Jack-Box system

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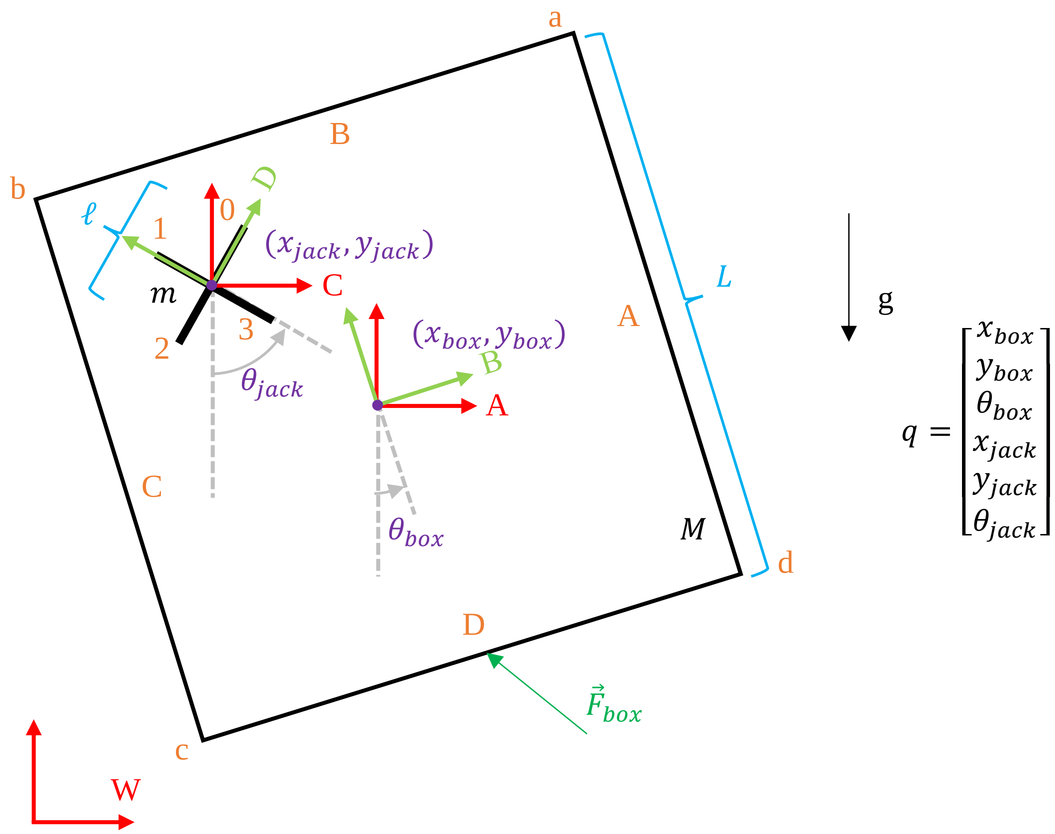

Fig 1. Drawing of system and all frames used.

For this project, I chose to do the default project of a jack in a box. The system and all the frames used are shown in Figure 1. The 6 system configurations include the x and y positions of the box and jack, as well as the angles with respect to the vertical of the box and jack.

Frame Transformations

Frame A is the translation from the "World Frame" (W) to the center of mass of the box. Frame B is the rotation from Frame A to the principal axes of the box. We desire a transformation from Frame B to the World Frame g_{WB} so that we can calculate a diagonal inertia matrix. The transformations for the box are as follows

$$g_{WA} = \begin{bmatrix} I_{3x3} & \begin{bmatrix} x_{box} \\ y_{box} \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix}, \quad g_{AB} = \begin{bmatrix} R(\theta_{box}) & 0 \\ 0 & 1 \end{bmatrix}$$

$$g_{WB} = g_{WA}\, g_{AB} \,,$$

where $R(\theta)$ is the rotation matrix expressed by

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and I_{3x3} is the 3x3 identity matrix.

