Proof Portfolio
Math 210

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Contents

Chapter 1

Question 1

We want to prove DeMorgan's laws for sets, using a double inclusion argument:
$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

The following theorems are both proved using the same structure. First you must know what double inclusion means, a way to prove that two sets are equal simply by showing that $A \subset B$ and $B \subset A$. This will show the sets are equal.

Definition 1.0.1. Complement, denoted as ($'$): all elements not in a given set, relative to another set. The symbol $\cup$ is used to show set union. The symbol $\cap$ is used to show set intersection.

Theorem 1.0.2. *For sets A and B,* $(A \cup B)' = A' \cap B'$.

Proof of 1.0.2. This proof is by double inclusion. DeMorgan's Laws states the complement of the union of two sets is the same as the intersection of their complements. Hence, $(A \cup B)' = (A \cap B)'$ is what we must prove. Suppose that there is some element a that is in the set $(A \cup B)'$, the complement set of $(A \cup B)$. Since $a \in (A \cup B)'$ then $a \notin (A \cup B)$. This means that $a \notin A$ and $a \notin B$. Thanks to the definition of complements we know now that $a \in A'$ and $a \in B'$. This shows $a \in A'$ and $a \in B'$ and therefore $a \in A' \cap B'$.

To prove by double inclusion we must also show that if there is a different element $a \in B'$ and $a \in A'$ then $a \in (B \cup A)'$. We will assume that there is some element $a \in B'$ and $a \in A'$. This means that $a \notin A$ and $a \notin B$. This also means that $a \notin (A \cup B)$ because a is not in either A nor B therefore it cant be in the union of those sets. We can see that since $a \notin (A \cup B)$ it must be in $a \in (A \cup B)'$. Now we have our proof by double inclusion. $\square$

Theorem 1.0.3. *For sets A and B,* $(A \cap B)' = A' \cup B'$

Proof of 1.0.3. This is a proof of DeMorgan's Laws using double inclusion. DeMorgan's Laws also state the complement of the intersection of two sets is the same as the union of their complements. Hence we want to show, $(A \cap B)' = A' \cup B'$. Suppose that a is an element in the set $(A \cap B)'$, $a \in (A \cap B)'$. Using the definition of the complement the original set $(A \cap B)$ will not have the element a. This means that $a \notin (A \cap B)$. This is just showing that a could be in either, or neither set A or B . Therefore, in order for a to be in $(A \cap B)'$ a must be in either A' or B' which means $a \in A' \cap B'$.

Now we will prove that if $a \in A' \cup B'$ then $a \in (A \cap B)'$. Suppose that there is some element $a \in A'$ or $a \in B'$. This means that $a \notin A$ or $a \notin B$. Therefore, $a \notin (A \cap B)$ because a is missing from at least one of the sets. However, $a \in (A \cap B)'$ because it is the complement set! This shows by double inclusion that $(A \cap B)' = A' \cup B'$. $\square$

For the following proofs I used double inclusion to state that if element $a \in A$ and $a \in B$ then the sets were the same. I also used the rules of *and* and *intersection*.

For any sets A, B, and C that are subsets of a universal set U,

$$A - (B \cap C) = (A - B) \cup (A - C)$$

Theorem 1.0.4. *For sets A, B, and C,* $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof of 1.0.4. This is a proof using double inclusion to show that these sets are equal. Assume a is an element in the set $A - (B \cap C)$. We can then see that $a \in A$ and $a \notin (B \cap C)$ because otherwise a would have been subtracted out. For a to be in $(A - B) \cup (A - C)$, a must be in either $(A - B)$ or $(A - C)$. In order for $a \notin (B \cap C)$ and for a to be in either $(A - B)$ or $(A - C)$, a can only be one of three conditions

1. $a \in B, a \notin C$,

2. $a \notin B, a \in C$,

3. $a \notin B, a \notin C$.

Any of those conditions would prove the statement $a \in (A-B) \cup (A-C)$.

1. The first condition states $a \in (A-C)$ and $a \notin (A-B)$,
2. the second condition states $a \in (A-B)$ and $a \notin (A-C)$,
3. the third condition states $a \in (A-B)$ and $a \in (A-C)$.

