

Aufgabe 2.7

$$f(\psi) = A_0 \cos(\psi + \delta) \quad \hat{=} a_1 f_1(\psi) + a_2 f_2(\psi) \quad \text{mit } f_1(\psi) = \cos \psi, f_2(\psi) = \sin \psi$$

$$a) A = \begin{pmatrix} f_1(x_1) & f_2(x_1) & \dots \\ f_1(x_2) & f_2(x_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad A = \begin{pmatrix} \cos(0^\circ) & \sin(0^\circ) \\ \cos(30^\circ) & \sin(30^\circ) \\ \cos(60^\circ) & \sin(60^\circ) \\ \cos(90^\circ) & \sin(90^\circ) \\ \cos(120^\circ) & \sin(120^\circ) \\ \cos(150^\circ) & \sin(150^\circ) \\ \cos(180^\circ) & \sin(180^\circ) \\ \cos(210^\circ) & \sin(210^\circ) \\ \cos(240^\circ) & \sin(240^\circ) \\ \cos(270^\circ) & \sin(270^\circ) \\ \cos(300^\circ) & \sin(300^\circ) \\ \cos(330^\circ) & \sin(330^\circ) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \\ 0 & 1 \\ -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \\ -1 & 0 \\ -\sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \\ 0 & -1 \\ 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

b) Lösungsvektor \hat{a} berechnen:

$$\hat{a} = (A^T A)^{-1} A^T \vec{y}$$

$$A^T \cdot A =$$

$$\begin{pmatrix} 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4} & 0 + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\ 0 & 0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$$

$$(A^T \cdot A)^{-1} = \frac{1}{\det B} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{36} \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$$

$$E \cdot A^T$$

2x2 2x12

$$\begin{pmatrix} 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$$

D 2x12

$$\hat{a} = D \cdot \vec{y}$$

$$\vec{y} = (-0,032; 0,068; 0,031; -0,090; 0,010; 0,076; 0,005; -0,088; 0,057; 0,080; -0,041; -0,074)$$

$$\hat{a} = \begin{pmatrix} -0,019 \\ 0,0008 \end{pmatrix}$$

c) Kovarianzmatrix $V[\hat{a}]$, Fehler von a_1 und a_2 , Korrelationskoeff. bestimmen

$$V[\hat{a}] = \sigma^2 (A^T A)^{-1} \quad (\text{aus Vorlesung}) \quad \sigma = 0,011$$

A : Designmatrix $(A^T A)^{-1}$ bereits in b) berechnet $\rightarrow \begin{pmatrix} 1/6 & 0 \\ 0 & 1/6 \end{pmatrix}$

$$\Rightarrow V[\hat{a}] = \begin{pmatrix} 11/6000 & 0 \\ 0 & 11/6000 \end{pmatrix} \quad \frac{1}{6000} \approx 1,67 \cdot 10^{-4}$$

$$\sigma_{a_1} = \sqrt{116000}$$

$$\sigma_{a_2} \approx 0,043$$

$$\rho = \frac{\text{cov}(a_1, a_2)}{\sigma_{a_1} \sigma_{a_2}}$$

$$= 0 \quad \text{weil} \quad \text{cor } v_{12} = v_{21} = 0$$

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d) A_0 , δ und Fehler & Korrelation aus a_1 und a_2 berechnen

$$f(\psi) = A_0 \cos(\psi + \delta) = A_0 \cos \psi \cos \delta - A_0 \sin \psi \sin \delta$$

$$a_1 = A_0 \cos \delta$$

$$a_2 = -A_0 \sin \delta$$

$$A_0 = \frac{a_1}{\cos \delta}$$

$$A_0 = -\frac{a_2}{\sin \delta}$$

$$\cos \delta = \frac{a_1}{A_0}$$

$$\sin \delta = \frac{-a_2}{A_0}$$

$$\Rightarrow a_1^2 + a_2^2 = A_0^2 \cos^2 \delta + A_0^2 \sin^2 \delta$$

$$\Rightarrow \sqrt{a_1^2 + a_2^2} = A_0 = (a_1^2 + a_2^2)^{\frac{1}{2}}$$

$$\tan \delta = \frac{\sin \delta}{\cos \delta} = \left(\frac{-a_2}{A_0} \frac{A_0}{a_1} \right) = -\frac{a_2}{a_1}$$

$$\Rightarrow \delta = \arctan\left(-\frac{a_2}{a_1}\right)$$

$$\sigma_{A_0} = \sqrt{\left(\frac{\partial A_0}{\partial a_1} \sigma_{a_1}\right)^2 + \left(\frac{\partial A_0}{\partial a_2} \sigma_{a_2}\right)^2}$$

$$= \sqrt{\left(\frac{1}{2} (a_1^2 + a_2^2)^{-\frac{1}{2}} \cdot 2a_1\right)^2 \sigma_{a_1}^2 + \left(\frac{a_2}{(a_1^2 + a_2^2)^{\frac{1}{2}}}\right)^2 \sigma_{a_2}^2}$$

$$a_1 = -0,019 \quad a_2 = 0,0008$$

$$\sigma_{A_0} = \sqrt{61663 \cdot 10^{-7} + 2,094 \cdot 10^{-12}}$$

$$= 8,1627 \cdot 10^{-4}$$

$$\sigma_{\delta} = \sqrt{\left(\frac{a_2}{a_1^2 + a_2^2}\right)^2 \sigma_{a_1}^2 + \left(\frac{a_1}{a_1^2 + a_2^2}\right)^2 \sigma_{a_2}^2}$$

$$= 0,104$$