ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 13

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University of Bath, 2019-20

LECTURE 13
Solving Transient Problems - Practice

- Understand how to implement transient FEM solver
- Know expected behaviour for different time integration schemes
- Able to select appropriate time integration parameters

THE TRANSIENT PROBLEM
Theta Scheme Fem Formulation

Transient diffusion/heat equation in 1D

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

subject to following conditions:

$$x = [0, 1], \quad c(x, 0) = 0, \quad c(0, t) = 0, \quad c(1, t) = 1$$

Domain Initial condition Dirichlet Boundary conditions

General theta scheme FEM formulation:

$$[M + \theta \Delta t K]\mathbf{c}^{n+1} = [M - (1-\theta)\Delta t K]\mathbf{c}^{n} + \Delta t \theta [\mathbf{F}^{n+1} + \mathbf{NBc}^{n+1}] + \Delta t (1-\theta)[\mathbf{F}^{n} + \mathbf{NBc}^{n}]$$

Global matrix to invert Previous time step information Source terms & Neumann BCs

IMPLEMENTING TRANSIENT FORM
The Pseudo-Code - Initialisation

1. Initialise mesh
2. Initialise time integration scheme: theta value, time step - dt, number of time steps, N
3. Define material coefficients: diffusion coefficient, lambda, source term f
4. Initialise Global Matrix (GM), Global Mass Matrix (M), Global Stiffness Matrix (K), and Global Vector (GV) to zero
5. Define two solution variable vectors: **Ccurrent,** c^n and **Cnext,** c^{n+1}
6. Set initial conditions on **Ccurrent**

IMPLEMENTING TRANSIENT FORM
The Pseudo-Code - Time Integration Loop

7. Loop over time from t=1 to T (where T=Ndt)

   1. Loop over elements:

      - calculate local element mass matrices and add to correct location in global mass matrix (M)

      - calculate local element stiffness matrices and add to correct location in global stiffness matrix (K)

   2. Calculate global matrix (GM) according to:

$$[M + \theta \Delta t K]$$

   3. Calculate matrix to multiply previous solution:

$$[M - (1-\theta)\Delta t K]$$

IMPLEMENTING TRANSIENT FORM
The Pseudo-Code - Time Integration Loop

4. Multiply this matrix by previous solution and store in Global Vector:

$$[M - (1-\theta)\Delta t K]\mathbf{c}^n$$

5. Loop over elements:

   - calculate local element sources vectors, multiply by time step, dt, and add to Global Vector in correct position

$$\Delta t[\theta \mathbf{F}^{n+1} + (1-\theta)\mathbf{F}^n]$$

   - if Neumann BCs have been specified, compute and add to the Global Vector

$$\Delta t[\theta \mathbf{NBC}^{n+1} + (1-\theta)\mathbf{NBC}^n]$$

IMPLEMENTING TRANSIENT FORM
The Pseudo-Code - Time Integration Loop

6. Set any Dirichlet Boundary Conditions in the usual way
7. Solve the final matrix system to obtain **Cnext**
8. Set **Ccurrent** equal to **Cnext**
9. Re-initialise global matrix, global mass matrix, global stiffness matrix, and global vector to zero
10. Plot/write to file the solution **Ccurrent**

IMPLEMENTING TRANSIENT FORM
A Note On Plotting/Storing The Solution

Often the time step needed for accuracy/stability in numerical solution smaller than needed for evaluating results

Can lead to excess data files & slower computation time due to time to write solution to file

Plot solution at a lower frequency using following commands:

```
%Plot the solution every 5 time steps
  if(mod(t,5)==0)
    x = msh.nvec;
    figure;
    plot(x,u_next);
  end
```

Modulo command

TRANSIENT HEAT EQUATION
An Analytical Solution

Transient diffusion/heat equation in 1D

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Has the following analytical solution:

$$c(x, t) = x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-n^2 \pi^2 t} \sin(n\pi x)$$