Any of these three conditions will make the statement $a \in A-(B \cap C)$ and $a \in (A-B) \cup (A-C)$ hold true. Now we must show this proof from the other direction. Assume a is an element is the set $(A-B) \cup (A-C)$. That means $a \in A$ and $a \in B$ or $a \in C$. The element a cannot be in both B and C as stated in the three conditions above. Following this, for a to be in $A-(B \cap C)$ a must be one of the three conditions again because otherwise we would be subtracting $a \in A$ from $a \in (B \cap C)$. This proves using double inclusion that $A-(B \cap C) = (A-B) \cup (A-C)$. $\square$

Chapter 2

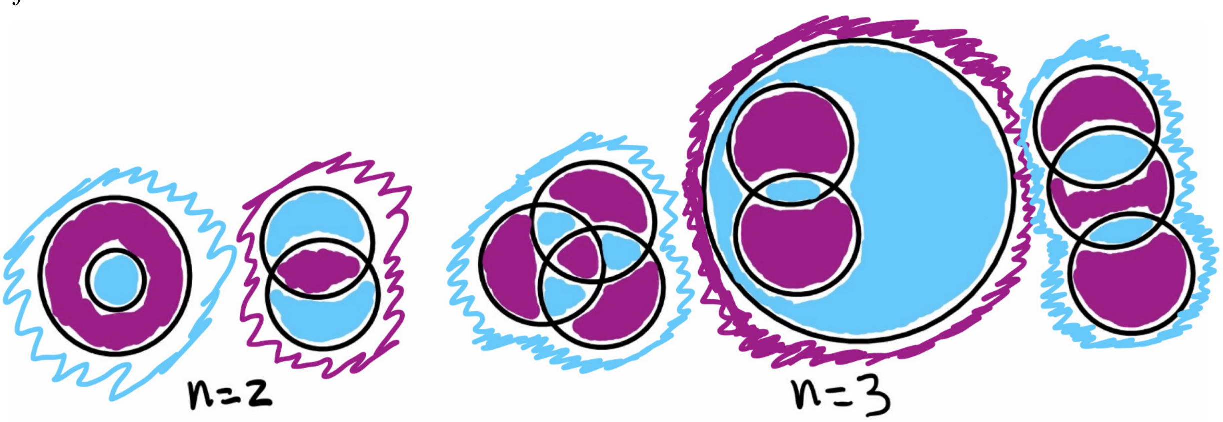
Question 2

Definition 2.0.1. 2-colored: An image can be 2-colored if each edge has an opposite color on each side. There can be no edges with the same color on either side. This includes the outside as a color

We will prove that n circles drawn on a plane can be 2-colored.

I approached this proof by thinking about how I could add and remove circles from drawings that I made and how I would have to adjust my 2 colors that I had created.

Proof. Base Cases: Consider n=2 and n=3.

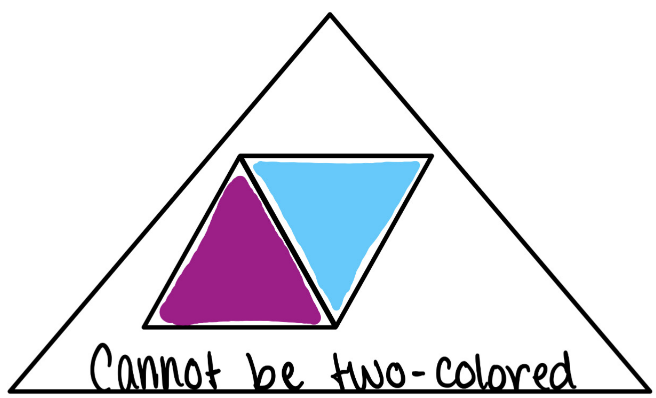


These 2 cases show that no matter how the circles are arranged they can be 2-colored.

Inductive Hypothesis: Assume $n \leq k$ overlapping circles drawn in a plane can be 2-colored.

Inductive Step: To begin we will start with $k+1$ circles and look at how we can get this to fit into our inductive hypothesis. If we have $k+1$ circles that are 2-colored then we remove one of those circles, we have k circles and we know that k circles can be 2-colored. Now, we will look at how we can add a circle to k to get $k+1$ circles. We know that k is 2-colored and we know that means that each edge must have an opposite color on either side of it. If we place a circle anywhere on our k circles, each place where a new edge is created must have opposite coloring. We can assume that because k is 2-colored everything on the outside of the new circle remains 2-colored and the only colors that need to change on the ones on the inside of the new circle. Each color that now has a match on the original side of the new circle must flip to the opposite color. This leaves us with a $k+1$ 2-colored circle. Meaning that n circles drawn on a plane can be 2-colored by proof by induction. $\square$

This proof will not work with triangles. Here is a counter example:



Chapter 3

Question 3

Definition 3.0.1. Fibonacci Sequence: Let $f_1 = f_2 = 1$ then $f_n = f_{n-1} + f_{n-2}$.

