Chapter 5

Matrix Algebra and Affine Transformations

5.1 The need for multiple coordinate frames

So far we have been dealing with cameras, lights and geometry in our scene by specifying everything with respect to a common origin and coordinate system. This turns out to be very limiting. For example, for animation we will want to be able to move the camera and models and render a sequence of images as time advances to create an illusion of motion. We may wish to build models separately and then combine them together to make a scene. We may wish to define one object relative to another – for example we may want to place a hand at the end of an arm. In order to provide this flexibility we need a good mechanism to provide for multiple coordinate systems and for easy transformation between them. We will call these coordinate systems, which we use to represent a scene, *coordinate frames*.

Figure 5.1 shows an example of what we mean. A cylinder has been built in its own modeling coordinate frame and then rotated and translated into the entire scene's coordinate frame. The set of operations providing for all transformations such as these, are known as the affine transforms. The affines include translations and all linear transformations, like scale, rotate, and shear.

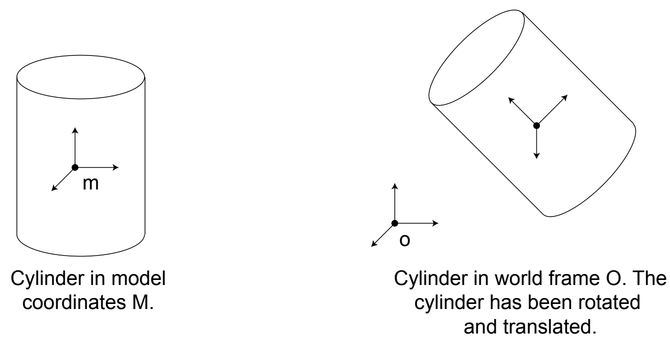


Figure 5.1: Object in *model* and *world* coordinate frames.

5.2 Affine transformations

Let us first examine the affine transforms in 2D space, where it is easy to illustrate them with diagrams, then later we will look at the affines in 3D.

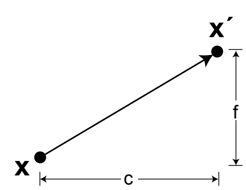
Consider a point $\mathbf{x} = (x, y)$. Affine transformations of $\mathbf{x}$ are all transforms that can be written

$$\mathbf{x}' = \begin{bmatrix} ax + by + c \\ dx + ey + f \end{bmatrix},$$

where a through f are scalars.

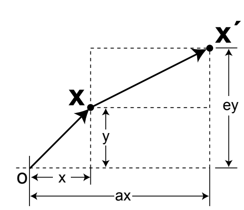
For example, if $a, e = 1$, and $b, d = 0$, then we have a pure translation

$$\mathbf{x}' = \begin{bmatrix} x + c \\ y + f \end{bmatrix}.$$



If $b, d = 0$ and $c, f = 0$ then we have a pure scale.

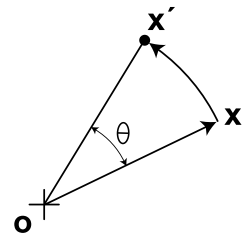
$$\mathbf{x}' = \begin{bmatrix} ax \\ ey \end{bmatrix}$$



And, if $a, e = \cos\theta$, $b = -\sin\theta$, $d = \sin\theta$, and $c, f = 0$, then we have a pure rotation about the origin

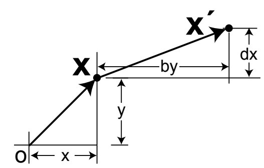
$$\mathbf{x}' = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

The rotation transformation can be derived with the use of some trigonometric identities.



Finally if $a, e = 1$, and $c, f = 0$ we have the shear transforms

$$\mathbf{x}' = \begin{bmatrix} x + by \\ y + dx \end{bmatrix}.$$



In summary, we have the four basic affine transformations shown in Figure 5.2:

- Translate moves a set of points a fixed distance in x and y,
- Scale scales a set of points up or down in the x and y directions,
- Rotate rotates a set of points about the origin,
- Shear offsets a set of points a distance proportional to their x and y coordinates.

