Chapter 4

Inverse geometric model of serial robots

4.1. Introduction

The direct geometric model of a robot provides the location of the end-effector in terms of the joint coordinates. The problem of computing the joint variables corresponding to a specified location of the end-effector is called the inverse geometric problem. This problem is at the center of computer control algorithms for robots. It has in general a multiple solution and its complexity is highly dependent on the geometry of the robot. The model that gives all the possible solutions for this problem is called the *Inverse Geometric Model* (IGM). In this chapter, we will present three methods to obtain the IGM of serial robots. First, we present the Paul method [Paul 81], which can be used to obtain an explicit solution for robots with relatively simple geometry that have many zero distances and parallel or perpendicular joint axes. Then, we develop a variation on the Pieper method [Pieper 68], which provides the analytical solution for the IGM of six degree-of-freedom robots with three prismatic joints or three revolute joints whose axes intersect at a point. Finally, we expose the Raghavan-Roth method [Raghavan 90], which gives the IGM for six degree-of-freedom robots with general (arbitrary) geometry using, at most, a sixteen degree polynomial.

When the inverse geometric model cannot be obtained or if it is difficult to implement in real time applications, iterative numerical techniques can be used. For this purpose, several algorithms can be found in the literature [Grudić 93]. Most of these algorithms use either the Newton-Raphson-based method [Pieper 68], [Goldenberg 85] or inverse Jacobian-based methods [Pieper 68], [Whitney 69], [Fournier 80], [Featherstone 83a]. In Chapter 6, we will present the second technique.

4.2. Mathematical statement of the problem

Let ${}^{f}\mathbf{T}_{E}^{d}$ be the homogeneous transformation matrix representing the desired location of the tool frame R_E relative to the world frame. In general, we can express ${}^{f}\mathbf{T}_{E}^{d}$ in the following form (§ 3.5):

$$ {}^{f}\mathbf{T}_{E}^{d} = \mathbf{Z}\, {}^{0}\mathbf{T}_{n}(\mathbf{q})\, \mathbf{E} \quad [4.1] $$

where (Figure 3.5):

- $\mathbf{Z}$ is the transformation matrix defining the location of the robot (frame R_0) relative to the world frame;
- ${}^{0}\mathbf{T}_{n}$ is the transformation matrix of the terminal frame R_n relative to frame R_0. It is a function of the joint variable vector $\mathbf{q}$;
- $\mathbf{E}$ is the transformation matrix defining the tool frame R_E relative to R_n.

Putting all the known matrices of relation [4.1] on the left side leads to:

$$ \mathbf{U}_0 = {}^{0}\mathbf{T}_{n}(\mathbf{q}) \quad [4.2] $$

with $\mathbf{U}_0 = \mathbf{Z}^{-1}\, {}^{f}\mathbf{T}_{E}^{d}\, \mathbf{E}^{-1}$

The problem is composed of a set of twelve nonlinear equations of n unknowns. The regular case has a finite number of solutions, whereas for redundant robots or in some singular configurations we obtain an infinite number of solutions. When the desired location is outside the reachable workspace there is no solution.

We say that a robot manipulator is *solvable* [Pieper 68], [Roth 76] when it is possible to compute all the joint configurations corresponding to a given location of the end-effector. Now, all non-redundant manipulators can be considered to be solvable [Lee 88], [Raghavan 90]. The number of solutions depends on the architecture of the robot manipulator and the amplitude of the joint limits. For six degree-of-freedom robots with only revolute joints (6R), or having five revolute joints and one prismatic joint (5R1P), the maximum number of solutions is sixteen. When the robot has three revolute joints whose axes intersect at a point, the maximum number of solutions is eight. For the 3P3R robots, this number reduces to two. In all cases, it decreases when the geometric parameters take certain particular values.

Robots with less than six degrees of freedom are not able to place the end-effector frame in an arbitrary location. Thus, we only specify the task in terms of placing some elements of the tool frame (points, axes) in the world frame. Under these conditions, the matrix $\mathbf{E}$ is not completely defined, and the equation to solve is given by:

$$Z^{-1}\ {}^{f}T_E^{d} = {}^{0}T_n(q)\ E \qquad [4.3]$$

4.3. Inverse geometric model of robots with simple geometry

For robots with simple geometry, where most of the distances (r_j and d_j) are zero and most of the angles (θ_j and α_j) are zero or $\pm\pi/2$, the inverse geometric model can be analytically obtained using the Paul method [Paul 81]. Most commercially available robots can be solved using this method.

4.3.1. *Principle*

Let us consider a robot manipulator whose transformation matrix has the expression:

$${}^{0}T_n = {}^{0}T_1(q_1)\ {}^{1}T_2(q_2) \ldots {}^{n-1}T_n(q_n) \qquad [4.4]$$

Let U_0 be the desired location such that:

$$U_0 = \begin{bmatrix} s_x & n_x & a_x & P_x \\ s_y & n_y & a_y & P_y \\ s_z & n_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [4.5]$$

The IGM is obtained by solving the following equation:

$$U_0 = {}^{0}T_1(q_1)\ {}^{1}T_2(q_2) \ldots {}^{n-1}T_n(q_n) \qquad [4.6]$$

To find the solutions of this equation, Paul [Paul 81] proposed to move each joint variable to the left side one after the other by successively premultiplying equation [4.6] by ${}^{j}T_{j-1}$, for j varying from 1 to n – 1. Then, the joint variables are determined by equating the elements of the two sides of each equation. For example, for a six degree-of-freedom robot, we proceed as follows:

– premultiply equation [4.6] by ${}^{1}T_0$:

$${}^{1}T_0\ U_0 = {}^{1}T_2\ {}^{2}T_3\ {}^{3}T_4\ {}^{4}T_5\ {}^{5}T_6 \qquad [4.7]$$

The elements of the left side are constants or functions of q_1. The elements of the right side are constants or functions of $q_2, \ldots, q_6$;

- try to solve q_1 by equating the elements of the two sides of equation [4.7];
- premultiply equation [4.7] by ${}^2\mathbf{T}_1$ and try to determine q_2;
- continue the process until all the variables are solved.

In summary, the equations used to obtain all the joint variables are written as:

$$\begin{aligned}
\mathbf{U}_0 &= {}^0\mathbf{T}_1\,{}^1\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4\,{}^4\mathbf{T}_5\,{}^5\mathbf{T}_6 \\
{}^1\mathbf{T}_0\,\mathbf{U}_0 &= {}^1\mathbf{T}_2\,{}^2\mathbf{T}_3\,{}^3\mathbf{T}_4\,{}^4\mathbf{T}_5\,{}^5\mathbf{T}_6 \\
{}^2\mathbf{T}_1\,\mathbf{U}_1 &= {}^2\mathbf{T}_3\,{}^3\mathbf{T}_4\,{}^4\mathbf{T}_5\,{}^5\mathbf{T}_6 \\
{}^3\mathbf{T}_2\,\mathbf{U}_2 &= {}^3\mathbf{T}_4\,{}^4\mathbf{T}_5\,{}^5\mathbf{T}_6 \\
{}^4\mathbf{T}_3\,\mathbf{U}_3 &= {}^4\mathbf{T}_5\,{}^5\mathbf{T}_6 \\
{}^5\mathbf{T}_4\,\mathbf{U}_4 &= {}^5\mathbf{T}_6
\end{aligned} \qquad [4.8]$$

with $\mathbf{U}_j = {}^j\mathbf{T}_6 = {}^j\mathbf{T}_{j-1}\,\mathbf{U}_{j-1}$

The resolution of equations [4.8] needs intuition, but the use of this method on a large number of industrial robots has shown that only few fundamental types of equations are encountered [Khalil 86b] (Table 4.1). The solutions of these equations are given in Appendix 1.

NOTES.–

- the matrices of the right side of equations [4.8] are already available when computing the direct geometric model (DGM) if the multiplication of the transformation matrices is started from the end of the robot;
- in certain cases, it may be more convenient to solve the robot by first determining q_n and ending with q_1. In this case, we postmultiply equation [4.6] by ${}^j\mathbf{T}_{j-1}$ for j varying from n to 2.

