Appendix 6

Recursive equations between the energy functions

In this appendix, we establish the recursive equation between the energy functions of the inertial parameters of two consecutive links in an open loop structure (serial or tree structured robots).

A6.1. Recursive equation between the kinetic energy functions of serial robots

The kinetic energy of link j can be written using equation [9.19] as:

$$E_j = \frac{1}{2}\,{}^j\mathbb{V}_j^T\,{}^j\mathbb{J}_j\,{}^j\mathbb{V}_j \tag{A6.1}$$

with:

$${}^j\mathbb{V}_j = \begin{bmatrix} {}^j\mathbf{V}_j \\ {}^j\omega_j \end{bmatrix} \tag{A6.2}$$

and:

$${}^j\mathbb{J}_j = \begin{bmatrix} M_j\mathbf{I}_3 & -{}^j\hat{\mathbf{M}}\mathbf{S}_j \\ {}^j\hat{\mathbf{M}}\mathbf{S}_j & {}^j\mathbf{J}_j \end{bmatrix} \tag{A6.3}$$

The recursive equation of the kinematic screw is written using equation [9.22] as:

$${}^j\mathbb{V}_j = {}^j\mathbb{T}_{j-1}\,{}^{j-1}\mathbb{V}_{j-1} + \dot{q}_j\,{}^j\mathbf{a}_j \tag{A6.4}$$

where ${}^{j}\mathbf{a}_j$ is defined by equation [9.23a].

The kinetic energy of link j is linear in the inertial parameters of link j. Consequently, it can be written as:

$$E_j = \mathbf{e}_j \mathbf{K}_j \qquad [A6.5]$$

where $\mathbf{e}_j$ is the (1x10) row matrix containing the energy functions of the inertial parameters of link j. The parameters of link j are given by:

$$\mathbf{K}_j = [\ XX_j \quad XY_j \quad XZ_j \quad YY_j \quad YZ_j \quad ZZ_j \quad MX_j \quad MY_j \quad MZ_j \quad M_j\ ]^T \qquad [A6.6]$$

By substituting for ${}^{j}\mathbb{V}_j$ from equation [A6.4] into equation [A6.1], we obtain:

$$E_j = \frac{1}{2}({}^{j}\mathbb{T}_{j-1}\, {}^{j-1}\mathbb{V}_{j-1} + \dot{q}_j\, {}^{j}\mathbf{a}_j)^T\, {}^{j}\mathbb{J}_j\, ({}^{j}\mathbb{T}_{j-1}\, {}^{j-1}\mathbb{V}_{j-1} + \dot{q}_j\, {}^{j}\mathbf{a}_j) \qquad [A6.7]$$

Developing equation [A6.7] gives:

$$E_j = \frac{1}{2}{}^{j-1}\mathbb{V}_{j-1}^T\, ({}^{j}\mathbb{T}_{j-1}^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbb{T}_{j-1})\, {}^{j-1}\mathbb{V}_{j-1} + \dot{q}_j\, {}^{j}\mathbf{a}_j^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbb{V}_j - \frac{1}{2}\dot{q}_j^2\, {}^{j}\mathbf{a}_j^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbf{a}_j \qquad [A6.8]$$

Let us set:

$$ {}^{j-1}\mathbb{J}_j = {}^{j}\mathbb{T}_{j-1}^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbb{T}_{j-1} \qquad [A6.9]$$

$$\dot{q}_j\, \eta_j\, {}^{j}\mathbf{K}_j = {}^{j}\mathbf{a}_j^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbb{V}_j - \frac{1}{2}\dot{q}_j\, {}^{j}\mathbf{a}_j^T\, {}^{j}\mathbb{J}_j\, {}^{j}\mathbf{a}_j \qquad [A6.10]$$

where the row vector η_j is given by:

$$\eta_j = \bar{\sigma}_j\,[0 \quad 0 \quad \omega_{1,j} \quad 0 \quad \omega_{2,j} \quad (\omega_{3,j} - \tfrac{1}{2}\dot{q}_j) \quad V_{2,j} \quad -V_{1,j} \quad 0 \quad 0] +$$
$$\sigma_j\,[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\omega_{2,j} \quad \omega_{1,j} \quad 0 \quad (V_{3,j} - \tfrac{1}{2}\dot{q}_j)] \qquad [A6.11]$$

with:

$${}^{j}\mathbf{V}_j = [V_{1,j} \quad V_{2,j} \quad V_{3,j}]^T$$
$${}^{j}\boldsymbol{\omega}_j = [\omega_{1,j} \quad \omega_{2,j} \quad \omega_{3,j}]^T$$

