Appendix 4

Solution of systems of linear equations

A4.1. Problem statement

Let us consider the following system of m linear equations in n unknowns:

$$\mathbf{Y} = \mathbf{W}\,\boldsymbol{\zeta} \qquad [A4.1]$$

where $\mathbf{W}$ is an (mxn) known matrix, $\mathbf{Y}$ is an (mx1) known vector and $\boldsymbol{\zeta}$ is the unknown (nx1) vector.

Let $\mathbf{W}_a$ be the augmented matrix defined by:

$$\mathbf{W}_a = [\mathbf{W} \; : \; \mathbf{Y}]$$

Let r and r_a denote the ranks of $\mathbf{W}$ and $\mathbf{W}_a$ respectively. The relation between r and r_a can be used to analyze the existence of solutions:

a) if $r = r_a$, the system has at least one solution:

- if $r = r_a = n$, there is a unique solution;
- if $r = r_a < n$, the number of solutions is infinite; the system is redundant. For example, this case is encountered with the inverse kinematic model (Chapter 6).

b) if $r \neq r_a$, the system [A4.1] is not compatible, meaning that it has no exact solution; it will be written as:

$$\mathbf{Y} = \mathbf{W}\,\boldsymbol{\zeta} + \boldsymbol{\rho} \qquad [A4.2]$$

where ρ is the residual vector or error vector. This case occurs when identifying the geometric and dynamic parameters (Chapters 11 and 12 respectively) or when solving the inverse kinematic model in the vicinity of singular configurations.

A4.2. Resolution based on the generalized inverse

A4.2.1. *Definitions*

The matrix $\mathbf{W}^{(-1)}$ is a *generalized inverse* of **W** if:

$$\mathbf{W}\,\mathbf{W}^{(-1)}\,\mathbf{W} = \mathbf{W} \qquad [A4.3]$$

If **W** is square and regular, then $\mathbf{W}^{(-1)} = \mathbf{W}^{-1}$. In addition, $\mathbf{W}^{(-1)}$ is said to be a left inverse or a right inverse respectively if:

$$\mathbf{W}^{(-1)}\,\mathbf{W} = \mathbf{I} \text{ or } \mathbf{W}\,\mathbf{W}^{(-1)} = \mathbf{I} \qquad [A4.4]$$

It can be shown that **W** has an infinite number of generalized inverses unless it is of dimension (nxn) and of rank n. A solution of the system [A4.1], when it is compatible, is given by:

$$\boldsymbol{\zeta} = \mathbf{W}^{(-1)}\,\mathbf{Y} \qquad [A4.5]$$

All the solutions are given by the general equation:

$$\boldsymbol{\zeta} = \mathbf{W}^{(-1)}\,\mathbf{Y} + (\mathbf{I} - \mathbf{W}^{(-1)}\,\mathbf{W})\,\mathbf{Z} \qquad [A4.6]$$

where **Z** is an arbitrary (nx1) vector. Note that:

$$\mathbf{W}\,(\mathbf{I} - \mathbf{W}^{(-1)}\,\mathbf{W})\,\mathbf{Z} = \mathbf{0} \qquad [A4.7]$$

Therefore, the term $(\mathbf{I} - \mathbf{W}^{(-1)}\,\mathbf{W})\,\mathbf{Z}$ is a projection of **Z** on the null space of **W**.

A4.2.2. *Computation of a generalized inverse*

The matrix **W** is partitioned in the following manner:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \\ \mathbf{W}_{21} & \mathbf{W}_{22} \end{bmatrix} \qquad [A4.8]$$

where $\mathbf{W}_{11}$ is a regular (rxr) matrix, and r is the rank of $\mathbf{W}$. Then, it can be verified that:

$$\mathbf{W}^{(-1)} = \begin{bmatrix} \mathbf{W}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad [A4.9]$$

This method gives the solution as a function of r components of $\mathbf{Y}$. Thus, the accuracy of the result may depend on the isolated minor. We will see in the next section that the pseudoinverse method allows us to avoid this limitation.

