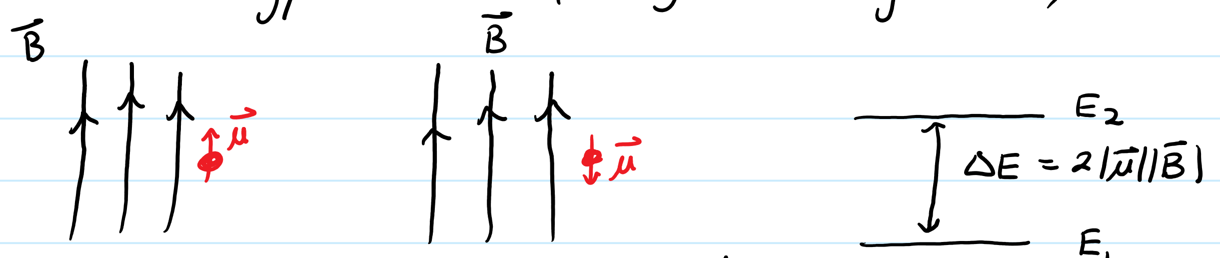
Spin Precession

I'm running the risk of "beating a dead horse" and boring you, but I want to do one more physical example of the two-state system involving spin.

So far, we've discussed the possible values of S_z for a spin $\frac{1}{2}$ particle, and we've discussed Schrödinger time evolution in the context of neutrino oscillations (but that didn't involve spin). Here, we're trying to bring the concepts of spin and time evolution together.

Spin $\frac{1}{2}$ particle in a magnetic field

Suppose you take a spin $\frac{1}{2}$ particle at rest in a uniform magnetic field. We know, because the particle has a magnetic moment, there is now an energy difference depending on the alignment of $\vec{\mu}$ and $\vec{B}$



spin aligned
$E = E_1$
(lowest energy)

spin anti-aligned
$E = E_2$
(highest energy)

Since $V = -\vec{\mu} \cdot \vec{B}$,
$$E_1 = -|\vec{\mu}||\vec{B}|\cos(0)$$
$$E_2 = -|\vec{\mu}||\vec{B}|\cos(180^\circ)$$

Now, we know that $\vec{\mu} = \gamma \vec{S}$, the magnetic moment is proportional to the spin. Where γ is called the gyromagnetic ratio.

For electrons, $\gamma = \frac{-ge}{2m}$ where g is a numerical factor which is very close to, but not exactly 2.

"Landé g-factor"

Spin Precession

The Hamiltonian, then is $H = -\vec{\mu}\cdot\vec{B} = -\gamma(\vec{S}\cdot\vec{B})$
and the Hamiltonian operator is $\hat{H} = -\gamma(\hat{\vec{S}}\cdot\vec{B})$

You might be wondering: where is the kinetic term $\hat{p}^2/2m$? Here I'm considering the particle to be at rest, so it only has potential energy.

We choose our z-axis to point along the magnetic field so $\vec{B} = B_0\hat{k}$
then:

$$\hat{H} = -\gamma B_0 \hat{S}_z \Rightarrow -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

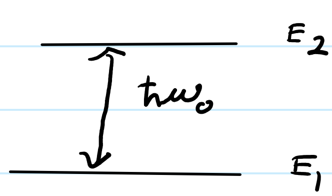
Poll Q: Thinking about units, what kind of quantity is $[\gamma B_0] = ?$

A.) frequency B.) time

C.) length D.) energy

$\therefore$ we can define $\omega_0 = B_0|\gamma|$ (absolute values are needed to keep ω_0 positive in general).
"Larmor Frequency"

The eigenvalues of $\hat{H}$ are clearly $\pm\frac{\hbar\omega_0}{2}$, so the energy difference between the two states is $\hbar\omega_0$



The eigenstates are clearly $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We now want to ask: if I start with the electron in an initial spin state, what happens as time goes on?

General Spin State - Any spin $\frac{1}{2}$ state can be written as

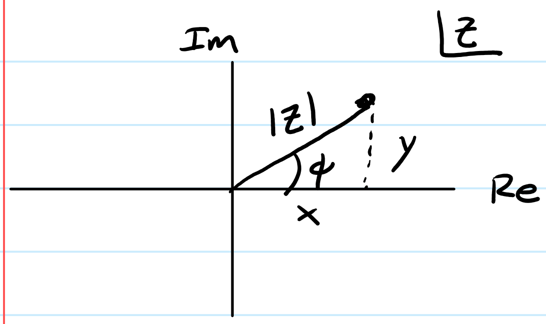
$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

where a and b are complex numbers. which must satisfy $|a|^2 + |b|^2 = 1$ in order to be normalized $\chi^{+}\chi = 1$.