Table 4.1. *Types of equations encountered in the Paul method*

Type 1	$X\, r_i = Y$
Type 2	$X\, S\theta_i + Y\, C\theta_i = Z$
Type 3	$X1\, S\theta_i + Y1\, C\theta_i = Z1$ $X2\, S\theta_i + Y2\, C\theta_i = Z2$
Type 4	$X1\, r_j\, S\theta_i = Y1$ $X2\, r_j\, C\theta_i = Y2$
Type 5	$X1\, S\theta_i = Y1 + Z1\, r_j$ $X2\, C\theta_i = Y2 + Z2\, r_j$
Type 6	$W\, S\theta_j = X\, C\theta_i + Y\, S\theta_i + Z1$ $W\, C\theta_j = X\, S\theta_i - Y\, C\theta_i + Z2$
Type 7	$W1\, C\theta_j + W2\, S\theta_j = X\, C\theta_i + Y\, S\theta_i + Z1$ $W1\, S\theta_j - W2\, C\theta_j = X\, S\theta_i - Y\, C\theta_i + Z2$
Type 8	$X\, C\theta_i + Y\, C(\theta_i + \theta_j) = Z1$ $X\, S\theta_i + Y\, S(\theta_i + \theta_j) = Z2$

r_i: prismatic joint variable,
$S\theta_i$, $C\theta_i$: sine and cosine of a revolute joint variable θ_i.

4.3.2. *Special case: robots with a spherical wrist*

Most six degree-of-freedom industrial robots have a spherical wrist composed of three revolute joints whose axes intersect at a point (Figure 4.1). This structure is characterized by the following set of geometric parameters:

$$\begin{cases} d_5 = r_5 = d_6 = 0 \\ \sigma_4 = \sigma_5 = \sigma_6 = 0 \\ S\alpha_5 \neq 0,\ S\alpha_6 \neq 0 \text{ (non-redundant robot)} \end{cases}$$

The position of the center of the spherical joint is obtained as a function of the joint variables q_1, q_2 and q_3. This type of structure allows the decomposition of the six degree-of-freedom problem into two three degree-of-freedom problems representing a position equation and an orientation equation. The position problem, which is a function of q_1, q_2 and q_3, is first solved, then the orientation problem allows us to determine θ_4, θ_5, θ_6.

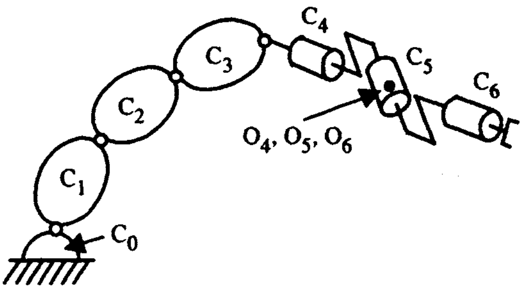


Figure 4.1. *Six degree-of-freedom robot with a spherical wrist*

4.3.2.1. *Position equation*

Since ${}^0\mathbf{P}_6 = {}^0\mathbf{P}_4$, the fourth column of the transformation matrix ${}^0\mathbf{T}_4$ is equal to the fourth column of $\mathbf{U}_0$:

$$\begin{bmatrix} Px \\ Py \\ Pz \\ 1 \end{bmatrix} = {}^0\mathbf{T}_1 {}^1\mathbf{T}_2 {}^2\mathbf{T}_3 {}^3\mathbf{T}_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad [4.9]$$

We obtain the variables q_1, q_2, q_3 by successively premultiplying this equation by ${}^j\mathbf{T}_0$, $j = 1, 2$, to isolate and determine sequentially the joint variables. The elements of the right side have already been calculated for the DGM.

4.3.2.2. *Orientation equation*

The orientation part of equation [4.2] is written as:

$$[\ \mathbf{s} \quad \mathbf{n} \quad \mathbf{a}\ ] = {}^0\mathbf{A}_6(\mathbf{q})$$

yielding:

$${}^3\mathbf{A}_0(q_1, q_2, q_3)\,[\ \mathbf{s} \quad \mathbf{n} \quad \mathbf{a}\ ] = {}^3\mathbf{A}_6(\theta_4, \theta_5, \theta_6)$$

which can be written as:

$$[\ \mathbf{F} \quad \mathbf{G} \quad \mathbf{H}\ ] = {}^3\mathbf{A}_6(\theta_4, \theta_5, \theta_6) \qquad [4.10]$$

Since q_1, q_2, q_3 have been determined, the left side elements are considered to be known. To obtain θ_4, θ_5, θ_6, we successively premultiply equation [4.10] by ${}^4\mathbf{A}_3$

then by ${}^5A_4$ and proceed by equating the elements of the two sides. Again, the elements of the right side have already been calculated for the DGM.

• **Example 4.1.** IGM of the Stäubli RX-90 robot. The geometric parameters are given in Table 3.1. The robot has a spherical wrist. The DGM is developed in Chapter 3.

a) Computation of $\theta_1, \theta_2, \theta_3$

i) by developing equation [4.9], we obtain:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = \begin{bmatrix} C1\,(-S23\,RL4 + C2\,D3) \\ S1\,(-S23\,RL4 + C2\,D3) \\ C23\,RL4 + S2\,D3 \\ 1 \end{bmatrix}$$

Note that the elements of the right side constitute the fourth column of ${}^0\mathbf{T}_6$, which have already been calculated for the DGM. No variable can be determined from this equation;

ii) premultiplying the previous equation by ${}^1\mathbf{T}_0$, we obtain the left side elements as:

$$\begin{aligned} U(1) &= C1\,P_x + S1\,P_y \\ U(2) &= -\,S1\,P_x + C1\,P_y \\ U(3) &= P_z \end{aligned}$$

The elements of the right side are obtained from the fourth column of ${}^1\mathbf{T}_6$:

$$\begin{aligned} T(1) &= -\,S23\,RL4 + C2\,D3 \\ T(2) &= 0 \\ T(3) &= C23\,RL4 + S2\,D3 \end{aligned}$$

By equating U(2) and T(2), we obtain the following two solutions for θ_1:

$$\begin{cases} \theta_1 = \operatorname{atan2}(P_y, P_x) \\ \theta'_1 = \theta_1 + \pi \end{cases}$$

iii) premultiplying by $^2\mathbf{T}_1$, we obtain the elements of the left side as:

$$
\begin{aligned}
U(1) &= C2\,(C1\,P_x + S1\,P_y) + S2\,P_z \\
U(2) &= -S2\,(C1\,P_x + S1\,P_y) + C2\,P_z \\
U(3) &= S1\,P_x - C1\,P_y
\end{aligned}
$$

The elements of the right side represent the fourth column of $^2\mathbf{T}_6$:

$$
\begin{aligned}
T(1) &= -S3\,RL4 + D3 \\
T(2) &= C3\,RL4 \\
T(3) &= 0
\end{aligned}
$$

We determine θ_2 and θ_3 by considering the first two elements, which constitute a type-6 system of equations (Table 4.1). First, an equation in θ_2 is obtained:

$$X\,S2 + Y\,C2 = Z$$

with:

$$
\begin{aligned}
X &= -2P_z\,D3 \\
Y &= -2\,B1D3 \\
Z &= (RL4)^2 - (D3)^2 - (P_z)^2 - (B1)^2 \\
B1 &= P_x\,C1 + P_y\,S1
\end{aligned}
$$

from which we deduce that:

$$
\begin{cases}
C2 = \dfrac{YZ - \varepsilon X\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2} \\[2ex]
S2 = \dfrac{XZ + \varepsilon Y\sqrt{X^2 + Y^2 - Z^2}}{X^2 + Y^2}
\end{cases}
\qquad \text{with } \varepsilon = \pm 1
$$

This gives two solutions of the following form:

$$\theta_2 = \operatorname{atan2}(S2, C2)$$

θ_2 being known, we obtain:

$$\theta_3 = \operatorname{atan2}(S3, C3)$$

with:

$$\begin{cases} S3 = \dfrac{-\,Pz\,S2 - B1C2 + D3}{RL4} \\ C3 = \dfrac{-\,B1S2 + Pz\,C2}{RL4} \end{cases}$$

b) Computation of θ_4, θ_5, θ_6

Once the variables θ_1, θ_2, θ_3 are determined, we define the (3×3) orientation matrix ${}^3A_6$ as follows:

$$[\,\mathbf{F} \quad \mathbf{G} \quad \mathbf{H}\,] = {}^3A_0\,[\,\mathbf{s} \quad \mathbf{n} \quad \mathbf{a}\,]$$

