Complex Fourier Series

Derivation and examples

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Chapter 1

Derivation

1.1 The Series

For a function $f(x)$ with range $-\frac{a}{2} < x < \frac{a}{2}$ we assume that it can be reproduced by a sum of complex exponentials of the form

$$exp(ik_n x) \tag{1.1.1}$$

Where $k = \frac{2n\pi}{a}$ so that n periods of a complex exponential fit's into the range $-\frac{a}{2} < x < \frac{a}{2}$. Along with complex coefficients for each term, we get the equation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a} x\right) \tag{1.1.2}$$

This is the complex Fourier series.

1.2 An interesting result

To calculate any particular coefficient, it will be important to evaluate the following intergral.

$$I_{nm} = int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a} x\right) \exp\left(\frac{-2im\pi}{a} x\right) dx \tag{1.2.1a}$$

$$= int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2i(n-m)\pi}{a} x\right) dx \tag{1.2.1b}$$

letting $p = (n - m)$

$$I_{nm} = int^{\frac{a}{2}}_{-\frac{a}{2}} \exp\left(\frac{2ip\pi}{a}x\right) dx \tag{1.2.1c}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(\frac{2ip\pi}{a}x\right)\right]^{\frac{a}{2}}_{-\frac{a}{2}} \tag{1.2.1d}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(\frac{2ip\pi}{a}\frac{a}{2}\right) - \exp\left(-\frac{2ip\pi}{a}\frac{a}{2}\right)\right] \tag{1.2.1e}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(ip\pi\right) - \exp\left(-ip\pi\right)\right] \tag{1.2.1f}$$

using $\exp(ix) = \cos(x) + i\sin(x)$

$$I_{nm} = \frac{a}{2ip\pi}\left[\cos(p\pi) + i\sin(p\pi) - (\cos(p\pi) - i\sin(p\pi))\right] \tag{1.2.1g}$$

$$= \frac{a}{2ip\pi}\left[\cos(p\pi) + i\sin(p\pi) - \cos(p\pi) + i\sin(p\pi)\right] \tag{1.2.1h}$$

$$= \frac{a}{2ip\pi}\left[2i\sin(p\pi)\right] \tag{1.2.1i}$$

$$= \frac{a}{p\pi}\sin(p\pi) \tag{1.2.1j}$$

$$= a\frac{\sin(p\pi)}{p\pi} \tag{1.2.1k}$$

$$= a\mathrm{sinc}(p\pi) \tag{1.2.1l}$$

$$= a\delta_{nm} \tag{1.2.1m}$$

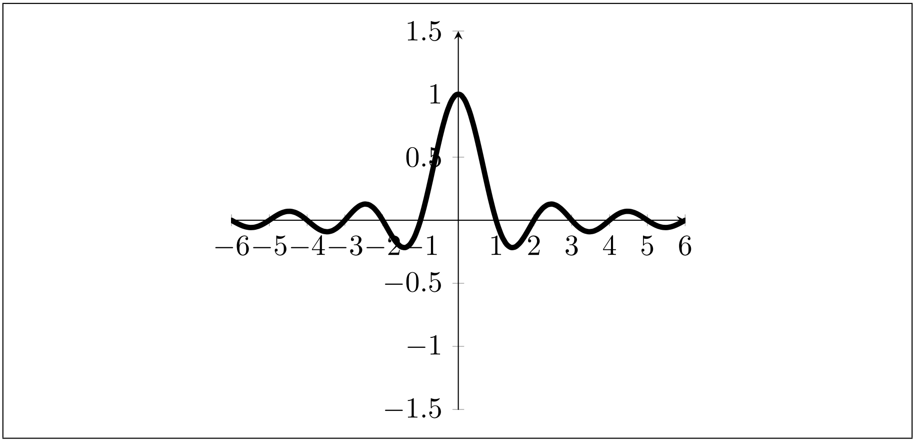

Figure 1.1: plot of $\mathrm{sinc}(x\pi)$

1.2.1 Summary

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right)\exp\left(\frac{-2im\pi}{a}x\right)dx = a\delta_{nm}$$

Where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n! = m \end{cases}$$

1.3 Extracting cooefficents

Multiplying equation 1.1.2 by $\exp\left(-\frac{2mi\pi}{a}\right)$ and integrating over the range $-\frac{a}{2} < x < \frac{a}{2}$ gives

$$\begin{aligned}
I_m &= \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}x\right) dx && (1.3.1a)\\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}x\right) dx && (1.3.1b)\\
&= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}x\right) dx && (1.3.1c)\\
&= \sum_{n=-\infty}^{\infty} c_n I_{nm} && (1.3.1d)\\
&= \sum_{n=-\infty}^{\infty} c_n a \delta_{nm} && (1.3.1e)\\
&= a c_m && (1.3.1f)
\end{aligned}$$

