ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 12

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LECTURE 12
Solving Transient Problems - Theory

- Understand the general reasoning behind numerical time stepping schemes
- Ability to apply these to the FEM formulation
- Able to relate this to the static problem FEM formulation

DIFFUSION-REACTION EQUATION
Bringing Back The Transient Form

Everything up to Lecture 8 allowed us to solve the static diffusion-reaction equation:

$$D\nabla^2 c + \lambda c + f = 0$$

Now we need to solve the transient version of this equation, which was derived in Lecture 3:

$$\frac{\partial c}{\partial t} = D\nabla^2 c + \lambda c + f$$

GENERAL TRANSIENT EQUATIONS
A General Approach To Solving Them

Consider a general transient equation:

$$\frac{\partial c}{\partial t} = F\left(x, c, t\right)$$

Integrate both sides wrt to time, between to time points

$$\int_{t_n}^{t_{n+1}} \frac{\partial c}{\partial t} dt = \int_{t_n}^{t_{n+1}} F\left(x, c, t\right) dt$$

Left hand side is trivial to evaluate:

$$c\left(t_{n+1}\right) - c\left(t_n\right) = \int_{t_n}^{t_{n+1}} F\left(x, c, t\right) dt$$

GENERAL TRANSIENT EQUATIONS
Using The Trapezium Rule

Integrate RHS using the trapezium rule:

$$\int_{t_n}^{t_{n+1}} F\left(x, c, t\right) dt = \frac{1}{2}\underbrace{\left(t_{n+1} - t_n\right)}_{\text{width}}\underbrace{\left(F\left(x, c, t_{n+1}\right) + F\left(x, c, t_n\right)\right)}_{\text{heights}}$$

Different between t_n and t_{n+1} is Δt

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \frac{1}{2}\left(F^{n+1} + F^{n}\right), \qquad \boxed{\theta = 1/2}$$

The trapezium rule creates what is known as the Crank-Nicolson scheme

SOLVING TRANSIENT EQUATIONS
Three Different Methods

Crank-Nicolson:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \frac{1}{2}\left(F^{n+1} + F^n\right), \qquad \boxed{\theta = 1/2}$$

Forward Euler:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = F^n(x, c, t), \qquad \boxed{\theta = 0}$$

Backward Euler:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = F^{n+1}(x, c, t), \qquad \boxed{\theta = 1}$$

All these three schemes can be generalised in the theta scheme formulation:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \theta F^{n+1}(x, c, t) + (1 - \theta) F^n(x, c, t)$$

TRANSIENT FEM FORMULATION
A Generalised Method

The weighted residual form, after integration by parts, for the domain x=[0,1] is now:

$$\int_0^1 \left(v\frac{\partial c}{\partial t} + D\frac{\partial v}{\partial x}\frac{\partial c}{\partial x} - \lambda cv \right) dx = \int_0^1 vf\,dx + \left[vD\frac{\partial c}{\partial x} \right]_0^1$$

The new term in the equation is:

$$v\frac{\partial c}{\partial t}$$

which, as before, can be represented by:

$$c = c_n\psi_n, v = \psi_m$$

LOCAL ELEMENT MASS MATRIX
Element Integral For Time Derivative

The local element integral is thus:

$$\int_{-1}^{1} \frac{d(c_n\psi_n)}{dt}.\psi_m J d\xi$$

Take the derivative outside of the integral:

$$\frac{dc_n}{dt}\int_{-1}^{1} \psi_n\psi_m J d\xi$$

But know that time derivative can be written:

$$\frac{dc_n}{dt} = \frac{c_n^{n+1} - c_n^n}{\Delta t}$$

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local element
mass matrix

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GLOBAL ELEMENT MATRICES
The Mass And Stiffness Matrices

Local element mass matrix:

$$M_{element} = \int_{-1}^{1} \psi_n \psi_m J d\xi$$

Local element stiffness matrix:

$$K_{element} = \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_m}{d\xi} \frac{d\xi}{dx} J d\xi - \int_{-1}^{1} \lambda \psi_n \psi_m J d\xi$$

Looping over all element, assemble these into global mass, M, matrix, and global stiffness, K, matrix

GLOBAL ELEMENT MATRICES
The Transient Fem Equation

Write the transient FEM equation in terms of these global matrices, using general theta scheme:

$$M\left(\frac{\mathbf{c}^{n+1}-\mathbf{c}^{n}}{\Delta t}\right)+K[\theta\mathbf{c}^{n+1}+(1-\theta)\mathbf{c}^{n}]=RHS$$

Global solution vector at time step n+1

Global solution vector at time step n

and:

$$RHS=\theta\mathbf{F}^{n+1}+(1-\theta)\mathbf{F}^{n}+\theta\mathbf{NBc}^{n+1}+(1-\theta)\mathbf{NBc}^{n}$$

Rearrange this to have unknowns on LHS:

$$[M + \theta \Delta t K]\mathbf{c}^{n+1} = [M - (1-\theta)\Delta t K]\mathbf{c}^{n} + \Delta t \theta [\mathbf{F}^{n+1} + \mathbf{NBc}^{n+1}] + \Delta t (1-\theta)[\mathbf{F}^{n} + \mathbf{NBc}^{n}]$$

Final global matrix to invert to solve the system

Solution at previous time step

Source terms & Neumann BCs

The Crank-Nicolson scheme is thus:

$$[M + \frac{1}{2}\Delta t K]\mathbf{c}^{n+1} = [M - \frac{1}{2}\Delta t K]\mathbf{c}^{n} + \frac{1}{2}\Delta t [\mathbf{F}^{n+1} + \mathbf{F}^{n} + \mathbf{NBc}^{n+1} + \mathbf{NBc}^{n}]$$