Review - Complex #s

Any complex number can be written in terms of real & imaginary parts $z = x + iy$, which can be represented as a vector in the complex plane:

$$z = \underbrace{x}_{\text{real part}} + \underbrace{iy}_{\text{imaginary part.}}$$



In analogy with vectors we can write

$$x = |z| \cdot \cos\phi$$
$$y = |z| \sin\phi$$

where $|z|$ is the magnitude (modulus) of the complex # and ϕ is the phase.

$$z = x + iy = |z|\cos\phi + |z|(i\sin\phi) = |z|\left(\cos\phi + i\sin\phi\right)$$
$$= |z| e^{i\phi}$$

So any complex # can be written in terms of a <u>real</u> magnitude and a phase ϕ.

So, $a = |a| e^{i\phi_a}$ and $|a|^2 + |b|^2 = 1$

$b = |b| e^{i\phi_b}$

Now, $|a|$ is an arbitrary real number with $0 \leq |a| \leq 1$ without loss of generality, we can call $|a| = \cos(\theta/2)$

with $0 \le \theta \le \pi$.

[I didn't do anything... $|a|$ is just the name for a quantity and now I'm calling that quantity $\cos(\theta/2)$

Why did I call it $\theta/2$ and not $\cos(\theta)$? That's just a convention. I could call it θ and require $0 \le \theta \le \pi/2$ ... it would be equivalent.]

But then, $|b| = \sqrt{1-|a|^2} = \sqrt{1-\cos^2(\theta/2)} = \sin(\theta/2)$

which is convenient!

So, any spinor can, in general, be written as

$$\chi = \begin{pmatrix} \cos(\theta/2)\, e^{i\phi_a} \\ \sin(\theta/2)\, e^{i\phi_b} \end{pmatrix}$$

Finally, we can factor out $e^{i\phi_a}$

$$\chi = e^{i\phi_a} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i(\phi_b-\phi_a)} \end{pmatrix}$$

This overall phase is not observable. Any observable involves $\chi^{+}\chi$ (we have to square the wavefunction to get observables) so, we can take $\phi_a = 0$. And, let's call $\phi_b \to \phi$ to save writing.

$$\boxed{\begin{array}{l} \text{Any normalized spinor can be written as} \\ \chi = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\, e^{i\phi} \end{pmatrix} \qquad \begin{array}{l} 0 \le \theta \le \pi \\ 0 \le \phi \le 2\pi \end{array} \end{array}}$$

Spin Precession

Based on Example 3.4 (Griffiths). If a spin $\frac{1}{2}$ particle with

$$\chi(0) = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix}$$

is put in a $\vec{B}$-field $\vec{B} = B_0\hat{k}$, calculate $\langle S_x\rangle$, $\langle S_y\rangle$, $\langle S_z\rangle$ as functions of t.

① Express $\chi(0)$ in terms of energy eigenstates.

$$\chi(0) = \cos(\theta/2)\,|1\rangle + \sin(\tfrac{\theta}{2})e^{i\phi}\,|2\rangle$$

$|1\rangle = |\uparrow\rangle$ spin eigenstates are energy eigenstates.

Poll Q: True / False The spin-eigenstates in the z-basis $|\uparrow\rangle$ and $|\downarrow\rangle$ will always be equal to the energy eigenstates in all problems.

A.) True

B.) False

Spin Precession

② Evolve in time: $\chi(t) = \cos(\theta/2)\, e^{-iE_1 t/\hbar}|1\rangle + \sin(\frac{\theta}{2}) e^{i\phi} e^{-iE_2 t/\hbar}|2\rangle$

$$E_1 = -\frac{\hbar\omega_0}{2} \qquad E_2 = +\frac{\hbar\omega_0}{2}$$

$$\chi(t) = \cos(\theta/2)\, e^{\frac{i\omega_0 t}{2}}|1\rangle + \sin(\theta/2)\, e^{-i(\frac{\omega_0 t}{2} - \phi)}|2\rangle$$

$$\chi(t) = \begin{pmatrix} \cos(\theta/2)\, e^{i\omega_0 t/2} \\ \sin(\theta/2)\, e^{-i(\frac{\omega_0 t}{2} - \phi)} \end{pmatrix}$$

③ Calculate Expectation values.

