Chapter 7

Geometric and kinematic models of complex chain robots

7.1. Introduction

In this chapter, we develop a method to describe the geometry of complex robots with tree or closed chain structures. This method constitutes the extension of the notation presented in Chapter 3 for serial robots [Khalil 86a]. We also present the computation of the direct and inverse geometric models of such mechanisms. Finally, we establish their direct and inverse kinematic models. The results are illustrated using the AKR-3000 robot, which contains two closed loops, and the Acma SR400 robot, which contains a parallelogram closed loop.

7.2. Description of tree structured robots

A tree structured robot is composed of n mobile links and n joints. The links are assumed to be perfectly rigid. The joints are either revolute or prismatic and assumed to be ideal (no backlash, no elasticity). A complex joint can be represented by an equivalent combination of revolute and prismatic joints with zero-length massless links.

The links are numbered consecutively from the base to the terminal links. Thus, link 0 is the fixed base and link n is one of the terminal links (Figure 7.1). Joint j connects link j to link a(j), where a(j) denotes the link antecedent to link j, and consequently $a(j) < j$. We define a main branch as the set of links on the path between the base and a terminal link. Thus, a tree structure has as many main branches as the number of terminal links.

The topology of the system is defined by a(j) for j = 1, ..., n. In order to compute the relationship between the locations of the links, we attach a frame R_i to each link i such that:

- $\mathbf{z}_i$ is along the axis of joint i;
- $\mathbf{x}_i$ is taken along the common normal between $\mathbf{z}_i$ and one of the succeeding joint axes, which are fixed on link i. Figure 7.2 shows the case where links j and k are articulated on link i.

Two cases are considered for computing the transformation matrix ${}^i\mathbf{T}_j$, which defines the location of frame R_j relative to frame R_i with i = a(j):

1) if $\mathbf{x}_i$ is along the common normal between $\mathbf{z}_i$ and $\mathbf{z}_j$, then ${}^i\mathbf{T}_j$ is the same as the transformation matrix between two consecutive frames of serial structure. It is obtained as a function of the four geometric parameters (α_j, d_j, θ_j, r_j) as defined in § 3.2 (equation [3.2]):

$$
{}^i\mathbf{T}_j = \mathbf{Rot}(\mathbf{x}, \alpha_j)\, \mathbf{Trans}(\mathbf{x}, d_j)\, \mathbf{Rot}(\mathbf{z}, \theta_j)\, \mathbf{Trans}(\mathbf{z}, r_j)
$$

$$
= \begin{bmatrix} C\theta_j & -S\theta_j & 0 & d_j \\ C\alpha_j S\theta_j & C\alpha_j C\theta_j & -S\alpha_j & -r_j S\alpha_j \\ S\alpha_j S\theta_j & S\alpha_j C\theta_j & C\alpha_j & r_j C\alpha_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [7.1]
$$

2) if $\mathbf{x}_i$ is not along the common normal between $\mathbf{z}_i$ and $\mathbf{z}_j$, then the matrix ${}^i\mathbf{T}_j$ must be defined using six geometric parameters. This case is illustrated in Figure 7.2, where $\mathbf{x}_i$ is along the common normal between $\mathbf{z}_i$ and $\mathbf{z}_k$. To obtain the six parameters defining frame R_j relative to frame R_i, we define $\mathbf{u}_j$ as the common normal between $\mathbf{z}_i$ and $\mathbf{z}_j$. The transformation from frame R_i to frame R_j can be obtained as a function of the six geometric parameters (γ_j, b_j, α_j, d_j, θ_j, r_j) where:
   - γ_j is the angle between $\mathbf{x}_i$ and $\mathbf{u}_j$ about $\mathbf{z}_i$;
   - b_j is the distance between $\mathbf{x}_i$ and $\mathbf{u}_j$ along $\mathbf{z}_i$.

The parameters γ_j and b_j permit to define $\mathbf{u}_j$ with respect to $\mathbf{x}_i$, whereas the classical parameters α_j, d_j, θ_j, r_j permit to define frame R_j with respect to the intermediate frame whose $\mathbf{x}$ axis is along $\mathbf{u}_j$ and $\mathbf{z}$ axis is along $\mathbf{z}_j$.

The transformation matrix ${}^i\mathbf{T}_j$ is obtained as:

$$
{}^i\mathbf{T}_j = \mathbf{Rot}(\mathbf{z}, \gamma_j)\, \mathbf{Trans}(\mathbf{z}, b_j)\, \mathbf{Rot}(\mathbf{x}, \alpha_j)\, \mathbf{Trans}(\mathbf{x}, d_j)\, \mathbf{Rot}(\mathbf{z}, \theta_j)\, \mathbf{Trans}(\mathbf{z}, r_j) \qquad [7.2]
$$

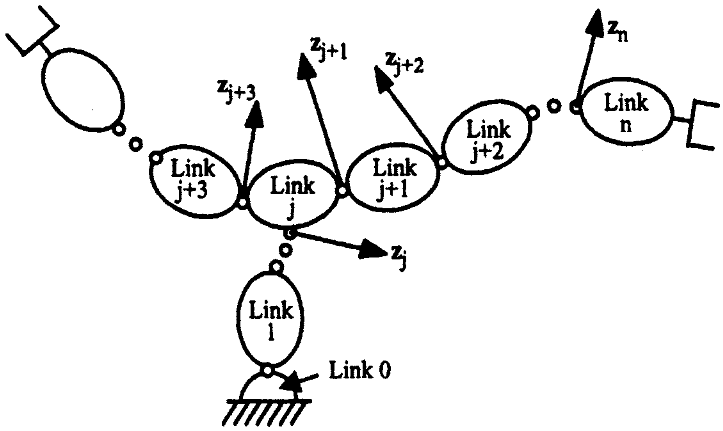

Figure 7.1. *Notations for a tree structured robot*

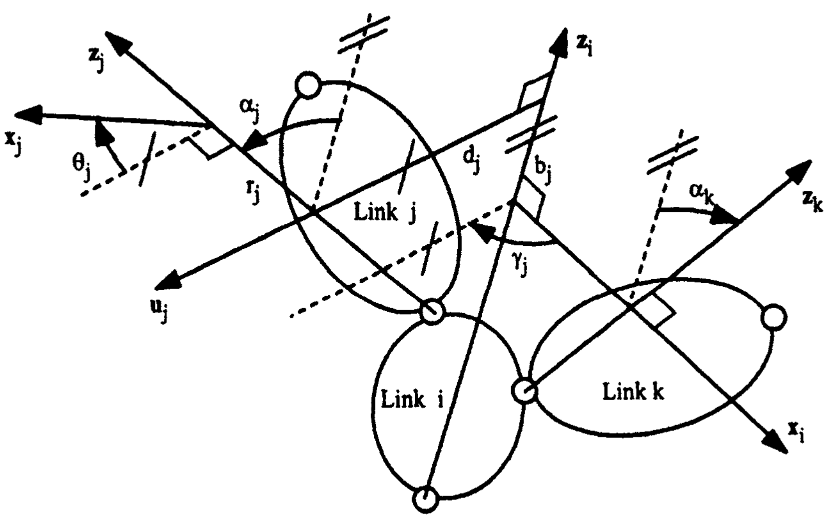

Figure 7.2. *Geometric parameters for a link with more than two joints*

After development, we obtain:

$$
{}^{i}\mathbf{T}_j = \begin{bmatrix} C\gamma_jC\theta_j-S\gamma_jC\alpha_jS\theta_j & -C\gamma_jS\theta_j-S\gamma_jC\alpha_jC\theta_j & S\gamma_jS\alpha_j & d_jC\gamma_j+r_jS\gamma_jS\alpha_j \\ S\gamma_jC\theta_j+C\gamma_jC\alpha_jS\theta_j & -S\gamma_jS\theta_j+C\gamma_jC\alpha_jC\theta_j & -C\gamma_jS\alpha_j & d_jS\gamma_j-r_jC\gamma_jS\alpha_j \\ S\alpha_jS\theta_j & S\alpha_jC\theta_j & C\alpha_j & r_jC\alpha_j+b_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [7.3]
$$

The inverse transformation ${}^{j}\mathbf{T}_i$ is expressed by:

$$
{}^{j}\mathbf{T}_i = \begin{bmatrix} & & & -b_jS\alpha_jS\theta_j-d_jC\theta_j \\ & {}^{i}\mathbf{A}_j^T & & -b_jS\alpha_jC\theta_j+d_jS\theta_j \\ & & & -b_jC\alpha_j-r_j \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [7.4]
$$

NOTES.–

- equation [7.3] represents the general form of the transformation matrix. The special case of serial robots (equation [7.1]) can be obtained from it by setting $b_j = 0$ and $\gamma_j = 0$;
- as for the serial structure, the joint variable q_j is given by:

$$
q_j = \bar{\sigma}_j\,\theta_j + \sigma_j\,r_j \qquad [7.5]
$$

  where $\sigma_j = 0$ if joint j is revolute, $\sigma_j = 1$ if joint j is prismatic and $\bar{\sigma}_j = 1 - \sigma_j$;
- we set $\sigma_j = 2$ to define a frame R_j with constant position and orientation with respect to frame a(j). In this case, q_j and $\bar{\sigma}_j$ are not defined;
- the definition of frame R_0 and the frames fixed to the terminal links can be made as in the case of serial robots.

