ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 7

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LECTURE 7
FEM: Assembly Of Global Matrix

- Understand link between nodes, elements and element matrices
- Ability to assemble global matrix and vector, including source terms
- Ability to evaluate a simple global matrix

REVISION
The Local Element Matrix

$$\int_{-1}^{1} D \frac{\partial c}{\partial x} \frac{\partial v}{\partial x} J d\xi \longrightarrow c_n \int_{-1}^{1} D \frac{d\psi_n}{d\xi} \frac{d\xi}{dx} \frac{d\psi_m}{d\xi} \frac{d\xi}{dx} J d\xi$$

$$Int_{00} = \int_{-1}^{1} D \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} \frac{d\psi_0}{d\xi} \frac{d\xi}{dx} J d\xi$$

Evaluating for n,m = 0,1 produces four integrals, assembled into matrix form:

$$\begin{bmatrix} Int_{00} & Int_{01} \\ Int_{10} & Int_{11} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

REVISION
The Local Element Matrix

$$\int_{-1}^{1} D \frac{\partial c}{\partial x}\frac{\partial v}{\partial x} J d\xi$$

$\xi = -1$ $\xi = 1$

standard element

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$x = 0$ $x = 1/3$

local element: e1

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$x = 1/3$ $x = 2/3$

local element: e2

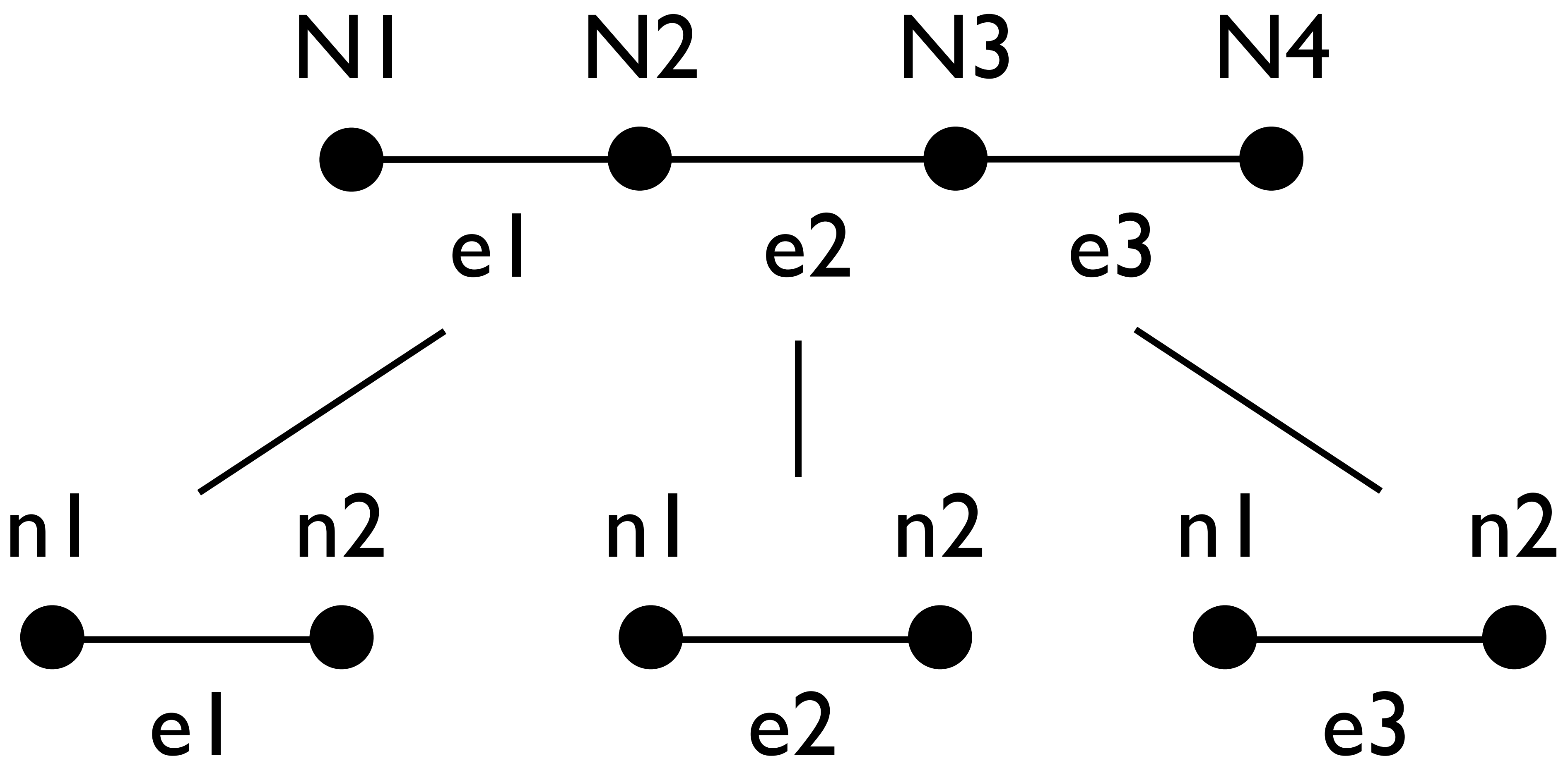
$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$$

