Ellipse

Derivation of the equations for an Ellipse

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Contents

List of Tables

List of Figures

1 Introduction

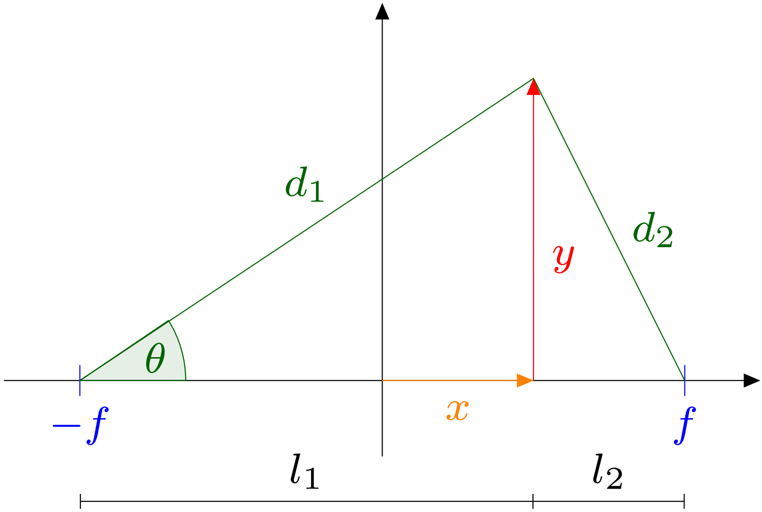

Figure 1: Two lines from two equally spaced focal points

An ellipse can be formed by keeping the sum of the lengths of two lines from two equally spaced focal points constant. This text however will not assume that is the case and will derive the equations for an ellipse in both Cartesian and polar forms from this setup. Figure 1 shows the geometry we will start with to derive the equations. Starting from this we note a couple of things. Firstly the sum of the lengths of each line is defined to be constant (D). Also the lengths of the projections onto the x axis must sum to $2f$.

$$d_1 + d_2 = D \tag{1.1}$$

$$l_1 + l_2 = 2f \tag{1.2}$$

From the geometry in figure 1

$$d_1 = \sqrt{{l_1}^2 + y^2} = \sqrt{(f+x)^2 + y^2} \tag{1.3}$$

$$d_2 = \sqrt{{l_2}^2 + y^2} = \sqrt{(f-x)^2 + y^2} \tag{1.4}$$

2 Carteisan form

Starting with equation 1.1 and subtituting equations 1.3 for d_1 and 1.4 for d_2 gives

$$D = \sqrt{(f+x)^2 + y^2} + \sqrt{(f-x)^2 + y^2} \tag{2.1}$$

taking one square root onto one side and squaring gives

$$\left(D - \sqrt{(f-x)^2 + y^2}\right)^2 = (f+x)^2 + y^2 \tag{2.2a}$$

$$D^2 - 2D\sqrt{(f-x)^2 + y^2} + (f-x)^2 + \cancel{y^2} = (f+x)^2 + \cancel{y^2} \tag{2.2b}$$

$$D^2 - 2D\sqrt{(f-x)^2 + y^2} + \cancel{f^2} - 2fx + \cancel{x^2} = \cancel{f^2} + 2fx + \cancel{x^2} \tag{2.2c}$$

$$D^2 - 4fx = 2D\sqrt{(f-x)^2 + y^2} \tag{2.2d}$$

$$\frac{D}{2} - \frac{2f}{D}x = \sqrt{(f-x)^2 + y^2} \tag{2.2e}$$

now we have isolated the second root, we can now square again to give

$$\left(\frac{D}{2} - \frac{2f}{D}x\right)^2 = (f-x)^2 + y^2 \tag{2.3a}$$

$$\left(\frac{D}{2}\right)^2 \cancel{-2fx} + \left(\frac{2f}{D}\right)^2 x^2 = f^2 \cancel{-2fx} + x^2 + y^2 \tag{2.3b}$$

$$\left(\frac{D}{2}\right)^2 + \left(\frac{2f}{D}\right)^2 x^2 = f^2 + x^2 + y^2 \tag{2.3c}$$

Finally group terms

$$\left(\frac{D}{2}\right)^2 - f^2 = \left(1 - \left(\frac{2f}{D}\right)^2\right) x^2 + y^2 \tag{2.4a}$$

$$\left(\frac{D}{2}\right)^2 - f^2 = \frac{\left(\frac{D}{2}\right)^2 - f^2}{\left(\frac{D}{2}\right)^2} x^2 + y^2 \tag{2.4b}$$

$$1 = \frac{x^2}{\left(\frac{D}{2}\right)^2} + \frac{y^2}{\left(\frac{D}{2}\right)^2 - f^2} \tag{2.4c}$$

$$\tag{2.4d}$$

finally giving us

$$1 = \frac{x^2}{\left(\frac{D}{2}\right)^2} + \frac{y^2}{\left(\frac{D}{2}\right)^2 - f^2} \tag{2.5}$$

Which is of the typical form for the Cartesian equation for an ellipse, with $A = \frac{D}{2}$ and $B = \sqrt{A^2 + f^2}$. Both of which can (and will) be shown to arise from the geometry of the situation.