7.3. Description of robots with closed chains

A closed chain structure consists of a set of rigid links connected to each other with joints where at least one closed loop exists. This structure enhances both the accuracy and the load-carrying capacity of the robot. The system is composed of L joints and n + 1 links, where link 0 is the fixed base and L > n. It may contain several terminal links. The number of independent closed loops is equal to:

$$
B = L - n \qquad [7.6]
$$

The joints are either active or passive. The N active joints are provided with actuators. We assume that the number of actuated joints is equal to the number of degrees of freedom of the robot. Thus, the position and orientation of all the links can be determined as a function of the active joint variables.

We introduce the parameter μ_j such that:

- $\mu_j = 1$ if joint j is actuated (active joint);
- $\mu_j = 0$ if joint j is non-actuated (passive joint).

To determine the geometric parameters of a mechanism with closed chains, we proceed as follows:

a) construct an equivalent tree structure having n joints by virtually cutting each closed chain at one of its passive joints. Since a closed loop contains several passive joints, select the joint to be cut in such a way that the difference between the number of links of the two branches from the root of the loop[1] to the links connected to the cut joint is as small as possible. This choice reduces the computational complexity of the dynamic model [Kleinfinger 86a]. The geometric parameters of the equivalent tree structure are determined as described in the previous section;

b) number the cut joints from n + 1 to L. For each cut joint k, assign a frame R_k fixed on one of the links connected to this joint, for instance link j. The $\mathbf{z}_k$ axis is taken along the axis of joint k, and the $\mathbf{x}_k$ axis is aligned with the common normal between $\mathbf{z}_k$ and $\mathbf{z}_j$ (Figure 7.3). Let i = a(k) where link i denotes the other link of joint k. The transformation matrix from frame R_i to frame R_k can be obtained as a function of the usual six (or four) geometric parameters γ_k, b_k, α_k, d_k, θ_k, r_k, where q_k is equal to θ_k or r_k;

c) since frame R_k is fixed on link j, the transformation matrix between frames R_j and R_k is constant. To avoid any confusion, this transformation will be denoted by ${}^{j}T_{k+B}$, with j = a(k+B). The geometric parameters defining this transformation will have as a subscript k + B. Note that frame R_{k+B} is aligned with frame R_k, and that both r_{k+B} and θ_{k+B} are zero.

In summary, the geometric description of a structure with closed loops is defined by an equivalent tree structure that is obtained by cutting each closed loop at one of its joints and by adding two frames at each cut joint. The total number of frames is equal to n + 2B and the geometric parameters of the last B frames are constants.

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[1] The root of a loop is the first common link when going from the links of the cut joint to the base of the robot.

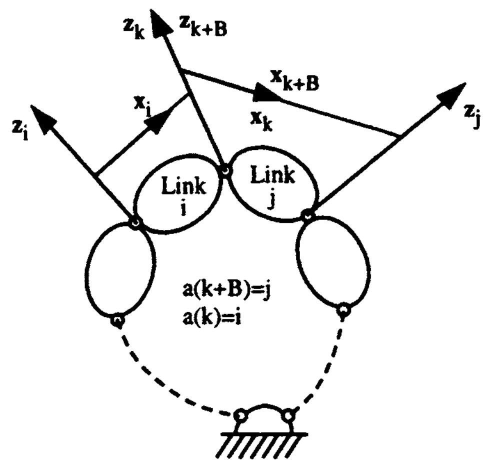

Figure 7.3. *Frames of a cut joint*

The (Lx1) joint variable vector $\mathbf{q}$ is written as:

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_a \\ \mathbf{q}_p \\ \mathbf{q}_c \end{bmatrix} \qquad [7.7]$$

with:

- $\mathbf{q}_a$: vector containing the N active joint variables;
- $\mathbf{q}_p$: vector containing the p=n−N passive joint variables of the equivalent tree structure;
- $\mathbf{q}_c$: vector containing the B variables of the cut joints. When a cut joint has several degrees of freedom (spherical, universal, ...), we can consider all of its joint variables to be belonging to $\mathbf{q}_c$.

Only the N active variables $\mathbf{q}_a$ are independent. Thus, there are c=L−N independent constraint equations between the joint variables $\mathbf{q}$. These relations form the geometric constraint equations satisfying the closure of the loops. Since R_k and R_{k+B} are aligned, the geometric constraint equations for each loop can be written as:

$$^{k+B}\mathbf{T}_j \ldots {}^{i}\mathbf{T}_k = \mathbf{I}_4 \qquad [7.8]$$

where $\mathbf{I}_4$ is the (4x4) identity matrix.

For a spatial loop, the maximum number of independent geometric constraint equations is six, while for a planar loop this number reduces to three. The geometric constraint equations can be obtained from relation [7.8]. They are represented by the nonlinear equation:

$$\boldsymbol{\phi}(\mathbf{q}) = \begin{bmatrix} \phi_1(\mathbf{q}) \\ \phi_2(\mathbf{q}) \\ \cdots \\ \phi_{(L-N)}(\mathbf{q}) \end{bmatrix} = \mathbf{0}_{(L-N)\times 1} \qquad [7.9]$$

To determine the locations of all the links of the closed chain structure, we have to compute the passive joint variables in terms of the active joint variables. For simple mechanisms, equation [7.9] may be solved in an analytical closed-form such that:

$$\mathbf{q}_p = \mathbf{g}_p(\mathbf{q}_a) \qquad [7.10a]$$

$$\mathbf{q}_c = \mathbf{g}_c(\mathbf{q}_a, \mathbf{q}_p) \qquad [7.10b]$$

Otherwise, numerical methods based on the inverse differential model can be used (§ 7.9) [Uicker 69], [Wang 91].

• **Example 7.1.** Description of the geometry of the Acma SR400 robot. This mechanism has six degrees of freedom, eight moving links and nine revolute joints. It contains a parallelogram closed loop. Joints 3, 8 and 9 are passive. The equivalent tree structure is obtained by cutting the loop at joint 9, which connects link 3 and link 8. The link coordinate frames are shown in Figure 7.4. The geometric parameters are given in Table 7.1.

• **Example 7.2.** Description of the AKR-3000 painting robot. This six degree-of freedom robot has 12 joints and 10 links. It contains two independent closed loops. Figure 7.5 shows the link coordinate frames. The first loop is cut at the joint connecting links 5 and 7, and the second loop is cut at the joint connecting links 2 and 6. Joints 1, 5, 6, 8, 9 and 10 are active, while joints 2, 3, 4, 7, 11 and 12 are passive. The geometric parameters are given in Table 7.2.

