# Chapter 2

# Transformation matrix between vectors, frames and screws

## 2.1. Introduction

In robotics, we assign one or more frames to each link of the robot and each object of the workcell. Thus, transformation of frames is a fundamental concept in the modeling and programming of a robot. It enables us to:

- compute the location, position and orientation of robot links relative to each other;
- describe the position and orientation of objects;
- specify the trajectory and velocity of the end-effector of a robot for a desired task;
- describe and control the forces when the robot interacts with its environment;
- implement sensory-based control using information provided by various sensors, each having its own reference frame.

In this chapter, we present a notation that allows us to describe the relationship between different frames and objects of a robotic cell. This notation, called *homogeneous transformation,* has been widely used in computer graphics [Roberts 65], [Newman 79] to compute the projections and perspective transformations of an object on a screen. Currently, this is also being used extensively in robotics [Pieper 68], [Paul 81]. We will show how the points, vectors and transformations between frames can be represented using this approach. Then, we will define the differential transformations between frames as well as the representation of velocities and forces using screws.

## 2.2. Homogeneous coordinates

### 2.2.1. *Representation of a point*

Let (${}^iP_x$, ${}^iP_y$, ${}^iP_z$) be the Cartesian coordinates of an arbitrary point P with respect to the frame $R_i$, which is described by the origin $O_i$ and the axes $x_i$, $y_i$, $z_i$ (Figure 2.1). The homogeneous coordinates of P with respect to frame $R_i$ are defined by ($w{}^iP_x$, $w{}^iP_y$, $w{}^iP_z$, w), where w is a scaling factor. In robotics, w is taken to be equal to 1. Thus, we represent the homogeneous coordinates of P by the (4x1) column vector:

$$
{}^i\mathbf{P} = \begin{bmatrix} {}^iP_x \\ {}^iP_y \\ {}^iP_z \\ 1 \end{bmatrix} \qquad [2.1]
$$

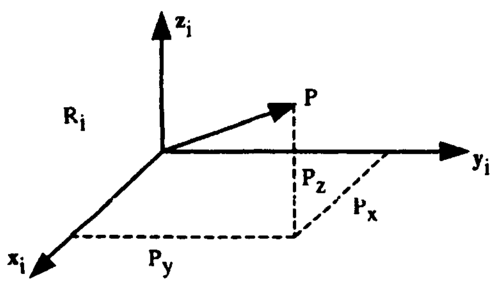


**Figure 2.1.** *Representation of a point vector*

### 2.2.2. *Representation of a direction*

A direction (free vector) is also represented by four components, but the fourth component is zero, indicating a vector at infinity. If the Cartesian coordinates of a unit vector **u** with respect to frame $R_i$ are (${}^iu_x$, ${}^iu_y$, ${}^iu_z$), its homogeneous coordinates will be:

$$
{}^i\mathbf{u} = \begin{bmatrix} {}^iu_x \\ {}^iu_y \\ {}^iu_z \\ 0 \end{bmatrix} \qquad [2.2]
$$

### 2.2.3. *Representation of a plane*

The homogeneous coordinates of a plane Q, whose equation with respect to a frame $R_i$ is ${}^i\alpha x + {}^i\beta y + {}^i\gamma z + {}^i\delta = 0$, are given by:

$$ {}^i\mathbf{Q} = [\ {}^i\alpha \quad {}^i\beta \quad {}^i\gamma \quad {}^i\delta\ ] \tag{2.3}$$

If a point P lies in the plane Q, then the matrix product ${}^i\mathbf{Q}\ {}^i\mathbf{P}$ is zero:

$$ {}^i\mathbf{Q}\ {}^i\mathbf{P} = [\ {}^i\alpha \quad {}^i\beta \quad {}^i\gamma \quad {}^i\delta\ ] \begin{bmatrix} {}^iP_x \\ {}^iP_y \\ {}^iP_z \\ 1 \end{bmatrix} = 0 \tag{2.4}$$

## 2.3. Homogeneous transformations [Paul 81]

### 2.3.1. *Transformation of frames*

The transformation, translation and/or rotation, of a frame $R_i$ into frame $R_j$ (Figure 2.2) is represented by the (4x4) homogeneous transformation matrix ${}^i\mathbf{T}_j$ such that:

$$ {}^i\mathbf{T}_j = [{}^i\mathbf{s}_j \quad {}^i\mathbf{n}_j \quad {}^i\mathbf{a}_j \quad {}^i\mathbf{P}_j] = \begin{bmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5a}$$

where ${}^i\mathbf{s}_j$, ${}^i\mathbf{n}_j$ and ${}^i\mathbf{a}_j$ contain the components of the unit vectors along the $\mathbf{x}_j$, $\mathbf{y}_j$ and $\mathbf{z}_j$ axes respectively expressed in frame $R_i$, and where ${}^i\mathbf{P}_j$ is the vector representing the coordinates of the origin of frame $R_j$ expressed in frame $R_i$.

We can also say that the matrix ${}^i\mathbf{T}_j$ defines frame $R_j$ relative to frame $R_i$. Thereafter, the transformation matrix [2.5a] will occasionally be written in the form of a partitioned matrix:

$$ {}^i\mathbf{T}_j = \begin{bmatrix} {}^i\mathbf{A}_j & & & {}^i\mathbf{P}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^i\mathbf{s}_j & {}^i\mathbf{n}_j & {}^i\mathbf{a}_j & {}^i\mathbf{P}_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.5b}$$

Apparently, this is in violation of the homogeneous notation since the vectors have only three components. In any case, the distinction in the representation with either three or four components will always be clear in the text.

In summary:

- the matrix ${}^{i}\mathbf{T}_j$ represents the transformation from frame $R_i$ to frame $R_j$;
- the matrix ${}^{i}\mathbf{T}_j$ can be interpreted as representing the frame $R_j$ (three orthogonal axes and an origin) with respect to frame $R_i$.

