

Complex Fourier Series

Hannah Michelle Ellis

June 2017

Contents

1	Derivation	5
1.1	The Series	5
1.2	An interesting result	6
1.2.1	Summary	8
1.3	Extracting cooefficients	9
1.3.1	Summary	10
2	Examples	11
2.1	Square Wave	12

Chapter 1

Derivation

1.1 The Series

For a function $f(x)$ with range $-\frac{a}{2} < x < \frac{a}{2}$ we assume that it can be reproduced by a sum of complex exponentials of the form

$$\exp(ik_n x) \tag{1.1}$$

Where $k = \frac{2n\pi}{a}$ so that n periods of a complex exponential fit's into the range $-\frac{a}{2} < x < \frac{a}{2}$. Along with complex coefficients for each term, we get the equation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \tag{1.2}$$

This is the complex Fourier series.

1.2 An interesting result

To calculate any particular coefficient, it will be important to evaluate the following integral.

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right) \exp\left(\frac{-2im\pi}{a}x\right) dx \quad (1.3a)$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2i(n-m)\pi}{a}x\right) dx \quad (1.3b)$$

letting $p = (n - m)$

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ip\pi}{a}x\right) dx \quad (1.3c)$$

$$= \frac{a}{2ip\pi} \left[\exp\left(\frac{2ip\pi}{a}x\right) \right]_{-\frac{a}{2}}^{\frac{a}{2}} \quad (1.3d)$$

$$= \frac{a}{2ip\pi} \left[\exp\left(\frac{2ip\pi}{a} \frac{a}{2}\right) - \exp\left(-\frac{2ip\pi}{a} \frac{a}{2}\right) \right] \quad (1.3e)$$

$$= \frac{a}{2ip\pi} [\exp(ip\pi) - \exp(-ip\pi)] \quad (1.3f)$$

using $\exp(ix) = \cos(x) + i \sin(x)$

$$I_{nm} = \frac{a}{2ip\pi} [\cos(p\pi) + i \sin(p\pi) - (\cos(p\pi) - i \sin(p\pi))] \quad (1.3g)$$

$$= \frac{a}{2ip\pi} [\cos(p\pi) + i \sin(p\pi) - \cos(p\pi) + i \sin(p\pi)] \quad (1.3h)$$

$$= \frac{a}{2ip\pi} [2i \sin(p\pi)] \quad (1.3i)$$

$$= \frac{a}{p\pi} \sin(p\pi) \quad (1.3j)$$

$$= a \frac{\sin(p\pi)}{p\pi} \quad (1.3k)$$

$$= a \operatorname{sinc}(p\pi) \quad (1.3l)$$

$$= a \delta_{nm} \quad (1.3m)$$

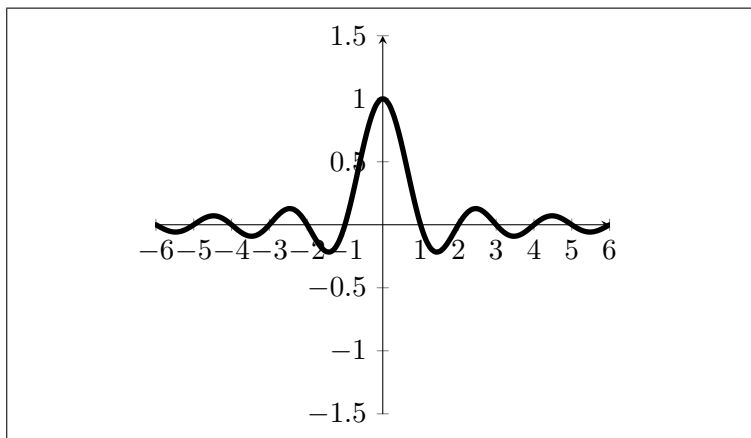


Figure 1.1: plot of $\text{sinc}(x\pi)$

1.2.1 Summary

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right) \exp\left(\frac{-2im\pi}{a}x\right) dx = a\delta_{nm}$$