2.1 The semi major axis

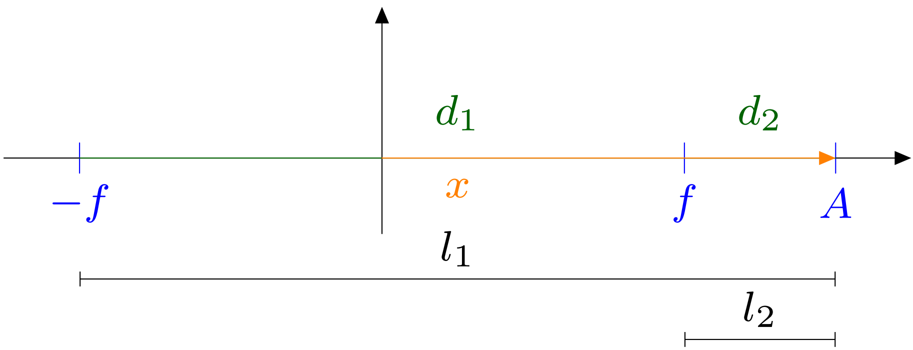

Figure 2: Orientating the two lines to form the semi major axis

In the situation shown in figure 2 we have that

$$d_1 = A + f \tag{2.6a}$$
$$d_2 = A - f \tag{2.6b}$$
$$D = d_1 + d_2 = A + f + A - f = 2A \tag{2.6c}$$

It imediatly follows from equation 2.6c that $A = \frac{D}{2}$

2.2 The semi minor axis

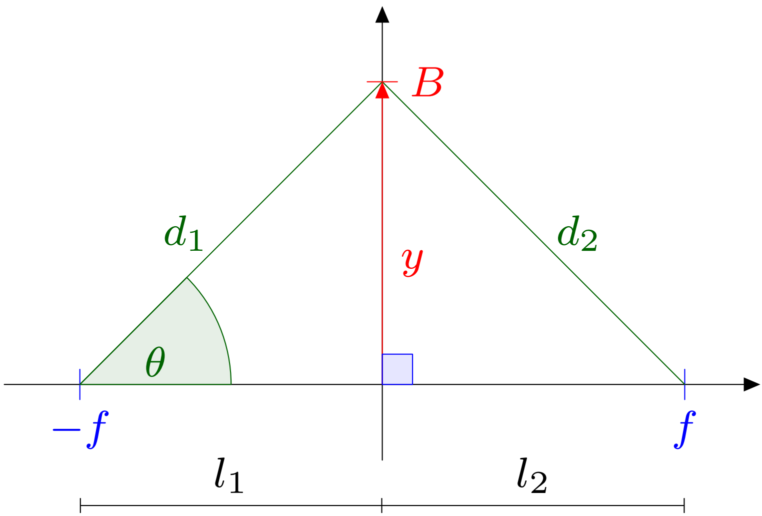

Figure 3: Orientating the two lines to form the semi major axis

In the situation shown in figure 3 we have that

$$d_2 = \frac{D}{2} = \sqrt{f^2 + B^2} \tag{2.7a}$$
$$B = \sqrt{\left(\frac{D}{2}\right)^2 - f^2} \tag{2.7b}$$

as obtained above

2.3 Summary

In this section we have found that the equation of an ellipse in Cartesian form is

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

Where A and B can be related to parameters of the constant length line formation as follows

$$A = \frac{D}{2}$$
$$B = \sqrt{\left(\frac{D}{2}\right)^2 - f^2}$$

Where D is the total length of the two lines connecting a point on the ellipse to the two focal points, and f is the distance from origin to the two focal points.

3 Polar Form

The polar form can be found directly from figure 1 as it is just a matter of finding the length of d_1 in terms of θ. Starting from equation 1.1 and substituting in 1.4 and rearranging to get d_1

$$d_1 = D - \sqrt{{l_2}^2 + y^2} \tag{3.1}$$

using equation 1.2 to substitute for l_2

$$d_1 = D - \sqrt{(2f - l_1)^2 + y^2} \tag{3.2a}$$
$$d_1 = D - \sqrt{4f^2 - 4fl_1 + {l_1}^2 + y^2} \tag{3.2b}$$

A quick glance at figure 1 will show that $d_1^2 = {l_1}^2 + y^2$ which we can use in equation 3.2b.

