Abstract Maths Cheat Sheet

1 Ket-vector Axioms

1. The sum of any two ket-vectors is also a ket-vector:

$$|A\rangle + |B\rangle = |C\rangle \tag{1.1}$$

2. Vector addition is commutative:

$$|A\rangle + |B\rangle = |B\rangle + |A\rangle \tag{1.2}$$

3. Vector addition is associative:

$$(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle) \tag{1.3}$$

4. There is a unique vector 0 such that when you add it to any ket, it gives the same ket back:

$$|A\rangle + 0 = |A\rangle \tag{1.4}$$

5. Given any ket $|A\rangle$, there is a unique ket $-|A\rangle$ such that

$$|A\rangle + (-|A\rangle) \tag{1.5}$$

6. Given any $|A\rangle$ and any complex number z, you can multiply them to get a new ket. Also, multiplication by a scalar is linear:

$$|zA\rangle = z\,|A\rangle = |B\rangle \tag{1.6}$$

7. The distributive property holds:

$$z\left(|A\rangle + |B\rangle\right) = z\,|A\rangle + z\,|B\rangle \tag{1.7}$$

$$(z+w)\,|A\rangle = z\,|A\rangle + w\,|A\rangle \tag{1.8}$$

Taken together 6 and 7 are often call linearity

2 Bra-vector Axioms

1. The sum of any two bra-vectors is also a bra-vector:

$$\langle A| + \langle B| = \langle C| \tag{2.1}$$

2. Vector addition is commutative:

$$\langle A| + \langle B| = \langle B| + \langle A| \tag{2.2}$$

3. Vector addition is associative:

$$\left(\langle A| + \langle B|\right) + \langle C| = \langle A| + \left(\langle B| + \langle C|\right) \tag{2.3}$$

4. There is a unique vector 0 such that when you add it to any bra, it gives the same bra back:

$$\langle A| + 0 = \langle A| \tag{2.4}$$

5. Given any bra $\langle A|$, there is a unique bra $-\langle A|$ such that

$$\langle A| + (-\langle A|) \tag{2.5}$$

6. Given any $\langle A|$ and any complex number z, you can multiply them to get a new bra. Also, multiplication by a scalar is linear:

$$\langle zA| = z\,\langle A| = \langle B| \tag{2.6}$$

7. The distributive property holds:

$$z\left(\langle A| + \langle B|\right) = z\,\langle A| + z\,\langle B| \tag{2.7}$$

$$(z+w)\,\langle A| = z\,\langle A| + w\,\langle A| \tag{2.8}$$

Taken together 6 and 7 are often call linearity

3 Bras and Kets

1. For every ket-vector $|A\rangle$, there is a bra-vector in the dual space, denoted by $\langle A|$.

2. Bra and Ket vectors together form inner products of bras and kets, using expressions like $\langle A|B\rangle$ to form bra-kets or brackets.

3. Suppose $\langle A|$ is the bra corresponding to the ket $|A\rangle$, and $\langle B|$ is the bra corresponding to the ket $|B\rangle$. Then the bra corresponding to $|A\rangle + |B\rangle$ is $\langle A| + \langle B|$

4. If z is a complex number, then it is not true that the bra corresponding to $z\,|A\rangle$ is $\langle A|\, z$. You have to remember to complex-conjugate. Thus, the bra corresponding to $z\,|A\rangle$ is $\langle A|\, z^*$

4 Inner Products

1. Inner products are linear:

$$\langle C|\,(|A\rangle + |B\rangle) = \langle C|A\rangle + \langle C|B\rangle \tag{4.1}$$

2. Interchanging bras and kets corresponds to complex conjugation:

$$\langle B|A\rangle = \langle B|A\rangle^* \tag{4.2}$$

Additionally we can show that $(\langle A| + \langle B|)\,|C\rangle = \langle A|C\rangle + \langle B|C\rangle$ using axiom 2 above we get

$$(\langle C|\,(|A\rangle + |B\rangle))^* = (\langle C|A\rangle + \langle C|B\rangle)^* \tag{4.3}$$

$$\langle C|\,(|A\rangle + |B\rangle) = \langle C|A\rangle + \langle C|B\rangle \tag{4.4}$$

which is the same as axiom 1

Also applying axiom 2 to $\langle A|A\rangle$ give that $\langle A|A\rangle = \langle A|A\rangle^*$ which can only be true if $\langle A|A\rangle$ is a real number.

Normalized Vector A vector is said to be normalized if its inner product with itself is 1 ie $\langle A|A\rangle = 1$

Orthogonal Vectors Two vectors are said to be orthogonal if their inner product is zero ie $|A\rangle$ and $|B\rangle$ are orthogonal if $\langle B|A\rangle = 0$.

5 Orthonormal Bases

Let's consider an N-dimensional space (one with N possible states) and a particular orthonormal basis of ket-vectors labeled $|i\rangle$. An orthonormal basis is one where each of the basis ket-vectors are normalised and orthogonal to each other, this leads means that inner products between two basis kets obey the following

$$\langle i|j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{5.1}$$