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Interaction of swell with localized current

Han Wang, Bia Villas Bôas, Jacques Vanneste and Bill Young

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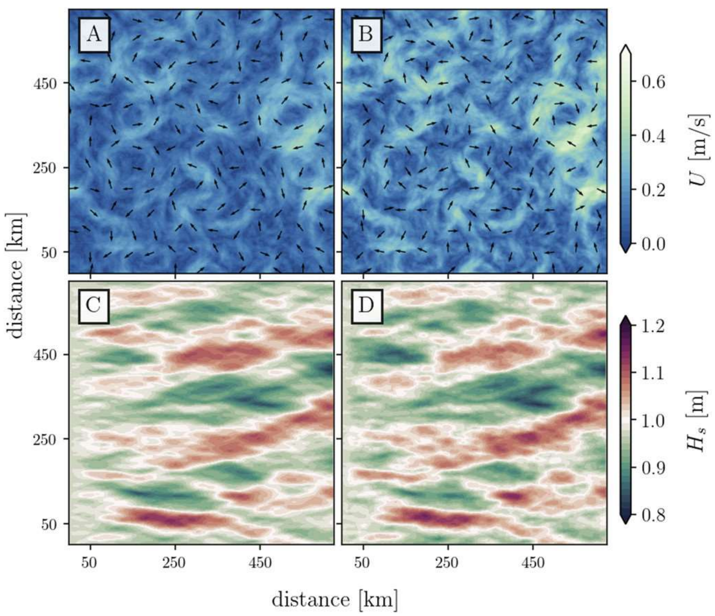
Motivation



Bôas, A. B. V., Cornuelle, B. D., Mazloff, M. R., Gille, S. T., & Ardhuin, F. (2020).

- Swell-like waves interact with currents
- How to explain significant wave height patterns?

Setup and assumptions

- Wave action conservation

$$\cancel{\partial_t \mathcal{A}} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}}\mathcal{A} - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}}\mathcal{A} = 0$$

wave action spectrum:

$$\mathcal{A} = \mathcal{A}(\boldsymbol{x}, \boldsymbol{k}), \text{ where}$$
$$\boldsymbol{x} = (x, y), \quad \boldsymbol{k} = (k, \theta)$$

frequency:

$$\omega(\boldsymbol{x}, \boldsymbol{k}) = \sigma(k) + \boldsymbol{k} \cdot \boldsymbol{U}(\boldsymbol{x}),$$

- Swell-like incoming wave:

$$\mathcal{A}(x, y, k, \theta) = \mathcal{A}_\star(K, \Theta) \quad \text{as} \quad x \to -\infty$$

localized around $k = k_\star$ and $\theta = 0$

$$K = \frac{k - k_\star}{\delta} \quad \text{and} \quad \Theta = \frac{\theta}{\delta}, \text{ where } \boxed{\delta \ll 1}$$

K and Θ are both $O(1)$.

- Weak current

$$\boxed{\varepsilon \stackrel{\text{def}}{=} U/c_\star \ll 1,}$$

where $c_\star$ is intrinsic group velocity at $(k_\star, 0)$

- Localized current

$$U << 1 \text{ at } |\boldsymbol{x}| >> 1$$

$$\cancel{\partial_t \mathcal{A}} + \nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}}\mathcal{A} - \nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}}\mathcal{A} = 0$$

Matched Asymptotics

- Distinguished limit

$$\gamma = \varepsilon/\delta = O(1)$$

- Advection

$$\nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}}\mathcal{A} \approx c_\star\partial_x\mathcal{A} + \delta c_\star\Theta\partial_y\mathcal{A}$$

Other terms ignored due to weak current

- Refraction

$$\nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}}\mathcal{A} \approx \gamma k_\star\partial_x U\partial_K\mathcal{A} + \gamma\partial_y U\partial_\Theta\mathcal{A}$$

Simplified from $K = \frac{k-k_\star}{\delta}$ and $\Theta = \frac{\theta}{\delta}$.

$$\cancel{\partial_t \mathcal{A}} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}}\mathcal{A} - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}}\mathcal{A} = 0$$

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Simplified from $K = \frac{k - k_\star}{\delta} \quad \text{and} \quad \Theta = \frac{\theta}{\delta}.$

- For $x = O(1)$

$$\cancel{\partial_t \mathcal{A}} + \nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}}\mathcal{A} - \nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}}\mathcal{A} = 0$$

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Simplified from $K = \frac{k-k_\star}{\delta}$ and $\Theta = \frac{\theta}{\delta}$.

