Recap

The advection - diffusion - reaction equation

$$\frac{\partial c}{\partial t} + \underbrace{\underline{u} \cdot \nabla c}_{\text{advection}} = \underbrace{D \nabla^2 c}_{\text{diffusion}} + \underbrace{\lambda c}_{\text{linear reaction}} + \underbrace{f}_{\text{source}} \qquad (1)$$

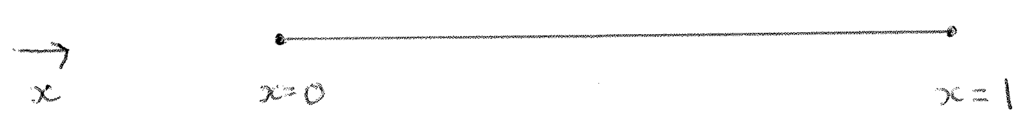
Simplify – static, not transient
– no fluid flow
– no reaction processes.

$$\Rightarrow \qquad D \nabla^2 c + f = 0 \qquad (2)$$

Move to a 1D situation:

$$D \frac{\partial^2 c}{\partial x^2} + f = 0 \qquad (3)$$

This is the static/steady state diffusion equation.
For $f = 0$ known as Laplace's equation
For $f \neq 0$ known as Poisson's equation
We have written here what is known as the strong form of the equation.
Want to solve it on the following domain:



So, how do we solve:

$$D \frac{\partial^2 c}{\partial x^2} + f = 0 \quad ? \qquad (4)$$

Suppose we had an approximate solution to eqn(4), given by c^A

Substituting c^A into (4) gives:

$$D \frac{\partial^2 c^A}{\partial x^2} + f = R \qquad (5)$$

R = Residual i.e. the overall error in the equation.

IF $c^A = c$, then $R = 0$.

$\therefore$ Aim is to minimise R, to make c^A as close to c as possible. How?

<u>Method of weighted residuals.</u>

1. Multiply by a weighting function, v
2. Integrate over domain

$$\Rightarrow \left[\int_0^1 v \cdot \left(D \frac{\partial^2 c^A}{\partial x^2} + f \right) dx = 0 \right] \qquad (7)$$

$$\int_0^1 R \cdot v \cdot dx = 0 \qquad (6)$$

Weighted average of the error, to distribute it evenly over the domain

[Show slides on approximating data points]

Reminder: integration by parts.

Functions $u(x)$ and $w(x)$

Derivatives: $u'(x) = \dfrac{du}{dx}$, $w'(x) = \dfrac{dw}{dx}$.

$$\int u(x)\, w'(x)\, dx = u(x)\, w(x) - \int w(x)\, u'(x)\, dx. \quad (8)$$

In eqn (7) identify terms $u(x)$ and $w(x)$.

$u(x) = v$ and $w(x) = D\dfrac{\partial c}{\partial x}$

$$\therefore \int_0^1 v.\, D\frac{\partial^2 c}{\partial x^2}\, dx = \left[v.\, D\frac{\partial c}{\partial x} \right]_0^1 - \int_0^1 D\frac{\partial c}{\partial x} . \frac{\partial v}{\partial x}\, dx. \quad (9)$$

$$\Rightarrow \quad -\int_0^1 D\frac{\partial c}{\partial x} . \frac{\partial v}{\partial x}\, dx + \left[v.\, D\frac{\partial c}{\partial x} \right]_0^1 + \int_0^1 vf\, dx = 0 \quad (10)$$

$$\Rightarrow \quad \underbrace{\int_0^1 D\frac{\partial c}{\partial x} . \frac{\partial v}{\partial x}\, dx}_{\text{unknown}} = \underbrace{\int_0^1 v.f\, dx + \left[v.\, D\frac{\partial c}{\partial x} \right]_0^1}_{\text{known}} \quad (11)$$

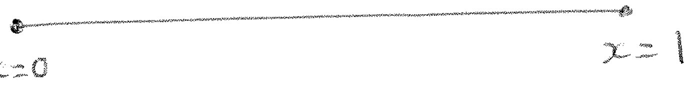
Note

$$\underbrace{\left[v.\, D\frac{\partial c}{\partial x} \right]_0^1}_{\text{Fickian diffusive flux}} = \text{Neumann or flux boundary condition.}$$

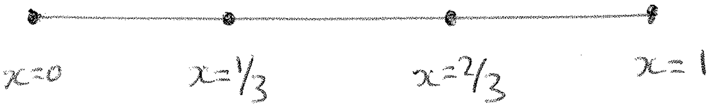
i.e. diffusive flux at the boundaries $x = 0$ and $x = 1$

What happened to the finite element method?

We have the domain:



Split into 3 equal elements.



Perform integral in each element i.e.

$$\int_0^1 dx = \int_0^{1/3} dx + \int_{1/3}^{2/3} dx + \int_{2/3}^{1} dx.$$

How to integrate in 1 element?

Represent space, x, and solution, c, using the basis functions ψ_0 and ψ_1 introduced in Lecture 4

$$c = c_0\,\psi_0(\xi) + c_1\,\psi_1(\xi)$$

$$x = x_0\,\psi_0(\xi) + x_1\,\psi_1(\xi)$$

where $\psi_0 = \dfrac{1-\xi}{2}$, $\psi_1 = \dfrac{1+\xi}{2}$

[Show slide]

Equally, the inverse holds:

$$\xi(x) = 2\left(\frac{x - x_0}{x_1 - x_0}\right) - 1$$

For x in that element between x_0 and x_1

Before integrating, need to define v.

The Galerkin Assumption

Choose v to be the basis functions ψ_0 and ψ_1 i.e. same functions that represent the solution.

This can be thought of as being "optimal" for reducing error.

Finally, we perform integration in ξ coordinate rather than x.

$$\int_{x_0}^{x_1} dx = \int_{-1}^{1} J \, d\xi$$

Where $J = \left|\frac{dx}{d\xi}\right|$ is the Jacobian or scaling between coordinates.

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