Stochastic processes

(1.)

So far, $(X_i)_{i=1}^n$ a collection of random variables, indexed by $i=1,\dots,n$.

"can think of as discrete time steps".

Def: A stochastic process is a family of random variables $X(t)$,

$$\mathbb{X} = \{ X(t) : t \in T \}$$

on a probability space $(\Omega, \cancel{\mathcal{A}}, \mathbb{P})$ where T is the index set of the process. //

Often $T = \mathbb{R}$ or $T = [0,\infty)$, but also $T = \mathbb{R}^2$ or $T = \mathbb{N}$.

3 ways of thinking about $\underline{X}$:

- For each fixed $t \in T$, $X(t)$ is a random variable $\Omega \to \mathbb{R}$.
- $\underline{X}$ is a (measurable) function

$$\underline{X} : T \times \Omega \to \mathbb{R}$$
$$(t, \omega) \longmapsto X(t, \omega).$$

- For each fixed $\omega \in \Omega$, $t \longmapsto X(t, \omega)$ is function of t called the sample path.

Examples:
- Brownian motion / Wiener process. → continuous in time.
- Poisson process → discrete increments.
- Let $f : [0, \infty) \to \mathbb{R}$ be deterministic, and Y a random variable, then

$$X(t, \omega) := f(t)\, Y(\omega), \quad T = [0, \infty)$$

is a stochastic process (if f is Borel measurable).

3.

- Let $\varphi \in U(0, 2\pi)$ and set

$$X(t) := A \cdot \sin(\omega \cdot t + \varphi) \quad , \; t \in \mathbb{R} = T.$$

A: constant, amplitude

ω: constant angular frequency

φ: random phase

"random wave"

"Phase shift keying"!

What about the distribution of $\underline{X}$?

Issue is that T is typically uncountable, e.g. $T=[0,\infty)$

Way out is to look at finite-dimensional distributions:

For each $n\in\mathbb{N}$ and times $t_1,t_2,\dots,t_n\in T$ we set

$$F_{t_1,t_2,\dots,t_n}(x_1,x_2,\dots,x_n) := \mathbb{P}(X(t_1)\le x_1, X(t_2)\le x_2,\dots, X(t_n)\le x_n)$$

for $x_1,x_2,\dots,x_n\in\mathbb{R}$.

$$\mathbf{F} = \{F_{t_1,t_2,\dots,t_n}\}_{(t_1,t_2,\dots,t_n)\in T^n,\, n\in\mathbb{N}}$$

Collection of all finite-dimensional distributions.

Notations: • Mean function of a stochastic process (5.)

$$\mu_X(t) := \mathbb{E}[X(t)] \qquad t \in T$$

deterministic, depend on one time.

• Variance function:

$$Var_X(t) := \mathbb{E}[(X(t))^2] - (\mu_X(t))^2, \quad t \in T$$

• Auto correlation function: $R_X(t,s)$, $t,s \in T$

$$R_X(t,s) := \mathbb{E}[X(t) \cdot X(s)]$$

deterministic depend on two "times" $s,t \in T$.

• Auto covariance function: $Cov_X(t,s)$, $t,s \in T$

$$Cov_X(t,s) := R_X(t,s) - \mu_X(t) \cdot \mu_X(s)$$

• A stochastic process is called weakly stationary if

1.) $\mu_X(t) = \text{constant} = \mu$

2.) $R_X(t,s)$ is a function of $|t-s|$ only, i.e.

$$R_X(t,s) = h(|t-s|) \text{ for some function } h. \quad //$$

Remark: To compute the autocorrelation function, it suffices to know $F_{t_1,t_2}(x_1,x_2)$

Lemma: $R_{\underline{X}}(t,s)$ is the autocorrelation function of a stochastic process $\underline{X} = \{X(t) : t \in T\}$ if and only if

1.) Symmetry: $R_X(t,s) = R_{\underline{X}}(s,t)$ $\forall t,s \in T$.

2.) Non-negative definiteness

$$\sum_{i,j=1}^{n} x_i \underbrace{R_{\underline{X}}(t_i,t_j)}_{} x_j \geq 0$$

for all $x_1, x_2, \ldots, x_n \in \mathbb{R}$
for all $t_1, t_2, \ldots, t_n \in T$
and all $n \in \mathbb{N}$.

"n by n matrix with $\vec{x}' R_X \vec{x} \geq 0$"

$\rightarrow$ c.f. non-negative definiteness for covariance matrices of random vectors.

Proof: Only necessity of (*)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j R_{\underline{X}}(t_i,t_j)$$

definition of $R_{\underline{X}}$

$$= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} x_i X(t_i)\cdot x_j X(t_j)\right]$$

$$= \mathbb{E}\Big[\underbrace{\Big(\sum_{i=1}^{n} x_i \cdot X(t_i)\Big)^2}_{\geq 0}\Big] \geq 0$$

∎

Exercises: • Show that

$$|R_{\underline{X}}(t,s)| \leq \sqrt{R_X(t,t)}\cdot\sqrt{R_X(s,s)}$$

Hint: $|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]}\cdot\sqrt{\mathbb{E}[Y^2]}$ Cauchy-Schwarz.

• $X(t) = A\sin(\omega t+\varphi)$, $\varphi \in U(0,2\pi)$.

Find $\mu_{\underline{X}}(t)$ and $R_{\underline{X}}(t,s)$.

Used: $\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$

Answers: $\mu_X(t) = \mathbb{E}[A\sin(\omega t+\varphi)] = \int_0^{2\pi}\frac{1}{2\pi}A\sin(\omega t+\varphi)\,d\varphi = 0$ $\forall t\in\mathbb{R}$

$R_X(t,s) = \frac{A^2}{2}\cos(\omega(t-s)) = R_X(s,t)$

$X(t)$ is weakly stationnary !

Suppose we are given a family of "finite-dimensional distribution functions" $F_{t_1,t_2,\dots,t_n}$, for $n \in \mathbb{N}$, all $t_1, \dots, t_n \in T$,

$$\mathbb{F} = \{F_{t_1,\dots,t_n}\}_{(t_1,\dots,t_n)\in T^n,\, n\in\mathbb{N}}$$

Is there a stochastic process $\underline{X}$ with $\mathbb{F}$ as its family of finite dimensional distributions?

<u>Thm (Kolmogorov consistency)</u>

Assume that for any $n < m$ and any times $(t_i)_{i=1}^m$ it holds that

$$\lim_{x_{n+1}, x_{n+2}, \dots, x_m \to +\infty} F_{t_1,t_2,\dots,t_n,t_{n+1},\dots,t_m}(x_1,x_2,\dots,x_n,x_{n+1},\dots,x_m) = F_{t_1,t_2,\dots,t_n}(x_1,x_2,\dots,x_n), \quad \forall x_1,\dots,x_n \in \mathbb{R}$$

then there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a stochastic process $\underline{X}$ on that probability space, having $\mathbb{F}$ as finite-dimensional distributions. //