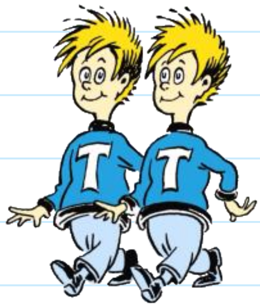
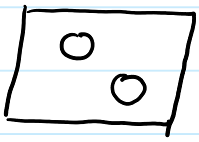
Reference: Griffiths 5.1

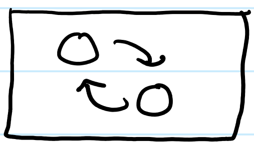
Identical Particles



There is possibly something which might have been bothering you in our previous discussion of two particle wavefunctions. If I have a system of e^+, e^-, then it's easy to say "the electron is particle #1, and the positron is #2. But what about a state with two electrons? How do I decide which one is which?

The answer is you can't. I could measure the position of one electron, but then if I look away for a second & then look back, I'd have no way of knowing which electron was which. Think of a pool table with two perfectly white cue balls on it. Suppose you find them in some configuration like this , and then you leave & go to class.

Later that day, you come back and see the same configuration... you have no way of knowing whether someone switched their positions.



The way to handle this situation for two particle states is to construct a state which is non-committal about which electron is which.

Let's write a combined spin/position state of two particles as

$$\Psi_{12} = \underbrace{\psi(\vec{r}_1, \vec{r}_2)}_{\text{position space wave function}} \underbrace{\chi(1,2)}_{\text{spin state, for example } |\uparrow\downarrow\rangle}$$

Identical Particles

We can construct the symmetrized state as:

$$\Psi_S = \Psi_{12} + \Psi_{21} = \psi(\bar{r}_1, \bar{r}_2)\chi(1,2) + \psi(\bar{r}_2, \bar{r}_1)\chi(2,1)$$

Notice this symmetric in $1 \leftrightarrow 2$, so it doesn't matter which particle is labeled as #1 or #2.

There is also the anti-symmetric combination

$$\Psi_A = \Psi_{12} - \Psi_{21} = \psi(\bar{r}_1, \bar{r}_2)\chi(1,2) - \psi(\bar{r}_2, \bar{r}_1)\chi(2,1)$$

At first sight this doesn't seem to do the job since switching $1 \leftrightarrow 2$ gives an overall minus sign. But remember that adding a minus sign to the wave function does not affect physics. All physical observables and probabilities involve $|\psi|^2$ and so overall minus signs do not matter.

To describe identical particles, then, we must have either a symmetrized or antisymmetrized state. How do I know which one to use? I have to quote a result that can only be derived using relativistic quantum mechanics, called the "spin-statistics theorem".

Identical Particles

A state describing 2 particles with $s_1, s_2 = n$ $(n = 0, 1, 2 \ldots)$ (i.e for particles with integer spin) must be symmetric. These particles are called bosons.

A state describing 2 particles with $s_1, s_2 = (n + \frac{1}{2})$ $(n = 0, 1, 2 \ldots)$ (i.e. for particles with half integer spin) must be anti-symmetric. These particles are called fermions.

Poll Q:

Consider 2 non-interacting particles which are in the same spin state χ. The particles are placed in a 1D infinite square well of length L. If their state is:

$$\frac{2}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)\right]|\chi\rangle|\chi\rangle$$

These particles are

A.) Distinguishable particles

B.) Identical Fermions

C.) Identical Bosons

Identical Particles

Poll Q: Consider 2 non-interacting spin $\frac{1}{2}$ particles. The particles are placed in a 1D infinite square well of length L. Which of the following is a possible state for the particles?

A.) $\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)+\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]|\uparrow\uparrow\rangle$

B.) $\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)-\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]\frac{\left[|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\right]}{\sqrt{2}}$

C.) $\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)-\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]|\downarrow\downarrow\rangle$

Remember, the whole state must be symmetric or antisymmetric. So it is possible to have:

Fermions	Bosons
$\psi \cdot \chi$	$\psi \cdot \chi$
symmetric · antisymmetric	symmetric · symmetric ·
or antisymmetric · symmetric	antisymmetric · antisymmetric

Pauli Exclusion Principle

Two identical fermions cannot be in the same state. For example, take a look at two fermions in the same spin state

$$\Psi = \left[\psi_a(x_1)\psi_b(x_2)-\psi_a(x_2)\psi_b(x_1)\right]|\chi\rangle|\chi\rangle$$

we cannot have $\psi_a=\psi_b$, or the whole thing vanishes!