The elements of **F** are written as:

$$\begin{aligned} U(1,1) &= C23\,(C1\,s_x + S1\,s_y) + S23\,s_z \\ U(2,1) &= -\,S23\,(C1\,s_x + S1\,s_y) + C23\,s_z \\ U(3,1) &= S1\,s_x - C1\,s_y \end{aligned}$$

The elements of **G** and **H** are obtained from **F** by replacing (s_x, s_y, s_z) by (n_x, n_y, n_z) and (a_x, a_y, a_z) respectively.

i) equating the elements of $[\,\mathbf{F} \quad \mathbf{G} \quad \mathbf{H}\,] = {}^3A_6$

The elements of ${}^3A_6$ are obtained from ${}^3T_6$, which is calculated for the DGM:

$${}^3A_6 = \begin{bmatrix} C6C5C4-S6S4 & -S6C5C4-C6S4 & -S5C4 \\ C6S5 & -S6S5 & C5 \\ -C6C5S4-S6C4 & S6C5S4-C6C4 & S5S4 \end{bmatrix}$$

We can determine θ_5 from the (2, 3) elements using an arccosine function. But this solution is not retained, considering that another one using an atan2 function may appear in the next equations;

ii) equating the elements of ${}^4A_3\,[\,\mathbf{F} \quad \mathbf{G} \quad \mathbf{H}\,] = {}^4A_6$

The elements of the first column of the left side are written as:

$$\begin{aligned} U(1,1) &= C4\,F_x - S4\,F_z \\ U(2,1) &= -\,C4\,F_z - S4\,F_x \\ U(3,1) &= F_y \end{aligned}$$

The elements of the second and third columns are obtained by replacing (F_x, F_y, F_z) with (G_x, G_y, G_z) and (H_x, H_y, H_z) respectively. The elements of ${}^4A_6$ are obtained from ${}^4T_6$, which has already been calculated for the DGM:

$$ {}^4A_6 = \begin{bmatrix} C6C5 & -S6C5 & -S5 \\ S6 & C6 & 0 \\ C6S5 & -S6S5 & C5 \end{bmatrix} $$

From the (2, 3) elements, we obtain a type-2 equation in θ_4:

$$ -C4\,H_z - S4\,H_x = 0 $$

which gives two solutions:

$$ \begin{cases} \theta_4 = \operatorname{atan2}(H_z, -H_x) \\ \theta'_4 = \theta_4 + \pi \end{cases} $$

From the (1, 3) and (3, 3) elements, we obtain a type-3 system of equations in θ_5:

$$ \begin{aligned} -S5 &= C4\,H_x - S4\,H_z \\ C5 &= H_y \end{aligned} $$

whose solution is:

$$ \theta_5 = \operatorname{atan2}(S5, C5) $$

Finally, by considering the (2, 1) and (2, 2) elements, we obtain a type-3 system of equations in θ_6:

$$ \begin{aligned} S6 &= -C4\,F_z - S4\,F_x \\ C6 &= -C4\,G_z - S4\,G_x \end{aligned} $$

whose solution is:

$$ \theta_6 = \operatorname{atan2}(S6, C6) $$

NOTES.– By examining the IGM solution of the Stäubli RX-90 robot, it can be observed that:

a) The robot has the following singular positions:

i) shoulder singularity: takes place when the point O_6 lies on the z_0 axis (Figure 4.2a). Thus $P_x = P_y = 0$, which corresponds to $S23RL4 - C2D3 = 0$. In this case, both the two arguments of the atan2 function used to determine θ_1 are zero, thus leaving θ_1 undetermined. We are free to choose any value for θ_1, but frequently the current value is assigned. This means that one can always find a solution, but when leaving this configuration, a small change in the desired location may require a significant variation in θ_1, impossible to realize due to the speed and acceleration limits of the actuator;

ii) wrist singularity: takes place when $C23(C1a_x + S1a_y) + S23a_z = H_x = 0$ and $(S1ax - C1ay) = H_z = 0$. The two arguments of the atan2 function used to determine θ_4 are zero. From the (2, 3) element of ${}^3A_6$, we notice that in this case $C\theta_5 = \pm 1$. Thus, the axes of joints 4 and 6 are collinear and it is the sum $\theta_4 \pm \theta_6$ that can be determined (Figure 4.2b). For example, when $\theta_5 = 0$, the orientation equation becomes:

$$[\,F \quad G \quad H\,] = {}^3A_6 = \begin{bmatrix} C46 & -S46 & 0 \\ 0 & 0 & 1 \\ -S46 & -C46 & 0 \end{bmatrix}$$

Thus, $\theta 4 + \theta 6 = \mathrm{atan2}(-G_x, -G_z)$. We can arbitrarily assign θ_4 to its current value and calculate the corresponding θ_6. We can also calculate the values of θ_4 and θ_6 for which the joints 4 and 6 move away from their limits;

iii) elbow singularity: occurs when $C3 = 0$. This singularity will be discussed in Chapter 6. It does not affect the inverse geometric model computation (Figure 4.2c).

b) The above-mentioned singularities are classified as first order singularities. Singularities of higher order may occur when several singularities of first order take place simultaneously.

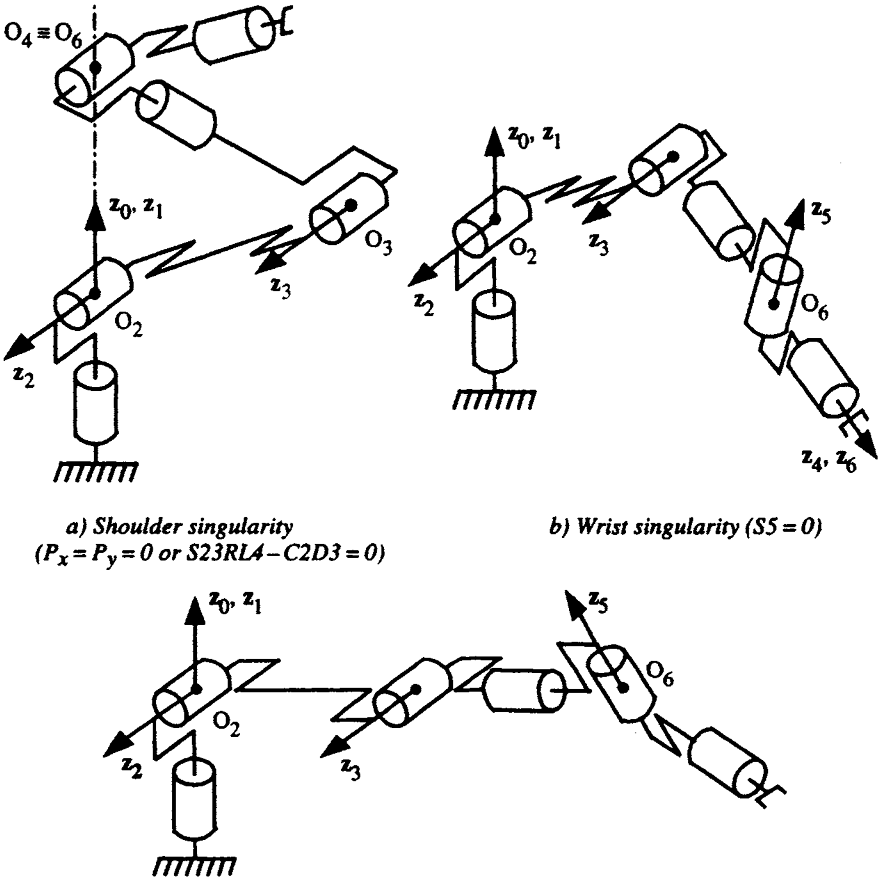
c) Number of solutions: in the regular case, the Stäubli RX-90 robot has eight solutions for the IGM (product of the number of possible solutions for each joint). Some of these configurations may not be accessible because of the joint limits.

