A Criterion for the Uniform Continuity of Univariate Functions

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Abstract

The uniform continuity of univariate functions can usually be determined by quantitative relations, Cantor's theorem, Lipschitz criterion, etc., which are suitable for a narrow range and are weak in operability for complex problems. Based on the theorem of higher mathematics and aimed at univariate functions defined on real number intervals, this paper gives a criterion for the uniform continuity of univariate functions by judging the bounded rate of change of the function. Meanwhile, proofs are provided along with examples, offering a convenient approach for solving practical problems.

Keywords: function; uniform continuity; criterion

1 Introduction

Uniform continuity is one of the important concepts in functional analysis and higher mathematics, which is a generalization and extension of the continuity, reflecting the overall property of the function. Therefore, it is particularly important to judge whether a function is uniformly continuous on its domain. The commonly used methods are not very convenient for solving unbounded problems of some functions on their domains. Hence, this paper proposes a new criterion for functions defined on the real number interval $[a, +\infty)$.

2 Criterion for the Uniform Continuity

2.1 Review

To facilitate the reader's understanding, we introduce several definitions of uniform continuity and non-uniform continuity of univariate functions[2].

1. If a function is uniformly continuous on an interval I, then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for any $x', x'' \in I$, if $|x' - x''| < \delta$, then $|f(x') - f(x'')| < \varepsilon$.
2. If a function is not uniformly continuous on an interval, then there exists $\varepsilon_0 > 0$, for any chosen δ (no matter how small), there always exist $x', x'' \in I$, such that $|x' - x''| < \delta$, but $|f(x') - f(x'')| \geq \varepsilon_0$.

2.2 Criterion for Determination

In particular, we consider the limit behavior on interval $[a, +\infty)$. If the function $f(x)$ is defined on interval $[a, +\infty)$ and for all $n \in R$, the limit of $f(x+n) - f(x)$ as x approaches $+\infty$ exists.

Then we have the following criterion for uniform continuity:

$$\lim_{x \to +\infty} f(x+n) - f(x) = \begin{cases} +\infty \\ \text{A} \qquad (A \neq 0) \\ o(n) \\ 0 \end{cases}$$

Further, we can deduce the following four circumstances for judgment:

1. If $\lim_{x \to +\infty} f(x+n) - f(x) = +\infty$, then $f(x)$ is not uniformly continuous.
2. If $\lim_{x \to +\infty} f(x+n) - f(x) = A$, then $f(x)$ is not uniformly continuous.
3. If $\lim_{x \to +\infty} f(x+n) - f(x) = (n)$, then $f(x)$ is uniformly continuous.
4. If $\lim_{x \to +\infty} f(x+n) - f(x) = 0$, then $f(x)$ is uniformly continuous.

However, if none of the above circumstances apply, we cannot determine uniform continuity.

Case 1: For any $\varepsilon > 0$, there exists $\delta > 0$, such that:

$$\lim_{x \to +\infty} f(x+n) - f(x) = +\infty,$$

it follows that there exists $G > \max(a, 0)$, such that for $x_1 > G$, $x_2 = x_1 + n$, and the distance $|x_2 - x_1| = n < \delta$, we have:

$$|f(x_2) - f(x_1)| > 1 = \varepsilon_0,$$

thus $f(x)$ is not uniformly continuous.

Case 2: When A is not 0 and we assume $A > 0$, let $\varepsilon_0 = \frac{A}{2}$, for any $\delta > 0$, there exists $0 < n < \delta$, such that:

$$\lim_{x \to +\infty} f(x+n) - f(x) = A,$$

it follows that there exists $G > \max(a, 0)$, such that for $x_1 > G$, $x_2 = x_1 + n$, and the distance $|x_2 - x_1| = n < \delta$, we have:

$$|f(x_2) - f(x_1)| > \frac{A}{2} = \varepsilon_0,$$

thus $f(x)$ is not uniformly continuous.

Case 3: For all $\varepsilon > 0$, there exists $\delta(\varepsilon) < \varepsilon$ such that:

$$\lim_{x \to +\infty} f(x+n) - f(x) = o(n),$$

and there exists $G > \max(a, 0)$, such that for $x > G$, and for all n, when $x_1, x_2 > G$ and $|x_2 - x_1| < \delta$, we have:

$$|f(x_2) - f(x_1)| < \varepsilon,$$

thus $f(x)$ is uniformly continuous.

