ME40064: System Modelling & Simulation
ME50344: Engineering Systems Simulation
Lecture 8

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University of Bath, 2019-20

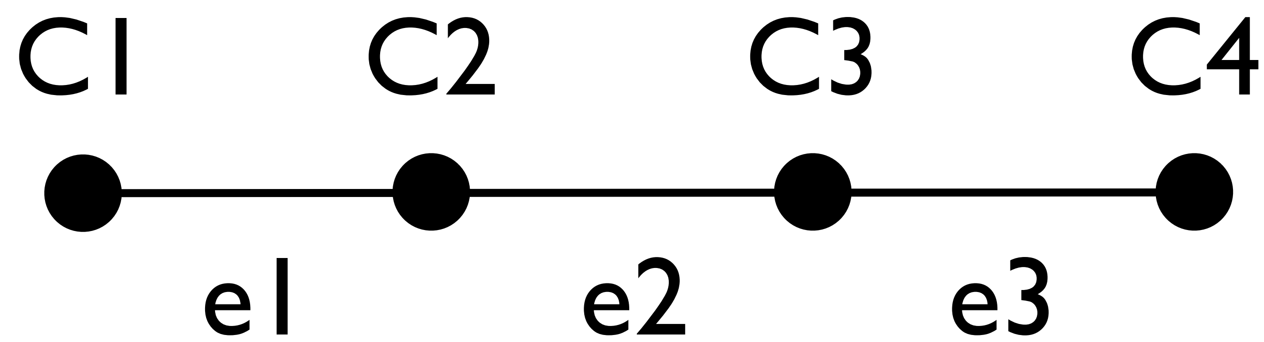
LECTURE 8
FEM: Boundary Conditions

- Understand physical interpretation of different BCs
- Able to implement these BCs in FEM formulation
- Able to solve FEM linear system

BOUNDARY CONDITIONS
Recap On Linear System

We want to impose boundary conditions on, and then solve, the following system:

$$D\frac{\partial^2 c}{\partial x^2} + f = 0$$



$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} BCs \end{bmatrix}$$

BOUNDARY CONDITIONS
Neumann Boundary Conditions

- represent a flux boundary condition
- e.g. a prescribed amount of the variable entering/leaving the domain - via Fickian diffusion:

$$D\frac{\partial c}{\partial x}$$

- naturally captured via the Galerkin method due to integration by parts

Dirichlet Boundary Conditions

Dirichlet conditions

- directly set value of solution, if this is known e.g. a set temperature
- using a value of 0 models a perfect sink
- for Laplace equation need at least one Dirichlet condition - why?

BOUNDARY CONDITIONS
A Note On Dirichlet Bcs

For Laplace equation (static diffusion), if only Neumann imposed, only restricting gradient of solution

- no unique solutions i.e. c(X) + constant, where constant can't be found
- Note: later in the course, when studying *transient* diffusion, this constraint is lifted

BOUNDARY CONDITIONS
Enforcing Dirichlet Bcs In Fem

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} BCs \end{bmatrix}$$

Want to impose the following boundary conditions:

$$c = 0 \quad at \quad x = 0$$

$$c = 1 \quad at \quad x = 1$$

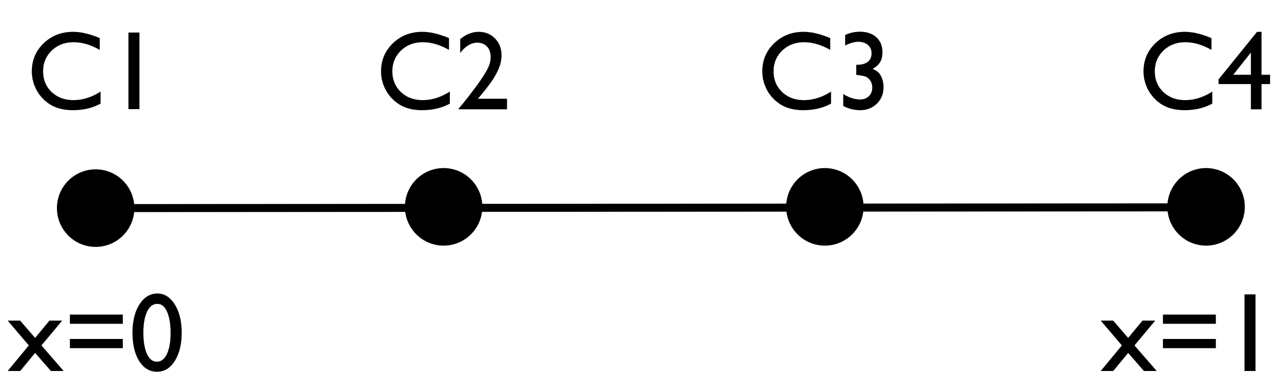
BOUNDARY CONDITIONS
Enforcing Dirichlet Bcs In Fem