- For $x = O(\delta^{-1})$

$$X = \delta x = O(1)$$

Matched Asymptotics

- Incident:

$$\mathcal{A}(x,y,k,\theta) = \mathcal{A}_\star(K,\Theta) \quad \text{as} \quad x \to -\infty$$ (1)

- For $x = O(1)$: Matching (1)

→ $$\mathcal{A} = \mathcal{A}_\star\left(K + \frac{\gamma k_\star}{c_\star} U(x,y)\,, \Theta + \frac{\gamma}{c_\star} V(x,y) - \frac{\gamma}{c_\star}\int_{-\infty}^{x} Z(x',y)\,\mathrm{d}x'\right),$$ (2)

where $Z \stackrel{\text{def}}{=} V_x - U_y$

- For $X = \delta x = O(1)$: Matching (2) at $x \to \infty$ as $X \to 0$

→ $$\mathcal{A} = \mathcal{A}_\star\left(K, \Theta - \gamma\Delta(y - X\Theta)\right).$$

where $\Delta(y) \stackrel{\text{def}}{=} \frac{1}{c_\star}\int_{-\infty}^{\infty} Z(x',y)\,\mathrm{d}x'.$

Matched Asymptotics

- Incident:

$$\boxed{\mathcal{A}(x,y,k,\theta)=\mathcal{A}_\star(K,\Theta) \quad \text{as} \quad x\to-\infty}$$ 1

- For $x=O(1)$: Matching 1

$$\to \boxed{\mathcal{A}=\mathcal{A}_\star\left(K+\frac{\gamma k_\star}{c_\star}U(x,y)\,,\Theta+\frac{\gamma}{c_\star}V(x,y)-\frac{\gamma}{c_\star}\int_{-\infty}^{x}Z(x',y)\,\mathrm{d}x'\right)}\,, \quad 2$$

where $Z\stackrel{\text{def}}{=}V_x-U_y$

- For $X=\delta x=O(1)$: Matching 2 at $x\to\infty$ as $X\to 0$

$$\to \boxed{\mathcal{A}=\mathcal{A}_\star\left(K,\Theta-\gamma\Delta(y-X\Theta)\right)}\,.$$

where $\Delta(y)\stackrel{\text{def}}{=}\frac{1}{c_\star}\int_{-\infty}^{\infty}Z(x',y)\,\mathrm{d}x'.$

Dysthe's formula

$$\boxed{\text{ray curvature}=\frac{\text{vorticity}}{\text{group velocity}}}.$$

$\to$ $\Delta(y)\stackrel{\text{def}}{=}\frac{1}{c_\star}\int_{-\infty}^{\infty}Z(x',y)\,\mathrm{d}x'$ is ray curvature accumulated over $x=O(1)$

Comparison with WAVEWATCH III

- Current: Gaussian vortex localized within $|\boldsymbol{x}| = r_v$

$$Z = \frac{\kappa}{2\pi r_v^2} \mathrm{e}^{-(x^2+y^2)/(2r_v^2)}.$$

- Swell: LHCS [Longuet-Higgins, Cartwright & Smith (1963)] model

$$\mathcal{A}_\star = F(k) \times N(s) \cos^{2s}(\theta/2)$$

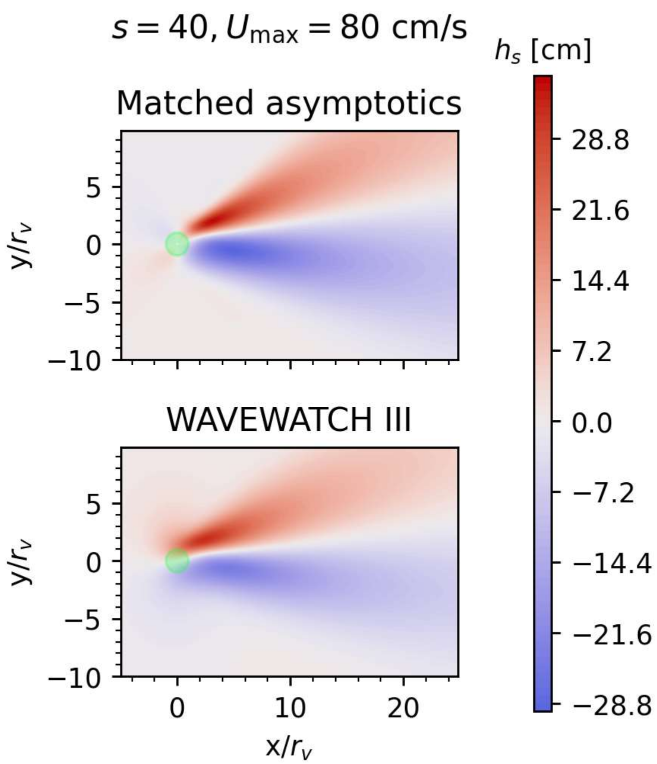
$$\mathcal{A}_\star \approx \delta^{-1} F(k) G(\Theta), \text{ where } \delta = 2/\sqrt{s}$$

