Kinematics Final Assignment

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Problem One

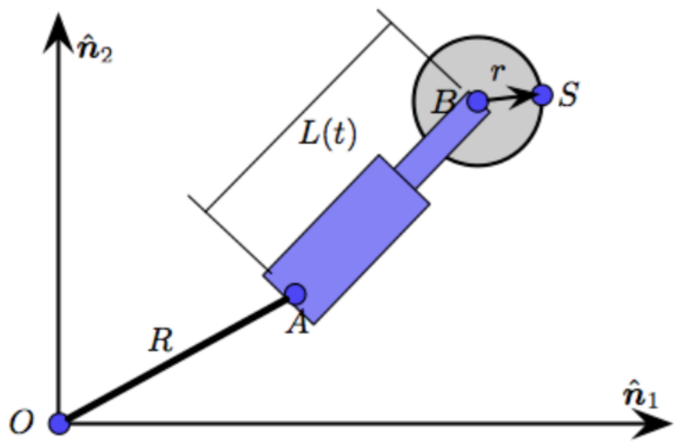

Figure 1: Illustration of Planar Dynamical System

Consider the dynamical system shown in Figure 1. A level with constant length R has a variable length piston attached to its end point A. At the end of piston is a disk with a constant radius r attached. The level and piston are free to rotate generally with the inertial spin rates $\dot{\alpha}$ and $\dot{\beta}$ respectively, while the disk is controlled to have the constant spin vector $\dot{\theta}\mathbf{\hat{n}_3}$ relative to the piston. All motions occurs in the $\mathbf{\hat{n}_1} - \mathbf{\hat{n}_2}$ plane. To find the solution, use the transport theorem and rotating frame axes. Don't project everything onto a common frame using $sin()$ and $cos()$ functions.

a) Determine the inertial velocity of point S.

b) Determine the inertial acceleration of point S.

c) Determine the velocity of point A as seen by an observer attached on the rotating disk sitting at location S.

Answer

First, we draw the rotating frame axes for better visualization.

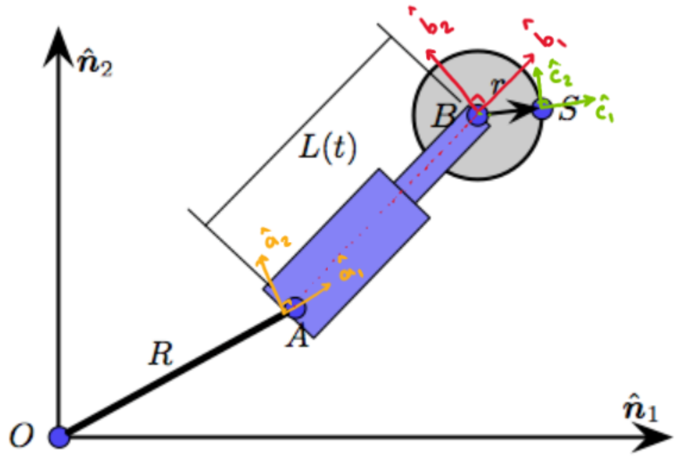

Figure 2: Rotating Frame Axes

We now have our inertial frame $\mathcal{N}$:$\{\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3\}$ and $\mathcal{A}$:$\{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$, $\mathcal{B}$:$\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ and finally $\mathcal{S}$:$\{\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2, \hat{\mathbf{c}}_3\}$. Since the motion is planar, we know that $\hat{\mathbf{n}}_3 = \hat{\mathbf{a}}_3 = \hat{\mathbf{b}}_3 = \hat{\mathbf{n}}_3$. We also need to determine all our point vectors and angular velocity vectors in the inertial frame:

$$\mathbf{r}_{\mathcal{A}/\mathcal{O}} = R\hat{\mathbf{a}}_1$$

$$\mathbf{r}_{\mathcal{B}/\mathcal{A}} = L(t)\hat{\mathbf{b}}_1$$

$$\mathbf{r}_{\mathcal{S}/\mathcal{B}} = r\hat{\mathbf{c}}_1$$

$$\omega_{\mathcal{A}/\mathcal{O}} = \dot{\alpha}\hat{\mathbf{n}}_3$$

$$\omega_{\mathcal{B}/\mathcal{A}} = \dot{\beta}\hat{\mathbf{n}}_3$$

$$\omega_{\mathcal{S}/\mathcal{B}} = \dot{\theta}\hat{\mathbf{n}}_3$$

So for $\mathbf{r}_{\mathcal{S}/\mathcal{O}}$ we'll have:

$$\begin{aligned}\mathbf{r}_{\mathcal{S}/\mathcal{O}} &= \mathbf{r}_{\mathcal{S}/\mathcal{B}} + \mathbf{r}_{\mathcal{B}/\mathcal{A}} + \mathbf{r}_{\mathcal{A}/\mathcal{O}} \\ &= r\hat{\mathbf{c}}_1 + L(t)\hat{\mathbf{b}}_1 + R\hat{\mathbf{a}}_1\end{aligned}$$