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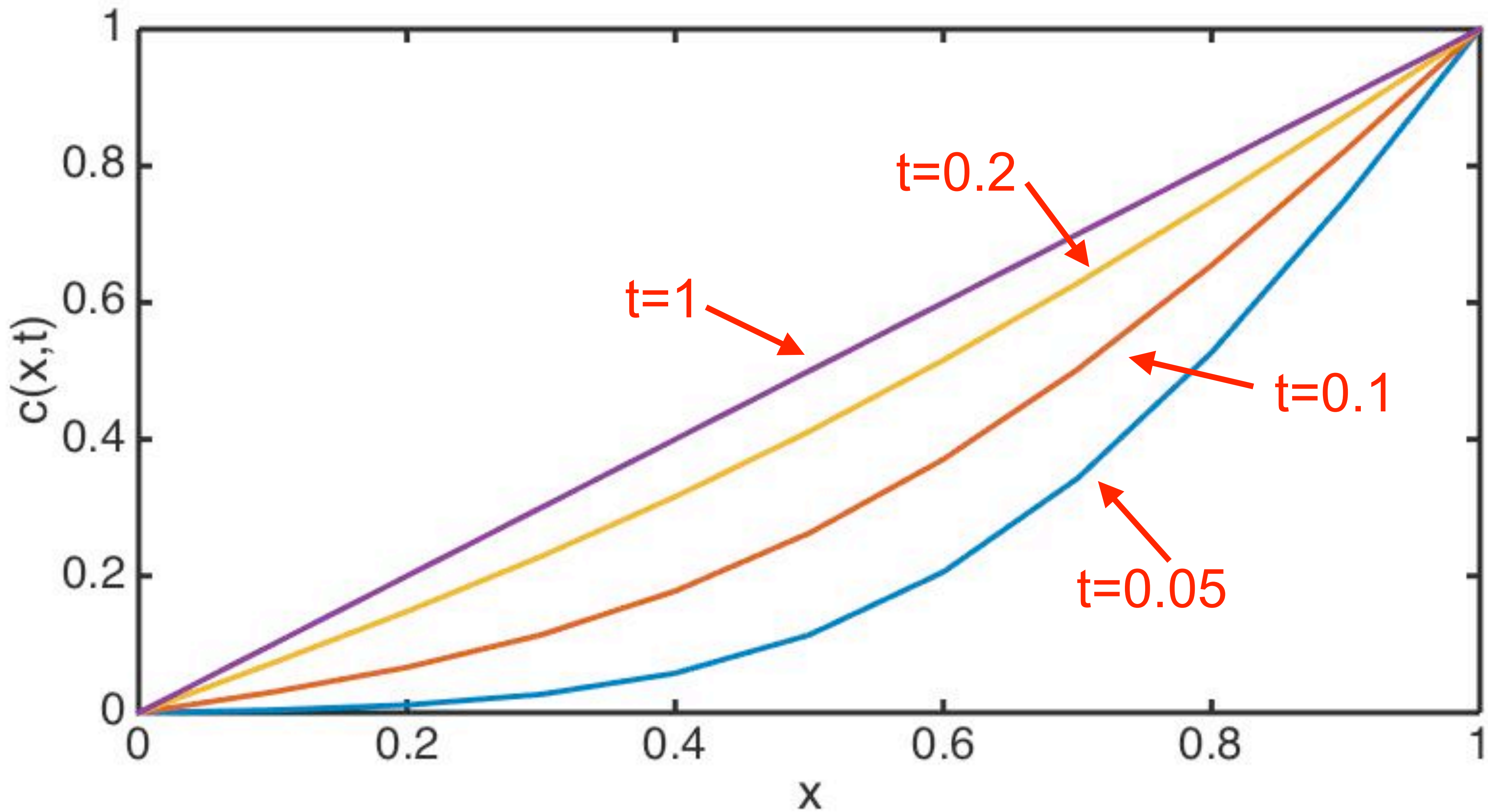
Steady state solution

