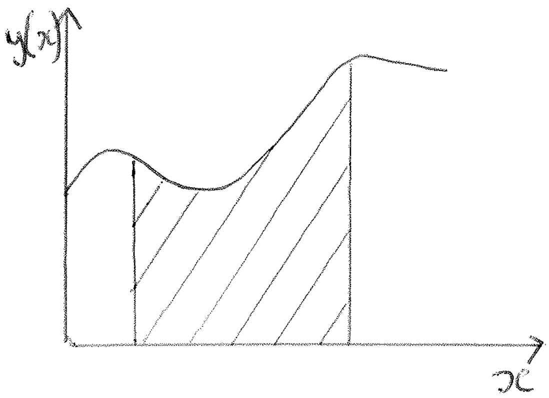
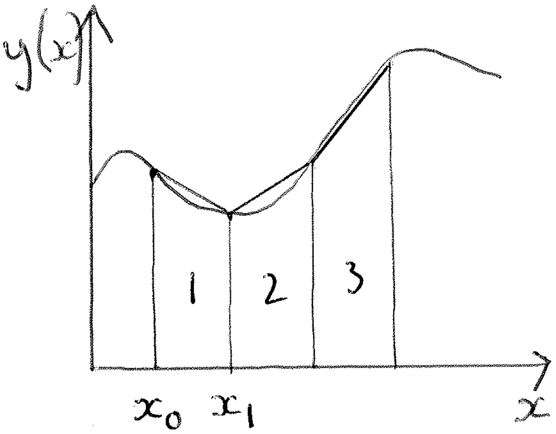
Numerical Integration

Revision : Trapezium Rule.



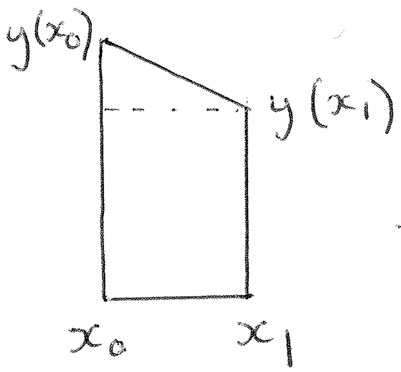
Integration = Finding area under the curve

Trapezium rule splits this area into several smaller pieces, the trapezia, the area of each of which is easily calculated.



$$\int = \text{Area } 1 + \text{Area } 2 + \text{Area } 3 .$$

Area of trapezium 1 :



$$\longrightarrow \quad \frac{1}{2}(x_1 - x_0) \cdot (y(x_0) - y(x_1))$$

$$+$$

$$\longrightarrow \quad (x_1 - x_0) \cdot y(x_1)$$

$$= \frac{1}{2}(x_1 - x_0)(y(x_0) + y(x_1))$$

Use more and smaller trapezia to get a more accurate answer.

Gaussian Quadrature

A more advanced, more accurate method for numerical integration

Quadrature → numerical integration.

Stated generally, to integrate a function, $f(x)$; the integral is transformed into a series summation

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{N} w_i f(x_i)$$

Note the limits of the integral are $-1, 1$. This is one reason the basis functions φ, were defined between $\xi = -1$ and $\xi = 1$.

Definition of the terms:

w_i = Gauss weights

x_i = Gauss points.

The values of w_i and x_i depend on N.

Important

Gaussian quadrature will integrate a polynomial of order $2N-1$ exactly.

For example,

$N=1 \Rightarrow 2N-1=1$

$\therefore$ G.Q. will integrate constant and linear functions exactly, ie:

$f(x)=a$

$f(x)=ax+b$

$N=2 \Rightarrow 2N-1=3$

$\therefore$ G.Q. will integrate constant, linear, quadratic, and cubic functions exactly i.e.

$f(x)=a$

$f(x)=ax+b$

$f(x)=ax^2+bx+c$

$f(x)=ax^3+bx^2+cx+d.$

N can be 3, 4, 5, ...

