Brownian motion continued

Def: Wiener process/Brownian motion is a stochastic process

$$\bar{W} = \{ W(t) : t \geq 0 \} \text{ such that}$$

i) $W(0) = 0$ almost surely (i.e. $\mathbb{P}(W(0)=0) = 1$.)

ii) for any $n \in \mathbb{N}$ and times $0 < t_1 < t_2 < \cdots < t_n$, and $x_1, x_2, \ldots, x_n \in \mathbb{R}$ the joint probability density of $W(t_1), W(t_2), \ldots, W(t_n)$ is

$$f_{W(t_1), W(t_2), \ldots, W(t_n)}(x_1, x_2, \ldots, x_n) = p(t_1, x_1, 0) \cdot p(t_2 - t_1, x_2, x_1) \times p(t_3 - t_2, x_3, x_2) \cdots p(t_n - t_{n-1}, x_n, x_{n-1})$$ //

where $p(t, y, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}, \quad t > 0, \; x, y \in \mathbb{R}.$

"transition probabilities"

Claim: $\tilde{W}$ is a Gaussian process.

Lemma: Autocorrelation function of $\tilde{W}$

$$R_{\tilde{W}}(t,s) = \min(t,s) \,, \quad \text{for } t,s \geq 0 .$$

Proof: Assume that $t > s$

law of total expectation

$$R_{\tilde{W}}(t,s) = \mathbb{E}[W(t) \cdot W(s)] = \mathbb{E}\Big[\mathbb{E}[W(t)W(s) \mid W(s)]\Big]$$

$$= \mathbb{E}\big[W(s)\,\underbrace{\mathbb{E}[W(t) \mid W(s)]}_{= W(s)}\big]$$

see first video (**)

$$= \mathbb{E}[(W(s))^2] = s \qquad \text{used that } W(s) \in \mathcal{N}(0,s)$$

In case $t \leq s$: $R_{\tilde{W}}(t,s) = \ldots = t$.

$$\Rightarrow \quad R_W(t,s) = \min(t,s)$$

$\blacksquare$

Hence the covariance matrix of $(W(t_1), W(t_2), \dots, W(t_n))'$, with $t_1 < t_2 < \dots < t_n$.

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is given by

$$\Lambda = \begin{pmatrix} R_W(t_1,t_1) & R_W(t_1,t_2) & R_W(t_1,t_3) & \cdots \\ R_W(t_2,t_1) & R_W(t_2,t_2) & \cdots & \\ \vdots & & \ddots & \end{pmatrix}$$

$$= \begin{pmatrix} t_1 & t_1 & t_1 & \cdots \\ t_1 & t_2 & t_2 & t_2 & \cdots \\ t_1 & t_2 & t_3 & t_3 & \cdots \\ t_1 & t_2 & t_3 & & \\ \vdots & \vdots & & \ddots & \\ & & & & t_n \end{pmatrix}$$

Corollary:

$$E\left[(W(t) - W(s))^2\right] \overset{(***)}{=} |t-s| \qquad t, s \geq 0$$

Proof: Exercise, choose first $t > s$, expand the square. //

④

Lemma: Let $t > s \geq 0$, then the <u>increment</u> $W(t) - W(s)$ satisfies

$$W(t) - W(s) \in \mathcal{N}(0, t-s)$$

$\blacksquare$

Proof: Since W is a Gaussian process, we know that $W(t) - W(s)$ is Gaussian (Cramér-Wold device).

Mean is zero, variance is given (***), i.e. $t-s$. $\blacksquare$

Thm: Let $0 = t_0 < t_1 < t_2 < \dots < t_n$, then the increments

$$W(t_1) - W(t_0), \; W(t_2) - W(t_1), \; \dots, \; W(t_n) - W(t_{n-1})$$

are <u>independent</u> Gaussian random variables with mean zero and variance $(t_{i+1} - t_i)$. //

Proof: It suffices to show that they are <u>uncorrelated</u> r.v. $\quad 0 \leq i, k \leq n-1$

$$\mathrm{Cov}\big(W(t_{k+1}) - W(t_k), W(t_{i+1}) - W(t_i)\big) \qquad \underline{k > i}$$

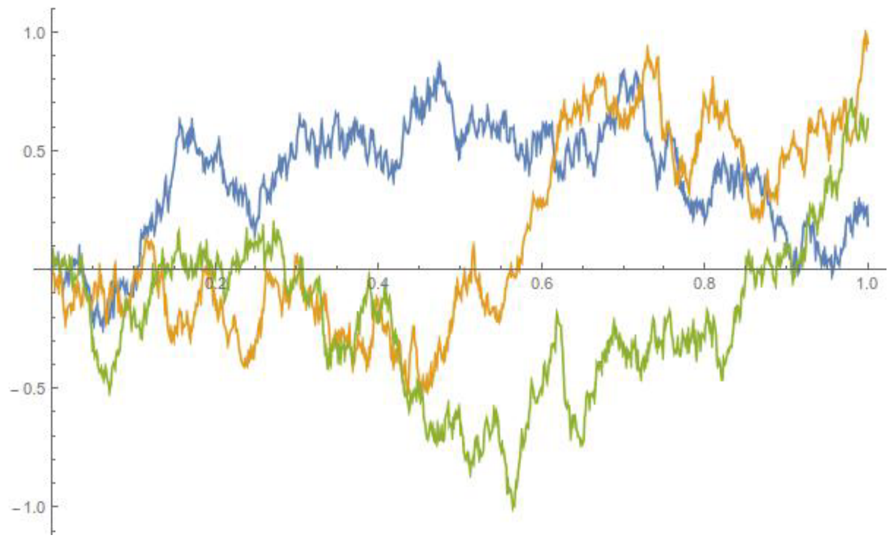
$$= \mathbb{E}\Big[\big(W(t_{k+1}) - W(t_k)\big) \cdot \big(W(t_{i+1}) - W(t_i)\big)\Big]$$

