Chapter 10

# Dynamics of robots with complex structure

## 10.1. Introduction

In this chapter, we present the dynamic modeling of tree structured robots and of closed chain mechanisms. We also derive the base inertial parameters for these structures. The algorithms given constitute a generalization of the results developed for serial robots in Chapter 9. We make use of the notations of § 9.2 and we assume that the reader is familiar with the geometric description of complex structures exposed in Chapter 7.

## 10.2. Dynamic modeling of tree structured robots

### 10.2.1. *Lagrange equations*

Since the joint variables are independent in a tree structure, we can make use of the Lagrange formulation in a similar way as for a serial structure. Thus, the kinetic energy and the potential energy will be computed by equations [9.16] and [9.25]. The recursive equations for computing the angular and linear velocities of link j must take into consideration that the antecedent of link j is link i, denoted as $i = a(j)$, and not $j - 1$ as in the case of simple open chain structures. The vector $L_j$ denotes the position of frame $R_j$ with respect to its antecedent frame $R_i$. Consequently, the linear and angular velocities of link j can be obtained from equations [9.17] and [9.18] by replacing $j - 1$ by i:

$$^{j}\omega_j = {}^{j}A_i\,{}^{i}\omega_i + \bar{\sigma}_j\,\dot{q}_j\,{}^{j}\mathbf{a}_j \qquad [10.1]$$

$$^{j}V_j = {}^{j}A_i\,({}^{i}V_i + {}^{i}\omega_i \times {}^{i}P_j) + \sigma_j\,\dot{q}_j\,{}^{j}\mathbf{a}_j \qquad [10.2]$$

### 10.2.2. *Newton-Euler formulation*

The forward recursive equations of the Newton-Euler inverse dynamic model (§ 9.5) can be generalized for tree structured robots by replacing $j-1$ by i, with $i = a(j)$:

$$^{j}\boldsymbol{\omega}_i = {}^{j}\mathbf{A}_i\, {}^{i}\boldsymbol{\omega}_i \qquad [10.3]$$

$$^{j}\boldsymbol{\omega}_j = {}^{j}\boldsymbol{\omega}_i + \bar{\sigma}_j\, \dot{q}_j\, {}^{j}\mathbf{a}_j \qquad [10.4]$$

$$^{j}\dot{\boldsymbol{\omega}}_j = {}^{j}\mathbf{A}_i\, {}^{i}\dot{\boldsymbol{\omega}}_i + \bar{\sigma}_j\, (\ddot{q}_j\, {}^{j}\mathbf{a}_j + {}^{j}\boldsymbol{\omega}_i \times \dot{q}_j\, {}^{j}\mathbf{a}_j) \qquad [10.5]$$

$$^{j}\dot{\mathbf{V}}_j = {}^{j}\mathbf{A}_i\, ({}^{i}\dot{\mathbf{V}}_i + {}^{i}\mathbf{U}_i\, {}^{i}\mathbf{P}_j) + \sigma_j\, (\ddot{q}_j\, {}^{j}\mathbf{a}_j + 2\, {}^{j}\boldsymbol{\omega}_i \times \dot{q}_j\, {}^{j}\mathbf{a}_j) \qquad [10.6]$$

$$^{j}\mathbf{F}_j = M_j\, {}^{j}\dot{\mathbf{V}}_j + {}^{j}\mathbf{U}_j\, {}^{j}\mathbf{MS}_j \qquad [10.7]$$

$$^{j}\mathbf{M}_j = {}^{j}\mathbf{J}_j\, {}^{j}\dot{\boldsymbol{\omega}}_j + {}^{j}\boldsymbol{\omega}_j \times ({}^{j}\mathbf{J}_j\, {}^{j}\boldsymbol{\omega}_j) + {}^{j}\mathbf{MS}_j \times {}^{j}\dot{\mathbf{V}}_j \qquad [10.8]$$

with $\boldsymbol{\omega}_0 = \mathbf{0}$, $\dot{\boldsymbol{\omega}}_0 = \mathbf{0}$, $\dot{\mathbf{V}}_0 = -\mathbf{g}$ and ${}^{j}\mathbf{U}_j = {}^{j}\hat{\dot{\boldsymbol{\omega}}}_j + {}^{j}\hat{\boldsymbol{\omega}}_j\, {}^{j}\hat{\boldsymbol{\omega}}_j$.

These equations will be computed recursively for $j = 1, \ldots, n$.

Let us suppose that k denotes all the links such that $a(k) = j$ (Figure 10.1). The backward recursive equations for $j = n, \ldots, 1$ are written as follows:

$$^{j}\mathbf{f}_j = {}^{j}\mathbf{F}_j + {}^{j}\mathbf{f}_{ej} + \sum_{k/a(k)=j} {}^{j}\mathbf{f}_k \qquad [10.9]$$

$$^{i}\mathbf{f}_j = {}^{i}\mathbf{A}_j\, {}^{j}\mathbf{f}_j \qquad [10.10]$$

$$^{j}\mathbf{m}_j = {}^{j}\mathbf{M}_j + {}^{j}\mathbf{m}_{ej} + \sum_{k/a(k)=j} ({}^{j}\mathbf{A}_k\, {}^{k}\mathbf{m}_k + {}^{j}\mathbf{P}_k \times {}^{j}\mathbf{f}_k) \qquad [10.11]$$

$$\Gamma_j = (\sigma_j\, {}^{j}\mathbf{f}_j + \bar{\sigma}_j\, {}^{j}\mathbf{m}_j)^T\, {}^{j}\mathbf{a}_j + F_{sj}\, \mathrm{sign}\,(\dot{q}_j) + F_{vj}\, \dot{q}_j + Ia_j\, \ddot{q}_j \qquad [10.12]$$

For a terminal link, ${}^{j}\mathbf{m}_k$ and ${}^{j}\mathbf{f}_k$ are zero.

### 10.2.3. *Direct dynamic model of tree structured robots*

Similarly, the computation of the direct dynamic model of tree structured robots can be obtained using the two methods presented in § 9.7 without any particular difficulty.

### 10.2.4. *Determination of the base inertial parameters*

All the results concerning the base inertial parameters of serial robots can be generalized for tree structured robots [Khalil 89b], [Khalil 95a]. Thereby, equations [9.37] and [9.38], or [9.48] and [9.49], giving the conditions of elimination or grouping of the inertial parameters, are valid. To expose the computation of the base parameters of tree structured robots, we recall that a main branch is composed of the set of links of a path connecting the base to a terminal link. Thus, there are as many main branches as the number of terminal links.

The parameters having no effect on the dynamic model for a tree structure can be obtained by applying the rules derived for serial robots to each main branch.

As in the case of serial robots (§ 9.4.2.2), the general grouped parameters for tree structures will concern the parameters $YY_j$, $MZ_j$ and $M_j$ if joint j is revolute, and the elements of the inertia tensor $J_j$ if joint j is prismatic. The general grouping equations are different than those of Chapter 9, because certain frames may be defined by six geometric parameters.