Taking the result from equation 1.3.1 and rearranging gives us

$$c_m = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}x\right) dx \qquad (1.3.2)$$

1.3.1 Summary

Assuming that a function on the range $-\frac{a}{2} < x < \frac{a}{2}$ can be expressed as a sum of complex exponentials of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right)$$

The value of a cooefficent c_n is given by the following intergral

$$c_n = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\exp\left(-\frac{2ni\pi}{a}x\right)dx$$

Chapter 2

Examples

2.1 Square Wave

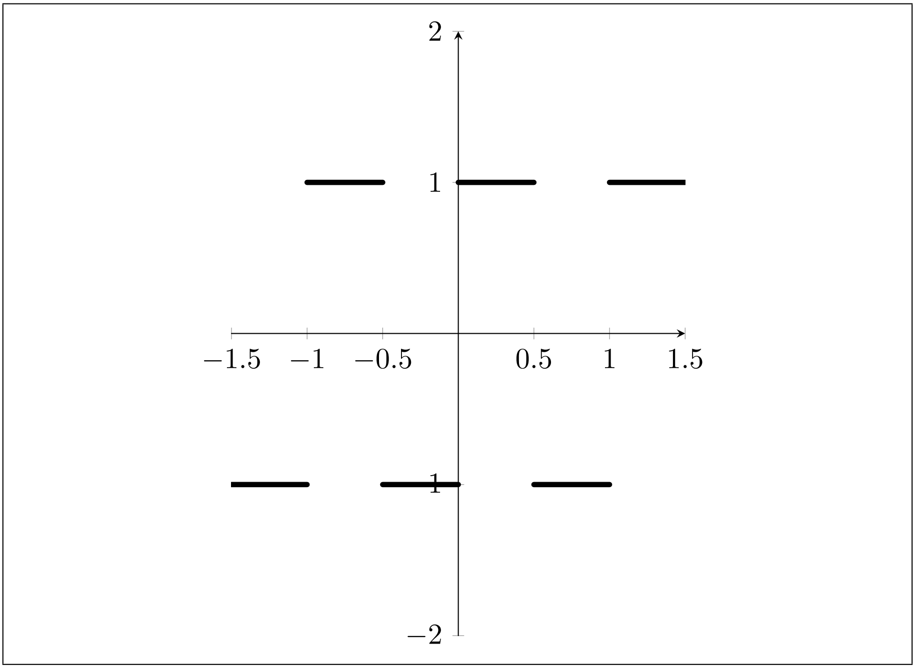

Figure 2.1: square wave

$$c_n = \frac{1}{a}\left[\int_{-\frac{a}{2}}^{0} -A\exp\left(-\frac{2ni\pi}{a}x\right)dx + \int_{0}^{\frac{a}{2}} A\exp\left(-\frac{2ni\pi}{a}x\right)dx\right] \tag{2.1.1a}$$

$$= \frac{A}{a}\left[\int_{0}^{\frac{a}{2}} \exp\left(-\frac{2ni\pi}{a}x\right)dx - \int_{-\frac{a}{2}}^{0} \exp\left(-\frac{2ni\pi}{a}x\right)dx\right] \tag{2.1.1b}$$

$$= \frac{A}{a}\left[-\frac{a}{2ni\pi}\left[\exp\left(-\frac{2ni\pi}{a}x\right)\right]_{0}^{\frac{a}{2}} + \frac{a}{2ni\pi}\left[\exp\left(-\frac{2ni\pi}{a}x\right)\right]_{-\frac{a}{2}}^{0}\right] \tag{2.1.1c}$$

$$= \frac{A}{2ni\pi}\left[\left[\exp\left(-\frac{2ni\pi}{a}x\right)\right]_{-\frac{a}{2}}^{0} - \left[\exp\left(-\frac{2ni\pi}{a}x\right)\right]_{0}^{\frac{a}{2}}\right] \tag{2.1.1d}$$

$$c_n = \frac{A}{2ni\pi}\left[\exp(0) - \exp(-ni\pi) - \exp(-ni\pi) + \exp(0)\right] \tag{2.1.1e}$$

$$= \frac{A}{ni\pi}\left[\exp(0) - \exp(-ni\pi)\right] \tag{2.1.1f}$$

$$= \frac{A}{ni\pi}\left[1 - \cos(n\pi) + i\sin(n\pi)\right] \tag{2.1.1g}$$

$$= \frac{A}{ni\pi}\left[1 + i\sin(n\pi)\right] \tag{2.1.1h}$$

Putting this result back in

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \tag{2.1.2}$$