NOTE.– If the (r,r) matrix $\mathbf{W}_{11}$ built up with the first r rows and the first r columns is not regular, it is always possible to define a matrix $\mathbf{W}'$ such that:

$$\mathbf{W}' = \mathbf{R}\,\mathbf{W}\,\mathbf{C} = \begin{bmatrix} \mathbf{W}'_{11} & \mathbf{W}'_{12} \\ \mathbf{W}'_{21} & \mathbf{W}'_{22} \end{bmatrix} \quad [A4.10]$$

where $\mathbf{W}'_{11}$ is a regular (rxr) matrix. The orthogonal matrices $\mathbf{R}$ and $\mathbf{C}$ permute the rows and columns of $\mathbf{W}$ respectively. The generalized inverse of $\mathbf{W}$ is derived from that of $\mathbf{W}'$ as:

$$\mathbf{W}^{(-1)} = \mathbf{C}\,(\mathbf{W}')^{(-1)}\,\mathbf{R} \quad [A4.11]$$

A4.3. Resolution based on the pseudoinverse

A4.3.1. *Definition*

The *pseudoinverse* of the matrix $\mathbf{W}$ is the generalized inverse $\mathbf{W}^+$ that satisfies [Penrose 55]:

$$\begin{cases} \mathbf{W}\,\mathbf{W}^+\,\mathbf{W} = \mathbf{W} \\ \mathbf{W}^+\,\mathbf{W}\,\mathbf{W}^+ = \mathbf{W}^+ \\ (\mathbf{W}^+\,\mathbf{W})^T = \mathbf{W}^+\,\mathbf{W} \\ (\mathbf{W}\,\mathbf{W}^+)^T = \mathbf{W}\,\mathbf{W}^+ \end{cases} \quad [A4.12]$$

It can be shown that the pseudoinverse always exists and is unique. All the solutions of the system [A4.1] are given by:

$$\boldsymbol{\zeta} = \mathbf{W}^+\,\mathbf{Y} + (\mathbf{I} - \mathbf{W}^+\,\mathbf{W})\,\mathbf{Z} \quad [A4.13]$$

The first term $\mathbf{W}^+\,\mathbf{Y}$ is the solution minimizing the Euclidean norm $\|\boldsymbol{\zeta}\|$. The second term $(\mathbf{I} - \mathbf{W}^+\,\mathbf{W})\,\mathbf{Z}$, also called *optimization term* or *homogeneous solution*, is the projection of an arbitrary vector $\mathbf{Z}$ of $\Re^n$ on $\mathcal{N}(\mathbf{W})$, the null space of $\mathbf{W}$, and therefore, does not change the value of $\mathbf{Y}$. It can be shown that $(\mathbf{I} - \mathbf{W}^+ \mathbf{W})$ is of rank $(n - r)$. Consequently, when the robot is redundant, this term may be used to optimize additional criteria satisfying the primary task. This property is illustrated by examples in Chapter 6.

When the system [A4.1] is not compatible, it can be shown that the solution $\mathbf{W}^+ \mathbf{Y}$ gives the least-squares solution minimizing the error $\|\mathbf{W}\,\boldsymbol{\zeta} - \mathbf{Y}\|^2 = \|\boldsymbol{\rho}\|^2$.

A4.3.2. *Pseudoinverse computation methods*

A4.3.2.1. *Method requiring explicit computation of the rank [Gorla 84]*

Let the matrix $\mathbf{W}$ be partitioned as indicated in equation [A4.8] such that $\mathbf{W}_{11}$ is of full rank r. Using the following notations:

$$\mathbf{W}_1 = \begin{bmatrix} \mathbf{W}_{11} \\ \mathbf{W}_{21} \end{bmatrix} \text{and } \mathbf{W}_2 = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{W}_{12} \end{bmatrix}$$

it can be shown that:

$$\mathbf{W}^+ = \mathbf{W}_2^T (\mathbf{W}_1^T \mathbf{W}\, \mathbf{W}_2^T)^{-1} \mathbf{W}_1^T \qquad [A4.14]$$

When $\mathbf{W}$ is of full rank, this equation may be simplified as follows:

- if $m > n$: $\mathbf{W} = \mathbf{W}_1 \rightarrow \mathbf{W}^+ = (\mathbf{W}^T \mathbf{W})^{-1} \mathbf{W}^T$, ($\mathbf{W}^+$ is then the left inverse of $\mathbf{W}$);
- if $m < n$: $\mathbf{W} = \mathbf{W}_2 \rightarrow \mathbf{W}^+ = \mathbf{W}^T (\mathbf{W}\,\mathbf{W}^T)^{-1}$, ($\mathbf{W}^+$ is then the right inverse of $\mathbf{W}$);
- if $m = n$: $\mathbf{W} = \mathbf{W}_1 = \mathbf{W}_2 \rightarrow \mathbf{W}^+ = \mathbf{W}^{-1}$.

