Orbits

Notes on calculations on classical orbits under gravity

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1 Plane Polar Coordinate System

In this section we will derive the equations of motion for an object acted upon by gravity in plane polar coordinates. To do this we will first consider plane polar coordinates themselves and the derivatives of the position vector within the coordinate system.

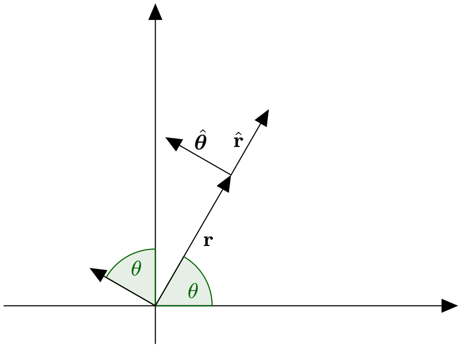

Figure 1: Position and unit vectors in plane polar coordinates

1.1 Polar unit vectors

The unit vectors of plane polar coordinates can be determined from the standard unit vectors as

$$\hat{\boldsymbol{r}} = \hat{\mathbf{i}}\cos(\theta) + \hat{\mathbf{j}}\sin(\theta) \tag{1.1}$$

$$\hat{\boldsymbol{\theta}} = -\hat{\mathbf{i}}\sin(\theta) + \hat{\mathbf{j}}\cos(\theta) \tag{1.2}$$

Where r is the distance from the origin to the position or alternatively the length of the position vector, and θ is the angle between $\hat{\mathbf{i}}$ and $\boldsymbol{r}$. Since the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ depend on θ which changes with time, the unit vectors also change with time. We will begin by looking at the derivative of $\hat{\boldsymbol{r}}$ with respect to time.

$$\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}\hat{\boldsymbol{r}} = \frac{d\theta}{dt}\left[-\hat{\mathbf{i}}\sin(\theta) + \hat{\mathbf{j}}\cos(\theta)\right] = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \tag{1.3}$$

where we have used equation 1.2 to replace the typical Cartesian coordinate unit vectors.

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\theta}{dt}\frac{d}{dt}\hat{\boldsymbol{\theta}} = \frac{d\theta}{dt}\left[-\hat{\mathbf{\imath}}\cos(\theta) - \hat{\mathbf{\jmath}}\sin(\theta)\right] = -\frac{d\theta}{dt}\hat{\boldsymbol{r}} \tag{1.4}$$

Since $\frac{d\hat{\boldsymbol{r}}}{dt}$ too depends on θ so is also time dependent.

$$\frac{d^2\hat{\boldsymbol{r}}}{dt^2} = \frac{d}{dt}\left[\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}\right] = \frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + \frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} - \left(\frac{d\theta}{dt}\right)^2\hat{\boldsymbol{r}} \tag{1.5}$$

1.1.1 Polar unit vectors summary

The polar coordinate unit vectors are

$$\begin{aligned}\hat{\boldsymbol{r}} &= \hat{\mathbf{\imath}}\cos(\theta) + \hat{\mathbf{\jmath}}\sin(\theta)\\ \hat{\boldsymbol{\theta}} &= -\hat{\mathbf{\imath}}\sin(\theta) + \hat{\mathbf{\jmath}}\cos(\theta)\end{aligned}$$

With the first derivatives being

$$\begin{aligned}\frac{d\hat{\boldsymbol{r}}}{dt} &= \omega\hat{\boldsymbol{\theta}}\\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\omega\hat{\boldsymbol{r}}\end{aligned}$$

And the second derivative of $\hat{\boldsymbol{r}}$ being

$$\frac{d^2\hat{\boldsymbol{r}}}{dt^2} = \frac{d\omega}{dt}\hat{\boldsymbol{\theta}} - \omega^2\hat{\boldsymbol{r}}$$

with $\omega = \frac{d\theta}{dt}$

1.2 Acceleration in polar coordinates

Here we will calculate the acceleration in terms of the polar unit vectors. Starting with the position vector

$$\mathbf{r} = r\hat{\mathbf{r}} \tag{1.6}$$

This is the length of the position vector in the direction of $\hat{\boldsymbol{r}}$. The first derivative will be the velocity of the position vector in terms of the polar unit vectors

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\frac{d\hat{\boldsymbol{r}}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}} \tag{1.7}$$

The velocity is the sum of the radial velocity in the radial direction $\dot{r}$ and the angular part of the velocity $r\omega$ in the angular direction.