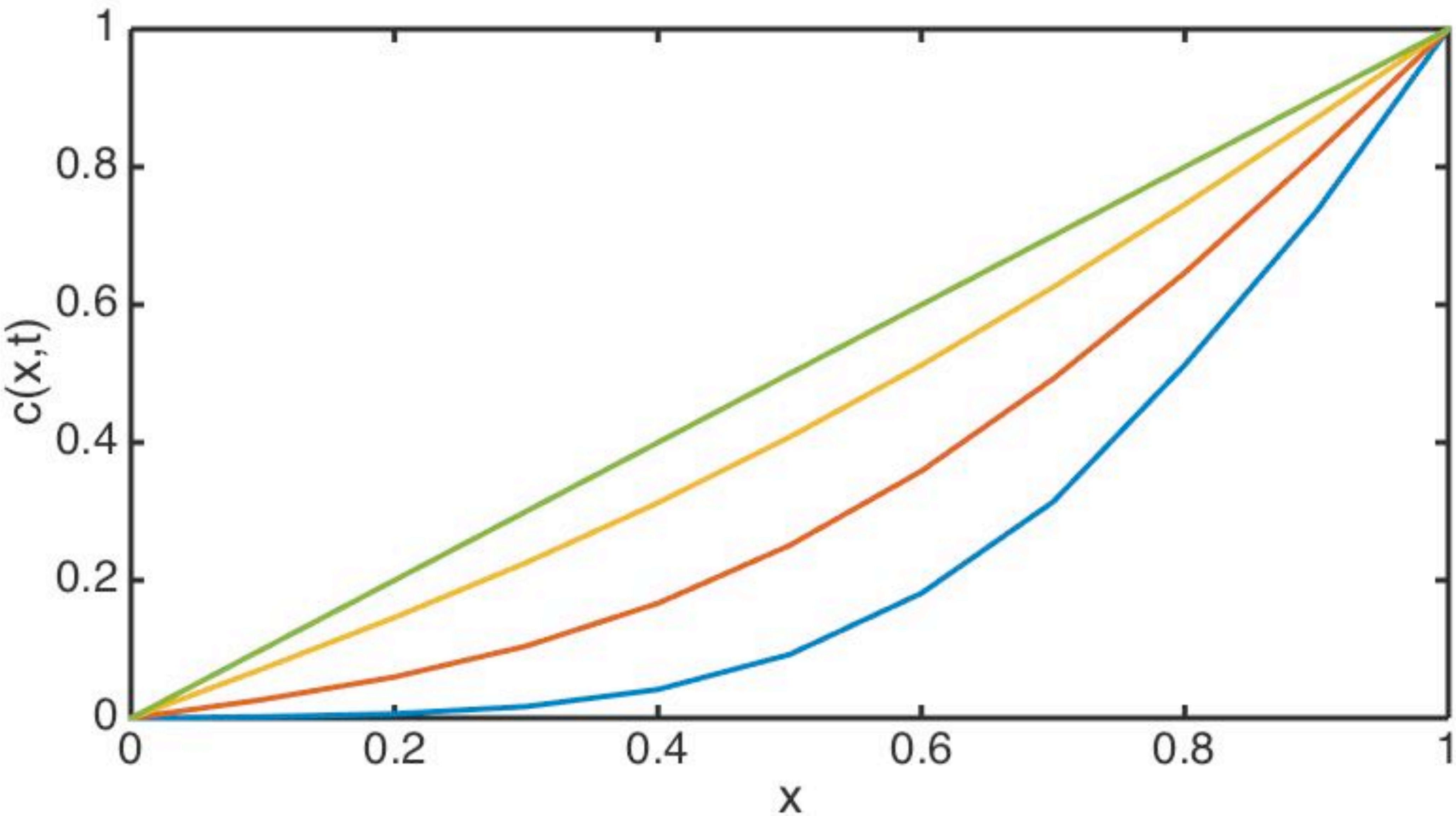
Transient solution - tends to zero as t goes to infinity

