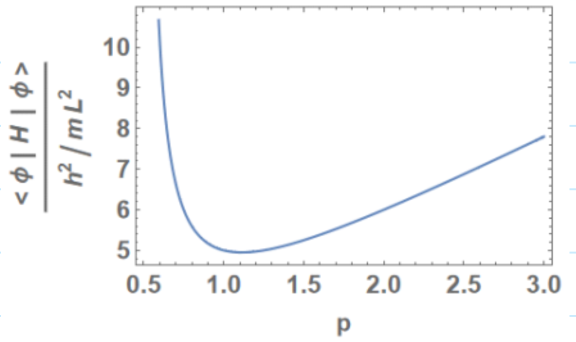


$$\frac{d}{dp}\left[\frac{p(1+4p)}{2p-1} \right] = 0$$

$$\frac{1+8p}{(2p-1)} - \frac{p(1+4p)}{(2p-1)^2} \cdot 2 = 0$$

$$(1+8p)(2p-1) - 2p - 8p^2 = 0$$
$$16p^2 - 6p - 1 - 2p - 8p^2 = 0$$
$$8p^2 - 8p - 1 = 0 \qquad p = \frac{8 \pm \sqrt{64+32}}{16} = \frac{8 \pm 4\sqrt{6}}{16} = \frac{1}{2} \pm \frac{\sqrt{6}}{4}$$

$$\approx -0.11 \text{ or } 1.11$$

only the + sign is a minimum, see graph above.

$$\text{So, } \langle \phi | \hat{H} | \phi \rangle = \frac{\hbar^2}{mL^2}\left[\frac{p(1+4p)}{2p-1} \right] = \frac{\hbar^2}{mL^2}\left[\frac{(\frac{1}{2}+\frac{\sqrt{6}}{4})(1+2+\sqrt{6})}{1+\frac{\sqrt{6}}{2} - 1} \right]$$

$$= \frac{\hbar^2}{mL^2}\left[\frac{(\frac{1}{2}+\frac{\sqrt{6}}{4})(3+\sqrt{6})}{\sqrt{6}/2} \right] = \frac{\hbar^2}{mL^2}\left[\frac{(1+\frac{\sqrt{6}}{2})(3+\sqrt{6})}{\sqrt{6}} \right]$$

$$= \frac{\hbar^2}{mL^2}\left[\frac{6+\frac{5\sqrt{6}}{2}}{\sqrt{6}} \right] = \frac{\hbar^2}{mL^2}\left[\sqrt{6} + \frac{5}{2} \right]$$

$$\boxed{\therefore \quad E_{gs} \leq \frac{\hbar^2}{mL^2} \cdot (4.949)}$$

$$\left[\text{Actual value} = \frac{\hbar^2}{mL^2}(4.935) \right]$$

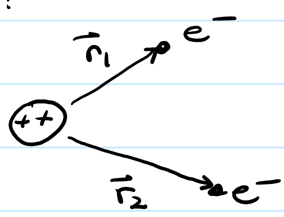
We are within about 0.3% of the true value!

Application: The Ground State energy of Helium

We analyzed the Helium atom earlier in the course:

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} - \frac{2ke^2}{r_1} - \frac{2ke^2}{r_2} + \frac{ke^2}{|\vec{r}_1 - \vec{r}_2|}$$

electron repulsion



Previously, we ignored the e-e repulsion term to get

$$\psi_{11}(\vec{r}_1, \vec{r}_2) = \frac{8}{\pi a^3} e^{-\frac{2}{a}(r_1 + r_2)}$$

and $E_{11} = 4 \cdot (-13.6\,eV)\left(\frac{1}{1^2} + \frac{1}{1^2}\right)$

$= -109\,eV$

This doesn't compare very well with the experimental value of $E_{gs} \cong -79\,eV$. So how can we do better?

We know $\psi_{11}(\vec{r}_1, \vec{r}_2)$ is <u>not</u> the actual ground state energy; we calculated it by ignoring the e-e repulsion. But, we <u>can</u> use this in the variational principle.

$\langle \psi_{11} | \hat{H} | \psi_{11} \rangle \geq E_{gs}$ to get an upper bound on E_{gs}.