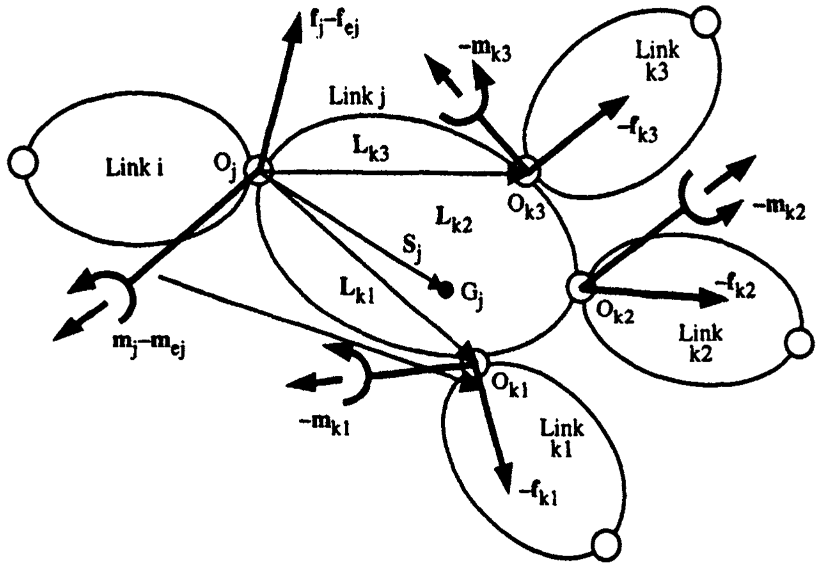


**Figure 10.1.** *Forces and moments acting on a link of a tree structure*

#### 10.2.4.1. *General grouping equations*

The recursive equation between the energy functions of two successive links is given by:

$$\mathbf{h}_j = \mathbf{h}_i \, {}^i\boldsymbol{\lambda}_j + \dot{q}_j \, \boldsymbol{\eta}_j \qquad [10.13]$$

where $\mathbf{h}_j$ is the (1x10) row matrix containing the energy functions of link j denoted by $[\, h_{XXj} \;\; h_{XYj} \;\; \cdots \;\; h_{Mj} \,]$, while ${}^i\boldsymbol{\lambda}_j$ is the (10x10) matrix expressing the transformation of the inertial parameters of a link from frame $R_j$ to frame $R_i$. The general form of $\boldsymbol{\eta}_j$ and ${}^i\boldsymbol{\lambda}_j$, with i = a(j), is developed in Appendix 6. We deduce that equations [9.50] and [9.54] are valid for tree structures after replacing $j-1$ by i, which leads to the following theorem:

**Theorem 10.1.** If joint j is revolute, then the parameters $YY_j$, $MZ_j$ and $M_j$ can be grouped with the parameters of link i and link j, with i = a(j). The general grouping equations are the following:

$$XXR_i = XX_i - YY_j \qquad [10.14a]$$

$$\mathbf{KR}_i = \mathbf{K}_i + YY_j \,({}^i\boldsymbol{\lambda}_j^1 + {}^i\boldsymbol{\lambda}_j^4) + MZ_j \, {}^i\boldsymbol{\lambda}_j^9 + M_j \, {}^i\boldsymbol{\lambda}_j^{10} \qquad [10.14b]$$

where ${}^i\boldsymbol{\lambda}_j^k$ is the $k^{th}$ column of the matrix ${}^i\boldsymbol{\lambda}_j$, which is a function of the geometric parameters defining frame $R_j$. In the following formulas, the corresponding subscript j has been dropped for simplicity. From Appendix 6, we obtain:

$${}^i\boldsymbol{\lambda}_j^1 + {}^i\boldsymbol{\lambda}_j^4 = \begin{bmatrix} 1-SS\gamma SS\alpha \\ CS\gamma SS\alpha \\ -S\gamma CS\alpha \\ 1-CC\gamma SS\alpha \\ C\gamma CS\alpha \\ SS\alpha \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [10.15]$$

$$
{}^{i}\lambda_j^9 = \begin{bmatrix} 2P_zC\alpha-2P_yC\gamma S\alpha \\ P_xC\gamma S\alpha-P_yS\gamma S\alpha \\ -P_xC\alpha-P_zS\gamma S\alpha \\ 2P_xS\gamma S\alpha+2P_zC\alpha \\ -P_yC\alpha+P_zC\gamma S\alpha \\ 2P_xS\gamma S\alpha-2P_yC\gamma S\alpha \\ S\gamma S\alpha \\ -C\gamma S\alpha \\ C\alpha \\ 0 \end{bmatrix} \quad [10.16], \qquad {}^{i}\lambda_j^{10} = \begin{bmatrix} P_y^2+P_z^2 \\ -P_xP_y \\ -P_xP_z \\ P_x^2+P_z^2 \\ -P_yP_z \\ P_x^2+P_y^2 \\ P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \quad [10.17]
$$

where $P_x$, $P_y$ and $P_z$ denote the coordinates of the vector ${}^{i}\mathbf{P}_j$, which can be obtained from equation [7.3] giving the general transformation matrix ${}^{i}\mathbf{T}_j$ such that:

$$
{}^{i}\mathbf{P}_j = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} d_j\, C\gamma_j + r_jS\gamma_jS\alpha_j \\ d_j\, S\gamma_j - r_jC\gamma_jS\alpha_j \\ r_jC\alpha_j + b_j \end{bmatrix} \qquad [10.18]
$$

After expanding equations [10.14], we obtain:

$$
\begin{aligned}
XXR_j &= XX_j - YY_j \\
XXR_i &= XX_i + YY_j\,(1 - SS\gamma_jSS\alpha_j) + 2\,MZ_j\,(P_zC\alpha_j - P_yC\gamma_jS\alpha_j) + M_j\,(P_y^2 + P_z^2) \\
XYR_i &= XY_i + YY_j\,(CS\gamma_jSS\alpha_j) + MZ_j\,(P_xC\gamma_jS\alpha_j - P_yS\gamma_jS\alpha_j) + M_j\,(-P_xP_y) \\
XZR_i &= XZ_i - YY_j\,(S\gamma_jCS\alpha_j) + MZ_j\,(-P_xC\alpha_j - P_zS\gamma_jS\alpha_j) + M_j\,(-P_xP_z) \\
YYR_i &= YY_i + YY_j\,(1 - CC\gamma_jSS\alpha_j) + 2\,MZ_j\,(P_xS\gamma_jS\alpha_j + P_zC\alpha_j) + M_j\,(P_x^2 + P_z^2) \\
YZR_i &= YZi + YY_j\,(C\gamma_jCS\alpha_j) + MZ_j\,(-P_y\,C\alpha_j + P_zC\gamma_jS\alpha_j) + M_j\,(-P_yP_z) \qquad [10.19] \\
ZZR_i &= ZZ_i + YY_jSS\alpha_j + 2MZ_j\,(P_x\,S\gamma_jS\alpha_j - P_yC\gamma_jS\alpha_j) + M_j(P_x^2 + P_y^2) \\
MXR_i &= MX_i + MZ_j\,(S\gamma_jS\alpha_j) + M_j\,P_x \\
MYR_i &= MY_i - MZ_j\,(C\gamma_jS\alpha_j) + M_j\,P_y \\
MZR_i &= MZ_i + MZ_j\,C\alpha_j + M_j\,P_z \\
MR_i &= M_i + M_j
\end{aligned}
$$

with SS(*) = sin(*) sin(*), CC(*) = cos(*) cos(*) and CS(*) = cos(*) sin(*).

**Theorem 10.2.** If joint j is prismatic, then the elements of the inertia tensor ${}^{j}\mathbf{J}_j$ can be grouped with those of ${}^{i}\mathbf{J}_i$ using the following equation:

$$
\mathbf{KR}_i = \mathbf{K}_i + {}^{i}\lambda_j^1\, XX_j + {}^{i}\lambda_j^2\, XY_j + \ldots + {}^{i}\lambda_j^6\, ZZ_j \qquad [10.20a]
$$

which is equivalent to:

$$ {}^{i}\mathbf{JR}_i = {}^{i}\mathbf{J}_i + {}^{i}\mathbf{A}_j\,{}^{j}\mathbf{J}_j\,{}^{j}\mathbf{A}_i \qquad [10.20b] $$

The expanded expressions of these equations are too complicated to be developed here.