If $\mathbf{W}_{11}$ is not of rank r, the orthogonal permutation matrices $\mathbf{R}$ and $\mathbf{C}$ of equation [A4.10] should be used, yielding:

$$\mathbf{W}^+ = \mathbf{C}\,(\mathbf{W}')^+ \mathbf{R} \qquad [A4.15]$$

A4.3.2.2. *Greville method [Greville 60], [Fournier 80]*

This recursive algorithm is based on the pseudoinverse properties of a partitioned matrix. It does not require the explicit computation of the rank of $\mathbf{W}$. Let $\mathbf{W}$ be a partitioned matrix such that:

$$\mathbf{W} = [\, \mathbf{U} \;:\; \mathbf{V} \,] \qquad [A4.16]$$

Its pseudoinverse $\mathbf{W}^+$ can be written as [Boullion 71]:

$$W^+ = \begin{bmatrix} U^+ - U^+VC^+ - U^+V(I - C^+C)\,M\,V^T(U^+)^TU^+(I - VC^+) \\ C^+ + (I - C^+C)\,M\,V^T(U^+)^TU^+(I - VC^T) \end{bmatrix} \qquad [A4.17]$$

with:

$$C = (I - UU^+)\,V$$
$$M = [I + (I - C^+C)\,V^T(U^+)^TU^+V\,(I - C^+C)]^{-1}$$

If the matrix V reduces to a single column, a recursive algorithm that does not require any matrix inversion may be employed.

Let W_k contain the first k columns of W. If W_k is partitioned such that the first $(k-1)$ columns are denoted by W_{k-1} and the k^{th} column is w_k, then:

$$W_k = [\ W_{k-1}\ :\ w_k\ ] \qquad [A4.18]$$

The pseudoinverse W_k^+ is derived from W_{k-1}^+ and from the k^{th} column of W:

$$W_k^+ = \begin{bmatrix} W_{k-1}^+ - d_k b_k \\ b_k \end{bmatrix} \qquad [A4.19]$$

where:

$$d_k = W_{k-1}^+\,w_k \qquad [A4.20]$$

In order to evaluate b_k, we define:

$$c_k = w_k - W_{k-1}\,d_k \qquad [A4.21]$$

then, we compute:

$$\begin{aligned} b_k = c_k^+ &= (c_k^T c_k)^{-1} c_k^T && \text{if } c_k \neq 0 \\ &= (1 + d_k^T d_k)^{-1} d_k^T W_{k-1}^+ && \text{if } c_k = 0 \end{aligned} \qquad [A4.22]$$

This recursive algorithm is initialized by calculating W_1^+ using equation [A4.14]:

$$W_1^+ = w_1^+ = (w_1^T w_1)^{-1} w_1^T \qquad (\text{if } w_1 = 0, \text{ then } W_1^+ = 0^T). \qquad [A4.23]$$

The pseudoinverse of W can also be calculated by handling recursively rows instead of columns: physically, it comes to consider the equations sequentially.

• **Example A4.1.** Computation of the pseudoinverse using the Greville method. Let us consider the following matrix:

$$\mathbf{W} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

i) first iteration (initialization):

$$\mathbf{W}_1^+ = [\ 1/5 \quad 2/5\ ]$$

ii) second iteration:

$$\mathbf{d}_2 = 8/5,\ \mathbf{c}_2 = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix},\ \mathbf{b}_2 = [\ 2 \quad -1\ ],\ \mathbf{W}_2^+ = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

iii) third iteration:

$$\mathbf{d}_3 = \begin{bmatrix} -1 \\ 2 \end{bmatrix},\ \mathbf{c}_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{b}_3 = [\ 7/6 \quad -2/3\ ]$$

Finally, the pseudoinverse is:

$$\mathbf{W}^+ = \begin{bmatrix} -11/6 & 4/3 \\ -1/3 & 1/3 \\ 7/6 & -2/3 \end{bmatrix}$$

A4.3.2.3. *Method based on the singular value decomposition of* $\mathbf{W}$

The singular value decomposition theory [Lawson 74], [Dongarra 79], [Klema 80] states that for an (mxn) matrix $\mathbf{W}$ of rank r, there exist orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ of dimensions (mxm) and (nxn) respectively, such that:

$$\mathbf{W} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^T \qquad [A4.24]$$