The acceleration

$$\begin{aligned}
\frac{d^2\boldsymbol{r}}{dt^2} &= \frac{d^2r}{dt^2}\hat{\boldsymbol{r}} + \frac{dr}{dt}\frac{d\hat{\boldsymbol{r}}}{dt} + \frac{dr}{dt}\omega\hat{\boldsymbol{\theta}} + r\frac{d\omega}{dt}\hat{\boldsymbol{\theta}} + r\omega\frac{d\hat{\boldsymbol{\theta}}}{dt} \\
&= \frac{d^2r}{dt^2}\hat{\boldsymbol{r}} + \frac{dr}{dt}\omega\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\omega\hat{\boldsymbol{\theta}} + r\frac{d\omega}{dt}\hat{\boldsymbol{\theta}} - r\omega^2\hat{\boldsymbol{r}} \\
&= \left[\frac{d^2r}{dt^2} - r\omega^2\right]\hat{\boldsymbol{r}} + \left[2\frac{dr}{dt}\omega + r\frac{d\omega}{dt}\right]\hat{\boldsymbol{\theta}} \qquad (1.8)
\end{aligned}$$

1.2.1 Acceleration in polar coordinates summary

The position vector

$$\boldsymbol{r} = r\hat{\boldsymbol{r}}$$

Velocity

$$\frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}$$

Acceleration

$$\frac{d^2\boldsymbol{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\omega^2\right]\hat{\boldsymbol{r}} + \left[2\frac{dr}{dt}\omega + r\frac{d\omega}{dt}\right]\hat{\boldsymbol{\theta}}$$

2 Angular Momentum

The angular momentum of a particle is given by

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} \\
&= m\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt} \\
&= mr\hat{\boldsymbol{r}} \times \left[\dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}\right] \\
&= m\left[r\dot{r}\hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}} + r^2\omega\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}}\right] \\
&= mr^2\omega\hat{\boldsymbol{k}} \qquad (2.1)
\end{aligned}$$

We shall see later that the angular momentum of a particle in orbit is conserved.

3 Areal Velocity

The areal velocity is the rate of change in the area swept out between a point (typically the origin) and a particle as it moves.

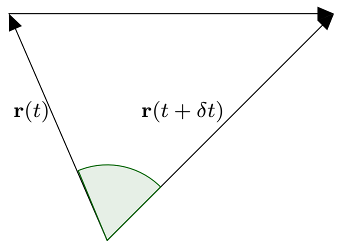

Figure 2: Areal velocity about a point

It can be seen from figure 2 that the area swept out is approximatly the area of the triangle formed between $\boldsymbol{r}(t)$ and $\boldsymbol{r}(t+\delta t)$ and would be equal in the limit $\delta t \to 0$

$$\begin{aligned}
\delta A &\approx \frac{1}{2} h |\boldsymbol{r}(t+\delta t)| \\
h &= |\boldsymbol{r}(t)| \sin\theta \\
\delta A &\approx \frac{1}{2} |\boldsymbol{r}(t)||\boldsymbol{r}(t+\delta t)| \sin\theta \\
\delta A &\approx \frac{1}{2} |\boldsymbol{r}(t) \times \boldsymbol{r}(t+\delta t)| \qquad (3.1)
\end{aligned}$$

Using the expansion

$$\boldsymbol{r}(t+\delta t) = \boldsymbol{r}(t) + \delta t \frac{d\boldsymbol{r}(t)}{dt} + \frac{\delta t}{2} \frac{d^2\boldsymbol{r}(t)}{dt^2} + \dots \qquad (3.2)$$

in equation 3.1

4 Gravity

4.1 Gravity as a central force

The force between two particles a distance r apart with mass M and m respectivly is given by

$$\boldsymbol{F} = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} \tag{4.1}$$

The acceleration on the particle of mass m due to the force of gravity between the two particles is

$$\ddot{\boldsymbol{r}} = \frac{\boldsymbol{F}}{m} = -\frac{GM}{r^2}\hat{\boldsymbol{r}} \tag{4.2}$$

Using equation 1.8 we can split the acceleration into radial and angular parts

$$-\frac{GM}{r^2} = \ddot{r} - r\omega^2 \tag{4.3}$$

$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{4.4}$$

We can use equation 4.4 to show that angular momentum is conserved

4.2 Conservation of angular momentum

Taking the first deriviative of angular momentum

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \left[m\frac{dr^2}{dt}\omega + mr^2\frac{d\omega}{dt}\right]\hat{\boldsymbol{k}} \\ &= m\left[\frac{dr^2}{dr}\dot{r}\dot{\theta} + r^2\ddot{\theta}\right]\hat{\boldsymbol{k}} \\ &= mr\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\boldsymbol{k}} \\ &= 0 \end{aligned} \tag{4.5}$$

where in the last line we have used equation 4.4. Since the angular momenum does not change with time, it is therefore a constant.