TRANSIENT HEAT EQUATION
Plotting The Analytical Solution

$$c(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(n\pi x)$$

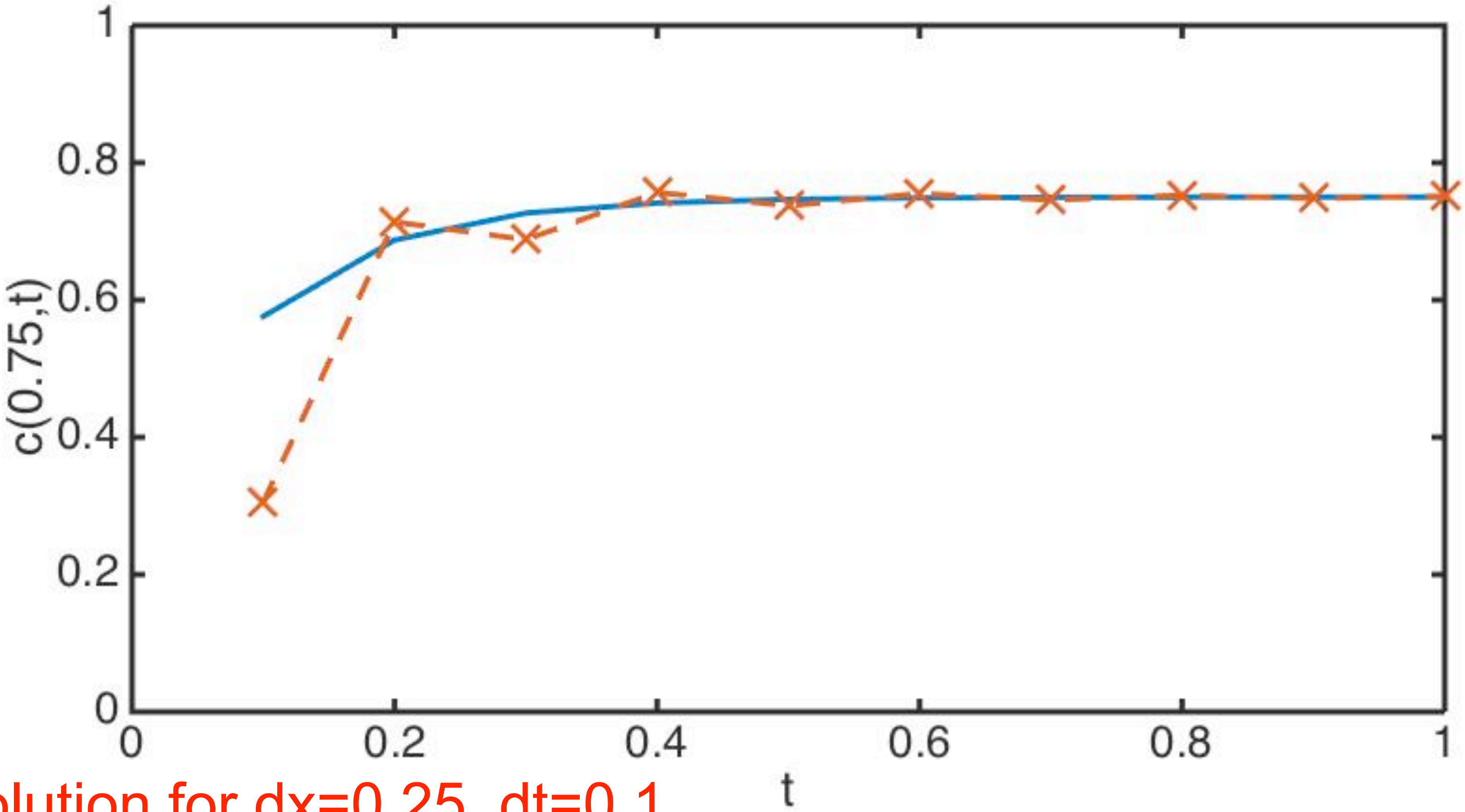


Numerical solution using dt=0.01, h=0.1 (i.e. 10 elements) and Crank-Nicolson scheme



