Revision and Activity

Master Theorem

Revision and Activity:

- Master Theorem

Solve the following recurrence equation using Master Theorem (if possible)

- $T(n) = 3T(^n/_2) + n^2$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2 + \mathrm{n} + 3$
- $T(n) = 16T\left(\frac{n}{4}\right) + n$
- $T(n) = 16T\left(\frac{n}{2}\right) + 10n^3$
- $T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$
- $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$
- $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$

$$T(n) = 3T\left({}^{n}/_{2}\right) + n^2$$

- $a = 3, b = 2, c = 2, f(n) = n^2$
- $\left(\frac{a}{b^c}\right) = \left(\frac{3}{2^2}\right) < 1, \textit{hence may be able to test using case } 3.$

Test using case 3:

$$n^2 \in \Omega\left(n^{\log_2 3+\varepsilon}\right) \text{ for some } \varepsilon > 1.$$

$$n^2 \in \Omega(n^{1.6+\varepsilon}) \text{ true for } \varepsilon > 0.4.$$

Test if $af\left(\frac{n}{b}\right) \leq kf(n)$ for some constant k < 1.

$$3\left(\frac{n}{2}\right)^2 \leq kn^2$$

$$\frac{3}{4}n^2 \leq kn^2, \text{and this is true for } k = \frac{3}{4} < 1.$$

$$\therefore \ \Theta(f(n)) = \boldsymbol{\Theta}\left(\boldsymbol{n^2}\right)$$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 + n + 3$$

- $a = 4, b = 2, c = 2, f(n) = n^2 + n + 3$
- $\left(\frac{a}{b^c}\right) = \left(\frac{4}{2^2}\right) = 1, \textit{hence may be able to test using case } 2.$

Test using case 2:

$$n^2 + n + 3 \in \Theta\left(n^{\log_2 4}\right)$$

$n^2 + n + 3 \in \Theta(n^2)$ this is true.

$$\therefore\ \Theta\left(n^{\log_b a} \lg n\right) = \Theta\left(n^{\log_2 4} \lg n\right) = \boldsymbol{\Theta\left(n^2 lg\, n\right)}$$

$$T(n) = 16T\left(\frac{n}{4}\right) + n$$

- $a = 16, b = 4, c = 1, f(n) = n$
- $\left(\frac{a}{b^c}\right) = \left(\frac{16}{4^1}\right) > 1, \mathit{hence\ may\ be\ able\ to\ test\ using\ case}\ 1.$

Test using case 1:

$n \in O\left(n^{\log_4 16-\varepsilon}\right)$ for some $\varepsilon > 1$.

$n \in O(n^{2-1})$ true for $\varepsilon = 1$.

$$\therefore\ \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_4 16}\right) = \boldsymbol{\Theta}\left(\boldsymbol{n^2}\right)$$

$$T(n) = 16T\left(\frac{n}{2}\right) + 10n^3$$

- $a = 16, b = 2, c = 3, f(n) = 10n^3$
- $\left(\frac{a}{b^c}\right) = \left(\frac{16}{2^3}\right) = \left(\frac{16}{8}\right) > 1,$ hence may be able to test using case 1.

Test using case 1:

$10n^3 \in O\left(n^{\log_2 16-\varepsilon}\right)$ for some $\varepsilon > 1$.

$10n^3 \in O(n^{4-1})$ true for $\varepsilon = 1$.

$$\therefore T(n) = \Theta\left(n^{\log_b a}\right) = \Theta\left(n^{\log_2 16}\right) = \boldsymbol{\Theta(n^4)}$$

$$T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$$

- $a = 0.5, b = 2, c = -1, f(n) = \frac{1}{n}$
- a is less than 1, hence master theorem cannot be applied to determine the running time complexity of this recurrence relation. Expansion and substitution will be used instead.

$$T(n) = \frac{1}{2}T\left(\frac{n}{2}\right) + \frac{1}{n}$$

$$T(n) = \frac{1}{2}\left[\frac{1}{2}T\left(\frac{n}{2^2}\right) + \frac{1}{\frac{n}{2}}\right] + \frac{1}{n}$$

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$$T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$$

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$$T(n) = \frac{1}{2}\left[\frac{1}{2}T\left(\frac{n}{2^2}\right) + \frac{1}{\frac{n}{2}}\right] + \frac{1}{n}$$

$$T(n) = \frac{1}{2^2}T\left(\frac{n}{2^2}\right) + \left(\frac{1}{2}\right)\frac{2}{n} + \frac{1}{n}$$