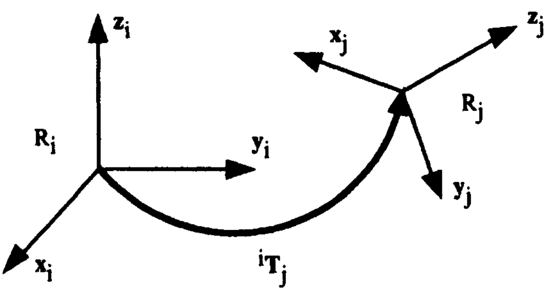


**Figure 2.2.** *Transformation of frames*

### 2.3.2. *Transformation of vectors*

Let the vector ${}^{j}\mathbf{P}$ define the homogeneous coordinates of the point P with respect to frame $R_j$ (Figure 2.3). Thus, the homogeneous coordinates of P with respect to frame $R_i$ can be obtained as:

$$ {}^{i}\mathbf{P} = {}^{i}(\mathbf{O}_i\mathbf{P}) = {}^{i}\mathbf{s}_j\,{}^{j}P_x + {}^{i}\mathbf{n}_j\,{}^{j}P_y + {}^{i}\mathbf{a}_j\,{}^{j}P_z + {}^{i}\mathbf{P}_j = {}^{i}\mathbf{T}_j\,{}^{j}\mathbf{P} \qquad [2.6] $$

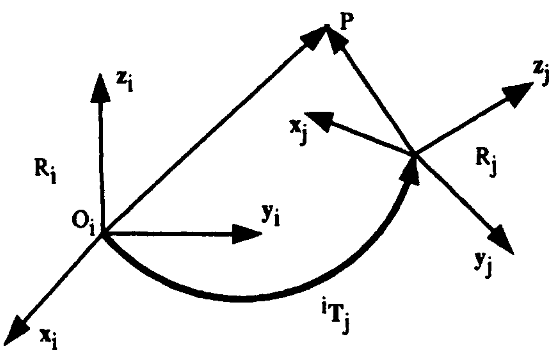


**Figure 2.3.** *Transformation of a vector*

Thus the matrix ${}^{i}\mathbf{T}_{j}$ allows us to calculate the coordinates of a vector with respect to frame $R_i$ in terms of its coordinates in frame $R_j$.

• **Example 2.1.** Deduce the matrices ${}^{i}\mathbf{T}_{j}$ and ${}^{j}\mathbf{T}_{i}$ from Figure 2.4. Using equation [2.5a], we directly obtain:

$$
{}^{i}\mathbf{T}_{j} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 12 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{j}\mathbf{T}_{i} = \begin{bmatrix} 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & -12 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

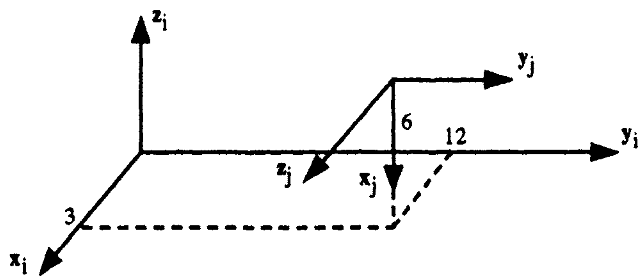


**Figure 2.4.** *Example 2.1*

### 2.3.3. *Transformation of planes*

The relative position of a point with respect to a plane is invariant with respect to the transformation applied to the set of {point, plane}. Thus:

$$
{}^{j}\mathbf{Q}\,{}^{j}\mathbf{P} = {}^{i}\mathbf{Q}\,{}^{i}\mathbf{P} = {}^{i}\mathbf{Q}\,{}^{i}\mathbf{T}_{j}\,{}^{j}\mathbf{P}
$$

leading to:

$$
{}^{j}\mathbf{Q} = {}^{i}\mathbf{Q}\,{}^{i}\mathbf{T}_{j} \qquad [2.7]
$$

### 2.3.4. *Transformation matrix of a pure translation*

Let **Trans**(a, b, c) be this transformation, where a, b and c denote the translation along the **x**, **y** and **z** axes respectively. Since the orientation is invariant, the transformation **Trans**(a, b, c) is expressed as (Figure 2.5):

$$ {}^{i}\mathbf{T}_j = \mathbf{Trans}(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [2.8] $$

From now on, we will also use the notation **Trans(u, d)** to denote a translation along an axis **u** by a value d. Thus, the matrix **Trans**(a, b, c) can be decomposed into the product of three matrices **Trans(x**, a) **Trans(y**, b) **Trans(z**, c), taking any order of the multiplication.

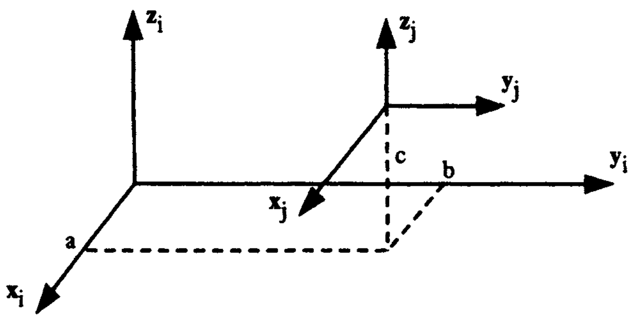


**Figure 2.5.** *Transformation of pure translation*

### 2.3.5. *Transformation matrices of a rotation about the principle axes*

#### 2.3.5.1. *Transformation matrix of a rotation about the x axis by an angle θ*

Let **Rot**(x, θ) be this transformation. From Figure 2.6, we deduce that the components of the unit vectors ${}^{i}\mathbf{s}_j$, ${}^{i}\mathbf{n}_j$, ${}^{i}\mathbf{a}_j$ along the axes $\mathbf{x}_j$, $\mathbf{y}_j$ and $\mathbf{z}_j$ respectively of frame $R_j$ expressed in frame $R_i$ are as follows:

$$ \begin{cases} {}^{i}\mathbf{s}_j = [1 \quad 0 \quad 0 \quad 0]^T \\ {}^{i}\mathbf{n}_j = [0 \quad C\theta \quad S\theta \quad 0]^T \\ {}^{i}\mathbf{a}_j = [0 \quad -S\theta \quad C\theta \quad 0]^T \end{cases} \qquad [2.9] $$

where Sθ and Cθ represent sin(θ) and cos(θ) respectively, and the superscript T indicates the transpose of the vector.

$$
{}^{i}\mathbf{T}_j = \mathbf{Rot}(\mathbf{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\theta & -S\theta & 0 \\ 0 & S\theta & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{rot}(\mathbf{x}, \theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [2.10]
$$

where **rot(x, θ)** denotes the (3x3) orientation matrix.

