
FILTERS

NOTES ON PASSIVE ELECTRIC FILTER CIRCUITS

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1 RC Low Pass Filter

1.1 Filter Attenuation

A simple low pass filter can be formed from a resistor and a capacitor in series.

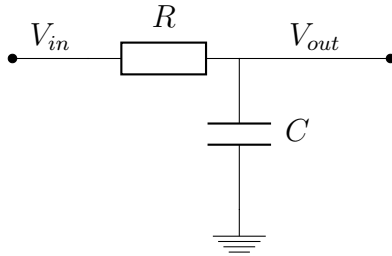


Figure 1: Circuit diagram for a RC low pass filter.

We can see from figure 1, that the circuit forms a potential divider just with a reactive element instead of purely resistive. The attenuation is then given by the standard potential divider result

$$\frac{V_{out}}{V_{in}} = \frac{-iX_C}{R - iX_C} \quad (1.1)$$

1.1.1 Cutoff Frequency

Let's introduce a new variable called u , where

$$\begin{aligned} u &= \frac{R}{X_C} \\ &= \omega RC \end{aligned} \quad (1.2)$$

where $\omega = 2\pi f$. If we look at the frequency when the resulting $u = 1$, which we will label f_0 or ω_0

$$\begin{aligned} \omega_0 RC &= 1 \\ \omega_0 &= \frac{1}{RC} \end{aligned} \quad (1.3)$$

We call the frequency when $u = 1$ the *cutoff frequency*, for reasons that will be clear later on. This frequency is when the resistance of the resistor is equal to the reactance of the capacitor¹ You can see that we can use the cutoff frequency as a replacement for our RC value, in equation 1.2.

$$\begin{aligned} u &= \omega RC \\ &= \frac{\omega}{\omega_0} = \frac{f}{f_0} \end{aligned} \tag{1.4}$$

1.1.2 Attenuation revisited

Now we have some understanding of the variable we introduced u , we can substitute it into our equation for the attenuation (equation 1.1), by noting that from equation 1.2 $R = uX_C$

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{-iX_C}{R - iX_C} \\ &= \frac{-iX_C}{uX_C - iX_C} \\ &= \frac{-i}{u - i} \\ &= \frac{1 - iu}{u^2 + 1} \end{aligned} \tag{1.5}$$

Normally, we don't consider the attenuation as a complex value, instead we care more about the magnitude and phase shift of an attenuation.

$$\begin{aligned} \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \frac{\sqrt{1 + u^2}}{1 + u^2} \\ &= \frac{1}{\sqrt{1 + u^2}} \\ &= \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^2}} \end{aligned} \tag{1.6}$$

¹by equal here, we mean the magnitudes are equal. If not the phase shift.

Where we have used equation 1.4 in place of u . For the phase shift of the filter,

$$\phi = -\arctan u = -\arctan \frac{f}{f_0} \quad (1.7)$$

Summary

In the last section we discovered the cutoff frequency was given by

$$f_0 = \frac{1}{2\pi RC}$$

and that the ratio of resistance to reactance can be given by

$$u = \frac{R}{X_C} = 2\pi RCf = \frac{f}{f_0}$$

and that the attenuation of the filter is given by

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{-iX_C}{R - iX_C} \\ &= \frac{1 - iu}{u^2 + 1} \end{aligned}$$

or in terms of magnitude and phase shift

$$\begin{aligned} \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \frac{1}{\sqrt{1 + u^2}} \\ &= \frac{1}{\sqrt{1 + \left(\frac{f}{f_0}\right)^2}} \end{aligned}$$

$$\phi = -\arctan u = -\arctan \frac{f}{f_0}$$

1.2 Log-Log Form

You won't often see attenuation given in the form seen earlier. It is more likely to be seen in Log-Log form, due to wanting to see the behaviour over a large range of frequencies and the fact the attenuation itself can get very small very fast. However it helps to look at the logirthm of u before looking at the attenuation straight away.

$$\ln u = \ln \frac{f}{f_0} = \ln f - \ln f_0 = F - F_0 \quad (1.8)$$

where we have used $F = \ln f$ and $F_0 = \ln f_0$

Now looking at the attenuation

$$\begin{aligned} \ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \ln \frac{1}{\sqrt{1+u^2}} \\ &= -\frac{1}{2} \ln (1+u^2) \\ &= -\frac{1}{2} \ln \left(u^2 \left(1 + \frac{1}{u^2} \right) \right) \\ &= -\frac{1}{2} \ln u^2 - \frac{1}{2} \ln \left(1 + \frac{1}{u^2} \right) \\ &= -\ln u - \frac{1}{2} \ln \left(1 + \frac{1}{u^2} \right) \\ &= F_0 - F - \frac{1}{2} \ln \left(1 + \left(\frac{f_0}{f} \right)^2 \right) \end{aligned} \quad (1.9)$$

