ME40064: Systems Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 5

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- Able to convert Poisson’s equation into weak form suitable for FEM
- Understand basic concept of discretisation of equations in FEM
- Understand the Galerkin assumption

REVIEW
Governing Equations

In lecture 3 we derived the transient advection-diffusion-reaction equation

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = D\nabla^2 c + \lambda c + f$$

To illustrate principles of FEM start with simpler version of this, Poisson's equation

$$D\nabla^2 c + f = 0$$

THE POISSON EQUATION
A 1D Formulation

Steady state diffusion with a source term

$$D\frac{\partial^2 c}{\partial x^2} + f = 0$$

This is the *strong* form of the equation

We will solve an integral or *weak* form

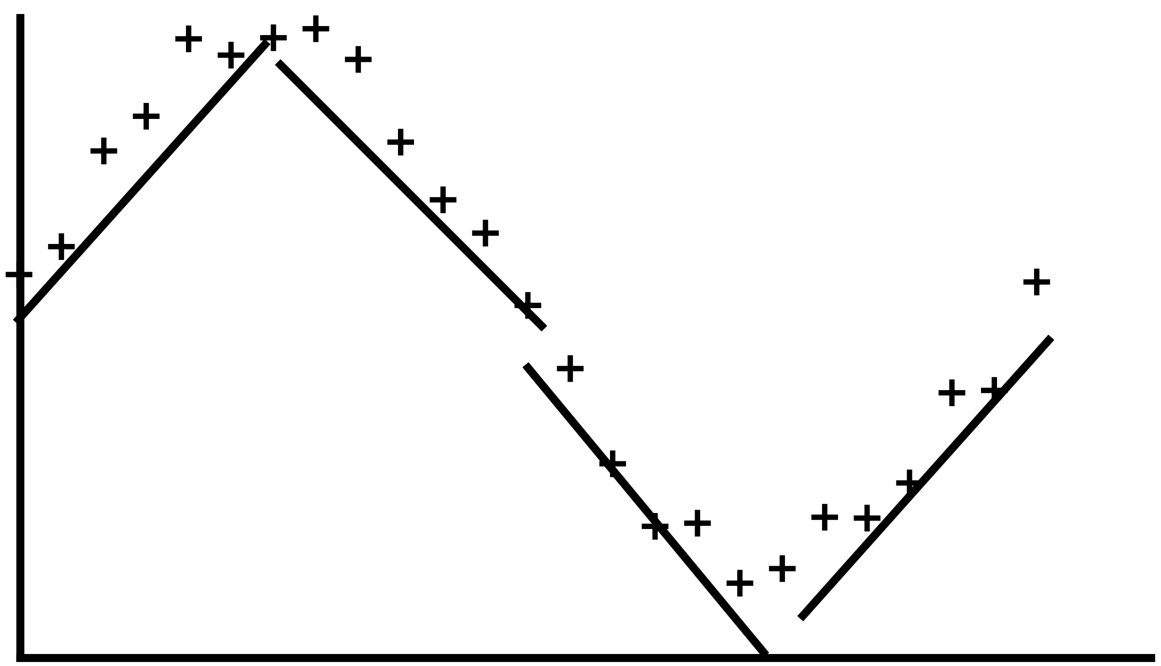
$$\int_\Omega v \left(D\frac{\partial^2 c}{\partial x^2} + f \right) dx = 0$$

where v is a weighting function and integral is defined over domain omega

WEAK FORM
What Is The Reasoning Behind It?