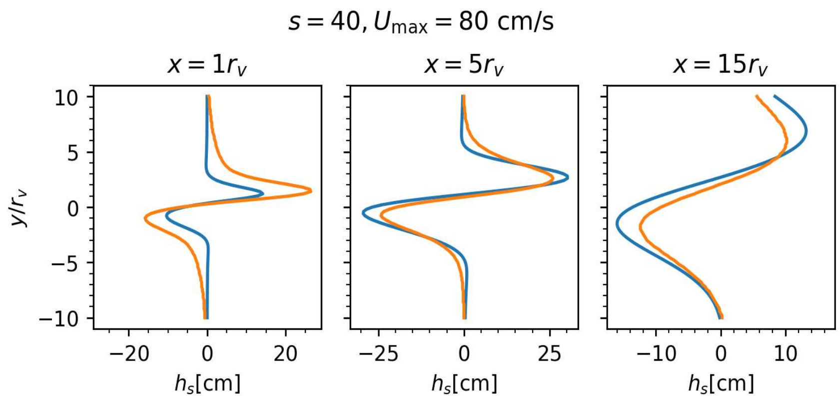
- Typical values of s for ocean swells: $s = 10 - 20$ [Ewans, Kevin C. (2001)]
- Compare significant wave height disturbance

$$h_s(\boldsymbol{x}) = 4 \left(g^{-1} \int \sigma(k) \mathcal{A}(\boldsymbol{x}, \boldsymbol{k}, t) \, \mathrm{d}\boldsymbol{k} \right)^{1/2} - \bar{H}_s, \text{ where } \bar{H}_s \text{ is basic state}$$