### 10.2.4.2. *Particular grouped parameters*

Particular grouped parameters can be obtained by applying the results of serial robots to each main branch b. Let $r_{1b}$ be the first revolute joint of branch b, and $r_{2b}$ be the subsequent revolute joint whose axis is not parallel to the $r_{1b}$ axis. Additional grouping and/or elimination of certain elements among $MX_j$, $MY_j$ and $MZ_j$ takes place if j is prismatic and lies between $r_{1b}$ and $r_{2b}$. For simplicity, the subscript b is dropped in the remainder of this section. Two cases are considered:

*i) the axis of the prismatic joint j is not parallel to the $r_1$ axis.* In this case, the coefficients $h_{MXj}$, $h_{MYj}$ and $h_{MZj}$ satisfy the equation:

$$ {}^{j}a_{xr1}\,h_{MXj} + {}^{j}a_{yr1}\,h_{MYj} + {}^{j}a_{zr1}\,h_{MZj} = \text{constant} \qquad [10.21] $$

where ${}^{j}\mathbf{a}_{r1} = [\ {}^{j}a_{xr1}\ \ {}^{j}a_{yr1}\ \ {}^{j}a_{zr1}\ ]^T$ is the unit vector of the $\mathbf{z}_{r1}$ axis referred to frame $R_j$. The corresponding grouping equations are given in Table 10.1;

*ii) the axis of the prismatic joint j is parallel to the $r_1$ axis.* The following equation is satisfied:

$$ [\mathbf{h}_{MSj}]^T = {}^{j}\mathbf{A}_i\,[\mathbf{h}_{MSi}]^T - [\ 2P_x\dot{h}_{ZZk}\ \ 2P_y h_{ZZk}\ \ 0\ ]^T \qquad [10.22] $$

where k denotes the nearest revolute joint from j back to the base, $k \geq r_1$; $\mathbf{h}_{MSj} = [\ h_{MXj}\ \ h_{MYj}\ \ h_{MZj}\ ]$; $P_x$ and $P_y$ are the first and second coordinates of ${}^{j}\mathbf{P}_i$ respectively, with i = a(j). Using equation [7.4], we obtain:

$$ {}^{j}\mathbf{P}_i = [\ P_x\ \ P_y\ \ P_z\ ]^T = [\ -b_jS\theta_jS\alpha_j - d_jC\theta_j\ \ -b_jC\theta_jS\alpha_j + d_jS\theta_j\ \ -b_jC\alpha_j - r_j\ ]^T $$

Therefore, we deduce that the parameter $MZ_j$ has no effect on the dynamic model and the parameters $MX_j$ and $MY_j$ can be grouped with the first moments of link i, and with the parameter $ZZ_k$ of link k using the following equations:

$$ \begin{aligned} MXR_i &= MX_i + (C\gamma_jC\theta_j - S\gamma_jC\alpha_jS\theta_j)\,MX_j - (C\gamma_jS\theta_j + S\gamma_jC\alpha_jC\theta_j)\,MY_j \\ MYR_i &= MY_i + (S\gamma_jC\theta_j + C\gamma_jC\alpha_jS\theta_j)\,MX_j + (-S\gamma_jS\theta_j + C\gamma_jC\alpha_jC\theta_j)\,MY_j \\ MZR_i &= MZ_i + S\theta_j\,S\alpha_j\,MX_j + C\theta_j\,S\alpha_j\,MY_j \\ ZZR_k &= ZZ_k + 2\,(d_j\,C\theta_j + b_j\,S\theta_j\,S\alpha_j)\,MX_j - 2\,(d_j\,S\theta_j + b_j\,C\theta_j\,S\alpha_j)\,MY_j \end{aligned} \qquad [10.23] $$

**Table 10.1.** *Grouped parameters if $r_1 < j < r_2$, $\sigma_j = 1$, and joint j axis is not parallel to the $r_1$ axis*

| Conditions | Grouping or elimination |
|---|---|
| ${}^{j}a_{zr1} \neq 0$ | $MXR_j = MX_j - \frac{{}^{j}a_{xr1}}{{}^{j}a_{zr1}} MZ_j$ <br> $MYR_j = MY_j - \frac{{}^{j}a_{yr1}}{{}^{j}a_{zr1}} MZ_j$ |
| ${}^{j}a_{zr1} = 0, {}^{j}a_{xr1}{}^{j}a_{yr1} \neq 0$ | $MXR_j = MX_j - \frac{{}^{j}a_{xr1}}{{}^{j}a_{yr1}} MY_j$ |
| ${}^{j}a_{zr1} = 0, {}^{j}a_{xr1} = 0$ | $MY_j \equiv 0$ |
| ${}^{j}a_{zr1} = 0, {}^{j}a_{yr1} = 0$ | $MX_j \equiv 0$ |

Therefore, the practical rules for computing the base inertial parameters given in § 9.4.2.4 can be applied for the tree structure case. The only difference is that the joints $r_1$ and $r_2$ should be defined for each main branch b as $r_{1b}$ and $r_{2b}$ respectively. Thus, a rule like "if $j$ is such that $r_1 < j < r_2$" means in the tree structure case "if $j$ is lying between $r_{1b}$ and $r_{2b}$". From this algorithm, it occurs that the number of minimum inertial parameters for the links (without considering the inertia of rotors) of a general robot is less than $b_m$, such that:

$$b_m \leq 7\, n_r + 4\, n_p - 4\, n_{r0} - 3\, n_{p0} - 2\, n_{g0} \qquad [10.24]$$

with:

- $n_r$: number of revolute joints $= \sum \bar{\sigma}_j$;
- $n_p$: number of prismatic joints $= \sum \sigma_j$;
- $n_{r0}$: number of revolute joints connected directly to the base;
- $n_{p0}$: number of prismatic joints connected directly to the base;
- $n_{g0}$: number of revolute joints connected directly to the base and whose axes are parallel to gravity.

The grouped parameters of the rotor inertias concern those of the actuators of joints ($p_{1b}$, $r_{1b}$ and $r_{2b}$), where $p_{1b}$ denotes the first prismatic joint of the main branch b. They can be obtained by applying the results of serial robots (§ 9.4.2.5) for each branch of the tree structure.

### 10.3. Dynamic model of robots with closed kinematic chains

Many methods have been proposed in the literature to compute the dynamic models of robots containing closed kinematic chains. Among them, let us mention the works of [Chace 67], [Uicker 69], [Chace 71], [Baumgarte 72], [Wittenburg 77], [Megahed 84], [Touron 84], [Luh 85b], [Wittenburg 85], [Kleinfinger 86b], [Giordano 86]. The dynamic model developed in this section is based on firstly computing the dynamic model of an equivalent tree structure, then by multiplying it by the Jacobian matrix representing the derivative of the tree structure variables with respect to the actuated variables [Kleinfinger 86b].

#### 10.3.1. *Description of the system*

The geometry of the robot is described using the method presented in Chapter 7. The system is composed of L joints and n + 1 links, where link 0 is the base. N joints are actuated (active) and the other L – N joints are unactuated (passive). The number of independent closed loops B is equal to L – n. We assume that the structure is controllable and has the minimum number of actuators, thus the number of actuated joints that represent the independent variables is equal to the number of degrees of freedom of the mechanism.

We construct an equivalent tree structure by virtually cutting each loop at one of its passive joints as has already been explained in § 7.3. Since a closed loop contains several unactuated joints, we select the cut joint in such a way that the difference between the number of links of the branches from the root of the loop to the cut joint is as small as possible. This choice reduces the computational complexity of the dynamic model [Kleinfinger 86a].