Note that only shear and scale change the shape determined by a set of points.

5.3 Matrix representation of the linear transformations

The affine transforms scale, rotate and shear are actually linear transforms and can be represented by a matrix multiplication of a point

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ax + by \\ dx + ey \end{bmatrix} = \begin{bmatrix} a & b \\ d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or $\mathbf{x}' = M\mathbf{x}$, where M is the matrix.

One very nice feature of the matrix representation is that we can use it to factor a complex transform into a set of simpler transforms. For example, suppose we

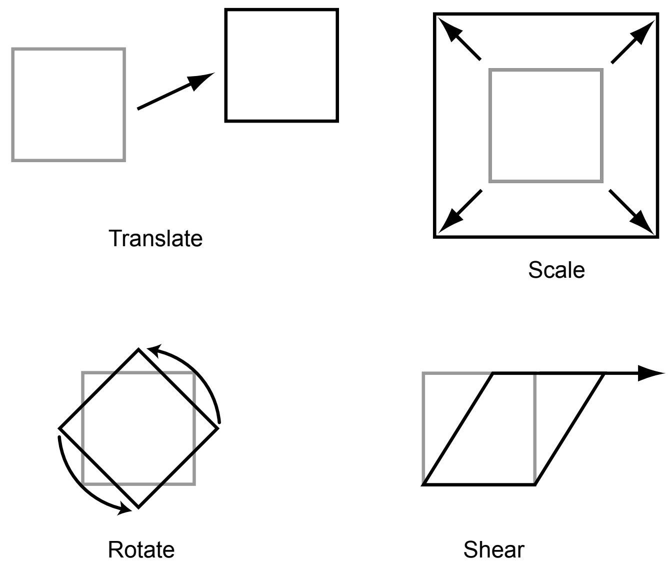


Figure 5.2: The basic affine transforms.

want to scale an object up to a new size, shear the object to a new shape, and finally rotate the object. Let S be the scale matrix, H be the shear matrix and R be the rotation matrix. Then

$$\mathbf{x}' = R(H(S\mathbf{x}))$$

defines a sequence of three transforms: 1st-scale, 2nd-shear, 3rd-rotate. Because matrix multiplication is associative, we can remove the parentheses and multiply the three matrices together, giving a new matrix $M = RHS$. Now we can rewrite our transform

$$\mathbf{x}' = (RHS)\mathbf{x} = M\mathbf{x}$$

If we have to transform thousands of points on a complex model, it is clearly easier to do one matrix multiplication, rather than three, each time we want to transform a point. Thus, matrices are a very powerful way to encapsulate a complex transform and to store it in a compact and convenient form.

In matrix form, we can catalog the linear transforms as

$$\text{Scale: } \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}, \text{ Rotate: } \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \text{ Shear: } \begin{bmatrix} 1 & h_x \\ h_y & 1 \end{bmatrix},$$

where s_x and s_y scale the x and y coordinates of a point, θ is an angle of counterclockwise rotation around the origin, h_x is a horizontal shear factor, and h_y is a vertical shear factor.

5.4 Homogeneous coordinates

Since the matrix form is so handy for building up complex transforms from simpler ones, it would be very useful to be able to represent all of the affine transforms by matrices. The problem is that translation is not a linear transform. The way out of this dilemma is to turn the 2D problem into a 3D problem, but in *homogeneous coordinates*.