Identical Particles

Poll Q: Two electrons are in a Helium atom. As an approximation, assume the electrons do not interact (no repulsion). The ground state wave function is

$$\psi_0(\vec{r}) = \frac{8}{\pi a^3} e^{-\frac{2(r_1+r_2)}{a}}$$

where $a = 0.5 \times 10^{-10}$ m (Bohr radius).

The spin state of the electrons must be:

A.) $|\uparrow\uparrow\rangle$

B.) $|\downarrow\downarrow\rangle$

C.) $\frac{1}{2}\left[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right]$

D.) $\frac{1}{2}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right]$

Identical Particles

Example 5.1 (Griffiths)

Two non-interacting particles are placed in an infinite square well

. Assume the particles are in the same spin state so the overall spin state is symmetric. Assume the mass of each is m.

Find the first two energy levels and wave functions if the particles are

a.) Distinguishable

b.) Identical bosons

c.) Identical fermions.

The Schrödinger eqn is $\dfrac{\hat{p}_1^2}{2m}\psi(x_1,x_2) + \dfrac{\hat{p}_2^2}{2m}\psi(x_1,x_2) = E\psi(x_1,x_2)$

Separate variables: $\psi(x_1,x_2) = \psi_1(x_1)\psi_2(x_2)$

$$\psi_2 \frac{\hat{p}_1^2}{2m}\psi_1 + \frac{\psi_1 \hat{p}_2^2 \psi_2}{2m} = E\psi_1\psi_2$$

$$\frac{1}{\psi_1}\frac{\hat{p}_1^2}{2m}\psi_1 = \frac{1}{\psi_2}\left(E - \frac{\hat{p}_2^2}{2m}\psi_2\right)$$

Both sides must be constant call it "E_1"

$$\left.\begin{aligned} \frac{\hat{p}_1^2}{2m}\psi_1 &= E_1\psi_1 \\ \frac{\hat{p}_2^2\psi_2}{2m} &= (E-E_1)\psi_2 \end{aligned}\right\}$$

Single particle Schrödinger eqn for infinite square well.

$$\left.\begin{aligned} \psi_1(x_1) &= \sqrt{\tfrac{2}{L}}\sin\left(\frac{n_1\pi x_1}{L}\right) & E_1 &= \frac{n_1^2\pi^2\hbar^2}{2mL^2} \\ \psi_2(x_2) &= \sqrt{\tfrac{2}{L}}\sin\left(\frac{n_2\pi x_2}{L}\right) & E-E_1 &= \frac{n_2^2\pi^2\hbar^2}{2mL^2} \end{aligned}\right\} \quad E = \frac{\pi^2\hbar^2}{2mL^2}(n_1^2+n_2^2)$$

So $\psi_{n_1 n_2}(x_1,x_2) = \psi_1(x_1)\psi_2(x_2) = \frac{2}{L}\sin\left(\frac{n_1\pi x_1}{L}\right)\sin\left(\frac{n_2\pi x_2}{L}\right)$

with $E = \frac{\pi^2\hbar^2}{2mL^2}(n_1^2+n_2^2)$ solves the 2-particle Schrödinger eqn.

Identical Particles

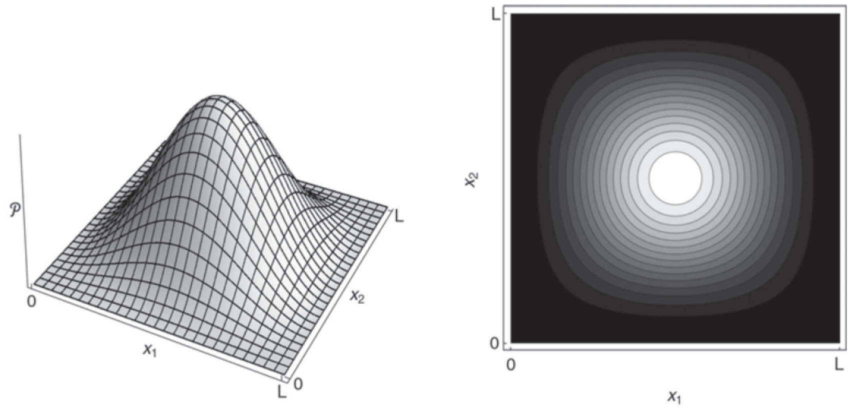
a.) If particles are distinguishable, lowest energy is $n_1=1, n_2=1$
the first excited state is $n_1=1, n_2=2$ or $n_2=2, n_1=1$
(2-fold degenerate).

