Appendix 9

Stability analysis using Lyapunov theory

In this appendix, we present some results about the stability analysis of nonlinear systems using Lyapunov theory. It is largely based on [Slotine 91] and [Zodiac 96].

A9.1. Autonomous systems

Let us consider the autonomous system (i.e. time-invariant) represented by the following state equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \qquad [A9.1]$$

A9.1.1. *Definition of stability*

An equilibrium point $\mathbf{x} = \mathbf{0}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ is said to be:

a) stable if for any $\varepsilon > 0$, there exists $R > 0$ such that if $\|\mathbf{x}(0)\| < \varepsilon$, then $\|\mathbf{x}(t)\| < R$;

b) asymptotically stable if for any $\varepsilon > 0$ and if $\|\mathbf{x}(0)\| < \varepsilon$, then $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$;

c) exponentially stable if there exist two strictly positive numbers α and λ such that: $\|\mathbf{x}(t)\| \leq \alpha \exp(-\lambda t) \|\mathbf{x}(0)\|$

d) an equilibrium point is globally asymptotically (exponentially) stable if it is asymptotically (exponentially) stable for any initial value $\mathbf{x}(0)$. A linear system is always globally exponentially stable or unstable.

Some of these definitions are illustrated in Figure A9.1.

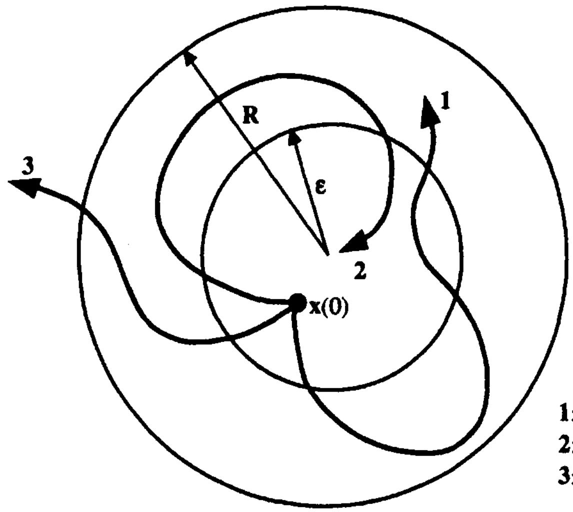

Figure A9.1. *Stability definition*

A9.1.2. *Positive definite and positive semi-definite functions*

The real function V($\mathbf{x}$) is positive definite (PD) in a ball B at the equilibrium point $\mathbf{x} = \mathbf{0}$ if V($\mathbf{x}$) > 0 and V($\mathbf{0}$) = 0. The function V($\mathbf{x}$) should have continuous partial derivatives. Moreover, for some $\varepsilon > 0$, V($\mathbf{x}$) should be less than ε in a finite region at the origin.

If V($\mathbf{x}$) ≥ 0, then the function is positive semi-definite (PSD).

A9.1.3. *Lyapunov direct theorem (sufficient conditions)*

If there exists V($\mathbf{x}$) PD in a ball B around the equilibrium point $\mathbf{x} = \mathbf{0}$ and if:

- $\dot{V}(\mathbf{x})$ is negative semi-definite (NSD), then $\mathbf{0}$ is a stable equilibrium point;
- $\dot{V}(\mathbf{x})$ is negative definite (ND), then $\mathbf{0}$ is asymptotically stable;
- $\dot{V}(\mathbf{x})$ is NSD and $\neq 0$ along all the trajectory, then $\mathbf{0}$ is asymptotically stable.

Moreover, if V($\mathbf{x}$) is PD all over the state space $\forall \mathbf{x} \neq \mathbf{0}$, $V(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{0}$, $\lim V(\mathbf{x}) \to \infty$ as $\|\mathbf{x}\| \to \infty$ and if $\dot{V}(\mathbf{x})$ is ND, then $\mathbf{0}$ is globally asymptotically stable.

A Lyapunov function can be interpreted as an energy function.

A9.1.4. *La Salle theorem and invariant set principle*

If $\dot{V}(\mathbf{x})$ is only NSD, it is yet possible to prove that the system is asymptotically stable, thanks to La Salle theorem [Hahn 67].

Definition. The set G is invariant for a dynamic system if every trajectory starting in G remains in G $\forall t$.