4.3 Energy

Here we consider the energy of a particle in orbit.

4.3.1 Potential Energy

Firstly let us consider the potential energy of the particle question. The potential energy is the work done in moving the particle from a radius of ∞ to r. So the work done is given by

$$\begin{aligned}
U(r) &= \int_{\infty}^{r} \boldsymbol{F}(r') \cdot d\boldsymbol{r}' \\
U(r) &= \int_{\infty}^{r} -\frac{GMm}{r'^2}\hat{\boldsymbol{r}} \cdot dr'\hat{\boldsymbol{r}} \\
U(r) &= -GMm \int_{\infty}^{r} \frac{dr'}{r'^2} \\
U(r) &= -GMm \left[\frac{-1}{r'}\right]_{\infty}^{r} \\
U(r) &= GMm \left[\frac{1}{\infty} - \frac{1}{r}\right] \\
U(r) &= -\frac{GMm}{r} \qquad (4.6)
\end{aligned}$$

4.3.2 Kinetic Energy

The kinetic energy of a particle is given by

$$T(\dot{r}) = \frac{1}{2}m\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \qquad (4.7)$$

using eqation 1.2.1 gives

$$\begin{aligned}
T(\dot{r}) &= \frac{1}{2}m\left[\dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}\right] \cdot \left[\dot{r}\hat{\boldsymbol{r}} + r\omega\hat{\boldsymbol{\theta}}\right] \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 \qquad (4.8)
\end{aligned}$$

we can also make the subsitution using equation 2.1 with the notation that $l = mr^2\omega$

$$
\begin{aligned}
l &= mr^2\omega \\
\omega &= \frac{l}{mr^2} \qquad (4.9)
\end{aligned}
$$

using equation 4.9 in equation 4.8 leads to

$$
\begin{aligned}
T(\dot{r}) &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 \\
&= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\frac{l^2}{m^2r^4} \\
&= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} \qquad (4.10)
\end{aligned}
$$

4.3.3 Total Energy

Finally the total energy can be found by putting together the equation for kinetic energy and potential energy

$$
\begin{aligned}
E(r,\dot{r}) &= T(\dot{r}) + U(r) \\
E(r,\dot{r}) &= \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r} \qquad (4.11)
\end{aligned}
$$

sometimes the last two terms of equation 4.11 are put together to form a new potential energy type term that is dependent only on the radius and the angular momentum.

4.3.4 Conservation of Total Energy

4.4 Summary

The equations of motion under gravity are

$$
\begin{aligned}
-\frac{GM}{r^2} &= \ddot{r} - r\omega^2 \\
0 &= r\ddot{\theta} + 2\dot{r}\dot{\theta}
\end{aligned}
$$

and the total energy of a particle acted on by gravity is

$$
E(r, \dot{r}) = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GMm}{r}
$$

5 Solving equations of motion

Here we will solve the system of eqations given in section 4.

5.1 Conservation of angular momentum revisited

Firstly we note that

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = r^2\frac{d^2\theta}{dt^2} + 2r\frac{dr}{dt}\frac{d\theta}{dt} \tag{5.1}$$

is the same as eqation 4.4 except for the r term in front of everything.

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \tag{5.2}$$

using equation 2.1

$$r^2\dot{\theta} = \frac{l}{m} \tag{5.3}$$

and is a constant of the motion. I.e. this is a restatement of the conservation of angular momentum. Using above we can get ω

$$\omega = \frac{l}{mr^2} \tag{5.4}$$

Now if we let $r = \frac{1}{u}$ we find that

$$\begin{aligned}\frac{dr}{dt} &= \frac{du^{-1}}{du}\frac{du}{dt} \\ &= -\frac{1}{u^2}\frac{du}{dt} \\ &= -r^2\frac{du}{dt} \\ &= -\left(r^2\frac{d\theta}{dt}\right)\frac{du}{d\theta}\end{aligned} \tag{5.5}$$

