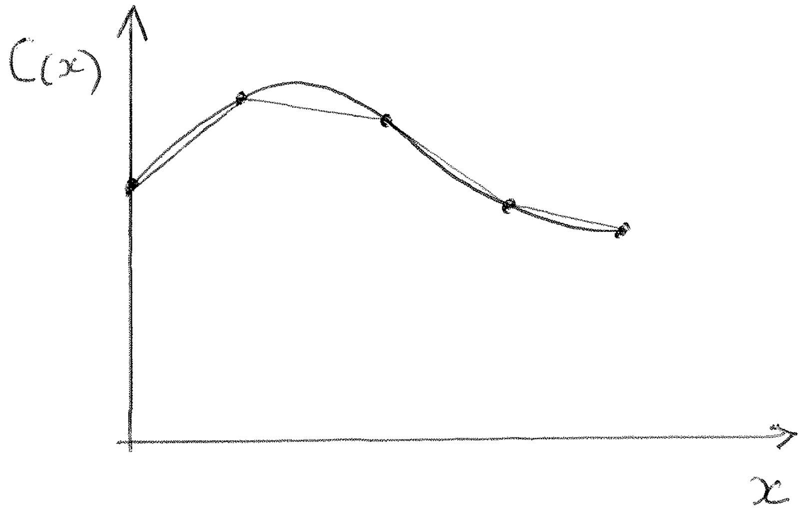
We know that for all but the simplest problems, with the most mathematically simple solutions, using a numerical solution technique, such as FEM, creates an approximation error.

Graphically, can think of it like this:



Red straight lines are the solution defined by the linear basis Functions of FEM.

How should we evaluate this error?

A simplistic way would compare the difference between the nodal FEM solution and the exact solution evaluated at those same nodal positions.

However, for certain problems these two values can be the same (or almost), implying that the error is zero.

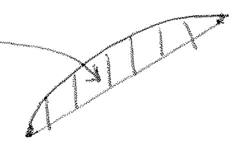
But this is incorrect.

Why?

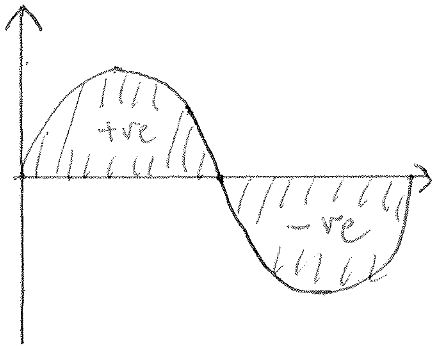
Consider the solution between just two nodes:

Throughout the element there is an error between the linear approximation and the exact solution.

One way to quantify this is to calculate this total area i.e



However, these areas could cancel each other out, as we add them from each element, just as it would when finding the total area under a sine curve:



$= 0.$

Therefore, take the area squared. This concept is used in electrical power and is known as the Root Mean Square.

In the finite element world, this type of error is called the L_2 norm.

Mathematically speaking a norm is a measure of length or size that is always positive (or zero).

If the exact solution to our problem is: $C_E(x)$ and our numerical solution is: $C(x)$, then the difference or error is:

$$\underline{\underline{E(x) = C_E(x) - C(x)}}$$

Now apply the RMS concept or L_2 norm to this error:

$$\|E\|_{L_2} = \left[\int_{\Omega} E^2(x)\, dx \right]^{1/2}$$

where Ω represents our solution domain that is defined by the finite element mesh.

The notation $\|\cdot\|_{L_2}$ indicates an L_2 norm of some quantity.

Therefore for the mesh we have used throughout this course, $x = [0, 1]$, this is:

$$\|E\|_{L_2} = \left[\int_0^1 E^2(x)\, dx \right]^{1/2}$$

As with our original weighted residual equation, this integral is split into integrals for each element of the mesh. For the 3-element mesh, that is:

$$\|E\|_{L_2} = \left[\int_0^{1/3} E^2(x)dx + \int_{1/3}^{2/3} E^2(x)\,dx + \int_{2/3}^{1} E^2(x)dx \right]^{1/2}$$

Each of these element integrals is evaluated within the standard element defined by ξ, e.g:

<u>Element 1</u>

$$\int_0^{1/3} E^2(x)\,dx = \int_{-1}^{1} E^2(x)\, J\, d\xi$$

$$= \int_{-1}^{1} \left(C_E(x) - C(x)\right)^2 J\, d\xi$$

This can be evaluated using Gaussian quadrature:

$$\int_{-1}^{1} \left(C_E(x) - C(x)\right)^2 J\, d\xi = \sum_{i=1}^{N} w_i \left(C_E(x(\xi_i)) - C(\xi_i)\right)^2 J$$

The total error is therefore the sum of these integrals in each element.