Lets quickly look at the term $\frac{f_0}{f}$ in equation 1.9. We'd like to express it in terms of our new variables F and F_0 . To do this, we note that since $F = \ln f$ then $f = \exp(f)$, and so

$$\frac{f_0}{f} = \frac{\exp(F_0)}{\exp(F)} = \exp(F_0 - F) \quad (1.10)$$

putting the result from equation 1.10 back into equation 1.9 gives us

$$\ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = F_0 - F - \frac{1}{2} \ln (1 + \exp(2(F_0 - F))) \quad (1.11)$$

This is as simple as it gets sadly, however we can study some particular values of this equation and the extreame cases. For $F \ll F_0$

$$\begin{aligned}
A(F \ll F_0) &= F_0 - F - \frac{1}{2} \ln \left(1 + \exp(2 \underbrace{(F_0 - F)}_{\text{large and +ve}}) \right) \\
&= F_0 - F - \frac{1}{2} \ln \left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{even larger and +ve}} \right) \\
&= F_0 - F - \frac{1}{2} \ln \left(\underbrace{1}_{\text{so this can be neglected}} + \exp(2(F_0 - F)) \right) \\
A(F \ll F_0) &\approx F_0 - F - \frac{1}{2} \ln (\exp(2(F_0 - F))) \\
&= F_0 - F - F_0 + F = 0
\end{aligned} \tag{1.12}$$

for $F \gg F_0$

$$\begin{aligned}
A(F \gg F_0) &= F_0 - F - \frac{1}{2} \ln \left(1 + \exp(2 \underbrace{(F_0 - F)}_{\text{large and -ve}}) \right) \\
&= F_0 - F - \frac{1}{2} \ln \left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{very small and +ve}} \right) \\
&= F_0 - F - \frac{1}{2} \ln \left(1 + \underbrace{\exp(2(F_0 - F))}_{\text{so this can be neglected}} \right) \\
A(F \gg F_0) &\approx F_0 - F - \frac{1}{2} \underbrace{\ln 1}_{=0} \\
&= F_0 - F
\end{aligned} \tag{1.13}$$

and finally when $F = F_0$

$$\begin{aligned} A(F = F_0) &= 0 - \frac{1}{2} \ln(1 + \exp(0)) \\ A(F = F_0) &= -\frac{\ln 2}{2} \end{aligned} \tag{1.14}$$

Summary

The equation for the log-log attenuation is given by

$$\ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = A(F) = F_0 - F - \frac{1}{2} \ln(1 + \exp(2(F_0 - F)))$$

where $F = \ln f$ and $F_0 = \ln f_0$, and has the following results

$$A(F \ll F_0) \approx 0$$

$$A(F = F_0) = -\frac{\ln 2}{2}$$

$$A(F \gg F_0) \approx F_0 - F$$

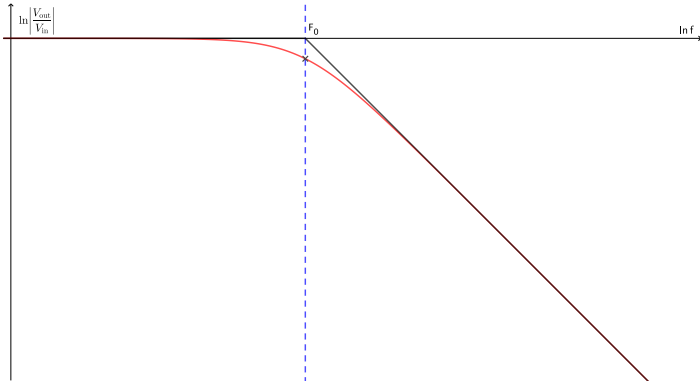


Figure 2: The log-log plot of the attenuation against frequency.

2 RL High Pass Filter

2.1 Filter Attenuation

A simple high pass filter can be formed from a resistor and an inductor in series.

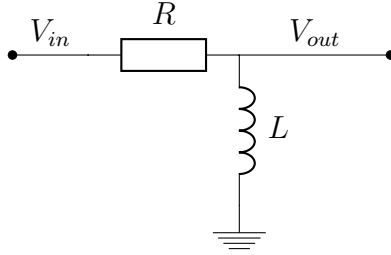


Figure 3: Circuit diagram for a RL high pass filter.