We represent the tree structure variables by the (nx1) vector $\mathbf{q}_{tr}$, and the cut joints by the (Bx1) vector $\mathbf{q}_c$. The total joint variables are given by equation [7.7]:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_{tr} \\ \mathbf{q}_c \end{bmatrix} \qquad [10.25]$$

The vector $\mathbf{q}_{tr}$ is partitioned into the (Nx1) vector of active joints $\mathbf{q}_a$ and the (px1) vector of passive joints $\mathbf{q}_p$:

$$\mathbf{q}_{tr} = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_p \end{bmatrix} \qquad [10.26]$$

The relation between $\mathbf{q}_a$ and $\mathbf{q}_p$ is obtained by solving the loop closure equations (§ 7.3). The constraint kinematic equations of first and second order have already been derived in § 7.8 and are rewritten here as:

$$[\ \mathbf{W}_a\ \ \mathbf{W}_p\ ]\begin{bmatrix}\dot{\mathbf{q}}_a\\ \dot{\mathbf{q}}_p\end{bmatrix} = \mathbf{0} \qquad [10.27]$$

$$[\ \mathbf{W}_a\ \ \mathbf{W}_p\ ]\begin{bmatrix}\ddot{\mathbf{q}}_a\\ \ddot{\mathbf{q}}_p\end{bmatrix} + \mathbf{\Psi} = \mathbf{0} \qquad [10.28]$$

where $\mathbf{W}_a$ and $\mathbf{W}_p$ are (pxN) and (pxp) matrices respectively. In regular configurations, the rank of $\mathbf{W}_p$ is equal to p. Thus, from equation [10.27], we obtain:

$$\dot{\mathbf{q}}_p = \mathbf{W}\,\dot{\mathbf{q}}_a \qquad [10.29]$$

where:

$$\mathbf{W} = -\mathbf{W}_p^{-1}\,\mathbf{W}_a \qquad [10.30]$$

## 10.3.2. *Computation of the inverse dynamic model*

If the joint positions and velocities can be expressed in terms of the independent actuated variables, we can use the standard Lagrange equation [9.4] to get the dynamic model of the closed chain structure [Desbats 90]. Otherwise, we have to use the Lagrange equation with constraints such that:

$$\Gamma_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \left[\frac{\partial \boldsymbol{\phi}(\mathbf{q}_{tr})}{\partial q_i}\right]^T \boldsymbol{\lambda} \quad i = 1, \ldots, n \qquad [10.31a]$$

where $L(\mathbf{q}_{tr}, \dot{\mathbf{q}}_{tr}, \ddot{\mathbf{q}}_{tr})$ is the Lagrangian of the equivalent tree structure; $\boldsymbol{\phi}(\mathbf{q}_{tr}) = \mathbf{0}$ is the vector containing the p independent constraint functions of the loop closure equations; $\boldsymbol{\lambda} = [\ \lambda_1\ \cdots\ \lambda_p\ ]^T$ is the Lagrange multiplier vector.

This equation can be rewritten as:

$$\mathbf{\Gamma} = \mathbf{\Gamma}_{tr}(\mathbf{q}_{tr}, \dot{\mathbf{q}}_{tr}, \ddot{\mathbf{q}}_{tr}) + \left[\frac{\partial \boldsymbol{\phi}(\mathbf{q}_{tr})}{\partial \mathbf{q}_{tr}}\right]^T \boldsymbol{\lambda} \qquad [10.31b]$$

where $\mathbf{\Gamma}_{tr}$ represents the inverse dynamic model of the equivalent tree structure. It is a function of $\mathbf{q}_{tr}$, $\dot{\mathbf{q}}_{tr}$ and $\ddot{\mathbf{q}}_{tr}$.

The general form of the dynamic model of the tree structure is given by:

$$\Gamma_{tr} = \mathbf{A}_{tr} \begin{bmatrix} \ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_p \end{bmatrix} + \mathbf{H}_{tr} \qquad [10.32]$$

where $\mathbf{A}_{tr}$ is the inertia matrix of the equivalent tree structure, and $\mathbf{H}_{tr}$ is the vector of centrifugal, Coriolis and gravity torques of the equivalent tree structure.

Using equation [10.27], we deduce that:

$$\frac{\partial \phi(\mathbf{q}_{tr})}{\partial \mathbf{q}_{tr}} = [\ \mathbf{W}_a \ \ \mathbf{W}_p\ ] \qquad [10.33]$$

The term containing the Lagrange multipliers represents the reaction forces transmitted by the cut joints to ensure that the loops remain closed. Let us decompose $\Gamma_{tr}$ in a similar way to equation [10.26]:

$$\Gamma_{tr} = \begin{bmatrix} \Gamma_a \\ \Gamma_p \end{bmatrix} \qquad [10.34]$$

where $\Gamma_a$ and $\Gamma_p$ denote the torques of the actuated and unactuated joints of the equivalent tree structure respectively.

The joint torques of the closed chain robot are given as:

$$\Gamma = \begin{bmatrix} \Gamma_{cl} \\ \mathbf{0}_{p\times 1} \end{bmatrix} \qquad [10.35]$$

where $\Gamma_{cl}$ denotes the torques of the N actuated joints, and the zero vector corresponds to the torques of the passive joints:

$$\Gamma = \begin{bmatrix} \Gamma_{cl} \\ \mathbf{0}_{p\times 1} \end{bmatrix} = \begin{bmatrix} \Gamma_a \\ \Gamma_p \end{bmatrix} + \begin{bmatrix} \mathbf{W}_a^T \lambda \\ \mathbf{W}_p^T \lambda \end{bmatrix} \qquad [10.36]$$

We have thus a system of n equations where the unknowns are $\Gamma_{cl}$ and $\lambda$. Computing the Lagrange multipliers from the lower part of equation [10.36] leads to:

$$\lambda = -[\mathbf{W}_p^T]^{-1}\,\Gamma_p \qquad [10.37]$$

Substituting [10.37] into the upper part of [10.36] yields:

$$\Gamma_{cl} = \Gamma_a - \mathbf{W}_a^T\,[\mathbf{W}_p^T]^{-1}\,\Gamma_p \qquad [10.38]$$

Using equation [10.30], the actuator torque vector of the closed chain robot is written as:

$$\Gamma_{cl} = \Gamma_a + \mathbf{W}^T\,\Gamma_p = [\ \mathbf{I}_N \quad \mathbf{W}^T\ ]\begin{bmatrix}\Gamma_a \\ \Gamma_p\end{bmatrix} \qquad [10.39]$$

which can be rewritten as:

$$\Gamma_{cl} = [\ \mathbf{I}_N \quad \mathbf{W}^T\ ]\,\Gamma_{tr} = \left[\ [\frac{\partial \mathbf{q}_a}{\partial \mathbf{q}_a}]^T \quad [\frac{\partial \mathbf{q}_p}{\partial \mathbf{q}_a}]^T\ \right]\Gamma_{tr} = \mathbf{G}^T\,\Gamma_{tr} \qquad [10.40]$$

where $\mathbf{I}_N$ is the (NxN) identity matrix; $\mathbf{W}$ is the Jacobian matrix representing the derivative of the passive joint positions with respect to the actuated ones; $\mathbf{G}$ is the Jacobian matrix representing the derivative of $\mathbf{q}_{tr}$ with respect to $\mathbf{q}_a$, equal to $\partial \mathbf{q}_{tr}/\partial \mathbf{q}_a$.

Equation [10.40] constitutes the inverse dynamic model of the closed chain structure. The vector $\Gamma_{tr}$ can be computed using the efficient recursive Newton-Euler algorithm described in § 10.2.2.

### 10.3.3. *Computation of the direct dynamic model*

To simulate the dynamics of a closed chain robot with a given input torque for the active joints $\Gamma_{cl}$ and a given state ($\mathbf{q}_a$, $\dot{\mathbf{q}}_a$), the dynamic equation [10.40] is formulated and solved for the independent accelerations $\ddot{\mathbf{q}}_a$. The accelerations are then numerically integrated to obtain the velocities and positions at the next sampling time. This process is repeated until integration through the time interval of interest is completed. The direct dynamic model can be obtained by formulating the Lagrange dynamic model as follows:

$$\Gamma_{cl} = \mathbf{A}_{cl}\,\ddot{\mathbf{q}}_a + \mathbf{H}_{cl} \qquad [10.41]$$

where $\mathbf{A}_{cl}$ is the inertia matrix and $\mathbf{H}_{cl}$ is the vector of centrifugal, Coriolis and gravity torques of the closed chain structure.

