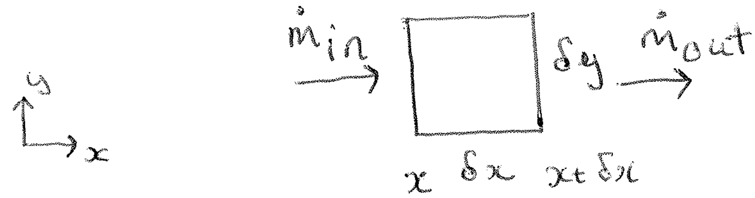
Derivation of general continuity equation



$\delta \text{Area} = \delta x \, \delta y$

$$\frac{dm}{dt} = \dot{m}_{in} - \dot{m}_{out} \quad \leftarrow \text{Change in mass.}$$

Change in concentration: $\quad c = \text{mass} / \text{volume} \quad (\text{area}).$

$$\delta c = \delta m / \delta A$$

$$\therefore \quad \frac{\delta c \, \delta A}{\delta t} = \frac{\delta c \, \delta x \, \delta y}{\delta t} = \dot{m}_{in} - \dot{m}_{out}$$

j = flux of mass = mass per unit length per unit time.

$$\frac{\delta c \, \delta x \, \delta y}{\delta t} = \underbrace{j(x) \cdot \delta y}_{\text{length of surface}} - j(x+\delta x) \cdot \delta y$$

$$= -\left(j(x+\delta x) - j(x)\right) \delta y.$$

Divide by $\delta x \, \delta y$ $\quad$ $\boxed{\delta y \text{ cancels.}}$

$$\frac{\delta c}{\delta t} = -\frac{\left(j(x+\delta x) - j(x)\right)}{\delta x}$$

As $\delta c, \delta x, \delta y, \delta t \rightarrow 0$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = -\frac{\partial j}{\partial x}$$

Flux operators

$$\frac{\partial c}{\partial t} + \nabla \cdot j = 0$$

general form where

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

j = total flux.

(tutorial exercise).

Diffusive flux.

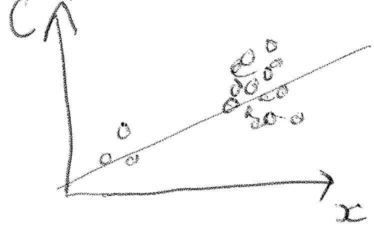
Fick's first law → molecular diffusion.

→ quantity moves in proportion to local gradient

→ ↑ the gradient, ↑ the flux.

$$j_{diff} = -D \nabla c$$

↑ negative because flux is in opposite direction to the gradient.



∴ +ve gradient
−ve flux.

Advective flux :- flux due to flow of fluid.

$$j_{advective} = \underline{v}\, c$$

$$= \frac{L}{T} \cdot \frac{M}{L^3} = \frac{M}{L^2 T} = \text{mass}/(\text{area} \times \text{time})$$

Put together

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(j_{diff} + j_{advec} \right) = 0$$

$$\frac{\partial c}{\partial t} + \nabla \cdot \left(-D\nabla c + \underline{v} c \right) = 0$$

Expanding

$$\frac{\partial c}{\partial t} + \nabla \cdot (\underline{v} c) = \nabla \cdot (D \nabla c)$$

Common simplifications

Fluid is incompressible i.e $\nabla \cdot \underline{v} = 0$

Diffusion coefficient is a constant scalar $\nabla D = 0$

$$\Rightarrow \frac{\partial c}{\partial t} + \underline{v} . \nabla c + \underbrace{c \nabla \cdot \underline{v}}_{= 0} = D \nabla^2 c + \underbrace{\nabla c . \nabla D}_{= 0}$$

$$\Rightarrow \frac{\partial c}{\partial t} + \underline{v} . \nabla c = D \nabla^2 c$$

Source / Sinks

Mass flux in / out not only way to increase mass.

Source / Sinks also do this.
+ve / −ve

$$\frac{\partial c}{\partial t} + \nabla \cdot j = S$$

Mass source

Straight forward $f = \left(\frac{\text{mass}}{\text{volume/time}}\right) = \underline{f(\underline{x}, t)}$.

e.g. modelling pollution from a chimney in environmental fluid mechanics.

Reaction

Source proportional to value of c.

$S = \lambda c$

$\lambda > 0$ = source → produces

$\lambda < 0$ = sink → destroys.

Used often in chemical reaction models.

Final Equation

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \nabla c = D \nabla^2 c + \lambda c + f$$