Example 3.0.2. Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

I approached this proof with the understanding that each element in the Fibonacci sequence determines the next the recursion of this sequence is important to remember. I proceed by using induction.

Theorem 3.0.3. *For all natural numbers n, f_{5n} is a multiple of 5.*

Proof. Base Cases:
For $n = 1$, we can see that $f_{5n} = f_5$ and $f_5 = 5$.
Also for $n = 2$, we can see that $f_{5n} = f_{10}$ and $f_{10} = 55$. Both of these base cases are shown in the example above.

Inductive Hypothesis: For all natural numbers n where $n \leq k$. The value f_{5n} is a multiple of 5.

Inductive Step: I will be inducting on the $5(k+1)st$ term, as in $f_{5(k+1)}$ term. We need to prove that this will be a multiple of 5 by showing that this can fit into the inductive hypothesis. So we will take $f_{5(k+1)}$ and move a step back until we can get f_{5k}.

$$\begin{aligned}
f_{5k+5} &= f_{5k+4} + f_{5k+3} \\
&= f_{5k+3} + f_{5k+2} + f_{5k+2} + f_{5k+1} \\
&= f_{5k+2} + f_{5k+1} + f_{5k} + f_{5k+1} + f_{5k} + f_{5k+1} + f_{5k+1} \\
&= f_{5k} + f_{5k+1} + f_{5k+1} + f_{5k} + f_{5k+1} + f_{5k} + f_{5k+1} + f_{5k+1} \\
&= 3f_{5k} + 5f_{5k+1}
\end{aligned}$$

We know from our inductive hypothesis that any term that has f_{5n} is divisible by 5. In order for $3f_{5k} + 5f_{5k+1}$ we now need to only look at if $5f_{5k+1}$ is divisible by 5, because there is a coefficient of 5 we know the term is divisible by 5. Therefore, both terms are divisible by 5 so the sum is as well. We can now see that for all natural numbers f_{5n} is a multiple of 5. $\square$

Theorem 3.0.4. *Prove that, for all natural numbers n, 6 divides $n^3 - n$*

Proof. Base Cases: For $n = 1$, we know that $\frac{1^3-1}{6} = 0$. Also for $n = 2$, $\frac{2^3-2}{6} = 1$.

These base cases show that 6 divides $n^3 - n$ in several cases we will now try to show that this is true for all cases not just these.

Inductive Hypothesis: For all natural numbers n where $n \leq k$, 6 divides $n^3 - n$. Inductive Step: I will be inducting on the nth term.To begin start with $k+1$ this looks like $6|(k+1)^3 - (k+1)$. In order to reach our inductive hypothesis we will expand out these terms in order to find like terms.

$$\begin{aligned}
k^3 + 3k^2 + 3k + 1 - k - 1 &= k^3 - k + (3k^2 + 3k) \\
&= (k^3 - k) + (3k(k+1)).
\end{aligned}$$

When we look at the expansion we can see that there is a $k^3 - k$ which is from our inductive hypothesis and we know is divisible by 6. Now we must show that the remaining terms are divisible by 2 and 3. We then pull out like terms again. This makes it obvious that our remaining term is divisible by 3 and a little difficult to see that it is divisible by 2. In order to show that it is divisible by 2 we will look at how consecutive numbers work. We know that numbers go even, odd, even, odd and so on, so if k is odd we know that because we are adding 1 to it we will then get an even number. We also know that any even number multiplied by any other number will remain even. Therefore, k could be any number and $3k(k+1)$ would remain divisible by 2 and 3 making the whole term divisible by 6. $\square$

Chapter 4

Question 4

For the following proofs I had to remember some simple things I had learned in math in elementary. The first was the simple tricks to division, especially for 4 and 9. Next, I had to remember how the places of digits work, 100's, 10's, and 1's. By adding in modular arithmetic I was then proceeded to prove with direct proofs