Typical ocean swells: $s = 10 - 20$

Comparison with WAVEWATCH III



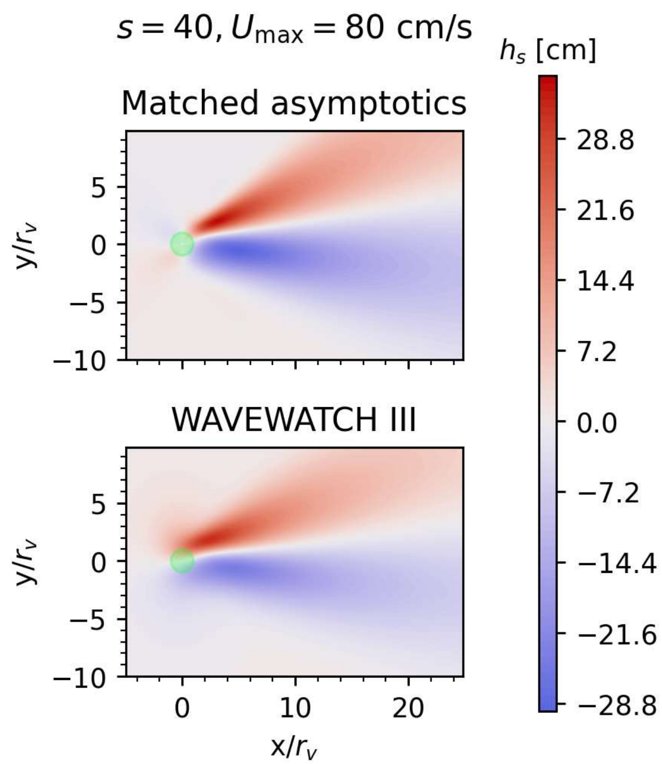


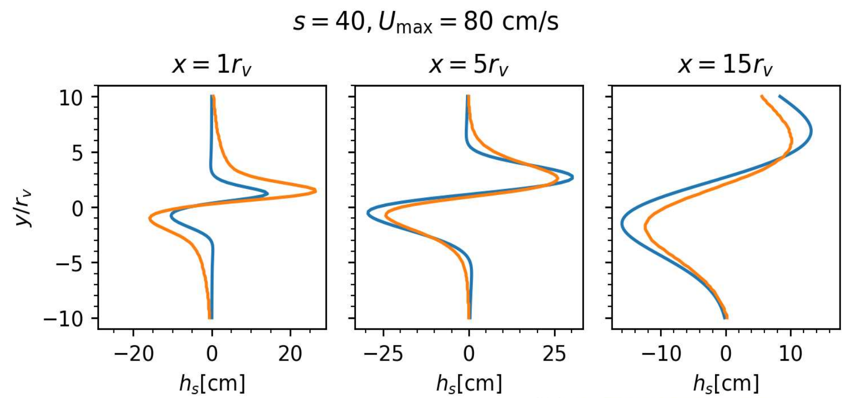
In this example,

$$\delta \approx 0.3,$$
$$\epsilon \approx 0.1,$$
$$\gamma \approx 0.3$$

Typical ocean swells: $s = 10 - 20$

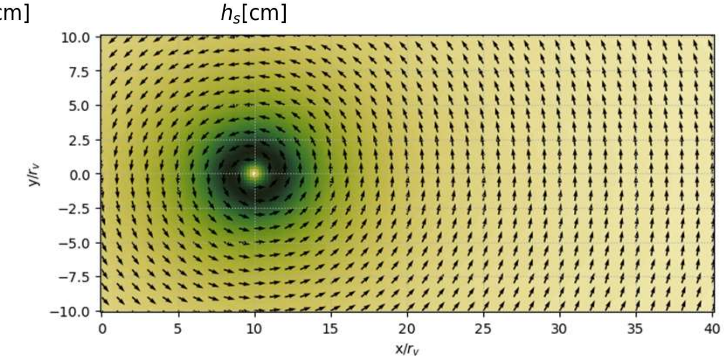
Comparison with WAVEWATCH III





Captures:

- Amplitude
  $\bar{H}_s = 100\text{cm}$
  → ~30% change
- Asymmetry



Typical ocean swells: $s = 10 - 20$

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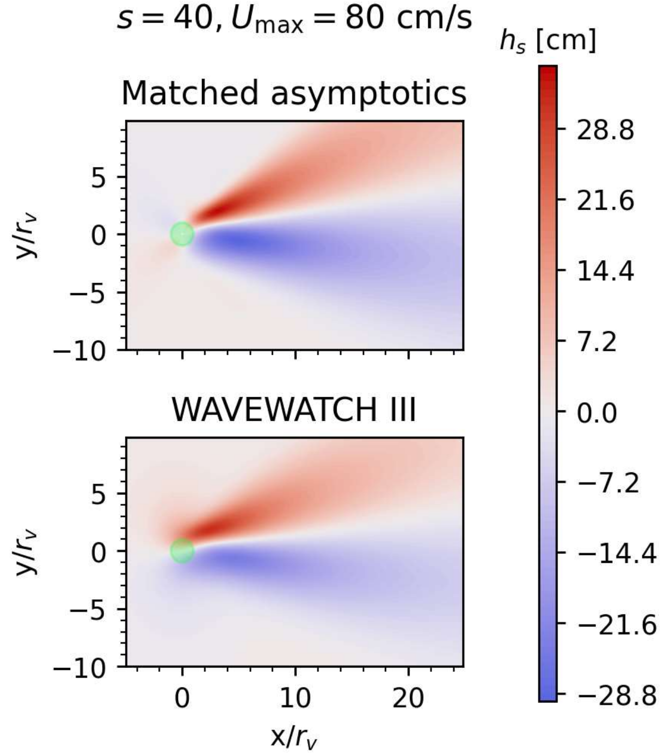

