

ME40064: System Modelling & Simulation

ME50344: Engineering Systems Simulation

Lecture 12

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LECTURE 12

Solving Transient Problems - Theory

- Understand the general reasoning behind numerical time stepping schemes
- Ability to apply these to the FEM formulation
- Able to relate this to the static problem FEM formulation

DIFFUSION-REACTION EQUATION

Bringing Back The Transient Form

Everything up to Lecture 8 allowed us to solve the static diffusion-reaction equation:

$$D\nabla^2 c + \lambda c + f = 0$$

Now we need to solve the transient version of this equation, which was derived in Lecture 3:

$$\frac{\partial c}{\partial t} = D\nabla^2 c + \lambda c + f$$

GENERAL TRANSIENT EQUATIONS

A General Approach To Solving Them

Consider a general transient equation:

$$\frac{\partial c}{\partial t} = F(x, c, t)$$

Integrate both sides wrt to time, between to time points

$$\int_{t_n}^{t_{n+1}} \frac{\partial c}{\partial t} dt = \int_{t_n}^{t_{n+1}} F(x, c, t) dt$$

Left hand side is trivial to evaluate:



$$c(t_{n+1}) - c(t_n) = \int_{t_n}^{t_{n+1}} F(x, c, t) dt$$

GENERAL TRANSIENT EQUATIONS

Using The Trapezium Rule

Integrate RHS using the trapezium rule:

$$\int_{t_n}^{t_{n+1}} F(x, c, t) dt = \frac{1}{2} (t_{n+1} - t_n) (F(x, c, t_{n+1}) + F(x, c, t_n))$$

 width  heights

Different between t_n and t_{n+1} is Δt

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \frac{1}{2} (F^{n+1} + F^n), \quad \boxed{\theta = 1/2}$$

The trapezium rule creates what is known as the Crank-Nicolson scheme

SOLVING TRANSIENT EQUATIONS

Three Different Methods

Crank-Nicolson:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \frac{1}{2} (F^{n+1} + F^n), \quad \theta = 1/2$$

Forward Euler:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = F^n(x, c, t), \quad \theta = 0$$

Backward Euler:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = F^{n+1}(x, c, t), \quad \theta = 1$$

SOLVING TRANSIENT EQUATIONS

A Generalised Method

All these three schemes can be generalised in the theta scheme formulation:

$$\frac{c(t_{n+1}) - c(t_n)}{\Delta t} = \theta F^{n+1}(x, c, t) + (1 - \theta) F^n(x, c, t)$$

TRANSIENT FEM FORMULATION

A Generalised Method

The weighted residual form, after integration by parts, for the domain $x=[0, l]$ is now:

$$\int_0^1 \left(v \frac{\partial c}{\partial t} + D \frac{\partial v}{\partial x} \frac{\partial c}{\partial x} - \lambda c v \right) dx = \int_0^1 v f dx + \left[v D \frac{\partial c}{\partial x} \right]_0^1$$

The new term in the equation is:

$$v \frac{\partial c}{\partial t}$$

which, as before, can be represented by:

$$c = c_n \psi_n, v = \psi_m$$

LOCAL ELEMENT MASS MATRIX

Element Integral For Time Derivative

The local element integral is thus:

$$\int_{-1}^1 \frac{d(c_n \psi_n)}{dt} \cdot \psi_m J d\xi$$

Take the derivative outside of the integral:

$$\frac{dc_n}{dt} \int_{-1}^1 \psi_n \psi_m J d\xi$$

But know that time derivative can be written:

$$\frac{dc_n}{dt} = \frac{c_n^{n+1} - c_n^n}{\Delta t}$$

LOCAL ELEMENT MASS MATRIX

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Take the derivative outside of the integral:

$$\frac{dc_n}{dt} \int_{-1}^1 \psi_n \psi_m J d\xi \rightarrow \text{local element mass matrix}$$

But know that time derivative can be written:

$$\frac{dc_n}{dt} = \frac{c_n^{n+1} - c_n^n}{\Delta t}$$

GLOBAL ELEMENT MATRICES

The Mass And Stiffness Matrices

Local element mass matrix:

$$M_{element} = \int_{-1}^1 \psi_n \psi_m J d\xi$$

Local element stiffness matrix:

$$K_{element} = \int_{-1}^1 D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_m}{d\xi} \frac{d\xi}{dx} J d\xi - \int_{-1}^1 \lambda \psi_n \psi_m J d\xi$$



Looping over all element, assemble these into global mass, M, matrix, and global stiffness, K, matrix

GLOBAL ELEMENT MATRICES

The Transient Fem Equation

Write the transient FEM equation in terms of these global matrices, using general theta scheme:

$$M \left(\frac{\mathbf{c}^{n+1} - \mathbf{c}^n}{\Delta t} \right) + K [\theta \mathbf{c}^{n+1} + (1 - \theta) \mathbf{c}^n] = RHS$$

 Global solution vector at time step n+1  Global solution vector at time step n

and:

$$RHS = \theta \mathbf{F}^{n+1} + (1 - \theta) \mathbf{F}^n + \theta \mathbf{NBc}^{n+1} + (1 - \theta) \mathbf{NBc}^n$$

GLOBAL ELEMENT MATRICES

The Transient Fem Equation

Rearrange this to have unknowns on LHS:

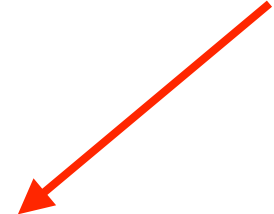
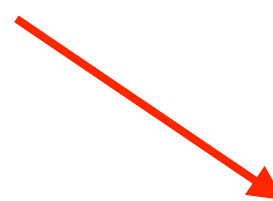
$$[M + \theta \Delta t K] \mathbf{c}^{n+1} = [M - (1 - \theta) \Delta t K] \mathbf{c}^n + \Delta t \theta [\mathbf{F}^{n+1} + \mathbf{NBc}^{n+1}] + \Delta t (1 - \theta) [\mathbf{F}^n + \mathbf{NBc}^n]$$



Final global
matrix to invert to
solve the system



Solution at
previous time step



Source terms &
Neumann BCs

The Crank-Nicolson scheme is thus:

$$[M + \frac{1}{2} \Delta t K] \mathbf{c}^{n+1} = [M - \frac{1}{2} \Delta t K] \mathbf{c}^n + \frac{1}{2} \Delta t [\mathbf{F}^{n+1} + \mathbf{F}^n + \mathbf{NBc}^{n+1} + \mathbf{NBc}^n]$$