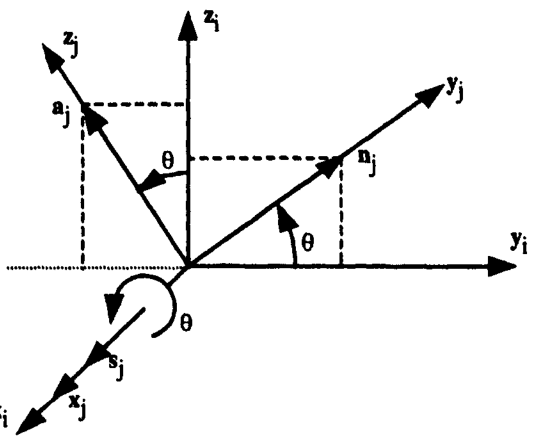


**Figure 2.6.** *Transformation of a pure rotation about the x-axis*

### 2.3.5.2. *Transformation matrix of a rotation about the y axis by an angle θ*

In the same way, we obtain:

$$
{}^{i}\mathbf{T}_j = \mathbf{Rot}(\mathbf{y}, \theta) = \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{rot}(\mathbf{y}, \theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [2.11]
$$

### 2.3.5.3. *Transformation matrix of a rotation θ about the z axis by an angle θ*

We can also verify that:

$$ {}^{i}\mathbf{T}_{j} = \mathbf{Rot}(\mathbf{z}, \theta) = \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{rot}(\mathbf{z}, \theta) & & 0 \\ & & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.12] $$

### 2.3.6. *Properties of homogeneous transformation matrices*

a) From equations [2.5], a transformation matrix can be written as:

$$ \mathbf{T} = \begin{bmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{A} & & \mathbf{P} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.13] $$

The matrix **A** represents the rotation whereas the column matrix **P** represents the translation. For a transformation of pure translation, $\mathbf{A} = \mathbf{I}_3$ ($\mathbf{I}_3$ represents the identity matrix of order 3), whereas $\mathbf{P} = \mathbf{0}$ for a transformation of pure rotation. The matrix **A** represents the direction cosine matrix. It contains three independent parameters (one of the vectors **s**, **n** or **a** can be deduced from the vector product of the other two, for example $\mathbf{s} = \mathbf{n} \mathrm{x} \mathbf{a}$; moreover, the dot product $\mathbf{n}.\mathbf{a}$ is zero and the magnitudes of **n** and **a** are equal to 1).

b) The matrix **A** is orthogonal, i.e. its inverse is equal to its transpose:

$$ \mathbf{A}^{-1} = \mathbf{A}^{T} \quad [2.14] $$

c) The inverse of a matrix ${}^{i}\mathbf{T}_{j}$ defines the matrix ${}^{j}\mathbf{T}_{i}$.

To express the components of a vector ${}^{i}\mathbf{P}_{1}$ into frame $R_j$, we write:

$$ {}^{j}\mathbf{P}_{1} = {}^{j}\mathbf{T}_{i}\, {}^{i}\mathbf{P}_{1} \quad [2.15] $$

If we postmultiply equation [2.6] by ${}^{i}\mathbf{T}_{j}^{-1}$ (inverse of ${}^{i}\mathbf{T}_{j}$), we obtain:

$$ {}^{i}\mathbf{T}_{j}^{-1}\, {}^{i}\mathbf{P}_{1} = {}^{j}\mathbf{P}_{1} \quad [2.16] $$

From equations [2.15] and [2.16], we deduce that:

$$ {}^{i}\mathbf{T}_{j}^{-1} = {}^{j}\mathbf{T}_{i} \quad [2.17] $$

d) We can easily verify that:

$$\mathbf{Rot}^{-1}(\mathbf{u}, \theta) = \mathbf{Rot}(\mathbf{u}, -\theta) = \mathbf{Rot}(-\mathbf{u}, \theta) \quad [2.18]$$

$$\mathbf{Trans}^{-1}(\mathbf{u}, d) = \mathbf{Trans}(-\mathbf{u}, d) = \mathbf{Trans}(\mathbf{u}, -d) \quad [2.19]$$

e) The inverse of a transformation matrix represented by equation [2.13] can be obtained as:

$$\mathbf{T}^{-1} = \begin{bmatrix} & -\mathbf{s}^T \mathbf{P} \\ \mathbf{A}^T & -\mathbf{n}^T \mathbf{P} \\ & -\mathbf{a}^T \mathbf{P} \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T & -\mathbf{A}^T \mathbf{P} \\ 0\ 0\ 0 & 1 \end{bmatrix} \quad [2.20]$$

f) Composition of two matrices. The multiplication of two transformation matrices gives a transformation matrix:

$$\mathbf{T}_1 \mathbf{T}_2 = \begin{bmatrix} \mathbf{A}_1 & \mathbf{P}_1 \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{A}_2 & \mathbf{P}_2 \\ 0\ 0\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1\mathbf{A}_2 & \mathbf{A}_1\mathbf{P}_2 + \mathbf{P}_1 \\ 0\quad 0\quad 0 & 1 \end{bmatrix} \quad [2.21]$$

Note that the matrix multiplication is non-commutative ($\mathbf{T}_1\mathbf{T}_2 \neq \mathbf{T}_2\mathbf{T}_1$).

g) If a frame $R_0$ is subjected to k consecutive transformations (Figure 2.7) and if each transformation i, (i = 1, ..., k), is defined with respect to the current frame $R_{i-1}$, then the transformation ${}^0\mathbf{T}_k$ can be deduced by multiplying all the transformation on the right as:

$${}^0\mathbf{T}_k = {}^0\mathbf{T}_1 \, {}^1\mathbf{T}_2 \, {}^2\mathbf{T}_3 \ldots {}^{k-1}\mathbf{T}_k \quad [2.22]$$

h) If a frame $R_j$, defined by ${}^i\mathbf{T}_j$, undergoes a transformation $\mathbf{T}$ that is defined relative to frame $R_j$, then $R_j$ will be transformed into $R_{j'}$ with ${}^i\mathbf{T}_{j'} = \mathbf{T}\ {}^i\mathbf{T}_j$ (Figure 2.8).

