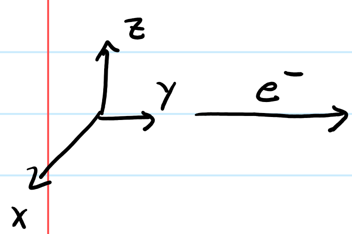
Reference: Griffiths 4.4
McIntyre 1.1

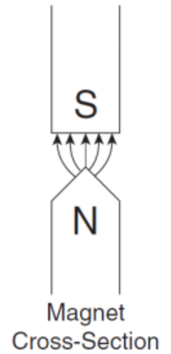
Stern-Gerlach Experiment

I want to discuss now one of the most famous experiments in QM, called the Stern-Gerlach experiment - performed by (surprize!) Stern and Gerlach in 1921-1922.

Here is my (simplified) version:

Take a beam of electrons, and shoot them through a magnetic field. But not just any $\vec{B}$ field, an <u>inhomogeneous</u> magnetic field. Here's the schematic:





$$B(x,y,z) = (B_0 - \alpha z)\hat{k}$$

Field is stronger near the N pole than the South pole.

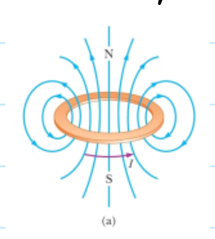
What will happen to the e^- beam? [Yes, there is the Lorentz Force out of the page ... but what happens in the z-direction?]

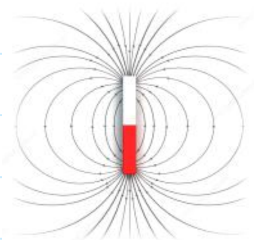
To approach this problem, let's first take a detour into <u>classical</u> physics.

Magnetic Forces on Current Loops

Imagine a small loop of wire carrying current & enclosing area A

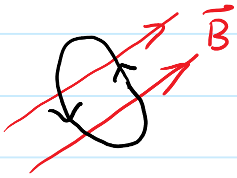
current loop

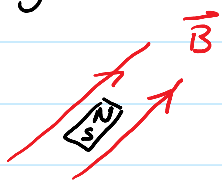




Bar magnet

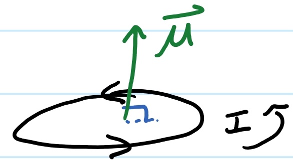
The $\vec{B}$ field from the wire looks just like a tiny magnet. If you put the loop in another $\vec{B}$ field, it aligns with it (just like a magnet would.)





Stern-Gerlach Experiment

The current carrying loop has a magnetic moment $\vec{\mu}$, which is perpendicular to the plane of the loop.
Think of the direction of $\vec{\mu}$ like a compass needle, it aligns with any external field.



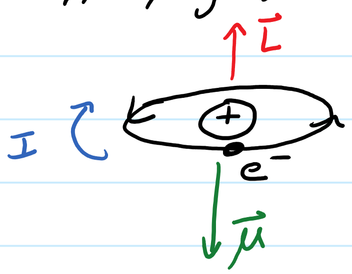
$|\vec{\mu}| = |\vec{I}| A$

Direction of μ is given by RHR $(\vec{r} \times \vec{I})$

The potential energy V of the dipole in a $\vec{B}$ field is $V = -\vec{\mu} \cdot \vec{B} = -|\vec{\mu}||\vec{B}|\cos\theta$
(The lowest energy is when $\vec{\mu}$ is $/\!/$ to $\vec{B}$)

Now $\vec{F} = -\nabla V = \nabla(\vec{\mu} \cdot \vec{B})$ so a dipole in a magnetic field experiences a force.

What does this have to do with our experiment? Classically, an atom (say, Hydrogen) can be thought of as a current loop of radius r



The current is: $\frac{\text{charge}}{\text{time}} = \frac{e}{2\pi r/v} = |\vec{I}|$

then, $|\vec{\mu}| = |\vec{I}|A = \frac{ev}{2\pi r} \cdot \pi r^2 = \frac{evr}{2}$

The angular momentum is $\vec{L} = \vec{r} \times \vec{p}$
$|\vec{L}| = |\vec{r}| m |\vec{v}| \cdot \sin(90°) = mvr$

So, $|\vec{\mu}| = \frac{e}{2m} \cdot |\vec{L}|$ so $\vec{\mu} = \frac{-e}{2m}\vec{L}$

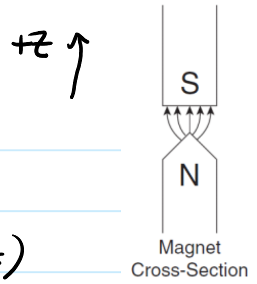
angular momentum.

magnetic dipole moment

In a $\vec{B}$-field, this "classical atom" would experience a force

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}) = \frac{-e}{2m} \nabla(\vec{L} \cdot \vec{B})$$

Stern-Gerlach Experiment



field gets weaker as z increases.