Let's write

$$\hat{H} = \underbrace{\frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} - \frac{2ke^2}{r_1} - \frac{2ke^2}{r_2}}_{\hat{H}^0} + \underbrace{\frac{ke^2}{|\vec{r}_1 - \vec{r}_2|}}_{\hat{H}_{int}}$$

(interaction)

Now, we know that $\hat{H}^0 |\psi_{11}\rangle = E_{11} |\psi_{11}\rangle$. This was the solution to the Schrödinger equation with $\hat{H}_{int} = 0$

$$\langle \psi_{11} | \hat{H} | \psi_{11} \rangle = \langle \psi_{11} | \overset{\text{act}}{\hat{H}^0} | \psi_{11} \rangle + \langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle \geq E_{gs}$$

$$= E_{11} \langle \psi_{11} | \psi_{11} \rangle + \langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle \geq E_{gs}$$

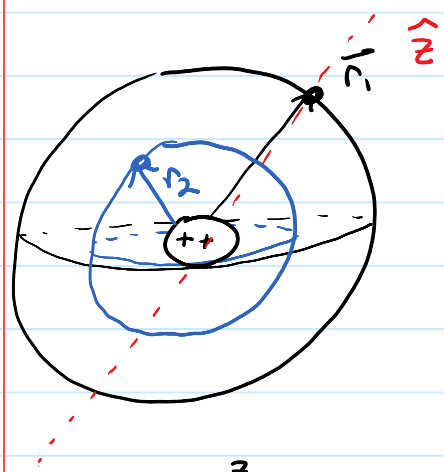
So, let's calculate $\langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle$

Remember $\psi_{11}(\vec{r}_1, \vec{r}_2)$ depends on both $\vec{r}_1$ and $\vec{r}_2$ so we need

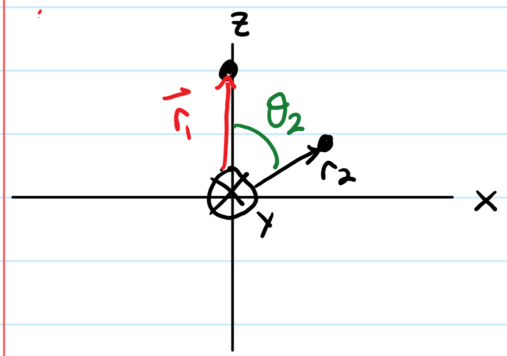
$$\int_0^\infty dr_1 \int_0^\infty dr_2 \underbrace{\int d\Omega_1} \int d\Omega_2 \; \psi_{11}^*(r_1,r_2) \frac{ke^2}{|\vec{r}_1 - \vec{r}_2|} \psi_{11}(r_1,r_2) \cdot r_1^2 r_2^2$$

Remember $\int d\Omega_1 = \int_0^{2\pi}\int_0^{\pi} \sin\theta_1 \, d\theta_1 \, d\phi_1$

from $\underline{\underline{r^2}} \sin\theta \, dr \, d\theta \, d\phi$ volume element.



The atom is spherically symmetric, so we have the freedom to choose our coordinates.
Let's chose $\hat{z}$ to point in the direction of $\vec{r}_1$
We can also rotate our axes so that $\vec{r}_2$ lies in the x-z plane. Then, we have



Now,
$$|\vec{r}_1 - \vec{r}_2| = \sqrt{(\vec{r}_1 - \vec{r}_2)\cdot(\vec{r}_1 - \vec{r}_2)}$$
$$= \sqrt{r_1^2 + r_2^2 - 2\vec{r}_1 \cdot \vec{r}_2}$$
$$= \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta_2}$$

Plug into the integrals!

Nothing depends on ϕ_2 so $\int_0^{2\pi} d\phi_2 = 2\pi$

$\langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle =$

$$ke^2 \left(\frac{64}{\pi^2 a^6}\right) \int_0^\infty dr_1 \int_0^\infty dr_2 \underbrace{\int d\Omega_1} \int d\Omega_2 \; e^{-\frac{2}{a}(r_1+r_2)} \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta_2}} e^{-\frac{2}{a}(r_1+r_2)} \cdot r_1^2 r_2^2$$

Nothing depends on θ_1 or ϕ_1, so $\int d\Omega = \int_0^{2\pi} d\phi_1 \int_0^{\pi} \sin\theta_1 \, d\theta_1 = 4\pi$

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Cleaning it up a bit:

$\langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle =$

$$\frac{ke^2 64}{\pi^2 a^6} \cdot 8\pi^2 \cdot \int_0^\infty dr_1 \int_0^\infty dr_2 \int_0^\pi \frac{e^{-\frac{4}{a}(r_1+r_2)} r_1^2 r_2^2}{\sqrt{r_1^2+r_2^2-2r_1r_2\cos\theta}} \sin\theta_2 \, d\theta_2$$