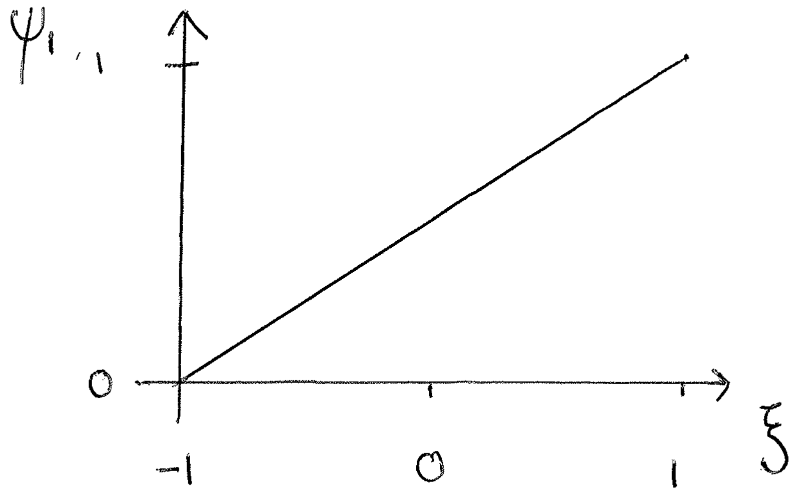
But what are w_i and x_i? How do we compute them?

Actually various formulae, involving the roots of complicated polynomials.

For the specific case of Gauss-Legendre Quadrature the values are as follows:

N	x_i	w_i
1	0	2
2	$-\sqrt{\frac{1}{3}}$	1
	$+\sqrt{1/3}$	1
3	$-\sqrt{3/5}$	$5/9$
	0	$8/9$
	$+\sqrt{3/5}$	$5/9$

A simple worked example using a linear function, ψ_1 i.e one of the basis functions we have been using:



$$\psi_1 = \left(\frac{1+\xi}{2}\right)$$

Using GQ for N=1

$$\int_{-1}^{1} \left(\frac{1+\xi}{2}\right) d\xi = \sum_{i=1}^{1} w_i \left(\frac{1+\xi_i}{2}\right)$$

$$= 2 \cdot \left(\frac{1+0}{2}\right) = \underline{\underline{1}} \checkmark$$

Using GQ for N=2

$$\int_{-1}^{1} \left(\frac{1+\xi}{2}\right) d\xi = \sum_{i=1}^{2} w_i \left(\frac{1+\xi_i}{2}\right)$$

$$= 1 \cdot \left(\frac{1-\sqrt{1/3}}{2}\right) + 1 \cdot \left(\frac{1+\sqrt{1/3}}{2}\right) = \frac{1}{2} + \frac{1}{2} = \underline{\underline{1}} \checkmark$$

Now lets apply the technique to the local element matrix integrals that were introduced in Lecture 6.

$$\text{Int}_{mn} = \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \cdot \frac{d\xi}{dx} \cdot \frac{d\psi_m}{d\xi} \cdot \frac{d\xi}{dx} \cdot J \, d\xi$$

For specific case $n = m = 0$.

$$\text{Int}_{00} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \cdot \frac{d\xi}{dx} \cdot \frac{d\psi_0}{d\xi} \cdot \frac{d\xi}{dx} \cdot J \, d\xi$$

Element size specific

In our 1D mesh $\frac{d\xi}{dx} = \frac{1}{J}$

$$\Rightarrow \int_{-1}^{1} D \cdot \frac{d\psi_0}{d\xi} \cdot \frac{d\psi_0}{d\xi} \cdot \frac{1}{J} \, d\xi$$

$$\frac{d\psi_0}{d\xi} = \frac{-1}{2} \qquad \therefore \text{Int}_{00} = \int_{-1}^{1} \frac{-1}{2} \cdot \frac{-D}{2J} \, d\xi = \int_{-1}^{1} \frac{D}{4J} \, d\xi$$

Using GQ with $N = 1$:

$$\text{Int}_{00} = \frac{D}{4J} \cdot 2 = \frac{D}{2J}$$

Using G.Q with $N=2$.

$$\text{Int}_{00} = 1 \cdot \left(\frac{D}{4J}\right) + 1 \cdot \left(\frac{D}{4J}\right) = \frac{D}{2J}$$

This is same as GQ with $N=1$, as expected.

Writing integrals as a sum of weighted function values is ideal for implementation as computer code. See lecture slides for some pseudo-code!