#1

General Form:

$$\underset{[n\times 1]}{y} = \underset{[n\cdot p]}{X}\,\underset{[p\cdot 1]}{\beta} + \underset{[n\cdot 1]}{\varepsilon}$$

$$f(\beta) = (y - X\hat{\beta})^T(y - X\beta)$$
$$= y^Ty - 2y^TX\beta + \beta^TX^TX\beta$$

Minimize $f(\beta)$, Take derivative

$$\frac{\partial f(\beta)}{\partial \beta} = -2X^Ty + 2X^TX\beta = 0$$
$$: X^TX\beta = X^Ty$$

Solve for $\hat{\beta}$:

$$X^TX\hat{\beta} = X^Ty$$
$$\hat{\beta} = (X^Ty)^{-1}X^Ty$$

#2 Two variables using MLE

Multiple Regression Model:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

Likelihood:

$$\mathcal{L}(\beta, \sigma^2 \mid x_1, x_2, x_3, y)$$
$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})^2$$

FOC's:

$$\frac{\partial \mathcal{L}}{\partial \beta_0} = +\frac{1}{\sigma^2}\sum(y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2}) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \beta_1} = +\frac{1}{\sigma^2}\sum(y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})x_{i1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \beta_2} = +\frac{1}{\sigma^2}\sum(y_i - \beta_0 - \beta_1 x_{i1} - \beta_2 x_{i2})x_{i2} = 0$$

$\hat{\beta}_0$:

$$n\bar{y} - n\hat{\beta}_0 - n\beta_1\bar{x}_1 - n\beta_2\bar{x}_2 = 0$$

$$\hat{\beta}_0 : \bar{y} - \beta_1\bar{x}_1 - \hat{\beta}_2\bar{x}_2$$

$\hat{\beta}_1$:

$$0 : \sum y_i x_{i1} - (n\hat{\beta}_0\bar{x}) - \hat{\beta}_1\sum x_{i1}^2 - \hat{\beta}\sum x_{i1}x_i$$

$$n\bar{x}_1(\bar{y} - \hat{\beta}\bar{x}_1 - \hat{\beta}_2\bar{x}_2)$$

$$0 : \sum y_i x_{i1} - n\bar{x}\bar{y} + n\bar{x}_1\hat{\beta}_1 + \hat{\beta}_2\bar{x}_2\bar{x}_1 n$$

$$\hat{\beta}_1(-n\bar{x}_1^2 + \sum x_{i1}^2) = (\sum y_i x_{i1} - n\bar{x}\bar{y})$$
$$-\hat{\beta}_2(\sum x_{i1}x_{i2} - n\bar{x}_1\bar{x}_2)$$

$$\hat{\beta}_1 = \frac{\sum(y_i - \bar{y})(x_i - \bar{x}) - \beta_2\left(\sum(x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2)\right)}{\sum(x_{i1} - \bar{x}_1)^2}$$