$x = 2/3$ $x = 1$

local element: e3

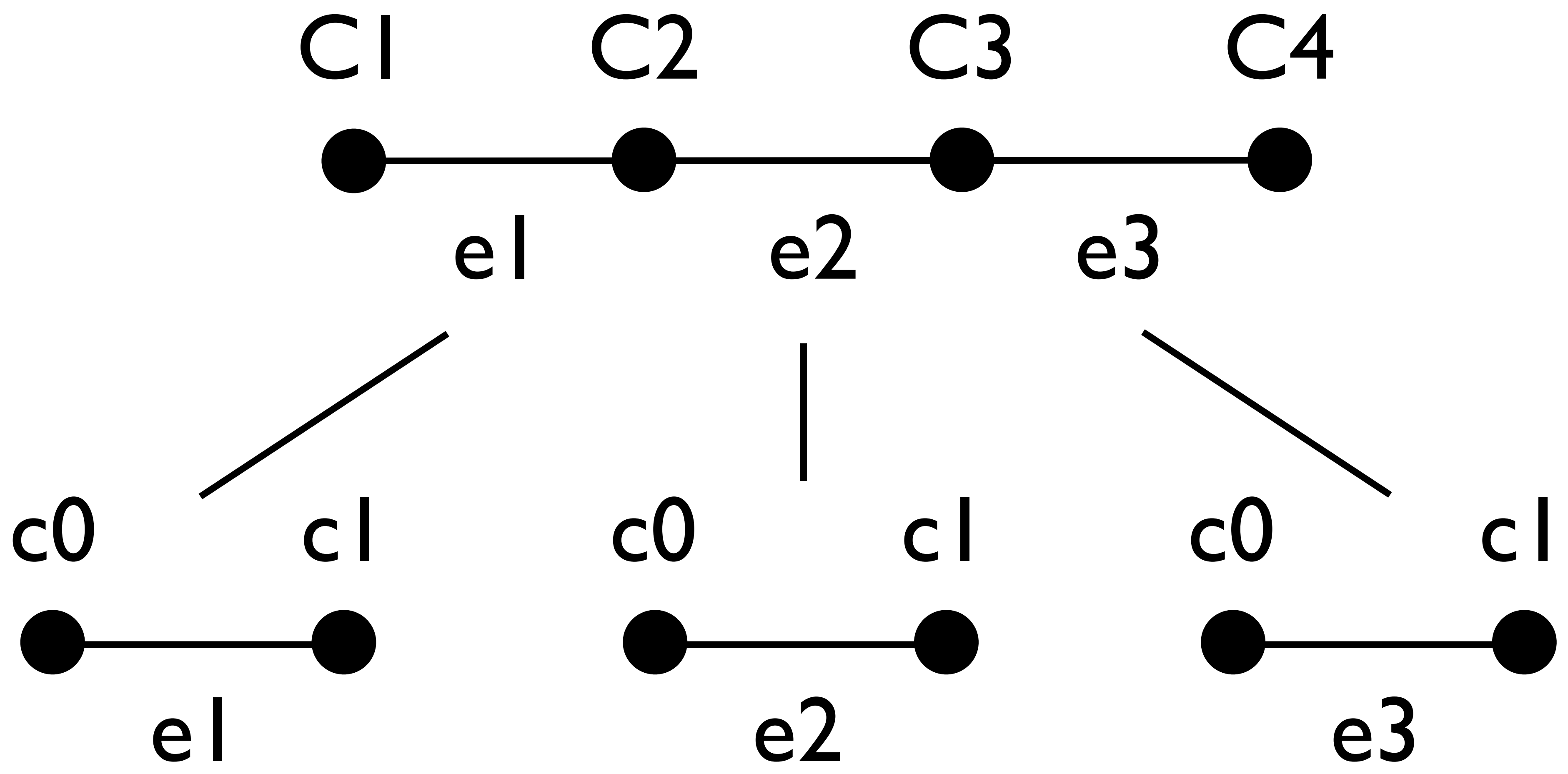
REVISION
Local And Global Mesh Nodes

These are not the same, c0 and c1, they are local to each element, assigned to the local nodes



ASSEMBLING GLOBAL MATRIX
Local And Global Solution Nodes

We have enforced continuity in our solution and write global solution nodes vs local solution:



ASSEMBLING GLOBAL MATRIX
Local And Global Solution Nodes

Identify the following links:

e1: c0 = C1

e1: c1 = e2:c0 = C2

e2: c1 = e3:c0 = C3

e3: c1 = C4

Therefore local element matrices can be written:

e1: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

e2: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$ e3: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$