Case 4: For all $\varepsilon > 0$, the function is uniformly continuous:

$$\lim_{x \to +\infty} f(x+n) - f(x) = 0,$$

and there exists $G > \max(a, 0)$, such that for $x > G$, and for all n, when $x_1, x_2 > G$ and $|x_2 - x_1| < \delta$, we have:

$$|f(x_2) - f(x_1)| < \varepsilon,$$

thus $f(x)$ is uniformly continuous.

3 Example

The uniform continuity of $f(x)$ can be determined when $f(x)$ is a monomial function where $a > 0$ and $\alpha \neq 1$. According to the definition of the Lipschitz condition (see reference [3] page 78), the function $f(x) = ax^\alpha$ is uniformly continuous on the interval $[a, +\infty)$ since α is greater than 1, implying that the change rate is bounded within that interval. To verify the limit $\lim_{x \to +\infty} f(x+n) - f(x)$, we use the definition of the function and properties of limits. Consider the function:

$$f(x+n) - f(x) = (x+n)^\alpha - x^\alpha = x^\alpha \left[\left(1 + \frac{n}{x}\right)^\alpha - 1 \right].$$

We know that as x approaches infinity, the limit $x \cdot \left[\left(1 + \frac{n}{x}\right)^\alpha - 1\right]$ equals $n\alpha$. We can demonstrate this as follows:

Firstly, we have

$$\lim_{x \to +\infty} x \cdot \left[\left(1 + \frac{n}{x}\right)^\alpha - 1 \right] = \lim_{t \to 0} n \cdot \frac{(1+t)^\alpha - 1}{t}.$$

According to the limit

$$\lim_{t \to 0} \frac{\ln(1+t)}{t} = 1,$$

we get

$$\lim_{t \to 0} n \cdot \frac{(1+t)^\alpha - 1}{\ln(1+t)} = \lim_{t \to 0} n \cdot \frac{(1+t)^\alpha - 1}{t / \ln(1+t)}.$$

For $\lim_{t \to 0} n \cdot \frac{(1+t)^\alpha - 1}{\ln(1+t)}$, since $(1+t)^\alpha - 1$ approaches 0 and the derivative $\to 0$, we have

$$\frac{\ln(1+t)}{1/\alpha} = \ln(1+t) \cdot k$$

and when $k \to 0$, $\ln(1+t) \cdot k/\alpha$ approaches 0.

From equations (1) and (2), we have

$$\lim_{t\to 0} n \cdot \frac{(1+t)^\alpha - 1}{\ln(1+t)} = \lim_{k\to 0} n \cdot \frac{k}{k/\alpha} = n\alpha.$$

Therefore, we have

$$\lim_{x\to +\infty} x \left[\left(1 + \frac{n}{x}\right)^\alpha - 1 \right] = n\alpha.$$

Special Cases

For $f(x+n) - f(x) = (x+n)^\alpha - x^\alpha = x^\alpha \left[\left(1 + \frac{n}{x}\right)^\alpha - 1 \right]$, we have

$$\lim_{x\to +\infty} x^{\alpha-1} = \begin{cases} 0, & 0 < \alpha < 1, \\ +\infty, & \alpha > 1 \end{cases}$$

Thus, we can deduce

$$\lim_{x\to +\infty} [(x+n)^\alpha - x^\alpha] = \begin{cases} 0, & 0 < \alpha < 1, \\ +\infty, & \alpha > 1 \end{cases}$$

The above results show that for $0 < \alpha \le 1$, $f(x)$ is uniformly continuous; and for $\alpha > 1$, $f(x)$ is not uniformly continuous.

4 Conclusion

For a monomial function defined on the positive real number interval, the function is uniformly continuous when $0 < \alpha \le 1$. For $\alpha > 1$, the function has unbounded variation and is not uniformly continuous.

The analysis of this article on the uniform continuity of a univariate function has been completed. The author has not found a similar research paper that studies the uniform continuity of univariate functions on the real number interval, especially the criteria for judging uniform continuity.

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