Complex Fourier Series

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June 2017

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Chapter 1

Derivation

1.1 The Series

For a function $f(x)$ with range $-\frac{a}{2} < x < \frac{a}{2}$ we assume that it can be reproduced by a sum of complex exponentials of the form

$$exp(ik_n x) \tag{1.1}$$

Where $k = \frac{2n\pi}{a}$ so that n periods of a complex exponential fit's into the range $-\frac{a}{2} < x < \frac{a}{2}$. Along with complex coefficients for each term, we get the equation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \tag{1.2}$$

This is the complex Fourier series.

1.2   An interesting result

To calculate any particular coefficient, it will be important
to evaluate the following intergral.

$$I_{nm} = int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right) \exp\left(\frac{-2im\pi}{a}x\right) dx \quad (1.3a)$$

$$= int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2i(n-m)\pi}{a}x\right) dx \quad (1.3b)$$

letting $p = (n - m)$

$$I_{nm} = int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ip\pi}{a}x\right) dx \quad (1.3c)$$

$$= \frac{a}{2ip\pi} \left[\exp\left(\frac{2ip\pi}{a}x\right)\right]_{-\frac{a}{2}}^{\frac{a}{2}} \quad (1.3d)$$

$$= \frac{a}{2ip\pi} \left[\exp\left(\frac{2ip\pi}{a}\frac{a}{2}\right) - \exp\left(-\frac{2ip\pi}{a}\frac{a}{2}\right)\right] \quad (1.3e)$$

$$= \frac{a}{2ip\pi} \left[\exp\left(ip\pi\right) - \exp\left(-ip\pi\right)\right] \quad (1.3f)$$

using $\exp(ix) = \cos(x) + i\sin(x)$

$$I_{nm} = \frac{a}{2ip\pi} \left[\cos(p\pi) + i\sin(p\pi) - (\cos(p\pi) - i\sin(p\pi))\right]$$

(1.3g)

$$= \frac{a}{2ip\pi} \left[\cos(p\pi) + i\sin(p\pi) - \cos(p\pi) + i\sin(p\pi)\right]$$
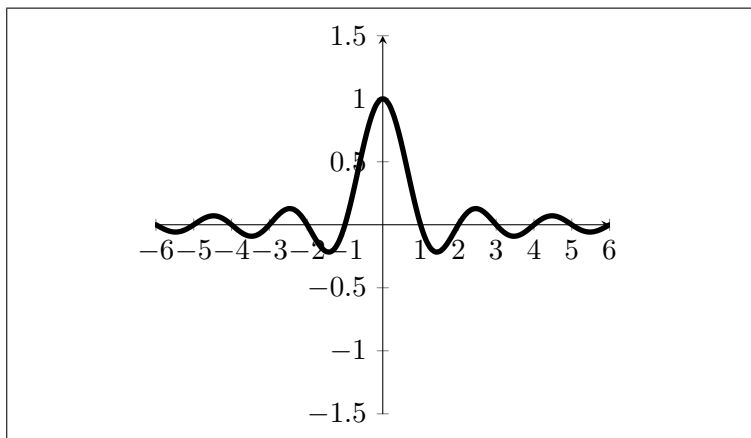
(1.3h)

$$= \frac{a}{2ip\pi} \left[2i\sin(p\pi)\right]$$

(1.3i)

$$= \frac{a}{p\pi}\sin(p\pi)$$

(1.3j)

$$= a\frac{\sin(p\pi)}{p\pi}$$

(1.3k)

$$= a\,\mathrm{sinc}(p\pi)$$

(1.3l)

$$= a\delta_{nm}$$

(1.3m)

Figure 1.1: plot of $\mathrm{sinc}(x\pi)$

1.2.1 Summary

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right) \exp\left(\frac{-2im\pi}{a}x\right) dx = a\delta_{nm}$$

Where

$$\delta_{nm} = \left\{ \begin{array}{ll} 1 & n = m \\ 0 & n! = m \end{array} \right.$$

1.3 Extracting cooefficents

Multiplying equation 1.2 by $\exp\left(-\frac{2mi\pi}{a}\right)$ and integrating over the range $-\frac{a}{2} < x < \frac{a}{2}$ gives

$$I_m = \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}\right) dx \qquad (1.4a)$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}\right) dx \qquad (1.4b)$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ni\pi}{a}x\right) \exp\left(-\frac{2mi\pi}{a}\right) dx \qquad (1.4c)$$

$$= \sum_{n=-\infty}^{\infty} c_n I_{nm} \qquad (1.4d)$$

$$= \sum_{n=-\infty}^{\infty} c_n a \delta_{nm} \qquad (1.4e)$$

$$= ac_m \qquad (1.4f)$$

Taking the result from equation 1.4 and rearranging gives us

$$c_m = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2mi\pi}{a}\right) dx \qquad (1.5)$$

1.3.1   Summary

Assuming that a function on the range $-\frac{a}{2} < x < \frac{a}{2}$ can be expressed as a sum of complex exponentials of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right)$$

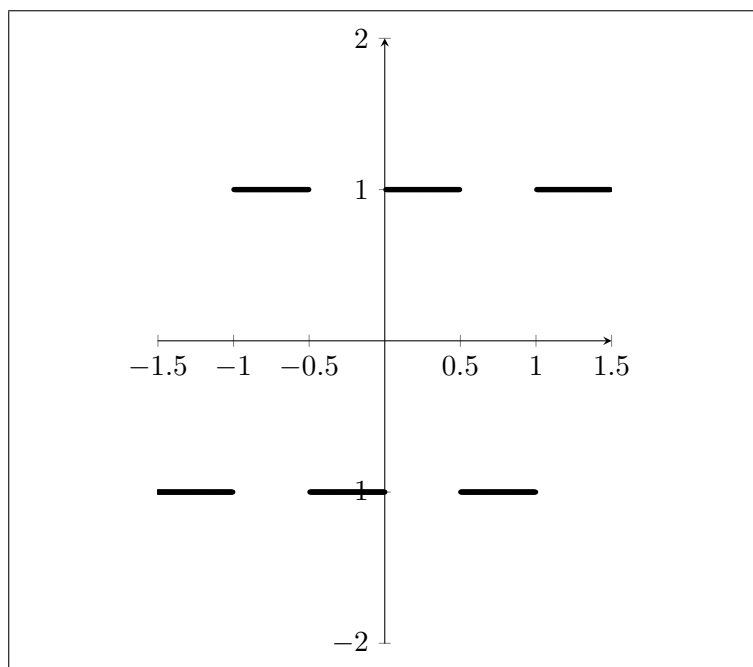The value of a cooefficent c_n is given by the following intergral

$$c_n = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2ni\pi}{a}\right) dx$$

Chapter 2

Examples

2.1 Square Wave

$$c_n = \frac{1}{a} \int_{-\frac{a}{2}}^{\frac{a}{2}} f(x) \exp\left(-\frac{2ni\pi}{a}\right) dx \qquad (2.1a)$$

Figure 2.1: square wave