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$$T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$$

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$$T(n) = \frac{1}{2^2}T\left(\frac{n}{2^2}\right) + \left(\frac{1}{2}\right)\frac{2}{n} + \frac{1}{n}$$

$$T(n) = \frac{1}{2^2}\left[\frac{1}{2}T\left(\frac{n}{2^3}\right) + \frac{1}{\frac{n}{2}}\right] + \left(\frac{1}{2}\right)\frac{2}{n} + \frac{1}{n}$$

$$T(n) = \frac{1}{2^3}T\left(\frac{n}{2^3}\right) + \left(\frac{1}{2^2}\right)\frac{2}{n} + \left(\frac{1}{2}\right)\frac{2}{n} + \frac{1}{n}$$

The recurrence relation can be generalized as

$$T(n) = \frac{1}{2^k} T\left(\frac{n}{2^k}\right) + \left[\frac{1}{2^k} + \frac{1}{2^{k-1}} + \frac{1}{2^{k-2}} + \cdots + \frac{1}{2^2} + \frac{1}{2^1}\right]\left(\frac{2}{n}\right) + \frac{1}{n} \qquad \cdots eq(1)$$

The recursive call will continue and stop when $\left(\frac{n}{2^k}\right) = 1$. Solving the equality, we have $n = 2^k$ and hence $k = \lg n$.

Substituting k into $eq(1)$, we have

$$T(n) = \frac{1}{2^{\lg n}} T\left(\frac{n}{2^{\lg n}}\right) + \left[\frac{1}{2^{\lg n}} + \frac{1}{2^{(\lg n)-1}} + \cdots + \frac{1}{2}\right]\left(\frac{2}{n}\right) + \frac{1}{n}$$

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$$T(n) = \frac{1}{2^{\lg n}} T\left(\frac{n}{2^{\lg n}}\right) + \left[\frac{1}{2^{\lg n}} + \frac{1}{2^{(\lg n)-1}} + \cdots + \frac{1}{2}\right]\left(\frac{2}{n}\right) + \frac{1}{n}$$

$$T(n) = \frac{1}{n} T\left(\frac{n}{n}\right) + \left[\frac{1}{2^{\lg n}} + \frac{1}{2^{(\lg n)-1}} + \cdots + \frac{1}{2}\right]\left(\frac{2}{n}\right) + \frac{1}{n}$$

$$T(n) = \frac{1}{n} T(1) + \left[\frac{1}{2^{\lg n}} + \frac{1}{2^{(\lg n)-1}} + \cdots + \frac{1}{2}\right]\left(\frac{2}{n}\right) + \frac{1}{n}$$

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$$T(n) = \frac{1}{n}T(1) + \left[\frac{1}{2^{\lg n}} + \frac{1}{2^{(\lg n)-1}} + \cdots + \frac{1}{2}\right]\left(\frac{2}{n}\right) + \frac{1}{n}$$

$\Theta\left(\frac{1}{2^{\lg n}}\right) = \Theta\left(\frac{1}{n}\right)$. Does not seem to be sensible algorithm complexity because as n increases, the complexity reduces (becomes less complex).

$$T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$$

- $a = 64, b = 8, c = 2, f(n) = -n^2 \lg n$
- f(n)is negative, hence master theorem cannot be applied to determine the running time complexity of this recurrence relation. Expansion and substitution will be used instead.

- The running time complexity of the algorithm is $\boldsymbol{\Theta\left(n^2 \log_2 n \log_8{}^2 n\right)}$.

$$T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$$

$$T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$$

$$= 64\left[64T\left(\frac{n}{8^2}\right) - \left(\frac{n}{8}\right)^2 \lg\left(\frac{n}{8}\right)\right] - n^2 \lg n$$

$$= 64^2T\left(\frac{n}{8^2}\right) - n^2(\lg n - 3) - n^2 \lg n$$

$$= 64^2T\left(\frac{n}{8^2}\right) - 2n^2 \lg n + 3n^2$$

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$$T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$$