Now, $\vec{B} = (B_0 - \alpha z)\hat{k}$, so $\vec{L} \cdot \vec{B} = L_z(B_0 - \alpha z)$

Then, the initial force on the atom as it enters the B-field is

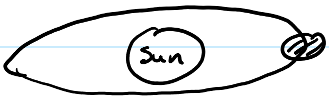
$$\vec{F} = \frac{-e}{2m} \nabla(\vec{L} \cdot \vec{B}) = \frac{-e}{2m} \frac{\partial}{\partial z}\left[L_z(B_0 - \alpha z)\right] = \frac{e\alpha}{2m} L_z$$

$$\boxed{F_z = (\text{constant}) \cdot L_z}$$

so the force on the loop is in the z-direction and is proportional to L_z.

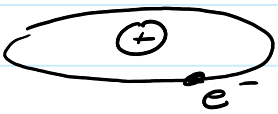
Spin

Now what about just an electron? Going back to our silly planetary model:



Earth

The Earth has angular momentum $\vec{L}$ about the sun, but the earth also has its own angular momentum since it spins on its axis.



Let's imagine the possibility that the electron itself has its own angular momentum called "spin". If the e^- has spin, it would have a magnetic moment $\vec{\mu}$, and would feel a force from the Stern-Gerlach apparatus.

constant ("gyromagnetic ratio").

Let's call $\vec{\mu}_e = \gamma \vec{S}$

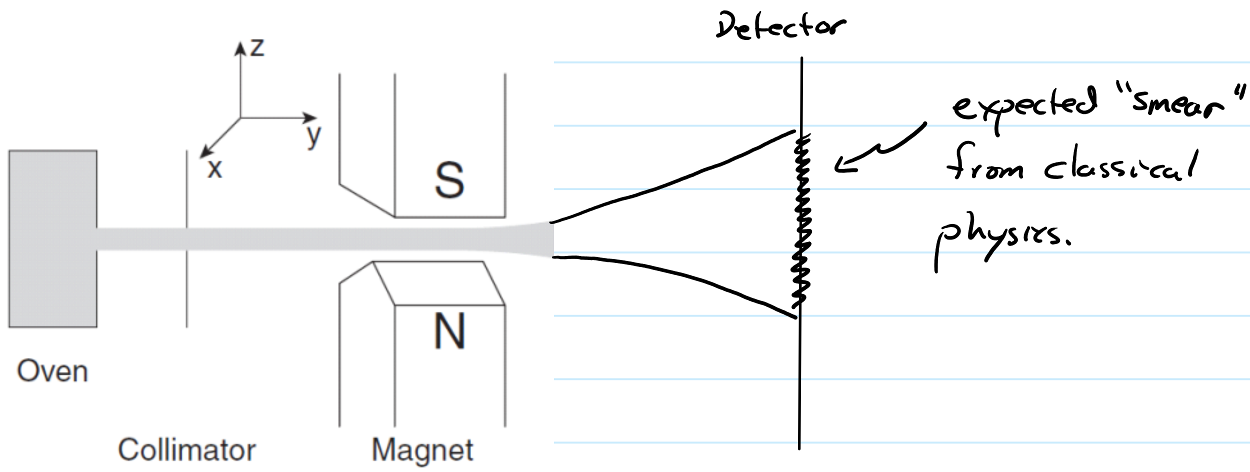
electron's spin.

electron's magnetic dipole moment

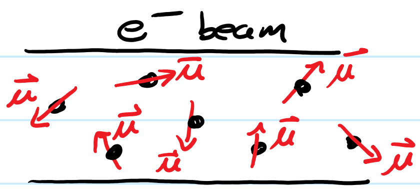
Stern-Gerlach Experiment

Then, the force on the electron is $F_z = (\text{constant}) \cdot S_z$ which depends on the z-component of its spin.

So, classically what would we expect for the experimental results?



If we boil electrons off of a metal, for example, and send them through the apparatus, there's no reason why the spin of the electron should point in any one direction. We'd expect them to be randomly oriented.



so some would be deflected up, some down, some not at all...

Classically we'd expect a <u>continuous</u> <u>smear</u> at our detector.