Notice that the brakects are the same as in equation 5.2 and so is constant. Now take the derivitive again with respect to time

$$
\begin{aligned}
\frac{d^2r}{dt^2} &= -\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\frac{du}{d\theta} - \left(r^2\frac{d\theta}{dt}\right)\frac{d}{dt}\frac{du}{d\theta} \\
&= -\left(r^2\frac{d\theta}{dt}\right)\frac{d\theta}{dt}\frac{d}{d\theta}\frac{du}{d\theta} \\
&= -r^2\left(\frac{d\theta}{dt}\right)^2\frac{d^2u}{d\theta^2} \qquad (5.6)
\end{aligned}
$$

Using equation 5.6 in equation 4.3 and continuing with the change of variables gives

$$
\begin{aligned}
-\frac{GM}{r^2} &= -r^2\left(\frac{d\theta}{dt}\right)^2\frac{d^2u}{d\theta^2} - r\left(\frac{d\theta}{dt}\right)^2 \\
GM &= r^4\left(\frac{d\theta}{dt}\right)^2\frac{d^2u}{d\theta^2} + r^3\left(\frac{d\theta}{dt}\right)^2 \\
GM &= \left[r^2\frac{d\theta}{dt}\right]^2\frac{d^2u}{d\theta^2} + \frac{1}{r}\left[r^2\frac{d\theta}{dt}\right]^2 \\
GM &= \left[r^2\frac{d\theta}{dt}\right]^2\left[\frac{d^2u}{d\theta^2} + u\right] \qquad (5.7)
\end{aligned}
$$

making the subsitution in equation 5.7 for the first set of brackets on the right hand side with equation 5.3 gives us

$$
\begin{aligned}
GM &= \left(\frac{l}{m}\right)^2\left[\frac{d^2u}{d\theta^2} + u\right] \\
\frac{GMm^2}{l^2} &= \frac{d^2u}{d\theta^2} + u \qquad (5.8)
\end{aligned}
$$

which leads to the solution being

$$
\begin{aligned}
u(\theta) &= \frac{GMm^2}{l^2} - c\cos\theta \\
r(\theta) &= \frac{1}{\frac{GMm^2}{l^2} - c\cos\theta} \qquad (5.9)
\end{aligned}
$$

equation 5.9 is a form of the polar equation for an ellipse. Rearanging to the more usual polar form will give us the semi-major axis and the eccentricity of the obit. Let $\gamma = \frac{l^2}{GMm^2}$

$$r(\theta) = \frac{1}{\frac{1}{\gamma} - c\cos\theta}$$
$$r(\theta) = \frac{\gamma}{1 - \frac{c}{\gamma}\cos\theta} \tag{5.10}$$

5.2 Summary

The equations of motion under gravity are

$$-\frac{GM}{r^2} = \ddot{r} - r\omega^2$$
$$0 = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Which has a solution for r in terms of θ which is

$$r(\theta) = \frac{\gamma}{1 - \frac{c}{\gamma}\cos\theta}$$

where $\gamma = \frac{l^2}{GMm^2}$. This is the equation for an ellipse in polar coordinates and so shows that orbits are elliptial.

6 Apoapsis and Periapsis

Starting with the equation of an ellipse in polar form

$$r(\theta) = \frac{A(1+e)(1-e)}{1-e\cos\theta} \tag{6.1}$$

When r is maximum then the particle is at the apoasis[1] of it's orbit. This occurs when $\cos\theta = 1$ i.e. when $\theta = 0$. We stall call the radius at apoasis a so using equation6.1 with $\cos\theta = 1$

$$a = \frac{A(1+e)(1-e)}{1-e} = A(1+e) \tag{6.2}$$

Also we can find the periapsis[2] when r is minimum. This occurs when $\cos\theta = -1$ i.e. when $\theta = \pi$. Again using equation 6.1 with $\cos\theta = -1$

$$p = \frac{A(1+e)(1-e)}{1+e} = A(1-e) \tag{6.3}$$

We can sum the equations 6.2 and 6.3 above to give

$$\begin{aligned} a + p &= A + eA + A - eA \\ a + p &= 2A \\ A &= \frac{a+p}{2} \end{aligned} \tag{6.4}$$

This also follows from the geometry of an ellipse.

finally taking the difference of the equations 6.2 and 6.3 above gives

$$\begin{aligned} a - p &= A + eA - A + eA \\ a - p &= 2eA \\ e &= \frac{a-p}{2A} \\ e &= \frac{a-p}{a+p} \end{aligned} \tag{6.5}$$

[1] the furthest point from the parent body

[2] the closest point from the parent body

Using the relationship for the focus and the ecentricity

$$
\begin{aligned}
f &= eA \\
&= \frac{a-p}{a+p}\frac{a+p}{2} \\
&= \frac{a-p}{2}
\end{aligned}
\tag{6.6}
$$

