Some Common Ordinary Differential Equations

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Contents

Chapter 1

Linear Ordinary Differential Equations

A linear ordinary differential equation is of the form

$$b(x) = \sum_0^n a_i(x)\frac{d^i y}{dx^i} = a_0(x)y + a_1(x)\frac{dy}{dx} + a_2(x)\frac{d^2 y}{dx^2} + \dots \quad (1.0.1)$$

When all of the $a_i(x)$ are constants and $b(x)$ is zero we get the homogeneous linear ordinary differential equation with constant coefficients which looks like this

$$0 = \sum_0^n a_i \frac{d^i y}{dx^i} = a_0 y + a_1\frac{dy}{dx} + a_2\frac{d^2 y}{dx^2} + \dots \quad (1.0.2)$$

1.1 First Order Homogenous Constant Linear Ordinary Differential Equation

In the case of first order, $n = 1$ and we get

$$a_1\frac{dy}{dx} + a_0 y = 0 \quad (1.1.1)$$

which can be rearranged into the more common form

$$\frac{dy}{dx} = ky \quad (1.1.2)$$

When using the trial equation $y = exp(\alpha x)$ we get the general solution of

$$y = c\exp(kx) \quad (1.1.3)$$

where c is the integration constant.

1.2 Second Order Homogenous Constant Linear Ordinary Differential Equation

In the case of second order, $n = 2$ and we get

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \tag{1.2.1}$$

1.2.1 In the case that $a_1 = 0$

In the case that $a_1 = 0$ we get the more familiar form as

$$\frac{d^2y}{dx^2} = ky \tag{1.2.2}$$

Using our typical trial equation $y = exp(\alpha x)$ gives

$$\alpha = \sqrt{k} \tag{1.2.3}$$

When $k > 0$

The solution becomes

$$\begin{aligned} y(x) &= A\exp(\alpha x) + B\exp(-\alpha x) \\ &= \tilde{A}\cosh(\alpha x) + \tilde{B}\sinh(\alpha x) \end{aligned} \tag{1.2.4}$$

Where we have used the result from here to change the form.

When $k = 0$

In the case that $k = 0$ the differential equation changes to be

$$\frac{d^2y}{dx^2} = 0 \tag{1.2.5}$$

which has the solution

$$y(x) = mx + c \tag{1.2.6}$$

When $k < 0$

When $k < 0$ then α is imaginary, so letting $\beta = i\sqrt{-k}$ gives a solution of the form

$$\begin{aligned} y(x) &= A\exp(\beta x) + B\exp(-\beta x) \\ &= \tilde{A}\cos(\beta x) + \tilde{B}\sin(\beta x) \end{aligned} \tag{1.2.7}$$