Appendix 11

Stability of passive systems

In this appendix, we present some useful results for the analysis and the design of passive and adaptive control laws [Landau 88]. For more details, the reader is referred to [Popov 73], [Desoer 75], [Landau 79].

A11.1. Definitions

Definition 1: positive real function

A rational function h(s) of the complex variable $s = \sigma + j\omega$ is positive real if:

a) h(s) is real when s is real;

b) h(s) has no poles in the Laplace right half plane $Re(s) > 0$ (Re denotes the real part of s);

c) the possible poles of h(s) along the axis $Re(s) = 0$ (when $s = j\omega$) are separate and the corresponding residuals are positive real or zero;

d) for any ω for which $s = j\omega$ is not a pole of h(s), then $Re(h(s)) \geq 0$.

Definition 2: strictly positive real function

A rational function h(s) of the complex variable $s = \sigma + j\omega$ is strictly positive real if:

a) h(s) is real when s is real;

b) h(s) has no poles in the Laplace right half plane $Re(s) \geq 0$;

c) $Re[h(j\,\omega)] > 0$ for any real value of ω, $-\infty < \omega < \infty$.

Definition 3: Hermitian matrix

A matrix $\mathbf{H}(s)$ of rational real functions in the complex variable $s = \sigma + j\omega$ is Hermitian if:

$$\mathbf{H}(s) = \mathbf{H}^T(s^*) \qquad [A11.1]$$

where s^* is the conjugate of s.

Definition 4: positive real matrix of functions

An (mxm) transfer matrix $\mathbf{H}(s)$ of rational real functions is positive real if:

a) no poles of the elements of $\mathbf{H}(s)$ are in the Laplace right half plane $Re(s) > 0$;

b) the possible poles of $\mathbf{H}(s)$ along the axis $Re(s) = 0$ are distinct and the corresponding matrix of residuals is Hermitian positive semi-definite;

c) the matrix $\mathbf{H}(j\omega) + \mathbf{H}^T(-j\omega)$ is Hermitian positive semi-definite for any real value ω that is not a pole of any element of $\mathbf{H}(s)$.

Definition 5: strictly positive real matrix of functions

An (mxm) transfer matrix $\mathbf{H}(s)$ of rational real functions is strictly positive real if:

a) no poles of the elements of $\mathbf{H}(s)$ are in the Laplace right half plane $Re(s) \geq 0$;

b) the matrix $\mathbf{H}(j\omega) + \mathbf{H}^T(-j\omega)$ is Hermitian positive definite for any real value of ω.

A11.2. Stability analysis of closed-loop positive feedback

Let us consider the closed-loop system of Figure A11.1, where the linear and time-invariant feed-forward block is described by the following state equations [Landau 88]:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}_1 = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{y}_2 \\ \mathbf{y}_1 = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}_1 = \mathbf{C}\mathbf{x} - \mathbf{D}\mathbf{y}_2 \end{cases} \qquad [A11.2]$$

in which (**A**, **B**) and (**A**, **C**) are controllable and observable respectively. The system is characterized by the transfer matrix $\mathbf{H}(s)$ defined by:

$$\mathbf{H}(s) = \mathbf{D} + \mathbf{C}[s\,\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} \qquad [A11.3]$$

The nonlinear time-varying feedback block is such that:

$$\mathbf{y}_2 = \mathbf{f}(\mathbf{u}_2, t, \tau) \quad \text{with } \tau \leq t \qquad [A11.4]$$

and satisfies the Popov inequality (proving the block passivity):

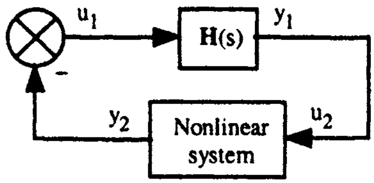

Figure A11.1. *Closed-loop positive feedback system*

$$\int_{t0}^{t1} y_2^T(t)\ u_2(t)\,dt \geq -\gamma_0^2 \quad \text{with } \gamma_0^2 < \infty, \text{ for } t_1 \geq t_0 \qquad [A11.5]$$

Theorem 1 (hyperstability)

For the closed-loop system of Figure A11.1 described by equations [A11.2], [A11.3] and [A11.4], and for any feedback block satisfying the inequality [A11.5], all solutions $\mathbf{x}(\mathbf{x}(0), t)$ satisfy the inequality:

$$\|\mathbf{x}(t)\| < \delta[\|\mathbf{x}(0)\| + \alpha_0] \quad \text{for } \delta > 0, \alpha_0 > 0, t \geq 0 \qquad [A11.6]$$

if, and only if, $\mathbf{H}(s)$ is a positive real transfer matrix.

Theorem 2 (asymptotic hyperstability)

For the closed-loop system of Figure A11.1 described by equations [A11.2], [A11.3] and [A11.4], and for any feedback block satisfying the inequality [A11.6], all solutions $\mathbf{x}(\mathbf{x}(0), t)$ satisfy both the inequality [A11.6] and $\lim \mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$ for any bounded input $u_1(t)$ if, and only if, $\mathbf{H}(s)$ is a strictly positive real transfer matrix.

NOTE.– Theorems 1 and 2 provide sufficient conditions to prove the stability and asymptotic stability respectively in the case where the Popov inequality is satisfied by the feedback block.

A11.3. Stability properties of passive systems

Lemma 1

A feedback combination of two strictly passive (positive) systems is always asymptotically stable.

Lemma 2

Any system obtained by a parallel combination of two passive (positive) blocks is itself a passive (positive) system.

Lemma 3

Any system obtained by a feedback combination of two passive (positive) blocks is itself a passive (positive) system.