Ground State

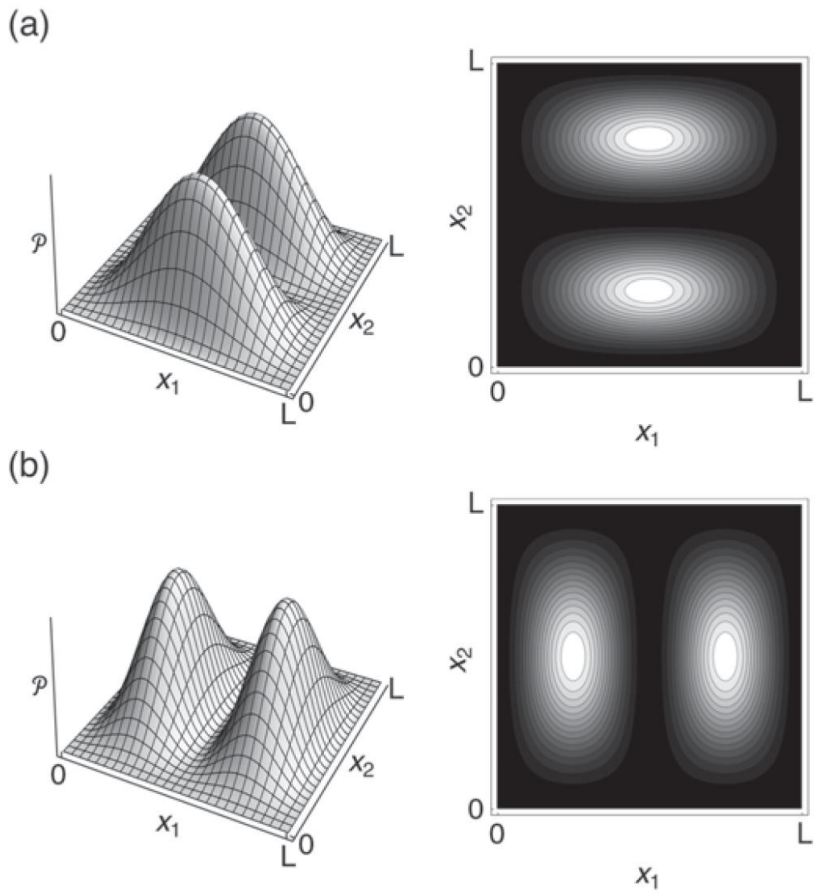
$$\psi_1(x_1,x_2) = \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right) \qquad E_1 = \frac{\pi^2\hbar^2}{2mL^2}(1^2+1^2) = \frac{\pi^2\hbar^2}{mL^2}$$

First Excited States.

$$\left\{\begin{aligned} \psi_{2a}(x_1,x_2) &= \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) \\ \psi_{2b}(x_1,x_2) &= \frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right) \end{aligned}\right\} \qquad E_2 = \frac{\pi^2}{2mL^2}(2^2+1^2) = \frac{5\pi^2\hbar^2}{2mL^2}$$



Ground state probability density $|\psi(x_1,x_2)|^2$ for infinite square well.



Probability density for the first excited state in the infinite square well for two distinguishable particles.

b.) If the particles are identical bosons, we must work with the symmetric combination

Normalization.

$$\psi^S_{n_1 n_2}(x_1,x_2) = A\left[\psi_{n_1 n_2}(x_1,x_2) + \psi_{n_1 n_2}(x_2,x_1)\right]$$

$$= \frac{2A}{L}\left[\sin\left(\frac{n_1\pi x_1}{L}\right)\sin\left(\frac{n_2\pi x_2}{L}\right) + \sin\left(\frac{n_1\pi x_2}{L}\right)\sin\left(\frac{n_2\pi x_1}{L}\right)\right]$$

