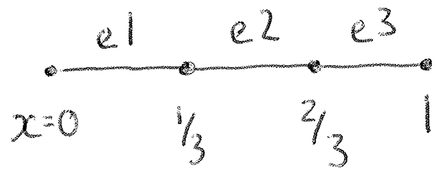
Lecture 7

Assembly of the global element matrix

Recap the local element matrix for the integral:

$$\int_{-1}^{1} D \frac{\partial c}{\partial x} \frac{\partial v}{\partial x} . J \, d\xi$$

This was evaluated for element 1.



In tutorial you saw that this was the same in all 3 elements, as all elements are the same size.

Note: not generally true.

Local element matrix 1

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Local element matrix 2.

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

Local element matrix 3

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

These are not the same c_0 and c_1

They are the c_0 and c_1 local to each element and represent the local nodes in each element.

$\Rightarrow$ Total of 6 local nodes

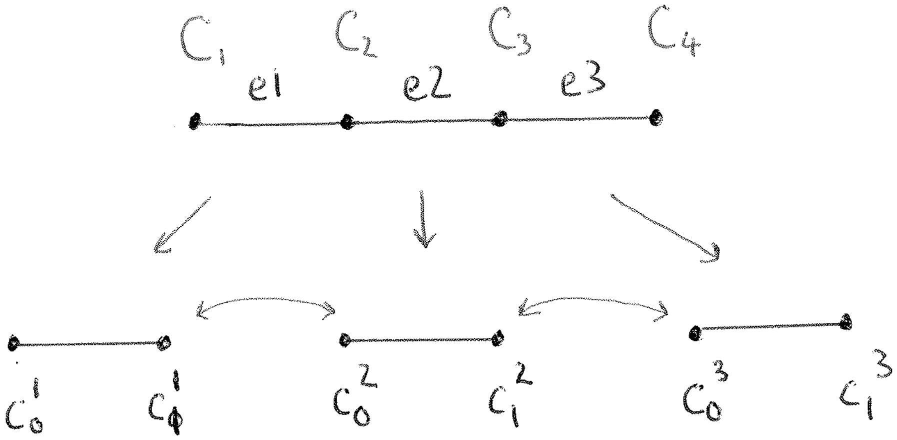
But clearly only 4 global nodes.

Remember that our integral is split in 3

$$\int_0^1 \cdot\, dx = \int_0^{1/3} \cdot\, dx + \int_{1/3}^{2/3} \cdot\, dx + \int_{2/3}^{1} \cdot\, dx$$

$\therefore$ Need to add together these local element matrices to represent this summation of integrals.

Do this in a way that represents the element structure of the mesh.



Can see that:

$$c_0^1 = C_1$$

$$c_1^1 = c_0^2 = C_2$$

$$c_1^2 = c_0^3 = C_3$$

$$c_1^3 = C_4$$

Therefore our local element matrices are:

e1 : $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

e2: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$

e3: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$

What we want to do is gather together the C_2 terms and the C_3 terms so that we have a final vector:

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

How do we do this? Recognise that we are solving a set of linear simultaneous equations. Add together the rows of the local element matrices that share a global node value.

i.e.

$$3DC_1 - 3DC_2 = A_1 \qquad \text{Line 1}$$

$$\left.\begin{aligned} -3DC_1 + 3DC_2 &= B_1 \\ 3DC_2 - 3DC_3 &= A_2 \end{aligned}\right\} \text{Add these together.} \quad \text{Line 2}$$

$$\left.\begin{aligned} -3DC_2 + 3DC_3 &= B_2 \\ 3DC_3 - 3DC_4 &= A_3 \end{aligned}\right\} \text{Add these together.} \quad \text{Line 3}$$

$$-3DC_3 + 3DC_4 = B_3 \qquad \text{Line 4}$$

That deals with the C_2 lines, do the same for C_3.

<u>For C_2:</u>

$$-3DC_1 + 6DC_2 - 3DC_3 = B_1 + A_2$$

But we want a matrix and a vector.

$$\begin{bmatrix} -3D & 6D & -3D & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Putting those 4 lines into the same form:

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Overlap of nodes happens at corners of the local matrices.

Four global nodes $\Rightarrow$ 4×4 global matrix but sparse $\rightarrow$ lots of zeros!

The vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ represents the values of our solution c, at the four global nodes of the mesh.

The final step in solving for c will be shown in Lecture 8.

In tutorial 3, calculated local element matrix for reaction operator

$\Rightarrow$ simply add the two element matrices together to specify a diffusion-reaction term

Some pseudo-code for assembling the global matrix:

0. Create empty global matrix of size $N_{nodes} \times N_{nodes}$.
1. Loop over the elements from 1 to N

   a. For each element calculate the local element matrices for
      - diffusion operator
      - reaction term (if needed)

   b. Add up the local element matrices to form a single 2 by 2 element matrix

   c. Add this local element matrix into the global matrix in the appropriate position.

Moving back to the original problem in weighted residual form:

$$\int_0^1 \left(D\frac{\partial^2 c}{\partial x^2} + \lambda c + f \right).v \, dx = 0$$

Integrating by parts + rearranging

$$\int_0^1 D\frac{\partial c}{\partial x}.\frac{\partial v}{\partial x}\,dx - \int_0^1 \lambda c.v.dx = \left[v.D\frac{\partial c}{\partial x} \right]_0^1 + \int_0^1 v.f.dx.$$

Lecture 6	Tutorial 3	Lecture 8	Lecture 7
Diffusion operator element matrix	Reaction term element matrix	Boundary condition	Source term element vector

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} - \begin{bmatrix} \lambda/9 & \lambda/18 \\ \lambda/18 & \lambda/9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} ? \\ \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix}$$

Source term :- move to element integral.

$$\int_{-1}^{1} v.f.J\,d\xi$$

$v = \psi_0, \psi_1$

if f is a constant, then take out of integral.

$$\text{Int}_0 = \int_{-1}^{1} \left(\frac{1-\xi}{2}\right).f.J\,d\xi = \frac{fJ}{2}\int_{-1}^{1} 1-\xi\,.d\xi = \frac{fJ}{2}\left[\xi - \frac{\xi^2}{2}\right]_{-1}^{1}$$

$$= \frac{fJ}{2}\left[\left(1-\frac{1}{2}\right) - \left(-1-\frac{1}{2}\right)\right] = f.J$$

$$\text{Int}_1 = \int_{-1}^{1}\left(\frac{1+\xi}{2}\right).f.J\,d\xi = \frac{fJ}{2}\left[\xi + \frac{\xi^2}{2}\right]_{-1}^{1} = \frac{fJ}{2}\left[\left(1+\frac{1}{2}\right) - \left(-1+\frac{1}{2}\right)\right]$$

$$= fJ.$$

Assembly of the global vector.

↳ proceeds along same lines as for global matrix.

→ add contributions together at the common nodes.

For that constant source term, local vector same in each element:

$$\begin{bmatrix} f.J \\ f.J \end{bmatrix}$$

∴ For our 3-element mesh, global vector is

$$\begin{bmatrix} \begin{bmatrix} fJ \\ fJ \end{bmatrix} + \begin{bmatrix} fJ \\ f.J \end{bmatrix} + \begin{bmatrix} fJ \\ fJ \end{bmatrix} \end{bmatrix} = \begin{bmatrix} fJ \\ 2fJ \\ 2fJ \\ fJ \end{bmatrix}$$

∴ Once we have boundary conditions, can solve this system