Now, we know that angular momentum is <u>quantized</u>. If you've developed some intuition about quantum mechanics, you'd expect the beam will <u>split</u> into several distinct ones depending on the possible values of S_z.

It's probably a good time to review the quantum theory of angular momentum...

Stern-Gerlach Experiment

Review of Angular Momentum in QM

Last semester when studying the hydrogen atom, we found the wavefunctions were labeled by 3 quantum #s n, ℓ, m $[\psi_{n\ell m}(r,\theta,\phi)]$

$E_n = -13.6eV/n^2$ so n (the principle quantum #) is related to energy.

And, ℓ and m were related to angular momentum. Do you remember how?

Poll Q: Suppose a beam of hydrogen atoms are prepared in states with $n=2, \ell=1$. This is all the information you have about the state. If this state is sent through a Stern-Gerlach apparatus, the single beam would split into how many pieces?

A.) 2 B.) 3

C.) 4 D.) 5

Quick Review of solution of Schrödinger equation for hydrogen:

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{k e^2}{r}\psi = E\psi \qquad \text{where } \psi(r,\theta,\phi)$$

We then separated variables:

$$\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$$

Radial Equation for $R(r)$ $\longleftrightarrow$ Angular Equation for $Y(\theta,\phi)$

Both depend on Seperation constant called "$\ell(\ell+1)$"

$$Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

Θ equation $\longleftrightarrow$ Φ equation

both depend on separation constant "m"

The Θ equation depends on both ℓ and m.
In solving the Θ equation we find ℓ must be a non-negative integer, or else the solution $\Theta_\ell(\theta)$ blows up at $\theta = 0$ or $\theta = \pi$. It was also found that $|m| \le \ell$ to get anything nontrivial

$$\Theta(\theta) \propto P_\ell^m(\cos\theta) \quad \text{(Associated Legendre functions)}$$

So $\ell = 0$ $\quad m = 0$ $\quad$ [1 state]

$\ell = 1$ $\quad m = \pm 1, 0$ $\quad$ [3 states]

$\ell = 2$ $\quad m = \pm 2, \pm 1, 0$ $\quad$ [5 states]

General: $m = \pm\ell, \pm(\ell-1), \pm(\ell-2), \ldots \pm 1, 0$ $\quad$ [$2\ell+1$ states]

Stern-Gerlach Experiment

Later on in Phys 405, we learned to interpret the meaning of ℓ, m

$\hat{L}^2 \psi_{n\ell m} = \hbar^2 \ell(\ell+1) \psi_{n\ell m}$ $\quad$ $\ell(\ell+1)$ is the eigenvalue of $\hat{L}^2$

$\hat{L}_z \psi_{n\ell m} = m\hbar \psi_{n\ell m}$ $\quad$ m is the eigenvalue of L_z

So, for an atom with a given ℓ (but m undetermined), the beam should split into $(2\ell+1)$ pieces since L_z can take on $2\ell+1$ values.

Now for electrons (or, any elementary particle) we'll label the eigenstates of <u>spin</u> with 2 quantum numbers "s" and "m"

"s" is related to total intrinsic angular momentum and "m" is related to the z-component of the spin. $\quad |s,m\rangle$

And the operators are $\hat{S}_x, \hat{S}_y, \hat{S}_z$ with $\hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$

by assumption they obey the same commutation relations as the angular momentum operators

$$[\hat{S}_x, \hat{S}_y] = i\hbar S_z \qquad [\hat{S}_y, \hat{S}_z] = i\hbar \hat{S}_x \qquad [\hat{S}_z, \hat{S}_x] = i\hbar \hat{S}_y$$

and $[\hat{S}^2, \hat{S}_x] = [\hat{S}^2, \hat{S}_y] = [\hat{S}^2, \hat{S}_z] = 0$

so

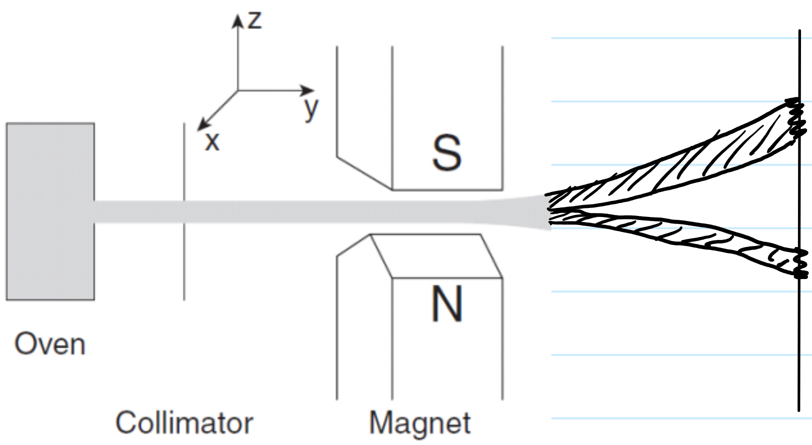
$$\hat{S}^2 |s,m\rangle = \hbar^2 s(s+1) \underbrace{|s,m\rangle} \qquad \hat{S}_z = m\hbar |s,m\rangle$$

eigenstate of $\hat{S}^2$ and $\hat{S}_z$. It has a definite total spin $= \hbar\sqrt{s(s+1)}$ and $S_z = m\hbar$