THE NUMERICAL SOLUTION
Some Practical Considerations

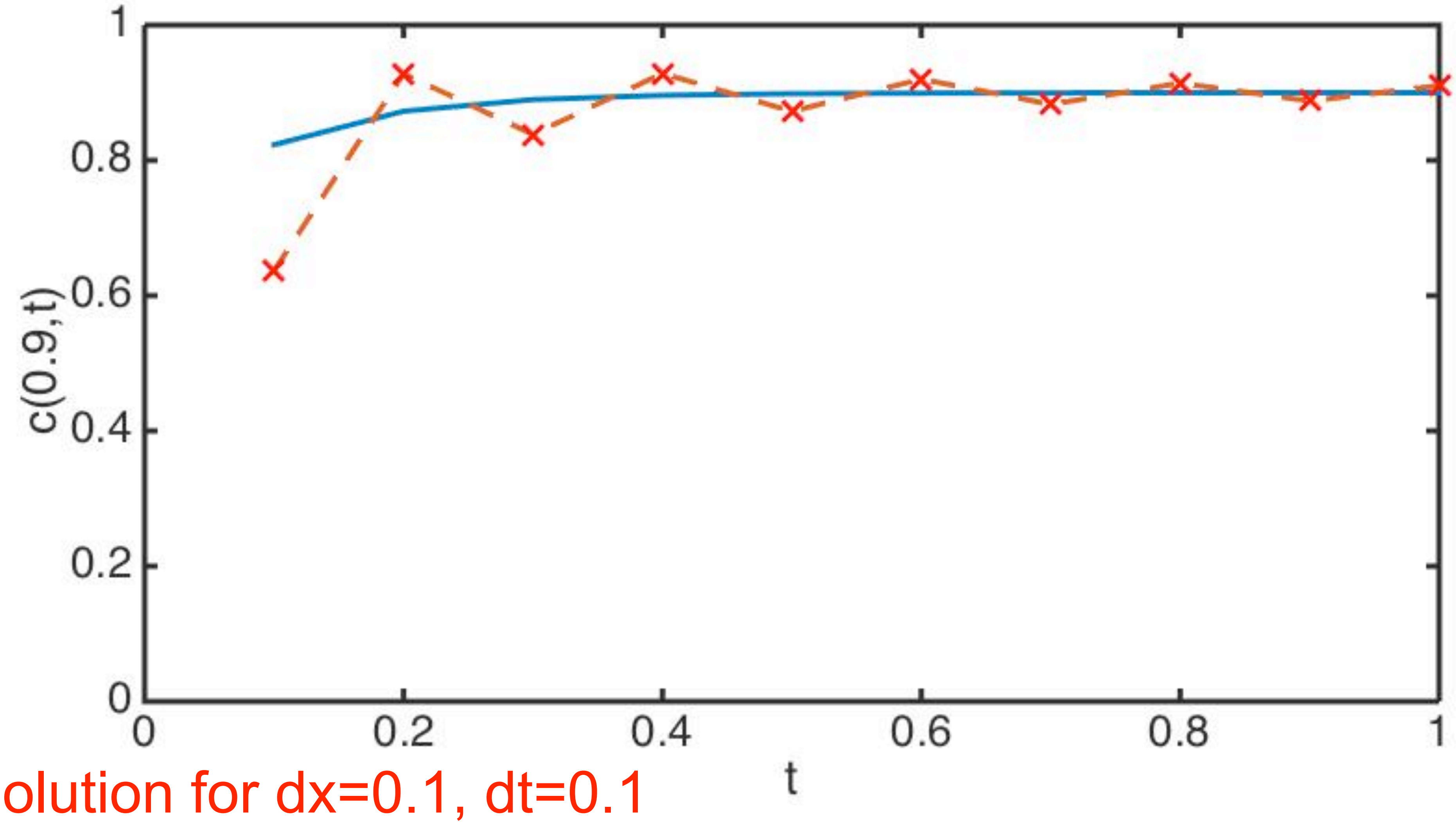
Crank-Nicolson is unconditionally stable: however, size of mesh and time step need to be balanced to prevent oscillations in the solution



Solution for dx=0.25, dt=0.1

THE NUMERICAL SOLUTION
Some Practical Considerations

Crank-Nicolson is unconditionally stable: however, size of mesh and time step need to be balanced to prevent oscillations in the solution



