Appendix 1

Solution of the inverse geometric model equations (Table 4.1)

A1.1. Type 2

The equation to be solved is:

$$X\ S\theta_i + Y\ C\theta_i = Z \qquad [A1.1]$$

Four cases are possible:

i) if $X = 0$ and $Y \neq 0$, we can write that:

$$C\theta_i = \frac{Z}{Y} \qquad [A1.2]$$

yielding:

$$\theta_i = \operatorname{atan2}(\pm\sqrt{1-(C\theta_i)^2},\ C\theta_i) \qquad [A1.3]$$

ii) if $Y = 0$ and $X \neq 0$, we obtain:

$$S\theta_i = \frac{Z}{X} \qquad [A1.4]$$

yielding:

$$\theta_i = \operatorname{atan2}(S\theta_i,\ \pm\sqrt{1-(S\theta_i)^2}) \qquad [A1.5]$$

iii) if X and Y are not zero, and Z = 0:

$$\begin{cases} \theta_i = \text{atan2}(-Y, X) \\ \theta_i' = \theta_i + \pi \end{cases} \qquad [A1.6]$$

(if X = Y = 0, the robot is in a singular configuration);

iv) if X, Y and Z are not zero, we can write that [Gorla 84]:

$$Y\,C\theta_i = Z - X\,S\theta_i \qquad [A1.7]$$

Squaring the equation leads to:

$$Y^2\,C^2\theta_i = Y^2\,(1 - S^2\theta_i) = Z^2 - 2Z\,X\,S\theta_i + X^2\,S^2\theta_i \qquad [A1.8]$$

Therefore, we have to solve a second degree equation in $S\theta_i$. Likewise, we can write an equation in $C\theta_i$. Finally, we obtain:

$$\begin{cases} S\theta_i = \dfrac{XZ + \varepsilon Y\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2} \\ C\theta_i = \dfrac{YZ - \varepsilon X\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2} \end{cases} \qquad [A1.9]$$

with $\varepsilon = \pm 1$ (it is straightforward to verify that two combinations of $S\theta_i$ and $C\theta_i$ can only satisfy the original equation). If $X^2 + Y^2 \leq Z^2$, there is no solution. Otherwise, the solution is given by:

$$\theta_i = \text{atan2}(S\theta_i, C\theta_i) \qquad [A1.10]$$

A1.2. Type 3

The system of equations to be solved is the following:

$$\begin{cases} X1\,S\theta_i + Y1\,C\theta_i = Z1 \\ X2\,S\theta_i + Y2\,C\theta_i = Z2 \end{cases} \qquad [A1.11]$$

Multiplying the first equation by Y2 and the second by Y1, under the condition that $X1Y2 - X2Y1 \neq 0$, yields:

$$S\theta_i = \frac{Z1\,Y2 - Z2\,Y1}{X1\,Y2 - X2\,Y1} \qquad [A1.12]$$

then, multiplying the first equation by X2 and the second by X1, yields:

$$C\theta_i = \frac{Z2\,X1 - Z1\,X2}{X1\,Y2 - X2\,Y1} \qquad [A1.13]$$

Thus:

$$\theta_i = \text{atan2}(S\theta_i, C\theta_i) \qquad [A1.14]$$

The condition X1Y2 - X2Y1 $\neq$ 0 means that the two equations of [A1.11] are independent. If it is not the case, we solve one of these equations as a type-2 equation.

In the frequent case where Y1 and X2 are zero, the system [A1.11] reduces to:

$$\begin{cases} X1\,S\theta_i = Z1 \\ Y2\,C\theta_i = Z2 \end{cases} \qquad [A1.15]$$

whose solution is straightforward:

$$\theta_i = \text{atan2}\left(\frac{Z1}{X1}, \frac{Z2}{Y2}\right) \qquad [A1.16]$$

A1.3. Type 4

The system of equations to be solved is given by:

$$\begin{cases} X1\,r_j\,S\theta_i = Y1 \\ X2\,r_j\,C\theta_i = Y2 \end{cases} \qquad [A1.17]$$