ASSEMBLING GLOBAL MATRIX
Global Solution Vector

Identify the following links:

e1: c0 = C1
e1: c1 = e2:c0 = C2
e2: c1 = e3:c0 = C3
e3: c1 = C4

$$\longrightarrow \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

Therefore local element matrices can be written:

e1: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

e2: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$ e3: $\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$

ASSEMBLY OF GLOBAL MATRIX

Solving Simultaneous Equations

Add together the rows of local element matrices that share a global node value

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

add together

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_2 \\ C_3 \end{bmatrix}$$

add together

$$\begin{bmatrix} 3D & -3D \\ -3D & 3D \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix}$$

ASSEMBLY OF GLOBAL MATRIX
Solving Simultaneous Equations

For the C2 matrix rows this gives:

$$-3DC_1 + 6DC_2 - 3DC_3$$

For the C3 matrix rows this gives:

$$-3DC_2 + 6DC_3 - 3DC_4$$

C1 and C4 rows are unchanged

ASSEMBLY OF GLOBAL MATRIX
Example For 3-Element Mesh

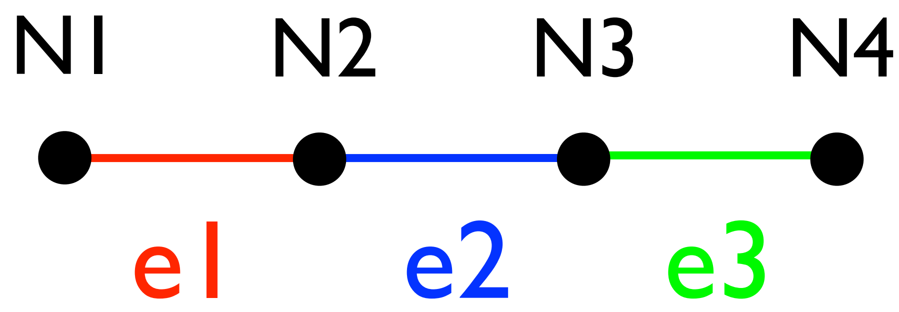
Putting this back into matrix form:

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix}$$

4 global nodes, therefore 4 by 4 global matrix

Note the number of zeros - these matrices are generally sparse

ASSEMBLY OF GLOBAL MATRIX
The Overlapping Nodes



$$\begin{bmatrix} \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

ASSEMBLY OF GLOBAL MATRIX
The Pseudo-Code

1. Create global matrix of zeros of size N_{nodes} by N_{nodes}
2. Loop over **elements** from 1 to $N_{elements}$

   2.i. For each element calculate local element matrices for diffusion operator & reaction term (if needed)

   2.ii. Sum the local element matrices to form a single 2-by-2 element matrix

   2.iii. **Add** this resultant element matrix into global matrix in the appropriate position

ASSEMBLY OF GLOBAL VECTOR

Back To Our Original Equation

$$\int_0^1 v\left(D\frac{\partial^2 c}{\partial x^2} + \lambda c + f\right)dx = 0$$

$$\int_0^1 D\frac{\partial v}{\partial x}\frac{\partial c}{\partial x}dx - \int_0^1 \lambda c v dx = \int_0^1 v f dx + \left[vD\frac{\partial c}{\partial x}\right]_0^1$$

Lecture 6: Diffusion term element matrix	Tutorial 3: Reaction term element matrix	Lecture 7: Source term element vector	Lecture 8: Boundary condition

THE LHS OF THE EQUATION
Integrating Sources Terms

Source term - integrate to form local element vector

$$\int_0^1 vf dx = \int_{-1}^1 vfJd\xi \qquad v = \psi_0, \psi_1$$

Integrate for the case that f is spatially constant:

$$Int_0 = \int_{-1}^1 \left(\frac{1-\xi}{2}\right) fJd\xi = \frac{fJ}{2}\left[\xi - \frac{\xi^2}{2}\right]_{-1}^1 = fJ$$

$$Int_1 = \int_{-1}^1 \left(\frac{1+\xi}{2}\right) fJd\xi = \frac{fJ}{2}\left[\xi + \frac{\xi^2}{2}\right]_{-1}^1 = fJ$$

ASSEMBLY OF GLOBAL VECTOR
Follow Principles Of Global Matrix

Add together vector rows that share a common node

$$\begin{bmatrix} fJ \\ fJ \end{bmatrix}$$

For the 3-element mesh, J = 1/6, therefore global vector is:

$$\begin{bmatrix} f/6 \\ f/6 + f/6 \\ f/6 + f/6 \\ f/6 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix}$$

ASSEMBLY OF GLOBAL VECTOR
Follow Principles Of Global Matrix

Once we have boundary conditions (Lecture 8) can solve this system:

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} BCs \end{bmatrix}$$