Frame C is the translation frame the World Frame to the center of mass of the jack. Frame D is the rotation from Frame C to the principal axes of the jack. Again, we want to calculate g_{WD}, the transformation from Frame D to the World Frame.

$$g_{WC} = \begin{bmatrix} I_{3x3} & \begin{bmatrix} x_{jack} \\ y_{jack} \\ 0 \end{bmatrix} \\ 0 & 1 \end{bmatrix} \ , \quad g_{CD} = \begin{bmatrix} R(\theta_{jack}) & 0 \\ 0 & 1 \end{bmatrix}$$

$$g_{WD} = g_{WC}\, g_{CD} \ .$$

Finally, the last important transformation is the transformation from the jack's frame to the box's frame g_{BD}. This is calculated as

$$g_{BD} = g_{WB}^{-1}\, g_{WD} \ .$$

Equations of Motion

The total kinetic energy of this system can now be calculated with the following equation

$$KE = \frac{1}{2} V_{WB}^{T} \begin{bmatrix} M I_{3x3} & 0 \\ 0 & \mathfrak{I}_{box} \end{bmatrix} V_{WB} + \frac{1}{2} V_{WD}^{T} \begin{bmatrix} m I_{3x3} & 0 \\ 0 & \mathfrak{I}_{jack} \end{bmatrix} V_{WD} \ ,$$

where $V_{WB} = (g_{WB}^{-1}\, \dot{g}_{WB})^{\vee}$ and $V_{WD} = (g_{WD}^{-1}\, \dot{g}_{WD})^{\vee}$. The moment of inertia matrices, $\mathfrak{I}$, were approximated by values from Wikipedia, in which the box is treated as a thin plate and the jack is treated as two rods crossing each other

$$\mathfrak{I}_{box} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} M L^2 \end{bmatrix} \ , \ \mathfrak{I}_{jack} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{6} m l^2 \end{bmatrix} \ .$$

The total potential energy of this system is

$$V = Mg\, [0 \ \ 1 \ \ 0 \ \ 0]\, g_{WB} \bar{r}_B + mg\, [0 \ \ 1 \ \ 0 \ \ 0]\, g_{WD} \bar{r}_D \ ,$$

where r_B and r_D are the locations of the center of masses of the box and jack in their respective frames, and $\bar{r}_B$ and $\bar{r}_D$ are the homogenous representations of these vectors. Because the center of masses are specifically centered at the origin of these frames, $\bar{r}_B$ and $\bar{r}_D$ are both $[0 \ \ 0 \ \ 0 \ \ 1]^T$.

Now, we can calculate the Lagrangian as

$$L = KE - V \ .$$

The Euler-Lagrange Equations are then expressed as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F \ ,$$

where F is the forcing vector which looks like

$$F = \begin{bmatrix} 0 \\ 1.1Mg \\ -20\left(\theta_{box} - \left[\frac{\pi}{20} + \frac{\pi}{45}\sin^2\left(\frac{t}{20}\right)\right]\right) \\ 0 \\ 0 \\ 0 \end{bmatrix} .$$

The second value in F was chosen to push the box upwards and oppose the force of gravity. It is slightly greater than the weight of the box because it is also opposing the weight of the mass, so that when the mass collides with the bottom floor of the box, the box does not suddenly accelerate downwards. A torque was also applied to the box to "shake" the box back and forth and allow more collisions between the box and the jack.