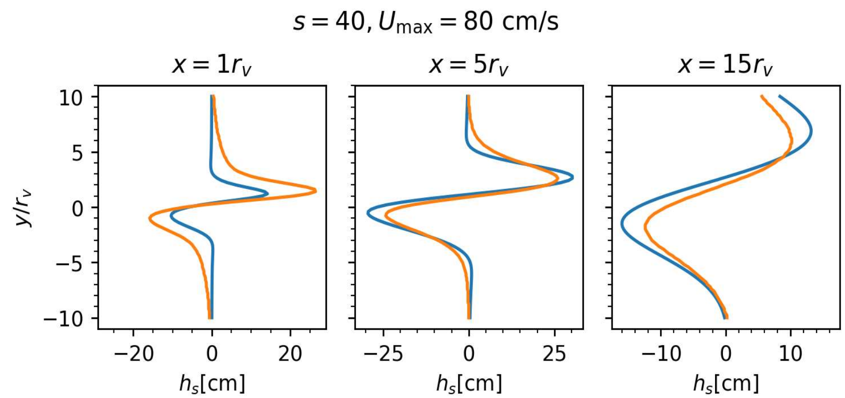

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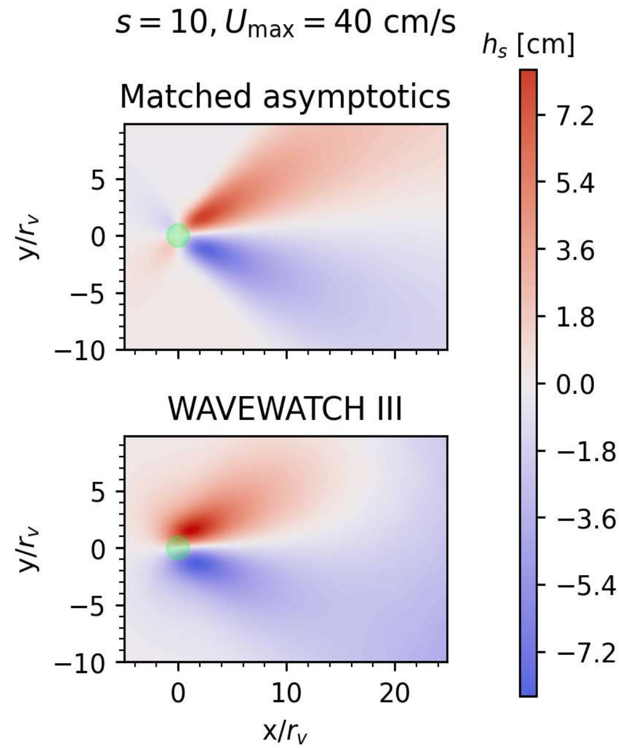
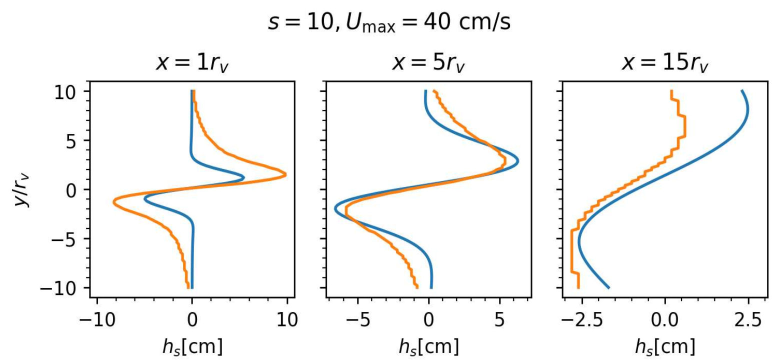




In this example,

$$\boxed{\delta \approx 0.6,}$$
$$\epsilon \approx 0.05,$$
$$\gamma \approx 0.1$$