To derive $\mathbf{A}_{cl}$ and $\mathbf{H}_{cl}$, we express $\ddot{\mathbf{q}}_p$ as a function of $\ddot{\mathbf{q}}_a$ in equation [10.40], then we identify the result with equation [10.41]. Using equations [10.28] and [10.30], we deduce that:

$$\ddot{\mathbf{q}}_p = \mathbf{W}\,\ddot{\mathbf{q}}_a - \mathbf{W}_p^{-1}\,\boldsymbol{\Psi} \qquad [10.42]$$

Using equation [10.32], we rewrite equation [10.40] as:

$$\boldsymbol{\Gamma}_{cl} = [\ \mathbf{I}_N \ \ \mathbf{W}^T\ ]\,\mathbf{A}_{tr}\begin{bmatrix}\ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_p\end{bmatrix} + [\ \mathbf{I}_N \ \ \mathbf{W}^T\ ]\,\mathbf{H}_{tr} \qquad [10.43]$$

Partitioning the matrix $\mathbf{A}_{tr}$ and the vector $\mathbf{H}_{tr}$ to explicit the terms corresponding to the active and passive joints gives:

$$\mathbf{A}_{tr} = \begin{bmatrix}\mathbf{A}_{aa} & \mathbf{A}_{ap} \\ \mathbf{A}_{pa} & \mathbf{A}_{pp}\end{bmatrix},\ \mathbf{H}_{tr} = \begin{bmatrix}\mathbf{H}_a \\ \mathbf{H}_p\end{bmatrix} \qquad [10.44]$$

where $\mathbf{A}_{pa} = [\mathbf{A}_{ap}]^T$.

By combining equations [10.43] and [10.44], we obtain:

$$\boldsymbol{\Gamma}_{cl} = \mathbf{A}_{aa}\,\ddot{\mathbf{q}}_a + \mathbf{A}_{ap}\,[\mathbf{W}\,\ddot{\mathbf{q}}_a - \mathbf{W}_p^{-1}\,\boldsymbol{\Psi}] + \mathbf{W}^T\mathbf{A}_{pa}\,\ddot{\mathbf{q}}_a + \\ \mathbf{W}^T\mathbf{A}_{pp}\,[\mathbf{W}\,\ddot{\mathbf{q}}_a - \mathbf{W}_p^{-1}\,\boldsymbol{\Psi}] + \mathbf{H}_a + \mathbf{W}^T\mathbf{H}_p \qquad [10.45]$$

Identifying equations [10.41] and [10.45] leads to:

$$\mathbf{A}_{cl} = \mathbf{A}_{aa} + \mathbf{A}_{ap}\,\mathbf{W} + \mathbf{W}^T\mathbf{A}_{pa} + \mathbf{W}^T\mathbf{A}_{pp}\,\mathbf{W} \qquad [10.46]$$

$$\mathbf{H}_{cl} = \mathbf{H}_a + \mathbf{W}^T\mathbf{H}_p - (\mathbf{A}_{ap} + \mathbf{W}^T\mathbf{A}_{pp})\,\mathbf{W}_p^{-1}\,\boldsymbol{\Psi} \qquad [10.47]$$

The solution of [10.41] gives the active joint accelerations, then the passive joint accelerations can be computed from equation [10.42]. Although the active joint accelerations are obtained by solving the linear system [10.41] without inverting the inertia matrix, we generally denote the direct dynamic model by:

$$\ddot{\mathbf{q}}_a = \mathbf{A}_{cl}^{-1}\,(\boldsymbol{\Gamma}_{cl} - \mathbf{H}_{cl}) \qquad [10.48]$$

• **Example 10.1.** Dynamic model of the Acma SR400 robot. The geometry and the constraint equations of the loop of this robot are treated in Example 7.1. The inverse dynamic model of the tree structured robot is computed using the recursive Newton-Euler algorithm quoted in § 10.2.2. To compute the dynamic model of the closed chain, we have to calculate the Jacobian matrix **G** representing the derivative of the variables $\mathbf{q}_{tr}$ with respect to the variables $\mathbf{q}_a$. We recall that:

$$\begin{aligned}
\mathbf{q}_a &= [\theta_1 \;\; \theta_2 \;\; \theta_4 \;\; \theta_5 \;\; \theta_6 \;\; \theta_7]^T \\
\mathbf{q}_p &= [\theta_3 \;\; \theta_8]^T \\
\mathbf{q}_{tr} &= [\theta_1 \;\; \theta_2 \;\; \theta_4 \;\; \theta_5 \;\; \theta_6 \;\; \theta_7 \;\; \theta_3 \;\; \theta_8]^T
\end{aligned}$$

The constraint equations are obtained in Example 7.1 as:

$$\begin{aligned}
\theta_3 &= \theta_7 + \pi/2 - \theta_2 \\
\theta_8 &= -\theta_7 + \theta_2
\end{aligned}$$

From these equations, we obtain:

$$\mathbf{G}^T = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & -1
\end{bmatrix}$$

The actuated torques of the closed chain robot is computed in terms of the joint torques of the tree structure as:

$$\begin{aligned}
\Gamma_{cl1} &= \Gamma_{tr1} \\
\Gamma_{cl2} &= \Gamma_{tr2} - \Gamma_{tr3} + \Gamma_{tr8} \\
\Gamma_{cl4} &= \Gamma_{tr4} \\
\Gamma_{cl5} &= \Gamma_{tr5} \\
\Gamma_{cl6} &= \Gamma_{tr6} \\
\Gamma_{cl7} &= \Gamma_{tr7} + \Gamma_{tr3} - \Gamma_{tr8}
\end{aligned}$$

where $\Gamma_{clj}$ and $\Gamma_{trj}$ denote the torque of joint j in the closed structure and in the tree structure respectively.

### 10.3.4. *Base inertial parameters of closed chain robots*

Since the matrix **G** is a function of the geometric parameters, we deduce from equation [10.40] that the minimum inertial parameters of the tree structure are valid to compute the dynamic model of the closed chain robot. The constraint equations of the loops may lead to additional elimination or grouping of certain inertial parameters. To compute the grouped parameters for the closed chain robot, we have to find the linear relations between the energy functions of the inertial parameters. We note that the expressions of the energy functions of the closed chain robot are obtained from those of the tree structure after expressing them as a function of the positions and velocities of the active joints. There is no complete symbolic solution for a general closed chain robot. Therefore, the numerical method developed in Appendix 5 [Gautier 91] can be used for this purpose. However, certain general grouped parameters can be obtained without solving the closure equations of the loops. These parameters belong to the links connected to the cut joints [Khalil 95a]. Furthermore, in § 10.3.5, we will show that the grouped parameters of a parallelogram closed loop can be computed explicitly.

Referring to the notations of the closed chain robots described in Chapter 7, we assume that frame $R_k$ and frame $R_{k+B}$ denote the frames placed on the cut joint k connecting link i to link j, with i = a(k) and j = a(k + B). Since frames $R_k$ and $R_{k+B}$ are aligned, then the kinematic screws of these frames are the same. Consequently, we deduce from equation [9.45] that the energy functions $\mathbf{h}_k$ are equal to $\mathbf{h}_{k+B}$:

$$\mathbf{h}_k = \mathbf{h}_{k+B} \tag{10.49}$$

Using the recursive equation of the energy functions [10.13] and by noting that i = a(k), j = a(k + B) and $\dot{q}_{k+B} = 0$, we obtain the following equation:

$$\mathbf{h}_i\, {}^i\boldsymbol{\lambda}_k + \dot{q}_k\, \boldsymbol{\eta}_k = \mathbf{h}_j\, {}^j\boldsymbol{\lambda}_{k+B} \tag{10.50}$$

Since the elements of the matrix ${}^j\boldsymbol{\lambda}_{k+B}$ are constants, two cases are considered to identify the linear combinations between the terms of $\mathbf{h}_i$ and $\mathbf{h}_j$:

*i) for a revolute cut joint,* we obtain the following three linear equations between the energy functions of links i and j:

$$\mathbf{h}_i\, ({}^i\lambda_k^1 + {}^i\lambda_k^4) = \mathbf{h}_j\, ({}^j\lambda_{k+B}^1 + {}^j\lambda_{k+B}^4) \tag{10.51a}$$

$$\mathbf{h}_i\, {}^i\lambda_k^9 = \mathbf{h}_j\, {}^j\lambda_{k+B}^9 \tag{10.51b}$$

$$\mathbf{h}_i\, {}^i\lambda_k^{10} = \mathbf{h}_j\, {}^j\lambda_{k+B}^{10} \tag{10.51c}$$

The left side terms of equations [10.51] are functions of the geometric parameters of frame $R_k$, while those of the right side are functions of the geometric parameters of frame $R_{k+B}$. The expressions of $\lambda$ are given by equations [10.15], [10.16] and [10.17] after considering the appropriate subscript.

