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Normalisation of Bessel functions

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~1 minute

Call the integral under inspection for I, and let all $J = J_n$. The substitution implies that we are bound to show that

$$\frac{1}{\alpha^2}\int_0^\alpha z(J(z))^2\,dz = \frac{1}{2}(J'(\alpha))^2\,.$$

I think it is more intuitive to start the work from the differential equation. First, we have the Bessel equation

$$z^2J'' + zJ' + (z^2 - n^2)J = 0,$$

which can be written

$$z(zJ')' = (n^2 - z^2)J\,.$$

Now multiply by $2J'$ to get

$$2(zJ')'zJ' = (n^2 - z^2)2JJ',$$

or

$$((zJ')^2)' = (n^2 - z^2)(J^2)'$$

Integrating from 0 to α, and then by parts, we get

$$\begin{aligned}\alpha^2(J'(\alpha))^2 = &\int (n^2 - z^2)(J^2)'\,dz \\ &= \left[(n^2 - z^2)(J(z))^2\right]_0^\alpha + 2\int_0^\alpha z(J(z))^2\,dz \\ &= 2\int_0^\alpha z(J(z))^2\,dz\end{aligned}$$

and we are done. Note that in the last step, we use that $(n^2 - 0^2) = 0$ if $n = 0$ and that $J_n(0) = 0$ if $n > 0$.

answered Feb 25, 2016 at 20:08

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