The (mxn) matrix $\mathbf{\Sigma}$ is diagonal and contains the singular values σ_i of $\mathbf{W}$. They are arranged in a decreasing order such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$. $\mathbf{\Sigma}$ has the following form:

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{S}_{rxr} & \mathbf{0}_{rx(n-r)} \\ \mathbf{0}_{(m-r)xr} & \mathbf{0}_{(m-r)x(n-r)} \end{bmatrix} \qquad [A4.25]$$

where **S** is a diagonal (rxr) matrix of rank r, formed by the non-zero singular values σ_i of **W**.

The singular values of **W** are the square roots of the eigenvalues of the matrices $\mathbf{W}^T\mathbf{W}$ or $\mathbf{W}\mathbf{W}^T$ depending on whether n < m or n > m respectively.

The columns of **V** are the eigenvectors of $\mathbf{W}^T\mathbf{W}$ and are called *right singular vectors* or *input singular vectors*. The columns of **U** are the eigenvectors of $\mathbf{W}\mathbf{W}^T$ and are called *left singular vectors* or *output singular vectors*.

The pseudoinverse is then written as:

$$\mathbf{W}^+ = \mathbf{V}\,\boldsymbol{\Sigma}^+\mathbf{U}^T \qquad [A4.26]$$

with:

$$\boldsymbol{\Sigma}^+ = \begin{bmatrix} \mathbf{S}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

This method, known as *Singular Value Decomposition* (SVD) [Maciejewski 89], is often implemented for rank determination and pseudoinverse computation in scientific software packages.

The SVD decomposition of **W** makes it possible to evaluate the 2-norm condition number, which can be used to investigate the sensitivity of the linear system to data variations on **Y** and **W**. Indeed, if **W** is a square matrix, and assuming uncertainties $\boldsymbol{\zeta} + d\boldsymbol{\zeta}$, the system [A4.1] may be written as:

$$\mathbf{Y} + d\mathbf{Y} = [\mathbf{W} + d\mathbf{W}]\,[\boldsymbol{\zeta} + d\boldsymbol{\zeta}] \qquad [A4.27]$$

The relative error of the solution may be bounded such that:

$$\frac{\|d\boldsymbol{\zeta}\|_p}{\|\boldsymbol{\zeta}\|_p} \leq \mathrm{cond}_p(\mathbf{W})\,\frac{\|d\mathbf{Y}\|_p}{\|\mathbf{Y}\|_p} \qquad [A4.28a]$$

$$\frac{\|d\boldsymbol{\zeta}\|_p}{\|\boldsymbol{\zeta} + d\boldsymbol{\zeta}\|_p} \leq \mathrm{cond}_p(\mathbf{W})\,\frac{\|d\mathbf{W}\|_p}{\|\mathbf{W}\|_p} \qquad [A4.28b]$$

$\mathrm{cond}_p(\mathbf{W})$ is the condition number of **W** with respect to the p-norm such that:

$$\mathrm{cond}_p(\mathbf{W}) = \|\mathbf{W}\|_p\,\|\mathbf{W}^+\|_p \qquad [A4.29]$$

where $\|*\|_p$ denotes a vector p-norm or a matrix p-norm.

The 2-norm condition number of a matrix **W** is given by:

$$\mathrm{cond}_2(\mathbf{W}) = \frac{\sigma_{max}}{\sigma_{min}} \qquad [A4.30]$$

Notice that the condition number is such that:

$$\mathrm{cond}_2(\mathbf{W}) \geq 1 \qquad [A4.31]$$

NOTES.–

– the p-norm of a vector $\boldsymbol{\zeta}$ is defined by:

$$\|\boldsymbol{\zeta}\|_p = (\sum_{i=1}^{n}|\zeta_i|^p)^{1/p} \quad \text{for } p \geq 1 \qquad [A4.32]$$

– the p-norm of a matrix $\mathbf{W}$ is defined by:

$$\|\mathbf{W}\|_p = \max\{\frac{\|\mathbf{W}\,\boldsymbol{\zeta}\|_p}{\|\boldsymbol{\zeta}\|_p} : \boldsymbol{\zeta} \neq \mathbf{0}_{n,1}\} = \max\{\|\mathbf{W}\,\boldsymbol{\zeta}\|_p : \|\boldsymbol{\zeta}\|_p = 1\} \qquad [A4.33]$$

– the 2-norm of a matrix is the largest singular value of $\mathbf{W}$. It is given by:

$$\|\mathbf{W}\|_2 = \sigma_{max}$$

– equations similar to [A4.28] can be derived for over determined linear systems.

