creating the hamiltonian of number partitioning problem

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Let's start with an example that will require only 2 qubits to implement. That means that our set of numbers should be also of the size 2.

Of course in this situation there is a solution to the problem (a valid partition of the set to 2 groups of the same sum) only if the set contains

2 equal numbers. Let's define the set S as:

$$S = \{1, 1\}$$

According to the explanation in section 2.1 in the paper "Ising Formulations of Many NP Problems", the problem can be phrased as Ising model as follows:

$$H = A\left(\sum_{i=1}^{N} n_i s_i\right)^2$$

When H is an energy function, $s_i = \pm 1$ is an Ising spin variable, N is the size of the numbers set and A $>$ 0 is some positive constant

(for simplicity we will scale this parameter later to 1).

If there is a solution to the Ising model with H $=$ 0, then there is a configuration of spins where the sum of the n_i for the $+1$ spins is the same for the

sum of the n_i for the -1 spins. Thus, if the ground state energy is H $=$ 0, there is a solution to the number partitioning problem.

I will comtinue and express the Hamiltonian in a quantum form. As it is described in section 1.1 in the paper "Ising Formulations of Many NP Problems",

a classical Ising model can be written as a quadratic function of a set of N spins $s_i = \pm 1$:

$$H(s_1, s_2, \ldots, s_N) = -\sum_{i<j} J_{ij} s_i s_j - \sum_{i=1}^{N} h_i s_i$$

The quantum version of this Hamiltonian is simply:

$$H_p = H(\sigma_1^z, \sigma_2^z, \ldots, \sigma_N^z)$$

So for the set S defined above, the Ising model equals to:

$$H = A\left(\sum_{i=1}^{N} n_i s_i\right)^2 = (1\cdot s_1 + 1\cdot s_2)^2 = 1{\cdot}s_1^2{+}1{\cdot}s_1 s_2{+}1{\cdot}s_2 s_1{+}1{\cdot}s_2^2 = 1{\cdot}\sigma_z^1\sigma_z^1{+}1{\cdot}\sigma_z^1\sigma_z^2{+}1{\cdot}\sigma_z^2\sigma_z^1{+}1{\cdot}\sigma_z^2\sigma_z^2$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can see that the Hamiltonian has 2 eigenvalues that equals to 0, which means, as we expected, that there are 2 possible partitions which satisfy the problem

(It doesn't matter in which group each number is, so there are 2 possible ways to devide the set).

Let's try a set that can't be partitioned: S = {1,2}. We get the Ising Model:

$$H = A\left(\sum_{i=1}^{N} n_i s_i\right)^2 = (1\cdot s_1 + 2\cdot s_2)^2 = 1{\cdot}s_1^2{+}2{\cdot}s_1 s_2{+}2{\cdot}s_2 s_1{+}4{\cdot}s_2^2 = 1{\cdot}\sigma_z^1\sigma_z^1{+}2{\cdot}\sigma_z^1\sigma_z^2{+}2{\cdot}\sigma_z^2\sigma_z^1{+}4{\cdot}\sigma_z^2\sigma_z^2$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that this Hamiltonian has no eigenvalues that equals to 0, and therefore, as expected, a partition that satisfies the problem doesn't exist.