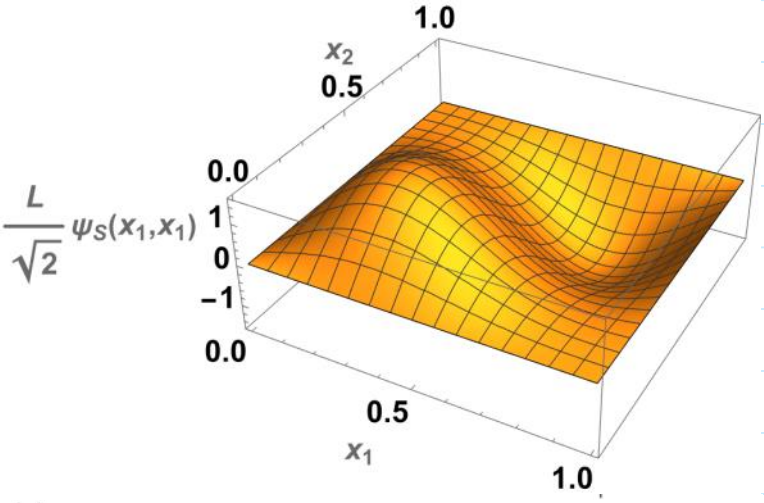
The lowest state is $n_1 = n_2 = 1$, the first excited state is $n_1 = 1, n_2 = 2$ or vice versa (these are identical).

$$\psi^S_1(x_1,x_2) = \frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right) \qquad E_1 = \frac{\pi^2\hbar^2}{2mL^2}$$

$$\psi^S_2(x_1,x_2) = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right] \qquad E_2 = \frac{5\pi^2\hbar^2}{2mL^2}$$

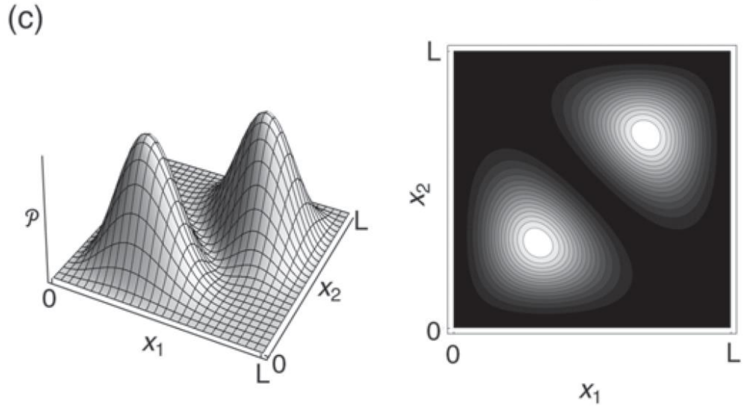
Normalized

Note the first excited state is not degenerate anymore.



First excited state for 2-particle infinite square well in the case of identical bosons.

(c)



Corresponding probability density $|\psi|^2$

c) If the particles are identical fermions, we must work with the anti-symmetric combination

Normalization.

$$\psi^A_{n_1 n_2}(x_1, x_2) = A\left[\psi_{n_1 n_2}(x_1, x_2) - \psi_{n_1 n_2}(x_2, x_1)\right]$$

$$= \frac{2A}{L}\left[\sin\left(\frac{n_1 \pi x_1}{L}\right)\sin\left(\frac{n_2 \pi x_2}{L}\right) - \sin\left(\frac{n_1 \pi x_2}{L}\right)\sin\left(\frac{n_2 \pi x_1}{L}\right)\right]$$

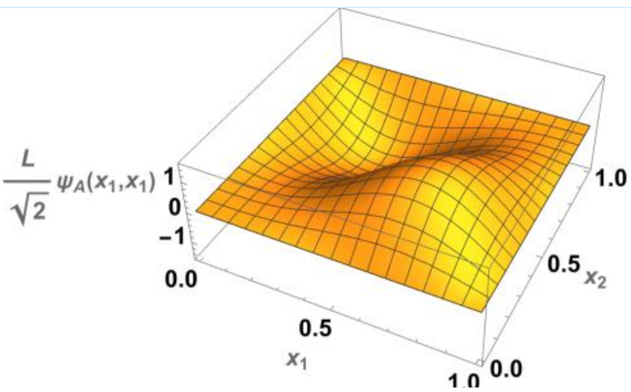
It is no longer possible to have $n_1 = 1$, $n_2 = 1$ since $\psi^A = 0$
the ground state is

$$\psi^A_{12}(x_1, x_2) = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right] \quad E_1 = \frac{5\pi^2\hbar^2}{2mL^2}$$