• **Example A4.2.** Computation of the pseudoinverse with the SVD method. Consider the same matrix as in Example A4.1:

$$\mathbf{W} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$$

It can be shown that:

$$\mathbf{V} = \begin{bmatrix} 0.338 & 0.848 & -0.408 \\ 0.551 & 0.174 & 0.816 \\ 0.763 & -0.501 & -0.408 \end{bmatrix}, \boldsymbol{\Sigma} = \begin{bmatrix} 6.55 & 0 & 0 \\ 0 & 0.374 & 0 \end{bmatrix}, \mathbf{U} = \begin{bmatrix} 0.57 & -0.822 \\ 0.822 & 0.57 \end{bmatrix}.$$

The pseudoinverse is obtained by applying equation [A4.26]:

$$\mathbf{W}^{+} = \begin{bmatrix} -1.83 & 1.33 \\ -0.333 & 0.333 \\ 1.17 & -0.667 \end{bmatrix}$$

A4.4. Resolution based on the QR decomposition

Given the system of equations [A4.1], two cases are to be considered depending on whether **W** is of full rank or not.

A4.4.1. *Full rank system*

Let us assume that **W** is of full rank. The QR decomposition of **W** consists of writing that [Golub 83]:

$$\mathbf{Q}^{T}\,\mathbf{W} = \begin{bmatrix} \mathbf{R} \\ \mathbf{0}_{(m-r),n} \end{bmatrix} \quad \text{for } m > n,\ r = n \qquad \text{[A4.34]}$$

$$\mathbf{Q}^{T}\,\mathbf{W} = [\ \mathbf{R} \quad \mathbf{0}_{m,n-r}\ ] \quad \text{for } n > m,\ r = m \qquad \text{[A4.35]}$$

where **R** is a regular and upper-triangular (rxr) matrix and where **Q** is an orthogonal (mxm) matrix.

For sake of brevity, let us only consider the case m > n, which typically occurs when identifying the geometric and dynamic parameters (Chapters 11 and 12 respectively). The case n > m can be similarly handled. The matrix **Q** is partitioned as follows:

$$\mathbf{Q} = [\ \mathbf{Q1} \quad \mathbf{Q2}\ ] \qquad \text{[A4.36]}$$

where the dimensions of **Q1** and **Q2** are (mxr) and mx(m−r) respectively.

Let us define:

$$\mathbf{G} = \mathbf{Q}^{T}\,\mathbf{Y} = \begin{bmatrix} \mathbf{Q1}^{T}\,\mathbf{Y} \\ \mathbf{Q2}^{T}\,\mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{G1} \\ \mathbf{G2} \end{bmatrix} \qquad \text{[A4.37]}$$

Since the matrix **Q** is orthogonal, it follows that [Golub 83]:

$$\|\mathbf{Y} - \mathbf{W}\,\zeta\|^{2} = \|\mathbf{Q}^{T}\,\mathbf{Y} - \mathbf{Q}^{T}\,\mathbf{W}\,\zeta\|^{2} = \|\mathbf{G1} - \mathbf{R}\,\zeta\,\|^{2} + \|\mathbf{G2}\|^{2} = \|\rho\|^{2} \qquad \text{[A4.38]}$$

From equation [A4.38], ζ is the unique solution of the system:

$$\mathbf{R}\,\zeta = \mathbf{G1} \qquad [A4.39]$$

Since **R** is a regular and upper-triangular (rxr) matrix, the system [A4.39] can be easily solved with a backward recursion technique (compute sequentially ζ_n, ζ_{n-1}, ...). The norm of the residual for the optimal solution is derived as:

$$\|\rho\|_{min} = \|\mathbf{G2}\| = \|\mathbf{Q2}^T\,\mathbf{Y}\| \qquad [A4.40]$$

This solution (when m > n and r = n) is identical to that obtained by the pseudoinverse. In order to speed up the computations for systems of high dimensions (for example, this is the case for the identification of the dynamic parameters), we can partition the system [A4.1] into k sub-systems such that:

$$\mathbf{Y}(i) = \mathbf{W}(i)\,\zeta \quad \text{for } i = 1, \ldots, k \qquad [A4.41]$$

