# Scattering of surface waves by oceanic currents: the U2H map

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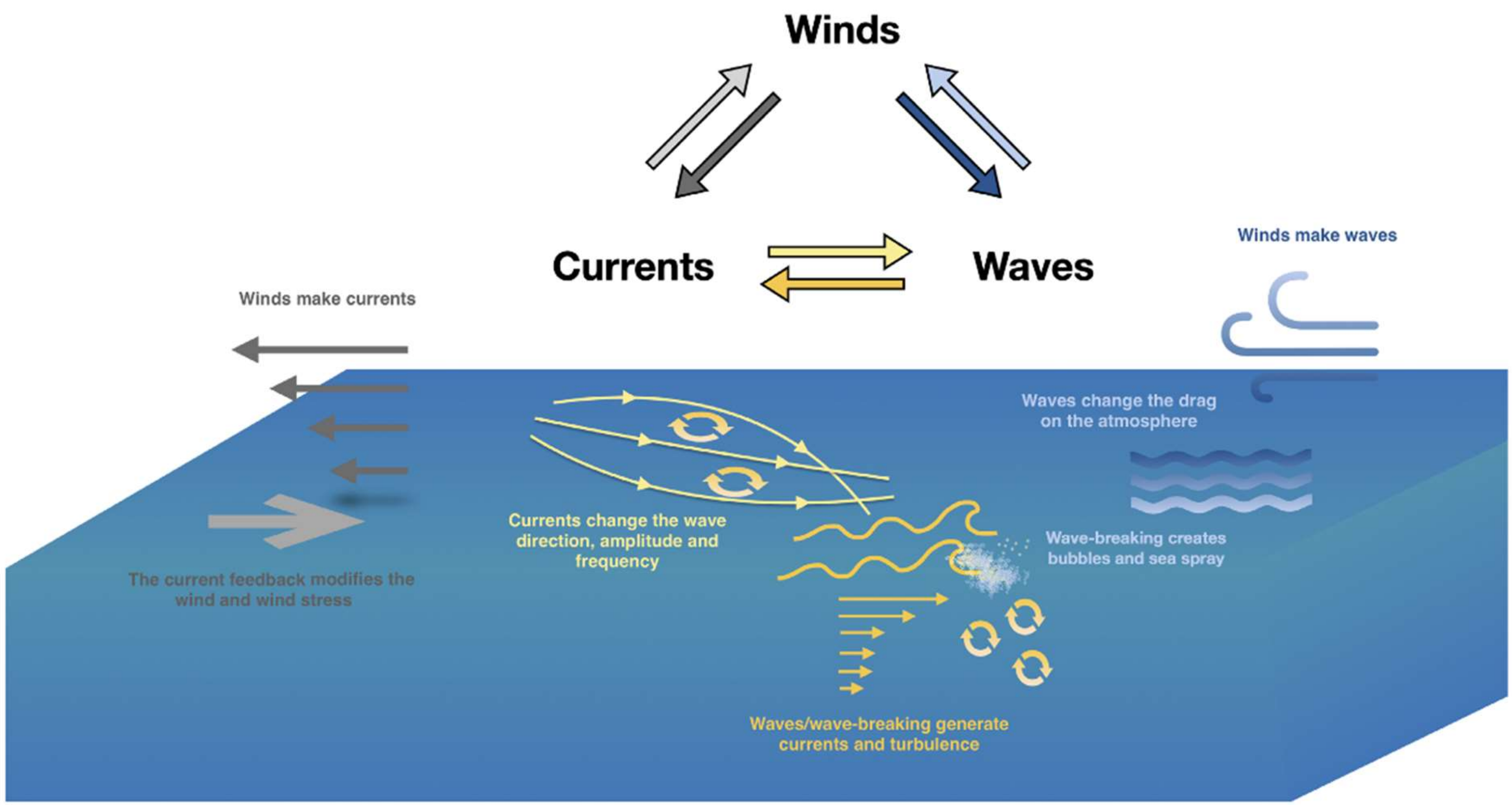
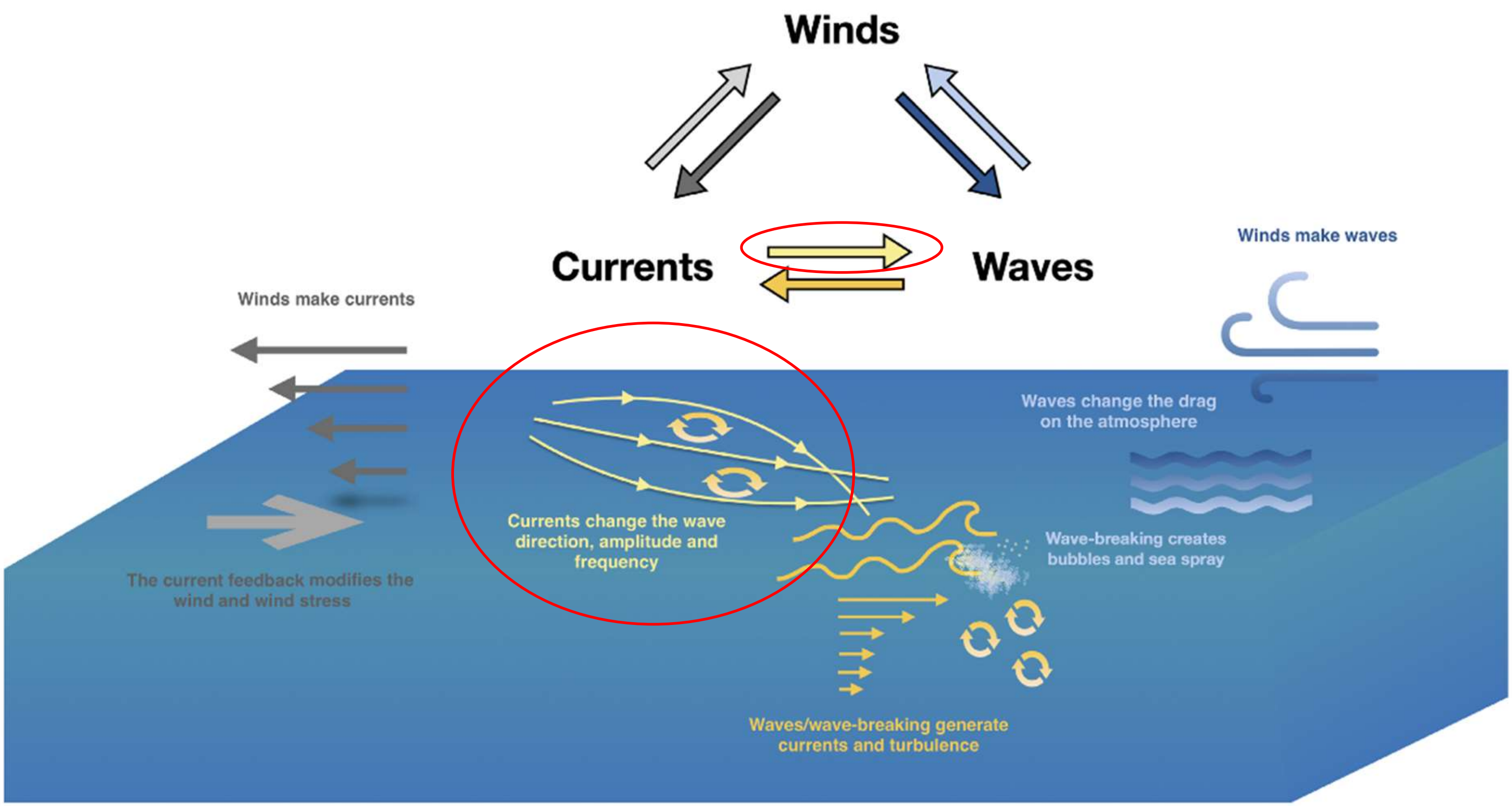


