A26

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a)

$$E(\hat{\mu}) = E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n} n\mu = \mu$$

b)

$$Var(\bar{x}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i,j=1}^{n} Cov(x_i, x_j) \underset{\text{unkorreliert}}{=} \frac{1}{n}\sum_{i=1}^{n} Var(x_i) = \frac{1}{n}\sigma^2$$

c)

$$E(S_0^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right) = \frac{1}{n}\sum_{i=1}^{n} E((x_i-\mu)^2) = \frac{1}{n}\sum_{i=1}^{n}\sigma^2 = \sigma^2$$

d)

$$E(S_n'^2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) = \frac{1}{n}E\left(\sum (x_i-\mu+\mu-\bar{x})^2\right)$$

$$\underset{\text{bin. Formel}}{=} \frac{1}{n}E\left(\sum (x_i-\mu)^2 - 2(x_i-\mu)(\bar{x}-\mu) + (\bar{x}-\mu)^2)\right)$$

$$= \frac{1}{n}E\left(\sum (x_i-\mu)^2 - 2\sum(x_i-\mu)(\bar{x}-\mu) + \sum(\bar{x}-\mu)^2\right)$$

$$= \frac{1}{n}E\left(\sum (x_i-\mu)^2 - 2n(\bar{x}-\mu)(\bar{x}-\mu) + n(\bar{x}-\mu)^2\right)$$

$$= \frac{1}{n}E\left(\sum (x_i-\mu)^2 - n(\bar{x}-\mu)^2\right)$$

$$= \frac{1}{n}\sum E(x_i-\mu)^2 - n\,E(\bar{x}-\mu)^2)$$

$$= \frac{1}{n}(Var(x_i) - n\,Var(\bar{x}))$$

$$= \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2 \quad \rightarrow \text{Verzerrt}$$

=> Korrektur

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

Blatt 13

Aufgabe 27

$f(\varphi) = A_0 \cos(\varphi + \delta) = a_1 f_1(\varphi) + a_2 f_2(\varphi)$ mit $f_1(\varphi) = \cos\varphi$, $f_2(\varphi) = \sin\varphi$

a) $A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \cdots \\ f_1(x_2) & f_2(x_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$

$$A = \begin{pmatrix} \cos(0°) & \sin(0°) \\ \cos(30°) & \sin(30°) \\ \cos(60°) & \sin(60°) \\ \cos(90°) & \sin(90°) \\ \cos(120°) & \sin(120°) \\ \cos(150°) & \sin(150°) \\ \cos(180°) & \sin(180°) \\ \cos(210°) & \sin(210°) \\ \cos(240°) & \sin(240°) \\ \cos(270°) & \sin(270°) \\ \cos(300°) & \sin(300°) \\ \cos(330°) & \sin(330°) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \\ 0 & 1 \\ -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \\ -1 & 0 \\ -\sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \\ 0 & -1 \\ 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

b) Lösungsvektor $\hat{a}$ berechnen:

$\hat{\vec{a}} = (A^T A)^{-1} A^T \vec{y}$

$$A^T \cdot A = \begin{pmatrix} 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4} & 0 + \cancel{\frac{\sqrt{3}}{4}} + \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} + \cancel{\frac{\sqrt{3}}{4}} + \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} \\ 0 & 0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \overset{a}{6} & \overset{b}{0} \\ \overset{c}{0} & \overset{d}{6} \end{pmatrix}$$

$$(\underbrace{A^T \cdot A}_{B})^{-1} = \frac{1}{\det B} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}}_{C}$$

$\underset{2\times 2}{C} \cdot \underset{2\times 12}{A^T}$

$$\begin{pmatrix} \frac{1}{6} & 0 \\ 0 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \frac{1}{6} & \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 & -\frac{1}{12} & -\frac{\sqrt{3}}{12} & -\frac{1}{6} & -\frac{\sqrt{3}}{12} & -\frac{1}{12} & 0 & \frac{1}{12} & \frac{\sqrt{3}}{12} \\ 0 & \frac{1}{12} & \frac{\sqrt{3}}{12} & \frac{1}{6} & \frac{\sqrt{3}}{12} & \frac{1}{12} & 0 & -\frac{1}{12} & -\frac{\sqrt{3}}{12} & -\frac{1}{6} & -\frac{\sqrt{3}}{12} & -\frac{1}{12} \end{pmatrix}}_{D}$$