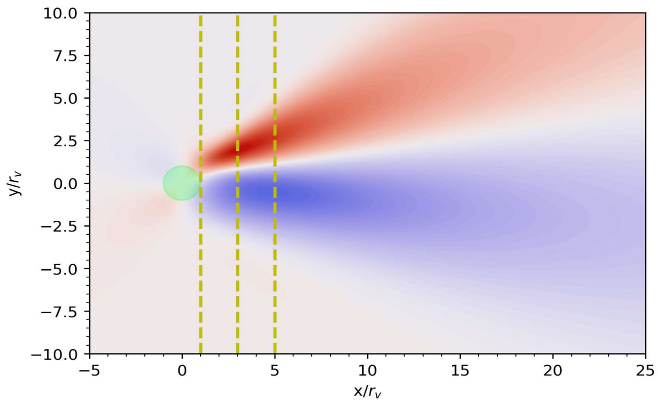
Typical ocean swells: $s = 10 - 20$

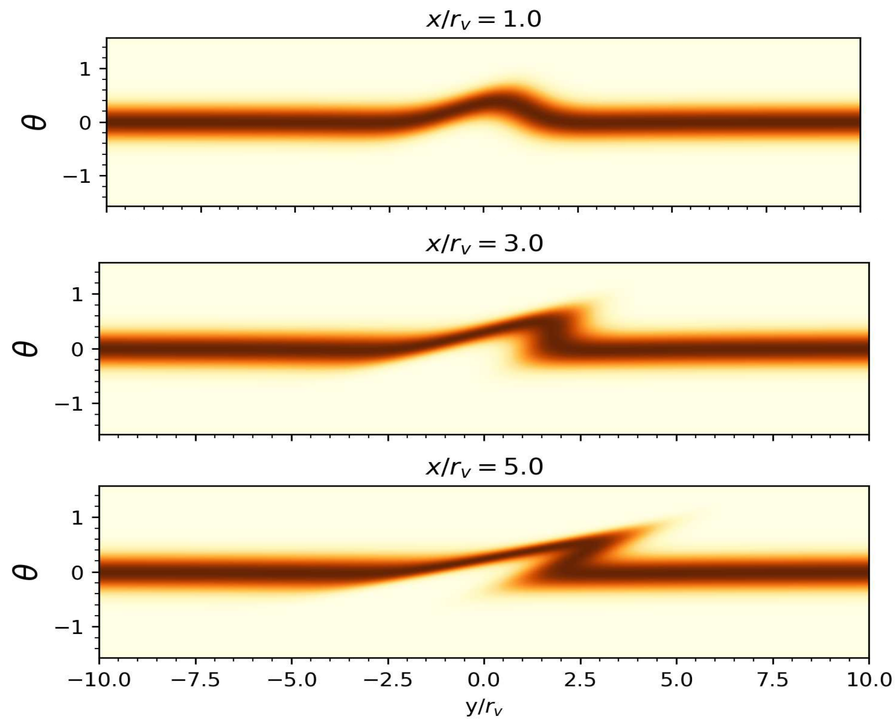
Caustics

$\int \mathcal{A}(x, y, k, \theta)\, \mathrm{d}k,$

Caustics: rays at different wavenumbers cross at the same location in real space

$$U_{\max} = 80\mathrm{cm/s}, s = 40$$





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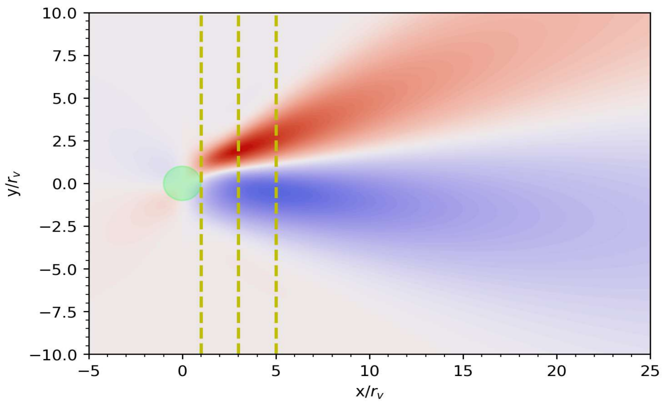
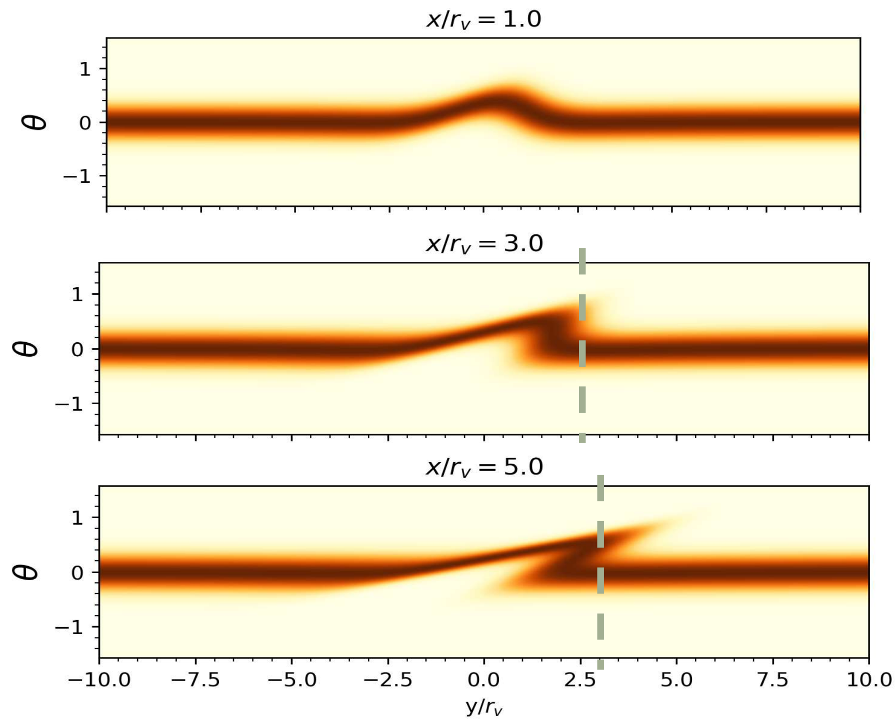