Where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

1.3 Extracting coefficients

Multiplying equation 1.2 by $\exp\left(-\frac{2mi\pi}{a}\right)$ and integrating over the range $-\frac{a}{2} < x < \frac{a}{2}$ gives

$$I_m = \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}x\right) dx \quad (1.4a)$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}x\right) dx \quad (1.4b)$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}x\right) dx \quad (1.4c)$$

$$= \sum_{n=-\infty}^{\infty} c_n I_{nm} \quad (1.4d)$$

$$= \sum_{n=-\infty}^{\infty} c_n a \delta_{nm} \quad (1.4e)$$

$$= ac_m \quad (1.4f)$$

Taking the result from equation 1.4 and rearranging gives us

$$c_m = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}x\right) dx \quad (1.5)$$

1.3.1 Summary

Assuming that a function on the range $-\frac{a}{2} < x < \frac{a}{2}$ can be expressed as a sum of complex exponentials of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right)$$

The value of a coefficient c_n is given by the following integral

$$c_n = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2ni\pi}{a}x\right) dx$$

Chapter 2

Examples

2.1 Square Wave

$$c_n = \frac{1}{a} \left[\int_{-\frac{a}{2}}^0 -A \exp\left(-\frac{2ni\pi}{a}x\right) dx + \int_0^{\frac{a}{2}} A \exp\left(-\frac{2ni\pi}{a}x\right) dx \right] \quad (2.1a)$$

$$= \frac{A}{a} \left[\int_0^{\frac{a}{2}} \exp\left(-\frac{2ni\pi}{a}x\right) dx - \int_{-\frac{a}{2}}^0 \exp\left(-\frac{2ni\pi}{a}x\right) dx \right] \quad (2.1b)$$

$$= \frac{A}{a} \left[-\frac{a}{2ni\pi} \left[\exp\left(-\frac{2ni\pi}{a}x\right) \right]_0^{\frac{a}{2}} + \frac{a}{2ni\pi} \left[\exp\left(-\frac{2ni\pi}{a}x\right) \right]_0^{-\frac{a}{2}} \right] \quad (2.1c)$$

$$= \frac{A}{2ni\pi} \left[\left[\exp\left(-\frac{2ni\pi}{a}x\right) \right]_{-\frac{a}{2}}^0 - \left[\exp\left(-\frac{2ni\pi}{a}x\right) \right]_0^{\frac{a}{2}} \right] \quad (2.1d)$$

$$c_n = \frac{A}{2ni\pi} [\exp(0) - \exp(-ni\pi) - \exp(-ni\pi) + \exp(0)] \quad (2.1e)$$

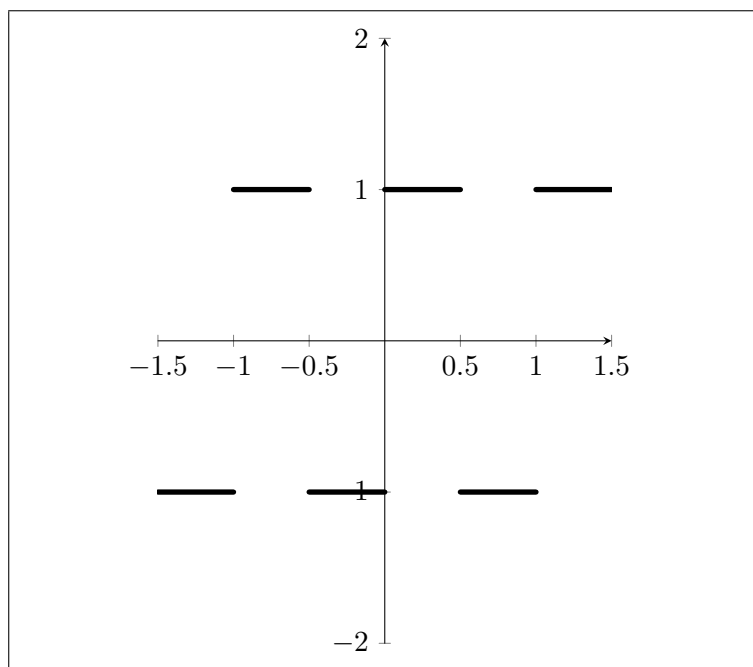
$$= \frac{A}{ni\pi} [\exp(0) - \exp(-ni\pi)] \quad (2.1f)$$

$$= \frac{A}{ni\pi} [1 - \cos(n\pi) + i \sin(n\pi)] \quad (2.1g)$$

$$= \frac{A}{ni\pi} [1 + i \sin(n\pi)] \quad (2.1h)$$

Putting this result back in

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \quad (2.2)$$

**Figure 2.1:** square wave