4.3.3. *Inverse geometric model of robots with more than six degrees of freedom*

A robot with more than six degrees of freedom is redundant and its inverse geometric problem has an infinite number of solutions. To obtain a closed form solution, $(n-6)$ additional relations are needed. Two strategies are possible:

- arbitrarily fixing $(n - 6)$ joint values to reduce the problem to six unknowns. The selection of the fixed joints is determined by the task specifications and the robot structure;

- introducing (n – 6) additional relations describing the redundancy, as is done in certain seven degree-of-freedom robots [Hollerbach 84b].



a) Shoulder singularity ($P_x = P_y = 0$ or $S23RL4 - C2D3 = 0$)

b) Wrist singularity ($S5 = 0$)

c) Elbow singularity ($C3 = 0$)

Figure 4.2. *Singular positions of the Stäubli RX-90 robot*

4.3.4. *Inverse geometric model of robots with less than six degrees of freedom*

When the robot has less than six degrees of freedom, the end-effector frame R_E cannot be placed at an arbitrary location except if certain elements of ${}^0T_E{}^d$ have specific values to compensate for the missing degrees of freedom. Otherwise, instead of realizing frame-to-frame contact, we consider tasks with less degrees of freedom such as point-to-point contact, or (point-axis) to (point-axis) contact [Manaoui 85].

In the next example, we will study this problem for the four degree-of-freedom SCARA robot whose geometric parameters are given in Table 3.2.

• **Example 4.2.** IGM of the SCARA robot (Figure 3.4).

i) frame-to-frame contact

In this case, the system of equations to be solved is given by equation [4.2] and U_0 is defined by equation [4.5]:

$$U_0 = {}^0T_4 = \begin{bmatrix} C123 & -S123 & 0 & C12D3+C1D2 \\ S123 & C123 & 0 & S12D3+S1D2 \\ 0 & 0 & 1 & r_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Examining the elements of this matrix reveals that frame-to-frame contact is possible if the third column of U_0 is equal to $[0 \quad 0 \quad 1 \quad 0]^T$. This implies two independent conditions, which compensate for the missing degrees of freedom. By equating the (3, 4) elements, we obtain:

$$r_4 = P_z$$

The (1, 4) and (2, 4) elements give a type-8 system of equations in θ_1 and θ_2 with the following solution:

$$\theta_2 = \text{atan2}(\pm\sqrt{1-(C2)^2}, C2)$$
$$\theta_1 = \text{atan2}(S1, C1)$$

with:

$$C2 = \frac{D^2-(D2)^2-(D3)^2}{2\,D2\,D3} \qquad D^2 = P_x{}^2 + P_y{}^2$$

$$S1 = \frac{B1\,P_y - B2\,P_x}{D^2} \qquad C1 = \frac{B1\,P_x + B2\,P_y}{D^2}$$

$$B1 = D2 + D3\,C2, \; B2 = D3\,S2$$

After determining θ_1 and θ_2, we obtain θ_3 as:

$$\theta_3 = \text{atan2}(s_y, s_x) - \theta 2 - \theta 1$$

ii) (point-axis) to (point-axis) contact

Let us suppose that the tool is defined by an axis of unit vector $\mathbf{a}_E$, passing by O_E such that:

$$^4\mathbf{P}_E = [Q_x \quad Q_y \quad Q_z]^T$$
$$^4\mathbf{a}_E = [W_x \quad W_y \quad W_z]^T$$

The task consists of placing the point O_E at a point of the environment while aligning the axis $\mathbf{a}_E$ with an axis of the environment, which are defined by:

$$^0\mathbf{P}_E^d = [P_x \quad P_y \quad P_z]^T$$
$$^0\mathbf{a}_E^d = [a_x \quad a_y \quad a_z]^T$$

The system to be solved is written as:

$$\begin{bmatrix} - & - & a_x & P_x \\ - & - & a_y & P_y \\ - & - & a_z & P_z \\ - & - & 0 & 1 \end{bmatrix} = {}^0\mathbf{T}_4 \begin{bmatrix} - & - & W_x & Q_x \\ - & - & W_y & Q_y \\ - & - & W_z & Q_z \\ - & - & 0 & 1 \end{bmatrix}$$

After simplifying, we obtain:

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} Q_x C123 - Q_y S123 + C12D3 + C1D2 \\ Q_x S123 + Q_y C123 + S12D3 + S1D2 \\ Q_z + r4 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} W_x C123 - W_y S123 \\ W_x S123 + W_y C123 \\ W_z \end{bmatrix}$$

Thus, we deduce that the condition $a_z = W_z$ must be satisfied to realize the task. The IGM solutions are obtained in the following way:

- from the a_x and a_y equations, we obtain $(\theta_1 + \theta_2 + \theta_3)$ by solving a type-3 system (Appendix 1):

$$\theta_1 + \theta_2 + \theta_3 = \text{atan2}(S123, C123)$$

$$\text{with } S123 = \frac{a_y W_x - a_x W_y}{W_x^2 + W_y^2} \text{ and } C123 = \frac{a_x W_x + a_y W_y}{W_x^2 + W_y^2} \text{ if } (W_x^2 + W_y^2) \neq 0;$$

- when $W_x = W_y = 0$, the axis of the end-effector is vertical and its orientation cannot be changed. Any value for θ_3 may be taken;
- from P_x and P_y equations, we obtain θ_1 and θ_2 by solving a type-8 system of equations;
- finally, from the third element of the position equation, we obtain $r_4 = P_z - Q_z$.

In summary, the task of a SCARA robot can be described in one of the following ways:

- placing the tool frame onto a specified frame provided that the third column of the matrix ${}^0T_4^d = {}^0T_E^d \, E^{-1} = [0 \;\; 0 \;\; 1 \;\; 0]^T$, in order to satisfy that z_4 is vertical;
- placing an axis and a point of the tool frame respectively onto an axis and a point of the environment provided that $a_z = W_z$. The obvious particular case is to locate a horizontal axis of the end-effector frame in a horizontal plane ($a_z = W_z = 0$).

4.4. Inverse geometric model of decoupled six degree-of-freedom robots

4.4.1. *Introduction*

The IGM of a six degree-of-freedom decoupled robot can be computed by solving two sub-problems, each having three unknowns [Pieper 68]. Two classes of structures are considered:

a) robots having a spherical joint given by one of the following four combinations: XXX(RRR), (RRR)XXX, XX(RRR)X, X(RRR)XX, where (RRR) denotes a spherical joint and X denotes either a revolute (R) or a prismatic (P) joint. Consequently, each combination results in eight structures;

b) robots having three revolute and three prismatic joints as given by one of the following 20 combinations: PPPRRR, PPRPRR, PPRRPR, ...

In this section, we present the inverse geometric model of these structures using two general equations [Khalil 90c], [Bennis 91a]. These equations make use of the six types of equations shown in Table 4.2. The first three types have already been used in the Paul method (Table 4.1). The explicit solution of a type-10 equation can be obtained symbolically using software packages like Maple or Mathematica. In general, however, the numerical solution is more accurate. We note that a type-11 equation can be transformed into type-10 using the half-angle transformation by writing $C\theta_i$ and $S\theta_i$ as:

$$C\theta_i = \frac{1 - t^2}{1 + t^2} \text{ and } S\theta_i = \frac{2t}{1 + t^2} \text{ with } t = \tan\frac{\theta_i}{2}$$

Table 4.2. *Types of equations encountered in the Pieper method*

Type 1	$X\, r_i = Y$
Type 2	$X\, C\theta_i + Y\, S\theta_i = Z$
Type 3	$X1\, S\theta_i + Y1\, C\theta_i = Z1$ $X2\, S\theta_i + Y2\, C\theta_i = Z2$
Type 9	$a_2\, r_i^2 + a_1\, r_i + a_0 = 0$
Type 10	$a_4\, r_i^4 + a_3\, r_i^3 + a_2\, r_i^2 + a_1\, r_i + a_0 = 0$
Type 11	$a_4\, S\theta_i^2 + a_3\, C\theta_i\, S\theta_i + a_2\, C\theta_i + a_1\, S\theta_i + a_0 = 0$

4.4.2. *Inverse geometric model of six degree-of-freedom robots having a spherical joint*

In this case, equation [4.6] is decoupled into two equations, each containing three variables:

- a position equation, which is a function of the joint variables that do not belong to the spherical joint;
- an orientation equation, which is a function of the joint variables of the spherical joint.