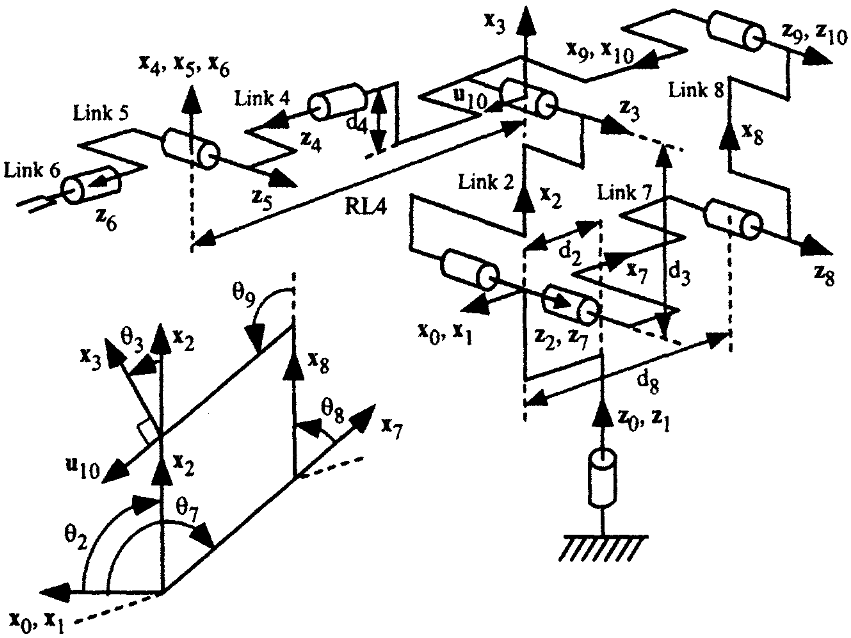

Figure 7.4. *Acma SR400 robot*

Table 7.1. *Geometric parameters of the Acma SR400 robot*

j	a(j)	μ_j	σ_j	γ_j	b_j	α_j	d_j	θ_j	r_j
1	0	1	0	0	0	0	0	θ1	0
2	1	1	0	0	0	$-\pi/2$	d2	θ_2	0
3	2	0	0	0	0	0	d3	θ_3	0
4	3	1	0	0	0	$-\pi/2$	d_4	θ_4	RL4
5	4	1	0	0	0	$\pi/2$	0	θ_5	0
6	5	1	0	0	0	$-\pi/2$	0	θ_6	0
7	1	1	0	0	0	$-\pi/2$	d2	θ_7	0
8	7	0	0	0	0	0	d_8	θ_8	0
9	8	0	0	0	0	0	$d_9=d_3$	θ_9	0
10	3	0	2	$\pi/2$	0	0	$d_{10}=-d_8$	0	0

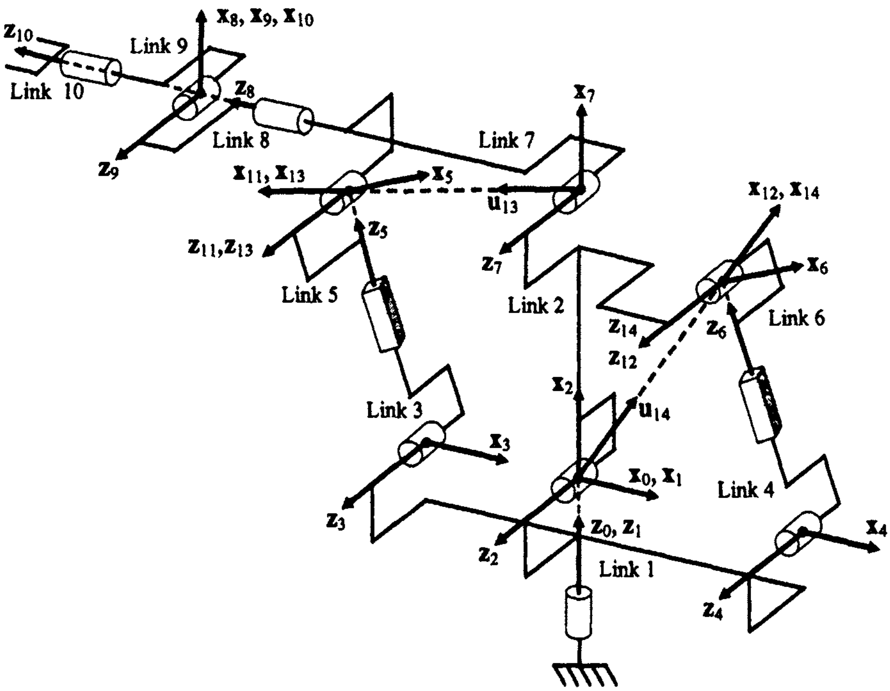

Figure 7.5. *AKR-3000 robot*

7.4. Direct geometric model of tree structured robots

We have shown in Chapter 3 that the DGM of a serial robot is obtained from the transformation matrix ${}^0T_n$ giving the location of the terminal link n relative to frame R_0. The extension to tree structured robots is straightforward. The transformation matrix ${}^0T_k$ specifying the location of the terminal link k relative to frame R_0 is obtained by multiplying the transformation matrices along the main branch connecting this terminal link to the base:

$$ {}^0T_k = {}^0T_i \cdots {}^{a(a(k))}T_{a(k)}\, {}^{a(k)}T_k \qquad [7.11] $$

Table 7.2. *Geometric parameters of the AKR-3000 robot*

j	a(j)	μ_j	σ_j	γ_j	b_j	α_j	d_j	θ_j	r_j
1	0	1	0	0	0	0	0	θ_1	0
2	1	0	0	0	0	$\pi/2$	0	θ_2	0
3	1	0	0	0	0	$\pi/2$	d_3	θ_3	0
4	1	0	0	0	0	$\pi/2$	d_4	θ_4	0
5	3	1	1	0	0	$-\pi/2$	0	0	r_5
6	4	1	1	0	0	$-\pi/2$	0	0	r_6
7	2	0	0	0	0	0	d_7	θ_7	0
8	7	1	0	0	0	$-\pi/2$	0	θ_8	r_8
9	8	1	0	0	0	$\pi/2$	0	θ_9	0
10	9	1	0	0	0	$-\pi/2$	0	θ_{10}	0
11	5	0	0	0	0	$\pi/2$	0	θ_{11}	0
12	6	0	0	0	0	$\pi/2$	0	θ_{12}	0
13	7	0	2	γ_{13}	0	0	d_{13}	0	0
14	2	0	2	γ_{14}	0	0	d_{14}	0	0

7.5. Direct geometric model of robots with closed chains

For a robot with closed chains, the DGM gives the location of the terminal(s) link(s) as a function of the active joint variables. The location of a terminal link k relative to the base ${}^0T_k$ is obtained, as usual, by multiplying the transformation matrices along the direct shortest path between the base and the terminal link as given by equation [7.11].

If the matrix ${}^0T_k$ contains passive joint variables, we have to compute these variables in terms of the active joint variables. This implies solution of the geometric constraint equations [7.8] as developed in § 7.7.

• **Example 7.3.** Direct geometric model of the Acma SR400 robot (Figure 7.4). The location of the terminal link relative to frame R_0 is obtained as:

$$ {}^0T_6 = {}^0T_1\,{}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_5\,{}^5T_6 $$

Since joint 3 is passive, we have to express q_3 in terms of the active variable q_7. This equation will be developed in Example 7.6.

• **Example 7.4.** Direct geometric model of the AKR-3000 robot (Figure 7.5). The transformation matrix through the direct path between the terminal link 10 and the base is written as:

$$ {}^{0}T_{10} = {}^{0}T_{1}\,{}^{1}T_{2}\,{}^{2}T_{7}\,{}^{7}T_{8}\,{}^{8}T_{9}\,{}^{9}T_{10} $$

The computation of the passive joint variables θ_2 and θ_7 in terms of the active joint variables r_5 and r_6 is developed in Example 7.5.

7.6. Inverse geometric model of closed chain robots

The IGM of a robot with a closed chain structure gives the active joint variables as a function of the location of the end-effector.