From the properties g and h, we deduce that:

- multiplication on the right (postmultiplication) of the transformation ${}^i\mathbf{T}_j$ indicates that the transformation is defined with respect to the current frame $R_j$;
- multiplication on the left (premultiplication) indicates that the transformation is defined with respect to the reference frame $R_i$.

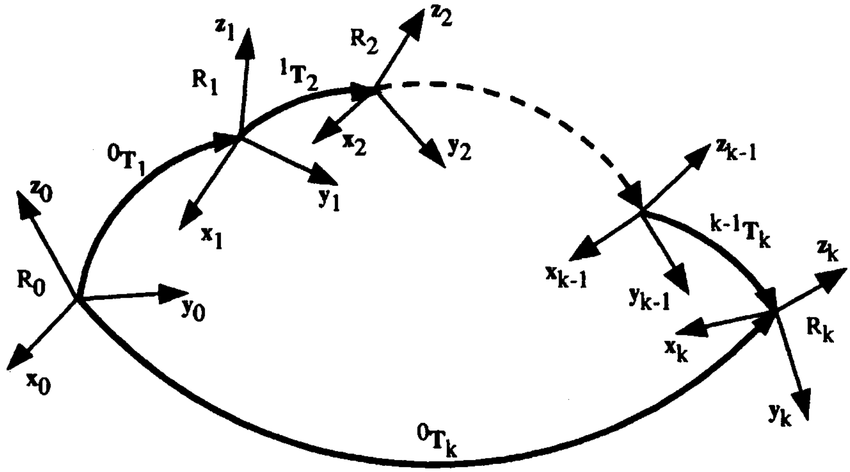


**Figure 2.7.** *Composition of transformations: multiplication on the right*

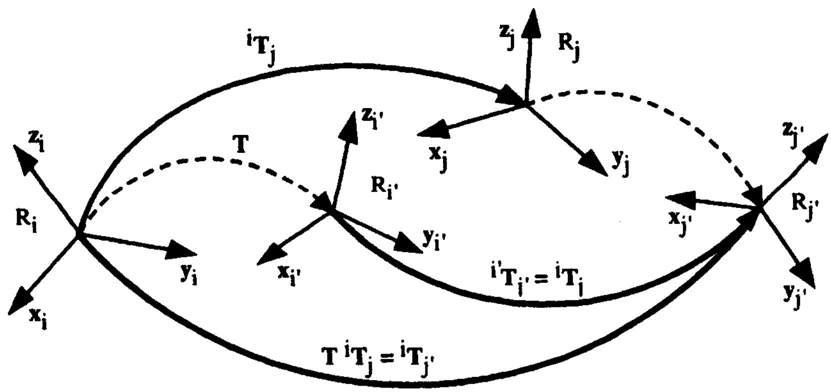


**Figure 2.8.** *Composition of transformations: multiplication on the left*

• **Example 2.2.** Consider the composite transformation illustrated in Figure 2.9 and defined by:

$$^0\mathbf{T}_2 = \mathbf{Rot}(\mathbf{x}, \frac{\pi}{6})\ \mathbf{Trans}(\mathbf{y}, \mathrm{d})$$

– reading $^0\mathbf{T}_2$ from left to right (Figure 2.9a): first, we apply the rotation; the new location of frame $R_0$ is denoted by frame $R_1$; then, the translation is defined with respect to frame $R_1$;

– reading ${}^0T_2$ from right to left (Figure 2.9b): first we apply the translation, then the rotation is defined with respect to frame $R_0$.

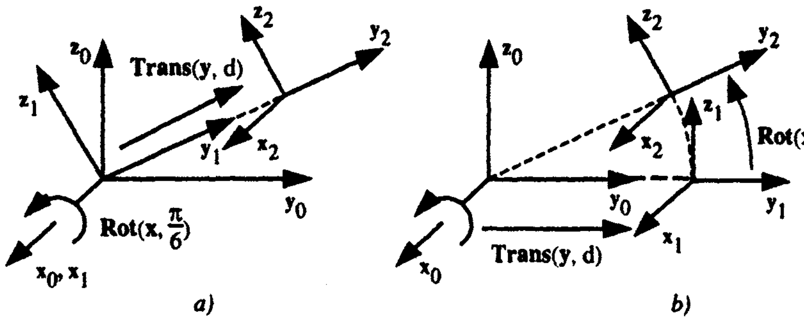


**Figure 2.9.** *Example 2.2*

i) Consecutive transformations about the same axis. We note the following properties:

$$\mathbf{Rot}(\mathbf{u}, \theta_1)\,\mathbf{Rot}(\mathbf{u}, \theta_2) = \mathbf{Rot}[\mathbf{u}, (\theta_1+\theta_2)] \quad [2.23]$$

$$\mathbf{Rot}(\mathbf{u}, \theta)\,\mathbf{Trans}(\mathbf{u}, \mathrm{d}) = \mathbf{Trans}(\mathbf{u}, \mathrm{d})\,\mathbf{Rot}(\mathbf{u}, \theta) \quad [2.24]$$

j) Decomposition of a transformation matrix. A transformation matrix can be decomposed into two transformation matrices, one represents a pure translation and the second a pure rotation:

$$\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{P} \\ 0\;0\;0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{P} \\ 0\;0\;0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ 0\;0\;0 & 1 \end{bmatrix} \quad [2.25]$$

### 2.3.7. *Transformation matrix of a rotation about a general vector located at the origin*

Let $\mathbf{Rot}(\mathbf{u}, \theta)$ be the transformation representing a rotation of an angle $\theta$ about an axis, with unit vector $\mathbf{u} = [u_x \quad u_y \quad u_z]^T$, located at the origin of frame $R_i$ (Figure 2.10). We define the frame $R_k$ such that $\mathbf{z}_k$ is along the vector $\mathbf{u}$ and $\mathbf{x}_k$ is along the common normal between $\mathbf{z}_k$ and $\mathbf{z}_i$. The matrix ${}^i\mathbf{T}_k$ can be obtained as:

$${}^i\mathbf{T}_k = \mathbf{Rot}(\mathbf{z}, \alpha)\,\mathbf{Rot}(\mathbf{x}, \beta) \quad [2.26]$$

where $\alpha$ is the angle between $\mathbf{x}_i$ and $\mathbf{x}_k$ about $\mathbf{z}_i$, and $\beta$ is the angle between $\mathbf{z}_i$ and $\mathbf{u}$ about $\mathbf{x}_k$.