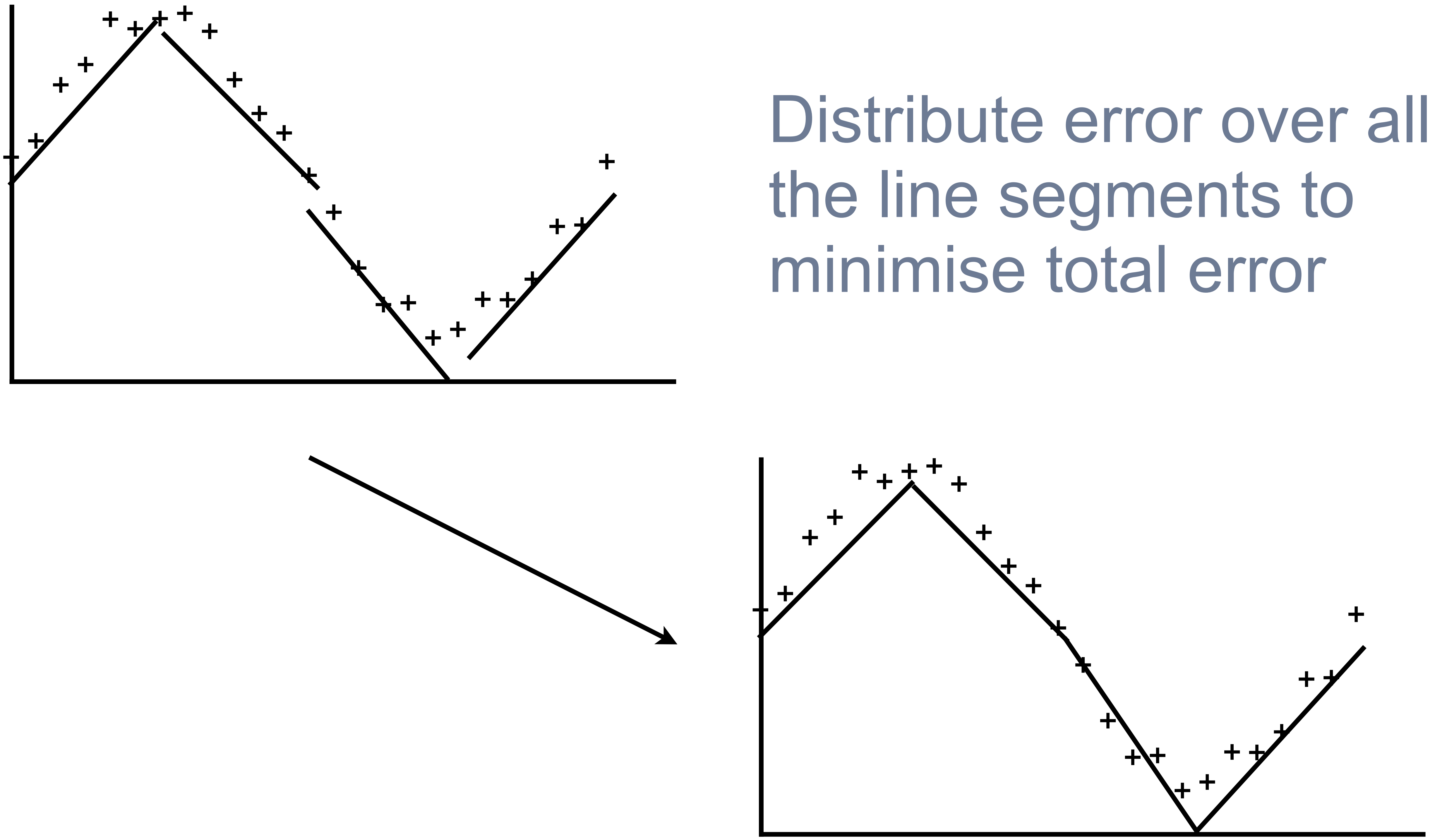
- Only solving equation in a “weak” sense
- Weighting the error distributes it over the domain
- Therefore spatially averaging the error to be zero
- Choice of weighting is very important

WEAK FORM
What Is The Reasoning Behind It?



WEAK FORM
What Is The Reasoning Behind It?

