Complex Fourier Series

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June 2017

1 "The Series"

For a function $f(x)$ with range $-\frac{a}{2} < x < \frac{a}{2}$ we assume that it can be reproduced by a sum of complex exponentials of the form

$$exp(ik_n x) \tag{1}$$

Where $k = \frac{2n\pi}{a}$ so that n periods of a complex exponential fit's into the range $-\frac{a}{2} < x < \frac{a}{2}$. Along with complex coefficients for each term, we get the equation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right) \tag{2}$$

This is the complex Fourier series.

2 "An interesting result"

To calculate any particular coefficient, it will be important to evaluate the following intergral.

$$I_{nm} = int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right) \exp\left(\frac{-2im\pi}{a}x\right) dx \tag{3a}$$

$$= int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2i(n-m)\pi}{a}x\right) dx \tag{3b}$$

letting $p = (n - m)$

$$I_{nm} = int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ip\pi}{a}x\right) dx \tag{3c}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(\frac{2ip\pi}{a}x\right)\right]_{-\frac{a}{2}}^{\frac{a}{2}} \tag{3d}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(\frac{2ip\pi}{a}\frac{a}{2}\right) - \exp\left(-\frac{2ip\pi}{a}\frac{a}{2}\right)\right] \tag{3e}$$

$$= \frac{a}{2ip\pi}\left[\exp\left(ip\pi\right) - \exp\left(-ip\pi\right)\right] \tag{3f}$$

using $\exp(ix) = \cos(x) + i\sin(x)$

$$\begin{aligned}
I_{nm} &= \frac{a}{2ip\pi}\left[\cos(p\pi) + i\sin(p\pi) - (\cos(p\pi) - i\sin(p\pi))\right] && \text{(3g)}\\
&= \frac{a}{2ip\pi}\left[\cos(p\pi) + i\sin(p\pi) - \cos(p\pi) + i\sin(p\pi)\right] && \text{(3h)}\\
&= \frac{a}{2ip\pi}\left[2i\sin(p\pi)\right] && \text{(3i)}\\
&= \frac{a}{p\pi}\sin(p\pi) && \text{(3j)}\\
&= a\frac{\sin(p\pi)}{p\pi} && \text{(3k)}\\
&= a\mathrm{sinc}(p\pi) && \text{(3l)}\\
&= a\delta_{nm} && \text{(3m)}
\end{aligned}$$

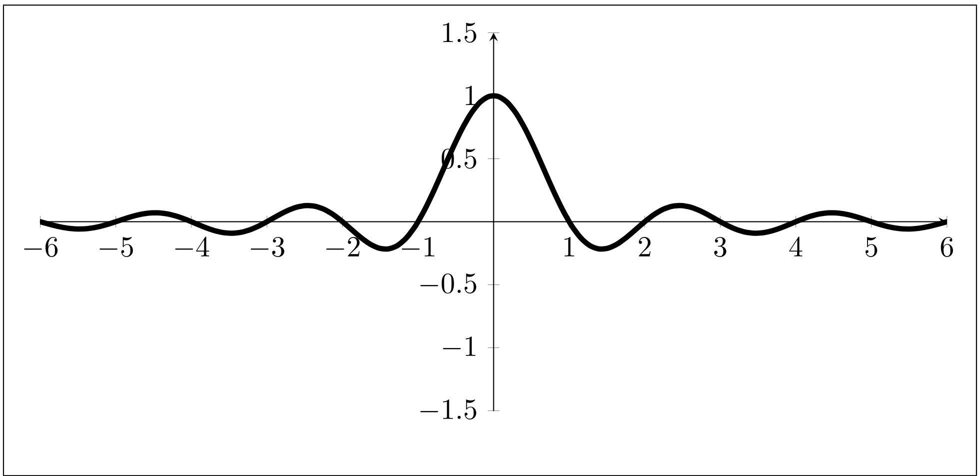

Figure 1: plot of $\mathrm{sinc}(x\pi)$

2.1 "Summary"

$$I_{nm} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2in\pi}{a}x\right)\exp\left(\frac{-2im\pi}{a}x\right)dx = a\delta_{nm}$$

Where

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n! = m \end{cases}$$

3 "Extracting cooefficents"

Multiplying equation 2 by $\exp\left(-\frac{2mi\pi}{a}\right)$ and integrating over the range $-\frac{a}{2} < x < \frac{a}{2}$ gives

$$\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\exp\left(-\frac{2mi\pi}{a}\right)dx = \int_{-\frac{a}{2}}^{\frac{a}{2}}\sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right)\exp\left(-\frac{2mi\pi}{a}\right)dx \tag{4a}$$

$$= \sum_{n=-\infty}^{\infty} c_n \int_{-\frac{a}{2}}^{\frac{a}{2}} \exp\left(\frac{2ni\pi}{a}x\right)\exp\left(-\frac{2mi\pi}{a}\right)dx \tag{4b}$$

$$= \sum_{n=-\infty}^{\infty} c_n I_{nm} \tag{4c}$$

$$= \sum_{n=-\infty}^{\infty} c_n a\delta_{nm} \tag{4d}$$

$$= ac_m \tag{4e}$$

Taking the result from equation 4 and rearranging gives us

$$c_m = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\exp\left(-\frac{2mi\pi}{a}\right)dx \tag{5}$$

3.1 "Summary"

Assuming that a function on the range $-\frac{a}{2} < x < \frac{a}{2}$ can be expressed as a sum of complex exponentials of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2ni\pi}{a}x\right)$$

The value of a cooefficent c_n is given by the following intergral

$$c_n = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} f(x)\exp\left(-\frac{2ni\pi}{a}\right)dx$$