Solution for dx=0.1, dt=0.1

THE NUMERICAL SOLUTION
Some Practical Considerations

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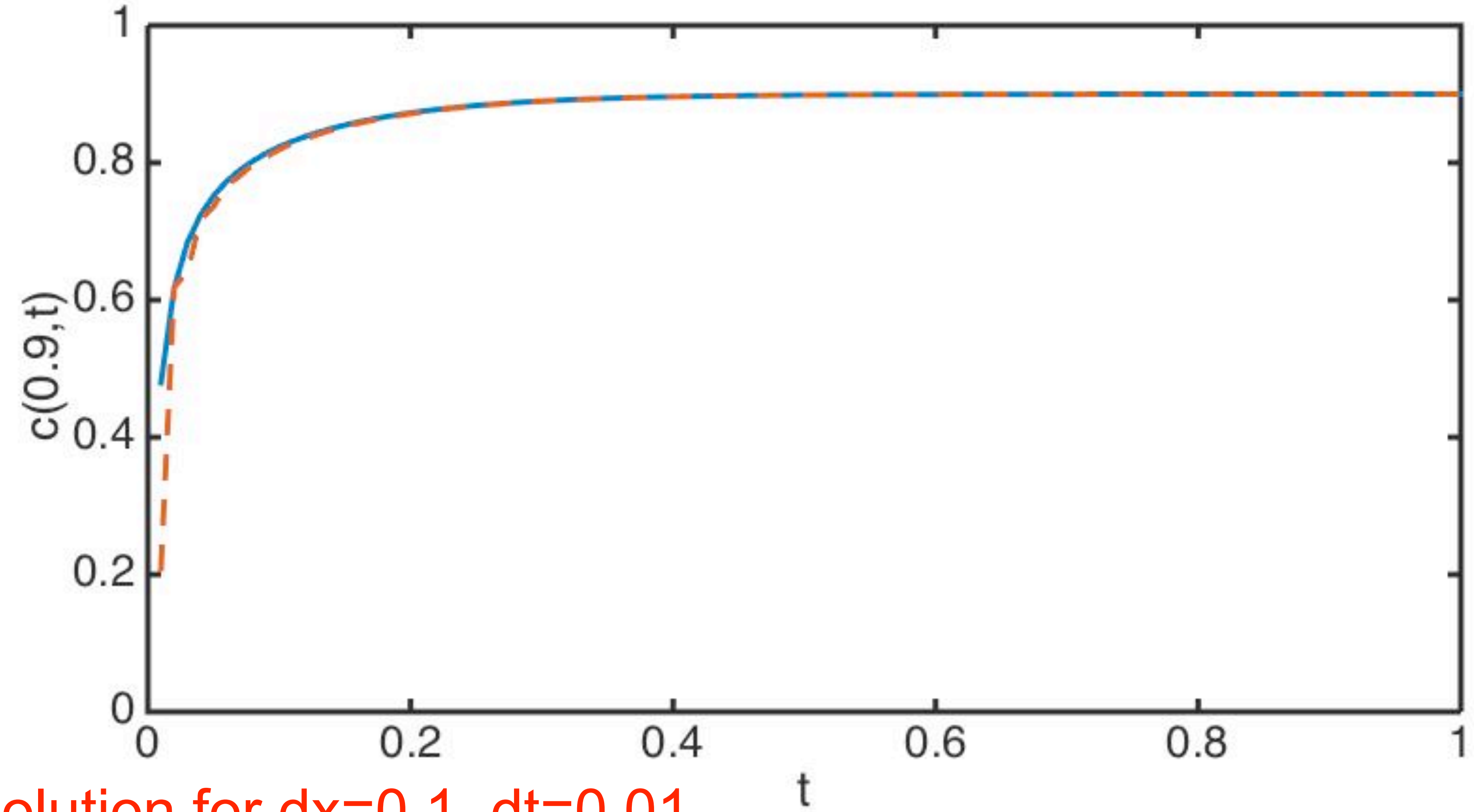


Solution for dx=0.1, dt=0.01

THE NUMERICAL SOLUTION
Some Practical Considerations

Theoretically, can show that to prevent these oscillations in the solution, observe the following condition:

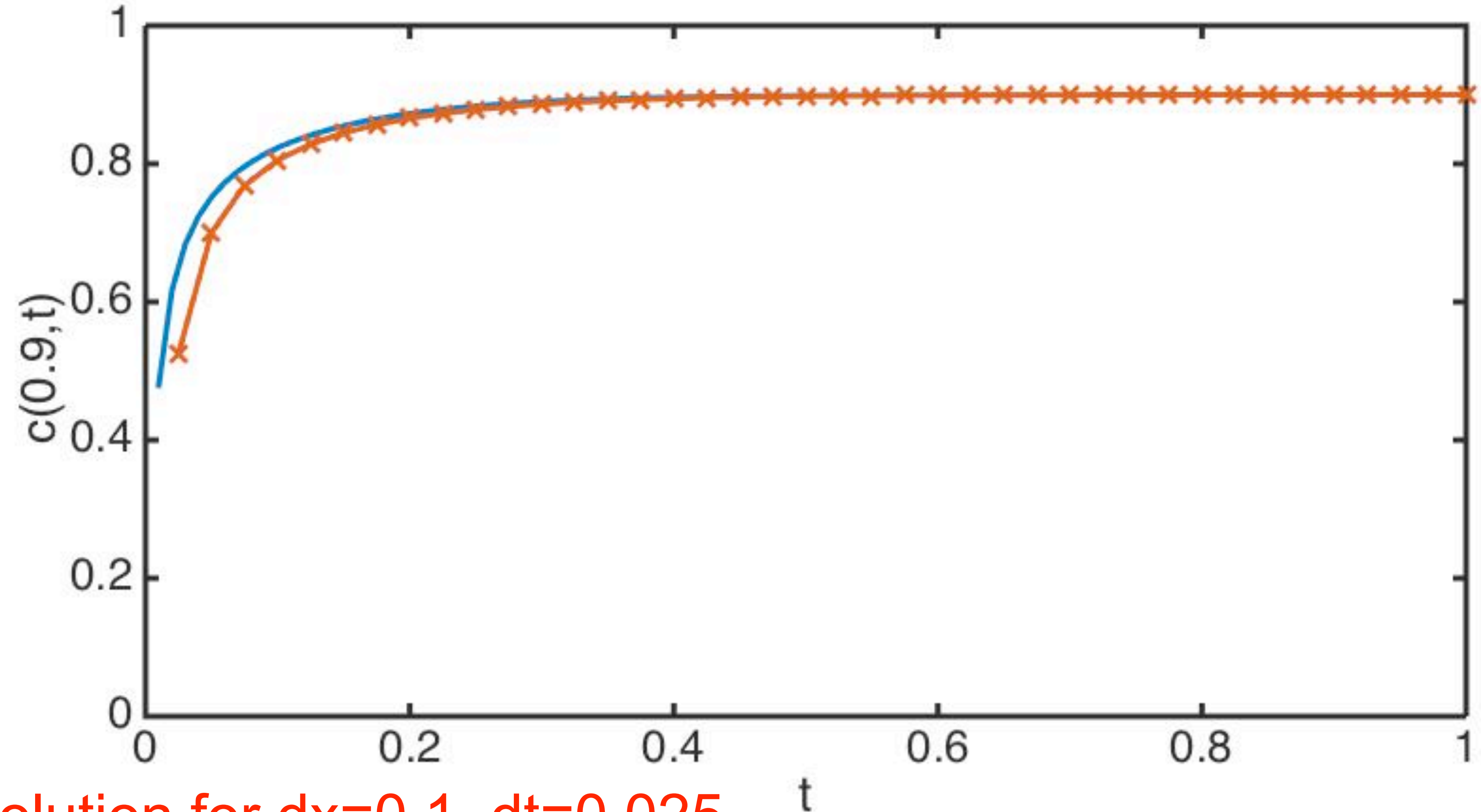
$$\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Use it as a rule of thumb - previous converged solution didn't quite satisfy this rule

If need a high resolution mesh, time step must be small - this can get computationally expensive