Distribute error over all the line segments to minimise total error



INTEGRATION BY PARTS

Formula for integration by parts

$$\int w(x)v'(x)dx = w(x)v(x) - \int w'(x)v(x)dx$$

Apply this to weak form equation

$$\int_\Omega D\frac{\partial v}{\partial x}\frac{\partial c}{\partial x}dx = \int_\Omega vf\,dx + \left[vD\frac{\partial c}{\partial x}\right]_{\partial\Omega_0}^{\partial\Omega_L}$$

2nd order operator is now linear, making the equation easier to solve

INTEGRATION BY PARTS
Incorporation Of Neumann Bcs

$$\int_{\Omega} D \frac{\partial v}{\partial x} \frac{\partial c}{\partial x} dx = \int_{\Omega} v f dx + \left[v D \frac{\partial c}{\partial x} \right]_{\partial \Omega_0}^{\partial \Omega_L}$$

INTEGRATION BY PARTS
Incorporation Of Neumann Bcs

$$\int_\Omega D \frac{\partial v}{\partial x}\frac{\partial c}{\partial x} dx = \int_\Omega v f dx + \boxed{\left[v D \frac{\partial c}{\partial x} \right]_{\partial\Omega_0}^{\partial\Omega_L}}$$

This term represents a flux at the boundary of the domain

- known as Neumann boundary conditions

This is the form of the equation that will be solved on the finite element mesh

GALERKIN ASSUMPTION
What And Why?

- First, we need appropriate choice of weighting function
- Galerkin showed that choosing weighting function to be same as that for the solution is “optimal” for convergence
- So what functions should we use?

BACK TO THE BASIS FUNCTIONS
Discretisation Of Galerkin Formulation

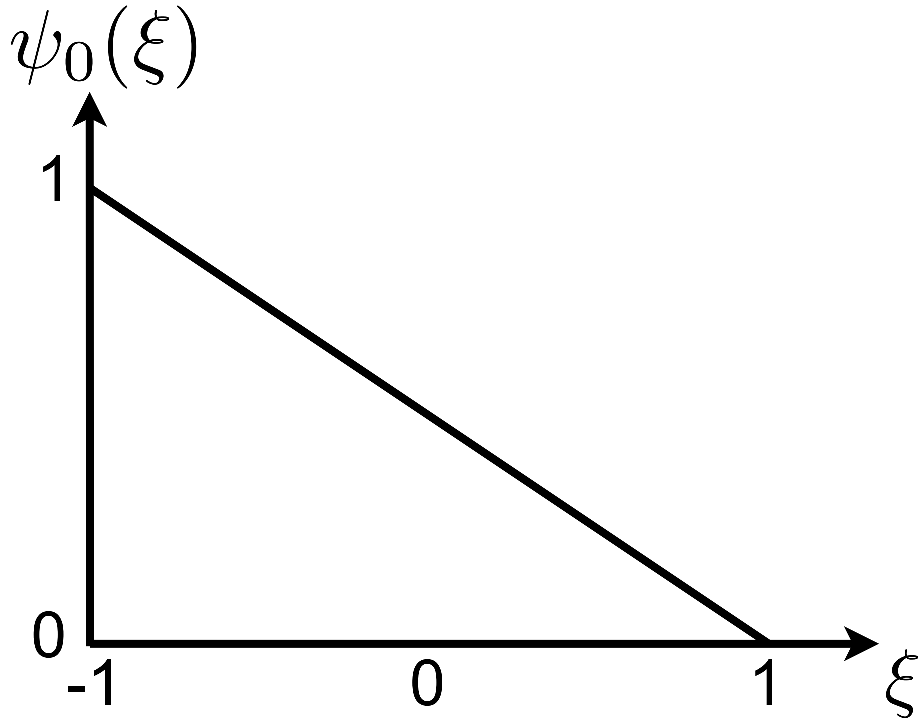
- Use linear Lagrange nodal basis functions to represent c and v
- c, representing solution = trial function
- v, representing the weighting = test function

