Gaussian processes

1.

Def: A stochastic process $X = \{X(t) : t \in T\}$ is called Gaussian, if every random vector

$$(X(t_1), X(t_2), \dots, X(t_n))'$$ is a

multivariate normal vector, for all n and all $t_1, t_2, \dots, t_n \in T$ //

In other words, the finite-dimensional distributions

$$F_{t_1, t_2, \dots, t_n} \in N\left(\vec{\mu}(t_1, t_2, \dots, t_n), \Lambda(t_1, \dots, t_n)\right)$$

mean vector

$$\vec{\mu}(t_1, t_2, \dots, t_n) = \left(E[X(t_1)], E[X(t_2)], \dots, E[X(t_n)]\right)'$$

Covariance matrix

$$\Lambda = (\lambda_{ij})_{ij=1}^{n} \text{ with } \lambda_{ij} = Cov_X(t_i, t_j)$$
$$= R_X(t_i, t_j) - \mu_X(t_i) \cdot \mu_X(t_j)$$

2.

Thm: Let $R(t,s)$ be a symmetric (i.e. $R(t,s)=R(s,t)$) and non-negative definite function

(i.e. $\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j R(t_i,t_j) \geq 0$ for all $x_1,\dots,x_n \in \mathbb{R}$, $t_1,\dots,t_n \in T$ and all $n \in \mathbb{N}$).

Then exists a Gaussian process $\underline{X}$ with $R(t,s)$ as its auto correlation function. //

Proof: See [TK].

"Meaning: A Gaussian process is uniquely characterized by its mean function and its autocorrelation function."

Lemma: A Gaussian process $\underline{X} = \{X(t) ; t \in \mathbb{R}\}$ is weakly stationary if and only if

$$(*) \quad \left(X(t_1+h), X(t_2+h), \dots, X(t_n+h) \right) \overset{d}{=} \left(X(t_1), X(t_2), \dots, X(t_n) \right)$$

holds for all $h \in \mathbb{R}$, $t_1, \dots, t_n \in \mathbb{R}$, and all n. //

Proof: Skipped.

Example: $\underline{X} = \{X(t) : t \in \mathbb{R}\}$ a weakly stationary Gaussian process that has mean zero and autocorrelation function

$$R_X(h) = e^{-\lambda |h|}, \quad \lambda > 0.$$

Find the distribution of $(X(t), X(t-1))'$ $\rightsquigarrow$ determine mean and covariance.

Mean: $E[X(t)] = E[X(t-1)] = 0$

Covariance: $Cov(X(t), X(t-1)) = E[X(t) X(t-1)] = R_X(1) = e^{-\lambda}$

$$Var(X(t)) = E[(X(t))^2] = R_X(0) = 1$$

$$\| $$

$(*)$ with $n=1$ $\quad Var(X(t-1))$

$$\Rightarrow \quad (X(t), X(t-1))' \in N\left(0, \begin{pmatrix} 1 & e^{-\lambda} \\ e^{-\lambda} & 1 \end{pmatrix}\right). \quad //$$

Example 2: $\underline{X} = \{X(t) : t \in \mathbb{R}\}$ weakly stationary Gaussian process with mean function zero, and auto correlation function

$$R_{\underline{X}}(h) = \frac{1}{1+h^2}.$$

Find the probability $\mathbb{P}(3X(1) > 1 - X(2))$.

Rewrite this as $\mathbb{P}(\underbrace{3X(1) + X(2)}_{=:Y} > 1)$

$$Y := 3X(1) + X(2) = \underset{1\times 2 \text{ matrix (row vector)}}{B} \begin{pmatrix} X(1) \\ X(2) \end{pmatrix} = (3, 1) \begin{pmatrix} X(1) \\ X(2) \end{pmatrix}$$

Clearly, $\mathbb{E}[Y] = 0$, Covariance $\Lambda_Y = \underset{1\times 2}{B} \underset{2\times 2}{\Lambda_X} \underset{2\times 1}{B'}$

But $\Lambda_X = \begin{pmatrix} R_X(0) & R_X(1) \\ R_X(1) & R_X(0) \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$. So $\Lambda_Y = (3, 1) \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 13$

$Y \in \mathcal{N}(0, 13)$

$$\mathbb{P}(3X(1) > 1 - X(2)) = \mathbb{P}(Y > 1) \overset{(**)}{=} \mathbb{P}\left(\frac{Y}{\sqrt{13}} > \frac{1}{\sqrt{13}}\right) = 1 - \underset{\text{cdf of } \mathcal{N}(0,1)}{\Phi}\left(\frac{1}{\sqrt{13}}\right) \quad //$$

(**) used $\frac{Y}{\sqrt{13}} \in \mathcal{N}(0, 1)$.