$$\langle S_x \rangle = \chi^\dagger(t) S_x \chi(t) = \chi^\dagger(t)\left[\frac{\hbar}{2}\sigma_x\right]\chi(t)$$

$$\frac{\hbar}{2}\begin{pmatrix} \cos(\theta/2)\, e^{-\frac{i\omega_0 t}{2}} & \sin(\frac{\theta}{2})\, e^{i(\frac{\omega_0 t}{2} - \phi)} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos(\theta/2)\, e^{\frac{i\omega_0 t}{2}} \\ \sin(\theta/2)\, e^{-i(\frac{\omega_0 t}{2} - \phi)} \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos(\theta/2)\, e^{-\frac{i\omega_0 t}{2}} & \sin(\frac{\theta}{2})\, e^{i(\frac{\omega_0 t}{2} - \phi)} \end{pmatrix}\begin{pmatrix} \sin(\theta/2)\, e^{-i(\frac{\omega_0 t}{2} - \phi)} \\ \cos(\theta/2)\, e^{i\omega_0 t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin(\theta/2)\cos(\theta/2)\, e^{-i(\omega_0 t - \phi)} + \sin(\theta/2)\cos(\frac{\theta}{2})\, e^{i(\omega_0 t - \phi)}$$

$$= \frac{\hbar}{2} \sin(\theta/2)\cos(\theta/2)\left[e^{i(\omega_0 t - \phi)} + e^{-i(\omega_0 t - \phi)} \right]$$

Double angle: $\sin(2\theta) = 2\sin\theta\cos\theta$

$\frac{1}{2}\sin(\theta) = \sin(\theta/2)\cos(\frac{\theta}{2})$

$$= \frac{\hbar}{4}\sin(\theta) \cdot 2\cos(\omega_0 t - \phi) = \underline{\underline{\frac{\hbar}{2}\sin\theta\cos(\omega_0 t - \phi)}}$$

Spin Precession

$$\langle S_y \rangle = \chi^\dagger(t)\, S_y\, \chi(t)$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos(\theta/2)e^{-\frac{i\omega_0 t}{2}} & \sin(\theta/2)e^{i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\cos(\theta/2)e^{\frac{i\omega_0 t}{2}}\\ \sin(\theta/2)e^{-i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos(\theta/2)e^{-\frac{i\omega_0 t}{2}} & \sin(\theta/2)e^{i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}\begin{pmatrix}-i\sin(\theta/2)e^{-i(\frac{\omega_0 t}{2}-\phi)}\\ i\cos(\theta/2)e^{+i\omega_0 t/2}\end{pmatrix}$$

$$= \frac{i\hbar}{2}\cos(\theta/2)\sin(\theta/2)\left[-e^{-i(\omega_0 t-\phi)} + e^{i(\omega_0 t-\phi)}\right]$$

$$= \frac{i\hbar}{2}\cdot\frac{1}{2}\sin(\theta)\left[-2i\sin(\omega_0 t-\phi)\right]$$

$$\langle S_y \rangle = \frac{\hbar}{2}\sin\theta\sin(\omega_0 t-\phi)$$

Finally:

$$\langle S_x \rangle = \chi^\dagger(t)\, S_z\, \chi(t) =$$

$$\frac{\hbar}{2}\begin{pmatrix}\cos(\theta/2)e^{-\frac{i\omega_0 t}{2}} & \sin(\theta/2)e^{i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos(\theta/2)e^{\frac{i\omega_0 t}{2}}\\ \sin(\theta/2)e^{-i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos(\theta/2)e^{-i\omega_0 t/2} & \sin(\theta/2)e^{i(\frac{\omega_0 t}{2}-\phi)}\end{pmatrix}\begin{pmatrix}\cos(\theta/2)e^{i\omega_0 t/2}\\ -\sin(\theta/2)e^{-i(\omega_0 t/2-\phi)}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] = \frac{\hbar}{2}\cos\theta$$ (time independent!)

Spin Precession

Summary:

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\theta \cos(\omega_0 t - \phi)$$
$$\langle S_y \rangle = \frac{\hbar}{2} \sin\theta \sin(\omega_0 t - \phi)$$
$$\langle S_z \rangle = \frac{\hbar}{2} \cos\theta$$

Geometrically, this looks like a precession around the $\vec{B}$ field.

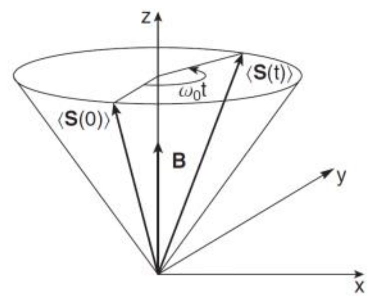


FIGURE 3.3 The expectation value of the spin vector precesses in a uniform magnetic field.

This is called Larmor Precession: the expectation value of the spin vector rotates around the $\vec{B}$ field axis at a frequency $\omega_0 = |\gamma| B_0$ where $\vec{\mu} = \gamma \vec{S}$

By the way, the precession of a magnetic dipole moment in a uniform $\vec{B}$ field is expected classically. (It's also related to why spinning gyroscopes precess in a uniform gravitational field).

This is yet another instance of Ehrenfest's Theorem: expectation values behave in the same way as classical physics predicts. The outcome of any one measurement of, say, S_x cannot be predicted though.