**ME40064: System Modelling & Simulation**
**ME50344: Engineering Systems Simulation**

## Tutorial 5: The Use of Numerical Integration Techniques within the Finite Element Method

## Example Solutions – PART A

1. Create Gauss scheme for N=1,2,3.

```
function [gauss] = CreateGaussScheme(npoints)

%Creates Gauss-Legendre integration weights & points for npoints
if(npoints < 1) || (npoints > 3)
    error('Gauss:argChk','Scheme not implemented.')
end

gauss.np = npoints;
gauss.wt = zeros(npoints,1);
gauss.xi = zeros(npoints,1);

if(npoints==1)

    gauss.wt(1) = 2.0;
    gauss.xi(1) = 0.0;

elseif(npoints==2)

    gauss.wt(:) = 1.0;
    gauss.xi(1) = -sqrt(1/3);
    gauss.xi(2) = sqrt(1/3);

elseif(npoints==3)

    gauss.wt(1) = 8/9;
    gauss.wt(2) = 5/9;
    gauss.wt(3) = 5/9;
    gauss.xi(1) = 0.0;
    gauss.xi(2) = -sqrt(3/5);
    gauss.xi(3) = sqrt(3/5);

end

end
```

2. The following is the worked solution for $Int_{01}$, the same process would follow for $Int_{00}$.

2. 2.1

$$Int_{01} = \int_{-1}^{1} \lambda \left(\frac{1-\xi}{2}\right) \cdot \left(\frac{1+\xi}{2}\right) J \, d\xi$$

Gauss scheme for $N = 1$ $\therefore$ $w_1 = 2$ , $\xi_1 = 0$.

$$Int_{01} = 2 \cdot \lambda \left(\frac{1-0}{2}\right) \cdot \left(\frac{1+0}{2}\right) \cdot J$$

$$= \frac{\lambda J}{2}$$

Gauss scheme for $N=2$ $\therefore$ $w_1 = 1$ , $\xi_1 = -\sqrt{1/3}$
$w_2 = 1$ , $\xi_2 = \sqrt{1/3}$

$$Int_{01} = 1 \cdot \lambda \left(\frac{1+\sqrt{1/3}}{2}\right)\left(\frac{1-\sqrt{1/3}}{2}\right) J$$
$$+ 1 \cdot \lambda \left(\frac{1-\sqrt{1/3}}{2}\right)\left(\frac{1+\sqrt{1/3}}{2}\right) J$$

$$= \lambda J \left[ \frac{1 - 1/3}{4} + \frac{1 - 1/3}{4} \right] = \frac{\lambda J}{4} \left[ \frac{2}{3} + \frac{2}{3} \right]$$

$$= \frac{\lambda J}{3}$$

Gauss scheme for $N=3$ $\therefore$

$w_1 = 8/9$ , $\xi_1 = 0$
$w_2 = 5/9$ , $\xi_2 = -\sqrt{3/5}$
$w_3 = 5/9$ , $\xi_3 = \sqrt{3/5}$

2.2

$$Int_{01} = \frac{8}{9} \cdot \lambda \left(\frac{1-0}{2}\right)\left(\frac{1+0}{2}\right) J$$

$$+ \frac{5}{9} \cdot \lambda \left(\frac{1+\sqrt{3/5}}{2}\right)\left(\frac{1-\sqrt{3/5}}{2}\right) J$$

$$+ \frac{5}{9} \cdot \lambda \left(\frac{1-\sqrt{3/5}}{2}\right)\left(\frac{1+\sqrt{3/5}}{2}\right) J$$

$$= \frac{8}{9} \lambda J \cdot \frac{1}{4} + \frac{5}{9} \lambda J \cdot \left(\frac{1-3/5}{4}\right) + \frac{5}{9} \lambda J \left(\frac{1-3/5}{4}\right)$$

$$= \frac{2}{9} \lambda J + \frac{5}{9} \lambda J \cdot \frac{1}{10} + \frac{5}{9} \lambda J \cdot \frac{1}{10}$$

$$= \frac{\lambda J}{3}$$

$\therefore$ When $N=2$ or $3$ GQ gives the exact answer for this integral, but for $N=1$, there is some error associated with the scheme.

3. The Matlab code to evaluate this integral is:

```
%% x^5 - 3x^4 + 2x^3 + x^2 + 4x + 8 %%


N=3;
gq= CreateGaussScheme(N);
Int=0;
for i=1:N
    xi = gq.xi(i);
    Int = Int + gq.wt(i)*(xi^5 - 3*xi^4 + 2*xi^3 + xi^2 + 4*xi + 8);
end
Int
```

The analytical derivation follows on the next page.

3. 3.1

Integrate:

$$\int_{-1}^{1} x^5 - 3x^4 + 2x^3 + x^2 + 4x + 8 \; dx$$

$$= \left[ \frac{x^6}{6} - \frac{3x^5}{5} + \frac{2x^4}{4} + \frac{x^3}{3} + \frac{4x^2}{2} + 8x \right]_{-1}^{1}$$

$$= \left( \cancel{\frac{1}{6}} - \frac{3}{5} + \cancel{\frac{2}{4}} + \frac{1}{3} + \cancel{\frac{4}{2}} + 8 \right)$$

$$- \left( \cancel{\frac{1}{6}} + \frac{3}{5} + \cancel{\frac{2}{4}} - \frac{1}{3} + \cancel{\frac{4}{2}} - 8 \right)$$

$$= -\frac{3}{5} - \frac{3}{5} + \frac{1}{3} + \frac{1}{3} + 8 + 8$$

$$= \underline{\underline{15.4667}}$$