$$= \mathbb{E}[W(t_{k+1}) \cdot W(t_{i+1})] - \mathbb{E}[W(t_{k+1}) W(t_i)] - \mathbb{E}[W(t_k) \cdot W(t_{i+1})] + \mathbb{E}[W(t_k) W(t_i)]$$

$$= \underbrace{\min(t_{k+1}, t_{i+1})}_{t_{i+1}} - \underbrace{\min(t_{k+1}, t_i)}_{t_i} - \underbrace{\min(t_k, t_{i+1})}_{t_{i+1}} + \underbrace{\min(t_k, t_i)}_{t_i} = 0 \qquad \blacksquare$$

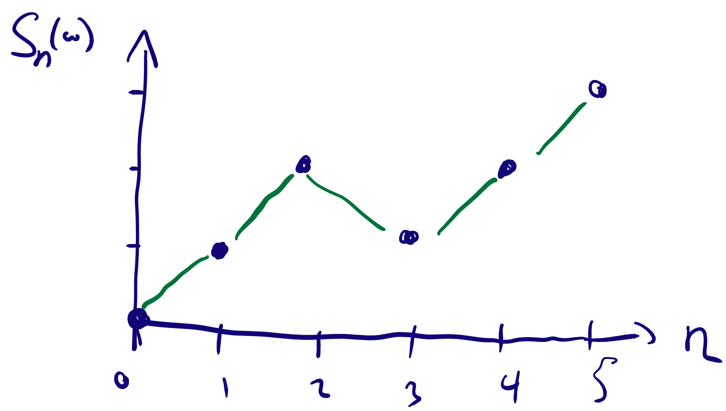
$$t_{i+1} - t_i - t_{i+1} + t_i$$

Remarks about sample paths



- The sample paths $\{W(t,\omega): t \geq 0\}$ are almost surely continuous.
  $\omega \in \Omega$ fixed.
- The sample paths are almost surely nowhere differentiable.

Donsker's invariance principle: Recall symmetric simple random walk: (X_i) iid
$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$. $S_n := \sum_{i=1}^{n} X_i$, $S_0 = 0$.



$$\frac{S_n}{\sqrt{n}} \xrightarrow[n \to \infty]{d} \mathcal{N}(0,1)$$

CLT.

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Two rescaling with n: Space: replace stepsize one by $\frac{1}{\sqrt{n}}$.

Time: replace unit time steps by $\frac{1}{n}$.

Let $(X_i^{(n)})_{i=1,\dots,n,\, n\geq 0}$, all i.i.d. with $\mathbb{P}(X_i^{(n)}=\frac{1}{\sqrt{n}})=\mathbb{P}(X_i^{(n)}=-\frac{1}{\sqrt{n}})=\frac{1}{2}$

Observe: $Var(X_i^{(n)})=\frac{1}{2}(\frac{1}{\sqrt{n}})^2+\frac{1}{2}(-\frac{1}{\sqrt{n}})^2=\frac{1}{n}$.

For $t\geq 0$ define $W^{(n)}(t):=\sum_{i=1}^{\lfloor tn\rfloor} X_i^{(n)}$

$\lfloor tn\rfloor$: integer part of tn.

In particular, $W^{(n)}(1)=\sum_{i=1}^{n} X_i^{(n)} \xrightarrow[n\to\infty]{d} W(1)\in N(0,1)$ CLT in disguise.

but also $W^{(n)}(t)\xrightarrow[n\to\infty]{d} W(t)\in N(0,t)$

In fact: For any $k\in\mathbb{N}$, $t_1<t_2<\dots<t_k$, we have

$$(W^{(n)}(t_1),W^{(n)}(t_2),\dots,W^{(n)}(t_k))$$
$$\xrightarrow[n\to\infty]{d} (W(t_1),W(t_2),\dots,W(t_n))$$

where W is a Wiener process. //

Why do we scale space and time in this manner?

Recall from (***) that

$$\mathbb{E}\left[\left(W(t_{k+1})-W(t_k)\right)^2\right] = \underbrace{t_{k+1}-t_k}_{\frac{1}{n}}$$

$\simeq$ typical size of the increments is of order $\frac{1}{\sqrt{n}}$.

"diffusive scaling: distance squared growth like the time"

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Thm: A stochastic process $\{W(t): t \geq 0\}$ is a Wiener process if and only if

1.) $W(0) = 0$ a.s.

2.) The sample paths $t \mapsto W(t)$ are almost surely continuous

3.) $\{W(t): t \geq 0\}$ has independent increments.

4.) $W(t) - W(s)$, $t > s$, is Gaussian with mean zero and variance $t-s$. //

Exercises: W a Brownian motion:

1.) Find $\mathbb{P}(W(2) > W(1)) = \mathbb{P}(W(2) - W(1) > 0) = 1/2$

since $W(2) - W(1) \in \mathcal{N}(0,1)$ and symmetry of $\mathcal{N}(0,1)$.

2.) Find $\mathbb{P}(W(2) > 2W(1))$

3.) Find $\mathbb{P}(W(2) > 0 \mid W(1) = 1)$

$$= \int_0^\infty p(1, y, 1)\, dy = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{(y-1)^2}{2}}\, dy$$

change of variable

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^\infty e^{-\frac{\tilde{y}^2}{2}}\, d\tilde{y} = 1 - \Phi(-1)$$

cdf of $\mathcal{N}(0,1)$.

4.) Find $\mathbb{P}(W(2) > 0 \mid W(1) > 1)$