We first determine the joint variables of the direct path between the base and the end-effector. This problem can be solved using the approaches developed for serial robots in Chapter 4. Then, we solve the geometric constraint equations of the loop to compute the passive joint variables belonging to this path in terms of the active joint variables (§ 7.7). To use the methods of Chapter 4, we have to define the link frames such that the geometric parameters b_j and γ_j of the frames of the direct path of the terminal link are zero. Otherwise, they can be eliminated by grouping them with the parameters r_i and θ_i, for $i = a(j)$, respectively. This can be proved by developing the elements of two consecutive transformation matrices ${}^{a(i)}T_i$ and ${}^{i}T_j$:

$$ {}^{a(i)}\mathbf{T}_i\,{}^{i}\mathbf{T}_j = \mathbf{Rot}(\mathbf{x}, \alpha_i)\,\mathbf{Trans}(\mathbf{x}, d_i)\,\mathbf{Rot}(\mathbf{z}, \theta_i)\,\mathbf{Trans}(\mathbf{z}, r_i)\,\mathbf{Rot}(\mathbf{z}, \gamma_j)\,\mathbf{Trans}(\mathbf{z}, b_j)\,\mathbf{Rot}(\mathbf{x}, \alpha_j)\,\mathbf{Trans}(\mathbf{x}, d_j)\,\mathbf{Rot}(\mathbf{z}, \theta_j)\,\mathbf{Trans}(\mathbf{z}, r_j) $$

This equation can be rewritten as:

$$ {}^{a(i)}\mathbf{T}_i\,{}^{i}\mathbf{T}_j = \mathbf{Rot}(\mathbf{x}, \alpha_i)\,\mathbf{Trans}(\mathbf{x}, d_i)\,\mathbf{Rot}(\mathbf{z}, \theta_i')\,\mathbf{Trans}(\mathbf{z}, r_i')\,\mathbf{Rot}(\mathbf{x}, \alpha_j)\,\mathbf{Trans}(\mathbf{x}, d_j)\,\mathbf{Rot}(\mathbf{z}, \theta_j)\,\mathbf{Trans}(\mathbf{z}, r_j) \qquad [7.12] $$

with $r_i' = r_i + b_j$ and $\theta_i' = \theta_i + \gamma_j$.

7.7. Resolution of the geometric constraint equations of a simple loop

7.7.1. *Introduction*

The computation of the geometric and dynamic models of robots with closed loop structure requires the resolution of the geometric constraint equations of the loops [7.8]. The objective is to compute the passive joint variables in terms of the active joint variables. Equation [7.8] constitutes a system of twelve nonlinear equations with up to six independent unknowns. Thus, a closed loop can have at most six passive joints, and a planar loop can have at most three passive joints. The problem of solving the geometric constraint equations is similar to that of the inverse geometric model of serial robots (Chapter 4). In this section, we develop an analytical method to obtain the solution for simple loops having three passive joints [Bennis 93], which is the case for most industrial robots with closed chains. This method can be extended to loops with four passive joints. We assume that the structure is compatible with the closure constraints of the loops, otherwise there would be no solution. This hypothesis will be developed in § 7.10 and will be verified when solving the constraint equations.

7.7.2. *General principle*

Let the three passive joints of the loop be denoted by i, j and k. We can obtain two systems of equations, one by using the position elements and the other by using the orientation elements of equation [7.8].

The general case is when a passive joint is situated between two active joints. Thus, equation [7.8] can be written as:

$$ {}^{k}\mathbf{T}_{a(i)}\, {}^{a(i)}\mathbf{T}_{i}(q_i)\, {}^{i}\mathbf{T}_{a(j)}\, {}^{a(j)}\mathbf{T}_{j}(q_j)\, {}^{j}\mathbf{T}_{a(k)}\, {}^{a(k)}\mathbf{T}_{k}(q_k) = \mathbf{I}_4 \qquad [7.13a] $$

The transformation matrices ${}^{k}\mathbf{T}_{a(i)}$, ${}^{i}\mathbf{T}_{a(j)}$ and ${}^{j}\mathbf{T}_{a(k)}$ are functions of the supposed known active joint variables. The matrices ${}^{a(i)}\mathbf{T}_{i}$, ${}^{a(j)}\mathbf{T}_{j}$ and ${}^{a(k)}\mathbf{T}_{k}$ are functions of the unknown variables q_i, q_j and q_k respectively.

Equation [7.13a] can also be written in the following forms:

$$ {}^{j}\mathbf{T}_{a(k)}\, {}^{a(k)}\mathbf{T}_{k}\, {}^{k}\mathbf{T}_{a(i)}\, {}^{a(i)}\mathbf{T}_{i}\, {}^{i}\mathbf{T}_{a(j)}\, {}^{a(j)}\mathbf{T}_{j} = \mathbf{I}_4 \qquad [7.13b] $$

$$ {}^{a(k)}\mathbf{T}_{k}\, {}^{k}\mathbf{T}_{a(i)}\, {}^{a(i)}\mathbf{T}_{i}\, {}^{i}\mathbf{T}_{a(j)}\, {}^{a(j)}\mathbf{T}_{j}\, {}^{j}\mathbf{T}_{a(k)} = \mathbf{I}_4 \qquad [7.13c] $$

$$ {}^{i}\mathbf{T}_{a(j)}\, {}^{a(j)}\mathbf{T}_{j}\, {}^{j}\mathbf{T}_{a(k)}\, {}^{a(k)}\mathbf{T}_{k}\, {}^{k}\mathbf{T}_{a(i)}\, {}^{a(i)}\mathbf{T}_{i} = \mathbf{I}_4 \qquad [7.13d] $$

We can rewrite equation [7.13a] such that the first transformation matrix contains the variable of joint i, namely θ_i or r_i.

i) position equation

We can obtain the position equation of the loop by postmultiplying equation [7.13a] by the vector $\mathbf{p}_0 = [\,0 \quad 0 \quad 0 \quad 1\,]^T$, and premultiplying it by $\mathbf{Rot}(\mathbf{x}, -\alpha_i)$ $\mathbf{Trans}(\mathbf{x}, -d_i)$ $\mathbf{Rot}(\mathbf{z}, -\gamma_i)$ $\mathbf{Trans}(\mathbf{z}, -b_i)$ ${}^{a(i)}\mathbf{T}_{k}$. This equation depends on r_k and not on θ_k, and can be rewritten as:

$$\mathbf{Rot}(\mathbf{z}, \theta_i)\, \mathbf{Trans}(\mathbf{z}, r_i)\, {}^{i}\mathbf{T}_{a(j)}\, \mathbf{Rot}(\mathbf{x}, \alpha_j)\, \mathbf{Trans}(\mathbf{x}, d_j)$$
$$\mathbf{Rot}(\mathbf{z}, \theta_j)\, \mathbf{Trans}(\mathbf{z}, r_j) \begin{bmatrix} \mathbf{f}(r_k) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix} \qquad [7.14]$$

where $\mathbf{g}$ is known.

ii) orientation equation

The orientation equation can be obtained by premultiplying relation [7.13a] by $\mathbf{Rot}(\mathbf{x}, -\alpha_j)\, \mathbf{Rot}(\mathbf{z}, -\gamma_i)\, {}^{a(i)}\mathbf{T}_k$ while only keeping the orientation terms:

$$\mathbf{rot}(\mathbf{z},\theta_i)\, [\, \mathbf{S}_1 \;\; \mathbf{N}_1 \;\; \mathbf{A}_1 \,]\, \mathbf{rot}(\mathbf{z}, \theta_j)\, [\, \mathbf{S}_2 \;\; \mathbf{N}_2 \;\; \mathbf{A}_2 \,]\, \mathbf{rot}(\mathbf{z}, \theta_k) = [\, \mathbf{S}_3 \;\; \mathbf{N}_3 \;\; \mathbf{A}_3 \,] \qquad [7.15]$$

where the (3x1) vectors $\mathbf{S}_i$, $\mathbf{N}_i$ and $\mathbf{A}_i$, for i = 1, 2, 3, are functions of the active joint variables.

Equations [7.14] and [7.15] have the same form as the position and orientation equations of serial robots (§ 4.4). However, as they are functions of the same variables, they must be resolved simultaneously [Bennis 91a]. An application of this method is given in Example 7.5 for the AKR-3000 robot.