We first take all of our points $\mathbf{x} = (x, y)$, express them as 2D vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and make these into 3D vectors with identical (thus the term homogeneous) 3rd coordinates set to 1:

$$\begin{bmatrix} x \\ y \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$

By convention, we call this third coordinate the w coordinate, to distinguish it from the usual 3D z coordinate. We also extend our 2D matrices to 3D homogeneous form by appending an extra row and column, giving

$$\text{Scale: } \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{Rotate: } \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{Shear: } \begin{bmatrix} 1 & h_x & 0 \\ h_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note what happens when we multiply our 3D homogeneous matrices by 3D homogeneous vectors:

$$\begin{bmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ dx + ey \\ 1 \end{bmatrix}.$$

This is the same result as in 2D, with the exception of the extra w coordinate, which remains 1. All we have really done is to place all of our 2D points on the plane $w = 1$ in 3D space, and now we do all the operations on this plane, as shown in Figure 5.3. Really, the operations are still 2D operations. But, the magic happens when we place the translation parameters c and f in the matrix in the 3rd column:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

All of a sudden we can now do translations as a linear operation in homogeneous coordinates! So, we can add a final matrix to our catalog:

$$\text{Translate: } \begin{bmatrix} 1 & 0 & \triangle x \\ 0 & 1 & \triangle y \\ 0 & 0 & 1 \end{bmatrix},$$

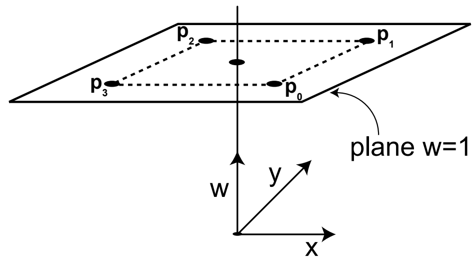


Figure 5.3: Affines in homogeneous coordinates take place on the plane $w = 1$.

where $\triangle x$ is the translation in the x direction and $\triangle y$ is the translation in the y direction.

Now, suppose we have a 2×2 square centered at the origin and we want to first rotate the square by 45° about its center and then move the square so its center is at $(3, 2)$. We can do this in two steps, as shown in Figure 5.4. In matrix form:

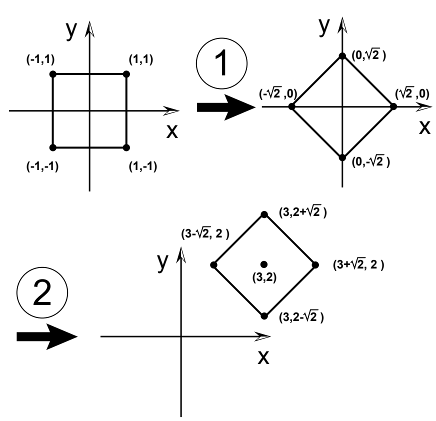


Figure 5.4: Example showing a rotation of 45° followed by a translation of $(3, 2)$.

$$M = T_{(3,2)} R_{45^\circ} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 0 \\ \sin 45^\circ & \cos 45^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ & 3 \\ \sin 45^\circ & \cos 45^\circ & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 3 \\ \sqrt{2}/2 & \sqrt{2}/2 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that

$$M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2+\sqrt{2} \\ 1 \end{bmatrix}, \text{and } M \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-\sqrt{2} \\ 2 \\ 1 \end{bmatrix},$$

verifying that we get the same result shown in Figure 5.4.

5.5 3D form of the affine transformations

Now, we can extend all of these ideas to 3D in the following way:

1. Convert all 3D points to homogeneous coordinates $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Longrightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$. The extra (4th) coordinate is again called the w coordinate.
2. Use matrices to represent the 3D affine transforms in homogeneous form.

The following matrices constitute the basic affine transforms in 3D, expressed in homogeneous form.

$$\text{Translate:} \begin{bmatrix} 1 & 0 & 0 & \triangle x \\ 0 & 1 & 0 & \triangle y \\ 0 & 0 & 1 & \triangle z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ Scale:} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\text{Shear:} \begin{bmatrix} 1 & h_{xy} & h_{xz} & 0 \\ h_{yx} & 1 & h_{yz} & 0 \\ h_{zx} & h_{zy} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In addition, there are three basic rotations in 3D,

$$\text{Rotation about the } x \text{ axis: } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\text{Rotation about the } y \text{ axis: } \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\text{Rotation about the } z \text{ axis: } \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$