4.4.2.1. *General solution of the position equation*

The revolute joint axes $m-1$, m and $m+1$ ($2 \leq m \leq 5$) form a spherical joint if:

$$\begin{cases} d_m = r_m = d_{m+1} = 0 \\ S\alpha_m \neq 0 \\ S\alpha_{m+1} \neq 0 \end{cases}$$

The position of the center of the spherical joint, O_m or O_{m-1}, is independent of the joint variables θ_{m-1}, θ_m and θ_{m+1}. Thus, we can show (Figure 4.3) that the position of O_m relative to frame R_{m-2} is given by:

$$^{m-2}\mathbf{T}_{m+1}\mathbf{Trans}(z,-r_{m+1})\mathbf{p}_0 = \begin{bmatrix} ^{m-2}\mathbf{P}_{m-1} \\ 1 \end{bmatrix} = \begin{bmatrix} d_{m-1} \\ -r_{m-1}S\alpha_{m-1} \\ r_{m-1}C\alpha_{m-1} \\ 1 \end{bmatrix} \qquad [4.11]$$

where $\mathbf{p}_0 = [\,0 \;\; 0 \;\; 0 \;\; 1\,]^T$ and ${}^{m-2}\mathbf{P}_{m-1}$ is obtained using equation [3.2].

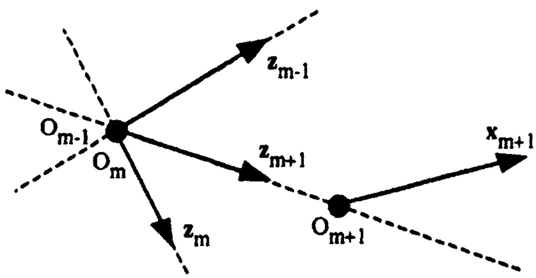


Figure 4.3. *Axes of a spherical joint*

To obtain the position equation, we write equation [4.6] in the following form:

$$ {}^0\mathbf{T}_{m-2}\,{}^{m-2}\mathbf{T}_{m+1}\,{}^{m+1}\mathbf{T}_6 = \mathbf{U}_0 \qquad [4.12] $$

Postmultiplying this relation by ${}^6\mathbf{T}_{m+1}$ **Trans**$(\mathbf{z}, -r_{m+1})\,\mathbf{p}_0$ and using equation [4.11], we obtain:

$$ {}^0\mathbf{T}_{m-2}\begin{bmatrix} {}^{m-2}\mathbf{P}_{m-1} \\ 1 \end{bmatrix} = \mathbf{U}_0\,{}^6\mathbf{T}_{m+1}\,\mathbf{Trans}(\mathbf{z}, -r_{m+1})\,\mathbf{p}_0 \qquad [4.13] $$

Equation [4.13] can be written in the general form:

$$ \mathbf{Rot}(\mathbf{z}, \theta_i)\,\mathbf{Trans}(\mathbf{z}, r_i)\,\mathbf{Rot}(\mathbf{x}, \alpha_j)\,\mathbf{Trans}(\mathbf{x}, d_j)\,\mathbf{Rot}(\mathbf{z}, \theta_j)\,\mathbf{Trans}(\mathbf{z}, r_j)\begin{bmatrix} \mathbf{f}(q_k) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix} \qquad [4.14] $$

where:

- the subscripts i, j and k represent the joints that do not belong to the spherical joint; i and j represent two successive joints;
- the vector **f** is a function of the joint variable q_k;
- the vector **g** is a constant.

By combining the parameters $\bar{q}_i$ and $\bar{q}_j$ with **g** and **f** respectively, equation [4.14] becomes:

$$ \mathbf{Rot/Trans}(\mathbf{z}, q_i)\,\mathbf{Rot}(\mathbf{x}, \alpha_j)\,\mathbf{Trans}(\mathbf{x}, d_j)\,\mathbf{Rot/Trans}(\mathbf{z}, q_j)\begin{bmatrix} \mathbf{F}(q_k) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ 1 \end{bmatrix} \qquad [4.15] $$

with:

- $\mathbf{Rot/Trans(z}, q_i) = \mathbf{Rot(z}, \theta_i)$ if $q_i = \theta_i$
  $= \mathbf{Trans(z}, r_i)$ if $q_i = r_i$

- $$\begin{bmatrix} \mathbf{F}(q_k) \\ 1 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_z \\ 1 \end{bmatrix} = \mathbf{Rot/Trans(z}, \bar{q}_j) \begin{bmatrix} \mathbf{f}(q_k) \\ 1 \end{bmatrix}$$

- $$\begin{bmatrix} \mathbf{G} \\ 1 \end{bmatrix} = \begin{bmatrix} G_x \\ G_y \\ G_z \\ 1 \end{bmatrix} = \mathbf{Rot/Trans(z}, -\bar{q}_i) \begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix}$$

- $\mathbf{Rot/Trans(z}, \bar{q}_i) = \mathbf{Trans(z}, r_i)$ if joint i is revolute
  $= \mathbf{Rot(z}, \theta_i)$ if joint i is prismatic

The components of **G** are constants and those of **F** are functions of the joint variable q_k. We note that if joint k is revolute, then:

$$\|\mathbf{F}\|^2 = a\, C\theta_k + b\, S\theta_k + c \qquad [4.16]$$

where a, b and c are constants.

Table 4.3 shows the equations that are used to obtain the joint variable q_k according to the types of joints i and j (columns 1 and 2). The variables q_i and q_j are then computed using equation [4.15]. Table 4.4 indicates the type of the obtained equations and the maximum number of solutions for each structure; the last column of the table indicates the order in which we calculate them. In Example 4.3, we will develop the solution for the case where joints i and j are revolute. We note that the maximum number of solutions for q_i, q_j and q_k is four.

NOTE.– The assignment of i, j and k for the joints that do not belong to the spherical joint is not unique. In order to get a simple solution for q_k, this assignment can be changed using the concept of the inverse robot (presented in Appendix 2). For instance, if the spherical joint is composed of the joints 4, 5 and 6, we can take i = 1, j = 2, k = 3. But we can also take i = 3, j = 2, k = 1 by using the concept of the inverse robot. We can easily verify that the second choice is more interesting if these joints are revolute and $S\alpha_2 \neq 0$, $d_2 \neq 0$ but $d_3 = 0$ or $S\alpha_3 = 0$.

Table 4.3. *Solutions of q_k and types of equations*

i	j	Conditions	Equations for q_k	Type θ_k	Type r_k
R	R	$S\alpha_j = 0$	$C\alpha_j F_z(q_k) = G_z$	2	1
		$d_j = 0$	$\lVert F\rVert^2 = \lVert G\rVert^2$	2	9
		$d_j \neq 0$ and $S\alpha_j \neq 0$	$\left[\frac{\lVert F\rVert^2 - \lVert G\rVert^2 - d_j^2}{2d_j}\right]^2 + \left[\frac{F_z - C\alpha_j G_z}{S\alpha_j}\right]^2 = G_x^2 + G_y^2$	11	10
R	P	$C\alpha_j = 0$	$F_y(q_k) = S\alpha_j G_z$	2	1
		$C\alpha_j \neq 0$	$(F_x + d_j)^2 + \left[\frac{F_y - S\alpha_j G_z}{C\alpha_j}\right]^2 = G_x^2 + G_y^2$	11	9
P	R	$C\alpha_j = 0$	$G_y = -S\alpha_j F_z(q_k)$	2	1
		$C\alpha_j \neq 0$	$(G_x - d_j)^2 + \left[\frac{G_y + S\alpha_j F_z}{C\alpha_j}\right]^2 = F_x^2 + F_y^2$	11	9
P	P		$F_x + d_j = G_x$	2	1

Table 4.4. *Type of equations and maximum number of solutions for q_i, q_j and q_k*