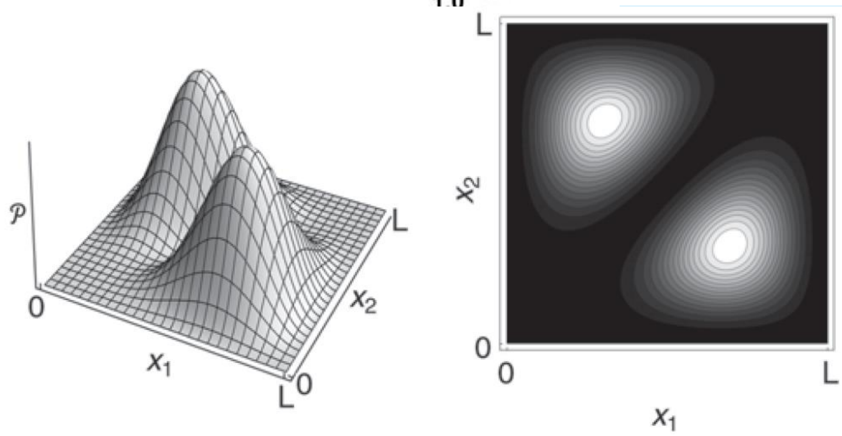
The first excite state is

$$\psi^A_{13}(x_1, x_2) = \frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{3\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{3\pi x_1}{L}\right)\right] \quad E_2 = \frac{10\pi^2\hbar^2}{2mL^2}$$

since $n_1^2 + n_2^2 = 1^2 + 3^2$



Ground State for 2-particle infinite square well in the case of identical fermions.



Corresponding probability density $|\psi|^2$

Exchange Forces

The plots from the previous question lead us to ask: how far apart are the particles on average? Of course, we know this is QM, so we can only talk in terms of averages. If you measure the position of each particle hundreds of times, calculate $(x_2-x_1)^2$ and average the result, what do you get?

[side note: why $(x_2-x_1)^2$ and not x_2-x_1? x_2-x_1 could be positive or negative, so on average $\langle x_2-x_1\rangle = 0$. But $(x_2-x_1)^2$ is always positive so it gives us a meaningful average.

Let's tackle this question in general for a symmetric/anti-symmetric state.

$$\Psi = \frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\right]$$

↑ This normalization assumes $a \neq b$.

where the + sign is for bosons, − for fermions, assuming the particles are in a symmetric spin state.

$$\langle (x_2-x_1)^2\rangle = \langle x_2^2\rangle - 2\langle x_1x_2\rangle + \langle x_1^2\rangle$$

Now, to calculate any expectation value:

$$\langle f(x_1,x_2)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi|^2 f(x_1,x_2)\,dx_1dx_2$$

Calculate $|\Psi|^2$: $\frac{1}{2}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\right]\left[\psi_a^*(x_1)\psi_b^*(x_2) \pm \psi_a^*(x_2)\psi_b^*(x_1)\right]$

$$|\Psi|^2 = \frac{1}{2}\Big(|\psi_a(x_1)|^2|\psi_b(x_2)|^2 + |\psi_a(x_2)|^2|\psi_b(x_1)|^2$$
$$\pm \psi_a(x_1)\psi_b^*(x_1)\,\psi_b(x_2)\psi_a^*(x_2) \pm \psi_a^*(x_1)\psi_b(x_1)\psi_b^*(x_2)\psi_a(x_2)\Big)$$

Identical Particles

Let's calculate $\langle x_1^2 \rangle$

$$\langle x_1^2 \rangle = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 \Big\{ \overbrace{|\psi_a(x_1)|^2 |\psi_b(x_2)|^2}^{1} + \overbrace{|\psi_b(x_1)|^2 |\psi_a(x_2)|^2}^{2}$$

$$\pm \underbrace{\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)}_{3} \pm \underbrace{\psi_a^*(x_1)\psi_b(x_1)\psi_b^*(x_2)\psi_a(x_2)}_{4} \Big\} \, dx_1 dx_2$$

There are 4 terms. For each term, seperate the x_1 and x_2 integrals.

1 $\quad \frac{1}{2} \underbrace{\int_{-\infty}^{\infty} x_1^2 |\psi_a(x_1)|^2 dx_1}_{= \langle x^2 \rangle_a} \cdot \underbrace{\int_{-\infty}^{\infty} |\psi_b(x_2)|^2 dx_2}_{1 \text{ (Normalized)}} = \frac{1}{2} \langle x^2 \rangle_a$

Average of $\langle x^2 \rangle$ in the single particle state ψ_a.

