Kinematics Final Assignment

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December 12, 2025

Problem Two

Using the $[B(\sigma)]$ matrix of the MRP differential kinematic equations,

$$[B(\sigma)] = (1-\sigma^2)[I_{3\times3}] + 2[\tilde{\sigma}] + 2\sigma\sigma^T$$

develop the simple inverse expression for $[B]^{-1}$.

Note the identity $[\tilde{\sigma}]^2 = \sigma\sigma^T - \sigma^2[I_{3\times3}]$

Hint: investigate what happens to $[B][B]^T$

Answer

Let us first make the matrix $[B]^T$:

$$\begin{aligned}[B]^T &= [(1-\sigma^2)[I_{3\times3}] + 2[\tilde{\sigma}] + 2\sigma\sigma^T]^T \\ &= [(1-\sigma^2)[I_{3\times3}]]^T + 2[\tilde{\sigma}]^T + 2[\sigma\sigma^T]^T\end{aligned}$$

From the three terms above, only $[\tilde{\sigma}]$ is skew-symmetric, for which we have:

$$[\tilde{\sigma}]^T = -[\tilde{\sigma}]$$

and so we have:

$$[B]^T = (1-\sigma^2)[I_{3\times3}] - 2[\tilde{\sigma}] + 2\sigma\sigma^T$$

Now we investigate what happens to $[B][B]^T$:

$$\begin{aligned}[B][B]^T &= [(1-\sigma^2)[I_{3\times3}] + 2[\tilde{\sigma}] + 2\sigma\sigma^T][(1-\sigma^2)[I_{3\times3}] - 2[\tilde{\sigma}] + 2\sigma\sigma^T] \\ &= (1-\sigma^2)^2[I_{3\times3}] - 2(1-\sigma^2)[\tilde{\sigma}] + 2(1-\sigma^2)\sigma\sigma^T \\ &+ 2(1-\sigma^2)[\tilde{\sigma}] - 4[\tilde{\sigma}]^2 + 4[\tilde{\sigma}]\sigma\sigma^T \\ &+ 2(1-\sigma^2)\sigma\sigma^T + 4\sigma\sigma^T[\tilde{\sigma}] + 4(\sigma\sigma^T)^2\end{aligned}$$

Looking at the derived expressions, we see that $-2(1-\sigma^2)[\tilde{\sigma}] + 2(1-\sigma^2)[\tilde{\sigma}] = 0$ and also $4[\tilde{\sigma}]\sigma\sigma^T = 4\sigma\sigma^T[\tilde{\sigma}] = 0$. Worth noting that $4(\sigma\sigma^T)^2 = 4[\sigma(\sigma^T\sigma)\sigma^T]$ and turns to $4\sigma^2[\sigma\sigma^T]$. Rewriting the remaining expressions we have:

$$\begin{aligned}[B][B]^T &= [(1-\sigma^2)^2[I_{3\times3}] + 4(1-\sigma^2)[\sigma\sigma^T] - 4[\tilde{\sigma}] + 4\sigma^2[\sigma\sigma^T] \\ &= [(1-\sigma^2)^2[I_{3\times3}] + 4[\sigma\sigma^T] - 4\sigma^2[\sigma\sigma^T] - 4[\tilde{\sigma}] + 4\sigma^2[\sigma\sigma^T] \\ &= [(1-\sigma^2)^2[I_{3\times3}] + 4[\sigma\sigma^T] - 4[\tilde{\sigma}]\end{aligned}$$

From the identity $[\tilde{\sigma}]^2 = \sigma\sigma^T - \sigma^2[I_{3\times3}]$, we put in $[\tilde{\sigma}]$:

$$\begin{aligned}
[B][B]^T &= [(1-\sigma^2)^2[I_{3\times3}] + 4[\sigma\sigma^T] - 4[\sigma\sigma^T - \sigma^2[I_{3\times3}]] \\
&= (1-\sigma^2)^2[I_{3\times3}] + 4[\sigma\sigma^T] - 4[\sigma\sigma^T] + 4\sigma^2[I_{3\times3}] \\
&= ((1-\sigma^2)^2 + 4\sigma^2)[I_{3\times3}] \\
&= (1 - 2\sigma^2 + \sigma^4 + 4\sigma^2)[I_{3\times3}] = (1 + 2\sigma^2 + \sigma^4)[I_{3\times3}] \\
&= (1+\sigma^2)^2[I_{3\times3}]
\end{aligned}$$

We end up with:

$$[B][B]^T = (1+\sigma^2)^2[I_{3\times3}]$$

So:

$$[B](\frac{1}{(1+\sigma^2)^2}[B]^T) = [I_{3\times3}]$$

$$=> [B]^{-1} = \frac{1}{(1+\sigma^2)^2}[B]^T$$