THE NUMERICAL SOLUTION
The Backward Euler Scheme

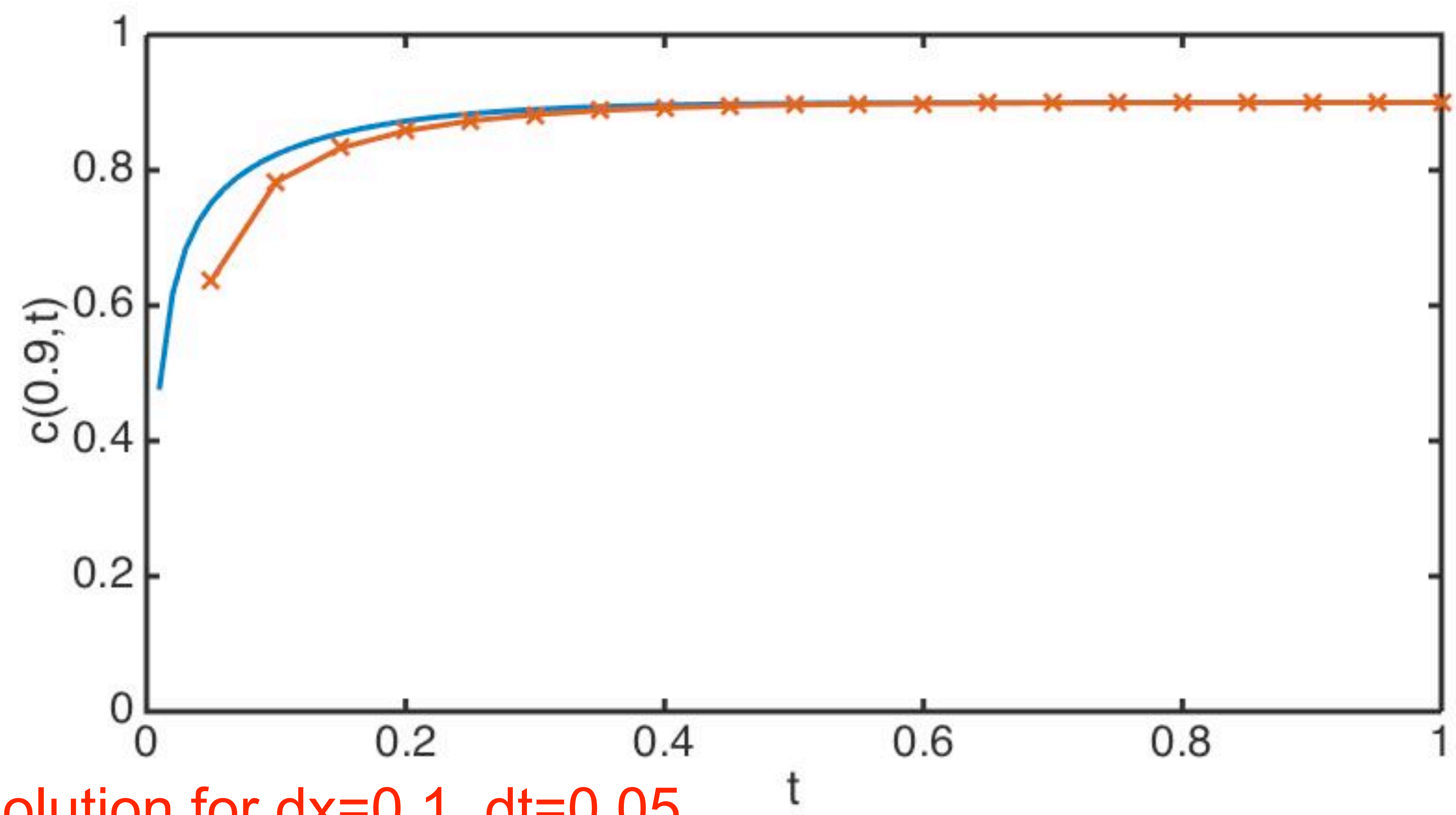
Backward Euler unconditionally stable:
- less accurate than Crank-Nicolson
- can take bigger time steps - computationally cheaper



Solution for dx=0.1, dt=0.025

THE NUMERICAL SOLUTION
The Backward Euler Scheme

Even for dt=0.05 the solution is still stable but error larger for t < 0.2, yet converges to steady state



Solution for dx=0.1, dt=0.05