Filters

Notes on passive electric filter circuits

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1 RC Low Pass Filter

1.1 Filter Attenuation

A simple low pass filter can be formed from a resistor and a capacitor in series.

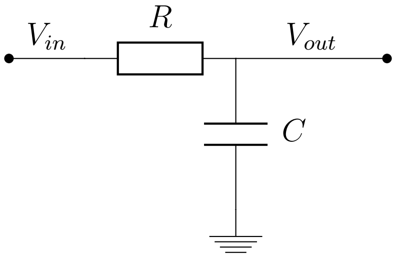

Figure 1: Circuit diagram for a RC low pass filter.

We can see from figure 1, that the circuit forms a potential divider just with a reactive element instead of purely resistive. The attenuation is then given by the standard potential divider result

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{-iX_{\text{C}}}{R - iX_{\text{C}}} \tag{1.1}$$

1.1.1 Cutoff Frequency

Let's introduce a new variable called u, where

$$\begin{aligned} u &= \frac{R}{X_{\text{C}}} \\ &= \omega RC \end{aligned} \tag{1.2}$$

where $\omega = 2\pi f$. If we look at the frequency when the resulting $u = 1$, which we will label f_0 or ω_0

$$\begin{aligned} \omega_0 RC &= 1 \\ \omega_0 &= \frac{1}{RC} \end{aligned} \tag{1.3}$$

We call the frequency when $u = 1$ the *cutoff frequency*, for reasons that will be clear later on. This frequency is when the resistance of the resistor is equal to the reactance of the capacitor[1] You can see that we can use the cutoff frequency as a replacement for our RC value, in equation 1.2.

$$\begin{aligned} u &= \omega RC \\ &= \frac{\omega}{\omega_0} = \frac{f}{f_0} \end{aligned} \tag{1.4}$$

1.1.2 Attenuation revisted

Now we have some understanding of the variable we introduced u, we can substitute it into our equation for the attenuation (equation 1.1), by noting that from equation 1.2 $R = uX_{\mathrm{C}}$

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{-iX_{\mathrm{C}}}{R - iX_{\mathrm{C}}} \\ &= \frac{-iX_{\mathrm{C}}}{uX_{\mathrm{C}} - iX_{\mathrm{C}}} \\ &= \frac{-i}{u - i} \\ &= \frac{1 - iu}{u^2 + 1} \end{aligned} \tag{1.5}$$

Normally, we don't consider the attenuation as a complex value, instead we care more about the magnitude and phase shift of an attenuation.

$$\begin{aligned} \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \frac{\sqrt{1 + u^2}}{1 + u^2} \\ &= \frac{1}{\sqrt{1 + u^2}} \\ &= \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^2}} \end{aligned} \tag{1.6}$$

[1] by equal here, we mean the magnitudes are equal. If not the phase shift.

Where we have used equation 1.4 in place of u. For the phase shift of the filter,

$$\phi = -\arctan u = -\arctan \frac{f}{f_0} \tag{1.7}$$

Sumary

In the last section we discovered the cutoff frequency was given by

$$f_0 = \frac{1}{2\pi RC}$$

and that the ratio of resistance to reactance can be given by

$$u = \frac{R}{X_{\mathrm{C}}} = 2\pi RCf = \frac{f}{f_0}$$

and that the attenuation of the filter is given by

$$\begin{aligned} \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} &= \frac{-iX_{\mathrm{C}}}{R - iX_{\mathrm{C}}} \\ &= \frac{1 - iu}{u^2 + 1} \end{aligned}$$

or in terms of magnitude and phase shift

$$\begin{aligned} \left| \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} \right| &= \frac{1}{\sqrt{1 + u^2}} \\ &= \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^2}} \end{aligned}$$

$$\phi = -\arctan u = -\arctan \frac{f}{f_0}$$

1.2 Log-Log Form

You won't often see attenuation given in the form seen earlier. It is more likely to be seen in Log-Log form, due to wanting to see the behaviour over a large range of frequencies and the fact the attenuation itself can get very small very fast. However it helps to look at the logirthm of u before looking at the attenuation straight away.