IF the exact solution $C_E(x)$ is known analytically, the L_2 norm can be numerically evaluated.

Using G.Q need to choose order of scheme N, based on the order of $C_E(x)^2$.

To evaluate $C_E(x)$ at the Gauss points must find x as a function of ξ, which in an element is:

$$x(\xi) = x_0\left(\frac{1-\xi}{2}\right) + x_1\left(\frac{1+\xi}{2}\right)$$

Therefore: $$C_E(x(\xi_i)) = C_E\left(x_0\left(\frac{1-\xi_i}{2}\right) + x_1\left(\frac{1+\xi_i}{2}\right)\right)$$

where x_0, x_1 are the positions of the local nodes in a given element.

To evaluate $C(\xi_i)$ is simply:

$$C(\xi_i) = c_0\left(\frac{1-\xi_i}{2}\right) + c_1\left(\frac{1+\xi_i}{2}\right)$$

where c_0, c_1 are the local solution values in a given element.

A simple example:

$$\frac{\partial^2 c}{\partial x^2} = 0 \qquad \text{subject to} \quad c(0) = 0, \quad c(1) = 1$$

Solution is: $c = x$

$\therefore$ To evaluate the L_2 norm should use G.Q with $N=2$.

For the 1st element in the 3-element mesh,
$x_0 = 0$, $x_1 = 1/3$

$\therefore$ The integral of $E^2(x)$ in this element is:

$$1.\left(\left(x_0\left(\frac{1-\xi_1}{2}\right) + x_1\left(\frac{1+\xi_1}{2}\right)\right) - \left(c_0\left(\frac{1-\xi_1}{2}\right) + c_1\left(\frac{1+\xi_1}{2}\right)\right)\right)^2$$

$$+ 1.\left(\left(x_0\left(\frac{1-\xi_2}{2}\right) + x_1\left(\frac{1+\xi_2}{2}\right)\right) - \left(c_0\left(\frac{1-\xi_2}{2}\right) + c_1\left(\frac{1+\xi_2}{2}\right)\right)\right)^2$$

For this problem, FEM determines the solution exactly, and $c_0 = 0$ and $c_1 = 1/3$ in the first element.

For $N=2$, $\xi_1 = -\sqrt{1/3}$ and $\xi_2 = +\sqrt{1/3}$

The integral therefore becomes:

$$1.\left(0.\left(\frac{1+\sqrt{1/3}}{2}\right) + \frac{1}{3}.\left(\frac{1-\sqrt{1/3}}{2}\right) - 0.\left(\frac{1+\sqrt{1/3}}{2}\right) - \frac{1}{3}\left(\frac{1-\sqrt{1/3}}{2}\right)\right)^2$$

$$+ 1.\left(0.\left(\frac{1-\sqrt{1/3}}{2}\right) + \frac{1}{3}.\left(\frac{1+\sqrt{1/3}}{2}\right) - 0.\left(\frac{1-\sqrt{1/3}}{2}\right) - \frac{1}{3}\left(\frac{1+\sqrt{1/3}}{2}\right)\right)^2$$

$$= 1 \cdot (0)^2 + 1 \cdot (0)^2 = \underline{\underline{0}}$$

$\therefore$ As expected the L_2 norm for the case when our FEM solution is exact, is zero.

Therefore to test convergence of your numerical solution against an analytical solution, run code for increasing mesh resolution (i.e. decreasing element size) and compute L_2 norm in each case.

If we say that for our linear mesh representation the exact solution $C_E(x)$ is:

$$C_E(x) = \underbrace{C(x)}_{\text{Linear}} + O(h^2) \leftarrow \text{terms of } h^2 \text{ and higher}$$

Take ln of each side where $E(x) = C_E(x) - C(x)$

$$\ln(E(x)) = \ln(O(h^2))$$

Write $O(h^2)$ as Gh^2 where G is a constant.

$$\therefore \ln(E(x)) = \ln G + 2 \ln h.$$

Plotting $\ln(E(x))$ against $\ln(h)$ is therefore a straightline with gradient 2.

This gradient of 2 is what we are hoping to observe when running the convergence test against the analytical solution.

If it is substantially different, could be a problem with our method or code!