			Type / Number of solutions				
i	j	Conditions	θ_k	r_k	q_i	q_j	Order
R	R	$S\alpha_j = 0$	2 / 2	1 / 1	3 / 1	2 / 2	θ_j then θ_i
		$d_j = 0$	2 / 2	9 / 2	3 / 1	3 / 2	θ_j then θ_i
		$d_j \neq 0$ and $S\alpha_j \neq 0$	11 / 4	10 / 4	3 / 1	3 / 1	θ_i then θ_j
R	P	$C\alpha_j = 0$	2 / 2	1 / 1	2 / 2	1 / 1	θ_i then r_j
		$C\alpha_j \neq 0$	11 / 4	9 / 2	3 / 1	1 / 1	θ_i then r_j
P	R	$C\alpha_j = 0$	2 / 2	1 / 1	1 / 1	2 / 2	θ_j then r_i
		$C\alpha_j \neq 0$	11 / 4	9 / 2	1 / 1	3 / 1	θ_j then r_i
P	P		2 / 2	1 / 1	1 / 1	1 / 1	r_j then r_i

• **Example 4.3.** Solving q_k when joints i and j are revolute. In this case, equation [4.15] is written as:

$$\mathbf{Rot}(\mathbf{z}, \theta_i)\, \mathbf{Rot}(\mathbf{x}, \alpha_j)\, \mathbf{Trans}(\mathbf{x}, d_j)\, \mathbf{Rot}(\mathbf{z}, \theta_j) \begin{bmatrix} \mathbf{F}(q_k) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ 1 \end{bmatrix} \quad [4.17]$$

Postmultiplying equation [4.17] by $\mathbf{Rot}(\mathbf{z}, -\theta_i)$, we obtain:

$$\begin{bmatrix} C\theta_j & -S\theta_j & 0 & d_j \\ C\alpha_j S\theta_j & C\alpha_j C\theta_j & -S\alpha_j & 0 \\ S\alpha_j S\theta_j & S\alpha_j C\theta_j & C\alpha_j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ 1 \end{bmatrix} = \begin{bmatrix} C\theta_i & S\theta_i & 0 & 0 \\ -S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_x \\ G_y \\ G_z \\ 1 \end{bmatrix} \quad [4.18]$$

Expanding equation [4.18] gives:

$$C\theta_j\, F_x - S\theta_j\, F_y + d_j = C\theta_i\, G_x + S\theta_i\, G_y \quad [4.18a]$$
$$C\alpha_j\, S\theta_j\, F_x + C\alpha_j\, C\theta_j\, F_y - S\alpha_j\, F_z = -S\theta_i\, G_x + C\theta_i\, G_y \quad [4.18b]$$
$$S\alpha_j\, S\theta_j\, F_x + S\alpha_j\, C\theta_j\, F_y + C\alpha_j\, F_z = G_z \quad [4.18c]$$

Three cases are considered depending on the values of the geometric parameters α_j and d_j:

a) $S\alpha_j = 0$ (thus $C\alpha_j = \pm 1$), $d_j \neq 0$. Equation [4.18c] can be written as:

$$C\alpha_j\, F_z(q_k) = G_z \quad [4.19]$$

We thus deduce that:

- if $q_k = \theta_k$, equation [4.19] is of type 2 in θ_k;
- if $q_k = r_k$, equation [4.19] is of type 1 in r_k.

Having determined q_k, the components of $\mathbf{F}$ are considered to be known. Adding the squares of equations [4.18a] and [4.18b] eliminates θ_i and gives a type-2 equation in θ_j:

$$F_x^2 + F_y^2 + d_j^2 + 2\, d_j\, (C\theta_j\, F_x - S\theta_j\, F_y) = G_x^2 + G_y^2 \quad [4.20]$$

After obtaining θ_j, equations [4.18a] and [4.18b] give a system of type-3 equations in θ_i.

b) $d_j = 0$ and $S\alpha_j \neq 0$. Adding the squares of equations [4.18] gives:

$$\|\mathbf{F}\|^2 = \|\mathbf{G}\|^2 \qquad [4.21]$$

Note that $\|\mathbf{F}\|^2$ is a function of q_k whereas $\|\mathbf{G}\|^2$ is a constant:

- if $q_k = \theta_k$, equation [4.21] is of type 2 in θ_k;
- if $q_k = r_k$, equation [4.21] is of type 9 in r_k.

Having obtained q_k and $\mathbf{F}$, equation [4.18c] gives θ_j using the type-2 equation. Finally, equations [4.18a] and [4.18b] give a system of type-3 equations in θ_i.

c) $d_j \neq 0$ and $S\alpha_j \neq 0$. Writing equation [4.17] in the form:

$$\begin{bmatrix} \mathbf{F}(q_k) \\ 1 \end{bmatrix} = \mathbf{Rot}(\mathbf{z}, -\theta_j)\ \mathbf{Trans}(\mathbf{x}, -d_j)\ \mathbf{Rot}(\mathbf{x}, -\alpha_j)\ \mathbf{Rot}(\mathbf{z}, -\theta_i) \begin{bmatrix} \mathbf{G} \\ 1 \end{bmatrix} \qquad [4.22]$$

after expanding, we obtain the third component as:

$$F_z = S\alpha_j\, S\theta_i\, G_x - S\alpha_j\, C\theta_i\, G_y + C\alpha_j\, G_z \qquad [4.23a]$$

Adding the squares of the components of equation [4.22] eliminates θ_j:

$$\|\mathbf{G}\|^2 + d_j^2 - 2\, d_j\, (C\theta_i\, G_x - S\theta_i\, G_y) = \|\mathbf{F}\|^2 \qquad [4.23b]$$

By eliminating θ_i from equations [4.23], we obtain:

$$\left[\frac{\|\mathbf{F}\|^2 - \|\mathbf{G}\|^2 - d_j^2}{2\, d_j}\right]^2 + \left[\frac{F_z - C\alpha_j\, G_z}{S\alpha_j}\right]^2 = G_x^2 + G_y^2 \qquad [4.24]$$

Here, we distinguish two cases:

- if $q_k = \theta_k$, equation [4.24] is of type 11 in θ_k;
- if $q_k = r_k$, equation [4.24] is of type 10 in r_k.

Knowing θ_k, equations [4.23a] and [4.23b] give a system of type-3 equations in θ_i. Finally, equations [4.18a] and [4.18b] are of type 3 in θ_j.

• **Example 4.4.** The variables θ_1, θ_2, θ_3 for the Stäubli RX-90 robot can be determined with the following equations using the Pieper method:

- equation for θ_3: $-2D3\ RL4\ S3 = (P_x)^2 + (P_y)^2 + (P_z)^2 - (D3)^2 - (RL4)^2$
- equation for θ_2: $(-RL4\ S3 + D3)\ S2 + (RL4\ C3)\ C2 = P_z$
- equations for θ_1: $[(-RL4\ S3 + D3)\ C2 - RL4\ C3\ S2]\ C1 = P_x$

  $[(-RL4\ S3 + D3)\ C2 - RL4\ C3\ S2]\ S1 = P_y$

4.4.2.2. *General solution of the orientation equation*

The spherical joint variables θ_{m-1}, θ_m and θ_{m+1} are determined from the orientation equation, which is deduced from equation [4.2] as:

$$^0\mathbf{A}_{m-2}\ ^{m-2}\mathbf{A}_{m+1}\ ^{m+1}\mathbf{A}_6 = [\ \mathbf{s}\quad \mathbf{n}\quad \mathbf{a}\ ] \tag{4.25}$$

The matrices $^0\mathbf{A}_{m-2}$ and $^{m+1}\mathbf{A}_6$ are functions of the variables that have already been obtained. Using equation [3.3] and after rearranging, equation [4.25] becomes:

$$\mathbf{rot}(\mathbf{z}, \theta_{m-1})\ \mathbf{rot}(\mathbf{x}, \alpha_m)\ \mathbf{rot}(\mathbf{z}, \theta_m)\ \mathbf{rot}(\mathbf{x}, \alpha_{m+1})\ \mathbf{rot}(\mathbf{z}, \theta_{m+1}) = [\ \mathbf{S}\quad \mathbf{N}\quad \mathbf{A}\ ] \tag{4.26}$$

with $[\ \mathbf{S}\quad \mathbf{N}\quad \mathbf{A}\ ] = \mathbf{rot}(\mathbf{x}, -\alpha_{m-1})\ ^{m-2}\mathbf{A}_0\ [\ \mathbf{s}\quad \mathbf{n}\quad \mathbf{a}\ ]\ ^6\mathbf{A}_{m+1}$

