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2019/2020

ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation

Assignment 2: Transient MATLAB-Based FEM Modelling

Summary

In this coursework you must extend your finite element code to be able to solve the transient diffusion-reaction equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + \lambda c + f$$

You will then use this code to model the effects of heat damage to human skin, thereby being able to estimate the level of thermal shielding that a piece of protective clothing must provide.

Throughout this assignment it is assumed that more thorough answers to the questions will achieve a higher mark i.e. by exploring the effect of element size, time step, basis function polynomial order, and time integration scheme on the accuracy of the results and conclusions.

*Note that you may re-use your existing functions from Assignment 1 to calculate the local element matrices & vectors etc. Alternatively, you can re-write your code to use Gaussian quadrature. However, you should **not** use MATLAB's in-built functions for numerical or symbolic integration to calculate these matrices.*

Part 1: Software Development & Verification

Check that your code is working correctly by solving the following transient diffusion equation, for the domain defined from x = 0 to 1, and subject to the following initial and Dirichlet boundary conditions:

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

$$x = [0, 1], \quad c(x, 0) = 0, \quad c(0, t) = 0, \quad c(1, t) = 1$$

In order to achieve good accuracy in your solution, use an element size of 0.1 (i.e. a 10 element mesh) and a time step, $\Delta t = 0.01$.

Compare your results to the following analytical solution:

$$c(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin(n\pi x)$$

A MATLAB function, `TransientAnalyticSoln.m`, to compute this analytical solution is provided on Moodle.

Create the following two figures to demonstrate that your code is correct:

a) Plot your solution c(x) vs. x, showing the solutions at t = 0.05, 0.1, 0.3, 1.0, in the format shown in Lecture 13. **[30%]**

b) Plot both the analytical solution and your numerical solution at x=0.8, for t = 0 to 1.0, to demonstrate that your solution is converged. **[5%]**

Extra credit is available for the following tasks (if performed, be sure to include appropriate description and figures in your report).

Important: note that Part 2 of this assignment can be carried out to a high-level without using any of these advanced features.

- Investigating accuracy and stability of forward Euler, backward Euler and Crank-Nicolson time stepping methods **[5%]**
- Using Gaussian quadrature to evaluate element integrals **[5%]**
- Implementing & testing performance of quadratic basis functions **[5%]**
- Investigating errors & convergence using the L2 norm **[5%]**

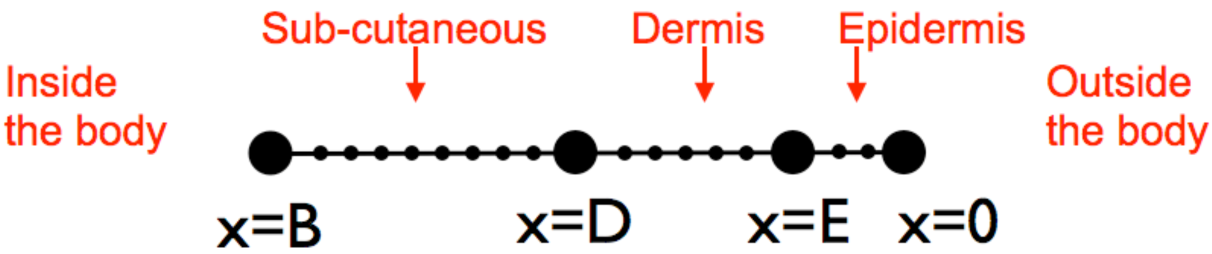
Part 2: Modelling & Simulation Results

1. Use your code to solve the tissue burn model from Lecture 14 **[15%]**.
   This is modelled using the equation:

$$\frac{\partial T}{\partial t} = \left(\frac{k}{\rho c}\right)\frac{\partial^2 T}{\partial x^2} - \left(\frac{G\rho_b c_b}{\rho c}\right)T + \left(\frac{G\rho_b c_b}{\rho c}\right)T_b$$

Note that you may find the backward Euler much faster than Crank-Nicolson for achieving stable results.