And the inertial angular velocities will be:

$$\omega_{\mathcal{B}/\mathcal{O}} = \omega_{\mathcal{B}/\mathcal{A}} + \omega_{\mathcal{A}/\mathcal{O}} = (\dot{\alpha} + \dot{\beta})\hat{\mathbf{n}}_3$$

$$\omega_{\mathcal{S}/\mathcal{O}} = \omega_{\mathcal{S}/\mathcal{B}} + \omega_{\mathcal{B}/\mathcal{O}} = (\dot{\theta} + \dot{\alpha} + \dot{\beta})\hat{\mathbf{n}}_3$$

a) Recalling Transport Theorem:

$$^{\mathcal{N}}(\frac{d\mathbf{r}}{dt}) = {^{\mathcal{P}}}(\frac{d\mathbf{r}}{dt}) + \omega_{\mathcal{P}/\mathcal{O}} \times {^{\mathcal{P}}}\mathbf{r}$$

To calculate $\mathbf{v}_{\mathcal{S}/\mathcal{O}}$:

$$\begin{aligned}\mathbf{v}_{\mathcal{S}/\mathcal{O}} = {}^{\mathcal{N}}(\frac{d\mathbf{r}_{\mathcal{S}/\mathcal{O}}}{dt}) &= {}^{\mathcal{N}}(\frac{d}{dt})(r\hat{\mathbf{c}}_\mathbf{1} + L(t)\hat{\mathbf{b}}_\mathbf{1} + R\hat{\mathbf{a}}_\mathbf{1}) \\ &= {}^{\mathcal{N}}(\frac{d}{dt})(r\hat{\mathbf{c}}_\mathbf{1}) + {}^{\mathcal{N}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_\mathbf{1}) + {}^{\mathcal{N}}(\frac{d}{dt})(R\hat{\mathbf{a}}_\mathbf{1})\end{aligned}$$

We'll use the Transport Theorem to solve each of these terms. First for ${}^{\mathcal{N}}(\frac{d}{dt})(r\hat{\mathbf{c}}_\mathbf{1})$:

$$\begin{aligned}{}^{\mathcal{N}}(\frac{d}{dt})(r\hat{\mathbf{c}}_\mathbf{1}) &= {}^{\mathcal{S}}(\frac{d}{dt})(r\hat{\mathbf{c}}_\mathbf{1}) + \omega_{\mathcal{S}/\mathcal{O}} \times {}^{\mathcal{S}}r\hat{c}_1 \\ &= r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{n}}_\mathbf{3} \times \hat{\mathbf{c}}_\mathbf{1} \\ &= r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2}\end{aligned}$$

And then, ${}^{\mathcal{N}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_\mathbf{1})$:

$$\begin{aligned}{}^{\mathcal{N}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_\mathbf{1}) &= {}^{\mathcal{B}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_\mathbf{1}) + \omega_{\mathcal{B}/\mathcal{O}} \times {}^{\mathcal{B}}L(t)\hat{\mathbf{b}}_\mathbf{1} \\ &= \dot{L}\hat{\mathbf{b}}_\mathbf{1} + L(\dot{\alpha} + \dot{\beta})\hat{\mathbf{n}}_\mathbf{3} \times \hat{\mathbf{b}}_\mathbf{1} \\ &= \dot{L}\hat{\mathbf{b}}_\mathbf{1} + L(\dot{\alpha} + \dot{\beta})\hat{\mathbf{b}}_\mathbf{2}\end{aligned}$$

At last, we have ${}^{\mathcal{N}}(\frac{d}{dt})(R\hat{\mathbf{a}}_\mathbf{1})$:

$$\begin{aligned}{}^{\mathcal{N}}(\frac{d}{dt})(R\hat{\mathbf{a}}_\mathbf{1}) &= {}^{\mathcal{A}}(\frac{d}{dt})(R\hat{\mathbf{a}}_\mathbf{1}) + \omega_{\mathcal{A}/\mathcal{O}} \times {}^{\mathcal{A}}R\hat{\mathbf{a}}_\mathbf{1} \\ &= R\dot{\alpha}\hat{\mathbf{n}}_\mathbf{3} \times \hat{\mathbf{a}}_\mathbf{1} \\ &= R\dot{\alpha}\hat{\mathbf{a}}_\mathbf{2}\end{aligned}$$

