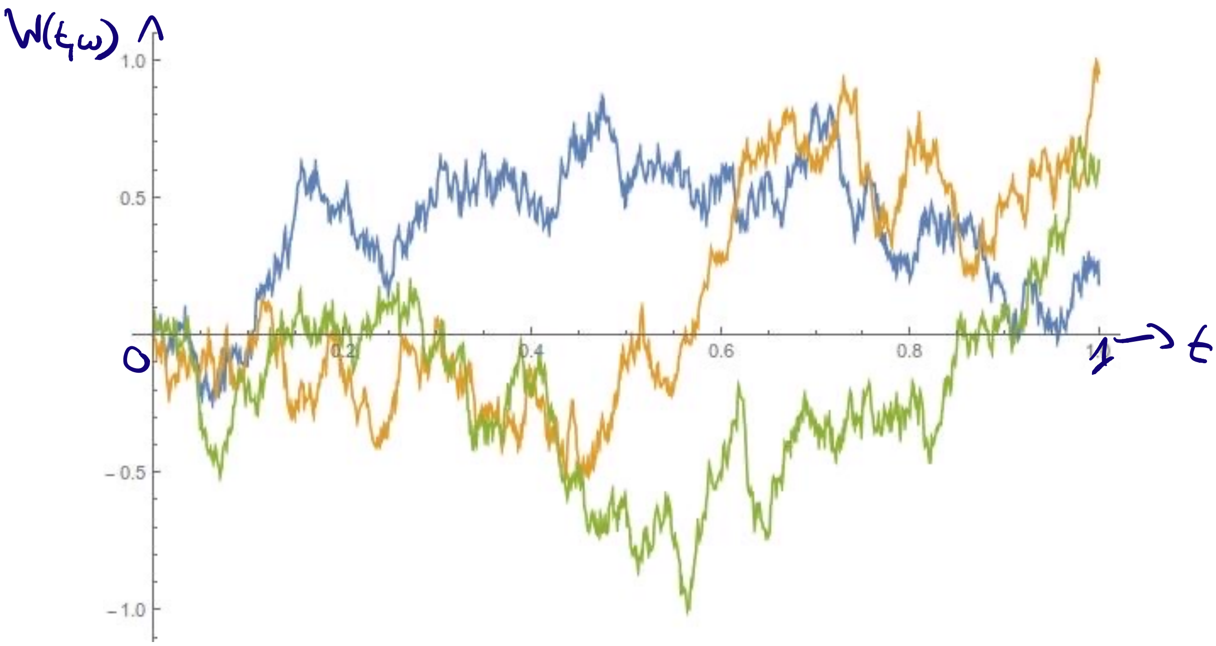
Brownian motion



Three sample paths $\{W(t,\omega) : t \in [0,1]\}$ of a Brownian motion.

Transition probability density (a.k.a. heat kernel)

$$p(t,y,x) := \frac{1}{\sqrt{2\pi t}} \, e^{\frac{-(y-x)^2}{2t}}, \quad t>0,\ x,y \in \mathbb{R}$$

This is, for fixed x, the p.d.f. of a random variable with distribution $\mathcal{N}(x,t)$ mean x, variance t

Obviously, $\int_{-\infty}^{\infty} p(t,y,x)\,dy = 1, \quad \forall t>0,\ x \in \mathbb{R}$

Exercise: $\frac{\partial}{\partial t} p(t,y,x) = \frac{1}{2} \frac{\partial^2}{\partial y^2} p(t,y,x)$ Heat equation.

Def: Wiener process/Brownian motion is a stochastic process

$$\mathbb{W} = \{W(t) : t \geq 0\} \text{ such that}$$

i.) $W(0) = 0$ almost surely (i.e. $\mathbb{P}(W(0)=0) = 1$.)

ii) for any $n \in \mathbb{N}$ and times $0 < t_1 < t_2 < \cdots < t_n$, and $x_1, x_2, \ldots, x_n \in \mathbb{R}$ the joint probability density of $W(t_1), W(t_2), \ldots, W(t_n)$ is

$$f_{W(t_1),W(t_2),\ldots,W(t_n)}(x_1,x_2,\ldots,x_n) = p(t_1,x_1,0)\cdot p(t_2-t_1,x_2,x_1) \times p(t_3-t_2,x_3,x_2)\cdots p(t_n-t_{n-1},x_n,x_{n-1})$$ //

Fact of life: Brownian motion exists, i.e. there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a stochastic process $\mathbb{W}$,

$$\mathbb{W} : [0,\infty) \times \Omega \to \mathbb{R}$$
$$(t,\omega) \mapsto W(t,\omega),$$

such that i.) and ii) hold.

Consequences: • Choose $n=1$ and $t_1=t$, then

$$W(t) \in \mathcal{N}(0,t) \quad , t>0 \quad (*)$$

• Choose $n=2$, $t_1=s<t_2=t$. Then the joint p.d.f of $(W(s),W(t))'$ is

$$f_{W(s),W(t)}(x,y) = p(\underline{s,x,0}) \cdot p(t-s,y,x)$$

marginal p.d.f. of $W(s)$: $f_{W(s)}(x)$

So

$$f_{W(t)|W(s)=x}(y) = \frac{f_{W(s)W(t)}(x,y)}{f_{W(s)}(x)} = p(t-s,y,x)$$

$$= \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \quad , t>s.$$

That means $W(t)/W(s)=x \in \mathcal{N}(x,t-s)$.

hence $E[W(t)/W(s)] \overset{(**)}{=} W(s)$, $t>s$.

- Let A be an interval on $\mathbb{R}$ (or any Borel set)

$$\mathbb{P}\left(W(t) \in A \mid W(s)=x\right) = \int_A p(t-s, y, x)\, dy$$

$t > s$

"Interpretation: transition probability density".

Thm: The Wiener process W is a Gaussian process.

Proof: See [TK].

Question: What is the autocorrelation function $R_W(t,s)$ of W?

Mean function $\mu_W(t) = \mathbb{E}[W(t)] \overset{(*)}{=} 0$

Lemma: $R_W(t,s) = \min(t,s)$ for $t,s \geq 0$.

Hints: Assume $t > s$: $R_W(t,s) = \mathbb{E}[W(t)\,W(s)] = \ldots$

- Total law of expectation, by conditioning on $W(s)$.
- Use (**).
- Use (*).