$$\int \mathcal{A}(x, y, k, \theta)\, \mathrm{d}k,$$



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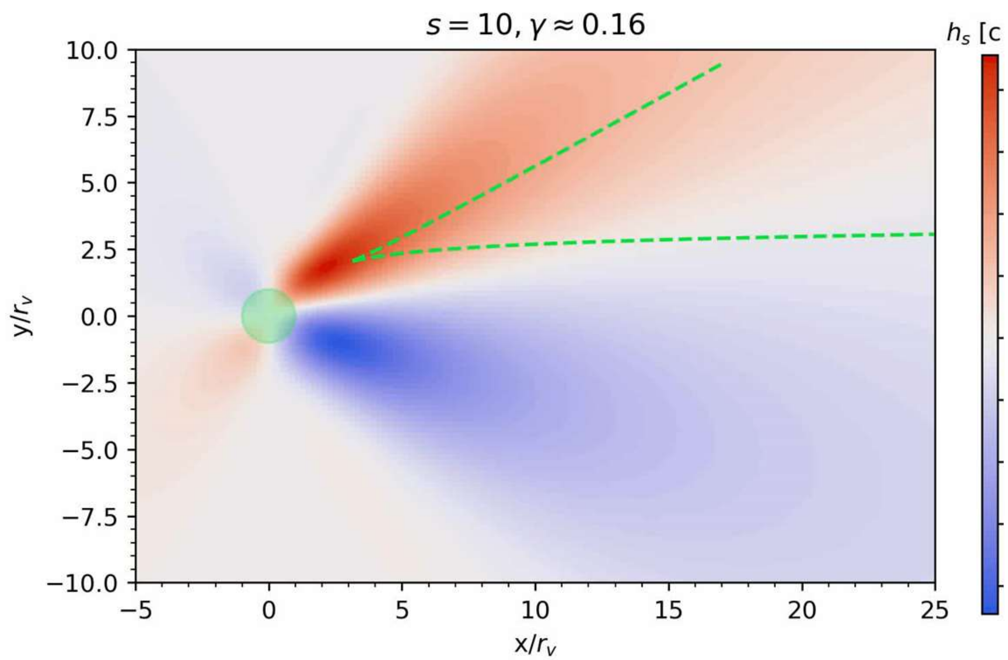
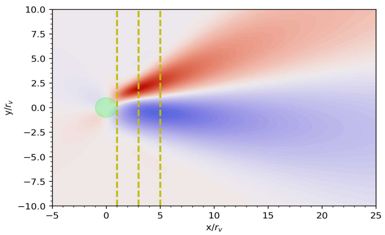
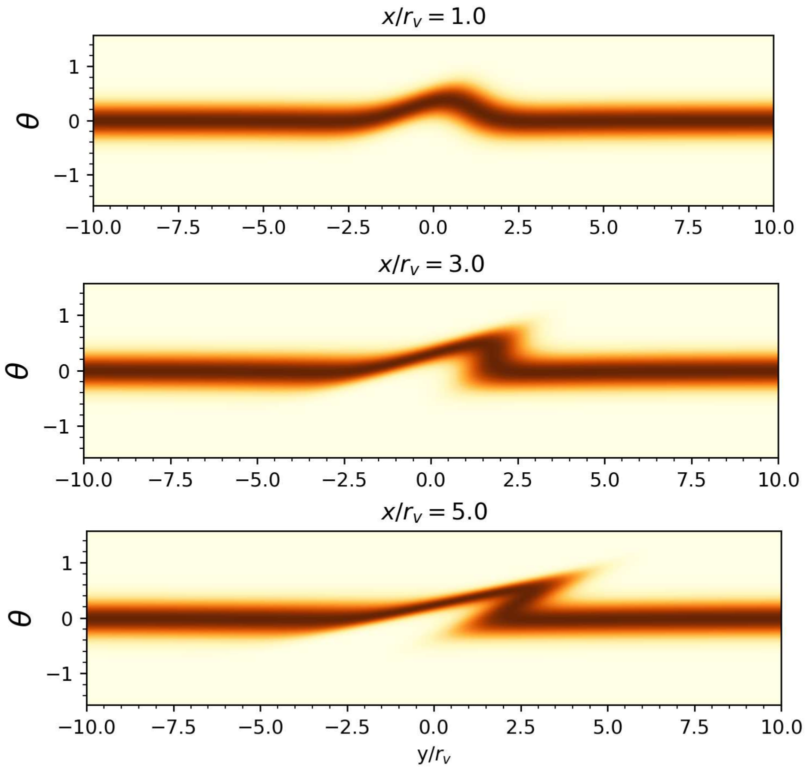
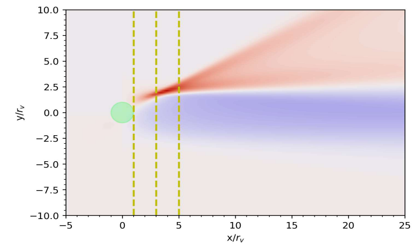


