ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 9

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LECTURE 9
FEM: Numerical Integration

- Understand need for numerical integration
- Ability to evaluate FEM expressions using Gaussian quadrature
- Appreciation of how to implement Gaussian quadrature in code

NUMERICAL INTEGRATION
Why Do We Need This?

- Helps generalise the finite element method to 2D & 3D
- Removes need for manual integration prior to implementation in code
- Greater flexibility in specifying material parameters & basis functions
- Can be as accurate as analytical integration

NUMERICAL INTEGRATION
The Gaussian Quadrature Method

Advanced, accurate method of numerical integration

Stated generally:

Gauss weights

Limits of integral match standard element

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{N} w_i . f(x_i)$$

Gauss points

<u>Integrates polynomial of order 2N -1 exactly</u>

NUMERICAL INTEGRATION
The Gaussian Quadrature Method

For N=1, 2N-1 = 1, i.e. can integrate constant and linear functions exactly:

$$f(x) = a$$
$$f(x) = ax + b$$

For N=2, 2N-1 = 3, i.e. can integrate constant, linear, quadratic & cubic functions exactly:

$$f(x) = a$$
$$f(x) = ax + b$$
$$f(x) = ax^2 + bx + c$$
$$f(x) = ax^3 + bx^2 + cx + d$$

NUMERICAL INTEGRATION
The Gaussian Quadrature Method

Gauss weights & points vary with N:

N	x_i	w_i
1	0	2
2	$-\sqrt{\frac{1}{3}}$	1
	$+\sqrt{\frac{1}{3}}$	1
3	$-\sqrt{\frac{3}{5}}$	$\frac{5}{9}$
	0	$\frac{8}{9}$
	$+\sqrt{\frac{3}{5}}$	$\frac{5}{9}$

GAUSSIAN QUADRATURE
Worked Example 1

Using GQ for N=1:

$$\int_{-1}^{1}\left(\frac{1+\xi}{2}\right)d\xi=\sum_{i=1}^{1}w_i.\left(\frac{1+\xi_i}{2}\right)$$

$$=2.\left(\frac{1+0}{2}\right)=1$$

Using GQ for N=2:

$$\int_{-1}^{1}\left(\frac{1+\xi}{2}\right)d\xi=\sum_{i=1}^{2}w_i.\left(\frac{1+\xi_i}{2}\right)$$

$$=1.\left(\frac{1-\sqrt{1/3}}{2}\right)+1.\left(\frac{1+\sqrt{1/3}}{2}\right)=1$$

GAUSSIAN QUADRATURE
Applied To Diffusion Operator

General integral for diffusion operator:

$$Int_{mn} = \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_m}{d\xi} \frac{d\xi}{dx} J d\xi$$

For specific case, n=m=0:

$$Int_{00} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

In the 1D mesh:

$$\frac{d\xi}{dx} = \frac{1}{J}$$

GAUSSIAN QUADRATURE
Applied To Diffusion Operator

Therefore integral is:

$$Int_{00} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\psi_0}{d\xi} \frac{1}{J} d\xi$$

Evaluating the basis function gradients:

$$\frac{d\psi_0}{d\xi} = -\frac{1}{2}$$

And substituting into integral:

$$Int_{00} = \int_{-1}^{1} D . \frac{-1}{2} . \frac{-1}{2} . \frac{1}{J} d\xi = \int_{-1}^{1} \frac{D}{4J} d\xi$$

GAUSSIAN QUADRATURE
Applied To Diffusion Operator

Evaluating this integral using Gaussian Quadrature

Using GQ for N=1:

$$Int_{00} = 2.\frac{D}{4J} = \frac{D}{2J}$$

Using GQ for N=2:

$$Int_{00} = 1.\frac{D}{4J} + 1.\frac{D}{4J} = \frac{D}{2J}$$

As expected, the answers are the scheme for these two schemes

GAUSSIAN QUADRATURE
A Matlab Data Structure

A data structure to represent a Gaussian Quadrature scheme:

- `gq.npts; %number of Gauss points`
- `gq.gsw(:); %array of Gauss weights`
- `gq.xipts(:); %array of Gauss points`

Function to create a Gaussian Quadrature structure of order N:

- `gq = CreateGQScheme(N);`

GAUSSIAN QUADRATURE
A Matlab Data Structure

```
function [ gq ] = CreateGQScheme(N)
%CreateGQScheme Creates GQ Scheme of order N
%   Creates and initialises a data structure
    gq.npts = N;
    if (N > 0) && (N < 4)
        %order of quadrature scheme i.e. %number of Gauss points
        gq.gsw = zeros(N,1); %array of Gauss weights
        gq.xipts = zeros(N,1); %array of Gauss points
        switch N
            case 1
              gq.gsw(1) = 2;
              gq.xipts(1) = 0;
            case 2
              %gq.gsw(1) = ;
              %gq.gsw(2) = ;
              %gq.xipts(1) = ;
              %gq.xipts(2) = ;
            case 3
        end

    else
      fprintf('Invalid number of Gauss points specified');
    end
end
```

GAUSSIAN QUADRATURE
The Pseudo-Code

1. Initialise quadrature scheme - number of points, generate weights and Gauss points
2. Initialise integral value to zero
3. Loop over number of Gauss points:
    1. At each Gauss point call functions to evaluate:
        1. Basis functions & gradients (as appropriate)
        2. Material parameters
    2. Multiply together & multiply by matching Gauss weight
    3. Add to integral value