$$\ln u = \ln \frac{f}{f_0} = \ln f - \ln f_0 = F - F_0 \tag{1.8}$$

where we have used $F = \ln f$ and $F_0 = \ln f_0$

Now looking at the attenuation

$$\begin{aligned} \ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \ln \frac{1}{\sqrt{1+u^2}} \\ &= -\frac{1}{2} \ln \left(1+u^2\right) \\ &= -\frac{1}{2} \ln \left(u^2 \left(1 + \frac{1}{u^2}\right)\right) \\ &= -\frac{1}{2} \ln u^2 - \frac{1}{2} \ln \left(1 + \frac{1}{u^2}\right) \\ &= -\ln u - \frac{1}{2} \ln \left(1 + \frac{1}{u^2}\right) \\ &= F_0 - F - \frac{1}{2} \ln \left(1 + \left(\frac{f_0}{f}\right)^2\right) \end{aligned} \tag{1.9}$$

Lets quickly look at the term $\frac{f_0}{f}$ in equation 1.9. We'd like to express it in terms of our new variables F and F_0. To do this, we note that since $F = \ln f$ then $f = \exp(f)$, and so

$$\frac{f_0}{f} = \frac{\exp(F_0)}{\exp(F)} = \exp(F_0 - F) \tag{1.10}$$

putting the result from equation 1.10 back into equation 1.9 gives us

$$\ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = F_0 - F - \frac{1}{2} \ln \left(1 + \exp(2(F_0 - F))\right) \tag{1.11}$$

This is as simple as it gets sadly, however we can study some particular values of this equation and the extreame cases. For $F << F_0$

$$
\begin{aligned}
A(F << F_0) &= F_0 - F - \frac{1}{2}\ln\left(1 + \exp(2\underbrace{(F_0 - F)}_{\text{large and +ve}})\right) \\
&= F_0 - F - \frac{1}{2}\ln\left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{even larger and +ve}}\right) \\
&= F_0 - F - \frac{1}{2}\ln\left(\underbrace{1}_{\text{so this can be neglected}} + \exp(2(F_0 - F))\right) \\
A(F << F_0) &\approx F_0 - F - \frac{1}{2}\ln\left(\exp(2(F_0 - F))\right) \\
&= F_0 - F - F_0 + F = 0 \qquad (1.12)
\end{aligned}
$$

for $F >> F_0$

$$
\begin{aligned}
A(F >> F_0) &= F_0 - F - \frac{1}{2}\ln\left(1 + \exp(2\underbrace{(F_0 - F)}_{\text{large and -ve}})\right) \\
&= F_0 - F - \frac{1}{2}\ln\left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{very small and +ve}}\right) \\
&= F_0 - F - \frac{1}{2}\ln\left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{so this can be neglected}}\right) \\
A(F << F_0) &\approx F_0 - F - \frac{1}{2}\underbrace{\ln 1}_{=0} \\
&= F_0 - F \qquad (1.13)
\end{aligned}
$$

and finally when $F = F_0$

$$A(F = F_0) = 0 - \frac{1}{2}\ln\left(1 + \exp(0)\right)$$
$$A(F = F_0) = -\frac{\ln 2}{2} \tag{1.14}$$

Sumary

The equation for the log-log attenuation is given by

$$\ln\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = A(F) = F_0 - F - \frac{1}{2}\ln\left(1 + \exp(2(F_0 - F))\right)$$

where $F = \ln f$ and $F_0 = \ln f_0$, and has the following results

$$A(F << F_0) \approx 0$$
$$A(F = F_0) = -\frac{\ln 2}{2}$$
$$A(F >> F_0) \approx F_0 - F$$

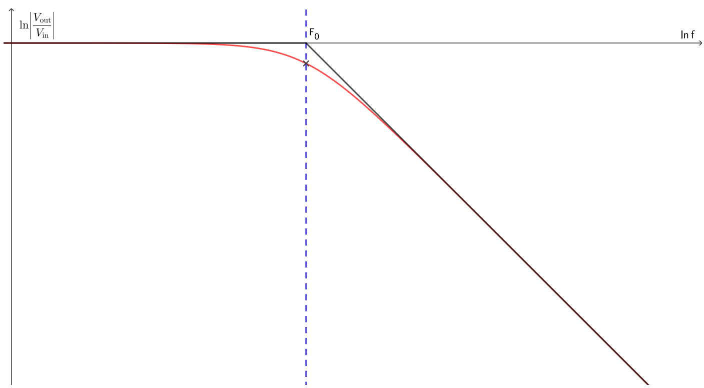

Figure 2: The log-log plot of the attenuation against frequency.

2 RL High Pass Filter

2.1 Filter Attenuation

A simple high pass filter can be formed from a resistor and an inductor in series.

