Sinc Transform Calculations

1 Conventions

$$\mathcal{F}(h(\boldsymbol{x}))(\boldsymbol{f}) = \int_{-\infty}^{\infty} h(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\boldsymbol{f}} dx$$

$$\mathcal{F}^{-1}(H(\boldsymbol{f}))(\boldsymbol{x}) = \int_{-\infty}^{\infty} H(\boldsymbol{f}) e^{2\pi i \boldsymbol{x}\boldsymbol{f}} df$$

$$\operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

$$\operatorname{sinc}(\boldsymbol{x}) = \operatorname{sinc}(x_1)\operatorname{sinc}(x_2)\operatorname{sinc}(x_3)$$

2 Sinc Transform Pairs

1D

$$h(x) = \begin{cases} b & |x| \leq a \\ 0 & |x| > a \end{cases}$$

$$\mathcal{F}(h(x))(f) = \int_{-a}^{a} b e^{-2\pi i x f} dx = 2ab\operatorname{sinc}(2\pi a f)$$

Using symmetry of Fourier transform, ie $\mathcal{F}(H(x))(f) = h(-f)$, and that $h(x)$ is even:

$$\mathcal{F}(H(x))(f) = \mathcal{F}(2ab\operatorname{sinc}(2\pi a x))(f) = h(x)$$

Letting $a = \frac{1}{2\pi}$ and $b = \pi$,

$$\mathcal{F}(\operatorname{sinc}(x))(f) = h(x)$$

where $h(x)$ is π on $|x| \leq \frac{1}{2\pi}$ and 0 elsewhere.

2D

$$h(\boldsymbol{x}) = \begin{cases} b & |x_1| \leq a, |x_2| \leq a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}(h(\boldsymbol{x}))(\boldsymbol{f}) = \int_{-a}^{a}\int_{-a}^{a} b e^{-2\pi i \boldsymbol{x}\boldsymbol{f}} d\boldsymbol{x} = 4a^2 b\operatorname{sinc}(2\pi a f_1)\operatorname{sinc}(2\pi a f_2) = 4a^2 b\operatorname{sinc}(2\pi a \boldsymbol{f})$$

Letting $a = \frac{1}{2\pi}$ and $b = \pi^2$,

$$\mathcal{F}(\operatorname{sinc}(\boldsymbol{x}))(\boldsymbol{f}) = h(\boldsymbol{x})$$

3D

$$h(\boldsymbol{x}) = \begin{cases} b & |x_1| \leq a, |x_2| \leq a, |x_3| \leq a \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}(h(\boldsymbol{x}))(\boldsymbol{f}) = \int_{-a}^{a}\int_{-a}^{a}\int_{-a}^{a} be^{-2\pi i \boldsymbol{x}\boldsymbol{f}}\, d\boldsymbol{x} = 8a^3 b \operatorname{sinc}(2\pi a f_1)\operatorname{sinc}(2\pi a f_2)\operatorname{sinc}(2\pi a f_3) = 8a^3 b \operatorname{sinc}(2\pi a \boldsymbol{f})$$

Letting $a = \frac{1}{2\pi}$ and $b = \pi^3$,

$$\mathcal{F}(\operatorname{sinc}(\boldsymbol{x}))(\boldsymbol{f}) = h(\boldsymbol{x})$$

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3 Sinc^2 Transform Pairs

Note: since these functions are piecewise continuous, it will be necessary to find quadrature points separately for each piece

1D By the convolution theorem,

$$\mathcal{F}(\operatorname{sinc}^2(x))(f) = (\mathcal{F}(\operatorname{sinc}(x)) * \mathcal{F}(\operatorname{sinc}(x)))(f) = \begin{cases} \pi(1 - \pi|x|) & |x| \leq \frac{1}{\pi} \\ 0 & \text{otherwise} \end{cases}$$