• **Example 7.5.** Resolution of the loop constraint equations of the AKR-3000 robot. From Table 7.2, we deduce that:

$$\mathbf{q}_a = [\; \theta_1 \;\; r_5 \;\; r_6 \;\; \theta_8 \;\; \theta_9 \;\; \theta_{10} \;]^T$$
$$\mathbf{q}_p = [\; \theta_2 \;\; \theta_3 \;\; \theta_4 \;\; \theta_7 \;]^T$$
$$\mathbf{q}_c = [\; \theta_{11} \;\; \theta_{12} \;]^T$$

<u>a) Equation of the loop composed of links 1, 2, 4 and 6</u>

Frames R_{12} and R_{14} are placed on the cut joint between links 2 and 6. The loop contains three revolute passive joints with parallel axes and one prismatic active joint. We need to calculate the passive variables θ_2, θ_4 and θ_{12} in terms of the active variable r_6. For convenience, let us group γ_{14} with θ_2 such that:

$$\theta_2' = \theta_2 + \gamma_{14} \qquad [7.16]$$

The geometric constraint equation of the loop can be written as:

$$ {}^{1}\mathbf{T}_4\, {}^{4}\mathbf{T}_6\, {}^{6}\mathbf{T}_{12} = {}^{1}\mathbf{T}_2\, {}^{2}\mathbf{T}_{14} \qquad [7.17]$$

i) orientation equation

$$^{1}\mathbf{A}_4\,^{4}\mathbf{A}_6\,^{6}\mathbf{A}_{12} = {}^{1}\mathbf{A}_2\,^{2}\mathbf{A}_{14} \tag{7.18}$$

Since the axes of the passive joints are parallel, equation [7.18] gives:

$$\mathbf{rot}(\mathbf{z}, \theta_4)\,\mathbf{rot}(\mathbf{z}, \theta_{12}) = \mathbf{rot}(\mathbf{z}, \theta_2') \tag{7.19}$$

We deduce that:

$$\theta_4 + \theta_{12} = \theta_2 + \gamma_{14} \tag{7.20}$$

ii) position equation

We choose to eliminate the passive joint variable θ_{12}. Premultiplying equation [7.17] by $^{4}\mathbf{T}_1$ and postmultiplying it by the vector $\mathbf{p}_0 = [\,0 \;\; 0 \;\; 0 \;\; 1\,]^T$ leads to:

$$^{4}\mathbf{T}_6\,^{6}\mathbf{T}_{12}\,\mathbf{p}_0 = {}^{4}\mathbf{T}_1\,^{1}\mathbf{T}_2\,^{2}\mathbf{T}_{14}\,\mathbf{p}_0 \tag{7.21}$$

which gives:

$$\mathbf{Rot}(\mathbf{z}, -\theta_4)\begin{bmatrix}\mathbf{F}(\theta_2') \\ 1\end{bmatrix} = \begin{bmatrix}\mathbf{G}(r_6) \\ 1\end{bmatrix} \tag{7.22}$$

with:

$$\begin{bmatrix}\mathbf{F}(\theta_2') \\ 1\end{bmatrix} = \mathbf{Trans}(\mathbf{x}, -d_4)\,\mathbf{Rot}(\mathbf{z}, \theta_2')\,^{2}\mathbf{T}_{14}\,\mathbf{p}_0 = \begin{bmatrix} C\theta_2' d_{14} - d_4 \\ S\theta_2' d_{14} \\ 0 \\ 1 \end{bmatrix} \tag{7.23}$$

$$\begin{bmatrix}\mathbf{G}(r_6) \\ 1\end{bmatrix} = {}^{4}\mathbf{T}_6\,^{6}\mathbf{T}_{12}\,\mathbf{p}_0 = \begin{bmatrix} 0 \\ r_6 \\ 0 \\ 1 \end{bmatrix} \tag{7.24}$$

θ_2' can be obtained in terms of r_6 by developing the expression:

$$\| \mathbf{F} \|^2 = \| \mathbf{G} \|^2 \tag{7.25}$$

which leads to:

$$d_{14}^2 + d_4^2 - 2\,d_4\,d_{14}\,C(\theta_2') = r_6^2 \tag{7.26}$$

Having determined θ_2' from equation [7.26], the components of $\mathbf{F}$ are known. The variable θ_4 can be obtained from equation [7.22] as:

$$\begin{cases} -S\theta_4 \, r_6 = F_x \\ C\theta_4 \, r_6 = F_y \end{cases} \tag{7.27}$$

where $\mathbf{F} = [F_x \quad F_y \quad F_z]^T$.

The variable θ_{12} can be determined from equation [7.20].

b) Equation of the loop composed of links 1, 2, 7, and 3

Frames R_{11} and R_{13} are placed on the cut joint between links 5 and 7. The loop contains four revolute passive joints with parallel axes and one prismatic active joint, but the passive variable θ_2 has already been obtained from the first loop. Thus, the three unknowns θ_{11}, θ_3 and θ_7 have to be computed in terms of the active variable r_5 and the variable θ_2. To simplify the development, let us group γ_{13} with θ_7 such that:

$$\theta_7' = \theta_7 + \gamma_{13} \tag{7.28}$$

The loop equation is written as:

$$^1\mathbf{T}_3 \, ^3\mathbf{T}_5 \, ^5\mathbf{T}_{11} = {}^1\mathbf{T}_2 \, ^2\mathbf{T}_7 \, ^7\mathbf{T}_{13} \tag{7.29}$$

i) orientation equation

By proceeding as for the first loop, we obtain the equation:

$$\theta_2 + \theta_7' = \theta_3 + \theta_{11} \tag{7.30}$$

ii) position equation

We choose to eliminate the passive variable θ_3 from equation [7.29], which can be rewritten as:

$$^2\mathbf{T}_7 \, ^7\mathbf{T}_{13} \, ^{13}\mathbf{T}_{11} \, ^{11}\mathbf{T}_5 \, ^5\mathbf{T}_3 \, \mathbf{p}_0 = {}^2\mathbf{T}_1 \, ^1\mathbf{T}_3 \, \mathbf{p}_0 \tag{7.31}$$

where the vector $^1\mathbf{T}_3 \, \mathbf{p}_0$ is devoid of θ_3. Equation [7.31] becomes:

$$\mathbf{Rot}(\mathbf{z}, \theta_7') \begin{bmatrix} \mathbf{F}(\theta_{11}, r_5) \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{G}(\theta_2) \\ 1 \end{bmatrix} \tag{7.32}$$

with:

$$\begin{bmatrix} \mathbf{F} \\ 1 \end{bmatrix} = \mathbf{Trans(x}, d_{13}) \, \mathbf{Rot(z}, -\theta_{11}) \, \mathbf{Rot(x}, -\alpha_{11}) \, {}^5\mathbf{T}_3 \, \mathbf{p}_0 = \begin{bmatrix} -S\theta_{11} r_5 + d_{13} \\ -S\theta_{11} r_5 \\ 0 \\ 1 \end{bmatrix} \qquad [7.33]$$

$$\begin{bmatrix} \mathbf{G} \\ 1 \end{bmatrix} = \mathbf{Trans(x}, -d_7) \, {}^2\mathbf{T}_1 \, {}^1\mathbf{T}_3 \, \mathbf{p}_0 = \begin{bmatrix} C\theta_2 d_3 - d_7 \\ -S\theta_2 d_3 \\ 0 \\ 1 \end{bmatrix} \qquad [7.34]$$

The variable θ_{11} can be obtained in terms of the variables r_5 and θ_2 by developing the equality $\| \mathbf{F} \|^2 = \| \mathbf{G} \|^2$:

$$r_5^2 + d_{13}^2 - 2 d_{13} r_5 S\theta_{11} = d_7^2 + d_3^2 - 2 d_3 d_7 C\theta_2 \qquad [7.35]$$

Having determined θ_{11}, we calculate θ_7' thanks to equation [7.32]:

$$\begin{cases} C\theta_7' F_x - S\theta_7' F_y = G_x \\ S\theta_7' F_x + C\theta_7' F_y = G_y \end{cases} \qquad [7.36]$$

The variable θ_3 can be determined from equation [7.30].

7.7.3. *Particular case of a parallelogram loop*

A parallelogram loop has three revolute passive joints and one revolute active joint. In this section, we propose a specific method, which is simpler than the general approach exposed previously. We use the fact that the orientation matrix between the frames of two parallel links is constant. Thus, if links k_1, k_2, k_3 and k_4 constitute a parallelogram, where links k_1 and k_2 are parallel to links k_3 and k_4 respectively (Figure 7.6), then [Bennis 91a]:

$$ {}^{k1}\mathbf{A}_{k3} = \mathbf{rot(u}, \theta_{c1}) = \text{constant} \qquad [7.37a]$$

$$ {}^{k2}\mathbf{A}_{k4} = \mathbf{rot(u}, \theta_{c2}) = \text{constant} \qquad [7.37b]$$

$$ {}^{k}\mathbf{A}_{k+B} = {}^{k}\mathbf{A}_{k3} \, {}^{k3}\mathbf{A}_{k2} \, {}^{k2}\mathbf{A}_{k4} \, {}^{k4}\mathbf{A}_{k+B} = \mathbf{I}_3 \qquad [7.37c]$$

The constants θ_{c1} and θ_{c2} can be obtained from a particular configuration of the robot. Equation [7.37c] allows us to compute the cut joint variable in terms of the other joint variables.