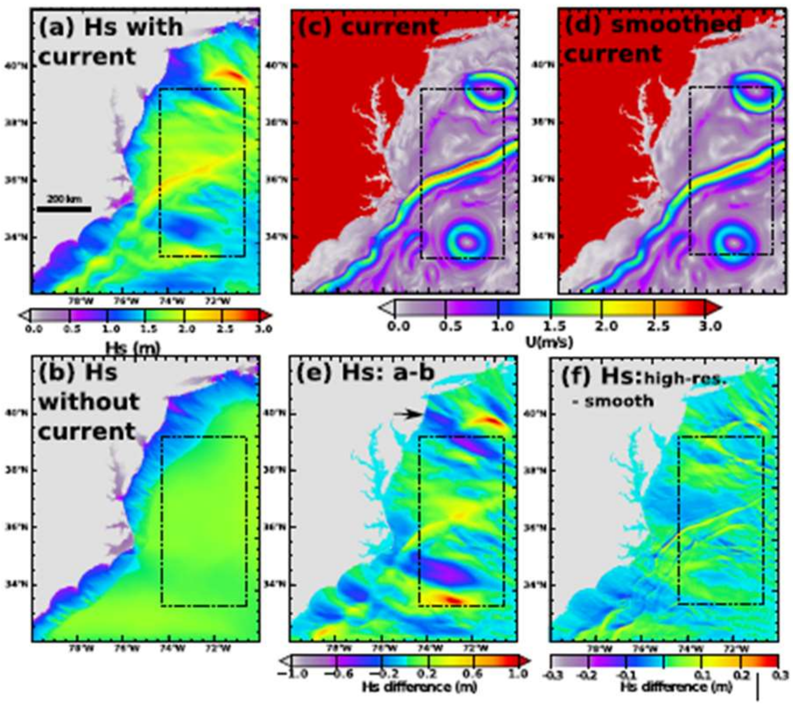

Villas Bôas and Pizzo (2021)