Theorem 4.0.1. *A number, a is divisible by for, $4|a$, if and only if the last two digits of the number, the 10's and 1's place, are divisible by 4.*

Proof. We will pick any number a with k digits. We will start by assuming that the last two digits are divisible by 4. Now we will then write out the base 10 representation of our number $a = (a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. We will then look at each digit and see if it is divisible by 4. Starting at the highest digit, $a_k * 10^k$, we check the divisibility. If $a_k * 10^k$ mod 4 is equal to 0 we can move on. As long as the exponent is greater than 2 the digit will be divisible by 4 so we can say with certainty that any number between 10^k and 10^2 is divisible by 4. Now we are left with the last two digits, 10's and 1's, to check for divisibility and since we already already assumed that the last two digits were divisible by 4 our proof holds.

Now we must prove the other direction by assuming again that a is divisible by 4. So once again we will write out the base 10 representation of a, $(a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. We know that any digit that has a multiplier of $10^{x>2}$ is divisible by 4. Which leaves us with the last two digits! Since we already assumed the number is divisible by 4 and that they are the last two digits of our number a so the last two digits must be divisible by 4. $\square$

Theorem 4.0.2. *A number, a is divisible by 9, $9|a$, if and only if the sum of all the digits is divisible by 9.*

Proof. We will pick any number a with k digits. We will start by assuming that the sum of the digits $a_k + ... + a_3 + a_2 + a_1$ of our number a is divisible by 9. We will then write out the base 10 representation of our number a. This will look like $(a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. Now, we will expand out this representation in order to get multiples of 9. It is clear that 10 is not a multiple of 9 therefore we will write write it as $9+1$. We now have $(a_k * (9+1)^k) + ... + (a_2 * (9+1)^2) + (a_1 * (9+1)) + a_0$ as our expanded base 10 representation. We will remove the exponents from the equation now because all terms that are previously divisible by 9 to any exponent will continue to be divisible by 9. We will also mod 9 at this point to remove any multiples of 9. We are then left with $(a_k) + ... + (a_3) + (a_2) + a_1$. These remaining numbers, a_k, a_3, a_2, and a_1 we are left with are the digits for our number a! Since we began by assuming that $a_k + ... + a_2 + a_1 + a_0$ we can state for sure that $9|(a_k + ... + a_2 + a_1 + a_0)$.

Now we must prove this the other direction assuming that the number a is again divisible by 9. This time we will assume that the base 10 representation of a has multiples of 9 that we can remove and be left with the original digits of a and that these all add up to a number that is divisible by 9. We will write out the base 10 representation of our number a. This looks the same as it did above, $(a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. We again want to pull out multiples of 9 so we will manipulate this representation to do so, $(a_k * (9+1)^k) + ... + (a_2 * (9+1)^2) + (a_1 * (9+1)) + a_0$. When we take mod 9 we are left with $a_k + ... + a_2 + a_1 + a_0$ since we assumed that $9|a$ the sum of $a + ... + b + c + d$ must add up to a number that is divisible by 9. As we saw before the numbers we pull out from the base 10 representation are the same as our original number a therefore we have proof that if a number is divisible by 9 the sum of the digits of a number add up to another number that is also divisible by 9. $\square$

Theorem 4.0.3. *A number, a is divisible by 11 if and only if the total after alternating addition and subtraction with each digit in the number is a multiple of 11.*

Proof. We will start with any number a with k digits and assume that our theorem is true by saying that if we alternate addition and subtraction for each digit in a the sum will equal a multiple of 11. We will show this by first writing out the base 10 representation of our number $a = (a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. In order to find multiples of 11 in this representation we will manipulate the representation writing 10 as $(11-1)$. This looks like, $(a_k * (11-1)^k) + ... + (a_2 * (11-1)^2) + (a_1 * (11-1)) + a_0$. We cannot ignore the exponents from this point forward because of how our rule alternates between addition and subtraction. So we have $(a_k * (11-1)^k) + ... + (a_2 * (11-1)^2) + (a_1 * (11-1)) + a_0$ and we will now mod by 11. This leaves us with $(a_k * (-1^k)) + ... + (a_2 * (-1)^2) + (a_1 * 1(-1)) + a_0$. which shows the alternation addition and subtraction! These remaining digits from our base 10 representation are our original digits of a with alternating addition and subtraction, $a_k(-1)^k + ... + a_2 - a_1 + a_0$. Since we assumed at the beginning that this case would be true our proof holds!

