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ME40064: System Modelling & Simulation
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Extra information: L2 norms

When computing an L2 norm for a transient problem, such as in the second coursework assignment, the process is not quite as straightforward as for static problems. Here are some pointers to keep in mind:

1. Don't forget to include the Jacobian when calculating the integrals for the L2 norm. Also remember to take the square root at the end of your calculation, otherwise the gradient of your log-log plot will not be the value you expect.

   The elemental contribution to the L2 norm is:

$$\int_{-1}^{1} (C_E(x) - C(x))^2 J d\xi = \sum_{i=1}^{N} w_i (C_E(x(\xi_i)) - C(\xi_i))^2 J$$

2. When you are computing the L2 norm, you are using basis function interpolation in two ways to obtain the approximate and exact solution at a Gauss point.
   a. To calculate the FEM solution at the Gauss point, interpolate directly using the basis functions and the nodal values of the solution, that is:

$$C(\xi_i) = c_0 \left(\frac{1-\xi_i}{2}\right) + c_1 \left(\frac{1+\xi_i}{2}\right)$$

   b. To calculate the exact, i.e. analytical, solution at the Gauss point, you must first determine the corresponding *x* coordinate of that Gauss point in that element. Do this by using basis function interpolation on the nodal position values, that is:

$$x(\xi_i) = x_0 \left(\frac{1-\xi_i}{2}\right) + x_1 \left(\frac{1+\xi_i}{2}\right)$$

   Then substitute this value of *x* into the analytical solution function to determine the exact value at that Gauss point:

$$C_E(x(\xi_i))$$

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   c. The difference between the values interpolated in (a) and (b) is the error at that Gauss point.

3. The error in the finite element solution for a given time value is now dependent on both the mesh element size and the time step size. Given that the L2 norm theory presented in the lecture covered only the spatial discretisation error, it is necessary to ensure that the temporal discretisation error is small compared to the spatial error, in order to obtain the correct convergence rate with element size.

4. This time step selection will need to be much smaller than that suggested by the rule-of-thumb in Lecture 13 (which is based only on stability criterion), and will need to be small enough for the finest mesh resolution that you plan to include in your convergence study. Otherwise you may find that the convergence rate is correct for the first few meshes but then starts to diverge.

5. In the tutorial exercise for L2 norms, a gradient of exactly equal to -2 could be obtained, as the function to be approximated was quadratic. In general, one does not obtain exactly these gradients and this will be true for these temporal problems. If you can obtain a gradient that is close to the expected value for the particular basis functions, then this is sufficient.

6. Ensure you are using a sufficiently accurate quadrature scheme – I would suggest trying at least N=3. You may wish to explore if using N=4 or 5 makes an appreciable difference to your convergence results.

7. The precise results you obtain may depend on which time point in the solution you choose to perform the L2 norm calculation for. In these first few time steps there is more of an appreciable error in the solution due to the initial transients arising from a zero initial condition that suddenly has a non-zero Dirichlet BC imposed at one end. The closer the solution is to the steady state solution, the better the convergence plot is likely to be. You may wish to try the process for several different time points to see if the L2 norm and convergence

rate changes appreciably or not.