From equation [2.26], we obtain:

$$\mathbf{u} = {}^i\mathbf{a}_k = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} S\alpha\, S\beta \\ -C\alpha\, S\beta \\ C\beta \end{bmatrix} \qquad [2.27]$$

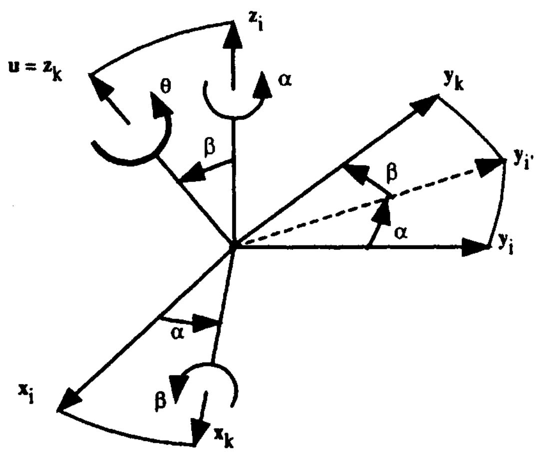


**Figure 2.10.** *Transformation of pure rotation about any axis*

The rotation about $\mathbf{u}$ is equivalent to the rotation about $\mathbf{z}_k$. From properties g and h of § 2.3.6, we deduce that:

$$\mathbf{Rot(u, \theta)}\ {}^i\mathbf{T}_k = {}^i\mathbf{T}_k\ \mathbf{Rot(z, \theta)} \qquad [2.28]$$

thus:

$$\begin{aligned} \mathbf{Rot(u, \theta)} &= {}^i\mathbf{T}_k\ \mathbf{Rot(z, \theta)}\ {}^i\mathbf{T}_k^{-1} \\ &= \mathbf{Rot(z, \alpha)\ Rot(x, \beta)\ Rot(z, \theta)\ Rot(x, -\beta)\ Rot(z, -\alpha)} \end{aligned} \qquad [2.29]$$

From this relation and using equation [2.27], we obtain:

$$\mathbf{Rot\ (u, \theta)} = \begin{bmatrix} & & & 0 \\ & \mathbf{rot(u, \theta)} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} u_x^2(1-C\theta)+C\theta & u_xu_y(1-C\theta)-u_zS\theta & u_xu_z(1-C\theta)+u_yS\theta & 0 \\ u_xu_y(1-C\theta)+u_zS\theta & u_y^2(1-C\theta)+C\theta & u_yu_z(1-C\theta)-u_xS\theta & 0 \\ u_xu_z(1-C\theta)-u_yS\theta & u_yu_z(1-C\theta)+u_xS\theta & u_z^2(1-C\theta)+C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.30]$$

We can easily remember this relation by writing it as:

$$\mathbf{rot}(\mathbf{u}, \theta) = \mathbf{u}\,\mathbf{u}^T (1 - C\theta) + \mathbf{I}_3\, C\theta + \hat{\mathbf{u}}\, S\theta \quad [2.31]$$

where $\hat{\mathbf{u}}$ indicates the skew-symmetric matrix defined by the components of the vector **u** such that:

$$\hat{\mathbf{u}} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} \quad [2.32]$$

Note that the vector product $\mathbf{u}\mathrm{x}\mathbf{V}$ is obtained by $\hat{\mathbf{u}}\,\mathbf{V}$.

### 2.3.8. *Equivalent angle and axis of a general rotation*

Let **T** be any arbitrary rotational transformation matrix such that:

$$\mathbf{T} = \begin{bmatrix} s_x & n_x & a_x & 0 \\ s_y & n_y & a_y & 0 \\ s_z & n_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad [2.33]$$

We solve the following expression for **u** and $\theta$:

$$\mathbf{Rot}\,(\mathbf{u}, \theta) = \mathbf{T} \qquad \text{with } 0 \le \theta \le \pi$$

Adding the diagonal terms of equations [2.30] and [2.33], we obtain:

$$C\theta = \frac{1}{2}(s_x + n_y + a_z - 1) \quad [2.34]$$

From the off-diagonal terms, we obtain:

$$\begin{cases} 2\, u_x\, S\theta = n_z - a_y \\ 2\, u_y\, S\theta = a_x - s_z \\ 2\, u_z\, S\theta = s_y - n_x \end{cases} \qquad [2.35]$$

yielding:

$$S\theta = \frac{1}{2}\sqrt{(n_z - a_y)^2 + (a_x - s_z)^2 + (s_y - n_x)^2} \qquad [2.36]$$

From equations [2.34] and [2.36], we deduce that:

$$\theta = \text{arctg}\,(S\theta/C\theta) \qquad \text{with } 0 \le \theta \le \pi \qquad [2.37]$$

$u_x$, $u_y$ and $u_z$ are calculated using equation [2.35] if $S\theta \neq 0$. When $S\theta$ is small, the elements $u_x$, $u_y$ and $u_z$ cannot be determined with good accuracy by this equation. However, in the case where $C\theta < 0$, we obtain $u_x$, $u_y$ and $u_z$ more accurately using the diagonal terms of **Rot(u, θ)** as follows:

$$u_x = \pm\sqrt{\frac{s_x - C\theta}{1 - C\theta}},\ u_y = \pm\sqrt{\frac{n_y - C\theta}{1 - C\theta}},\ u_z = \pm\sqrt{\frac{a_z - C\theta}{1 - C\theta}} \qquad [2.38]$$

From equation [2.35], we deduce that:

$$\begin{cases} u_x = \text{sign}(n_z - a_y)\sqrt{\dfrac{s_x - C\theta}{1 - C\theta}} \\ u_y = \text{sign}(a_x - s_z)\sqrt{\dfrac{n_y - C\theta}{1 - C\theta}} \\ u_z = \text{sign}(s_y - n_x)\sqrt{\dfrac{a_z - C\theta}{1 - C\theta}} \end{cases} \qquad [2.39]$$

where sign(.) indicates the sign function of the expression between brackets, thus $\text{sign}(e) = +1$ if $e \ge 0$, and $\text{sign}(e) = -1$ if $e < 0$.