Now, do the θ integral:

$$= \frac{512 \cdot ke^2}{a^6} \int_0^\infty dr_1 \int_0^\infty dr_2 \cdot \frac{e^{-\frac{4}{a}(r_1+r_2)} \cdot r_1^2 r_2^2}{\sqrt{r_1^2+r_2^2}} \int_0^\pi \frac{\sin\theta}{\sqrt{1 - \frac{2r_1r_2\cos\theta}{(r_1^2+r_2^2)}}} d\theta$$

$$\int_0^\pi \frac{\sin\theta}{\sqrt{1-b\cos\theta}} d\theta \qquad \text{Let } u = 1-b\cos\theta$$
$$du = b \cdot \sin\theta \, d\theta$$

$$\int_{1-b}^{1+b} \frac{\frac{1}{b} \cdot du}{\sqrt{u}} = \frac{2}{b} \sqrt{u} \Big|_{1-b}^{1+b} = \frac{2}{b}\left(\sqrt{1+b} - \sqrt{1-b}\right)$$

Now, $b \equiv \dfrac{2r_1r_2}{r_1^2+r_2^2}$

$$1+b = \frac{r_1^2+r_2^2+2r_1r_2}{r_1^2+r_2^2} = \frac{(r_1+r_2)^2}{(r_1^2+r_2^2)}$$

$$1-b = \frac{r_1^2+r_2^2-2r_1r_2}{r_1^2+r_2^2} = \frac{(r_1-r_2)^2}{r_1^2+r_2^2}$$

$$2\left(\frac{\sqrt{1+b} - \sqrt{1-b}}{b}\right) = \left[\frac{(r_1+r_2)}{\sqrt{r_1^2+r_2^2}} - \frac{|r_1-r_2|}{\sqrt{r_1^2+r_2^2}}\right] \frac{(r_1^2+r_2^2)}{r_1r_2}$$

$$2\left(\frac{\sqrt{1+b}-\sqrt{1-b}}{b}\right)=\left[\frac{(r_1+r_2)}{\sqrt{r_1^2+r_2^2}}-\frac{|r_1-r_2|}{\sqrt{r_1^2+r_2^2}}\right]\frac{(r_1^2+r_2^2)}{r_1 r_2}$$

$$=\frac{\sqrt{r_1^2+r_2^2}}{r_1 r_2}\left[(r_1+r_2)-|r_1-r_2|\right]$$

Now, put it together:

$$\frac{512\cdot ke^2}{a^6}\int_0^\infty dr_1\int_0^\infty dr_2\cdot\frac{e^{-\frac{4}{a}(r_1+r_2)}\cdot \cancel{r_1^2 r_2^2}}{\cancel{\sqrt{r_1^2+r_2^2}}}\left[\frac{\cancel{\sqrt{r_1^2+r_2^2}}}{\cancel{r_1 r_2}}\left(r_1+r_2-|r_1-r_2|\right)\right]$$

$$=\frac{512\, ke^2}{a^6}\int_0^\infty dr_1\int_0^\infty dr_2\; e^{-\frac{4}{a}(r_1+r_2)}\, r_1 r_2\underbrace{\left(r_1+r_2-|r_1-r_2|\right)}$$

since $|x|=\begin{cases}x & \text{if } x>0\\ -x & \text{if } x<0\end{cases}\Rightarrow\quad=\begin{cases}2r_2 & \text{if } r_1>r_2\\ 2r_1 & \text{if } r_2>r_1\end{cases}$

So now, lets break the r_2 integral into two intervals:

$$=\frac{1024\, ke^2}{a^6}\int_0^\infty e^{-\frac{4}{a}r_1}dr_1\left[\underbrace{\int_0^{r_1}}_{r_2<r_1} e^{-\frac{4}{a}r_2}\, r_1 r_2^2\, dr_2+\underbrace{\int_{r_1}^\infty}_{r_2>r_1} e^{-\frac{4}{a}r_2}\, r_1^2 r_2\, dr_2\right]$$

$$=\frac{1024\, ke^2}{a^a}\int_0^\infty e^{-\frac{4r_1}{a}}r_1^2\left[\frac{1}{r_1}\int_0^{r_1}e^{-\frac{4r_2}{a}}r_2^2\,dr_2+\int_{r_1}^\infty e^{-\frac{4r_2}{a}}r_2\,dr_2\right]dr_1$$