2 $\quad$ Same as 1, but change $a \leftrightarrow b$ $\quad = \frac{1}{2} \langle x^2 \rangle_b$

3 $\quad \pm \frac{1}{2} \int_{-\infty}^{\infty} \psi_a(x_1)\psi_b^*(x_1)\, x_1^2 \, dx_1 \underbrace{\int_{-\infty}^{\infty} \psi_b(x_2)\psi_a^*(x_2)\, dx_2}_{=0 \text{ (orthogonal)}}$

4 $\quad = 0$ for the same reason.

$$\langle x_1^2 \rangle = \frac{\langle x^2 \rangle_a + \langle x^2 \rangle_b}{2}$$

Now, how about $\langle x_2^2 \rangle$? Same as above, just take $x_1 \rightarrow x_2$
The final result doesn't depend on which one is x_1 or x_2.

$$\langle x_2^2 \rangle = \frac{\langle x^2 \rangle_a + \langle x^2 \rangle_b}{2}. \qquad \text{and} \qquad \langle x_1^2 \rangle + \langle x_2^2 \rangle = \langle x^2 \rangle_a + \langle x^2 \rangle_b$$

Finally, how about $\langle x_1 x_2 \rangle$?

$$\langle x_1 x_2\rangle = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 \Big\{ \underbrace{|\psi_a(x_1)|^2|\psi_b(x_2)|^2}_{1} + \underbrace{|\psi_b(x_1)|^2|\psi_a(x_2)|^2}_{2}$$

$$\pm \underbrace{\psi_a(x_1)\psi_b^*(x_1)\psi_b(x_2)\psi_a^*(x_2)}_{3} \pm \underbrace{\psi_a^*(x_1)\psi_b(x_1)\psi_b^*(x_2)\psi_a(x_2)}_{4}\Big\}\, dx_1 dx_2$$

1 $\quad \frac{1}{2}\int_{-\infty}^{\infty} x_1 |\psi_a(x_1)|^2 dx_1 \int_{-\infty}^{\infty} x_2 |\psi_b(x_2)|^2 dx_2 \quad = \frac{1}{2}\langle x\rangle_a \langle x\rangle_b$

2 $\quad$ Same as above, change $a \leftrightarrow b$ $\quad = \frac{1}{2}\langle x\rangle_a \langle x\rangle_b$.

3 $\quad \pm\frac{1}{2}\int_{-\infty}^{\infty} \psi_a(x_1)\psi_b^*(x_1)\, x_1\, dx_1 \underbrace{\int_{-\infty}^{\infty}\psi_b(x_2)\psi_a^*(x_2)\, x_2\, dx_2}$

Integration variable can be renamed to x_1!

$$= \pm\frac{1}{2}\left|\int_{-\infty}^{\infty}\psi_a(x_1)\psi_b^*(x_1)\, x_1\, dx_1\right|^2$$

$$= \pm\frac{1}{2}\left|\int_{-\infty}^{\infty}\psi_a(x)\psi_b^*(x)\cdot x\, dx\right|^2$$

4 $\quad$ This term is the complex conjugate of 3, so we can just take the conjugate. But 3 is already real. So, 4 is the same as 3.

$$\langle x_1 x_2\rangle = \langle x\rangle_a \langle x\rangle_b \pm \left|\int_{-\infty}^{\infty}\psi_a(x)\psi_b^*(x)\, x\, dx\right|^2$$

In all, then:

$$\langle x_1^2\rangle - 2\langle x_1 x_2\rangle + \langle x_2^2\rangle =$$

$$\langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b \mp 2\left|\int_{-\infty}^{\infty}\psi_a^*\psi_b\, x\, dx\right|^2$$

bosons ($-$)

fermions ($+$)

Identical Particles

$$\boxed{\langle (x_2-x_1)^2\rangle = \langle x^2\rangle_a - 2\langle x\rangle_a\langle x\rangle_b + \langle x^2\rangle_b \mp 2\left|\int_{-\infty}^{\infty}\psi_a\psi_b^* x\,dx\right|^2}$$

Average distance squared between identical particles in state

$$\frac{1}{\sqrt{2}}\left[\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)\right]$$

Note: the last term is the "exchange" term. If the particles were distinguishable, we wouldn't have the last term.