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} BCs \end{bmatrix}$$

Want to impose the following boundary conditions:

$$c = 0 \quad at \quad x = 0 \longrightarrow node: \quad C_1$$

$$c = 1 \quad at \quad x = 1 \longrightarrow node: \quad C_4$$



BOUNDARY CONDITIONS
Enforcing Dirichlet Bcs In Fem

$$\begin{bmatrix} \cancel{3D} & \cancel{-3D} & \cancel{0} & \cancel{0} \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ \cancel{0} & \cancel{0} & \cancel{-3D} & \cancel{3D} \end{bmatrix} \begin{bmatrix} \cancel{C_1} \\ C_2 \\ C_3 \\ \cancel{C_4} \end{bmatrix} = \begin{bmatrix} \cancel{f/6} \\ f/3 \\ f/3 \\ \cancel{f/6} \end{bmatrix} + \begin{bmatrix} BCs \end{bmatrix}$$

1. Set matrix rows to zero that correspond to Dirichlet boundary nodes

2. Set those rows of the RHS vector equal to the boundary condition value

3. Set the diagonal entry of the BC matrix rows to 1

This produces the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ f/3 \\ f/3 \\ \underline{1} \end{bmatrix}$$

The altered rows effectively enforce the following:

$$1 \times C_1 + 0 \times C_2 + 0 \times C_3 + 0 \times C_4 = 0$$
$$0 \times C_1 + 0 \times C_2 + 0 \times C_3 + 1 \times C_4 = 1$$

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

Arise naturally from integration by parts of diffusion operator:

$$\int_0^1 v\left(D\frac{\partial^2 c}{\partial x^2} + \lambda c + f\right) dx = 0$$

$$\int_0^1 D\frac{\partial v}{\partial x}\frac{\partial c}{\partial x} dx - \int_0^1 \lambda c v dx = \int_0^1 v f dx + \left[v D\frac{\partial c}{\partial x}\right]_0^1$$

Neumann BC

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} \\ BCs \\ \\ \end{bmatrix}$$

Want to impose the following boundary conditions:

$$D\frac{\partial c}{\partial x} = 1 \quad at \quad x = 0$$ Neumann BC

$$c = 1 \quad at \quad x = 1$$ Dirichlet BC

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

Therefore need to evaluate this term:

$$\left[D\frac{\partial c}{\partial x}v\right]_0^1$$

$$=\left[\left(D\frac{\partial c}{\partial x}|_{x=1}.v(x=1)\right)-\left(D\frac{\partial c}{\partial x}|_{x=0}.v(x=0)\right)\right]$$

Due to the shape of the basis functions, we know that:

$$v(x=1)=\psi_1(x=1)=1$$

$$v(x=0)=\psi_0(x=0)=1$$

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

Rest of the terms are specified as our BC:

$$= \left[\left(D\frac{\partial c}{\partial x}|_{x=1} . v(x=1) \right) - \left(D\frac{\partial c}{\partial x}|_{x=0} . v(x=0) \right) \right]$$

Writing in the vector form to add to our system:

$$\begin{bmatrix} -D\frac{\partial c}{\partial x}|_{x=0} \\ 0 \\ 0 \\ D\frac{\partial c}{\partial x}|_{x=1} \end{bmatrix}$$

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

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$$= \left[\left(D\frac{\partial c}{\partial x}|_{x=1}.v(x=1) \right) - \left(D\frac{\partial c}{\partial x}|_{x=0}.v(x=0) \right) \right]$$

For our particular example:

Note the sign

$$\begin{bmatrix} -D\frac{\partial c}{\partial x}|_{x=0} \\ 0 \\ 0 \\ D\frac{\partial c}{\partial x}|_{x=1} \end{bmatrix} \longrightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This vector is added to the source vector, f

BOUNDARY CONDITIONS
Enforcing Neumann Bcs In Fem

$$\begin{bmatrix} 3D & -3D & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & -3D & 3D \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} f/6 \\ f/3 \\ f/3 \\ f/6 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

1. Neumann BC now enforced at node 1
2. Proceed with enforcing Dirichlet BC as before

System with 2 Dirichlet boundary conditions applied:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3D & 6D & -3D & 0 \\ 0 & -3D & 6D & -3D \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ f/3 \\ f/3 \\ 1 \end{bmatrix}$$

Written in a compact notation:

$$M\mathbf{c} = \mathbf{f}$$

Fem: Solution Process In Matlab

For the linear system:

$$M\mathbf{c} = \mathbf{f}$$

To find solution vector c, multiply by inverse of the global matrix, M:

$$\mathbf{c} = M^{-1}\mathbf{f}$$

In Matlab, this can be done by:

```
c = M\f;
```

BOUNDARY CONDITIONS
Fem: Solving The Linear System Ii

Note that for large finite element meshes, specialised matrix solvers such as PETSc are needed to solve these linear systems, particularly for parallel processors

For this course, Matlab is perfectly adequate

SOLVING THE LINEAR SYSTEM
Fem: An Example Solution

Source term, f, set to zero, and D, set to 1, to give final system:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 \\ 0 & -3 & 6 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Solving in Matlab gives:

$$\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.3333 \\ 0.6667 \\ 1.000 \end{bmatrix}$$

SOLVING THE LINEAR SYSTEM
Fem: An Example Solution

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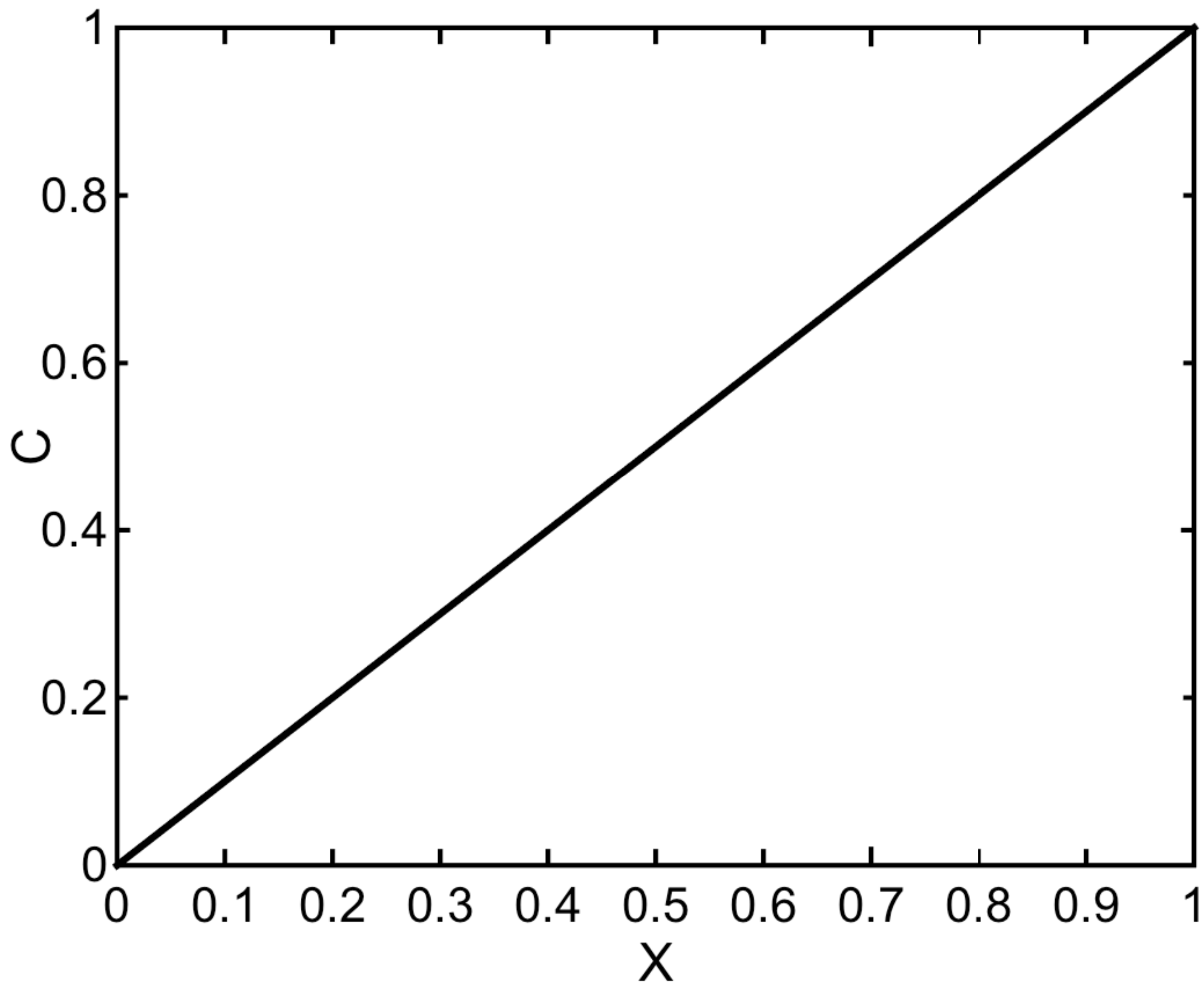
$$c = 0 \quad at \quad x = 0$$