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$$= \ 64^2 T\left(\frac{n}{8^2}\right) - 2n^2 \lg n + 3n^2$$

$$= 64^2 \left[64T\left(\frac{n}{8^3}\right) - \left(\frac{n}{8^2}\right)^2 \lg\left(\frac{n}{8^2}\right)\right] - 2n^2 \lg n + 3n^2$$

$$= \ 64^3 T\left(\frac{n}{8^3}\right) - n^2(\lg n \ - 6) - 2n^2 \lg n + 3n^2$$

$$= \ 64^3 T\left(\frac{n}{8^3}\right) - n^2 \lg n + 6n^2 - 2n^2 \lg n + 3n^2$$

$$= \ 64^3 T\left(\frac{n}{8^3}\right) - 3n^2 \lg n + 9n^2$$

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$$T(n) = 64T\left(\frac{n}{8}\right) - n^2 \lg n$$

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$$= 64^3 T\left(\frac{n}{8^3}\right) - 3n^2 \lg n + 9n^2$$

$$= 64^3 \left[64T\left(\frac{n}{8^4}\right) - \left(\frac{n}{8^3}\right)^2 \lg\left(\frac{n}{8^3}\right)\right] - 3n^2 \lg n + 9n^2$$

$$= 64^4 T\left(\frac{n}{8^4}\right) - n^2(\lg n - 9) - 3n^2 \lg n + 9n^2$$

$$= 64^4 T\left(\frac{n}{8^4}\right) - n^2 \lg n + 9n^2 - 3n^2 \lg n + 9n^2$$

$$= 64^4 T\left(\frac{n}{8^4}\right) - 4n^2 \lg n + 18n^2$$

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The recurrence relation can be generalized as

$$T(n) = 64^k T\left(\frac{n}{8^k}\right) - kn^2 \lg n + 3\left(\frac{k(k-1)}{2}\right)n^2 \qquad \cdots\cdots\cdots eq(1)$$

The recursive call will continue and stop when $\left(\frac{n}{8^k}\right) = 1$. Solving the equality, we have $n = 8^k$ and hence $k = \log_8 n$.

Substituting k into $eq(1)$, we have

$$T(n) = 64^{\log_8 n} T\left(\frac{n}{8^{\log_8 n}}\right) - (\log_8 n) n^2 \lg \mathrm{n} + 3\left(\frac{\log_8 n(\log_8 n - 1)}{2}\right)n^2.$$

$$T(n) = (8^2)^{\log_8 n} T\left(\frac{n}{n}\right) - n^2 \log_n n \log_2 n + 3n^2\left(\frac{{\log_8}^2 n - \log_8 n}{2}\right)$$

$$T(n) = (8^2)^{\log_8 n} T\left(\frac{n}{n}\right) - n^2 \log_8 n \log_2 n + 3n^2 \left(\frac{\log_8{}^2 n - \log_8 n}{2}\right)$$

$$T(n) = \left(8^{\log_8 n}\right)^2 T(1) - n^2 \log_8 n \log_2 n + \frac{3n^2}{2} \log_8{}^2 n - \frac{3n^2}{2} \log_8 n$$

$\therefore$ *The running time complexity is* $\boldsymbol{\Theta\left(n^2 \log_2 n \log_8{}^2 n\right)}$

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$$

- $a = 4, b = 2, c = 2, f(n) = n^2 \lg n$
- Case 1:

$$is\ f(n) \in O\left(n^{\log_b a - \varepsilon}\right)?$$
$$n^2 \lg n \in O\left(n^{\log_2 4 - \varepsilon}\right)$$
$$n^2 \lg n \in O(n^{2-\varepsilon})$$
$$n^2 \lg n \notin O(n^1) \quad \text{if } \varepsilon = 1$$

$$\therefore case\ 1\ of\ the\ master\ theorem\ cannot\ be\ used.$$

- Case 2:

$$is\ f(n) \in O\left(n^{\log_b a}\right)?$$

$$n^2 \lg n \in O\left(n^{\log_2 4}\right)$$

$$n^2 \lg n \in O(n^2)$$

$$n^2 \lg n \notin O(n^2)$$

$$\therefore\ case\ 2\ of\ the\ master\ theorem\ cannot\ be\ used.$$

- Case 3:

$$is\ f(n) \in O\left(n^{\log_b a+\varepsilon}\right)?$$

$$n^2 \lg n \in O\left(n^{\log_2 4+\varepsilon}\right)$$

$$n^2 \lg n \in O(n^3) \quad \text{if } \varepsilon = 1$$

$$is\ af\left(\frac{n}{2}\right) \leq C \times f(n)?$$

$$4 \times f\left(\frac{n}{2}\right) \leq C \times n^2 \lg n$$

$$4 \times \left(\frac{n}{2}\right)^2 \lg\left(\frac{n}{2}\right) \leq C \times n^2 \lg n$$

$$4 \times \frac{n^2}{4}(\lg n - \lg 2) \leq C \times n^2 \lg n$$

$$n^2 \lg n - n^2 \leq C \times n^2 \lg n \quad (Note\colon \lg 2 = 1)$$

Since C must be < 1, the above inequality failed and hence, case 3 of master theorem cannot be applied.
Expansion and substitution will be used instead.