We deduce that for any closed loop, if the cut joint is revolute, then we have three linear relations between the elements of $\mathbf{h}_i$ and $\mathbf{h}_j$. There is no general systematic choice for the parameters to be grouped. They must be studied on a case-by-case basis. Furthermore, in some cases, these relations may not lead to three additional grouping parameters with respect to those obtained for the equivalent tree structure;

*ii) for a prismatic cut joint,* we obtain the following six linear equations between the energy functions of links i and j:

$$\mathbf{h}_i \, {}^i\lambda_k^r = \mathbf{h}_j \, {}^j\lambda_{k+B}^r \quad \text{for } r = 1, \ldots, 6 \qquad [10.52]$$

In this case, we can group the parameters of the inertia tensor of link j with those of link i, with j > i, using the following equation:

$$^i\mathbf{JR}_i = {}^i\mathbf{J}_i + {}^i\mathbf{A}_{k+B} \, {}^k\mathbf{A}_j \, {}^j\mathbf{J}_j \, {}^j\mathbf{A}_k \, {}^{k+B}\mathbf{A}_i \qquad [10.53]$$

### 10.3.5. *Base inertial parameters of parallelogram loops*

For parallelogram loops and planar loops, additional general linear relations between the energy functions can be deduced [Khalil 95a]. We develop in this section the grouping relations for parallelogram loops, where all the grouped parameters can be obtained systematically without computing explicitly the energy functions [Bennis 91b]. In fact, we can prove that one parameter among the first moments (MX or MY) of one link of the parallelogram can be grouped using equation [10.51c], and that the inertia tensors of two links can be grouped.

Let us consider a parallelogram loop composed of links k1, k2, k3, k4. We assume that the loop is cut between links k3 and k4 and that link k1 is parallel to link k3, and link k2 is parallel to link k4 (Figure 10.2). Thus:

$$\begin{cases} \omega_{k1} = \omega_{k3} \\ \omega_{k2} = \omega_{k4} \end{cases} \qquad [10.54]$$

Consequently, we can group the inertia tensor ${}^{k3}\mathbf{J}_{k3}$ with ${}^{k1}\mathbf{J}_{k1}$ using the equation:

$$^{k1}\mathbf{JR}_{k1} = {}^{k1}\mathbf{J}_{k1} + {}^{k1}\mathbf{A}_{k3} \, {}^{k3}\mathbf{J}_{k3} \, {}^{k3}\mathbf{A}_{k1} \qquad [10.55]$$

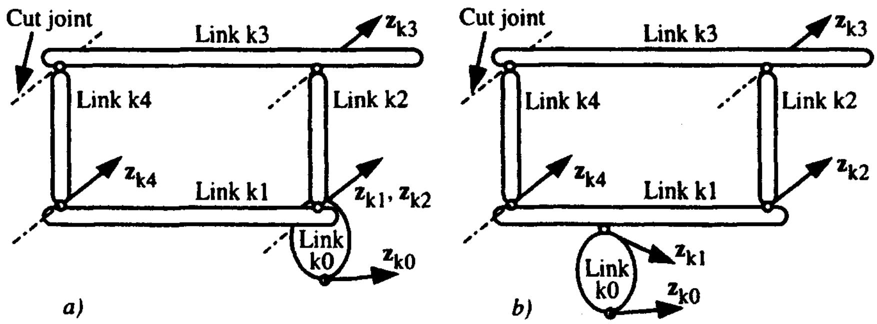


**Figure 10.2.** *Examples of parallelograms*

Similarly, ${}^{k4}\mathbf{J}_{k4}$ can be grouped with ${}^{k2}\mathbf{J}_{k2}$ using the following equation:

$$ {}^{k2}\mathbf{JR}_{k2} = {}^{k2}\mathbf{J}_{k2} + {}^{k2}\mathbf{A}_{k4}\,{}^{k4}\mathbf{J}_{k4}\,{}^{k4}\mathbf{A}_{k2} \qquad [10.56] $$

### 10.3.6. *Practical computation of the base inertial parameters*

Most of the parameters to be eliminated or grouped can be computed by applying the following rules. Firstly, the joints $r_1$ and $r_2$ for each main branch of the equivalent tree structure must be determined. Then, apply the following rules for $j = n, ..., 1$:

1) if link j constitutes a link that is connected to a cut joint, that is to say $j = a(k)$ with $k > n$, then apply either the general grouping equations or the grouping equations of the parallelogram depending on the type of the corresponding loop;

2) group:
   a) $YY_j$, $MZ_j$ and $M_j$ if $\sigma_j = 0$, using Theorem 10.1;
   b) $XX_j$, $XY_j$, $XZ_j$, $YY_j$, $YZ_j$ and $ZZ_j$ if $\sigma_j = 1$, using Theorem 10.2;

3) if joint j is prismatic and $\mathbf{a}_j$ is parallel to $\mathbf{a}_{r1}$ for $r_1 < j < r_2$, then eliminate $MZ_j$ and group $MX_j$ and $MY_j$ using equation [10.23];

4) if joint j is prismatic and $\mathbf{a}_j$ is not parallel to $\mathbf{a}_{r1}$ for $r_1 < j < r_2$, then group or eliminate one of the parameters $MX_j$, $MY_j$, $MZ_j$ using Table 10.1;

5) if joint j is revolute and $r_1 \le j < r_2$, then eliminate $XX_j$, $XY_j$, $XZ_j$ and $YZ_j$. Notice that the axis of this joint is parallel to the axis of joint $r_1$, and that the parameter $YY_j$ has been eliminated by rule 2;

6) if j is revolute and $r_1 \le j < r_2$, and $\mathbf{a}_j$ is along $\mathbf{a}_{r1}$, and if $\mathbf{a}_{r1}$ is parallel to both $\mathbf{a}_i$ and gravity $\mathbf{g}$ for all $i < j$, then eliminate the parameters $MX_j$, $MY_j$. Notice that $MZ_j$ has been eliminated by rule 2;

7) if j is prismatic and $j < r_1$, then eliminate the parameters $MX_j$, $MY_j$, $MZ_j$.

From this algorithm, we deduce that the number of minimum inertial parameters for the links (without considering the inertia of rotors) of a general robot is less than $b_m$, such that:

$$b_m \leq 7\, n_r + 4\, n_p - 4\, n_{r0} - 3\, n_{p0} - 11\, n_{par} - 2\, n_{g0} \qquad [10.57]$$

with:

- $n_r$: number of revolute joints of the equivalent tree structure;
- $n_p$: number of prismatic joints of the equivalent tree structure;
- $n_{g0}$: number of revolute joints that are directly connected to the base and whose axes are parallel to gravity;
- $n_{r0}$: number of revolute joints directly connected to the base;
- $n_{p0}$: number of prismatic joints directly connected to the base;
- $n_{par}$: number of parallelogram loops in the mechanism.