• **Example 2.3.** Suppose that the location of a frame $R_E$, which is fixed to the end-effector of a robot, relative to the reference frame $R_0$ is given by the matrix **Rot(x, – π/4)**. Determine the vector $^E\mathbf{u}$ and the angle of rotation θ that transforms frame $R_E$ to the location **Rot(y, π/4) Rot(z, π/2)**. We can write:

$$\mathbf{Rot}(\mathbf{x}, -\pi/4)\ \mathbf{Rot}(\mathbf{u}, \theta) = \mathbf{Rot}(\mathbf{y}, \pi/4)\ \mathbf{Rot}(\mathbf{z}, \pi/2)$$

Thus:

$$\mathbf{Rot(u, \theta)} = \mathbf{Rot(x, \pi/4)\ Rot(y, \pi/4)\ Rot(z, \pi/2)}$$

$$= \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/2 & -1/2 & 0 \\ 1/\sqrt{2} & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using equations [2.34] and [2.36], we get: $C\theta = -\frac{1}{2}$, $S\theta = \frac{\sqrt{3}}{2}$, giving $\theta = 2\pi/3$.

Equation [2.35] yields: $u_x = \frac{1}{\sqrt{3}}$, $u_y = 0$, $u_z = \sqrt{\frac{2}{3}}$.

## 2.4. Kinematic screw

In this section, we will use the concept of screw to describe the velocity of a body in space.

### 2.4.1. *Definition of a screw*

A vector field H on $\Re^3$ is a screw if there exist a point $O_i$ and a vector $\mathbf{\Omega}$ such that for all points $O_j$ in $\Re^3$:

$$\mathbf{H}_j = \mathbf{H}_i + \mathbf{\Omega} \times \mathbf{O}_i\mathbf{O}_j$$

where $\mathbf{H}_j$ is the vector of H at $O_j$ and the symbol x indicates the vector product; $\mathbf{\Omega}$ is called the vector of the *screw* of H.

Then, it is easy to prove that for every couple of points $O_k$ and $O_m$:

$$\mathbf{H}_m = \mathbf{H}_k + \mathbf{\Omega} \times \mathbf{O}_k\mathbf{O}_m$$

Thus, the screw at a point $O_i$ is well defined by the vectors $\mathbf{H}_i$ and $\mathbf{\Omega}$, which can be concatenated in a single (6x1) vector.

### 2.4.2. *Representation of velocity (kinematic screw)*

Since the set of velocity vectors at all the points of a body defines a screw field, the screw at a point $O_i$ can be defined by:

- $\mathbf{V}_i$ representing the linear velocity at $O_i$ with respect to the fixed frame $R_0$, such that $\mathbf{V}_i = \frac{d}{dt}(\mathbf{O}_0\mathbf{O}_i)$;
- $\boldsymbol{\omega}_i$ representing the angular velocity of the body with respect to frame $R_0$. It constitutes the vector of the screw of the velocity vector field.

Thus, the velocity of a point $O_j$ is calculated in terms of the velocity of the point $O_i$ by the following equation:

$$\mathbf{V}_j = \mathbf{V}_i + \boldsymbol{\omega}_i \times \mathbf{O}_i\mathbf{O}_j \qquad [2.40]$$

The components of $\mathbf{V}_i$ and $\boldsymbol{\omega}_i$ can be concatenated to form the *kinematic screw* vector $\mathbb{V}_i$, i.e.:

$$\mathbb{V}_i = [\ \mathbf{V}_i^T \ \ \boldsymbol{\omega}_i^T\ ]^T \qquad [2.41]$$

The kinematic screw is also called *twist* or *spatial velocity.*

### 2.4.3. *Transformation of screws*

Let ${}^i\mathbf{V}_i$ and ${}^i\boldsymbol{\omega}_i$ be the vectors representing the kinematic screw in $O_i$, origin of frame $R_i$, expressed in frame $R_i$. To calculate ${}^j\mathbf{V}_j$ and ${}^j\boldsymbol{\omega}_j$ representing the kinematic screw in $O_j$ expressed in frame $R_j$, we first note that:

$$\boldsymbol{\omega}_j = \boldsymbol{\omega}_i \qquad [2.42]$$

$$\mathbf{V}_j = \mathbf{V}_i + \boldsymbol{\omega}_i \times \mathbf{L}_{i,j} \qquad [2.43]$$

$\mathbf{L}_{i,j}$ being the position vector connecting $O_i$ to $O_j$.

Equations [2.42] and [2.43] can be rewritten as:

$$\begin{bmatrix} \mathbf{V}_j \\ \boldsymbol{\omega}_j \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & -\hat{\mathbf{L}}_{i,j} \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_i \\ \boldsymbol{\omega}_i \end{bmatrix} \qquad [2.44]$$

where $\mathbf{I}_3$ and $\mathbf{0}_3$ represent the (3x3) identity matrix and zero matrix respectively. Projecting this relation in frame $R_i$, we obtain:

$$\begin{bmatrix} {}^i\mathbf{V}_j \\ {}^i\omega_j \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & -{}^i\hat{\mathbf{P}}_j \\ \mathbf{0}_3 & \mathbf{I}_3 \end{bmatrix} \begin{bmatrix} {}^i\mathbf{V}_i \\ {}^i\omega_i \end{bmatrix} \qquad [2.45]$$

Since ${}^j\mathbf{V}_j = {}^j\mathbf{A}_i\,{}^i\mathbf{V}_j$ and ${}^j\omega_j = {}^j\mathbf{A}_i\,{}^i\omega_j$, equation [2.45] gives:

$$ {}^j\mathbb{V}_j = {}^j\mathbb{T}_i\,{}^i\mathbb{V}_i \qquad [2.46]$$

where ${}^j\mathbb{T}_i$ is the (6x6) transformation matrix between screws:

$$ {}^j\mathbb{T}_i = \begin{bmatrix} {}^j\mathbf{A}_i & -{}^j\mathbf{A}_i\,{}^i\hat{\mathbf{P}}_j \\ \mathbf{0}_3 & {}^j\mathbf{A}_i \end{bmatrix} \qquad [2.47]$$

The transformation matrices between screws have the following properties:

*i) product:*

$$ {}^0\mathbb{T}_j = {}^0\mathbb{T}_1\,{}^1\mathbb{T}_2 \ldots {}^{j-1}\mathbb{T}_j \qquad [2.48]$$

*ii) inverse:*

$$ {}^j\mathbb{T}_i^{-1} = \begin{bmatrix} {}^i\mathbf{A}_j & {}^i\hat{\mathbf{P}}_j\,{}^i\mathbf{A}_j \\ \mathbf{0}_3 & {}^i\mathbf{A}_j \end{bmatrix} = {}^i\mathbb{T}_j \qquad [2.49]$$

Note that equation [2.49] gives another possibility, other than equation [2.45], to define the transformation matrix between screws.