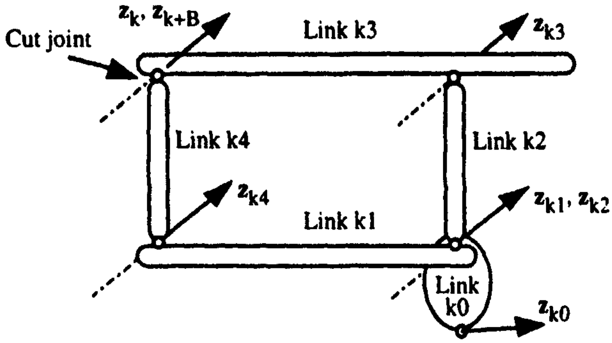

Figure 7.6. *Parallelogram closed loop*

• **Example 7.6.** Resolution of the geometric constraint equations of the Acma SR400 robot. From Table 7.1, we deduce that:

$$\mathbf{q}_a = [\ \theta_1\ \ \theta_2\ \ \theta_4\ \ \theta_5\ \ \theta_6\ \ \theta_7\ ]^T$$
$$\mathbf{q}_p = [\ \theta_3\ \ \theta_8\ ]^T$$
$$\mathbf{q}_c = [\ \theta_9\ ]$$

The loop composed of links 7, 2, 3 and 8 forms a parallelogram (Figure 7.4). Joints 3, 8 and 9 are passive, whereas joint 7 is active. In this loop, links 2 and 3 are parallel to links 8 and 7 respectively. Using equations [7.37] we obtain:

$$^7\mathbf{A}_3 = {}^7\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3 = \mathbf{rot}(\mathbf{z}, -\theta_7+\theta_2+\theta_3) = \mathbf{rot}(\mathbf{z}, \pi/2) \qquad [7.38]$$
$$^2\mathbf{A}_8 = {}^2\mathbf{A}_1\,{}^1\mathbf{A}_7\,{}^7\mathbf{A}_8 = \mathbf{rot}(\mathbf{z}, -\theta_2)\,\mathbf{rot}(\mathbf{z}, \theta_7)\,\mathbf{rot}(\mathbf{z}, \theta_8) = \mathbf{rot}(\mathbf{z}, 0) \qquad [7.39]$$
$$^9\mathbf{A}_8\,{}^8\mathbf{A}_7\,{}^7\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3\,{}^3\mathbf{A}_{10} = \mathbf{I}_3 \qquad [7.40]$$

From these equations we deduce that:

$$\theta_3 = \pi/2 - \theta_2 + \theta_7 \qquad [7.41]$$
$$\theta_8 = \theta_2 - \theta_7 \qquad [7.42]$$
$$\theta_9 = \pi/2 + \theta_3 = \pi - \theta_2 + \theta_7 \qquad [7.43]$$

7.8. Kinematic model of complex chain robots

The kinematic model provides the velocity of the terminal link corresponding to the specified velocities of the active joints. Since the joints of a tree structured robot are actuated and independent, the kinematic model for these robots can be obtained by applying the techniques developed for serial robots to each main branch. For robots with closed chains, we first compute the kinematic model of the direct chain between the end-effector and the base by proceeding as for a serial robot. Then, we compute the passive joint velocities in terms of the active joint velocities. The solution can be obtained either by differentiating the geometric constraint equations, or by resolving the kinematic constraint equations of the loops.

The kinematic constraint equations can be obtained by equating the velocities of frames R_k and R_{k+B} associated with each cut joint. They can be computed using the Jacobian matrix of the two branches of each loop as follows:

$$\begin{bmatrix} \mathbf{V}_k \\ \omega_k \end{bmatrix} = \mathbf{J}_k \, \dot{\mathbf{q}}_{b1} = \mathbf{J}_{k+B} \, \dot{\mathbf{q}}_{b2} \qquad [7.44]$$

where $\dot{\mathbf{q}}_{b1}$ and $\dot{\mathbf{q}}_{b2}$ are the joint velocities along each branch of the loop.

Equation [7.44] can be rewritten as:

$$\mathbf{J}_k \, \dot{\mathbf{q}}_{b1} - \mathbf{J}_{k+B} \, \dot{\mathbf{q}}_{b2} = \mathbf{0} \qquad [7.45]$$

Using equation [5.9] and taking into account Figure 7.7 leads to:

$$\mathbf{J}_k = \begin{bmatrix} \sigma_e \mathbf{a}_e + \bar{\sigma}_e(\mathbf{a}_e \times \mathbf{L}_{e,k}) & \cdots & \sigma_k \mathbf{a}_k + \bar{\sigma}_k(\mathbf{a}_k \times \mathbf{L}_{k,k}) \\ \bar{\sigma}_e \, \mathbf{a}_e & \cdots & \bar{\sigma}_k \mathbf{a}_k \end{bmatrix} \qquad [7.46]$$

where e indicates the first link, after the root of the loop, of the branch leading to frame R_k, and:

$$\mathbf{J}_{k+B} = \begin{bmatrix} \sigma_d \mathbf{a}_d + \bar{\sigma}_d(\mathbf{a}_d \times \mathbf{L}_{d,k+B}) & \cdots & \sigma_j \mathbf{a}_j + \bar{\sigma}_j(\mathbf{a}_j \times \mathbf{L}_{j,k+B}) \\ \bar{\sigma}_d \mathbf{a}_d & \cdots & \bar{\sigma}_j \mathbf{a}_j \end{bmatrix} \qquad [7.47]$$

where d indicates the first link, after the root of the loop, of the branch leading to frame R_{k+B}.

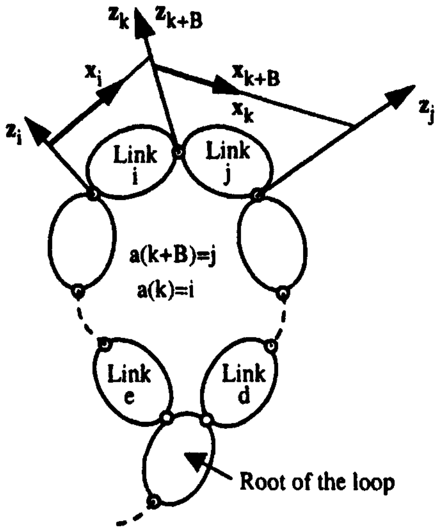

Figure 7.7. *General notations for a closed loop*

The elements of the Jacobian matrices J_k and J_{k+B} are generally computed with respect to frames R_k and R_{k+B} respectively, or with respect to the frame of the root of the loop. By combining the constraint equations of all the loops, and after eliminating the possible zero rows, the kinematic constraint equation can be written as [Zghaib 92]:

$$\mathbf{J}\,\dot{\mathbf{q}} = \mathbf{0} \qquad [7.48]$$

which can be developed as:

$$\begin{bmatrix} \mathbf{W}_a & \mathbf{W}_p & \mathbf{0} \\ \mathbf{W}_{ac} & \mathbf{W}_{pc} & \mathbf{W}_c \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}_a \\ \dot{\mathbf{q}}_p \\ \dot{\mathbf{q}}_c \end{bmatrix} = \mathbf{0} \qquad [7.49]$$

with:

- $\dot{\mathbf{q}}_a$: (Nx1) vector of the active joint velocities;
- $\dot{\mathbf{q}}_p$: (px1) vector of the passive joint velocities of the equivalent tree structure, where p=n−N;
- $\dot{\mathbf{q}}_c$: (Bx1) vector of the cut joint velocities;
- the dimensions of the matrices are the following: $\mathbf{W}_a$ (pxN), $\mathbf{W}_p$ (pxp), $\mathbf{W}_{ac}$ (BxN), $\mathbf{W}_{pc}$ (Bxp), $\mathbf{W}_c$ (BxB).