6.1 Relationship between apoasis and periapsis

Comparing equation 5.10 to the equation for an elipse in polar form

$$
r = \frac{A(1-e^2)}{1-e\cos\theta} \tag{6.7}
$$

$$
\Rightarrow A(1-e^2) = \gamma \tag{6.8}
$$

using equation 6.4 and equation 6.5 we get

$$
\begin{aligned}
\gamma &= \frac{a+p}{2}\left(1-\frac{(a-p)^2}{(a+p)^2}\right) \\
&= \frac{(a+p)^2-(a-p)^2}{2(a+p)} \\
&= \frac{a^2+2ap+p^2-a^2+2ap-p^2}{2(a+p)} \\
&= \frac{2ap}{a+p}
\end{aligned}
\tag{6.9}
$$

6.2 Summary

The apoasis a and periapsis p radius is given by

$$
\begin{aligned}
a &= A(1+e) \\
p &= A(1-e)
\end{aligned}
$$

where A is the semi-major axis of the ellipse and e is the ecentricity. A , e and f can be found in terms of a and p by the following

$$A = \frac{a+p}{2}$$
$$e = \frac{a-p}{a+p}$$
$$f = \frac{a-p}{2}$$

The relationship between the apoasis and periapsis is given by

$$\gamma = \frac{2ap}{a+p}$$

where $\gamma = \frac{l^2}{GMm^2}$

7 Circular Orbits

When an orbit is circular $r = const$. This means that $\dot{r} = \ddot{r} = 0$. Looking back at the equation 4.3

$$\begin{aligned} -\frac{GM}{r^2} &= \ddot{r} - r\omega^2 \\ -\frac{GM}{r^2} &= -r\omega^2 \end{aligned} \tag{7.1}$$

since $\ddot{r} = 0$.

7.1 Radius of orbit

Making the substitution here that $\omega = \frac{l}{mr^2}$ in equation 7.1 gives

$$\begin{aligned} GM &= \frac{1}{r}\left(\frac{l}{m}\right)^2 \\ r &= \frac{\left(\frac{l}{m}\right)^2}{GM} = \gamma \end{aligned} \tag{7.2}$$

so we find that the raduis is γ

7.2 Angular momentum

Since $r = \gamma$ the angular momentum is easy to find

$$\begin{aligned} r &= \frac{\left(\frac{l}{m}\right)^2}{GM} \\ \frac{l}{m} &= \sqrt{GMr} \end{aligned} \tag{7.3}$$

7.3 Period

Since the angular momentum is related to the areal velocity by equation **??**, finding the period is just a case of finding out how long it takes to sweep out the full area of a circle.

$$\begin{aligned}
\pi r^2 &= \frac{1}{2}\frac{l}{m}T \\
T &= \frac{2\pi r^2}{\frac{l}{m}} \\
T &= \frac{2\pi}{G^2M^2}\left(\frac{l}{m}\right)^3
\end{aligned} \tag{7.4}$$

Using equation 7.3 gives

$$\begin{aligned}
T &= \frac{2\pi}{G^2M^2}\sqrt{G^3M^3r^3} \\
T &= \frac{2\pi}{\sqrt{GM}}\sqrt{r^3}
\end{aligned} \tag{7.5}$$

7.4 Energy

The energy is given by equation 4.11, so rearranging to get energy per unit mass

$$\frac{E}{m} = \frac{1}{2}\dot{r}^2 + \frac{1}{2r^2}\left(\frac{l}{m}\right)^2 - \frac{GM}{r} \tag{7.6}$$

Since the orbit is circular $\dot{r} = 0$ so the first term disappears leaving only

$$\begin{aligned}
\frac{E}{m} &= \frac{1}{2r^2}\left(\frac{l}{m}\right)^2 - \frac{GM}{r} \\
\frac{E}{m}\frac{r^2}{GM} &= \frac{1}{2}\frac{\left(\frac{l}{m}\right)^2}{GM} - r \\
\frac{E}{m}\frac{r^2}{GM} &= \frac{1}{2}\gamma - r \\
\frac{E}{m}\frac{r^2}{GM} &= -\frac{1}{2}r \\
\frac{E}{m} &= -\frac{GM}{2}\frac{1}{r}
\end{aligned} \tag{7.7}$$