## 2.5. Differential translation and rotation of frames

The differential transformation of the position and orientation – or location – of a frame $R_i$ attached to any body may be expressed by a differential translation vector $\mathbf{dP}_i$ expressing the translation of the origin of frame $R_i$, and of a differential rotation vector $\boldsymbol{\delta}_i$, equal to $\mathbf{u}_i\, d\theta$, representing the rotation of an angle $d\theta$ about an axis, with unit vector $\mathbf{u}_i$, passing through the origin $O_i$.

Given a transformation ${}^{i}\mathbf{T}_j$, the transformation ${}^{i}\mathbf{T}_j + \mathbf{d}^{i}\mathbf{T}_j$ can be calculated, taking into account the property h of § 2.3.6, by the premultiplication rule as:

$$ {}^{i}\mathbf{T}_j + \mathbf{d}^{i}\mathbf{T}_j = \mathbf{Trans}({}^{i}dx_i, {}^{i}dy_i, {}^{i}dz_i)\, \mathbf{Rot}({}^{i}\mathbf{u}_i, d\theta)\, {}^{i}\mathbf{T}_j \qquad [2.50] $$

Thus, the differential of ${}^{i}\mathbf{T}_j$ is equal to:

$$ \mathbf{d}^{i}\mathbf{T}_j = [\mathbf{Trans}({}^{i}dx_i, {}^{i}dy_i, {}^{i}dz_i)\, \mathbf{Rot}({}^{i}\mathbf{u}_i, d\theta) - \mathbf{I}_4]\, {}^{i}\mathbf{T}_j \qquad [2.51] $$

In the same way, the transformation ${}^{i}\mathbf{T}_j + \mathbf{d}^{i}\mathbf{T}_j$ can be calculated, using the postmultiplication rule as:

$$ {}^{i}\mathbf{T}_j + \mathbf{d}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\, \mathbf{Trans}({}^{j}dx_j, {}^{j}dy_j, {}^{j}dz_j)\, \mathbf{Rot}({}^{j}\mathbf{u}_j, d\theta) \qquad [2.52] $$

and the differential of ${}^{i}\mathbf{T}_j$ becomes:

$$ \mathbf{d}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\, [\mathbf{Trans}({}^{j}dx_j, {}^{j}dy_j, {}^{j}dz_j)\, \mathbf{Rot}({}^{j}\mathbf{u}_j, d\theta) - \mathbf{I}_4] \qquad [2.53] $$

From equations [2.51] and [2.53], the differential transformation matrix $\mathbf{\Delta}$ is defined as [Paul 81]:

$$ \mathbf{\Delta} = [\mathbf{Trans}(dx, dy, dz)\, \mathbf{Rot}(\mathbf{u}, d\theta) - \mathbf{I}_4] \qquad [2.54] $$

such that:

$$ \mathbf{d}^{i}\mathbf{T}_j = {}^{i}\mathbf{\Delta}\, {}^{i}\mathbf{T}_j \qquad [2.55] $$

or:

$$ \mathbf{d}^{i}\mathbf{T}_j = {}^{i}\mathbf{T}_j\, {}^{j}\mathbf{\Delta} \qquad [2.56] $$

Assuming that $d\theta$ is sufficiently small so that $S(d\theta) \approx d\theta$ and $C(d\theta) \approx 1$, the transformation matrix of a pure rotation $d\theta$ about an axis of unit vector $\mathbf{u}$ can be calculated from equations [2.30] and [2.54] as:

$$ {}^{j}\mathbf{\Delta} = \begin{bmatrix} & {}^{j}\hat{\mathbf{\delta}}_j & & {}^{j}\mathbf{dP}_j \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} & {}^{j}\hat{\mathbf{u}}_j\, d\theta & & {}^{j}\mathbf{dP}_j \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad [2.57] $$

where $\hat{\mathbf{u}}$ and $\hat{\mathbf{\delta}}$ represent the skew-symmetric matrices defined by the vectors $\mathbf{u}$ and $\mathbf{\delta}$ respectively.

Note that the transformation matrix between screws can also be used to transform the differential translation and rotation vectors between frames:

$$\begin{bmatrix} {}^{j}\mathbf{dP}_j \\ {}^{j}\delta_j \end{bmatrix} = {}^{j}\mathbb{T}_i \begin{bmatrix} {}^{i}\mathbf{dP}_i \\ {}^{i}\delta_i \end{bmatrix} \qquad [2.58]$$

In a similar way as for the kinematic screw, we call the concatenation of $\mathbf{dP}_i$ and $\delta_i$ the *differential screw*.

• **Example 2.4.** Consider using the differential model of a robot to control its displacement. The differential model calculates the joint increments corresponding to the desired elementary displacement of frame $R_n$ fixed to the terminal link (Figure 2.11). However, the task of the robot is often described in the tool frame $R_E$, which is also fixed to the terminal link. The problem is to calculate ${}^{n}\mathbf{dP}_n$ and ${}^{n}\delta_n$ in terms of ${}^{E}\mathbf{dP}_E$ and ${}^{E}\delta_E$.