When the cut joint is complex with several degrees of freedom (spherical, universal, ...), we can consider the corresponding joint velocities to be belonging to $\dot{\mathbf{q}}_c$. From the first row of equation [7.49], we obtain:

$$\mathbf{W}_p\, \dot{\mathbf{q}}_p = -\mathbf{W}_a\, \dot{\mathbf{q}}_a \qquad [7.50]$$

If the system is compatible with the loop constraints, the rank of $\mathbf{W}_p$ will be equal to p (outside the possible singular positions). We deduce that:

$$\dot{\mathbf{q}}_p = -\mathbf{W}_p^{-1}\, \mathbf{W}_a\, \dot{\mathbf{q}}_a = \mathbf{W}\, \dot{\mathbf{q}}_a \qquad [7.51]$$

By differentiating equation [7.49] with respect to time, we obtain the acceleration constraint equation:

$$\begin{bmatrix} \mathbf{W}_a & \mathbf{W}_p & \mathbf{0} \\ \mathbf{W}_{ac} & \mathbf{W}_{pc} & \mathbf{W}_c \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_a \\ \ddot{\mathbf{q}}_p \\ \ddot{\mathbf{q}}_c \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Psi} \\ \boldsymbol{\Phi} \end{bmatrix} = \mathbf{0} \qquad [7.52]$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ represent the vector $\dot{\mathbf{J}}\, \dot{\mathbf{q}}$.

The components of the vectors $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ of the loop of the cut joint k can be determined with the recursive equations giving the terminal accelerations ${}^k\dot{\mathbf{V}}_k$ and ${}^k\dot{\boldsymbol{\omega}}_k$ relative to the root of the loop when setting $\ddot{\mathbf{q}}_{b1} = 0$ and $\ddot{\mathbf{q}}_{b2} = 0$ (§ 5.10). We obtain:

$$\begin{bmatrix} {}^k\dot{\mathbf{V}}_k(\ddot{\mathbf{q}}_{b1}=\mathbf{0}) \\ {}^k\dot{\boldsymbol{\omega}}_k(\ddot{\mathbf{q}}_{b1}=\mathbf{0}) \end{bmatrix} - \begin{bmatrix} {}^{k+B}\dot{\mathbf{V}}_{k+B}(\ddot{\mathbf{q}}_{b2}=\mathbf{0}) \\ {}^{k+B}\dot{\boldsymbol{\omega}}_{k+B}(\ddot{\mathbf{q}}_{b2}=\mathbf{0}) \end{bmatrix} = {}^k\dot{\mathbf{J}}_k\, \dot{\mathbf{q}}_{b1} - {}^{k+B}\dot{\mathbf{J}}_{k+B}\, \dot{\mathbf{q}}_{b2} \qquad [7.53]$$

The vectors $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are determined by applying equation [7.53] to all the loops, k = 1, ..., L, and by grouping them in the same order as those of the velocity constraint equation [7.49]. The computation of equation [7.53] can be carried out using the efficient recursive equations [5.47] after replacing j − 1 by a(j).

• **Example 7.7.** Calculation of the kinematic constraint equations of the Acma SR400 robot described in Example 7.1. By differentiating with respect to time the geometric constraint equations developed in Example 7. 6, we obtain:

$$\begin{cases} \dot{\theta}_3 = -\dot{\theta}_2 + \dot{\theta}_7 \\ \dot{\theta}_8 = \dot{\theta}_2 - \dot{\theta}_7 \\ \dot{\theta}_9 = -\dot{\theta}_2 + \dot{\theta}_7 \end{cases}$$

The acceleration constraint equations can be obtained by differentiating the kinematic constraint equations.

• **Example 7.8.** Computation of the kinematic constraint equations of the AKR-3000 robot described in Example 7.2. We recall that:

$$\dot{\mathbf{q}}_a = [\,\dot{\theta}_1 \quad \dot{r}_5 \quad \dot{r}_6 \quad \dot{\theta}_8 \quad \dot{\theta}_9 \quad \dot{\theta}_{10}]^T$$

$$\dot{\mathbf{q}}_p = [\,\dot{\theta}_2 \quad \dot{\theta}_3 \quad \dot{\theta}_4 \quad \dot{\theta}_7]^T$$

$$\dot{\mathbf{q}}_c = [\,\dot{\theta}_{11} \quad \dot{\theta}_{12}]^T$$

The kinematic constraint equations can be obtained either by projecting the Jacobian matrices in the terminal frame R_k of each loop or in the frame fixed to the root of each loop. We present these two cases in the following.

i) projection of the Jacobian matrices in the base frame of the loop.

The kinematic constraint equation of the first loop is written as:

$$^1\mathbf{J}_{11}\,\dot{\mathbf{q}}_{b1,1} - {}^1\mathbf{J}_{13}\,\dot{\mathbf{q}}_{b2,1} = \mathbf{0}$$

with $\dot{\mathbf{q}}_{b1,1} = [\dot{\theta}_3 \quad \dot{r}_5 \quad \dot{\theta}_{11}]^T$ and $\dot{\mathbf{q}}_{b2,1} = [\dot{\theta}_2 \quad \dot{\theta}_7]^T$, giving the following non-trivial equations:

$$-r_5\,C3\,\dot{\theta}_3 - S3\,\dot{r}_5 + (d_7\,S2 - d_{13}\,S\alpha)\,\dot{\theta}_2 + d_{13}\,S\alpha\,\dot{\theta}_7 = 0$$

$$-r_5\,S3\,\dot{\theta}_3 + C3\,\dot{r}_5 - (d_7\,C2 + d_{13}\,C\alpha)\,\dot{\theta}_2 - d_{13}\,C\alpha\,\dot{\theta}_7 = 0$$

$$-\dot{\theta}_3 - \dot{\theta}_{11} + \dot{\theta}_2 + \dot{\theta}_7 = 0$$

with $\alpha = \gamma_{13} + \theta_2 + \theta_7$.

The kinematic constraint equation of the second loop is written as:

$$^1\mathbf{J}_{12}\,\dot{\mathbf{q}}_{b1,2} - {}^1\mathbf{J}_{14}\,\dot{\mathbf{q}}_{b2,2} = \mathbf{0}$$

with $\dot{q}_{b1,2} = [\ \dot{\theta}_4 \quad \dot{r}_6 \quad \dot{\theta}_{12}]^T$ and $\dot{q}_{b2,2} = \dot{\theta}_2$, giving the following non-trivial equations:

$$- r_6\, C4\, \dot{\theta}_4 - S4\, \dot{r}_6 + d_{14}\, S(\gamma_{14} + \theta_2)\, \dot{\theta}_2 = 0$$

$$- r_6\, S4\, \dot{\theta}_4 + C4\, \dot{r}_6 - d_{14}\, C(\gamma_{14} + \theta_2)\, \dot{\theta}_2 = 0$$

$$- \dot{\theta}_4 - \dot{\theta}_{12} + \dot{\theta}_2 = 0$$

These six equations can easily be put in the form [7.49].

ii) projection of the Jacobian matrices in the terminal frame of the loop.

The kinematic constraint equation of the first loop is written as:

$$^{11}\mathbf{J}_{11}\, \dot{q}_{b1,1} - {}^{13}\mathbf{J}_{13}\, \dot{q}_{b2,1} = \mathbf{0}$$

After developing, we obtain the following non-trivial equations:

$$- r_5\, C\theta_{11}\, \dot{\theta}_3 + S\theta_{11}\, \dot{r}_5 - d_7\, S(\gamma_{13} + \theta_7)\, \dot{\theta}_2 = 0$$

$$r_5\, S\theta_{11}\, \dot{\theta}_3 + C\theta_{11}\, \dot{r}_5 - [d_{13} + d_7\, C(\gamma_{13}+\theta_7)]\, \dot{\theta}_2 - d_{13}\, \dot{\theta}_7 = 0$$

$$- \dot{\theta}_3 - \dot{\theta}_{11} + \dot{\theta}_2 + \dot{\theta}_7 = 0$$

The kinematic constraint equation of the second loop is written as:

$$^{12}\mathbf{J}_{12}\, \dot{q}_{b1,2} - {}^{14}\mathbf{J}_{14}\, \dot{q}_{b2,2} = \mathbf{0}$$

After developing, we obtain the following non-trivial equations:

$$- r_6\, C\theta_{12}\, \dot{\theta}_4 + S\theta_{12}\, \dot{r}_6 + d_{14}\, \dot{\theta}_2 = 0$$

$$r_6\, S\theta_{12}\, \dot{\theta}_4 + C\theta_{12}\, \dot{r}_6 = 0$$

$$- \dot{\theta}_4 - \dot{\theta}_{12} + \dot{\theta}_2 = 0$$

We note that the equations of the second solution are less complicated, but they have the disadvantage of being functions of the cut joint variables.