$$d_1 = D - \sqrt{4f^2 - 4fl_1 + d_1^2} \tag{3.3a}$$

$$D - d_1 = \sqrt{4f^2 - 4fl_1 + d_1^2} \tag{3.3b}$$

now squaring away the square root

$$(D - d_1)^2 = 4f^2 - 4fl_1 + d_1^2 \tag{3.4a}$$

$$D^2 - 2Dd_1 + \cancel{d_1^2} = 4f^2 - 4fl_1 + \cancel{d_1^2} \tag{3.4b}$$

$$D^2 - 2Dd_1 = 4f^2 - 4fl_1 \tag{3.4c}$$

$$d_1 = \frac{D^2 - 4f^2 + 4fl_1}{2D} \tag{3.4d}$$

$$d_1 = \frac{D}{2} - \frac{(2f)^2}{2D} + \frac{2f}{D}l_1 \tag{3.4e}$$

$$d_1 - \frac{2f}{D}l_1 = \frac{D}{2} - \frac{(2f)^2}{2D} \tag{3.4f}$$

Now we wish to get things in terms of θ by making the substitution $l_1 = d_1 \cos\theta$

$$d_1\left(1 - \frac{2f}{D}\cos\theta\right) = \frac{D}{2} - \frac{(2f)^2}{2D} \tag{3.5a}$$

$$d_1\left(1 - \frac{2f}{D}\cos\theta\right) = \frac{D}{2}\left(1 - \frac{(2f)^2}{2D}\frac{2}{D}\right) \tag{3.5b}$$

$$d_1\left(1 - \frac{2f}{D}\cos\theta\right) = \frac{D}{2}\left(1 - \left(\frac{2f}{D}\right)^2\right) \tag{3.5c}$$

finally we get the form

$$d_1 = \frac{\frac{D}{2}\left(1 - \left(\frac{2f}{D}\right)^2\right)}{1 - \frac{2f}{D}\cos\theta} \tag{3.6}$$

which is exactly the polar form

$$r = \frac{A\left(1 - e^2\right)}{1 - e\cos\theta} \tag{3.7}$$

with $A = \frac{D}{2}$ as before and $e = \frac{2f}{D}$

3.1 Polar form in terms of apses

Starting with the a slightly expanded form of equation 3.7

$$r(\theta) = \frac{A(1+e)(1-e)}{1 - e\cos\theta} \tag{3.8}$$

When r is maximums when $\cos\theta = 1$ i.e. when $\theta = 0$. We stall call the maximum radius a so using equation 3.8 with $\cos\theta = 1$

$$a = \frac{A(1+e)(1-e)}{1-e} = A(1+e) \tag{3.9}$$

Also we can find when r is minimum. This occurs when $\cos\theta = -1$ i.e. when $\theta = \pi$. Again using equation 3.8 with $\cos\theta = -1$

$$p = \frac{A(1+e)(1-e)}{1+e} = A(1-e) \tag{3.10}$$

We can sum the equations 3.9 and 3.10 above to give

$$\begin{aligned} a + p &= A + eA + A - eA \\ a + p &= 2A \\ A &= \frac{a+p}{2} \qquad (3.11) \end{aligned}$$

This also follows from the geometry of an ellipse.

finally taking the difference of the equations 3.9 and 3.10 above gives

$$\begin{aligned} a - p &= A + eA - A + eA \\ a - p &= 2eA \\ e &= \frac{a-p}{2A} \\ e &= \frac{a-p}{a+p} \qquad (3.12) \end{aligned}$$

Using equation 3.6 and comparing to equation 3.7 gives

$$\begin{aligned} e &= \frac{f}{A} \\ f &= eA \\ &= \frac{a-p}{a+p}\frac{a+p}{2} \\ &= \frac{a-p}{2} \qquad (3.13) \end{aligned}$$

Finally we can get B in terms of a and p by using equation 2.7b

$$B^2 = \left(\frac{D}{2}\right)^2 - f^2 \qquad (3.14a)$$

$$B^2 = A^2 - f^2 \qquad (3.14b)$$

$$B^2 = \frac{(a+p)^2}{4} - \frac{(a-p)^2}{4} \qquad (3.14c)$$

$$B^2 = \frac{a^2+2ap+p^2}{4} - \frac{a^2-2ap+p^2}{4} \qquad (3.14d)$$

$$B^2 = ap \qquad (3.14e)$$

$$B = \sqrt{ap} \qquad (3.14f)$$

3.2 Summary

Letting $a = r_{max}$ and $p = r_{min}$ we find that these relate to other properties of the ellipse as follows

$$\begin{aligned} A &= \frac{a+p}{2} \\ e &= \frac{a-p}{a+p} \\ f &= \frac{a-p}{2} \\ B &= \sqrt{ap} \end{aligned}$$