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$$

$$T(n) = 4\left[4T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2 \lg\left(\frac{n}{2}\right)\right] + n^2 \lg n$$

$$T(n) = 4\left[4T\left(\frac{n}{2^2}\right) + \frac{n^2}{4}\left(\lg n - \lg 2\right)\right] + n^2 \lg n$$

$$T(n) = 4^2T\left(\frac{n}{2^2}\right) + n^2(\lg n - 1) + n^2 \lg n$$

$$T(n) = 4^2T\left(\frac{n}{2^2}\right) + n^2 \lg n - n^2 + n^2 \lg n$$

$$T(n) = 4^2T\left(\frac{n}{2^2}\right) + 2n^2 \lg n - n^2$$

$$T(n) = 4^2 T\left(\frac{n}{2^2}\right) + 2n^2 \lg n - n^2$$

$$T(n) = 4^2 \left[4T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2 \lg\left(\frac{n}{2^2}\right)\right] + 2n^2 \lg n - n^2$$

$$T(n) = 4^2 \left[4T\left(\frac{n}{2^3}\right) + \frac{n}{4^2}^2 (\lg n - \lg 4)\right] + 2n^2 \lg n - n^2$$

$$T(n) = 4^3 T\left(\frac{n}{2^3}\right) + n^2(\lg n - 2) + 2n^2 \lg n - n^2$$

$$T(n) = 4^3 T\left(\frac{n}{2^3}\right) + n^2 \lg n - 2n^2 + 2n^2 \lg n - n^2$$

$$T(n) = 4^3 T\left(\frac{n}{2^3}\right) + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^3 T\left(\frac{n}{2^3}\right) + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^3 \left[4T\left(\frac{n}{2^4}\right) + \left(\frac{n}{2^3}\right)^2 \lg\left(\frac{n}{2^3}\right)\right] + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^3 \left[4T\left(\frac{n}{2^4}\right) + \frac{n^2}{8^2} (\lg n - \lg 8)\right] + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^4 T\left(\frac{n}{2^4}\right) + n^2(\lg n - 3) + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^4 T\left(\frac{n}{2^4}\right) + n^2 \lg n - 3n^2 + 3n^2 \lg n - 3n^2$$

$$T(n) = 4^4 T\left(\frac{n}{2^4}\right) + 4n^2 \lg n - 6n^2$$

$$T(n) = 4^4 T\left(\frac{n}{2^4}\right) + 4n^2 \lg n - 6n^2$$

...

$$T(n) = 4^k T\left(\frac{n}{2^k}\right) + kn^2 \lg n - \left(\frac{k(k-1)}{2}\right) n^2$$

The recursive call will continue and stop when $\frac{n}{2^k} = 1$. Solving the equality, we have $n = 2^k$ and hence, $k = \lg n$.

Substitute k into the above equation, we have

$$T(n) = 4^{\lg n} T\left(\frac{n}{2^{\lg n}}\right) + (\lg n) n^2 \lg n - \left(\frac{\lg n\,(\lg n - 1)}{2}\right) n^2$$

$$T(n) = (2^2)^{\lg n} T\left(\frac{n}{n}\right) + n^2\, lg^2\, n - \left(\frac{lg^2\, n - \lg n}{2}\right) n^2$$

$$T(n) = \left(2^{\lg n}\right)^2 T(1) + n^2\, lg^2\, n - \left(\frac{n^2 lg^2 n}{2}\right) + \left(\frac{n^2 \lg n}{2}\right)$$

$$T(n) = (n)^2 + \left(\frac{n^2 lg^2 n}{2}\right) + \left(\frac{n^2 \lg n}{2}\right)$$

∴ the running time complexity is $\boldsymbol{\Theta(n^2 lg^2 n)}$.