Equation [A6.9] transforms the inertial parameters of link j from frame R_j into frame R_{j-1}. It can be written as:

$$^{j-1}\mathbf{K}_j = {}^{j-1}\boldsymbol{\lambda}_j\,{}^j\mathbf{K}_j \qquad [A6.12]$$

The expression of $^{j-1}\boldsymbol{\lambda}_j$ is obtained by comparing equations [A6.9] and [A6.12]. It is given in Table 9.1 for serial robots.

Using equations [A6.5], [A6.9], [A6.10] and [A6.12], we rewrite equation [A6.8] as follows:

$$E_j = (\,\mathbf{e}_{j-1}\,{}^{j-1}\boldsymbol{\lambda}_j + \dot{q}_j\,\boldsymbol{\eta}_j\,)\,\mathbf{K}_j \qquad [A6.13]$$

Finally, from equations [A6.5] and [A6.13], we deduce that:

$$\mathbf{e}_j = \mathbf{e}_{j-1}\,{}^{j-1}\boldsymbol{\lambda}_j + \dot{q}_j\,\boldsymbol{\eta}_j \qquad [A6.14]$$

with $\mathbf{e}_0 = \mathbf{0}_{1\times10}$.

A6.2. Recursive equation between the potential energy functions of serial robots

The potential energy of link j is written as (equation [9.25b]):

$$U_j = -{}^0\mathbf{g}^T\,[{}^0\mathbf{P}_j\,M_j + {}^0\mathbf{A}_j\,{}^j\mathbf{MS}_j] \qquad [A6.15]$$

where $^0\mathbf{g} = [\;g_1\;\;g_2\;\;g_3\;]$ indicates the acceleration due to gravity.

This expression is linear in the inertial parameters. It can be written as:

$$U_j = \mathbf{u}_j\,\mathbf{K}_j \qquad [A6.16]$$

Using equations [A6.15] and [A6.12], we can write that:

$$U_j = \mathbf{g}_u\,{}^0\boldsymbol{\lambda}_j\,\mathbf{K}_j \qquad [A6.17]$$

where:

$$\mathbf{g}_u = [\;\mathbf{0}_{1\times6}\;\; -{}^0\mathbf{g}^T\;\; 0\;] \qquad [A6.18]$$

From equation [A6.17], we deduce that:

$$\mathbf{u}_j = \mathbf{g}_u\,{}^0\boldsymbol{\lambda}_j \qquad [A6.19]$$

Since $^0\boldsymbol{\lambda}_j = {}^0\boldsymbol{\lambda}_{j-1}\,{}^{j-1}\boldsymbol{\lambda}_j$, we obtain the following recursive equation for the potential energy functions:

$$\mathbf{u}_j = \mathbf{u}_{j-1}\, {}^{j-1}\boldsymbol{\lambda}_j \qquad [A6.20]$$

with $\mathbf{u}_0 = \mathbf{g}_u$.

A6.3. Recursive equation between the total energy functions of serial robots

The total energy of link j is written as:

$$H_j = E_j + U_j = (\mathbf{e}_j + \mathbf{u}_j)\, \mathbf{K}_j = \mathbf{h}_j\, \mathbf{K}_j \qquad [A6.21]$$

with:

$$\mathbf{h}_j = \mathbf{e}_j + \mathbf{u}_j \qquad [A6.22]$$

From equations [A6.14] and [A6.20], we obtain the following recursive equation:

$$\mathbf{h}_j = \mathbf{h}_{j-1}\, {}^{j-1}\boldsymbol{\lambda}_j + \dot{q}_j\, \boldsymbol{\eta}_j \qquad [A6.23]$$

with $\mathbf{h}_0 = \mathbf{g}_u$.