Impact

This system required 16 $\phi(q)$ impact equations to account for all four legs of the jack being able to collide with all four walls of the box. To compute the impact, we need to transform the coordinates of the ends of the jack's legs from Frame D to Frame B. Then, we can write equations for when the distance between any of the legs and any of the walls is 0. Notice in Figure 1 that each of the legs are numbered 0 through 3, while each wall of the box is labeled A through D. This notation is used in my code to differentiate the different impact conditions. An example of one of these equations is if we consider impact between leg 0 and side A. The coordinate of leg 0 in Frame D $r_{D,0}$ is $[\ell/2 \;\; 0 \;\; 0]^T$. Collision occurs when the horizontal distance in Frame B between the leg and wall A is zero, in which wall A is located at $L/2$. Thus we can write

$$\phi_{0A}(q) = \frac{L}{2} - [1 \;\; 0 \;\; 0 \;\; 0]\, g_{BD} \begin{bmatrix} \ell/2 \\ 0 \\ 0 \\ 1 \end{bmatrix} .$$

In my code, I have written my impact equations such that $\phi(q)$ is always positive when the jack is inside the box and negative when the jack is outside the box. This allows me to write a general impact condition function, in which impact is detected whenever any of the $\phi(q)$ are negative.

When an impact is detected, the update must satisfy the following conditions for an elastic collision

$$q(\tau^-) = q(\tau^+) \,,$$
$$p|_{\tau^-}^{\tau^+} = \lambda \frac{\partial \phi(q)}{\partial q} \,,$$
$$H|_{\tau^-}^{\tau^+} = 0 \,,$$

in which τ^- is the instant before impact and τ^+ is the instant after the impact. The Legendre transform p is expressed as

$$p = \frac{\partial L}{\partial \dot{q}} \,,$$

and the Hamiltonian H is

$$H = p\dot{q} - L(q, \dot{q}) \,.$$

These equations make sure that upon collision, the momentum normal to the surface must be conserved and the Hamiltonian (and the total energy of the system in this case) must also be conserved, which is expected of a perfectly elastic collision. Additionally, these equations tell us that only velocity is updated, while the positions are not. Intuitively, we can see that it would not make sense for the jack or box to instantaneously translate or rotate during impact.

When we solve these equations for $q(\tau^+)$, we will get two sets of solutions, one in which $\lambda = 0$ and another where $\lambda \neq 0$. The solution where $\lambda = 0$ is the solution where the jack passes through the wall. Thus, we throw this solution out and take the solution corresponding to $\lambda \neq 0$.

Final Simulation Results

For my simulation, I chose the values that $M = 10$, $m = 1$, $L = 1$, $\ell = 0.2$, $g = 9.8$. I chose the initial conditions $(x_{box}, y_{box}, \theta_{box}, x_{jack}, y_{jack}, \theta_{jack}) = (0, 0, -\pi/3, 0, 0.3, \pi/6)$ and started the system from rest. **I believe that my simulation is working properly.** I simulated this system for 10 seconds. My two trajectories are shown in Figure 2, in which we can see that the jack's displacement trajectories change a lot, while the box's displacement remains closer to the origin. Likewise, the jack's angular trajectories vary a lot while the box oscillates in a way representative of the torque on the box. The box's relative indifference to impact with the jack is due to the fact that the box is ten times heavier than the jack.