So, using rotating frame axes and Transport Theorem:

$$\mathbf{v}_{\mathcal{S}/\mathcal{O}} = r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2} + \dot{L}\hat{\mathbf{b}}_\mathbf{1} + L(\dot{\alpha} + \dot{\beta})\hat{\mathbf{b}}_\mathbf{2} + R\dot{\alpha}\hat{\mathbf{a}}_\mathbf{2}$$

b) To calculate the inertial velocity of point S, we use the Transport Theorem on the expression we derived in the last part for $\mathbf{v}_{\mathcal{S}/\mathcal{O}}$:

$$\begin{aligned}\mathbf{a}_{\mathcal{S}/\mathcal{O}} &= {}^{\mathcal{N}}(\frac{d\mathbf{v}_{\mathcal{S}/\mathcal{O}}}{dt}) \\ &= {}^{\mathcal{N}}(\frac{d}{dt})(r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2} + \dot{L}\hat{\mathbf{b}}_\mathbf{1} + L(\dot{\alpha} + \dot{\beta})\hat{\mathbf{b}}_\mathbf{2} + R\dot{\alpha}\hat{\mathbf{a}}_\mathbf{2})\end{aligned}$$

Just like last part, we start solving each term, starting with ${}^{\mathcal{N}}(\frac{d}{dt})(r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2})$:

$$\begin{aligned}{}^{\mathcal{N}}(\frac{d}{dt})(r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2}) &= {}^{\mathcal{S}}(\frac{d}{dt})(r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2}) + \omega_{\mathcal{S}/\mathcal{O}} \times r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2} \\ &= r(\ddot{\alpha} + \ddot{\beta} + \ddot{\theta})\hat{\mathbf{c}}_\mathbf{2} + r(\dot{\alpha} + \dot{\beta} + \dot{\theta})((\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{3} \times \hat{\mathbf{c}}_\mathbf{2}) \\ &= r(\ddot{\alpha} + \ddot{\beta} + \ddot{\theta})\hat{\mathbf{c}}_\mathbf{2} - r(\dot{\alpha} + \dot{\beta} + \dot{\theta})^2\hat{\mathbf{c}}_\mathbf{1}\end{aligned}$$

But since $\dot{\theta}$ is constant, $\ddot{\theta} = 0$ and the final expression for this term will be:

$${}^{\mathcal{N}}(\frac{d}{dt})(r(\dot{\alpha} + \dot{\beta} + \dot{\theta})\hat{\mathbf{c}}_\mathbf{2}) = -r(\dot{\alpha} + \dot{\beta} + \dot{\theta})^2\hat{\mathbf{c}}_\mathbf{1} + r(\ddot{\alpha} + \ddot{\beta})\hat{\mathbf{c}}_\mathbf{2}$$

Next term is ${}^{\mathcal{N}}(\frac{d}{dt})(\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} + L(\dot{\alpha}+\dot{\beta})\hat{\mathbf{b}}_{\mathbf{2}})$:

$$\begin{aligned}
{}^{\mathcal{N}}(\frac{d}{dt})(\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} + L(\dot{\alpha}+\dot{\beta})\hat{\mathbf{b}}_{\mathbf{2}}) &= {}^{\mathcal{B}}(\frac{d}{dt})(\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} + L(\dot{\alpha}+\dot{\beta})\hat{\mathbf{b}}_{\mathbf{2}}) + \omega_{\mathcal{B}/\mathcal{O}} \times (\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} + L(\dot{\alpha}+\dot{\beta})\hat{\mathbf{b}}_{\mathbf{2}}) \\
&= \ddot{L}\hat{\mathbf{b}}_{\mathbf{1}} + (\dot{L}(\dot{\alpha}+\dot{\beta}) + L(\ddot{\alpha}+\ddot{\beta}))\hat{\mathbf{b}}_{\mathbf{2}} + \dot{L}(\dot{\alpha}+\dot{\beta})\hat{\mathbf{b}}_{\mathbf{3}} \times \hat{\mathbf{b}}_{\mathbf{1}} + L(\dot{\alpha}+\dot{\beta})^2\hat{\mathbf{b}}_{\mathbf{3}} \times \hat{\mathbf{b}}_{\mathbf{2}} \\
&= (\ddot{L} - L(\dot{\alpha}+\dot{\beta})^2)\hat{\mathbf{b}}_{\mathbf{1}} + (L(\ddot{\alpha}+\ddot{\beta}) + 2\dot{L}(\dot{\alpha}+\dot{\beta}))\hat{\mathbf{b}}_{\mathbf{2}}
\end{aligned}$$