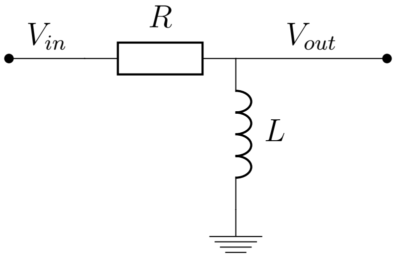

Figure 3: Circuit diagram for a RL high pass filter.

We can see from figure 3, that the circuit forms a potential divider just with a reactive element instead of purely resistive. The attenuation is then given by the standard potential divider result

$$\frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{iX_{\mathrm{L}}}{R + iX_{\mathrm{L}}} \tag{2.1}$$

2.1.1 Cutoff Frequency

Let's introduce a new variable called u, where

$$\begin{aligned} u &= \frac{R}{X_{\mathrm{L}}} \\ &= \frac{R}{\omega L} \end{aligned} \tag{2.2}$$

where $\omega = 2\pi f$. If we look at the frequency when the resulting $u = 1$, which we will label f_0 or ω_0

$$\begin{aligned} \frac{R}{\omega_0 L} &= 1 \\ \omega_0 &= \frac{R}{L} \end{aligned} \tag{2.3}$$

We call the frequency when $u = 1$ the *cutoff frequency*, for reasons that will be clear later on. This frequency is when the resistance of the resistor is equal to the reactance of the inductor[2] You can see that we can use the cutoff frequency as a replacement for our $\frac{R}{L}$ value, in equation 2.2.

$$\begin{aligned} u &= \frac{R}{\omega L} \\ &= \frac{\omega_0}{\omega} = \frac{f_0}{f} \end{aligned} \tag{2.4}$$

2.1.2 Attenuation revisted

Now we have some understanding of the variable we introduced u, we can substitute it into our equation for the attenuation (equation 2.1), by noting that from equation 2.2 $R = uX_{\mathrm{C}}$

$$\begin{aligned} \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} &= \frac{iX_{\mathrm{L}}}{R + iX_{\mathrm{L}}} \\ &= \frac{iX_{\mathrm{L}}}{uX_{\mathrm{L}} + iX_{\mathrm{L}}} \\ &= \frac{i}{u + i} \\ &= \frac{1 + iu}{u^2 + 1} \end{aligned} \tag{2.5}$$

Normally, we don't consider the attenuation as a complex value, instead we care more about the magnitude and phase shift of an

[2]by equal here, we mean the magnitudes are equal. If not the phase shift.

attenuation.

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{\sqrt{1+u^2}}{1+u^2} = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+\left(\frac{f_0}{f}\right)^2}} \tag{2.6}$$

Where we have used equation 2.4 in place of u. For the phase shift of the filter,

$$\phi = \arctan u = \arctan \frac{f_0}{f} \tag{2.7}$$

Sumary

In the last section we discovered the cutoff frequency was given by

$$f_0 = \frac{1}{2\pi}\frac{R}{L}$$

and that the ratio of resistance to reactance can be given by

$$u = \frac{R}{X_{\text{L}}} = \frac{1}{2\pi f}\frac{R}{L} = \frac{f_0}{f}$$

and that the attenuation of the filter is given by

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{iX_{\text{L}}}{R+iX_{\text{L}}} = \frac{1+iu}{u^2+1}$$

or in terms of magnitude and phase shift

$$\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = \frac{1}{\sqrt{1+u^2}} = \frac{1}{\sqrt{1+\left(\frac{f_0}{f}\right)^2}}$$

$$\phi = \arctan u = -\arctan\frac{f_0}{f}$$

2.2 Log-Log Form

You won't often see attenuation given in the form seen earlier. It is more likely to be seen in Log-Log form, due to wanting to see the behaviour over a large range of frequencies and the fact the attenuation itself can get very small very fast. However it helps to look at the logirthm of u before looking at the attenuation straight away.

$$\ln u = \ln\frac{f_0}{f} = \ln f_0 - \ln f = F_0 - F \tag{2.8}$$

where we have used $F = \ln f$ and $F_0 = \ln f_0$

Now looking at the attenuation

$$\begin{aligned}
\ln\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| &= \ln\frac{1}{\sqrt{1+u^2}} \\
&= -\frac{1}{2}\ln\left(1+u^2\right) \\
&= -\frac{1}{2}\ln\left(u^2\left(1+\frac{1}{u^2}\right)\right) \\
&= -\frac{1}{2}\ln u^2 - \frac{1}{2}\ln\left(1+\frac{1}{u^2}\right) \\
&= -\ln u - \frac{1}{2}\ln\left(1+\frac{1}{u^2}\right) \\
&= F - F_0 - \frac{1}{2}\ln\left(1+\left(\frac{f}{f_0}\right)^2\right) \qquad (2.9)
\end{aligned}$$