Your finite element mesh and material parameters must represent the three-layered structure of skin, as follows:



The parameter values for each layer are provided in Table 1. However, in this question we assume zero blood flow i.e. **G=0 everywhere**. The layers of the tissue are defined at the following x coordinates: **E=0.00166667, D=0.005, B=0.01.**

a) Run your code for the following initial conditions and Dirichlet boundary conditions, for a maximum of 50 seconds:

$$T(x,0) = 310.15K, \quad T(x=B,t) = 310.15K, \quad T(x=0,t) = 393.15K$$

Create a plot showing the spatial temperature distribution in the tissue for various points in time between 0 and 50 seconds. Explain the shape of the curves as they change through time based on your knowledge of the physics, material parameters, initial conditions, and boundary conditions.

b) Use these temperature profiles to determine whether tissue damage will occur or not. To do this, evaluate the following integral numerically at the mesh node for **x=E**:

$$\Gamma = \int_{t_{burn}}^{t} 2\times 10^{98} \exp\left(-\frac{12017}{(T-273.15)}\right) dt$$

The limits of the integral are $t=t_{burn}$, i.e. the time value at which the temperature T, at x=E, becomes greater than 317.15 Kelvin, and t=50.

If $\Gamma > 1$ at x=E, this produces a second-degree burn. State the value of Γ you have calculated at this node (don't be alarmed if it is very large or small), and hence state whether you expect a second-degree burn to occur.

Note that the MATLAB function, `trapz(x)`, uses the trapezium rule to integrate the vector x, assuming an interval of 1. Therefore, multiply the output of the function by your timestep, Δt, to obtain the final value of this integral.

2. Use your code to determine the minimum temperature reduction (to the nearest 0.5 degree K) that must be achieved by the protective clothing at the boundary x=0, in order to prevent a second-degree burn (as defined in Q1b).

   State the final Dirichlet boundary condition at x=0 that achieves this, and explain how you used your code, and any other calculations, to estimate this value. Include any data or figures that you feel are relevant in order to demonstrate your approach to estimating this. **[10%]**

3. Re-run your simulations for Questions 1 & 2, but now including the blood flow related reaction & source terms, using the parameter values in Table 1. How much does this change your results? Is blood flow an important effect to consider for future modelling of this problem? Finally, discuss how realistic you think the initial conditions, boundary conditions & model assumptions are. **[10%]**

Parameter	**Epidermis**	**Dermis**	**Sub-cutaneous**
k	25	40	20
G	0	0 (Q1a) 0.0375 (Q2)	0 (Q1a) 0.0375 (Q2)
ρ	1200	1200	1200
c	3300	3300	3300
ρ_b	-	1060	1060
c_b	-	3770	3770
T_b	-	310.15	310.15

Table 1: Parameter values for tissue & blood. Note that some are realistic and others are not, having been chosen to allow you to solve your model in a reasonable time frame.

Presentation [5%]
- Clear graphical presentation of equations, text, and results
- Quality of English, proper use of references, figures, and tables

Overall Code Quality [5%]
- Correctness, elegance, and readability, of MATLAB code
- Evidence of software verification and other good development practice

SUBMISSION GUIDELINES

You may structure your report as a set of answers to these questions – there is no requirement to write this in a lab report format. You also do not need to re-explain the entirety of the finite element method – a high-level overview of the method is sufficient. However, your report must not assume that the reader knows the content in this assignment document.

- You **must** include all your Matlab source code as **text** in the Appendices – failure to do so will cause you to lose marks. **Do not** paste your code into the document as an OLE item or as an image.
- **Do not** upload archived/zipped/compressed folders of these source files.
- **Do not** use MATLAB's symbolic algebra toolbox.
- Your code should use meaningful variable names and include comments, in line with good practice.
- Word limit of **2000** words (not including source code).

Submit your work using the online submission function on the unit's Moodle page.

Deadline: 4pm on Wednesday, 4th December 2019.