Theorem. Let R be the set of all points where $\dot{V} = 0$ and M be the largest invariant set of R; then every solution originating from R tends to M as $t \to \infty$.

A9.2. Non-autonomous systems

Let us consider the non-autonomous system (i.e. time-varying) represented by the following state equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \qquad [A9.2]$$

A9.2.1. *Definition of stability*

The equilibrium point $\mathbf{x} = \mathbf{0}$ such that $\mathbf{f}(\mathbf{x}, 0) = \mathbf{0}$ $\forall t \geq t_0$ is said to be:

a) stable at $t = t_0$, if for any $\varepsilon > 0$ there exists $R(\varepsilon, t_0) > 0$ such that if $\|\mathbf{x}(t_0)\| < \varepsilon$ then $\|\mathbf{x}(t)\| < R$ $\forall t \geq t_0$;

b) asymptotically stable at $t = t_0$, if it is stable and if there exists $R(t_0) > 0$ such that $\|\mathbf{x}(t_0)\| < R(t_0) \Rightarrow \mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$;

c) exponentially stable, if there exist two positive numbers α and λ such that:

$\|\mathbf{x}(t)\| \leq \alpha \exp(-\lambda\,(t-t_0))\,\|\mathbf{x}(t_0)\|$, $\forall t \geq t_0$ for $\mathbf{x}(t_0)$ sufficiently small;

d) globally asymptotically stable, if it is stable and if $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, $\forall \mathbf{x}(t_0)$;

e) uniformly stable, if $R = R(\varepsilon)$ can be chosen independently of t_0.

A9.2.2. *Lyapunov direct method*

Definition 1 (Function of class K)

A continuous function $\alpha : \Re^+ \to \Re^+$ is of class K if $\alpha(0) = 0$, $\alpha(\sigma) > 0$ $\forall \sigma > 0$, and α is non-decreasing.

Definition 2 (PD function)

A function $V(\mathbf{x}, t)$ is locally (globally) PD if and only if there exists a function α of class K such that $V(\mathbf{0}, t) = 0$, $V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|)$, $\forall t \geq 0$ and $\forall \mathbf{x}$ in a ball B.

Definition 3 (Decreasing function)

A function $V(\mathbf{x}, t)$ is locally (globally) decreasing if there exists a function α of class K such that $V(\mathbf{0}, t) = 0$ and $V(\mathbf{x}, t) \leq \alpha(\|\mathbf{x}\|)$, $\forall t > 0$ and $\forall \mathbf{x}$ in a ball B.

Lyapunov theorem

Let us assume that in a ball B around the equilibrium point $\mathbf{x} = \mathbf{0}$:

- there exists a Lyapunov function $V(\mathbf{x}, t)$ whose first derivatives are continuous;
- there exist functions α, β, γ of class K;

then, the equilibrium point is:

a) stable if $V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|)$, $\dot{V}(\mathbf{x}, t) \leq 0$;

b) uniformly stable if $\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$, $\dot{V}(\mathbf{x}, t) \leq 0$;

c) uniformly asymptotically stable if:

$\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$, $\dot{V}(\mathbf{x}, t) \leq -\gamma(\|\mathbf{x}\|) < 0$;

d) globally uniformly asymptotically stable if:

$\alpha(\|\mathbf{x}\|) \leq V(\mathbf{x}, t) \leq \beta(\|\mathbf{x}\|)$, $\dot{V}(\mathbf{x}, t) \leq -\gamma(\|\mathbf{x}\|) < 0$, $\alpha(\|\mathbf{x}\|) \to \infty$ as $\mathbf{x} \to \infty$.

Barbalat lemma. If $\dot{\mathbf{f}}(t)$ is a uniformly continuous function such that $\lim \mathbf{f}(t)$ is bounded as $t \to \infty$, then $\dot{\mathbf{f}}(t) \to 0$ as $t \to \infty$.

Barbalat theorem. If $V(\mathbf{x}, t)$ has a lower bound such that $V(\mathbf{x}, t) \geq \alpha(\|\mathbf{x}\|)$ and if $\dot{V}(\mathbf{x}, t) \leq 0$, then $\dot{V}(\mathbf{x}, t) \to 0$ as $t \to \infty$ if $\dot{V}(\mathbf{x}, t)$ is uniformly continuous with respect to time.