Let $\mathbf{Q}(i) = [\mathbf{Q1}(i) \quad \mathbf{Q2}(i)]$ and $\mathbf{R}(i)$ be the matrices obtained after a QR decomposition of the matrix $\mathbf{W}(i)$. The global system reduces to the following system of (nxk) equations in n unknowns:

$$\begin{bmatrix} \mathbf{Q1}^T(1)\mathbf{Y}(1) \\ \cdots \\ \mathbf{Q1}^T(k)\mathbf{Y}(k) \end{bmatrix} = \begin{bmatrix} \mathbf{Q1}^T(1)\mathbf{W}(1) \\ \cdots \\ \mathbf{Q1}^T(k)\mathbf{W}(k) \end{bmatrix} \zeta$$

$$\begin{bmatrix} \mathbf{Q1}^T(1)\mathbf{Y}(1) \\ \cdots \\ \mathbf{Q1}^T(k)\mathbf{Y}(k) \end{bmatrix} = \begin{bmatrix} \mathbf{R}(1) \\ \cdots \\ \mathbf{R}(k) \end{bmatrix} \zeta \qquad [A4.42]$$

A4.4.2. *Rank deficient system*

Again, let us assume that m > n but in this case r < n. We permute the columns of **W** in such a way that the first columns are independent (the independent columns correspond to the diagonal non-zero elements of the matrix **R** obtained after QR decomposition of **W**). We proceed by a QR decomposition of the permutation matrix and we obtain :

$$\mathbf{Q}^T\,\mathbf{W}\,\mathbf{P} = \begin{bmatrix} \mathbf{R1} & \mathbf{R2} \\ \mathbf{0}_{(m-r),r} & \mathbf{0}_{(m-r),(n-r)} \end{bmatrix} \qquad [A4.43]$$

where **P** is a permutation matrix obtained by permuting the columns of an identity matrix, **Q** is an orthogonal (mxm) matrix, and **R1** is a regular and upper-triangular (rxr) matrix.

Let:

$$\mathbf{P}^T \boldsymbol{\zeta} = \begin{bmatrix} \boldsymbol{\zeta 1} \\ \boldsymbol{\zeta 2} \end{bmatrix}$$

From equation [A4.37], we obtain:

$$\|\rho\|^2 = \|\mathbf{Y} - \mathbf{W}\,\boldsymbol{\zeta}\|^2 = \|\mathbf{Q}^T\,\mathbf{Y} - \mathbf{Q}^T\,\mathbf{W}\,\mathbf{P}\,\mathbf{P}^T\,\boldsymbol{\zeta}\|^2$$

$$= \left\| \begin{bmatrix} \mathbf{G1} \\ \mathbf{G2} \end{bmatrix} - \begin{bmatrix} \mathbf{R1}\,\boldsymbol{\zeta 1} + \mathbf{R2}\,\boldsymbol{\zeta 2} \\ \mathbf{0}_{(n-r),1} \end{bmatrix} \right\|^2$$

$$= \|\mathbf{G1} - [\mathbf{R1}\,\boldsymbol{\zeta 1} + \mathbf{R2}\,\boldsymbol{\zeta 2}]\|^2 + \|\mathbf{G2}\|^2 \qquad \text{[A4.44]}$$

$\boldsymbol{\zeta 1}$ is the unique solution of the system:

$$\mathbf{R1}\,\boldsymbol{\zeta 1} = \mathbf{G1} - \mathbf{R2}\,\boldsymbol{\zeta 2} \qquad \text{[A4.45]}$$

Then, we obtain a family of optimal solutions parameterized by the matrices **P** and $\boldsymbol{\zeta 2}$:

$$\boldsymbol{\zeta} = \mathbf{P} \begin{bmatrix} \boldsymbol{\zeta 1} \\ \boldsymbol{\zeta 2} \end{bmatrix} \qquad \text{[A4.46]}$$

All solutions provide the minimum norm residual given by equation [A4.40]. We obtain a base solution for $\boldsymbol{\zeta 2} = \mathbf{0}_{(n-r),1}$. Recall that the pseudoinverse solution provides the minimum norm residual together with the minimum norm $\|\boldsymbol{\zeta}\|^2$.