2D By the convolution theorem,

$$\mathcal{F}(\operatorname{sinc}^2(\boldsymbol{x}))(\boldsymbol{f}) = (\mathcal{F}(\operatorname{sinc}(\boldsymbol{x})) * \mathcal{F}(\operatorname{sinc}(\boldsymbol{x})))(\boldsymbol{f}) = \begin{cases} \pi^2(1 - \pi|x_1|)(1 - \pi|x_2|) & |x_1| \leq \frac{1}{\pi}, |x_2| \leq \frac{1}{\pi} \\ 0 & \text{otherwise} \end{cases}$$

3D By the convolution theorem,

$$\mathcal{F}(\operatorname{sinc}^2(\boldsymbol{x}))(\boldsymbol{f}) = \begin{cases} \pi^3(1 - \pi|x_1|)(1 - \pi|x_2|)(1 - \pi|x_3|) & |x_1| \leq \frac{1}{\pi}, |x_2| \leq \frac{1}{\pi}, |x_3| \leq \frac{1}{\pi} \\ 0 & \text{otherwise} \end{cases}$$

4 Sinc

Define $f(\boldsymbol{k}) = \sum_{j=1}^{m} \delta(\boldsymbol{k} - \boldsymbol{k_j}) q_j$

$$(\text{sinc}*f)(\boldsymbol{k_i}) = \int_{-\infty}^{\infty} \text{sinc}(\boldsymbol{k_i} - \boldsymbol{k}) f(\boldsymbol{k}) d\boldsymbol{k} = \sum_{j=1}^{m} q_j \int_{-\infty}^{\infty} \text{sinc}(\boldsymbol{k_i} - \boldsymbol{k}) \delta(\boldsymbol{k} - \boldsymbol{k_j}) d\boldsymbol{k} = \sum_{j=1}^{m} q_j \text{sinc}(\boldsymbol{k_i} - \boldsymbol{k_j})$$

By the convolution theorem, $\mathcal{F}(\text{sinc} * f) = \mathcal{F}(\text{sinc})\mathcal{F}(f)$. So, we are interested in

$$(\text{sinc} * f)(\boldsymbol{k_i}) = \mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(\boldsymbol{k_i})$$

$$\mathcal{F}(f(\boldsymbol{k})) = \int_{-\infty}^{\infty} \sum_{j=1}^{m} \delta(\boldsymbol{k} - \boldsymbol{k_j}) q_j e^{-2\pi i \boldsymbol{k}\boldsymbol{x}} d\boldsymbol{k} = \sum_{j=1}^{m} q_j e^{-2\pi i \boldsymbol{k_j}\boldsymbol{x}}$$

1D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(k_i) = \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \pi \Big(\sum_{j=1}^{m} q_j e^{-2\pi i k_j x} \Big) e^{2\pi i x k_i} dx$$

Change of variables: let $y = 2\pi x$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(k_i) = \frac{1}{2} \int_{-1}^{1} \Big(\sum_{j=1}^{m} q_j e^{-i k_j y} \Big) e^{i y k_i} dy$$

2D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(\boldsymbol{k_i}) = \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \pi^2 \Big(\sum_{j=1}^{m} q_j e^{-2\pi i \boldsymbol{k_j}\boldsymbol{x}} \Big) e^{2\pi i \boldsymbol{x}\boldsymbol{k_i}} d\boldsymbol{x}$$

Change of variables: let $\boldsymbol{y} = 2\pi \boldsymbol{x}$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(\boldsymbol{k_i}) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Big(\sum_{j=1}^{m} q_j e^{-i \boldsymbol{k_j}\boldsymbol{y}} \Big) e^{i \boldsymbol{y}\boldsymbol{k_i}} d\boldsymbol{y}$$

3D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(\boldsymbol{k_i}) = \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \int_{\frac{-1}{2\pi}}^{\frac{1}{2\pi}} \pi^3 \Big(\sum_{j=1}^{m} q_j e^{-2\pi i \boldsymbol{k_j}\boldsymbol{x}} \Big) e^{2\pi i \boldsymbol{x}\boldsymbol{k_i}} d\boldsymbol{x}$$

