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Tutorial 3: Constructing Element Matrices for FEM
Example Solutions

a. This example solution mainly uses argumentation to show that the two element matrices are identical. The key point to take from this is that these element matrices are only identical because the domain x = [0,1] has been divided into equal-sized elements, and hence J is the same in each. In general, this will not be the case, and hence in most applications, the local element matrices will vary between elements.

b. Note that the integration of the functions has been done using the chain rule in these example solutions. If you wish you can expand the polynomials and integrate each term individually.

Q1. 1.1

The integral, defined over the standard element is:

$$c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \cdot \frac{d\xi}{dx} \cdot \frac{d\psi_m}{d\xi} \cdot \frac{d\xi}{dx} \, J \, d\xi$$

Evaluating the terms:

$$J = \left| \frac{dx}{d\xi} \right| = \left| \frac{x_1 - x_0}{2} \right|$$

For the second element $x_1 = 2/3$ and $x_0 = 1/3$.

$$\therefore \; J = \left| \frac{2/3 - 1/3}{2} \right| = \frac{1}{6}$$

Similarly:

$$\frac{d\xi}{dx} = \frac{2}{2/3 - 1/3} = 6.$$

These two terms are the same as for element 1, because the element is the same length as element 1.

The derivatives of the basis function $\frac{d\psi_n}{d\xi}$ do not change with the local element, as these are defined purely in terms of the standard element.

1.2

For a local element matrix

$$\begin{bmatrix} Int_{00} & Int_{01} \\ Int_{10} & Int_{11} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

On page 1 it was shown that all the terms in the integral are the same as for element 1, $\therefore$ the value of the integrals when evaluated will be the same, as the limits of the integral are the same.

$\therefore$ The local element matrix is:

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

where c_0 and c_1 are the local node values in element 2.

Q2. 2.1

a) Want to evaluate the following integral

$$\int_{x_0}^{x_1} \lambda c v \, dx = \int_{-1}^{1} \lambda c v J \, d\xi \qquad (1)$$

where λ is a scalar coefficient that is known.

Represent: $c = C_n \psi_n$ where $\psi_0 = \frac{1-\xi}{2}$

$n,m = 0,1$ $\quad v = \psi_m$ $\quad \psi_1 = \frac{1+\xi}{2}$

$\therefore$ Can write eqn (1) as:

$$C_n \int_{-1}^{1} \lambda \psi_n \psi_m J \, d\xi \qquad (2)$$

b) For element 1, we have previously found that $J = 1/6$.

$\therefore$ Eqn (2) is $\quad C_n \int_{-1}^{1} \lambda \psi_n \psi_m \frac{1}{6} \, d\xi$

Want to fill out an element matrix of the form

$$\begin{bmatrix} Int_{00} & Int_{01} \\ Int_{10} & Int_{11} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \end{bmatrix}$$

2.2

Int$_{00}$

$$\int_{-1}^{1} \lambda \psi_0 \psi_0 \frac{1}{6} \, d\xi$$

$$= \int_{-1}^{1} \lambda \left(\frac{1-\xi}{2}\right)^2 \cdot \frac{1}{6} \cdot d\xi$$

$$= \left[\lambda \cdot \left(\frac{1-\xi}{2}\right)^3 \cdot \frac{1}{3} \cdot (-2) \cdot \frac{1}{6} \right]_{-1}^{1}$$

where the chain rule has been used to integrate $\left(\frac{1-\xi}{2}\right)^2$.

Evaluating the limits:

$$\left[\left(-\lambda \cdot \left(\frac{1-1}{2}\right)^3 \cdot \frac{1}{9}\right) - \left(-\lambda \cdot \left(\frac{1-(-1)}{2}\right)^3 \cdot \frac{1}{9}\right) \right]$$

$$= \left[0 + \lambda \cdot \left(\frac{2}{2}\right)^3 \cdot \frac{1}{9} \right] = \frac{\lambda}{9}$$

2.3

$\underline{Int_{01} = Int_{10}}$

$$\int_{-1}^{1} \lambda \psi_0 \psi_1 \cdot \frac{1}{6} \, d\xi$$

$$= \int_{-1}^{1} \lambda \left(\frac{1-\xi}{2}\right) \cdot \left(\frac{1+\xi}{2}\right) \cdot \frac{1}{6} \cdot d\xi$$

$$= \int_{-1}^{1} \lambda \left(\frac{1-\xi^2}{4}\right) \cdot \frac{1}{6} \cdot d\xi$$

$$= \left[\lambda \left(\frac{\xi - \xi^3/3}{4}\right) \cdot \frac{1}{6} \right]_{-1}^{1}$$

$$= \left[\lambda \left(\frac{3\xi - \xi^3}{12}\right) \cdot \frac{1}{6} \right]_{-1}^{1}$$

$$= \left[\frac{\lambda}{6} \left(\left(\frac{3-1}{12}\right) - \left(\frac{-3-(-1)}{12}\right) \right) \right]$$

$$= \left[\frac{\lambda}{6} \left(\frac{2}{12} + \frac{2}{12} \right) \right] = \frac{\lambda}{18}$$

2.4

Int$_{11}$

$$\int_{-1}^{1} \lambda \psi_1 \psi_1 \cdot \frac{1}{6} d\xi$$

$$= \int_{-1}^{1} \lambda \left(\frac{1+\xi}{2}\right)^2 \cdot \frac{1}{6} d\xi$$

$$= \left[\lambda \left(\frac{1+\xi}{2}\right)^3 \cdot \frac{1}{3} \cdot 2 \cdot \frac{1}{6} \right]_{-1}^{1}$$

$$= \left[\frac{\lambda}{6} \left(\left(\frac{1+1}{2}\right)^3 \cdot \frac{2}{3} - \cancel{\left(\left(\frac{1-1}{2}\right)^3 \cdot \frac{2}{3} \right)} \right) \right]$$

$$= \frac{\lambda}{6} \cdot 1 \cdot \frac{2}{3} = \frac{\lambda}{9}$$

$\therefore$ Local element matrix is:

$$\begin{bmatrix} \lambda/9 & \lambda/18 \\ \lambda/18 & \lambda/9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

c. In this case the Jacobian for the element defined by the points x0 = 1/6 & x1 = 1/3 is J = 1/12, i.e. half of that for the elements in the 3-element mesh. Therefore the local element matrix is half of that in Q2b, i.e:

$$\begin{bmatrix} \lambda/18 & \lambda/36 \\ \lambda/36 & \lambda/18 \end{bmatrix}$$

d. To fully create the Matlab function, `TestMatrixCreate`, the following code must be saved in a file named, `TestMatrixCreate.m` .

```
function [ SqMat ] = TestMatrixCreate(a,b,c)
%Creates a symmetric 2 by 2 matrix and assigns elements
with polynomial
%functions of a,b,c

  SqMat = zeros(2,2);
  SqMat(1,1) = c^3 + 2*b*c + a;
  SqMat(1,2) = b^2 + a;
  SqMat(2,1) = SqMat(1,2);
  SqMat(2,2) = 2*c^3 + 4*b*c + 5*a;

end
```