Let $u=\frac{4r_2}{a}$ $\quad du=\frac{4}{a}dr_2$

$$=\frac{1024ke^2}{a^6}\int_0^\infty e^{-\frac{4r_1}{a}}r_1^2\left[\frac{1}{r_1}\int_0^{\frac{4r_1}{a}}e^{-u}\left(\frac{ua}{4}\right)^2\left(\frac{a}{4}\right)du+\int_{4r_1/a}^\infty e^{-u}\left(\frac{ua}{4}\right)\left(\frac{a}{4}\right)du\right]dr_1$$

$$= \frac{1024ke^2}{a^6}\int_0^\infty e^{-\frac{4r_1}{a}} r_1^2\left[\frac{1}{r_1}\int_0^{4r_1/a} e^{-u}\left(\frac{ua}{4}\right)^2\left(\frac{a}{4}\right)du + \int_{4r_1/a}^\infty e^{-u}\left(\frac{ua}{4}\right)\left(\frac{a}{4}\right)du\right]dr_1$$

$$= \frac{1024ke^2}{a^6}\left(\frac{a}{4}\right)^2\int_0^\infty e^{-\frac{4r_1}{a}} r_1^2\left[\left(\frac{a}{4r_1}\right)\int_0^{\frac{4r_1}{a}} e^{-u}u^2du + \int_{\frac{4r_1}{a}}^\infty e^{-u}u\,du\right]dr_1$$

Now, let $z = \frac{4r_1}{a}$ $\quad dz = \frac{4}{a}dr_1$

$$= \frac{64ke^2}{a^4}\int_0^\infty e^{-z}\left(\frac{a}{4}z\right)^2\left[\frac{1}{z}\int_0^z e^{-u}u^2du + \int_z^\infty ue^{-u}du\right]\left(\frac{a}{4}\right)dz$$

$$= \frac{64ke^2}{a^4}\left(\frac{a}{4}\right)^3\int_0^\infty e^{-z}z^2\left[\frac{1}{z}\int_0^z e^{-u}u^2du + \int_z^\infty ue^{-u}du\right]dz$$

$$= \frac{ke^2}{a}\int_0^\infty e^{-z}z^2\Bigg[\underbrace{\frac{1}{z}\int_0^z e^{-u}u^2du}_{\frac{1}{z}\left[2-e^{-z}(2+2z+z^2)\right]} + \underbrace{\int_z^\infty ue^{-u}du}_{e^{-z}(1+z)}\Bigg]dz$$

$$= \frac{ke^2}{a}\int_0^\infty e^{-z}z^2\left[\frac{2}{z} + e^{-z}\left(1+z-\frac{2}{z}-2-z\right)\right]dz$$

$$= \frac{ke^2}{a}\int_0^\infty e^{-z}\left[2z + z^2e^{-z}\left(-\frac{2}{z}-1\right)\right]dz$$

$$= \frac{ke^2}{a}\left\{\underbrace{\int_0^\infty 2ze^{-z}dz}_{2\cdot 1! = 2} - \underbrace{\int_0^\infty e^{-2z}(2z+z^2)dz}_{-\int_0^\infty e^{-x}\left(x+\frac{x^2}{4}\right)\frac{1}{2}dx}\right\}$$

Let $x = 2z$
$dx = 2dz$

$$-\frac{1}{2}\left[1! + \frac{2!}{4}\right] = -\frac{3}{4}$$

$$\langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle = \frac{ke^2}{a}\left(2 - \frac{3}{4}\right) = \frac{5ke^2}{4a}$$

Recall $E_1 = 13.6\,eV = \frac{ke^2}{2a}$

$$\langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle = \frac{5}{2} E_1 = 34\,eV$$

And so, $E_{gs} \leq E_{11} + \langle \psi_{11} | \hat{H}_{int} | \psi_{11} \rangle$

$$E_{gs} \leq -109\,eV + 34\,eV$$

$$\boxed{E_{gs} \leq -75\,eV}$$

Much closer to the experimental value of $-79\,eV$, about 5% too high.

It is possible to get even better bounds by trying more complicated wave functions. Griffiths has an improved estimate with

$$\psi(r_1, r_2) = \frac{Z^3}{\pi a^3} e^{-\frac{Z(r_1+r_2)}{a}}$$

In this case, $\langle \psi | \hat{H} | \psi \rangle = \left[-2Z^2 + \left(\frac{27}{4}\right) Z\right] E_1$

which is minimized for $Z \approx 1.69$ and

$$E_{gs} \leq -77.5\,eV$$

which is about 2% away from the experimental value.