Lets quickly look at the term $\frac{f_0}{f}$ in equation 2.9. We'd like to express it in terms of our new variables F and F_0. To do this, we note that since $F = \ln f$ then $f = \exp(f)$, and so

$$\frac{f}{f_0} = \frac{\exp(F)}{\exp(F_0)} = \exp(F - F_0) \qquad (2.10)$$

putting the result from equation 2.10 back into equation 2.9 gives us

$$\ln\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = F - F_0 - \frac{1}{2}\ln\left(1 + \exp(2(F - F_0))\right) \qquad (2.11)$$

This is as simple as it gets sadly, however we can study some particular values of this equation and the extreame cases. For

$F << F_0$

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\begin{aligned}
A(F << F_0) &= F - F_0 - \frac{1}{2}\ln\left(1 + \exp(2\ \underbrace{(F - F_0)}_{\text{large and -ve}}\ )\right) \\
&= F - F_0 - \frac{1}{2}\ln\left(1 + \underbrace{\exp(2(F - F_0))}_{\text{very small and +ve}}\right) \\
&= F - F_0 - \frac{1}{2}\ln\left(1 + \ \underbrace{\exp(2(F - F_0))}_{\text{so this can be neglected}}\right) \\
A(F << F_0) &\approx F - F_0 - \frac{1}{2}\underbrace{\ln 1}_{=0} \\
&= F - F_0 \qquad (2.12)
\end{aligned}
$$

for $F >> F_0$

$$
\begin{aligned}
A(F >> F_0) &= F - F_0 - \frac{1}{2}\ln\left(1 + \exp(2\ \underbrace{(F - F_0)}_{\text{large and +ve}}\ )\right) \\
&= F - F_0 - \frac{1}{2}\ln\left(1 + \underbrace{\exp(2(F - F_0))}_{\text{even larger and +ve}}\right) \\
&= F - F_0 - \frac{1}{2}\ln\left(\underbrace{1}_{\text{so this can be neglected}} + \exp(2(F - F_0))\right) \\
A(F >> F_0) &\approx F - F_0 - \frac{1}{2}\ln\left(\exp(2(F - F_0))\right) \\
&= F - F_0 - F + F_0 = 0 \qquad (2.13)
\end{aligned}
$$

and finally when $F = F_0$

$$
\begin{aligned}
A(F = F_0) &= F_0 - F_0 - \frac{1}{2}\ln\left(1 + \exp(2(F_0 - F_0))\right) \\
&= -\frac{\ln 2}{2} \qquad (2.14)
\end{aligned}
$$

Sumary

The equation for the log-log attenuation is given by

$$\ln\left|\frac{V_{\text{out}}}{V_{\text{in}}}\right| = F - F_0 - \frac{1}{2}\ln\left(1+\exp(2(F-F_0))\right)$$

where $F = \ln f$ and $F_0 = \ln f_0$, and has the following results

$$A(F << F_0) \approx F - F_0$$
$$A(F = F_0) = -\frac{\ln 2}{2}$$
$$A(F >> F_0) \approx 0$$

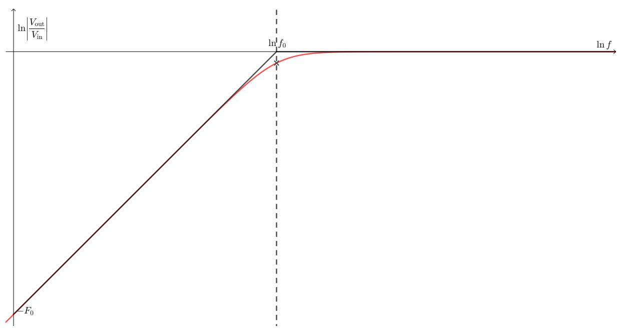

Figure 4: The log-log plot of the attenuation against frequency.

