ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 11

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LECTURE 11
Error Analysis

- Understand conceptually the meaning of an L2-norm of an error
- Ability to compute L2-norm using Gaussian quadrature
- Knowledge of expected results for the linear basis approximation

ERROR ANALYSIS
Why Do We Need This?

- Perform convergence analysis as we increase our mesh resolution
- Understand fundamental accuracy of our numerical methods
- Use the error analysis to automatically perform localised mesh refinement
- Understand accuracy of a particular simulation
- For comparison with analytical solutions

QUANTIFYING THE ERROR
The L2-Norm

A norm is a measure of length/size of a vector that is always positive (or zero)

The L2 norm of the error:

$$E(x) = C_E(x) - C(x)$$

is effectively the Root Mean Square (RMS) of the error:

$$||E||_{L_2} = \left[\int_\Omega E^2(x)dx\right]^{1/2}$$

QUANTIFYING THE ERROR
Calculating The L2-Norm

As always, integrate this quantity in each element, using the standard element domain for the integral, using Gaussian quadrature:

not represented on the FEM mesh
i.e. a specified analytical function

$$\int_{-1}^{1} (C_E(x) - C(x))^2 J d\xi = \sum_{i=1}^{N} w_i (C_E(x(\xi_i)) - C(\xi_i))^2 J$$

where x is represented by x_0, x_1 local to the element:

$$x(\xi_i) = x_0 \psi_0(\xi_i) + x_1 \psi_1(\xi_i)$$

QUANTIFYING THE ERROR
Calculating The L2-Norm

As always, integrate this quantity in each element, using the standard element domain for the integral, using Gaussian quadrature:

$$\int_{-1}^{1} (C_E(x) - C(x))^2 J d\xi = \sum_{i=1}^{N} w_i (C_E(x(\xi_i)) - C(\xi_i))^2 J$$

where C is represented by c_0, c_1 local to the element

$$C(\xi_i) = c_0 \left(\frac{1 - \xi_i}{2} \right) + c_1 \left(\frac{1 + \xi_i}{2} \right)$$

QUANTIFYING THE ERROR

Calculating The L2-Norm

As always, integrate this quantity in each element, using the standard element domain for the integral, using Gaussian quadrature:

$$\int_{-1}^{1} (C_E(x) - C(x))^2 J d\xi = \sum_{i=1}^{N} w_i (C_E(x(\xi_i)) - C(\xi_i))^2 J$$

Sum these integrals for all elements, then take the square root to give the L2 norm for the entire domain

PLOTTING THE ERROR
A Mesh Convergence Graph

Given that we use a linear mesh to represent the solution, can write:

$$C_E(x) = \underbrace{C(x)}_{\text{Linear}} + O(h^2)$$

Terms of h^2 and higher: where h is the element size

Take natural log of each side:

$$\ln(C_E(x) - C(x)) = \ln(E(x)) = \ln(O(h^2))$$

Writing the higher order terms as:

$$O(h^2) = Gh^2$$

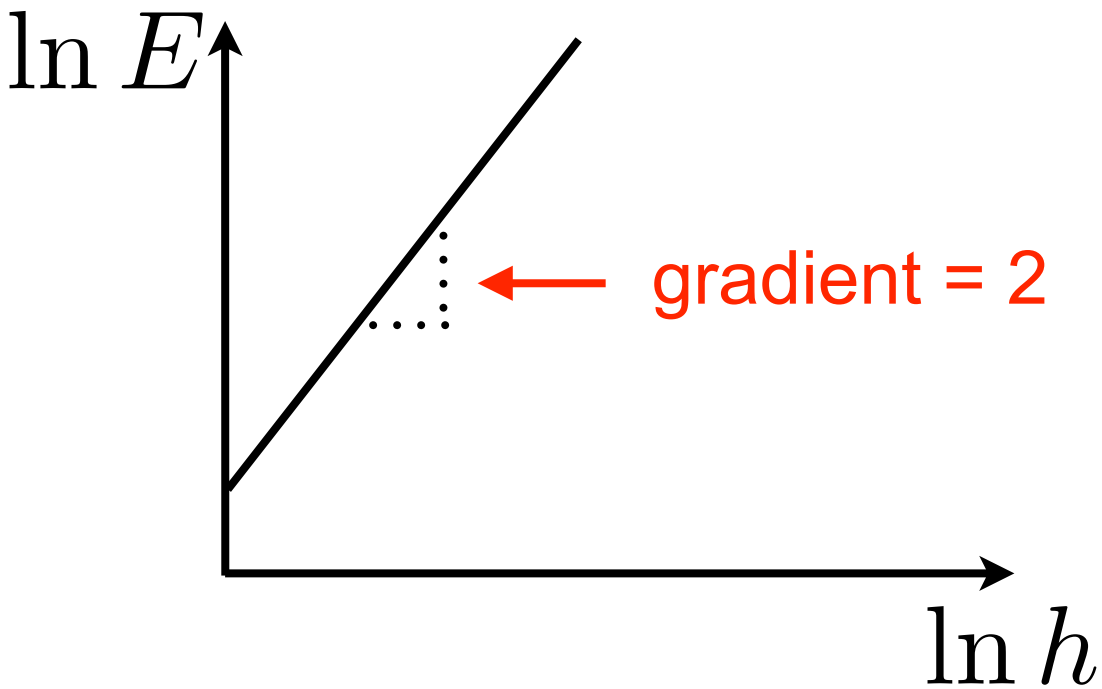
where G is a constant

PLOTTING THE ERROR
A Mesh Convergence Graph

Produces the following equation:

$$\ln(E(x)) = \ln G + 2\ln h$$

Plotting this gives a straight line with a gradient of 2:



When comparing against an analytical solution, <u>for a linear mesh</u>, should observe this feature