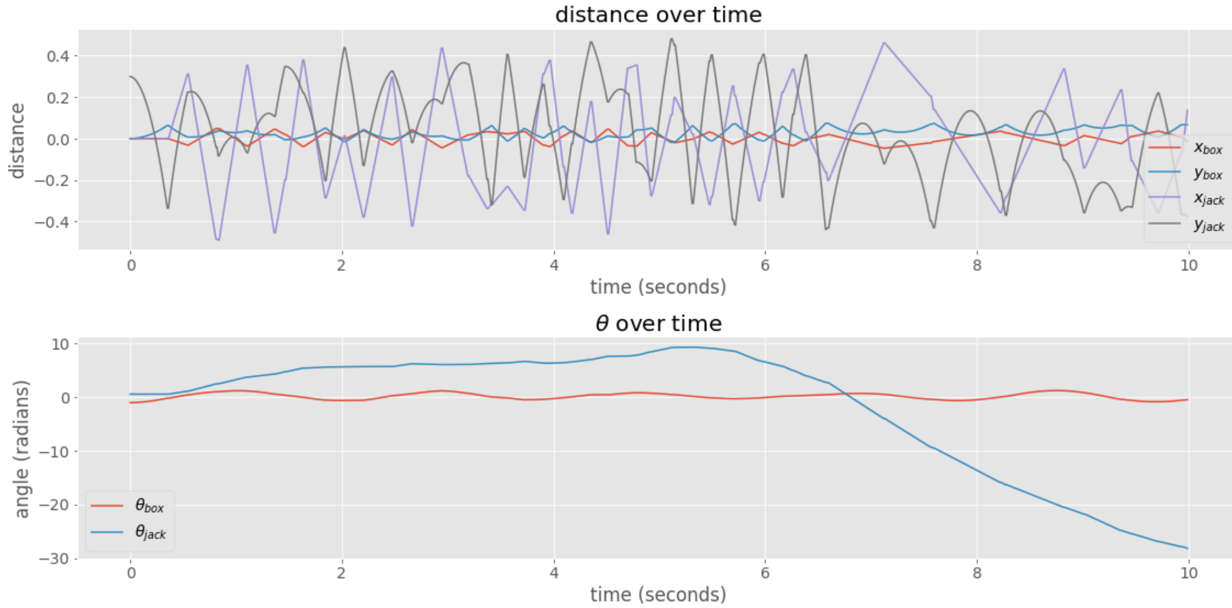

Fig 2. Displacement and angular trajectories over time for the box and the jack.

Because our system has an external force, we cannot simply check if the Hamiltonian is conserved at all time points. Therefore, I only focused on verifying that the Hamiltonian was conserved during each impact when the external forcing is removed. This has been graphed in Figure 3, in which we can see that the difference is not exactly zero as we expect, but we can notice that the axis is on the scale of 10^{-14}. This very small variation from 0 is due to error incurred by numerical integration.

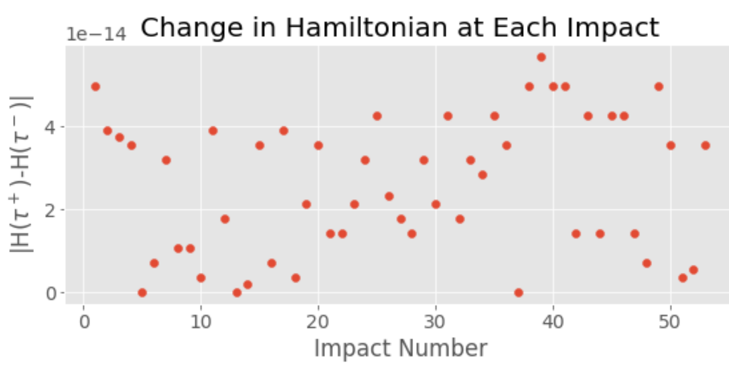

Fig 3. The Absolute Change in Hamiltonian $|H(\tau^+) - H(\tau^-)|$ at each impact.

The actual simulation looks pretty much as expected. The jack never goes through any of the walls, and it bounces off the walls as the box rotates clockwise and counter-clockwise from the applied torque. Each time the jack hits a wall, we can see that both the box and the jack are repelled in opposite directions (this is apparent in Figure 2, where the x positions of the two bodies after the first collision are almost always opposite, and the y position of the two bodies are also almost always opposite). Even though the box is a lot heavier than the jack, it should not be unaffected by the jack bouncing around inside it. Additionally, we can see that the jack spins a lot quicker than the box. This is also due to the difference in mass, which in turn makes the rotation inertia of the jack a lot smaller than the box. This means that the box's rotation is more resistant to a change in motion than the jack's rotation, so the jack ends up rotating a lot more than the box after impact.

Nonetheless, this simulation is not entirely accurate to how this would look in real life. Mainly, we can see that the jack continuously bounces a lot more chaotically than we would imagine a jack bouncing in a box with that speed of oscillation. This is due to our assumption of elastic collisions, meaning that the jack is not losing energy to friction on every bounce like it would in real life.