$$c = 1 \quad at \quad x = 1$$

Dirichlet conditions correctly enforced

Fem: An Example Solution

Plotting this solution for the whole mesh:



SOLVING THE LINEAR SYSTEM
Fem: An Example Solution

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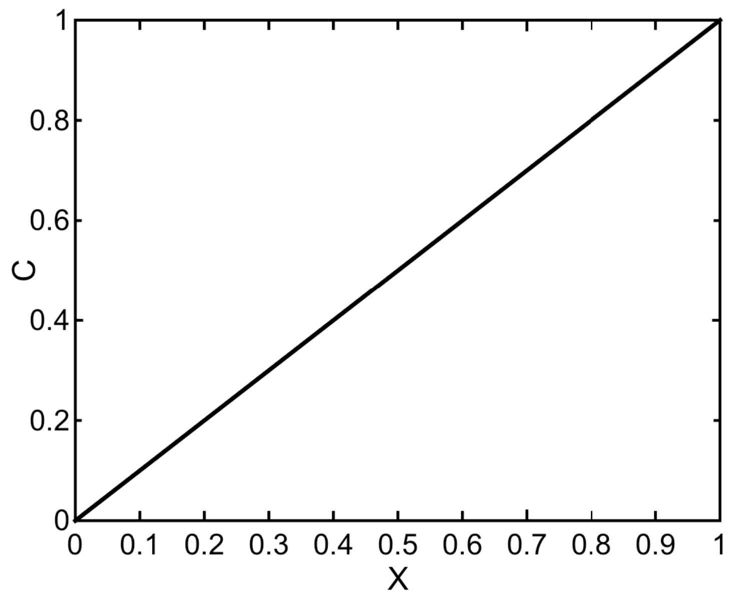

- vector, c, contains values of the solution at the global nodes

- interpolate in between nodes using basis function representation

SOLVING A STATIC FEM PROBLEM
The Pseudo-Code

1. Initialise mesh
2. Define material coefficients
3. Create & initialise the global matrix and vector to zero
4. Loop over elements: calculate local element matrices and add to correct location in global matrix
5. Loop over elements: calculate local element vectors and add to correct location in global vector
6. Apply boundary conditions to global matrix and/or vector
7. Solve the final matrix system
8. Plot the solution vector

COURSEWORK ASSIGNMENT ONE
Tips For Writing Up Your Results

1. Do not screen-grab your Matlab results - export the graphs using the command: `saveas(h,'filename.ext')`.
2. Make sure all plot titles, labels, and legends are in a large enough font size - the Matlab default is usually too small
3. Do not assume the reader knows what each plot/figure is - imagine you are writing it for a third party, not the person who set the questions (i.e. me).
4. Explain the meaning of your results, as well as their accuracy.
5. When deciding which plots to include, think about it in terms of evidence; what does the reader need to see in order to believe your conclusions?