3 RLC Filter

3.1 Filter Attenuation

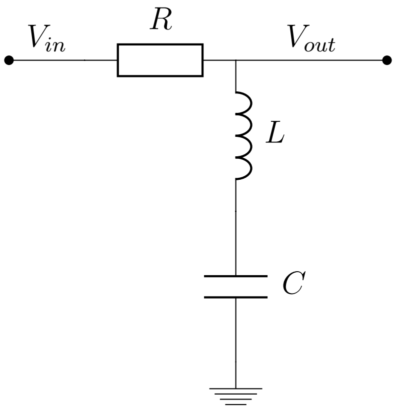

Figure 5: Circuit diagram for a RLC filter.

We can see from figure 5, that the circuit forms a potential divider just with a reactive element instead of purely resistive. The attenuation is then given by the standard potential divider result

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{iX}{R + iX} \tag{3.1}$$

Since the reactive part of the filter consists of both an inductor and a capacitor it's reactance is

$$\begin{aligned} X &= X_{\text{L}} - X_{\text{C}} \\ &= \omega L - \frac{1}{\omega C} \\ &= \frac{\omega^2 LC - 1}{\omega C} \end{aligned} \tag{3.2}$$

Looking at equation 3.2, you will notice it's possible for the reactance of the inductor to cancel out the reactance of the capacitor, leaving an overall zero reactance. This happens when

$$\begin{aligned}
0 &= X_{\mathrm{L}} - X_{\mathrm{C}} \\
&= \omega_r L - \frac{1}{\omega_r C} \\
\frac{1}{\omega_r C} &= \omega L \\
\omega_r^2 &= \frac{1}{LC} \\
\omega_r &= \frac{1}{\sqrt{LC}} \qquad (3.3)
\end{aligned}$$

Here we have labeled the frequency at which the reactance is zero as f_r, or ω_r, where the r stands for resonance, as this is the frequency at which the LC part would resonate. This will become useful later. Now if we do the usual thing of defining a new variable $u = \frac{R}{X}$ which is the ratio of the resistor value and the reactance.

$$\begin{aligned}
u &= \frac{R}{X} \\
&= \frac{\omega RC}{\omega^2 LC - 1} \\
&= \frac{\omega RC}{\left(\frac{\omega}{\omega_r}\right)^2 - 1} \qquad (3.4)
\end{aligned}$$

We have used the result from equation 3.3 to replace the LC part of equation 3.4.

Before going much further, notice the numerator of equation 3.4 might look familiar. It looks a little bit like equation 1.3, which we shall use here to deal with the RC part, but we will lable it ω_c

where the c stands for cutoff.

$$\begin{aligned} u &= \frac{\frac{\omega}{\omega_c}}{\left(\frac{\omega}{\omega_r}\right)^2 - 1} \\ &= \frac{\frac{f}{f_c}}{\left(\frac{f}{f_r}\right)^2 - 1} \end{aligned} \tag{3.5}$$

Again we can consider the frequency[3] when $u = 1$, or when the resistance is equal[4] to the reactance.

$$1 = \frac{\frac{f^*}{f_c}}{\left(\frac{f^*}{f_r}\right)^2 - 1}$$

$$\frac{1}{{f_r}^2} f^{*2} - \frac{1}{f_c} f^* - 1 = 0 \tag{3.6}$$

You will notice that equation 3.6 is a quadratic equation, which has a known solution of

$$\begin{aligned} f^* &= \frac{\frac{1}{f_c} \pm \sqrt{\frac{1}{{f_c}^2} + 4\frac{1}{{f_r}^2}}}{2\frac{1}{{f_r}^2}} \\ &= \frac{\frac{{f_r}^2}{f_c} \pm {f_r}^2\sqrt{\frac{1}{{f_c}^2} + 4\frac{1}{{f_r}^2}}}{2} \\ &= \frac{\frac{{f_r}^2}{f_c} \pm {f_r}^2\sqrt{\frac{1}{{f_c}^2} + 4\frac{{f_c}^2}{{f_c}^2{f_r}^2}}}{2} \\ &= \frac{\frac{{f_r}^2}{f_c} \pm \frac{{f_r}^2}{f_c}\sqrt{1 + 4\frac{{f_c}^2}{f_r^2}}}{2} \\ &= \frac{{f_r}^2}{f_c}\frac{1 \pm \sqrt{1 + 4\left(\frac{f_c}{f_r}\right)^2}}{2} \end{aligned} \tag{3.7}$$

[3]or due to the square term in equation 3.5, frequencies

[4]at least in magnitude