We first compute r_j by squaring both equations and adding them; then, we obtain θ_i by solving a type-3 system of equations:

$$\begin{cases} r_j = \pm\sqrt{(Y1/X1)^2 + (Y2/X2)^2} \\ \theta_i = \text{atan2}\left(\frac{Y1}{X1\,r_j}, \frac{Y2}{X2\,r_j}\right) \end{cases} \qquad [A1.18]$$

A1.4. Type 5

The system of equations to be solved is as follows:

$$\begin{cases} X1\ S\theta_i = Y1 + Z1\ r_j \\ X2\ C\theta_i = Y2 + Z2\ r_j \end{cases} \quad [A1.19]$$

Let us normalize the equations such that:

$$\begin{cases} S\theta_i = V1 + W1\ r_j \\ C\theta i = V2 + W2\ r_j \end{cases} \quad [A1.20]$$

After squaring both equations and adding them, we obtain a second degree equation in r_j, which can be solved if:

$$[W1^2 + W2^2 - (V1\ W2 - V2\ W1)^2] > 0 \quad [A1.21]$$

Then, we obtain θ_i by solving a type-3 system of equation.

A1.5. Type 6

The system of equations is given by:

$$\begin{cases} W\ S\theta_j = X\ C\theta_i + Y\ S\theta_i + Z1 \\ W\ C\theta_j = X\ S\theta_i - Y\ C\theta_i + Z2 \end{cases} \quad [A1.22]$$

with $Z1 \neq 0$ and/or $Z2 \neq 0$. By squaring both equations and adding them, we obtain a type-2 equation in θ_j:

$$B1\ S\theta_i + B2\ C\theta_i = B3 \quad [A1.23]$$

with:

$$B1 = 2\,(Z1\ Y + Z2\ X)$$
$$B2 = 2\,(Z1\ X - Z2Y)$$
$$B3 = W^2 - X^2 - Y^2 - Z1^2 - Z2^2$$

Knowing θ_i, we obtain θ_j by solving a type-3 system of equation.

A1.6. Type 7

The system of equations is the following:

$$\begin{cases} W1\ C\theta_j + W2\ S\theta_j = X\ C\theta_i + Y\ S\theta_i + Z1 \\ W1\ S\theta_j - W2\ C\theta_j = X\ S\theta_i - Y\ C\theta_i + Z2 \end{cases} \quad [A1.24]$$

It is a generalized form of a type-6 system. Squaring both equations and adding them gives a type-2 equation in θ_i:

$$B1\, S\theta_i + B2\, C\theta_i = B3 \qquad [A1.25]$$

where $B3 = W1^2 + W2^2 - X^2 - Y^2 - Z1^2 - Z2^2$. The terms B1 and B2 are identical to those of equation [A1.23].

After solving for θ_i, we compute θ_j as a solution of a type-3 system of equation.

A1.7. Type 8

The system of equations is the following:

$$\begin{cases} X\, C\theta_i + Y\, C(\theta_i + \theta_j) = Z1 \\ X\, S\theta_i + Y\, S(\theta_i + \theta_j) = Z2 \end{cases} \qquad [A1.26]$$

By squaring both equations and adding them, θ_i vanishes, yielding:

$$C\theta_j = \frac{Z1^2 + Z2^2 - X^2 - Y^2}{2XY} \qquad [A1.27]$$

hence:

$$\theta_j = \mathrm{atan2}(\pm\sqrt{1 - (C\theta j)^2}, C\theta j) \qquad [A1.28]$$

Then, [A1.26] reduces to a system of two equations in θ_i such that:

$$\begin{cases} S\theta_i = \dfrac{B1Z2 - B2\, Z1}{B1^2 + B2^2} \\ C\theta_i = \dfrac{B1Z1 + B2Z2}{B1^2 + B2^2} \end{cases} \qquad [A1.29]$$

with $B1 = X + Y\, C\theta_j$ and $B2 = Y\, S\theta_j$. Finally:

$$\theta_i = \mathrm{atan2}(S\theta_i, C\theta_i) \qquad [A1.30]$$