D 2×12

$\hat{\vec{a}} = \underset{2\times 12}{D} \cdot \underset{12 \times 1}{\vec{y}}$

$\vec{y} = (-0{,}032; 0{,}068; 0{,}031; -0{,}090; 0{,}010; 0{,}076; 0{,}005; -0{,}088; 0{,}057; 0{,}080; -0{,}041; -0{,}074)$

$$\hat{\vec{a}} = \begin{pmatrix} -0{,}019 \\ 0{,}0008 \end{pmatrix}$$

c) Kovarianzmatrix $V[\hat{a}]$, Fehler von a_1 und a_2, Korrelationskoeff bestimmen

$V[\hat{a}] = \sigma^2 (A^T A)^{-1}$ (aus Vorlesung) $\sigma = 0{,}011$

A: Designmatrix $(A^T A)^{-1}$ bereits in b) berechnet $\rightarrow \begin{pmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{pmatrix}$

$$\Rightarrow V[\hat{a}] = \begin{pmatrix} 11/6000 & 0 \\ 0 & 11/6000 \end{pmatrix}$$

$\frac{11}{6000} \approx 1{,}83 \cdot 10^{-3}$

$\sigma_{a_1} = \sqrt{11/60000} = \sigma_{a_2} \approx 0{,}043$

$\rho = \frac{\mathrm{cov}(a_1, a_2)}{\sigma_{a_1}\sigma_{a_2}} = \frac{0}{\sigma_{a_1}\sigma_2} = 0$ weil ~~cor~~ $V_{12} = V_{21} = 0$

d) A_0, δ und Fehler & Korrelation aus a_1 und a_2 berechnen

$f(\varphi) = A_0 \cos(\varphi + \delta) = A_0 \cos\varphi \cos\delta - A_0 \sin\varphi \sin\delta$

$a_1 = A_0 \cos\delta \qquad a_2 = -A_0 \sin\delta \qquad \Rightarrow a_1^2 + a_2^2 = A_0^2 \cos^2\delta + A_0^2 \sin^2\delta$

$A_0 = \frac{a_1}{\cos\delta} \qquad A_0 = \frac{-a_2}{\sin\delta} \qquad \Rightarrow \sqrt{a_1^2 + a_2^2} = A_0 = (a_1^2 + a_2^2)^{\frac{1}{2}}$

$\cos\delta = \frac{a_1}{A_0} \qquad \sin\delta = \frac{-a_2}{A_0} \qquad \tan\delta = \frac{\sin\delta}{\cos\delta} = \left(-\frac{a_2}{A_0}\frac{A_0}{a_1}\right) = -\frac{a_2}{a_1}$

$\Rightarrow \delta = \arctan\left(-\frac{a_2}{a_1}\right)$

$\sigma_{A_0} = \sqrt{\left(\frac{\partial A_0}{\partial a_1}\sigma_{a_1}\right)^2 + \left(\frac{\partial A_0}{\partial a_2}\sigma_{a_2}\right)^2} = \sqrt{\left(\frac{1}{2}(a_1^2 + a_2^2)^{-\frac{1}{2}} \cdot 2a_1\right)^2 \sigma_{a_1}^2 + \left(\frac{a_2}{(a_1 + a_2)^{\frac{1}{2}}}\right)^2 \sigma_{a_2}^2}$

$a_1 = -0{,}019 \qquad a_2 = 0{,}0008$

$\sigma_{A_0} = \sqrt{6{,}1663 \cdot 10^{-7} + 2{,}094 \cdot 10^{-12}} = 8{,}1627 \cdot 10^{-4}$

$\sigma_\delta = \sqrt{\left(\frac{a_2}{a_1^2 + a_2^2}\right)^2 \sigma_{a_1}^2 + \left(\frac{\cancel{a_1}}{a_2^2 + a_1^2}\right)^2 \sigma_{a_2}^2} = 0{,}104$