A6.4. Expression of ${}^{a(j)}\boldsymbol{\lambda}_j$ in the case of the tree structured robot

In the case of the tree structured robot, equation [A6.23] is valid after replacing $j-1$ by $i = a(j)$. The (10x10) matrix ${}^i\boldsymbol{\lambda}_j$ represents the matrix transforming the inertial parameters $\mathbf{K}_j$ from frame R_j to frame R_i and can be obtained by developing the following equation:

$${}^i\mathbb{J}_j = {}^j\mathbb{T}_i^T\, {}^j\mathbb{J}_j\, {}^j\mathbb{T}_i \qquad [A6.24]$$

which is equivalent to:

$${}^i\mathbf{K}_j = {}^i\boldsymbol{\lambda}_j\, {}^j\mathbf{K}_j \qquad [A6.25]$$

By comparing equations [A6.24] and [A6.25], we obtain the expressions of the elements of ${}^i\boldsymbol{\lambda}_j$ in terms of the elements of the matrix ${}^i\mathbf{T}_j$, which are functions of the geometric parameters (γ_j, b_j, α_j, d_j, θ_j, r_j) defining frame R_j relative to frame R_i (Chapter 7), as follows:

$${}^i\boldsymbol{\lambda}_j = \begin{bmatrix} \boldsymbol{\lambda}11 & \boldsymbol{\lambda}12 & \boldsymbol{\lambda}13 \\ \mathbf{0}_{3x6} & {}^i\mathbf{A}_j & {}^i\mathbf{P}_j \\ \mathbf{0}_{1x6} & \mathbf{0}_{1x3} & 1 \end{bmatrix} \qquad [A6.26]$$

The dimensions of the matrices **λ11**, **λ12** and **λ13** are (6x6), (6x3) and (6x1) respectively. To simplify the writing, let:

$$ {}^{i}A_j = [\ s \quad n \quad a\ ] \tag{A6.27} $$

$$ {}^{i}P_j = [\ P_x \quad P_y \quad P_z\ ]^T \tag{A6.28} $$

Thus:

$$ \lambda 11 = \begin{bmatrix} s_x s_x & 2s_x n_x & 2s_x a_x & n_x n_x & 2n_x a_x & a_x a_x \\ s_x s_y & s_y n_x + s_x n_y & s_y a_x + s_x a_y & n_x n_y & n_y a_x + n_x a_y & a_x a_y \\ s_z s_x & s_z n_x + s_x n_z & s_z a_x + s_x a_z & n_x n_z & n_z a_x + n_x a_z & a_z a_x \\ s_y s_y & 2n_y s_y & 2s_y a_y & n_y n_y & 2n_y a_y & a_y a_y \\ s_y s_z & s_z n_y + s_y n_z & a_y s_z + s_y a_z & n_y n_z & n_z a_y + n_y a_z & a_y a_z \\ s_z s_z & 2n_z s_z & 2s_z a_z & n_z n_z & 2n_z a_z & a_z a_z \end{bmatrix} \tag{A6.29} $$

$$ \lambda 12 = \begin{bmatrix} 2(s_z P_z + s_y P_y) & 2(n_z P_z + n_y P_y) & 2(a_z P_z + a_y P_y) \\ -s_y P_x - s_x P_y & -n_y P_x - n_x P_y & -a_y P_x - a_x P_y \\ -s_z P_x - s_x P_z & -n_z P_x - n_x P_z & -a_z P_x - a_x P_z \\ 2(s_z P_z + s_x P_x) & 2(n_z P_z + n_x P_x) & 2(a_z P_z + a_x P_x) \\ -s_z P_y - s_y P_z & -n_z P_y - n_y P_z & -a_z P_y - a_y P_z \\ 2(s_y P_y + s_x P_x) & 2(n_y P_y + n_x P_x) & 2(a_y P_y + a_x P_x) \end{bmatrix} \tag{A6.30} $$

$$ \lambda 13 = [\ P_z P_z + P_y P_y \quad -P_x P_y \quad -P_x P_z \quad P_z P_z + P_x P_x \quad -P_z P_y \quad P_x P_x + P_y P_y\ ]^T \tag{A6.31} $$