7.5 Summary

$$
\begin{aligned}
r &= \gamma \\
\frac{l}{m} &= \sqrt{GMr} \\
T &= \frac{2\pi}{\sqrt{GM}} r^{\frac{3}{2}} \\
\frac{E}{m} &= -\frac{GM}{2}\frac{1}{r}
\end{aligned}
$$

8 Elliptical Orbits

Unlike circular orbits, elliptical orbits do not have a constant radius and so we don't see the same simplification as we do for circular orbits[3]. However we already have a relationship bettween γ and the apses, equation 6.9

$$\gamma = \frac{2ap}{a+p} \tag{8.1}$$

8.1 Radial Acceleration

However we can calculate the radial acceleration at the apses by using equation 4.3

$$\begin{aligned}
\ddot{r} &= r\omega^2 - \frac{GM}{r^2} \\
\ddot{r} &= \frac{1}{r^3}\left(\frac{l}{m}\right)^2 - \frac{GM}{r^2} \\
\ddot{r} &= GM\left(\frac{\gamma}{r^3} - \frac{1}{r^2}\right)
\end{aligned} \tag{8.2}$$

8.1.1 Radial Acceleration at Apoapsis

At apoapsis $r = a$. Substituting this into equation 8.2 gives

$$\ddot{r}(a) = GM\left(\frac{\gamma}{a^3} - \frac{1}{a^2}\right) \tag{8.3}$$

8.2 Angular momentum

Since we already have a relationship bettween γ and the apses, the angular momentum is easy to find from γ

$$\frac{l}{m} = \sqrt{GM\frac{2ap}{a+p}} \tag{8.4}$$

[3]that $\ddot{r} = 0$

8.3 Period

Again the period of an orbit should be as simple as using the link between angular momentum and areal velocity and knowing the area of an ellipse

$$\pi AB = \frac{1}{2}\frac{l}{m}T \tag{8.5}$$

substituting in the relationship between A,B and the apses.

$$A = \frac{a+p}{2} \tag{8.6a}$$
$$B = \sqrt{ap} \tag{8.6b}$$

gives

$$\begin{aligned}
\pi\sqrt{ap}\frac{a+p}{2} &= \frac{1}{2}\frac{l}{m}T \\
\pi\sqrt{ap}(a+p) &= \sqrt{GM\frac{2ap}{a+p}}T \\
T &= \frac{2\pi}{\sqrt{GM\left(\frac{2}{a+p}\right)^2}} \\
T &= \frac{2\pi}{\sqrt{GM}}\left(\frac{a+p}{2}\right)^{\frac{3}{2}}
\end{aligned} \tag{8.7}$$

8.4 Energy

Again we use the energy equation 4.11, and rearranging to get energy per unit mass

$$\frac{E}{m} = \frac{1}{2}\dot{r}^2 + \frac{1}{2r^2}\left(\frac{l}{m}\right)^2 - \frac{GM}{r} \tag{8.8}$$

although our orbit isn't circular, there are two points in the orbit where $\dot{r} = 0$ allowing us to drop the first term. Since energy is conservered, the energy will be the same for all points along the orbit. The two points where $\dot{r} = 0$ are at the apopsis and periapsis. So substituting $r = a$ gives

$$\begin{aligned}
\frac{E}{m} &= \frac{1}{2a^2}\left(\frac{l}{m}\right)^2 - \frac{GM}{a} \\
\frac{E}{m} &= \frac{GM}{2a^2}\gamma - \frac{GM}{a} \\
\frac{E}{m} &= GM\left[\frac{2ap}{2a^2(a+p)} - \frac{1}{a}\right] \\
\frac{E}{m} &= GM\left[\frac{2ap - 2a(a+p)}{2a^2(a+p)}\right] \\
\frac{E}{m} &= GM\left[\frac{a^2}{a^2(a+p)}\right] \\
\frac{E}{m} &= \frac{GM}{2}\frac{1}{\frac{a+p}{2}} \qquad (8.9)
\end{aligned}$$

Which again shows that for elliptical orbits we get the same pattern as circular orbits, except we take the average of the orbital radius in place of r.

8.5 Summary

$$\begin{aligned}
\gamma &= \frac{2ap}{a+p} \\
\frac{l}{m} &= \sqrt{GM\frac{2ap}{a+p}} \\
T &= \frac{2\pi}{\sqrt{GM}}\left(\frac{a+p}{2}\right)^{\frac{3}{2}} \\
\frac{E}{m} &= \frac{GM}{2}\frac{1}{\frac{a+p}{2}}
\end{aligned}$$