Abstract Maths Cheat Sheet

1 Ket-vector Axioms

1. The sum of any two ket-vectors is also a ket-vector:
$$|A\rangle + |B\rangle = |C\rangle \tag{1.1}$$
2. Vector addition is commutative:
$$|A\rangle + |B\rangle = |B\rangle + |A\rangle \tag{1.2}$$
3. Vector addition is associative:
$$(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle) \tag{1.3}$$
4. There is a unique vector 0 such that when you add it to any ket, it gives the same ket back:
$$|A\rangle + 0 = |A\rangle \tag{1.4}$$
5. Given any ket $|A\rangle$, there is a unique ket $-|A\rangle$ such that
$$|A\rangle + (-|A\rangle) \tag{1.5}$$
6. Given any $|A\rangle$ and any complex number z, you can multiply them to get a new ket. Also, multiplication by a scalar is linear:
$$|zA\rangle = z|A\rangle = |B\rangle \tag{1.6}$$
7. The distributive property holds:
$$z(|A\rangle + |B\rangle) = z|A\rangle + z|B\rangle \tag{1.7}$$
$$(z+w)|A\rangle = z|A\rangle + w|A\rangle \tag{1.8}$$

Taken together 6 and 7 are often call linearity

2 Bra-vector Axioms

1. The sum of any two bra-vectors is also a bra-vector:
$$\langle A| + \langle B| = \langle C| \tag{2.1}$$
2. Vector addition is commutative:
$$\langle A| + \langle B| = \langle B| + \langle A| \tag{2.2}$$
3. Vector addition is associative:
$$(\langle A| + \langle B|) + \langle C| = \langle A| + (\langle B| + \langle C|) \tag{2.3}$$
4. There is a unique vector 0 such that when you add it to any bra, it gives the same bra back:
$$\langle A| + 0 = \langle A| \tag{2.4}$$
5. Given any bra $\langle A|$, there is a unique bra $-\langle A|$ such that
$$\langle A| + (-\langle A|) \tag{2.5}$$
6. Given any $\langle A|$ and any complex number z, you can multiply them to get a new bra. Also, multiplication by a scalar is linear:
$$\langle zA| = z\langle A| = \langle B| \tag{2.6}$$
7. The distributive property holds:
$$z(\langle A| + \langle B|) = z\langle A| + z\langle B| \tag{2.7}$$
$$(z+w)\langle A| = z\langle A| + w\langle A| \tag{2.8}$$

Taken together 6 and 7 are often call linearity

3 Bras and Kets

1. For every ket-vector $|A\rangle$, there is a bra-vector in the dual space, denoted by $\langle A|$.
2. Bra and Ket vectors together form inner products of bras and kets, using expressions like $\langle A|B\rangle$ to form bra-kets or brackets.
3. Suppose $\langle A|$ is the bra corresponding to the ket $|A\rangle$, and $\langle B|$ is the bra corresponding to the ket $|B\rangle$. Then the bra corresponding to $|A\rangle + |B\rangle$ is $\langle A| + \langle B|$
4. If z is a complex number, then it is not true that the bra corresponding to $z|A\rangle$ is $\langle A|z$. You have to remember to complex-conjugate. Thus, the bra corresponding to $z|A\rangle$ is $\langle A|z^*$

4 Inner Products

1. Inner products are linear:

$$\langle C| (|A\rangle + |B\rangle) = \langle C|A\rangle + \langle C|B\rangle \tag{4.1}$$

2. Interchanging bras and kets corresponds to complex conjugation:

$$\langle B|A\rangle = \langle B|A\rangle^* \tag{4.2}$$

Additionally we can show that $(\langle A| + \langle B|) |C\rangle = \langle A|C\rangle + \langle B|C\rangle$ using axiom 2 above we get

$$(\langle C| (|A\rangle + |B\rangle))^* = (\langle C|A\rangle + \langle C|B\rangle)^* \tag{4.3}$$
$$\langle C| (|A\rangle + |B\rangle) = \langle C|A\rangle + \langle C|B\rangle \tag{4.4}$$

which is the same as axiom 1

Also applying axiom 2 to $\langle A|A\rangle$ give that $\langle A|A\rangle = \langle A|A\rangle^*$ which can only be true if $\langle A|A\rangle$ is a real number.

Normalized Vector A vector is said to be normalized if its inner product with itself is 1 ie $\langle A|A\rangle = 1$

Orthogonal Vectors Two vectors are said to be orthogonal if their inner product is zero ie $|A\rangle$ and $|B\rangle$ are orthogonal if $\langle B|A\rangle = 0$.

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