Now we must prove this from the other direction by assuming that a is divisible by 11 therefore the base 10 representation will allow us to pull out multiples of 11 and leave us with the original digits with a total that is a multiple of 11. We will write out the base 10 representation of our number a. This looks the same as it did above, $(a_k * 10^k) + ... + (a_2 * 10^2) + (a_1 * 10) + a_0$. Once again, we will manipulate this representation in order to get a multiple of 11 in it, $(a_k * (11-1)^k) + ... + (a_2 * (11-1)^2) + (a_1 * (11-1)) + a_0$. We will mod by 11 of a leaving us with, $(a_k * (-1^k)) + ... + (a_2 * (-1)^2) + (a_1 * 1(-1)) + a_0$. Now we are left with $a_k(-1)^k + ... + a_2 - a_1 + a_0$. Since we assumed at the beginning that our number was divisible by 11 the summation of the numbers left from the base 10 representation are divisible by ll. As we saw before and can also see now the numbers remaining from our base 10 expansion are also our original digits of a. Now we have a proof that a number is divisible by 11 if and only if the total of the digits with alternating addition and subtraction is a multiple of 11. $\square$

Chapter 5

Reflection

5.1 Mid Semester Reflection

What is one skill you feel you've developed in 210 thus far?

I learned how to write a correct proof.

Which proof in this proof portfolio has been the biggest challenge for you thus far?

The first question was the hardest for me. Just keeping the math straight in my head made it really difficult for me to make a clear resolution on paper.

What insights ("Aha!" moments) have you had about the process of proving results and writing up proofs for your proof portfolio or throughout this course?

Induction has definitely been an "aha" moment as much as I hate making the steps real once I do it makes so much more sense. Also the proving from both directions, its always a struggle for me to understand what the difference between the two is but once I see it written I understand.

Of the proofs in the proof portfolio thus far, of which proof are you the most proud?

Honestly, I am proud of all of them. I hated writing them and for how much I despised doing this I feel like my proofs turned out very well done and that I have an actual understanding of what is going on with them.

Which aspects of this course are helping you learn? What would further help your learning?

I wish we had gone over the homework questions more than just sometimes in class and doing one as a group. I feel like this kind of wasted a lot of time and I would have rather had each question explained because most of the time I didn't understand 90 percent of the problems we had and they never got explained. Its much easier to learn from examples than it is from proofs.

What is one goal you have for yourself concerning 210? (It could be related to the proof portfolio, could be a skill you want to develop, or a study habit you want to have.)

I want to not get a C. This is the first semester I wont be getting a majority of A's in my classes and it would really hurt to get a C after how much effort I have put into this.

5.2 End of Semester Reflection

Which of your proofs do you feel is the best? Why do you think this is?

Same answer as my mid-semester reflection because I just wrote it.

Which of your proofs do you feel least confident about? Why do you think this is?

Same answer as my mid-semester reflection because I just wrote it.

What do you feel is your greatest strength when writing proofs for this portfolio?

Having complete explanantions.

Looking back at your mid-semester reflection, what progress have you made on the skill you hoped to develop further?

Well I wrote these at the same time so I'm still hoping for the same thing.

Reflect on the entire semester and comment on how you feel you have progressed in writing proofs. You should specifically address what you think has been your biggest improvement and what insights ("Aha!" moments) you have had about the process of proving results and writing up your proofs for your portfolio.

Same as mid semester reflection.

Do you think that the proof portfolio was valuable in helping you improve your skills in writing proofs? If so, please indicate which parts were most helpful. If not, please comment on why you feel it was not valuable. In both cases, please give any suggestions you might have on how the instructor could improve the implementation of the project in the future.

I hate the fact that math and writing were combined in this assignment mainly because I hate writing. It was hard for me to focus on both the math and the writing at the same time without shutting my brain off on one or the other, however I do think it helped me realize how to focus on both better. I wish you had spent more time on the proof portfolio through out the semester rather than just at the end. I don't feel like I was procrastinating because I had been so on top of everything because you said it wasn't something to leave to the end and I feel as though you did not assist in making it a project that could be completed throughout. It felt extremely rushed at the end.