And finally, ${}^{\mathcal{N}}(\frac{d}{dt})(R\dot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}})$:

$$\begin{aligned}
{}^{\mathcal{N}}(\frac{d}{dt})(R\dot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}}) &= {}^{\mathcal{A}}(\frac{d}{dt})(R\dot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}}) + \omega_{\mathcal{A}/\mathcal{O}} \times R\dot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}} \\
&= R\ddot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}} + R\dot{\alpha}^2\hat{\mathbf{a}}_{\mathbf{3}} \times \hat{\mathbf{a}}_{\mathbf{2}} \\
&= -R\dot{\alpha}^2\hat{\mathbf{a}}_{\mathbf{1}} + R\ddot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}}
\end{aligned}$$

So, for $\mathbf{a}_{\mathcal{S}/\mathcal{O}}$ we have:

$$\begin{aligned}
\mathbf{a}_{\mathcal{S}/\mathcal{O}} = &-r(\dot{\alpha}+\dot{\beta}+\dot{\theta})^2\hat{\mathbf{c}}_{\mathbf{1}} + r(\ddot{\alpha}+\ddot{\beta})\hat{\mathbf{c}}_{\mathbf{2}} \\
&+ (\ddot{L} - L(\dot{\alpha}+\dot{\beta})^2)\hat{\mathbf{b}}_{\mathbf{1}} + (L(\ddot{\alpha}+\ddot{\beta}) + 2\dot{L}(\dot{\alpha}+\dot{\beta}))\hat{\mathbf{b}}_{\mathbf{2}} \\
&- R\dot{\alpha}^2\hat{\mathbf{a}}_{\mathbf{1}} + R\ddot{\alpha}\hat{\mathbf{a}}_{\mathbf{2}}
\end{aligned}$$

c) To determine $\mathbf{v}_{\mathcal{A}/\mathcal{S}}$, we first need to find $\mathbf{r}_{\mathcal{A}/\mathcal{S}}$ and then take a $\mathcal{S}$-frame derivative from it:

$$\begin{aligned}
\mathbf{r}_{\mathcal{A}/\mathcal{S}} &= \mathbf{r}_{\mathcal{A}/\mathcal{O}} + \mathbf{r}_{\mathcal{O}/\mathcal{S}} \\
&= \mathbf{r}_{\mathcal{A}/\mathcal{O}} - \mathbf{r}_{\mathcal{S}/\mathcal{O}} \\
&= R\hat{\mathbf{a}}_{\mathbf{1}} - (r\hat{\mathbf{c}}_{\mathbf{1}} + L(t)\hat{\mathbf{b}}_{\mathbf{1}} + R\hat{\mathbf{a}}_{\mathbf{1}}) \\
&= -r\hat{\mathbf{c}}_{\mathbf{1}} - L(t)\hat{\mathbf{b}}_{\mathbf{1}}
\end{aligned}$$

Then, to find $\mathbf{v}_{\mathcal{A}/\mathcal{S}}$:

$$\begin{aligned}
\mathbf{v}_{\mathcal{A}/\mathcal{S}} &= {}^{\mathcal{S}}(\frac{d}{dt})(\mathbf{r}_{\mathcal{A}/\mathcal{S}}) \\
&= {}^{\mathcal{S}}(\frac{d}{dt})(-r\hat{\mathbf{c}}_{\mathbf{1}} - L(t)\hat{\mathbf{b}}_{\mathbf{1}}) \\
&= -{}^{\mathcal{S}}(\frac{d}{dt})(r\hat{\mathbf{c}}_{\mathbf{1}}) - {}^{\mathcal{S}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_{\mathbf{1}}) \\
&= -{}^{\mathcal{S}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_{\mathbf{1}}) \\
&= -[{}^{\mathcal{B}}(\frac{d}{dt})(L(t)\hat{\mathbf{b}}_{\mathbf{1}}) + \omega_{\mathcal{B}/\mathcal{S}} \times L(t)\hat{\mathbf{b}}_{\mathbf{1}}]
\end{aligned}$$

We know that $\omega_{\mathcal{B}/\mathcal{S}} = -\omega_{\mathcal{S}/\mathcal{B}} = -\dot{\theta}$:

$$\begin{aligned}
\mathbf{v}_{\mathcal{A}/\mathcal{S}} &= -[\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} - L\dot{\theta}\hat{\mathbf{c}}_{\mathbf{3}} \times \hat{\mathbf{b}}_{\mathbf{1}}] \\
&= -[\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} - L\dot{\theta}\hat{\mathbf{b}}_{\mathbf{2}}] \\
&= -\dot{L}\hat{\mathbf{b}}_{\mathbf{1}} + L\dot{\theta}\hat{\mathbf{b}}_{\mathbf{2}}
\end{aligned}$$