• **Example 10.2.** Computation of the base inertial parameters of the Acma SR400 robot. This structure has two main branches: the first contains links 1, ..., 6, while the second is composed of links 1, 7 and 8. For the first branch, we obtain $r_1 = 1$ and $r_2 = 2$; for the second branch, we find $r_1 = 1$ and $r_2 = 7$. Applying the general algorithm for $j = 8, ..., 1$, we obtain:

*Link 8.* This link constitutes a terminal link in a parallelogram loop. Equation [10.51c] gives:

$$h_{XX3}\, d_8^2 + h_{ZZ3}\, d_8^2 - h_{MY3}\, d_8 + h_{M3} = h_{YY8}\, d_3^2 + h_{ZZ8}\, d_3^2 + h_{MX8}\, d_3 + h_{M8}$$

We choose to group $MX_8$ as follows:

$$YYR_8 = YY_8 - d_3\, MX_8$$
$$ZZR_8 = ZZ_8 - d_3\, MX_8$$
$$MR_8 = M_8 - \frac{MX_8}{d_3}$$
$$XXR_3 = XX_3 + MX_8 \frac{d_8^2}{d_3}$$
$$ZZR_3 = ZZ_3 + MX_8 \frac{d_8^2}{d_3}$$
$$MYR_3 = MY_3 - MX_8 \frac{d_8}{d_3}$$

$$MR_3 = M_3 + \frac{MX_8}{d_3}$$

Equation [10.55] allows us to group $\mathbf{J}_8$ with $\mathbf{J}_2$:

$$^2\mathbf{JR}_2 = {}^2\mathbf{J}_2 + {}^2\mathbf{A}_8\, {}^8\mathbf{J}_8\, {}^8\mathbf{A}_2$$

Since the orientation matrix $^2\mathbf{A}_8 = \mathbf{I}_3$ (equation [7.39]), we obtain:

$$\begin{aligned}
XXR_2 &= XX_2 + XX_8 \\
XYR_2 &= XY_2 + XY_8 \\
XZR_2 &= XZ_2 + XZ_8 \\
YYR_2 &= YY_2 + (YY_8 - d_3\, MX_8) \\
YZR_2 &= YZ_2 + YZ_8 \\
ZZR_2 &= ZZ_2 + (ZZ_8 - d_3\, MX_8)
\end{aligned}$$

Finally, since joint 8 is revolute, we group the parameters $MZ_8$ and $MR_8$ with the parameters of link 7 using equations [10.19]:

$$\begin{aligned}
XZR_7 &= XZ_7 - MZ_8\, d_8 \\
YYR_7 &= YY_7 + M_8\, d_8 - MX_8 \frac{d_8^2}{d_3} \\
ZZR_7 &= ZZ_7 + d_8^2\, M_8 - MX_8 \frac{d_8^2}{d_3} \\
MXR_7 &= MX_7 - MX_8 \frac{d8}{d3} + M_8\, d_8 \\
MR_7 &= M_7 + M_8 - \frac{MX_8}{d_3}
\end{aligned}$$

Thus, concerning link 8, only the parameter $MY_8$ belongs to the base inertial parameters of the robot.

*Link 7.* This link is a terminal link of a parallelogram loop. We group $\mathbf{J}_7$ with $\mathbf{J}_3$ using equation [10.55]:

$$^3\mathbf{JR}_3 = {}^3\mathbf{J}_3 + {}^3\mathbf{A}_7\, {}^7\mathbf{J}_7\, {}^7\mathbf{A}_3$$

Since $^7\mathbf{A}_3 = \mathbf{rot}(\mathbf{z}, \frac{\pi}{2})$, (equation [7.38]), we obtain:

$$\begin{aligned}
XXR_3 &= XX_3 + YY_7 + M_8\, d_8^2 \\
XYR_3 &= XY_3 - XY_7 \\
XZR_3 &= XZ_3 + YZ_7 \\
YYR_3 &= YY_3 + XXR7 = YY_3 + XX_7
\end{aligned}$$

$$YZR_3 = YZ_3 - XZR_7 = YZ_3 - XZ_7 + MZ_8\, d_8$$
$$ZZR_3 = ZZ_3 + ZZ_7 + d_8^2\, M_8$$

We group the parameters $MR_7$ and $MZ_7$ with the parameters of link 1 using the following equation:

$$ZZR_1 = ZZ_1 + (M_7 + M_8)\, d_2^2 - MX_8 \frac{d_2^2}{d_3}$$

Note that $MZ_7$ does not appear in this expression. Thus, it has no effect on the dynamic model. The minimum parameters of link 7 are $MXR_7$ and $MY_7$.

*Link 6.* From Theorem 10.1 and since $a(6) = 5$, we group the parameters $YY_6$, $MZ_6$ and $MR_6$ as follows:

$$XXR_6 = XX_6 - YY_6$$
$$XXR_5 = XX_5 + YY_6$$
$$ZZR_5 = ZZ_5 + YY_6$$
$$MYR_5 = MY_5 + MZ_6$$
$$MR_5 = M_5 + M_6$$

The minimum parameters of link 6 are: $XXR_6$, $XY_6$, $XZ_6$, $YZ_6$, $ZZ_6$, $MX_6$ and $MY_6$.

*Link 5.* We group the parameters $YY_5$, $MZ_5$ and $MR_5$ with those of link 4:

$$XXR_5 = XX_5 + YY_6 - YY_5$$
$$XXR_4 = XX_4 + YY_5$$
$$ZZR_4 = ZZ_4 + YY_5$$
$$MYR_4 = MY_4 - MZ_5$$
$$MR_4 = M_4 + M_5 + M_6$$

The minimum parameters of link 5 are: $XXR_5$, $XY_5$, $XZ_5$, $YZ_5$, $ZZR_5$, $MX_5$ and $MYR_5$.

*Link 4.* We group the parameters $YY_4$, $MZ_4$ and $MR_4$ with those of link 3:

$$XXR_4 = XX_4 + YY_5 - YY_4$$
$$XXR_3 = XX3 + YY7 + M_8\, d_8^2 + YY_4 + 2\, RL_4\, MZ_4 + RL_4^2\, MR_4$$
$$XYR_3 = XY_3 - XY_7 - d_4\, MZ_4 - d_4\, RL_4\, MR_4$$
$$XZR_3 = XZ_3 + YZ_7$$
$$YYR_3 = YY3 + XX7 + d_4^2\, MR_4$$
$$YZR_3 = YZ_3 - XZ_7 + MZ_8\, d_8$$

$$\begin{aligned}
ZZR_3 &= ZZ_3 + ZZ_7 + d_8^2\, M_8 + YY_4 + 2\, RL_4\, MZ_4 + (d_4^2 + RL_4^2)\, MR_4 \\
MXR_3 &= MX_3 + d_4\, MR_4 \\
MYR_3 &= MY_3 - d_8 \frac{MX_8}{d_3} + MZ_4 + RL_4\, MR_4 \\
MR_3 &= M_3 + \frac{MX_8}{d_3} + M_4 + M_5 + M_6
\end{aligned}$$

The minimum parameters of link 4 are: $XXR_4$, $XY_4$, $XZ_4$, $YZ_4$, $ZZR_4$, $MX_4$ and $MYR_4$.

*Link 3.* We group the parameters $YYR_3$, $MZ_3$ and $MR_3$ with those of link 2:

$$\begin{aligned}
XXR_3 &= XX_3 + YY_7 + M8\, d_8^2 + YY_4 + 2RL_4MZ_4 + RL_4^2MR_4 - YY_3 - XX_7 - d_4^2MR_4 \\
XXR_2 &= XX_2 + XX_8 + YY_3 + XX_7 + d_4^2\, MR_4 \\
XZR_2 &= XZ_2 + XZ_8 - d_3\, MZ_3 \\
YYR_2 &= YY_2 + YY_8 + YY_3 - d_3\, MX_8 + XX_7 + d_4^2\, MR_4 + d3^2\, MR_3 \\
ZZR_2 &= ZZ_2 + ZZ_8 - d_3\, MX_8 + d_3^2\, MR_3 \\
MXR_2 &= MX_2 + d_3\, MR_3 \\
MR_2 &= M_2 + MR_3
\end{aligned}$$

The minimum parameters of link 3 are: $XXR_3$, $XYR_3$, $XZR_3$, $YZR_3$, $ZZR_3$, $MXR_3$ and $MYR_3$.