7.9. Numerical calculation of $\mathbf{q}_p$ and $\mathbf{q}_c$ in terms of $\mathbf{q}_a$

Based on equation [7.45], we derive the following differential model, which can be used to numerically compute the variables $\mathbf{q}_p$ and $\mathbf{q}_c$ for a given $\mathbf{q}_a$:

$$\begin{bmatrix} {}^{b1}\mathbf{J}_{n+1+B} & -{}^{b1}\mathbf{J}_{n+1} \\ \cdots & \cdots \\ {}^{bB}\mathbf{J}_{L+B} & -{}^{bB}\mathbf{J}_{L} \end{bmatrix} \begin{bmatrix} \mathbf{dq}_p \\ \mathbf{dq}_c \end{bmatrix} = \begin{bmatrix} {}^{b1}\mathbf{dX}^{n+1} \\ \cdots \\ {}^{bB}\mathbf{dX}^{L} \end{bmatrix} \qquad [7.54]$$

where $\mathbf{dX}^k$ corresponds to the vector of position and orientation errors between frame R_k and frame R_{k+B}, and b_j denotes the frame of the root of loop j.

The left side terms of equation [7.45] are deduced from equation [7.54] by combining the equations of all the loops, eliminating the trivial rows, and by eliminating the columns corresponding to $\mathbf{dq}_a$ since $\mathbf{q}_a$ is given.

To calculate $\mathbf{q}_p$ and $\mathbf{q}_c$ we use the following algorithm:

1) from the current configuration of $\mathbf{q}_p$ and $\mathbf{q}_c$ (which can be initialized by random values within the joint domain), compute ${}^{bj}\mathbf{T}_k(\mathbf{q})$ and ${}^{bj}\mathbf{T}_{k+B}(\mathbf{q})$ for k = n+1, ..., L;

2) compute the vector of position and orientation errors of all the loops:

$$\mathbf{dX} = \begin{bmatrix} \mathbf{dX}^{n+1} \\ \cdots \\ \mathbf{dX}^{L} \end{bmatrix}$$

where $\mathbf{dX}^k$ corresponds to the difference between the transformation matrices ${}^{bj}\mathbf{T}_k(\mathbf{q})$ and ${}^{bj}\mathbf{T}_{k+B}(\mathbf{q})$ as detailed in § 6.6;

3) – if $\mathbf{dX}$ is sufficiently small, then the desired $\mathbf{q}_p$ and $\mathbf{q}_c$ are equal to their current values. Stop the calculation;

– if $\|\mathbf{dX}\| > S$, then set $\mathbf{dX} = \dfrac{\mathbf{dX}}{\|\mathbf{dX}\|} S$, where S is a fixed threshold value;

4) calculate $\mathbf{dq}_p$ and $\mathbf{dq}_c$ corresponding to $\mathbf{dX}$ by solving equation [7.54]. Update the variables $\mathbf{q}_p$ and $\mathbf{q}_c$ using the equation:

$$\begin{cases} \mathbf{q}_p = \mathbf{q}_p + \mathbf{dq}_p \\ \mathbf{q}_c = \mathbf{q}_c + \mathbf{dq}_c \end{cases} \qquad [7.55]$$

5) return to the first step.

7.10. Number of degrees of freedom of robots with closed chains

The number of degrees of freedom N of a mechanism is equal to the number of independent joint variables required to specify the location of all the links with respect to the fixed base. Thus N is equal to the difference between the number of joints L and the number of independent geometric constraint equations c :

$$N = L - c \tag{7.56}$$

The problem consists in determining c. As mentioned in § 7.7.1, this number is, at most, six for a spatial loop and three for a planar loop.

The following simple formula is true for most robot structures:

$$N = L - \sum_{j=1}^{B} c_j \tag{7.57}$$

where c_j is the number of independent geometric constraint equations of the loop j, in general six for a spatial loop and three for a planar loop. The application of equation [7.57] for the SR400 robot and AKR robot gives N=6.

Several formulas like [7.57] have been proposed to systematically determine the number of degrees of freedom of a mechanism. These formulas are based on the number of links and joints and their degrees of freedom but do not take into account the geometric constraints that some mechanisms possess. Consequently, they may provide an erroneous result [Sheth 71]. For example, these formulas do not work with the Bennett mechanism [Bennett 03]. For this reason, an exact solution is obtained by analyzing the geometric and kinematic constraints using the mechanism theory techniques [Le Borzec 75], [Hervé 78], or by studying the rank of the Jacobian matrix as given in the following.

From equation [7.49] we deduce that $\dot{\mathbf{q}}$ belongs to the null space of $\mathbf{J}$. Therefore, at a given configuration, the number of degrees of freedom is equal to the dimension of the null space of $\mathbf{J}$. Consequently:

$$N = \min_{\mathbf{q}} (\dim (\mathcal{N}(\mathbf{J})) \tag{7.58}$$

where $\mathcal{N}(\mathbf{J})$ is the null space of the matrix $\mathbf{J}$.

Thus, the number of independent constraints c is given by the maximum value of the rank of the matrix $\mathbf{J}$ (equation 7.49) [Angeles 88], [Gosselin 90]:

$$c = \max_{\mathbf{q}} (\text{rank } \mathbf{J}(\mathbf{q})) \tag{7.59}$$

7.11. Classification of singular positions

The general kinematic equation of a closed chain robot is given by [Gosselin 90]:

$$\mathbf{J}_1(\mathbf{q})\,\dot{\mathbf{q}}_a = \mathbf{J}_2(\mathbf{q})\,\dot{\mathbf{X}} \qquad [7.60]$$

where $\dot{\mathbf{X}}$ is the velocity of the end-effector.

Input/output singularities occur in the configurations where $\mathbf{J}_1$ and/or $\mathbf{J}_2$ are singular. Three kinds of singularities are encountered:

i) $\mathbf{J}_1$ *is singular:* in this configuration, we can find the non-zero vector $\dot{\mathbf{q}}_a \neq \mathbf{0}$ for which $\dot{\mathbf{X}} = \mathbf{0}$. The terminal link loses one or more degrees of freedom and it cannot generate motion in some directions. This kind of singularity is the only one that occurs in serial robots. It is called *serial singularity*;

ii) $\mathbf{J}_2$ *is singular:* in this case, the structure is not in static equilibrium. Thus, $\dot{\mathbf{X}} \neq \mathbf{0}$ even though the actuated joints are locked ($\dot{\mathbf{q}}_a = \mathbf{0}$). This kind of singularity takes place in parallel robots (Chapter 8) and is known as *parallel singularity*. To avoid such singularity, redundant actuators may be used;

iii) $\mathbf{J}_1$ *and* $\mathbf{J}_2$ *are singular:* in this case, the structure has both serial and parallel singularities simultaneously. Thus, some links may move even though $\dot{\mathbf{q}}_a = \mathbf{0}$ and motions in some directions of the terminal link are unrealizable.

From equation [7.49], we can deduce another type of singularity that occurs when $\mathbf{W}_p$ is singular, giving $\dot{\mathbf{q}}_p \neq \mathbf{0}$ while $\dot{\mathbf{q}}_a = \mathbf{0}$. This singularity is termed *internal singularity*.

7.12. Conclusion

The method of description presented in this chapter allows extension of the results obtained for serial robots to complex chain robots. In fact, a serial robot can be considered as a special case of a tree structured robot. We showed that the computation of the direct and inverse geometric models of a closed chain robot could be formulated as the calculation of these models for a serial structure together with the resolution of the geometric constraint equations of the closure of the loops.

The geometric constraint equations of the loops can be solved using the methods for computing the IGM exposed in Chapter 4. We also presented a general analytical method for loops with less than four passive joints.

We also showed how to establish the kinematic model of such structures and how to obtain the kinematic constraint equations using the Jacobian matrix. The problem of the determination of the number of degrees of freedom of a closed chain robot has finally been addressed. For a survey of the manipulability, which we did not cover here, the reader is referred to [Bicchi 98].

The following chapter deals with the geometric and kinematic modeling of parallel robots, which require a specific approach for the description and the modeling.