The left side of equation [4.26] is a function of the joint variables θ_{m-1}, θ_m and θ_{m+1} whereas the right side is known. Since $\mathbf{rot}(\mathbf{z}, \theta)$ defines a rotation about the axis $\mathbf{z}_0 = [\ 0\quad 0\quad 1\ ]^T$, then $\mathbf{z}_0$ is invariant with this rotation, which results in:

$$\mathbf{rot}(\mathbf{z}, \theta)\ \mathbf{z}_0 = \mathbf{z}_0 \text{ and } \mathbf{z}_0^T\ \mathbf{rot}(\mathbf{z}, \theta) = \mathbf{z}_0^T \tag{4.27}$$

i) determination of θ_m

To eliminate θ_{m-1}, we premultiply equation [4.26] by $\mathbf{z}_0^T$ and postmultiply it by $\mathbf{z}_0$:

$$\mathbf{z}_0^T\ \mathbf{rot}(\mathbf{x}, \alpha_m)\ \mathbf{rot}(\mathbf{z}, \theta_m)\ \mathbf{rot}(\mathbf{x}, \alpha_{m+1})\ \mathbf{z}_0 = \mathbf{z}_0^T\ [\ \mathbf{S}\quad \mathbf{N}\quad \mathbf{A}\ ]\ \mathbf{z}_0 \tag{4.28}$$

thus, we obtain:

$$S\alpha_m\, S\alpha_{m+1}\, C\theta_m + C\alpha_m\, C\alpha_{m+1} = A_z$$

Equation [4.28] is of type 2 in θ_m and gives two solutions (Appendix 1);

ii) determination of θ_{m-1}

Having obtained θ_m, let us write:

$$[\ \mathbf{S1}\quad \mathbf{N1}\quad \mathbf{A1}\ ] = \mathbf{rot}(\mathbf{x}, \alpha_m)\ \mathbf{rot}(\mathbf{z}, \theta_m)\ \mathbf{rot}(\mathbf{x}, \alpha_{m+1}) \tag{4.29}$$

Postmultiplying equation [4.26] by $\mathbf{z}_0$ and using equation [4.29] gives:

$$\mathbf{rot}(\mathbf{z}, \theta_{m-1})\ \mathbf{A1} = \mathbf{A} \tag{4.30}$$

The first two elements of [4.30] give a type-3 system of equations in θ_{m-1};

iii) determination of θ_{m+1}

By premultiplying equation [4.26] by $\mathbf{z}_0^T$ and using equation [4.29], we obtain:

$$[\ S1_z \quad N1_z \quad A1_z\ ]\ \mathbf{rot}(\mathbf{z}, \theta_{m+1}) = [\ S_z \quad N_z \quad A_z\ ] \qquad [4.31]$$

This gives a type-3 system of equations in θ_{m+1}.

These equations yield two solutions for the spherical joint variables. Thus, the maximum number of solutions of the IGM for a six degree-of-freedom robot with a spherical joint is eight.

4.4.3. *Inverse geometric model of robots with three prismatic joints*

The IGM of this class of robots is obtained by solving firstly the three revolute joint variables using the orientation equation. After this, the prismatic joint variables are obtained using the position equation. The number of solutions for the IGM of such robots is two.

4.4.3.1. *Solution of the orientation equation*

Let the revolute joints be denoted by i, j and k. The orientation equation can be deduced from equations [4.2] and [3.3] as:

$$\mathbf{rot}(\mathbf{z}, \theta_i)\ [\ \mathbf{S1} \quad \mathbf{N1} \quad \mathbf{A1}\ ]\ \mathbf{rot}(\mathbf{z},\theta_j)\ [\ \mathbf{S2} \quad \mathbf{N2} \quad \mathbf{A2}\ ]\ \mathbf{rot}(\mathbf{z},\theta_k) = [\ \mathbf{S3} \quad \mathbf{N3} \quad \mathbf{A3}\ ] \qquad [4.32]$$

where the orientation matrices $[\ \mathbf{Si} \quad \mathbf{Ni} \quad \mathbf{Ai}\ ]$, for $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, are known. The solution of equation [4.32] is similar to that of § 4.4.2.2 and gives two solutions.

4.4.3.2. *Solution of the position equation*

Let the prismatic joints be denoted by i', j' and k'. The revolute joint variables being determined, the position equation is written as:

$$\mathbf{Trans}(\mathbf{z}, r_{i'})\ \mathbf{T1}\ \mathbf{Trans}(\mathbf{z}, r_{j'})\ \mathbf{T2}\ \mathbf{Trans}(\mathbf{z}, r_{k'}) = \mathbf{T3} \qquad [4.33]$$

with $\mathbf{Ti} = \begin{bmatrix} \mathbf{Si'} & \mathbf{Ni'} & \mathbf{Ai'} & \mathbf{Pi'} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The matrices **Ti**, for **i** = 1, 2, 3, are known. The previous equation gives a system of three linear equations in $r_{i'}$, $r_{j'}$ and $r_{k'}$.

4.5. Inverse geometric model of general robots

The Raghavan-Roth method [Raghavan 90] gives the solution to the inverse geometric problem for six degree-of-freedom robots with general geometry (the geometric parameters may have arbitrary real values). In this method, we first compute all possible solutions of one variable q_i using a polynomial equation, which is called the *characteristic polynomial*. Then, the other variables are uniquely derived for each q_i. This method is based on the *dyalitic elimination* technique presented in Appendix 3.

In order to illustrate this method, we consider the 6R robot and rewrite equation [4.2] as follows:

$$ {}^0\mathbf{T}_1\, {}^1\mathbf{T}_2\, {}^2\mathbf{T}_3\, {}^3\mathbf{T}_4 = \mathbf{U}_0\, {}^6\mathbf{T}_5\, {}^5\mathbf{T}_4 \qquad [4.34] $$

The left and right sides of equation [4.34] represent the transformation of frame R_4 relative to frame R_0 using two distinct paths. The joint variables appearing in the elements of the previous equation are:

$$ \begin{bmatrix} \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3 & \theta_1,\theta_2,\theta_3 \\ \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3 & \theta_1,\theta_2,\theta_3 \\ \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3,\theta_4 & \theta_1,\theta_2,\theta_3 & \theta_1,\theta_2,\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 \\ \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 \\ \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 & \theta_5,\theta_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

From this equation, we observe that the third and fourth columns of the left side are independent of θ_4. This is due to the fact that the elements of the third and fourth columns of the transformation matrix ${}^{i-1}\mathbf{T}_i$ are independent of θ_i (see equation [3.2]). From equation [4.34], we can thus establish the following equations devoid of θ_4:

$$ \mathbf{a}_l = \mathbf{a}_r \qquad [4.35a] $$
$$ \mathbf{P}_l = \mathbf{P}_r \qquad [4.35b] $$

where the vectors **a** and **P** contain the first three elements of the third and fourth columns of equation [4.34] respectively, and the subscripts "l" and "r" indicate the left and right sides respectively. Equations [4.35] give six scalar equations.

It is now necessary to eliminate four of the five remaining variables to obtain a polynomial equation in one variable. This requires the use of the following additional equations:

$$(\mathbf{a}^T \mathbf{P})_l = (\mathbf{a}^T \mathbf{P})_r \qquad [4.36a]$$

$$(\mathbf{P}^T \mathbf{P})_l = (\mathbf{P}^T \mathbf{P})_r \qquad [4.36b]$$

$$(\mathbf{a}\mathrm{x}\mathbf{P})_l = (\mathbf{a}\mathrm{x}\mathbf{P})_r \qquad [4.36c]$$

$$[\mathbf{a}(\mathbf{P}^T \mathbf{P}) - 2\mathbf{P}(\mathbf{a}^T \mathbf{P})]_l = [\mathbf{a}(\mathbf{P}^T \mathbf{P}) - 2\mathbf{P}(\mathbf{a}^T \mathbf{P})]_r \qquad [4.36d]$$

Equations [4.36a] and [4.36b] are scalar, whereas equations [4.36c] and [4.36d] are vectors. They do not contain $\sin^2(.)$, $\cos^2(.)$ or $\sin(.)\cos(.)$. We thus have fourteen scalar equations that may be written in the following matrix form:

$$\mathbf{A}\,\mathbf{X1} = \mathbf{B}\,\mathbf{Y} \qquad [4.37]$$

where:

- $\mathbf{X1} = [\ S2S3 \quad S2C3 \quad C2S3 \quad C2C3 \quad S2 \quad C2 \quad S3 \quad C3 \quad 1\ ]^T$ [4.38]
- $\mathbf{Y} = [\ S5S6 \quad S5C6 \quad C5S6 \quad C5C6 \quad S5 \quad C5 \quad S6 \quad C6\ ]^T$ [4.39]
- **A**: (14x9) matrix whose elements are linear combinations of S1 and C1;
- **B**: (14x8) matrix whose elements are constants.