*Link* 2. We group the parameters $YYR_2$, $MZ_2$ and $MR_2$ with those of link 1:

$$\begin{aligned}
XXR_2 &= XX_2 + XX_8 - YY_2 - YY_8 - d_3^2\, MR_3 + d_3\, MX_8 \\
ZZR_1 &= ZZ_1 + (M_7 + M_8)d_2^2 - d_2^2 \frac{MX_8}{d_3} + YY_2 + YY_8 + YY_3 + XX_7 + d_4^2MR_4 + \\
&\quad d_3^2MR_3 - d_3\, MX_8 + d_2^2MR_2
\end{aligned}$$

The minimum parameters of link 2 are: $XXR_2$, $XYR_2$, $XZR_2$, $YZR_2$, $ZZR_2$, $MXR_2$ and $MY_2$. Note that $MZ_2$ does not appear in this expression. Thus, it has no effect on the dynamic model.

*Link 1.* Only the parameter $ZZR_1$ belongs to the base inertial parameters.

Finally, the rotor inertias are treated as shown in § 9.4.2.5, leading to group $I_{a1}$, $I_{a2}$ and $I_{a7}$ with $ZZR_1$, $ZZR_2$ and $ZZR_3$ respectively.

The final result is summarized as follows:

- the following 11 parameters have no effect on the dynamic model: $XX_1$, $XY_1$, $XZ_1$, $YY_1$, $YZ_1$, $MX_1$, $MY_1$, $MZ_1$, $M_1$, $MZ_2$ and $MZ_7$;
- the following 33 parameters have been grouped: $I_{a1}$, $YY_2$, $M_2$, $I_{a2}$, $YY_3$, $MZ_3$, $M_3$, $YY_4$, $MZ_4$, $M_4$, $YY_5$, $MZ_5$, $M_5$, $YY_6$, $MZ_6$, $M_6$, $XX_7$, $XY_7$, $XZ_7$, $YY_7$, $YZ_7$, $ZZ_7$, $M_7$, $I_{a7}$, $XX_8$, $XY_8$, $XZ_8$, $YY_8$, $YZ_8$, $ZZ_8$, $MX_8$, $MZ_8$ and $M_8$;
- the SR400 robot has 42 base parameters (Table 10.2);
- the grouping equations are:

$$
\begin{aligned}
ZZR_1 &= Ia_1 + ZZ_1 + YY_2 + YY_3 + XX_7 + YY_8 + d_4^2(M_4 + M_5 + M_6) + \\
&\quad d_2^2(M_3 + M_4 + M_5 + M_6) + d_3^2(M_2 + M_3 + M_4 + M_5 + M_6) + d_2^2(M_7 + M_8) \\
XXR_2 &= XX_2 - YY_2 + XX_8 - YY_8 - d3^2(M_3 + M_4 + M_5 + M_6) \\
XYR_2 &= XY_2 + XY_8 \\
XZR_2 &= XZ_2 + XZ_8 - d_3MZ_3 \\
YZR_2 &= YZ_2 + YZ_8 \\
ZZR_2 &= Ia_2 + ZZ_2 + ZZ_8 + d_3^2(M_3 + M_4 + M_5 + M_6) \\
MXR_2 &= MX_2 + MX_8 + d_3(M_3 + M_4 + M_5 + M_6) \\
XXR_3 &= XX_3 - YY_3 + YY_4 - XX_7 + YY_7 - d4^2(M_4 + M_5 + M_6) + 2MZ_4RL_4 + \\
&\quad (M_4 + M_5 + M_6)RL_4^2 + d_8^2M_8 \\
XYR_3 &= XY_3 - XY_7 - d_4MZ_4 - d_4RL_4(M_4 + M_5 + M_6) \\
XZR_3 &= XZ_3 + YZ_7 \\
YZR_3 &= YZ_3 - XZ_7 + d_8MZ_8 \\
ZZR_3 &= Ia_7 + ZZ_3 + YY4 + ZZ_7 + d_8^2M_8 + 2MZ_4RL_4 + (M4 + M5 + M6)*(d4^2 + RL4^2) \\
MXR_3 &= MX_3 + d_4(M_4 + M_5 + M_6) \\
MYR_3 &= MY_3 + MZ_4 + (M_4 + M_5 + M_6)RL_4 \\
XXR_4 &= XX_4 - YY_4 + YY_5 \\
ZZR_4 &= YY_5 + ZZ_4 \\
MYR_4 &= MY_4 - MZ_5 \\
XXR_5 &= XX_5 - YY_5 + YY_6 \\
ZZR5 &= YY6 + ZZ_5 \\
MYR_5 &= MY_5 + MZ_6 \\
XXR_6 &= XX_6 - YY_6 \\
MXR_7 &= MX_7 - \frac{d_8}{d_3}MX_8 + d_8M8
\end{aligned}
$$

Table 10.3 illustrates the computational complexity of the inverse dynamic model for the Acma SR400 robot. Two cases are considered: general inertial parameters where all the parameters are assumed to have real values different from zero; and the simplified case where the links are assumed to be symmetric. For each case, the dynamic model is computed twice: firstly with the standard inertial parameters, and secondly with the base inertial parameters. We note that the real time computation of the inverse dynamic model for this robot can be realized using classical personal computers.

**Table 10.2.** *Base inertial parameters of the Acma SR400 robot*

| j | XXj | XYj | XZj | YYj | YZj | ZZj | MXj | MYj | MZj | Mj | Iaj |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | ZZR1 | 0 | 0 | 0 | 0 | 0 |
| 2 | XXR2 | XYR2 | XZR2 | 0 | YZR2 | ZZR2 | MXR2 | MY2 | 0 | 0 | 0 |
| 3 | XXR3 | XYR3 | XZR3 | 0 | YZR3 | ZZR3 | MXR3 | MYR3 | 0 | 0 | 0 |
| 4 | XXR4 | XY4 | XZ4 | 0 | YZ4 | ZZR4 | MX4 | MYR4 | 0 | 0 | Ia4 |
| 5 | XXR5 | XY5 | XZ5 | 0 | YZ5 | ZZR5 | MX5 | MYR5 | 0 | 0 | Ia5 |
| 6 | XXR6 | XY6 | XZ6 | 0 | YZ6 | ZZ6 | MX6 | MY6 | 0 | 0 | Ia6 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | MXR7 | MY7 | 0 | 0 | 0 |
| 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | MY8 | 0 | 0 | 0 |

**Table 10.3.** *Computational complexity of the inverse dynamic model of the Acma SR400 robot*

| Set of inertial parameters | Complete | | Simplified | |
|---|---|---|---|---|
| | Multiplicat. | Additions | Multiplicat. | Additions |
| **Standard parameters** | 430 | 420 | 295 | 245 |
| **Base parameters** | 304 | 326 | 243 | 118 |

## 10.4. Conclusion

In this chapter, we have developed the dynamics of robots with tree structure or containing closed chains. This treatment constitutes a generalization of the results presented in Chapter 9 for serial robots. We can use the efficient Newton-Euler method for computing the inverse and direct dynamic models of tree structured systems. The corresponding base inertial parameters can be determined using the symbolic algorithm, which is composed of simple rules and makes use of closed form grouping equations. Concerning the systems with closed chain, the inverse dynamic model is computed from the inverse dynamic model of the equivalent tree structure and the Jacobian matrix representing the derivative of the joint positions of the equivalent tree structure with respect to the actuated joint positions. The base parameters of general closed chain robots can be completely determined using the numerical method presented in Appendix 5. However, most of them and even all of them in many cases can be computed using the rules of the symbolic algorithm.

From this study, we can conclude that the computation of the dynamic model in real time is now possible using classical personal computers. In Chapters 11 and 12, we direct our attention toward the identification of the geometric and dynamic parameters appearing in the different models of robots.