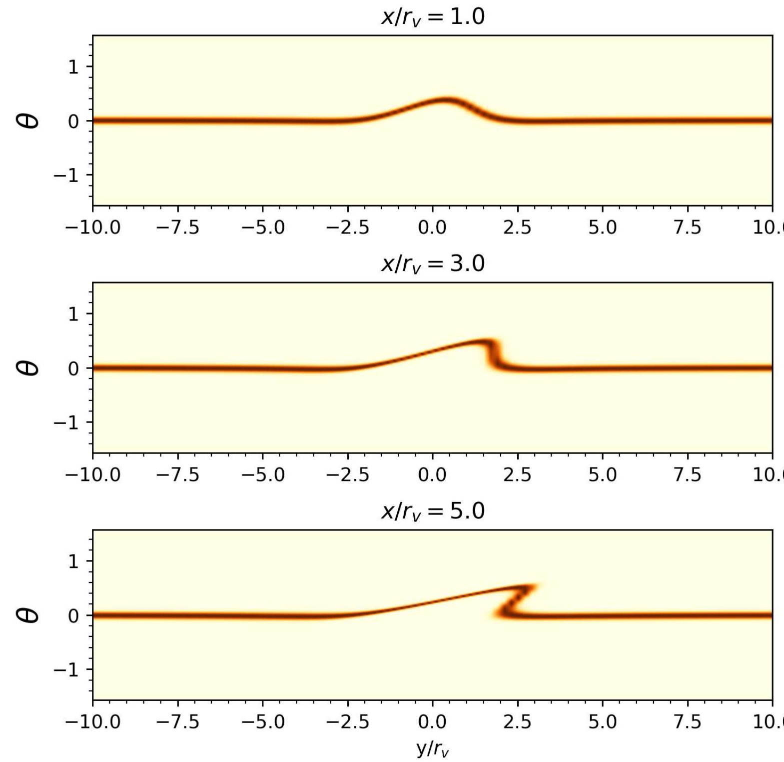
s=40
→more
spread out





s=1000
→more
localized





Main messages

- Solution found (swell & localized, weak current)

  Can modify for currents other than Gaussian vortex (e.g. Gaussian dipole)

- Caustics are not relevant (i.e. no singularity in h_s) under realistic parameters

  Key difference between e.g. [White and Fornberg (1997)]:

  $\mathcal{A}$ has angular spread; localized, but not plane wave

Next step: turbulent flows

- Have progress in the regime

$$\epsilon \ll 1 \text{ and } \boxed{\epsilon \ll \delta}$$

(δ doesn't need to be small)

- Current doesn't need to be localized
- Found deterministic and spectral relationship between current velocities and h_s

Thank you!

- Contact: hwang310@ed.ac.uk , or

491 - Observations and Modeling of Current Effects on Waves during the S-Mode Pilot Campaign

Tuesday, January 10, 2023

5:00 PM - 6:30 PM

Colorado Convention Center - Hall A (Exhibit Hall Level)