To eliminate θ5 and θ6, we select eight scalar equations out of equation [4.37]. The system [4.37] will be partitioned as:

$$\begin{bmatrix} \mathbf{A1} \\ \mathbf{A2} \end{bmatrix} \mathbf{X1} = \begin{bmatrix} \mathbf{B1} \\ \mathbf{B2} \end{bmatrix} \mathbf{Y} \qquad [4.40]$$

where **A1 X1 = B1 Y** gives six equations, and **A2 X1 = B2 Y** represents the remaining eight equations. By eliminating **Y**, we obtain the following system of equations:

$$\mathbf{D}\,\mathbf{X1} = \mathbf{0}_{6x1} \qquad [4.41]$$

where $\mathbf{D} = [\mathbf{A1} - \mathbf{B1}\ \mathbf{B2}^{-1}\ \mathbf{A2}]$ is a (6x9) matrix whose elements are functions of S1 and C1.

Using the half-angle transformation for the sine and cosine functions in equation [4.41] ($Ci = \frac{1 - x_i^2}{1 + x_i^2}$ and $Si = \frac{2x_i}{1 + x_i^2}$ with $x_i = \tan\frac{\theta_i}{2}$ for i = 1, 2, 3) yields the new homogeneous system of equations:

$$\mathbf{E}\,\mathbf{X2} = \mathbf{0}_{6x1} \qquad [4.42]$$

where **E** is a (6x9) matrix whose elements are quadratic functions of x_1, and:

$$\mathbf{X2} = [\ x_2^2x_3^2 \quad x_2^2x_3 \quad x_2^2 \quad x_2x_3^2 \quad x_2x_3 \quad x_2 \quad x_3^2 \quad x_3 \quad 1\ ]^T \qquad [4.43]$$

Thus, we have a system of six equations with nine unknowns. We now eliminate x_2 and x_3 dyalitically (see Appendix 3). Multiplying equation [4.42] by x_2, we obtain six additional equations with only three new unknowns:

$$\mathbf{E}\,\mathbf{X3} = \mathbf{0}_{6x1} \qquad [4.44]$$

with $\mathbf{X3} = [\ x_2^3x_3^2 \quad x_2^3x_3 \quad x_2^3 \quad x_2^2x_3^2 \quad x_2^2x_3 \quad x_2^2 \quad x_2x_3^2 \quad x_2x_3 \quad x_2\ ]^T$.

Combining equations [4.42] and [4.44], we obtain a system of twelve homogeneous equations:

$$\mathbf{S}\,\mathbf{X} = \mathbf{0}_{12x1} \qquad [4.45]$$

where:

$$\mathbf{X} = [\ x_2^3x_3^2 \ x_2^3x_3 \ x_2^3 \ x_2^2x_3^2 \ x_2^2x_3 \ x_2^2 \ x_2x_3^2 \ x_2x_3 \ x_2 \ x_3^2 \ x_3 \ 1\ ]^T \qquad [4.46]$$

and **S** is a (12x12) matrix whose elements are quadratic functions of x_1 and has the following form:

$$\mathbf{S} = \begin{bmatrix} \mathbf{E} & \mathbf{0}_{6x3} \\ \mathbf{0}_{6x3} & \mathbf{E} \end{bmatrix} \qquad [4.47]$$

In order that equation [4.45] has a non-trivial solution, the determinant of the matrix **S** must be zero. The characteristic polynomial of equation [4.47], which gives the solution for x_1, can be obtained from:

$$\det(\mathbf{S}) = 0 \qquad [4.48]$$

It can be shown that this determinant, which is a polynomial of degree 24, has $(1+x_1^2)^4$ as a common factor [Raghavan 90]. Thus, equation [4.48] is written as:

$$\det(\mathbf{S}) = f(x_1)\,(1+x_1^2)^4 = 0 \qquad [4.49]$$

The polynomial $f(x_1)$ is of degree sixteen and represents the characteristic polynomial of the robot. The real roots of this polynomial give all the solutions for

θ_1. For each value of θ_1, we can calculate the matrix S. The variables θ_2 and θ_3 are uniquely determined by solving the linear system of equation [4.45]. By substituting θ_1, θ_2 and θ_3 in equation [4.37] and using eight equations, we can calculate θ_5 and θ_6. Finally, we consider the following equation to calculate θ_4:

$$^4T_3 = {}^4T_6\, U_0\, {}^0T_3 \qquad [4.50]$$

By using the (1, 1) and (2, 1) elements, we obtain θ_4 using an atan2 function.

The same method can also be applied to six degree-of-freedom robots having prismatic joints. It this case, Si and Ci have to be replaced by r_i^2 and r_i in **X1** and **Y** respectively, i being the prismatic joint.

NOTE.– Equation [4.34] is a particular form of equation [4.2] that can be written in several other forms [Mavroidis 93], for example:

$$^4T_5\, {}^5T_6\, {}^6T_7\, {}^0T_1 = {}^4T_3\, {}^3T_2\, {}^2T_1 \qquad [4.51a]$$
$$^5T_6\, {}^6T_7\, {}^0T_1\, {}^1T_2 = {}^5T_4\, {}^4T_3\, {}^3T_2 \qquad [4.51b]$$
$$^6T_7\, {}^0T_1\, {}^1T_2\, {}^2T_3 = {}^6T_5\, {}^5T_4\, {}^4T_3 \qquad [4.51c]$$
$$^0T_1\, {}^1T_2\, {}^2T_3\, {}^3T_4 = {}^7T_6\, {}^6T_5\, {}^5T_4 \qquad [4.51d]$$
$$^1T_2\, {}^2T_3\, {}^3T_4\, {}^4T_5 = {}^1T_0\, {}^7T_6\, {}^6T_5 \qquad [4.51e]$$
$$^2T_3\, {}^3T_4\, {}^4T_5\, {}^5T_6 = {}^2T_1\, {}^1T_0\, {}^7T_6 \qquad [4.51f]$$

with $^7T_6 = U_0$ and $^6T_7 = U_0^{-1}$

The selection of the starting equation not only defines the variable of the characteristic equation but also the degree of the corresponding polynomial. For specific values of the geometric parameters, certain columns of the matrix S become dependent and it is necessary to either change the selected variables and columns [Khalil 94b], [Murareci 97] or choose another starting equation [Mavroidis 93].

When the robot is in a singular configuration, the rows of the matrix S are linearly dependent. In this case, it is not possible to find a solution. In fact, this method has proved the maximum number of solutions that can be obtained for the inverse geometric problem of serial robots, but it is hardly usable to develop a general numerical method to treat any robot architecture.

4.6. Conclusion

In this chapter, we have presented three methods for calculating the inverse geometric model. The Paul method is applicable to a large number of structures with particular geometrical parameters where most of the distances are zero and most of the angles are zero or $\pm \pi/2$. The Pieper method gives the solution for the six degree-of-freedom robots having three prismatic joints or three revolute joints whose axes intersect at a point. Finally, the general method provides the solution for the IGM of six degree-of-freedom robots with general geometry.

The analytical solution, as compared to the differential methods discussed in the next chapter, is useful for obtaining all the solutions of the inverse geometric model. Some of them may be eliminated because they do not satisfy the joint limits. Generally, the selected solution is left to the robot's user and depends on the task specifications: to avoid collisions between the robot and its environment; to ensure the continuity of the trajectory as required in certain tasks prohibiting configuration changes (machining, welding,...); to avoid as much as possible the singular configurations that may induce control problems (namely discontinuity of velocity), etc.