Change of variables: let $\boldsymbol{y} = 2\pi \boldsymbol{x}$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc})\mathcal{F}(f))(\boldsymbol{k_i}) = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \Big(\sum_{j=1}^{m} q_j e^{-i \boldsymbol{k_j}\boldsymbol{y}} \Big) e^{i \boldsymbol{y}\boldsymbol{k_i}} d\boldsymbol{y}$$

5 Sinc^2

Define $f(\boldsymbol{k}) = \sum_{j=1}^{m} \delta(\boldsymbol{k} - \boldsymbol{k_j})q_j$

$$(\text{sinc}^2 * f)(\boldsymbol{k_i}) = \int_{-\infty}^{\infty} \text{sinc}^2(\boldsymbol{k_i} - \boldsymbol{k}) f(\boldsymbol{k}) d\boldsymbol{k} = \sum_{j=1}^{m} q_j \int_{-\infty}^{\infty} \text{sinc}^2(\boldsymbol{k_i} - \boldsymbol{k}) \delta(\boldsymbol{k} - \boldsymbol{k_j}) d\boldsymbol{k} = \sum_{j=1}^{m} q_j \text{sinc}^2(\boldsymbol{k_i} - \boldsymbol{k_j})$$

By the convolution theorem, $\mathcal{F}(\text{sinc}^2 * f) = \mathcal{F}(\text{sinc}^2)\mathcal{F}(f)$. So, we are interested in

$$(\text{sinc}^2 * f)(\boldsymbol{k_i}) = \mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(\boldsymbol{k_i})$$

1D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(k_i) = \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \pi(1 - \pi|x|)\Big(\sum_{j=1}^{m} q_j e^{-2\pi i k_j x}\Big) e^{2\pi i x k_i} dx$$

Change of variables: let $y = 2\pi x$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(k_i) = \frac{1}{4} \int_{-2}^{2} (2 - |y|)\Big(\sum_{j=1}^{m} q_j e^{-i k_j y}\Big) e^{i y k_i} dy$$

2D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(\boldsymbol{k_i}) = \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \pi^2(1 - \pi|x_1|)(1 - \pi|x_2|)\Big(\sum_{j=1}^{m} q_j e^{-2\pi i \boldsymbol{k_j x}}\Big) e^{2\pi i \boldsymbol{x k_i}} d\boldsymbol{x}$$

Change of variables: let $\boldsymbol{y} = 2\pi \boldsymbol{x}$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(\boldsymbol{k_i}) = \frac{1}{16} \int_{-2}^{2} \int_{-2}^{2} (2 - |y_1|)(2 - |y_2|)\Big(\sum_{j=1}^{m} q_j e^{-i \boldsymbol{k_j y}}\Big) e^{i \boldsymbol{y k_i}} d\boldsymbol{y}$$

3D

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(\boldsymbol{k_i}) = \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \int_{\frac{-1}{\pi}}^{\frac{1}{\pi}} \pi^3(1 - \pi|x_1|)(1 - \pi|x_2|)(1 - \pi|x_3|)\Big(\sum_{j=1}^{m} q_j e^{-2\pi i \boldsymbol{k_j x}}\Big) e^{2\pi i \boldsymbol{x k_i}} d\boldsymbol{x}$$

Change of variables: let $\boldsymbol{y} = 2\pi \boldsymbol{x}$

$$\mathcal{F}^{-1}(\mathcal{F}(\text{sinc}^2)\mathcal{F}(f))(\boldsymbol{k_i}) = \frac{1}{64} \int_{-2}^{2} \int_{-2}^{2} \int_{-2}^{2} (2 - |y_1|)(2 - |y_2|)(2 - |y_3|)\Big(\sum_{j=1}^{m} q_j e^{-i \boldsymbol{k_j y}}\Big) e^{i \boldsymbol{y k_i}} d\boldsymbol{y}$$

Note: each final integral is approximated using Legendre-Gauss quadrature weights, shifted (if necessary) according to the continuity of the function and integration bounds. The sum within each integral is calculated by the first call to a non-uniform FFT program; this output is then weighted (in the case of sinc^2) and used as input again to a non-uniform FFT program to compute the final answer.