In analogy with what we did for hydrogen, there are $(2s+1)$ possible values of S_z for a given "s". If beam of particles with spin s is put into the Stern-Gerlach apparatus it should split into $(2s+1)$ pieces.

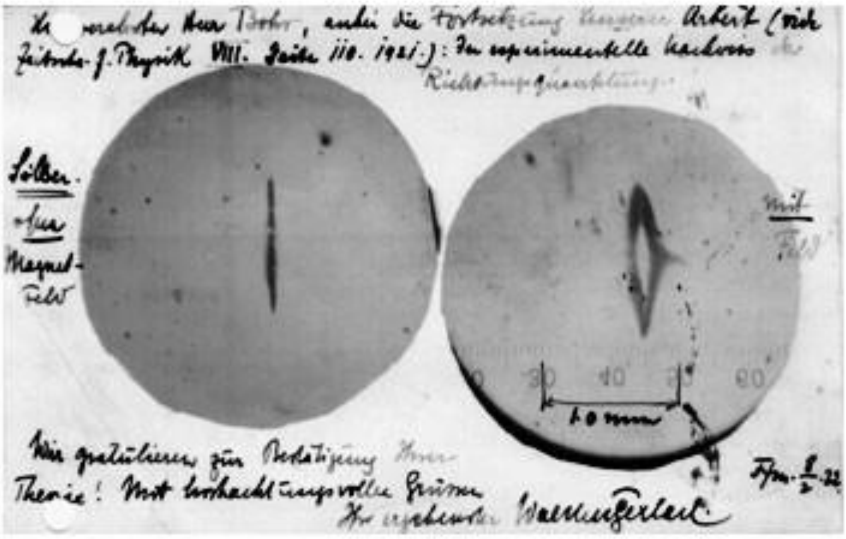
Stern-Gerlach Experiment

Now for the results of the Stern-Gerlach Experiment with electrons:



The beam splits into <u>two</u>

This is the postcard from Gerlach to Bohr demonstrating the results.



Apparently there are only <u>two</u> possible values of S_z. So what is "s"?

$(2s+1) = 2 \qquad s = \frac{1}{2} \qquad (!?)$

There are <u>two</u> eigenstates of spin for electrons

$|\frac{1}{2}, +\frac{1}{2}\rangle \Rightarrow$ total spin $= \frac{\sqrt{3}}{2}\hbar$, $S_z = +\hbar/2$ "Spin up"

$|\frac{1}{2}, -\frac{1}{2}\rangle \Rightarrow$ total spin $= \frac{\sqrt{3}}{2}\hbar$, $S_z = -\hbar/2$ "Spin down"

Stern-Gerlach Experiment

You may see these written as $|\frac{1}{2},\frac{1}{2}\rangle = |\uparrow\rangle$ or $|+\rangle$
$|\frac{1}{2},-\frac{1}{2}\rangle = |\downarrow\rangle$ or $|-\rangle$

What's going on here? For hydrogen, ℓ could only be an integer, but for spin s can be a half integer.
Why is this allowed?

Spin is actually different than "orbital" angular momentum. We call it spin in analogy with a spinning object like the Earth, but an electron is not a spinning ball. As far as we know the electron has zero size. It is a point particle without structure. And yet, it somehow still has angular momentum. There is really no way to picture this intuitively, it's fundamentally quantum with no classical analog.

Because the electron is a point, I can't describe its spin state as a function of $r, \theta,$ and ϕ at all. The issue of the "wave function" blowing up does not exist here in contrast to the hydrogen atom which is not a point. This is the fundamental difference between the two systems.

For a general particle; its states are labeled by $|s,m\rangle$, and possible values of S are:

Half integer values are allowed for S!

$$s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, 3, \ldots$$
$$m = 0, \pm 1, \pm 2, \ldots \pm S$$

To see where the half integer values come from, see Griffiths 4.3.1 which uses the ladder operator technique.