Example: Two non-interacting bosons are in an infinite square well of length L are in the state:

$$\frac{\sqrt{2}}{L}\left[\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right) + \sin\left(\frac{\pi x_2}{L}\right)\sin\left(\frac{2\pi x_1}{L}\right)\right]$$

determine $\langle (x_2-x_1)^2\rangle$. Then repeat for fermions.

a.) Bosons: $\psi_a(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$ $\quad \psi_b(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$

$$\langle x^2\rangle_a = \int_0^L |\psi_a|^2 x^2\,dx = \frac{2}{L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right)x^2\,dx$$

let $u = \frac{\pi x}{L}$ $\quad du = \frac{\pi}{L}dx \Rightarrow \frac{2}{L}\int_0^{\pi}\sin^2(u)\left(\frac{Lu}{\pi}\right)^2\left(\frac{L}{\pi}\right)du$

$$= \frac{2L^2}{\pi^3}\int_0^{\pi} u^2\sin^2(u)\,du$$

$$= \frac{2L^2}{\pi^3}\cdot\frac{\pi}{12}(2\pi^2-3)$$

$$= \frac{L^2}{6\pi^2}(2\pi^2-3)$$

$$\langle x^2\rangle_b = \int_0^L|\psi_b|^2x^2\,dx = \frac{2}{L}\int_0^L \sin^2\left(\frac{2\pi x}{L}\right)x^2\,dx = \frac{2L^2}{\pi^3}\int_0^{\pi}u^2\sin^2(2u)\,du$$

$$= \frac{2L^2}{\pi^3}\cdot\frac{\pi}{48}(8\pi^2-3)$$

$$= \frac{L^2}{24\pi^2}(8\pi^2-3)$$

$$\langle x^2\rangle_a + \langle x^2\rangle_b = \frac{L^2}{6\pi^2}\left(2\pi^2 - 3 + 2\pi^2 - \frac{3}{4}\right) = \frac{L^2}{6\pi^2}\left(4\pi^2 - \frac{15}{4}\right)$$

and $\langle x\rangle_a = \langle x\rangle_b = \frac{L}{2}$ expectation value of $x = \frac{L}{2}$ due to symmetry of square well.

$$\langle x^2\rangle_a - 2\langle x\rangle_a\langle x\rangle_b + \langle x^2\rangle_b = \frac{L^2}{6\pi^2}\left(4\pi^2 - \frac{15}{4}\right) - \frac{L^2}{2}$$

$$= L^2\left[\frac{2}{3} - \frac{15}{24\pi^2} - \frac{1}{2}\right] = L^2\left[\frac{1}{6} - \frac{15}{24\pi^2}\right]$$

Finally, we need the exchange term:

$$2\left|\int \psi_a\psi_b \cdot x\,dx\right|^2 = 2\cdot\left|\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)x\,dx\right|^2$$

$u = \frac{\pi x}{L}$ $du = \frac{\pi}{L}dx$ $\Rightarrow 2\left|\frac{2}{L}\int_0^\pi \sin(u)\sin(2u)\left(\frac{L}{\pi}\right)^2 du\right|^2$

$$= 2\cdot\left|\frac{2L}{\pi^2}\int_0^\pi \sin(u)\sin(2u)\,du\right|^2$$

$$= 2\cdot\left|\frac{2L}{\pi^2}\cdot\left(-\frac{8}{9}\right)\right|^2 = \frac{512L^2}{81\pi^4}$$

So, $\langle(x_2-x_1)^2\rangle = L^2\left[\frac{1}{6} - \frac{15}{24\pi^2} - \frac{512}{81\pi^4}\right]$

$$= L^2\left[0.103 - 0.065\right]$$

The average distance between the particles is

$\sqrt{\langle(x_2-x_1)^2\rangle} = L\sqrt{0.103 - 0.065} = 0.196\,L$ (Bosons)

$\sqrt{\langle(x_2-x_1)^2\rangle} = L\sqrt{0.103 + 0.065} = 0.410\,L$ (Fermions)

switch sign of exchange term.

$\sqrt{\langle(x_2-x_1)^2\rangle} = L\sqrt{0.103 + 0} = 0.321\,L$ (Distinguishable)

No exchange term

Conclusion: Compared with distinguishable particles, bosons prefer to cluster together and are on average closer together.

Fermions prefer to be further apart on average.

(Bosons are "friendly", fermions are "antisocial")
It's almost like there's an attractive force between bosons, and a repelling force between fermions, but this is a purely quantum phenomenon & is not a conventional force.

Meme of the week: