Ranking and Selection Procedures

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Abstract

This paper uses three Ranking and Selection Procedures to choose optimal systems for finding lowest Inventory costs, highest profit in a Bottle Filling System, and lowest cost at a Truck Weigh Station. The three procedures are: Selecting the best of k systems - Dudewicz and Dalal procedure, Selecting a subset of size m containing the best of k systems - Koening and Law procedure, and Nelson and Matejcik (NM) procedure.

Ultimately, the subset m of k systems proved to be the optimal method because it required the least amount of replications, and thus was the least computationally expensive. The NM method also saw advantages in its ease of use, while the DD method would have an advantage if it started from a place of expert opinion, thus negating the need for the first iteration in the best m of k procedure.

Contents

1 Intro

Ranking and selection ($R\&S$) procedures are statistical approaches used to compare and evaluate different scenarios or systems in order to identify the "best" one or a subset of good options. In this case, "best" refers to the alternative system with the maximum or minimum expected value of a given performance criteria.

There are three main approaches within ranking and selection:

- indifference-zone selection,
- subset selection,
- multiple comparisons.

In terms of indifference-zone selection methods, Dudewicz and Dalal ($\mathscr{DD}$) [1] developed a two-stage sampling procedure that involves sampling from each of the systems being evaluated. Rinott ($\mathscr{R}$) [2] introduced another popular procedure that uses sample means from the systems to make the selection. In both procedures the k systems are simulated independently. On the other hand, Nelson and Matejcik ($\mathscr{NM}$) [3] developed a two-stage procedure that allows for the use of common random numbers (CRN) in selecting the best system.

Subset selection involves screening a set of scenarios to identify a smaller subset that contains the best or at least a "good" option. The procedure has been derived by Koenig and Law [4]. Indifference-zone selection aims to select the best scenario while controlling the probability of selecting the true best option. Multiple comparisons, on the other hand, focus on forming confidence intervals to compare different scenarios simultaneously. Subset selection is a valuable approach, especially in the early stages of a simulation study when there is a large number (k) of alternative systems to evaluate. The goal of subset selection is to perform an initial screening to eliminate clearly inferior systems. This screening process helps save computational resources by avoiding the need to obtain precise estimates of the behaviour of these inferior systems, which would otherwise require a significant amount of computer time. By using subset selection, we can quickly identify a smaller subset of scenarios that gives multiple "good" options rather than one "best" option. This allows us to focus our resources and efforts on further analysing and refining this subset, rather than wasting time on clearly inferior systems. Subset selection serves as a useful tool to efficiently narrow down the options and prioritize the most promising scenarios for further evaluation in a simulation study.

Additionally, multiple comparisons with the best (MCB) is a technique that aims to form confidence intervals for comparing the differences between scenarios. Nelson and Matejcik demonstrated that the output of indifference-zone procedures can be used to construct MCB confidence intervals, ensuring both correct selection and coverage of the differences with a specified confidence level [5].

Overall, ranking and selection procedures provide a systematic and statistical approach to evaluate and compare different scenarios or systems, helping to identify the best option or subset of options.

In this project, we will explain and detail 3 procedures, namely:

1. Dudewicz and Dalal ($\mathscr{DD}$) procedure
2. Koening and Law procedure
3. Nelson and Matejcik ($\mathscr{NM}$) procedure

We will apply them to 3 separate examples and compare the output and computational cost of each.

2 Selecting the best of k systems - Dudewicz and Dalal ($\mathscr{DD}$) procedure

The goal of this procedure is to compare k systems and select a system with the smallest total cost μ_{i1}. Let "CS" denote this event of "correct selection". We want $P(CS) \geq P^*$ provided that $\mu_{i2} - \mu_{i1} \geq d^*$,

where the minimal CS probability $P^* > \frac{1}{k}$ and the "indifference" amount $d^* > 0$ are both specified by the analyst.

The statistical procedure for solving this problem (Dudewicz and Dalal ($\mathscr{DD}$)) involves "two-stage" sampling from each of the k systems. In the first stage we make a fixed number of replications of each system, then use the resulting variance estimates to determine how many more replications from each system are necessary in a second stage of sampling in order to reach a decision.

In the first-stage sampling, we make $n_0 \geq 2$ replications of each of the k systems and define the first-stage sample means and variances

$$\bar{X}_i^{(1)}(n_0) = \frac{\sum_{j=1}^{n_0} X_{ij}}{n_0}$$

and

$$S_i^2(n_0) = \frac{\sum_{j=1}^{n_0} \left[X_{ij} - \bar{X}_i^{(1)}(n_0)\right]^2}{n_0 - 1}$$

for $i = 1, 2, \ldots, k$.

The next step is to compute the total sample size N_i needed for system i as

$$N_i = \max\left\{n_0 + 1, \left\lceil \frac{h_1^2 S_i^2(n_0)}{(d^*)^2} \right\rceil\right\}$$

where $\lceil x \rceil$ is the smallest integer that is greater than or equal to the real number x, and h_1 (which depends on k, P^*, and n_0) is a constant that can be obtained from a table.

Next, make $N_i - n_0$ more replications of system $i (i = 1, 2, \ldots, k)$ and obtain the second-stage sample means

$$\bar{X}_i^{(2)}(N_i - n_0) = \frac{\sum_{j=n_0+1}^{N_i} X_{ij}}{N_i - n_0}$$

Then define the weights

$$W_{i1} = \frac{n_0}{N_i}\left\{1 + \sqrt{1 - \frac{N_i}{n_0}\left[1 - \frac{(N_i - n_0)(d^*)^2}{h_1^2 S_i^2(n_0)}\right]}\right\}$$

and $W_{i2} = 1 - W_{i1}$, for $i = 1, 2, \ldots, k$. Finally, define the weighted sample means

$$\tilde{X}_i(N_i) = W_{i1}\bar{X}_i^{(1)}(n_0) + W_{i2}\bar{X}_i^{(2)}(N_i - n_0)$$

and select the system with the smallest $\tilde{X}_i(N_i)$.

3 Selecting a subset of size m containing the best of k systems - Koening and Law ($\mathscr{KL}$) procedure

The optimal system in Ranking in Selection in Section 2 relies on an understanding of the system and expert knowledge to narrow down parameters. When expert opinion is not available, and a broader set of parameters and larger number of systems need to be compared.

A larger number of systems can be computationally expensive to rank, demanding the use of an initial "screening process", or a subset selection method. The procedure described builds upon the indifference zone procedure described by Rinott (1978) [6] [7], and uses the "two stage" sampling method similar to that for selecting the optimal mean μ_i.

The goal of selecting a subset is to find, with probability at least P*, the selected subset that will contain a system with the smallest mean response μ. Correct Selection (CS) is when as the subset of systems of size m in the total number of systems k contains the optimal mean $\mu_{[k]}$.

This method is an extension of the previous method, selecting a optimal mean in k systems. To that end, variables X_{ij}, μ_i, μ_{i_1},σ^2 described in the previous section are again used here.
m: number of systems in subset
k: total number of systems

All X_{ij}'s are i.i.d. Normal, with no correlated random variables. σ^2's are unknown and may or may not be equal.

The assumption of unknown and σ's that may or may not be equal is unique to Koenig and Law [4], and sets it apart from the formulation of the k-sample problem described by Paulson (1949), where σ^2 is known. One such formula is

$$n_1 = -\left(\frac{2\sigma^2}{\delta^{*2}}\right) log(1 - P*), \quad \text{Dudewicz (1969a) [8]}$$

Above, the number of samples is denoted by n_1, and it is reliant on knowing variance σ^2. In the Koenig and Law method, N_1 uses constant h_2 for subset selection in place of variance, as seen below.

Alternatively, Gupta (1956) [8] also describes a method assuming a common known variance σ^2. In this method, the seleciton of the subset is of random size, allowing the size to be determined by the observations themselves. The Koenig and Law approach uses a fixed sized subset.

The approach is as follows:

1. Take first-stage sample of $n_0 \geq 2$ replications from each system and define $\bar{X}_i^{(1)}(n_0)$ and $\bar{S}_i^2(n_0)$ for $i = 1, 2, .., k$

   (a)
   $$\bar{X}_i^{(1)}(n_0) = \frac{\sum_{j=1}^{n_0} X_{ij}}{n_0}$$

   (b)
   $$\bar{S}_i^2(n_0) = \frac{\sum_{j=1}^{n_0} [X_{ij} - \bar{X}_i^l(n_0)]^2}{(d*)^2}$$

2. **Next compute the total number of replications, N_i needed for the ith system, using h_2 instead of h_1.**
   $$N_i = max\left\{n_0 + 1, \left\lceil \frac{h_2^2 S_i^2(n_0)}{n_0 - 1} \right\rceil\right\}$$

   As seen previously in the selection of the optimal mean, we computed the total number of replications needed based on the current number of replications, the sample variance, and constant h_1. h_2 differs from h_1 because h_2 takes into account m, the subset, and k, the total number of systems, whereas h_1 only takes into account k.

   The derivations of h_1 and h_2 are complicated and beyond the scope of this project.

3. Make $N_i - n_0$ replications, form the second-stage sample means, weights, and weighted sample means, and define the selected subset

4 Nelson and Matejcik ($\mathscr{NM}$) procedure

The Nelson and Matejcik procedure builds on the Dudewicz and Dalal procedure, and is also used to determine the number of replications needed to ensure the best (highest or lowest mean outcome) simulation that will give the highest mean with a pre-determined certainty [1]. This method differs because it allows common random numbers to be used [9], which means this can be used to lower the overall variance off the system. The method is implemented step wise the following way:

1. Start by running each simulation 2 or more times, the numbers of times the simulation is run is called n_0. The results of the simulations will be used as input to determine to covariance and the differences between the results.

2. The pairwise variance between the systems can then be calculated using the following equation:
   $$S^2 = \frac{2\sum_{i=1}^{k}\sum_{j=1}^{n_0}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2}{(k-1)(n_0 - 1)}$$

   In this equation, k is the number of models that are compared and n_0 is the amount of replications per model that are initially performed. Notice that this comes down to a single pairwise variance value.

3. From this the required sample size can be calculated with the following equation:

$$N = max\left\{n_0, \left\lceil \frac{g^2 S^2}{(d^*)^2} \right\rceil\right\}$$

In this equation, the values of g can be looked up from a table (These can be found in the appendx7) and is dependent on the required accuracy (α) using the correlation coefficient between the simulations, the number of simulations k, and the number of initial iterations (n_0) [10]. If n_0 is the highest value, then the accuracy is already reached with the initial iterations. The value d^* is the minimum difference between replications that you would want to detect, and the constant can be selected by the modeller.

4. Based on the outcome of the previous equation, add $N - n_0$ iterations per model (with common random variables).

5. After running the new iterations the overall average per simulation method is calculated:

$$\frac{\sum_{j=1}^{N} \bar{X}_{ij}}{N}$$

6. The method or system with the smallest or largest $\bar{X}_i(N)$ can then be selected as the best alternative.

A general weakness of this method is the need to look up the g values in a table. These tables are only available in specific books like the book from Tamhane and Hochberg [10]. It also requires a common correlation coefficient between all the values [10].

5 Example simulations explanation

This paper uses three simulations, described here.

5.1 Inventory Simulation Model

The inventory system model used in this project was adapted from Arthurbdt's model found in their github repository [11].

1. Establish Inventory System class defined as "Single product inventory system"
2. Initialize system with order point, order size, when to start inventory, and initial ordering costs, shortage costs, holding costs. These constants can be established when calling upon the model during the execuation of the simulation.
3. Create function that updates order costs, determines when the order will be received, and updates the inventory level and costs
4. Create function to review inventory, checking the level at regular intervals (determined by inputs) and ordering if the level is below the determined "reorder point."
5. Create function to update holding and shortage costs at each inventory movement
6. Generate demand at random intervals and update the inventory level. The levels of demand can be specified when the function is called upon.
7. Before running the model, the following constants that are used in the model need to be established:
   (a) average time between customer demands (months)
   (b) possible customer demand sizes
   (c) probability of each demand size
   (d) units in inventory at simulation start

(e) fixed cost of placing an order

(f) variable cost of ordering an item

(g) monthly cost for each item in backlog

(h) monthly cost for each item in inventory

(i) duration of the simulation (in months)

8. The full model is then run with reorder-points, order-sizes, and num-reps as input.

5.2 Bottle filling system

The bottle filling system consists of a filler and labeller and is a typical manufacturing system with bottles coming in, and two machines, if running, handling them at a steady state. The machines are running at the same speed and thus are modelled by a single unit. This unit has 5 reasons for stopping, four are breakdowns and one is maintenance. All breakdowns frequencies are modelled to be exponential distributions and the breakdown times per breakdown type are also modelled as exponential distributions. The maintenance time and frequency are fixed values.

1. Big breakdown on the filler. The big breakdown frequency is dependent on the amount of maintenance performed on the machine. This represents component failure that can be prevented by in-time replacement.

2. Small breakdown on the filler. The distribution of the length of small breakdowns is dependent on the amount of personnel on the manufacturing line. Small breakdowns represent material failure or machine safety stops which ask for operator action but don't require the maintenance team.

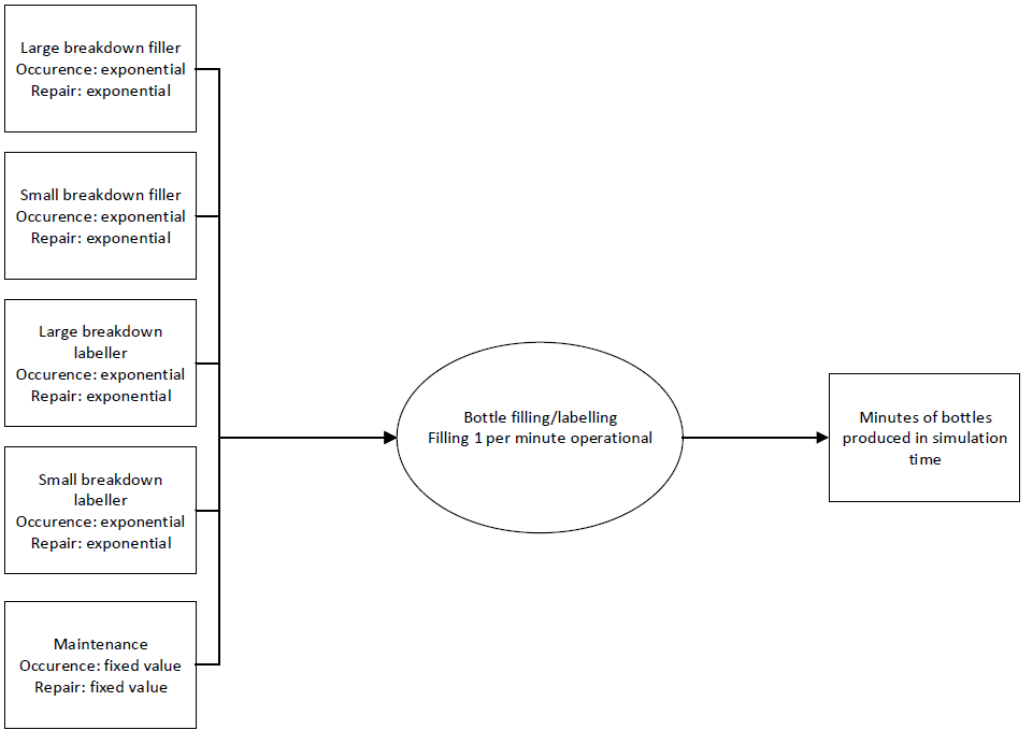
3. Big breakdown on the labeller. The big breakdown frequency is dependent on the amount of maintenance performed on the machine. This represents component failure that can be prevented by in-time replacement.

4. Small breakdown on the labeller. The big breakdown frequency is dependent on the amount of maintenance performed on the machine. This represents component failure that can be prevented by in-time replacement.

5. Maintenance. The maintenance is planned and therefore also has a fixed frequency and duration. The length of the maintenance determines the frequency of big breakdowns, but it also costs time and money in itself. Each hour of maintenance costs $1 representing the personnel, replacement parts and tools needed.

This system is illustrated in figure 1 and it represents a simplified version of a real bottling filling manufacturing system with 2 of the machines that are always on such a line.

Figure 1: Schematic overview of bottle filling system



As stated above, the system is an optimization between the personnel costs for production, the maintenance costs and the profit of each time unit of production. Each minute of production in which the system is not in maintenance or breakdown makes a profit of $100 (baseline) while, as stated above, an hour of maintenance costs $1, and adding additional personnel costs $30 per hour independent of the line producing or not. However, having an additional person reduces the length of small breakdowns by 50% and having each hour of additional maintenance per week reduces the long breakdown frequency by 12.5% up to a maximum of 50%.

The model is run over a 6 week period with weekly maintenance sessions. The various iterations are described in table 1. For ease of reading, abbreviations are used. The meaning of the abbreviations are descrebed below Table 1.

Table 1: Bottle filling configurations

i	Pol [-] fixed	Mh/w [hours] fixed	p/w [weeks] fixed	fbbf [hours] expo	fsbf [minutes] expo	lbbf [hours] expo	lsbf [minutes] expo	fbbd [hours] expo	fsbd [minutes] expo	lbbd [hours] expo	lsbd [minutes] expo
1	2	4	6	24	60	36	45	2	2	4	4
2	2	8	6	48	60	72	45	2	2	4	4
3	3	4	6	24	60	36	45	2	1	4	2
4	3	8	6	48	60	72	45	2	1	4	2

- Pol: amount of personnel on line, fixed value
- Mh/w: maintenance hours per week, fixed value
- p/w: amount of production weeks for the simulation, fixed value
- fbbf: Filler big breakdown frequency, exponentially distributed, the value given is the expected value
- fsbf: filler small breakdown frequency, exponentially distributed, the value given is the expected value

- lbbf: labeller big breakdown frequency, exponentially distributed, the value given is the expected value
- lsbf: labeller small breakdown frequency, exponentially distributed, the value given is the expected value
- fbbd: filler big breakdown duration, exponentially distributed, the value given is the expected value
- fsbd: filler small breakdown duration, exponentially distributed, the value given is the expected value
- lbbd: labeller big breakdown duration, exponentially distributed, the value given is the expected value
- lsbd: labeller small breakdown duration, exponentially distributed, the value given is the expected value

The output of the build model is in kilo dollars to make sure the variance does not go to too large amounts as it does when usual dollars would be used.

5.3 Truck Weigh Station queuing system

We are simulating a simplified truck weigh station queuing system on a busy interstate.

- Trucks arrive according to a Poisson distribution with $\lambda_1 = 15$ per minute (i.e., mean interarrival rate $\mu_1 = 0.067$ minutes) to the weigh station, where there are several servers who each have exponential service time with mean rate $\mu_2 = 0.75$ minutes.
- After that, the trucks are assigned to the shortest of the weight station queues, where they go through the system (time is uniformly distributed between 0.5 minutes and 1 minute).

The goal is to keep average wait times below 15 minutes, so there is a penalty if a truck waiting time is longer than 15 min. In addition, there is a cost associated with each weigh station employee and scale. We want to minimise the total cost of the system.

The original notebook was written for one of the assignments of ISYE 6501 by one of the teammembers and was modified for the needs of this project. The simulation runs as follows:

- System is created where we define check time according an exponential distribution (mean interarrival rate $\mu_1 = 1/(arr\ rate)$ minutes) and the weigh time according to a uniform distribution between 0.5 minutes and 1 minute.
- Tracks are processed through the System - going through the station check queue and then finding the shortest scale queue.
- For every track, several parameters are calculated, namely
  - total time in the system
  - truck check time
  - weighing time
  - waiting time
  - waiting cost - \$0.1 for every minute the truck wait for longer than 15 minutes
  - overhead cost - \$150 per checker or scale per hour of simulation
  - total cost - sum of waiting cost and overhead cost
- The total number of trucks through the system is also calculated

The 9 different systems investigated are detailed in Table 2 and calculate the total cost in $ for each (scaled by 1000).

Table 2: Truck Weigh Station queuing configurations

i	checkers	scales
0	10	10
1	10	11
2	10	12
3	11	10
4	11	11
5	11	12
6	12	12
7	12	12
8	12	12

6 Example applications of selection procedures

In this chapter the various selected methods described in chapters 2, 3 and 4 are applied to the simulations

6.1 Inventory system

6.1.1 Application of selection of the best of k systems method

In the first-stage sampling, we make n_0 replications of each of the 8 systems and calculate the first-stage sample means and variances. The number of initial iterations affects the total number of replications needed (the lower n_0, the higher h_1, the higher the number of replication in the second stage). Then, we select d^* the "indifference" amount. The smaller this value, the larger the number of replications needed. Lastly, the choice of P^* will affect the total number of replications, with higher value, corresponding to more replications.

In this case, we select $d^* = 1$ and $P^* = 0.95$ and derive $h_1 = 3.551$ from Table 16. With these parameters, a total of 1,342 replications were needed to establish the best system among the 8 alternatives. System 4 (reorder point = 20, order size = 50) was identified as being the best.

Table 3: Optimal System Selection

i	reorder point	order size	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	10	50	122.95	40.19	507	124.31	0.043	0.957	124.25
1	10	60	123.32	11.42	144	124.01	0.139	0.861	123.91
2	10	80	127.25	13.57	172	126.47	0.140	0.860	126.58
3	10	100	132.83	9.42	119	132.10	0.183	0.817	132.23
4	20	50	118.69	9.52	121	119.44	0.199	0.801	119.29
5	20	60	121.33	10.17	129	121.04	0.183	0.817	121.10
6	20	80	127.16	6.42	81	126.65	0.249	0.751	126.77
7	20	100	134.40	5.45	69	133.90	0.316	0.684	134.06

6.1.2 Application of subset m of k method

The subset method is an extension of the "best of k" method in Section 6.1.1.

Instead of calculated $N - i$ with $h_1 = 3.551$, h_2 is derived from the table with $k = 8$, $m = 3$, and $P^* = 0.95$ giving $h_2 = 2.267$ 16. In the subsets of size m system, the method is similar to the best of K systems, with the main difference being the h constant. For this model, the top 3 of 8 systems is found with constant $h_2 = 2.267$. With a constant lower than that of the "Optimal system Selection"

(Section 6.1.1), the number of necessary replications is fewer. A total of 496 replications were run for the first iteration.

Table 4: Best 3 of 8, First Iteration

i	reorder point	order size	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	10	50	124.975	24.094	124	123.912	0.175	0.825	124.098
1	10	60	123.675	11.139	58	122.897	0.399	0.601	123.208
2	10	80	126.995	13.170	68	127.971	0.325	0.675	127.653
3	10	100	132.960	8.675	45	130.756	0.492	0.508	131.841
4	20	50	120.345	14.748	76	119.621	0.286	0.714	119.828
5	20	60	121.900	8.623	45	120.892	0.506	0.494	121.402
6	20	80	127.350	10.342	54	127.256	0.431	0.569	127.296
7	20	100	134.490	5.054	26	133.383	0.783	0.217	134.250

From above, the best systems, unranked, are 1, 4, 5. While they hav different means, the best is not shown to have the lowest number, only that it is within the best 3. A second iteration is run to find the best of the three using the same methods for "Best of K Method," this time $h_1 = 2.872$ For the second iteration, a total of 286 replications were run.

Table 5: Best 3 of 8: Second Iteration

i	reorder point	order size	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
1	10	60	123.675	11.139	92	124.253	0.539	0.461	123.940
4	20	50	120.345	14.748	122	119.319	0.453	0.547	119.780
5	20	60	121.900	8.623	72	120.987	0.632	0.368	121.560

Ultimately, system four is selected. The final mean is the lowest of the three shown in Table 4.

6.1.3 Application of the NM method

The Nelson and Matejcik modified method for Dudewicz and Dalal as described in 4 is applied to the reorder system as summarized in table 11. For all of the iterations the correlation coefficient was found to be between 0.1 and 0.2, this influences g. All of the iterations are set with the same seed prior to using the random number generator to make sure of common random numbers.

Table 6: configurations for application of NM method to inventory system

d^*	CI [%]	n_0	g	S^2	N
1	90	20	2.36	22	123
1	95	20	2.7	22	160
5	90	20	2.36	22	20
5	95	20	2.7	22	20
1	90	5	2.89	12.7	107
1	95	5	3.54	12.7	160
5	90	5	2.89	12.7	5
5	95	5	3.54	12.7	7

General trends are as follows:

- A higher d^* leads to a lower amount of iterations that are needed because you accept a higher difference as the same system.
- More iterations were estimated to be needed with a lower amount of initial iterations. The main driver for this is the increasing g value with a decreasing amount of initial iterations.

- A broader confidence interval leads to more iterations being needed by increasing the g value (from table [10]).
- The baseline configuration with $d^* = 1$ and 20 initial iterations requires 160 iterations in total to reach 95% confidence. For 8 simulations this leads to a total amount of iterations needed of 1280. **After 160 iterations the system with the lowest cost was the (20,50) system and thus this method gave the same result as the methods above.**

6.2 Bottle filling system

6.2.1 Application of selection of the best of k systems method

In the first-stage sampling, start with making 20 replications of each of the 4 systems and calculate the first-stage sample means and variances. We select $d^* = 20$ and $P^* = 0.95$ and derive h_1 from the table 15. With these parameters, a total of 1716 replications are needed to establish the best system among the 4 alternatives. In this case, the system with the highest value (maximum profit) is System 3 (pol = 3, mh = 8).

Table 7: Optimal Selection System

i	pol	mh	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	2	4	4217.640	23189.73	558	4228.363	0.041	0.959	4227.918
1	2	8	4402.215	21175.64	510	4219.402	0.048	0.952	422.8085
2	3	4	4362.220	11201.22	270	4219.878	0.088	0.912	4232.345
3	3	8	4624.335	15719.93	378	4225.966	0.056	0.944	4248.382

6.2.2 Application of subset m of k method

The subset method is an extension of the "best of k" method in Section 6.2.1, with $d^* = 20$

Instead of calculated $N - i$ with $h_1 = 3.551$, h_2 is derived from the table with $k = 4$, $m = 2$, and $P^* = 0.95$ giving $h_2 = 2.071$ 16.

A total of 787 replications are run for the first iteration.

Table 8: Subset 2 of 4, First Iteration

i	pol	mh	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	2	4	4221.465	32702.48	351	4234.829	0.064	0.936	4233.974
1	2	8	4471.880	19075.22	205	4428.198	0.112	0.888	4433.090
2	3	3	4364.705	10714.85	115	4344.129	0.186	0.814	4347.957
3	3	8	4626.905	10809.88	116	4618.499	0.183	0.817	4620.038

The best subsets are for systems 1 and 3, and they are run using the best of k method to find the optimal system, this time with $h_1 = 2.453$. A total of 450 replications are needed in this iteration.

Table 9: Subset 2 of 4, Second Iteration

i	pol	mh	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
1	2	8	4471880	19075.22	287	4421459.03	0.231	0.769	4433993
3	3	8	4626905	10809.88	163	4643243.64	0.332	0.668	4638747

The optimal system is 3, which was also the optimal system in Table 9, at highest profit = $4,638,747 ($4638.747 x 1000) and pol = 3, mh = 8. This was not reached until this second iteration. With the two iterations, a total of 1237 replications were run.

6.2.3 Application of the NM method

The bottle filling simulations attempt to find the optimal way to allocate resources in a manufacturing system. The Nelson and Matejcik modified method for Dudewicz and Dalal helps to determine how many replications are needed to determine the optimal bottling line maintenance strategy. Again, the correlation coefficients found were between 0.1 and 0.2 so the values from the table for 0.1 were taken [10]. Also the same seed (1208) was used in each replication to make sure the common random numbers condition held up. Because the profit over the six week period is generally over $4,000,000 ($4000 x 1000), the minimum difference indicator d^* is set to be 10 and 20 thousand dollars. Without a large d^*, the replications would produce high sample variances leading to extremely high amounts of iterations needed to get to the final answer. Using 1000's of dollars and a higher d^* creates an interpretable result.

Table 10: configurations for application of NM method to bottling system

d^*	CI [%]	n_0	g	S^2	N
10	90	20	2.04	$40 * 10^3$	1649
10	95	20	2.39	$40 * 10^3$	2263
20	90	20	2.04	$40 * 10^3$	413
20	95	20	2.39	$40 * 10^3$	565
10	90	5	2.43	$56 * 10^3$	3268
10	95	5	3.83	$56 * 10^3$	8118
20	90	5	2.43	$56 * 10^3$	576
20	95	5	3.83	$56 * 10^3$	791

- Again a higher d^* leads to a lower amount of iterations that are needed because you accept a higher difference as the same system.
- A lower amount of initial iterations did lead to a higher total amount of iterations being estimated in this case because of the higher sample variance found.
- A broader confidence interval leads to more iterations being needed by increasing the g value (from table [10]).
- The baseline configuration with $d^* = 10^4$ and $n_0 = 20$ leads to an approximation of 565 iterations per simulation being needed to reach a confidence interval of 95%. This means 2260 iterations will be needed in total over the 4 simulations. **When the model is run for this amount of iterations the highest profit model is found to be with 3 people and 8 hours of maintenance per week leading to a profit of $4624475 per 6 weeks.**

6.3 Truck Weigh Station queuing system

6.3.1 Application of selection of the best of k systems method

In the first-stage sampling, start with making 20 replications of each of the 4 systems and calculate the first-stage sample means and variances. Next, select $d^* = 1$ and $P^* = 0.95$ and derive $h_1 = 3619$ from the Table 15. With these parameters, a total of 1493 replications were needed to establish the best system among the 9 alternatives. System 7 (checkers = 12, scales = 11) was identified as being the best.

Table 11: Optimal System Selection

i	checkers	scales	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	10	10	64.80	15.93	209	64.49	0.107	0.893	64.53
1	10	11	64.90	25.83	339	63.75	0.070	0.930	63.83
2	10	12	66.40	24.97	328	65.02	0.074	0.926	65.12
3	11	10	64.76	12.58	165	63.71	0.134	0.866	63.85
4	11	11	43.06	8.21	108	42.20	0.212	0.788	42.39
5	11	12	44.69	12.05	158	43.23	0.138	0.862	43.44
6	12	10	64.98	10.47	138	64.90	0.173	0.827	64.91
7	12	11	42.41	2.06	27	41.53	0.757	0.243	42.19
8	12	12	43.20	0.00	21	43.20	1.	0.	43.20

6.3.2 Application of subset m of k method

The subset method is an extension of the "best of k" method in Section 6.3.1.

Instead of calculated $N - i$ with $h_1 = 3.619$, h_2 is derived from the table with $k = 9$, $m = 5$, and $P^* = 0.95$ giving $h_2 = 1.720$ 16.

A total of 371 replications are run for the first iteration. Systems 1, 4, 7, and 8, are selected as the top systems

Table 12: Best 5 of 9: First Iteration

i	checkers	scales	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
0	10	10	64.8	15.93	48	64.5	0.484	0.516	64.65
1	10	11	64.9	25.83	77	64.45	0.298	0.702	64.58
2	10	12	66.4	24.97	74	64.94	0.289	0.711	65.37
3	11	10	64.76	12.58	38	65.16	0.599	0.401	64.92
4	11	11	43.06	8.21	25	43.37	0.869	0.131	43.10
5	11	12	44.69	12.05	36	45.01	0.605	0.395	44.82
6	12	10	64.98	10.47	31	65.61	0.657	0.343	65.19
7	12	11	42.41	2.06	21	41.91	1.286	-0.286	42.55
8	12	12	43.20	0.00	21	43.2	1.00	0.00	43.20

The second iteration is on systems 1,4,5,7,8 with $h_1 = 3.258$ using the method in Section 6.3.1. A total of 534 replications are needed to produce the final results.

Table 13: Best 5 of 9: Second Iteration

i	checkers	scales	$\bar{X}$	S_i^2	N_i	$\bar{X}(N_i - n_0)$	W_{i1}	W_{i2}	$\bar{X}(N_i)$
1	10	11	64.90	25.83	275	63.84	0.491	0.509	64.36
4	11	11	43.06	8.21	88	42.083	0.906	0.094	42.97
5	11	12	44.69	12.05	128	43.20	0.741	0.259	44.31
7	12	11	42.41	2.06	22	41.67	1.374	-0.374	42.68
8	12	12	43.20	0.00	21	43.20	1.000	0.000	43.20

System 7 (checkers = 12, scales = 11) was identified as being the best, found after a total of 905 replications over 2 replications.

6.3.3 Application of the NM method

For the application of the Nelson and Matejcik adjusted Dudewicz and Dalal method to the queuing system the correlation coefficient was found to be between 0.2 and 0.4 for between the 9 iterations described in 2. Because of this, the tables using a correlation coefficient of 0.3 were used to find the g

value for this application [10]. In this case the $d^* = 1$ because of the small values in the range of 40 to 60 (meaning \$40000 to \$60000), lending to a difference of more than \$1000 being detected. Exploration is done to show how little iterations need to be done when a range of \$5000 would be enough. In this case the same seed (451) was used.

Table 14: configurations for application of NM method to the queuing system

d^* (In thousands)	CI [%]	n_0	g	S^2	N
1	90	20	2.34	19.2	106
1	95	20	2.69	19.2	139
5	90	20	2.34	19.2	20
5	95	20	2.69	19.2	20
1	90	5	2.89	21.64	264
1	95	5	3.49	21.64	181
5	90	5	2.89	21.64	8
5	95	5	3.49	21.64	11

- A lower amount of initial iterations did lead to a higher total amount of iterations being estimated in this case because of the higher sample variance found and the increased g value due to the lower amount of iterations.
- A broader confidence interval leads to more iterations being needed by increasing the g value (from table [10]).
- The baseline configuration with $d^* = 1$ and 20 initial iterations requires 139 iterations in total to reach 95% confidence. For 9 simulations this leads to a total amount of iterations needed of 1251. **After running each simulation type 139 times the system with 12 checkers and 11 scales came out as the best system with an average cost of \$4,225,000.**

7 Conclusions/Discussion

For each model, each procedure chose the same system as the best, thus one system did not outperform another in terms of accuracy. To choose the best method, other factors are taken into account.

The "subset m of k" method is the best procedure overall due to its low cost of complexity. It consistently outperformed the "best of k, DD" procedure in terms of number of total replications while giving the same result. The "subset m of k" will thus always be quicker to run.

The cost of the NM method is high, as it uses the same number of replications for each system. This can lead to more replications than necessary for comparison. However, this also means that the procedure is easy to use. If being done by hand, using one calculation will take less time than having to recalculate N_i for multiple systems. As these methods are no longer done by hand, this advantage is not relevant. Thus, it is not an optimal method.

The DD method gives the same solution as "subset m of k", but uses more replications. The DD method is a good choice when expert opinion already creates the first subset. In this scenario, less systems are being compared, or the systems being compared are already similar. Thus, the advantage of using a first screening are no longer necessary.

Regardless of the system in this paper, "subset m of k" method is determined as optimal, and can be recommended for simulations of similar size and use.

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Appendix

Table 15: Values of h_1 for the procedure of Section 2 [6]

P *	n_0	k=2	k = 3	k = 4	k = 5	k = 6	k=7	k = 8	k = 9	k = 10
0.9	20	1.896	2.342	2.583	2.747	2.87	2.969	3.051	3.121	3.182
0.9	40	1.852	2.283	2.514	2.669	2.785	2.878	2.954	3.019	3.076
0.95	20	2.453	2.872	3.101	3.258	3.377	3.472	3.551	3.619	3.679
0.95	40	2.386	2.786	3.003	3.15	3.26	3.349	3.422	3.484	3.539

Table 16: Values of h_2 for the procedure of Section 3 [6]

m	k = 3	k z 4	k = 5	k = 6	k = 7	k= 8	k = 9	k = 10
				P* = 0.90, n_0 = 20				
2	1.137	1.601	1.86	2.039	2.174	2.282	2.373	2.45
3		0.782	1.243	1.507	1.69	1.83	1.943	2.038
4			0.556	1.012	1.276	1.461	1.603	1.718
5				0.392	0.843	1.105	1.291	1.434
6					0.265	0.711	0.971	1.156
7						0.162	0.603	0.861
8							0.075	0.512
9								
				P* = 0.90, n_0 = 40				
2	1.114	1.57	1.825	1.999	2.131	2.237	2.324	2.399
3		0.763	1.219	1.479	1.66	1.798	1.909	2.002
4			0.541	0.991	1.251	1.434	1.575	1.688
5				0.381	0.824	1 .083	1.266	1.408
6					0.257	0.693	0.95	1.133
7						0.156	0.587	0.841
8							0.072	0.497
9								
				P* = 0.95, n_0 = 20				
2	1.631	2.071	2.321	2.494	2.625	2.731	2.819	2.894
3		1.256	1.697	1.952	2.131	2.267	2.378	2.47
4			1.021	1.458	1.714	1.894	2.033	2.146
5				0.852	1.284	1.539	1.72	1.86
6					0.721	1.149	1.402	1.583
7						0.615	1.038	1.29
8							0.526	0.945
9								0.449
				P* = 0.95, n_0 = 40				
2	1.591	2.023	2.267	2.435	2.563	2.665	2.75	2.823
3		1.222	1.656	1.907	2.082	2.217	2.325	2.415
4			0.99	1.42	1.672	1.85	1.987	2.098
5				0.824	1.248	1.499	1.678	1.816
6					0.695	1.114	1.363	1.541
7						0.591	1.004	1.252
8							0.505	0.913
9								0.43

TABLE 4. One-Sided Upper α Equicoordinate Point of the k-Variate Equicorrelated t-Distribution with Common Correlation Coefficient ρ and Degrees of Freedom ν ($T^{(\alpha)}_{k,\nu,\rho}$)

$\rho = 0.1$

ν	α \ k	2	3	4	5	6	7	8	9	10	12	14	16	18	20
2	.01	9.36	10.9	12.1	13.0	13.0	14.4	15.0	15.4	15.9	16.6	17.2	17.7	18.2	18.6
	.05	4.83	4.75	5.28	5.70	6.04	6.32	6.57	6.79	6.98	7.34	7.58	7.82	8.02	8.20
	.10	2.71	3.23	3.61	3.91	4.15	4.36	4.53	4.69	4.82	5.08	5.25	5.42	5.56	5.69
	.20	1.71	2.18	2.38	2.59	2.77	2.92	3.04	3.15	3.25	3.42	3.55	3.67	3.77	3.86
3	.01	5.68	6.40	6.93	7.35	7.69	7.98	8.23	8.46	8.65	9.00	9.28	9.53	9.75	9.94
	.05	3.07	3.51	3.83	4.08	4.29	4.46	4.61	4.74	4.86	5.06	5.23	5.37	5.50	5.61
	.10	2.24	2.61	2.87	3.08	3.24	3.38	3.50	3.61	3.70	3.87	4.00	4.12	4.22	4.30
	.20	1.52	1.84	2.06	2.23	2.36	2.48	2.57	2.66	2.74	2.86	2.97	3.06	3.14	3.21
4	.01	4.53	5.01	5.36	5.64	5.87	6.06	6.23	6.38	6.51	6.74	6.93	7.18	7.25	7.38
	.05	2.71	3.05	3.38	3.49	3.65	3.79	3.90	4.00	4.09	4.25	4.38	4.49	4.59	4.68
	.10	2.06	2.36	2.58	2.75	2.89	3.00	3.10	3.19	3.27	3.40	3.51	3.61	3.69	3.76
	.20	1.44	1.72	1.92	2.07	2.19	2.29	2.38	2.45	2.52	2.63	2.73	2.81	2.88	2.94
5	.01	3.99	4.36	4.64	4.85	5.03	5.18	5.31	5.43	5.53	5.71	5.86	5.99	6.10	6.20
	.05	2.52	2.82	3.03	3.20	3.33	3.45	3.54	3.63	3.71	3.84	3.95	4.05	4.13	4.21
	.10	1.96	2.23	2.43	2.58	2.70	2.80	2.89	2.97	3.03	3.15	3.25	3.34	3.41	3.47
	.20	1.40	1.66	1.84	1.98	2.09	2.19	2.27	2.34	2.40	2.50	2.59	2.67	2.73	2.79
6	.01	3.68	4.08	4.23	4.41	4.56	4.68	4.79	4.89	4.98	5.13	5.25	5.36	5.46	5.54
	.05	2.41	2.68	2.87	3.02	3.14	3.24	3.33	3.41	3.47	3.59	3.69	3.78	3.85	3.92
	.10	1.89	2.15	2.33	2.47	2.58	2.68	2.76	2.83	2.89	3.00	3.09	3.17	3.24	3.30
	.20	1.37	1.62	1.80	1.93	2.03	2.12	2.20	2.26	2.32	2.42	2.50	2.58	2.64	2.69
7	.01	3.48	3.76	3.97	4.13	4.26	4.37	4.46	4.55	4.62	4.76	4.87	4.96	5.05	5.12
	.05	2.33	2.58	2.76	2.90	3.01	3.10	3.19	3.26	3.32	3.43	3.52	3.60	3.67	3.73
	.10	1.85	2.10	2.27	2.40	2.50	2.59	2.67	2.74	2.80	2.90	2.98	3.06	3.12	3.18
	.20	1.35	1.59	1.76	1.89	1.99	2.08	2.15	2.21	2.27	2.37	2.44	2.51	2.57	2.62
8	.01	3.34	3.60	3.78	3.93	4.05	4.15	4.24	4.31	4.38	4.50	4.60	4.69	4.76	4.83
	.05	2.28	2.52	2.68	2.81	2.92	3.01	3.08	3.15	3.21	3.31	3.40	3.47	3.54	3.60
	.10	1.82	2.06	2.22	2.35	2.45	2.53	2.61	2.67	2.73	2.82	2.90	2.97	3.03	3.09
	.20	1.33	1.57	1.74	1.86	1.96	2.05	2.12	2.18	2.23	2.32	2.40	2.47	2.52	2.58
9	.01	3.23	3.48	3.65	3.79	3.90	3.99	4.07	4.14	4.20	4.31	4.41	4.49	4.56	4.62
	.05	2.24	2.47	2.63	2.75	2.85	2.93	3.01	3.07	3.13	3.23	3.31	3.38	3.44	3.49
	.10	1.80	2.03	2.19	2.31	2.40	2.49	2.56	2.62	2.67	2.77	2.84	2.91	2.97	3.02
	.20	1.32	1.56	1.72	1.84	1.94	2.02	2.09	2.15	2.20	2.29	2.37	2.43	2.49	2.54
10	.01	3.16	3.39	3.55	3.68	3.78	3.87	3.94	4.01	4.07	4.17	4.26	4.33	4.40	4.46
	.05	2.20	2.43	2.58	2.70	2.80	2.88	2.95	3.01	3.06	3.16	3.24	3.30	3.36	3.42
	.10	1.78	2.00	2.16	2.28	2.37	2.45	2.52	2.58	2.63	2.72	2.80	2.86	2.92	2.97
	.20	1.31	1.55	1.71	1.83	1.92	2.00	2.07	2.13	2.18	2.27	2.34	2.40	2.46	2.51

Figure 2: Caption

TABLE 4. Continued.

$\rho = 0.1$

ν	α \ k	2	3	4	5	6	7	8	9	10	12	14	16	18	20
12	.01	3.04	3.26	3.40	3.52	3.61	3.69	3.76	3.82	3.88	3.97	4.05	4.12	4.18	4.23
	.05	2.16	2.37	2.52	2.63	2.72	2.80	2.86	2.92	2.97	3.06	3.13	3.20	3.26	3.30
	.10	1.75	1.97	2.12	2.23	2.32	2.40	2.46	2.52	2.57	2.66	2.73	2.79	2.85	2.89
	.20	1.30	1.53	1.69	1.80	1.89	1.97	2.04	2.09	2.15	2.23	2.30	2.36	2.42	2.46
16	.01	2.91	3.10	3.24	3.34	3.42	3.49	3.55	3.61	3.65	3.74	3.80	3.86	3.92	3.96
	.05	2.10	2.30	2.44	2.54	2.63	2.70	2.76	2.81	2.86	2.94	3.01	3.07	3.12	3.16
	.10	1.72	1.93	2.07	2.18	2.26	2.33	2.40	2.45	2.50	2.58	2.65	2.71	2.76	2.80
	.20	1.28	1.51	1.66	1.77	1.86	1.94	2.00	2.05	2.10	2.19	2.26	2.31	2.36	2.41
20	.01	2.84	3.02	3.14	3.24	3.31	3.38	3.44	3.48	3.53	3.60	3.67	3.72	3.77	3.81
	.05	2.07	2.26	2.39	2.49	2.58	2.64	2.70	2.75	2.80	2.87	2.94	2.99	3.04	3.08
	.10	1.70	1.90	2.04	2.14	2.23	2.30	2.36	2.41	2.45	2.53	2.60	2.65	2.70	2.75
	.20	1.28	1.50	1.64	1.75	1.84	1.92	1.98	2.03	2.08	2.16	2.23	2.28	2.33	2.38
24	.01	2.79	2.96	3.08	3.17	3.25	3.31	3.36	3.41	3.45	3.52	3.58	3.63	3.68	3.72
	.05	2.05	2.24	2.37	2.46	2.54	2.61	2.66	2.71	2.75	2.83	2.89	2.94	2.99	3.03
	.10	1.69	1.89	2.02	2.12	2.21	2.27	2.33	2.38	2.43	2.50	2.57	2.62	2.67	2.71
	.20	1.27	1.49	1.63	1.74	1.83	1.90	1.96	2.02	2.06	2.14	2.21	2.26	2.31	2.36
30	.01	2.75	2.91	3.02	3.11	3.18	3.24	3.29	3.33	3.37	3.44	3.50	3.54	3.59	3.63
	.05	2.03	2.21	2.34	2.43	2.51	2.57	2.62	2.67	2.71	2.79	2.85	2.90	2.94	2.98
	.10	1.67	1.87	2.00	2.10	2.18	2.25	2.31	2.36	2.40	2.47	2.54	2.59	2.64	2.68
	.20	1.26	1.48	1.62	1.73	1.82	1.89	1.95	2.00	2.05	2.13	2.19	2.25	2.29	2.34
40	.01	2.70	2.86	2.96	3.05	3.11	3.17	3.22	3.26	3.30	3.36	3.41	3.46	3.50	3.54
	.05	2.01	2.19	2.31	2.40	2.47	2.54	2.59	2.63	2.67	2.74	2.80	2.85	2.89	2.93
	.10	1.66	1.85	1.98	2.08	2.16	2.23	2.28	2.33	2.37	2.44	2.51	2.56	2.60	2.64
	.20	1.26	1.47	1.61	1.72	1.81	1.88	1.93	1.99	2.03	2.11	2.17	2.23	2.27	2.31
60	.01	2.66	2.81	2.91	2.99	3.05	3.10	3.15	3.19	3.22	3.28	3.33	3.38	3.42	3.45
	.05	1.99	2.16	2.28	2.37	2.44	2.50	2.55	2.60	2.63	2.70	2.76	2.80	2.84	2.88
	.10	1.65	1.84	1.97	2.06	2.14	2.20	2.26	2.30	2.35	2.42	2.47	2.52	2.57	2.61
	.20	1.25	1.46	1.60	1.71	1.79	1.86	1.92	1.97	2.02	2.09	2.15	2.21	2.25	2.29
120	.01	2.62	2.76	2.86	2.93	2.99	3.04	3.08	3.12	3.15	3.21	3.26	3.30	3.34	3.37
	.05	1.97	2.14	2.25	2.34	2.41	2.47	2.52	2.56	2.60	2.66	2.71	2.76	2.80	2.84
	.10	1.64	1.82	1.95	2.04	2.12	2.18	2.23	2.28	2.32	2.39	2.44	2.49	2.54	2.57
	.20	1.25	1.45	1.59	1.70	1.78	1.85	1.91	1.96	2.00	2.07	2.14	2.19	2.23	2.27
∞	.01	2.57	2.71	2.80	2.87	2.93	2.98	3.02	3.06	3.09	3.14	3.18	3.22	3.26	3.29
	.05	1.95	2.12	2.23	2.31	2.38	2.43	2.48	2.52	2.56	2.62	2.67	2.72	2.75	2.79
	.10	1.63	1.81	1.93	2.02	2.10	2.16	2.21	2.25	2.29	2.36	2.41	2.46	2.50	2.54
	.20	1.24	1.45	1.59	1.69	1.77	1.84	1.89	1.94	1.98	2.06	2.12	2.17	2.21	2.25

Figure 3: Caption

ρ = 0.3

ν	α \ k	2	3	4	5	6	7	8	9	10	12	14	16	18	20
2	.01	9.15	10.5	11.5	12.3	13.0	13.5	14.0	14.4	14.7	15.4	15.9	16.3	16.7	17.1
	.05	3.93	4.56	5.02	5.37	5.66	5.91	6.12	6.30	6.46	6.74	6.97	7.17	7.35	7.50
	.10	2.63	3.09	3.46	3.68	3.88	4.06	4.21	4.34	4.45	4.65	4.81	4.95	5.08	5.19
	.20	1.64	1.99	2.23	2.41	2.56	2.69	2.80	2.89	2.97	3.11	3.23	3.33	3.42	3.49
3	.01	5.68	6.24	6.71	7.08	7.38	7.63	7.84	8.04	8.21	8.50	8.74	8.95	9.14	9.30
	.05	3.81	4.41	4.89	5.31	5.69	6.04	6.33	6.58	6.80	7.17	7.49	7.75	7.98	8.19
	.10	2.62	3.02	3.35	3.63	3.88	4.09	4.27	4.43	4.58	4.83	5.04	5.22	5.38	5.52
	.20	1.67	1.95	2.17	2.36	2.52	2.66	2.78	2.89	2.99	3.15	3.29	3.41	3.51	3.60
4	.01	4.48	4.92	5.24	5.48	5.69	5.86	6.00	6.13	6.25	6.45	6.61	6.75	6.88	6.99
	.05	3.27	3.68	3.98	4.23	4.44	4.63	4.78	4.92	5.04	5.25	5.42	5.57	5.69	5.80
	.10	2.47	2.80	3.07	3.30	3.49	3.66	3.80	3.93	4.04	4.23	4.39	4.53	4.65	4.75
	.20	1.74	2.00	2.21	2.38	2.53	2.66	2.77	2.87	2.96	3.11	3.24	3.35	3.44	3.52
5	.01	3.95	4.38	4.73	5.01	5.24	5.43	5.60	5.74	5.87	6.08	6.26	6.41	6.54	6.65
	.05	3.01	3.40	3.72	3.98	4.20	4.39	4.55	4.69	4.81	5.02	5.20	5.35	5.48	5.59
	.10	2.36	2.68	2.93	3.14	3.32	3.48	3.62	3.74	3.85	4.03	4.19	4.33	4.45	4.55
	.20	1.72	1.95	2.14	2.30	2.44	2.56	2.67	2.76	2.85	2.99	3.12	3.23	3.33	3.41
6	.01	3.65	3.95	4.19	4.39	4.56	4.70	4.83	4.94	5.04	5.22	5.38	5.52	5.65	5.76
	.05	2.83	3.14	3.40	3.62	3.81	3.97	4.12	4.25	4.36	4.56	4.73	4.88	5.01	5.12
	.10	2.24	2.50	2.72	2.90	3.06	3.20	3.33	3.44	3.54	3.72	3.88	4.02	4.14	4.24
	.20	1.70	1.91	2.08	2.23	2.36	2.47	2.57	2.66	2.74	2.89	3.02	3.13	3.23	3.31
7	.01	3.45	3.72	3.94	4.12	4.28	4.42	4.54	4.65	4.75	4.93	5.08	5.22	5.35	5.46
	.05	2.78	3.08	3.33	3.54	3.73	3.89	4.03	4.16	4.28	4.48	4.65	4.80	4.94	5.06
	.10	2.20	2.44	2.65	2.83	2.98	3.12	3.25	3.36	3.46	3.64	3.80	3.94	4.06	4.17
	.20	1.68	1.87	2.03	2.17	2.30	2.41	2.51	2.60	2.68	2.83	2.96	3.07	3.17	3.26
8	.01	3.32	3.56	3.76	3.93	4.08	4.21	4.33	4.43	4.53	4.70	4.85	4.98	5.10	5.21
	.05	2.75	3.04	3.28	3.48	3.65	3.80	3.93	4.05	4.15	4.34	4.50	4.64	4.77	4.88
	.10	2.17	2.40	2.59	2.75	2.89	3.01	3.12	3.22	3.31	3.47	3.61	3.74	3.85	3.95
	.20	1.67	1.85	2.00	2.13	2.24	2.34	2.43	2.51	2.59	2.73	2.85	2.96	3.05	3.13
9	.01	3.22	3.45	3.64	3.80	3.94	4.06	4.17	4.27	4.35	4.51	4.65	4.78	4.89	4.99
	.05	2.72	3.01	3.24	3.43	3.59	3.73	3.85	3.96	4.06	4.24	4.39	4.52	4.64	4.74
	.10	2.15	2.38	2.56	2.71	2.85	2.97	3.07	3.17	3.26	3.42	3.56	3.68	3.78	3.87
	.20	1.66	1.84	1.98	2.11	2.21	2.31	2.40	2.48	2.56	2.69	2.81	2.91	3.00	3.08
10	.01	3.14	3.36	3.55	3.71	3.85	3.97	4.08	4.18	4.27	4.42	4.56	4.68	4.79	4.89
	.05	2.70	2.98	3.22	3.41	3.58	3.71	3.83	3.93	4.03	4.21	4.35	4.48	4.59	4.69
	.10	2.13	2.36	2.54	2.70	2.83	2.95	3.05	3.15	3.23	3.38	3.52	3.64	3.74	3.83
	.20	1.65	1.83	1.97	2.09	2.19	2.28	2.37	2.44	2.52	2.65	2.77	2.87	2.96	3.04
12	.01	3.03	3.23	3.39	3.53	3.66	3.77	3.87	3.96	4.04	4.19	4.32	4.44	4.54	4.64
	.05	2.68	2.94	3.15	3.32	3.47	3.59	3.70	3.80	3.88	4.04	4.17	4.28	4.38	4.47
	.10	2.10	2.32	2.50	2.65	2.78	2.89	2.99	3.08	3.16	3.30	3.42	3.53	3.62	3.70
	.20	1.64	1.81	1.95	2.07	2.17	2.26	2.34	2.41	2.48	2.60	2.71	2.80	2.88	2.95

Figure 4: Caption

TABLE 4. Continued.

ρ = 0.3

ν	α \ k	2	3	4	5	6	7	8	9	10	12	14	16	18	20
16	.01	2.90	3.09	3.21	3.31	3.39	3.46	3.51	3.56	3.61	3.68	3.75	3.80	3.85	3.89
	.05	2.69	2.87	3.00	3.12	3.22	3.30	3.37	3.44	3.50	3.61	3.71	3.79	3.86	3.92
	.10	2.19	2.37	2.50	2.61	2.70	2.78	2.85	2.91	2.96	3.06	3.14	3.21	3.27	3.32
	.20	1.69	1.84	1.96	2.06	2.14	2.21	2.27	2.32	2.37	2.46	2.53	2.60	2.65	2.69
20	.01	2.83	3.00	3.12	3.21	3.29	3.35	3.40	3.45	3.49	3.56	3.62	3.67	3.72	3.76
	.05	2.65	2.84	2.96	3.07	3.15	3.23	3.29	3.34	3.39	3.48	3.56	3.62	3.68	3.73
	.10	2.15	2.31	2.44	2.54	2.63	2.70	2.76	2.82	2.87	2.96	3.03	3.09	3.15	3.20
	.20	1.68	1.81	1.92	2.01	2.09	2.16	2.22	2.27	2.32	2.40	2.47	2.53	2.58	2.62
24	.01	2.78	2.95	3.08	3.19	3.28	3.36	3.42	3.48	3.53	3.62	3.70	3.76	3.82	3.87
	.05	2.64	2.82	2.95	3.06	3.15	3.23	3.30	3.36	3.41	3.51	3.59	3.66	3.72	3.77
	.10	2.13	2.29	2.42	2.52	2.61	2.68	2.74	2.80	2.85	2.94	3.01	3.07	3.13	3.17
	.20	1.67	1.80	1.90	1.99	2.06	2.13	2.18	2.23	2.28	2.36	2.43	2.49	2.54	2.58
30	.01	2.75	2.92	3.04	3.15	3.24	3.31	3.38	3.43	3.48	3.56	3.63	3.69	3.74	3.78
	.05	2.62	2.79	2.92	3.02	3.11	3.19	3.25	3.31	3.36	3.45	3.52	3.58	3.63	3.68
	.10	2.11	2.27	2.39	2.49	2.57	2.64	2.70	2.75	2.80	2.88	2.95	3.01	3.06	3.10
	.20	1.66	1.78	1.88	1.96	2.03	2.09	2.15	2.19	2.24	2.32	2.38	2.43	2.48	2.52
40	.01	2.73	2.89	3.01	3.11	3.19	3.26	3.32	3.37	3.42	3.50	3.57	3.63	3.68	3.72
	.05	2.60	2.76	2.89	2.99	3.07	3.14	3.20	3.25	3.30	3.38	3.45	3.51	3.56	3.60
	.10	2.08	2.24	2.36	2.45	2.53	2.60	2.66	2.71	2.76	2.84	2.91	2.96	3.01	3.05
	.20	1.65	1.76	1.86	1.94	2.01	2.07	2.12	2.17	2.21	2.29	2.35	2.40	2.45	2.49
60	.01	2.71	2.86	2.98	3.08	3.16	3.23	3.29	3.34	3.39	3.47	3.53	3.59	3.64	3.68
	.05	2.58	2.73	2.86	2.96	3.04	3.11	3.17	3.22	3.27	3.35	3.42	3.47	3.52	3.56
	.10	2.06	2.22	2.33	2.42	2.50	2.57	2.63	2.68	2.73	2.81	2.87	2.92	2.97	3.01
	.20	1.64	1.74	1.83	1.91	1.98	2.04	2.09	2.14	2.18	2.25	2.32	2.37	2.41	2.45
120	.01	2.61	2.75	2.85	2.92	2.98	3.03	3.07	3.10	3.14	3.19	3.24	3.28	3.31	3.34
	.05	1.96	2.12	2.23	2.31	2.38	2.43	2.48	2.52	2.56	2.62	2.67	2.71	2.75	2.78
	.10	1.62	1.79	1.91	2.00	2.07	2.13	2.18	2.22	2.26	2.32	2.38	2.42	2.46	2.50
	.20	1.22	1.41	1.54	1.63	1.71	1.77	1.83	1.87	1.91	1.98	2.04	2.08	2.13	2.16
∞	.01	2.57	2.70	2.80	2.86	2.92	2.97	3.01	3.04	3.07	3.12	3.17	3.20	3.24	3.27
	.05	1.94	2.10	2.21	2.28	2.35	2.40	2.45	2.49	2.52	2.58	2.63	2.67	2.71	2.74
	.10	1.61	1.78	1.90	1.98	2.05	2.11	2.16	2.20	2.24	2.30	2.35	2.40	2.43	2.47
	.20	1.21	1.40	1.53	1.63	1.70	1.76	1.81	1.86	1.90	1.97	2.02	2.07	2.11	2.15

Figure 5: Caption