Let the transformation describing the tool frame in frame $R_n$ be:

$${}^{n}T_E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0.1 \\ 0 & 0 & 1 & -0.3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and that the value of the desired elementary displacement is:

$${}^{E}\mathbf{dP}_E = [\ 0 \quad 0 \quad -0.01\ ]^T,\ {}^{E}\delta_E = [\ 0 \quad -0.05 \quad 0\ ]^T$$

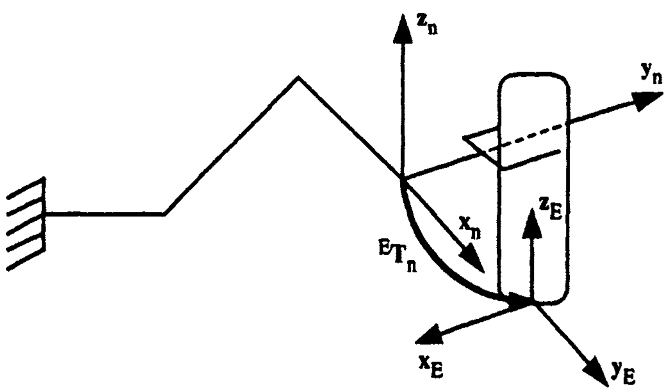


**Figure 2.11.** *Example 2.4*

Using equation [2.58], we obtain:

$$^{n}\boldsymbol{\delta}_n = {}^{n}\mathbf{A}_E \, {}^{E}\boldsymbol{\delta}_E \,, \quad {}^{n}\mathbf{dP}_n = {}^{n}\mathbf{A}_E \, ({}^{E}\boldsymbol{\delta}_E \mathrm{x} {}^{E}\mathbf{P}_n + {}^{E}\mathbf{dP}_E)$$

The numerical application gives:

$$^{n}\mathbf{dP}_n = [\ 0 \quad 0.015 \quad -0.005\ ]^T, \ {}^{n}\boldsymbol{\delta}_n = [\ -0.05 \quad 0 \quad 0\ ]^T$$

In a similar way, we can evaluate the error in the location of the tool frame due to errors in the position and orientation of the terminal frame. Suppose that the position error is equal to 10 mm in all directions and that the rotation error is estimated as 0.01 radian about the **x** axis:

$$^{n}\mathbf{dP}_n = [\ 0.01 \quad 0.01 \quad 0.01\ ]^T, \ {}^{n}\boldsymbol{\delta}_n = [\ 0.01 \quad 0 \quad 0\ ]^T$$

The error on the tool frame is calculated by:

$$^{E}\boldsymbol{\delta}_E = {}^{E}\mathbf{A}_n \, {}^{n}\boldsymbol{\delta}_n, \quad {}^{E}\mathbf{dP}_E = {}^{E}\mathbf{A}_n \, ({}^{n}\boldsymbol{\delta}_n \mathrm{x} {}^{n}\mathbf{P}_E + {}^{n}\mathbf{dP}_n)$$

which results in:

$$^{E}\mathbf{dP}_E = [\ -0.013 \quad 0.01 \quad 0.011\ ]^T, \ {}^{E}\boldsymbol{\delta}_E = [\ 0 \quad 0.01 \quad 0\ ]^T$$

## 2.6. Representation of forces (wrench)

A collection of forces and moments acting on a body can be reduced to a *wrench* $\mathbb{f}_i$ at point $O_i$, which is composed of a force $\mathbf{f}_i$ at $O_i$ and a moment $\mathbf{m}_i$ about $O_i$:

$$\mathbb{f}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{m}_i \end{bmatrix} \qquad [2.59]$$

Note that the vector field of the moments constitutes a screw where the vector of the screw is $\mathbf{f}_i$. Thus, the wrench forms a screw.

Consider a given wrench $^{i}\mathbb{f}_i$, expressed in frame $R_i$. For calculating the equivalent wrench $^{j}\mathbb{f}_j$, we use the transformation matrix between screws such that:

$$\begin{bmatrix} {}^{j}\mathbf{m}_j \\ {}^{j}\mathbf{f}_j \end{bmatrix} = {}^{j}\mathbb{T}_i \begin{bmatrix} {}^{i}\mathbf{m}_i \\ {}^{i}\mathbf{f}_i \end{bmatrix} \qquad [2.60]$$

which gives:

$$^{j}\mathbf{f}_j = {}^{j}\mathbf{A}_i\,{}^{i}\mathbf{f}_i \qquad [2.61]$$
$$^{j}\mathbf{m}_j = {}^{j}\mathbf{A}_i\,({}^{i}\mathbf{f}_i \mathrm{x}\, {}^{i}\mathbf{P}_j + {}^{i}\mathbf{m}_i) \qquad [2.62]$$

It is often more practical to permute the order of $\mathbf{f}_i$ and $\mathbf{m}_i$. In this case, equation [2.60] becomes:

$$\begin{bmatrix} {}^{j}\mathbf{f}_j \\ {}^{j}\mathbf{m}_j \end{bmatrix} = {}^{i}\mathbb{T}_j^{T} \begin{bmatrix} {}^{i}\mathbf{f}_i \\ {}^{i}\mathbf{m}_i \end{bmatrix} \qquad [2.63]$$

• **Example 2.5.** Let the transformation matrix $^{n}\mathbf{T}_E$ describing the location of the tool frame with respect to the terminal frame be:

$$^{n}\mathbf{T}_E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0.1 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Supposing that we want to exert a wrench $^{E}\mathbb{f}_E$ with this tool such that $^{E}\mathbf{f}_E = [0\ \ 0\ \ 5]^T$ and $^{E}\mathbf{m}_E = [0\ \ 0\ \ 3]^T$, determine the corresponding wrench $^{n}\mathbb{f}_n$ at the origin $O_n$ and referred to frame $R_n$. Using equations [2.61] and [2.62], it follows that:

$$^{n}\mathbf{f}_n = {}^{n}\mathbf{A}_E\,{}^{E}\mathbf{f}_E$$
$$^{n}\mathbf{m}_n = {}^{n}\mathbf{A}_E\,({}^{E}\mathbf{f}_E \mathrm{x}\, {}^{E}\mathbf{P}_n + {}^{E}\mathbf{m}_E)$$

The numerical application leads to:

$$^{n}\mathbf{f}_n = [\ 0\quad 0\quad 0.5\ ]^T$$
$$^{n}\mathbf{m}_n = [\ 0.5\quad 0\quad 3\ ]^T$$

## 2.7. Conclusion

In the first part of this chapter, we have developed the homogeneous transformation matrix. This notation constitutes the basic tool for the modeling of robots and their environment. Other techniques have been used in robotics: quaternion [Yang 66], [Castelain 86], (3x3) rotation matrices [Coiffet 81] and the Rodrigues formulation [Wang 83]. Readers interested in these techniques can consult the given references.

We have also recalled some definitions about screws, and transformation matrices between screws, as well as differential transformations. These concepts will be used extensively in this book. In the following chapter, we deal with the problem of robot modeling.