We can see from figure 3, that the circuit forms a potential divider just with a reactive element instead of purely resistive. The attenuation is then given by the standard potential divider result

$$\frac{V_{out}}{V_{in}} = \frac{iX_L}{R + iX_L} \quad (2.1)$$

2.1.1 Cutoff Frequency

Let's introduce a new variable called u , where

$$\begin{aligned} u &= \frac{R}{X_L} \\ &= \frac{R}{\omega L} \end{aligned} \quad (2.2)$$

where $\omega = 2\pi f$. If we look at the frequency when the resulting $u = 1$, which we will label f_0 or ω_0

$$\begin{aligned} \frac{R}{\omega_0 L} &= 1 \\ \omega_0 &= \frac{R}{L} \end{aligned} \quad (2.3)$$

We call the frequency when $u = 1$ the *cutoff frequency*, for reasons that will be clear later on. This frequency is when the resistance of the resistor is equal to the reactance of the inductor² You can see that we can use the cutoff frequency as a replacement for our $\frac{R}{L}$ value, in equation 2.2.

$$\begin{aligned} u &= \frac{R}{\omega L} \\ &= \frac{\omega_0}{\omega} = \frac{f_0}{f} \end{aligned} \tag{2.4}$$

2.1.2 Attenuation revisited

Now we have some understanding of the variable we introduced u , we can substitute it into our equation for the attenuation (equation 2.1), by noting that from equation 2.2 $R = uX_C$

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{iX_L}{R + iX_L} \\ &= \frac{iX_L}{uX_L + iX_L} \\ &= \frac{i}{u + i} \\ &= \frac{1 + iu}{u^2 + 1} \end{aligned} \tag{2.5}$$

Normally, we don't consider the attenuation as a complex value, instead we care more about the magnitude and phase shift of an

²by equal here, we mean the magnitudes are equal. If not the phase shift.

attenuation.

$$\begin{aligned}
 \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \frac{\sqrt{1+u^2}}{1+u^2} \\
 &= \frac{1}{\sqrt{1+u^2}} \\
 &= \frac{1}{\sqrt{1+\left(\frac{f_0}{f}\right)^2}}
 \end{aligned} \tag{2.6}$$

Where we have used equation 2.4 in place of u . For the phase shift of the filter,

$$\phi = \arctan u = \arctan \frac{f_0}{f} \tag{2.7}$$

Summary

In the last section we discovered the cutoff frequency was given by

$$f_0 = \frac{1}{2\pi} \frac{R}{L}$$

and that the ratio of resistance to reactance can be given by

$$u = \frac{R}{X_L} = \frac{1}{2\pi f} \frac{R}{L} = \frac{f_0}{f}$$

and that the attenuation of the filter is given by

$$\begin{aligned}
 \frac{V_{\text{out}}}{V_{\text{in}}} &= \frac{iX_L}{R + iX_L} \\
 &= \frac{1 + iu}{u^2 + 1}
 \end{aligned}$$

or in terms of magnitude and phase shift

$$\begin{aligned}\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \frac{1}{\sqrt{1+u^2}} \\ &= \frac{1}{\sqrt{1+\left(\frac{f_0}{f}\right)^2}}\end{aligned}$$

$$\phi = \arctan u = -\arctan \frac{f_0}{f}$$

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$$\ln u = \ln \frac{f_0}{f} = \ln f_0 - \ln f = F_0 - F \quad (2.8)$$

where we have used $F = \ln f$ and $F_0 = \ln f_0$

Now looking at the attenuation

$$\begin{aligned}
\ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| &= \ln \frac{1}{\sqrt{1+u^2}} \\
&= -\frac{1}{2} \ln (1+u^2) \\
&= -\frac{1}{2} \ln \left(u^2 \left(1 + \frac{1}{u^2} \right) \right) \\
&= -\frac{1}{2} \ln u^2 - \frac{1}{2} \ln \left(1 + \frac{1}{u^2} \right) \\
&= -\ln u - \frac{1}{2} \ln \left(1 + \frac{1}{u^2} \right) \\
&= F - F_0 - \frac{1}{2} \ln \left(1 + \left(\frac{f}{f_0} \right)^2 \right) \tag{2.9}
\end{aligned}$$

Lets quickly look at the term $\frac{f_0}{f}$ in equation 2.9. We'd like to express it in terms of our new variables F and F_0 . To do this, we note that since $F = \ln f$ then $f = \exp(f)$, and so

$$\frac{f}{f_0} = \frac{\exp(F)}{\exp(F_0)} = \exp(F - F_0) \tag{2.10}$$

putting the result from equation 2.10 back into equation 2.9 gives us

$$\ln \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = F - F_0 - \frac{1}{2} \ln (1 + \exp(2(F - F_0))) \tag{2.11}$$

This is as simple as it gets sadly, however we can study some particular values of this equation and the extreame cases.