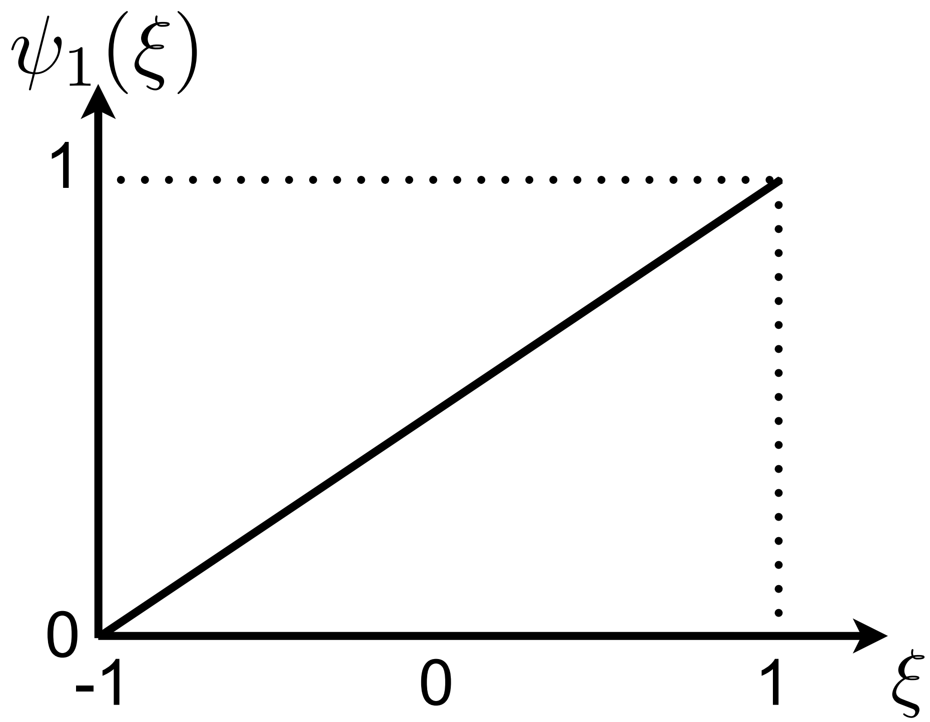
$$c = c_0\psi_0(\xi) + c_1\psi_1(\xi)$$

$$x = x_0\psi_0(\xi) + x_1\psi_1(\xi)$$

$$\xi = 2\left(\frac{x - x_0}{x_1 - x_0}\right) - 1$$

BASIS FUNCTIONS
Linear Nodal Lagrange





$$\psi_0(\xi) = \begin{cases} \frac{1-\xi}{2} & \xi \in \Omega_{st} \\ 0 & \xi \notin \Omega_{st} \end{cases} \qquad \psi_1(\xi) = \begin{cases} \frac{1+\xi}{2} & \xi \in \Omega_{st} \\ 0 & \xi \notin \Omega_{st} \end{cases}$$

Use sum of two linear functions to represent each linear segment in the line fitted to the data

BACK TO THE BASIS FUNCTIONS
Moving From One Element To Three

Now equation has been discretised, need to discretise the domain into multiple elements:

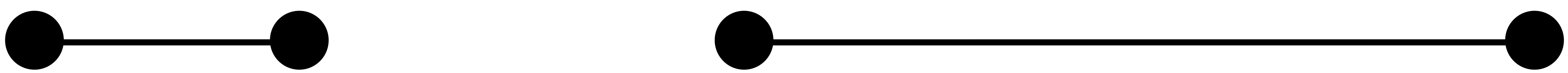
$$\int_0^1 v f\,dx = \int_0^{1/3} v f\,dx + \int_{1/3}^{2/3} v f\,dx + \int_{2/3}^{1} v f\,dx$$

$$\boxed{\int_{1/3}^{2/3} v f\,dx = \int_{-1}^{1} v f J\,d\xi}$$

BACK TO THE BASIS FUNCTIONS
Mapping Of Local To Standard Element

$$\int_{1/3}^{2/3} v f \, dx = \int_{-1}^{1} v f J \, d\xi$$

$x = 1/3$ $\quad x = 2/3$ $\qquad \xi = -1$ $\quad \xi = 1$



local element

standard element

Jacobian is the transformation between coordinates of local and standard elements

$$J = \left| \frac{dx}{d\xi} \right|$$