Villas Bôas and Pizzo (2021)

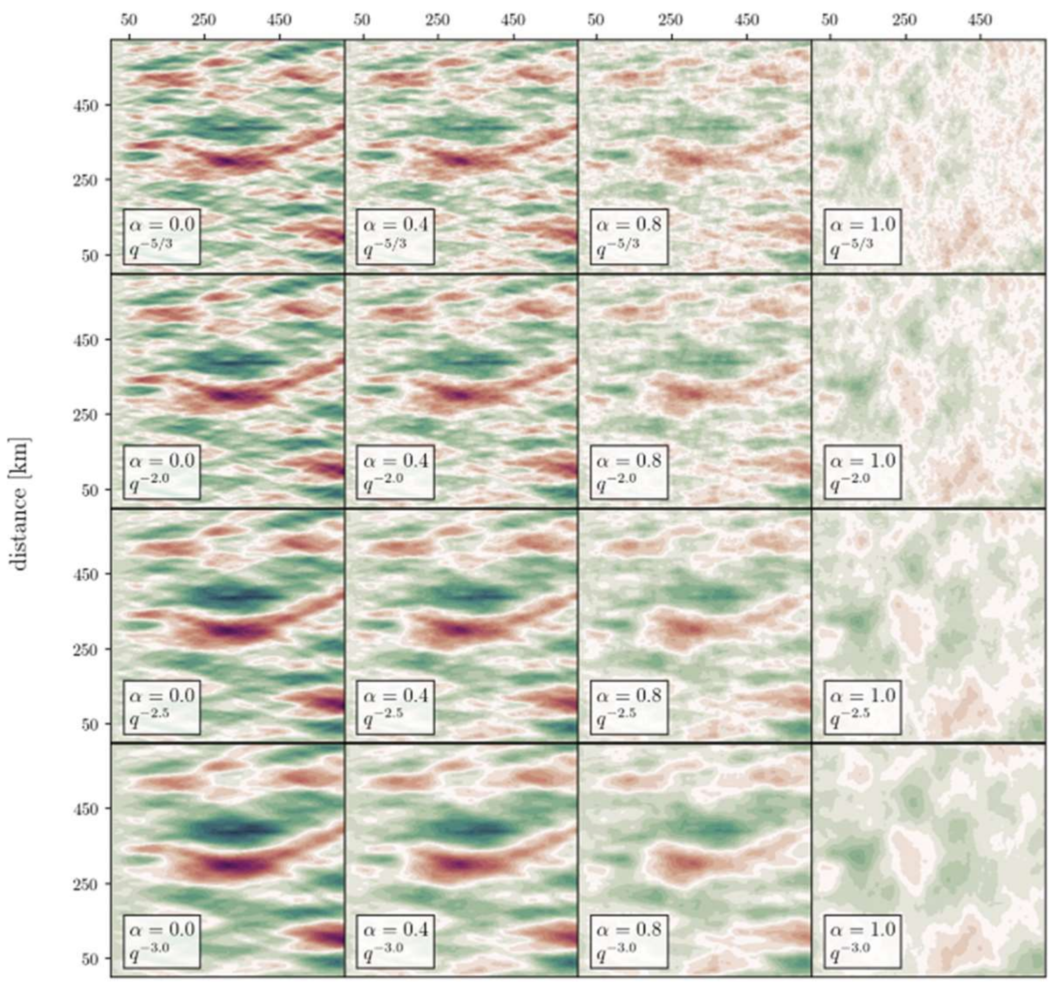
## Small-scale open ocean currents have large effects on wind wave heights

Fabrice Ardhuin[1], Sarah T. Gille[2], Dimitris Menemenlis[3], Cesar B. Rocha[2], Nicolas Rascle[1], Bertrand Chapron[1], Jonathan Gula[1], and Jeroen Molemaker[1]




- Significant Wave Height (Hs)
  Hs = 4 × sea surface height rms
- Spatial variability of the Hs at scales between 10-100km is driven by currents

More vorticity → More divergence




Shallower Spectral Slope

Steeper Spectral Slope

- Different impacts from rotational and divergent currents

$$\boldsymbol{U} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}^{\perp}\psi\,,$$

- Rotational currents induce more changes than divergent currents, Hs more streaky
- Shallower spectral slope in KE -> finer structures in Hs

Bôas, A. B. V., Cornuelle, B. D., Mazloff, M. R., Gille, S. T., & Ardhuin, F. (2020).

# The U2H map

- Map current velocity ("U") to Hs ("H")
- Assumptions:

1. scale separation between waves and currents

$$\partial_t \mathcal{A} + \boldsymbol{\nabla}_{\boldsymbol{k}}\omega \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}\mathcal{A} - \boldsymbol{\nabla}_{\boldsymbol{x}}\omega \cdot \boldsymbol{\nabla}_{\boldsymbol{k}}\mathcal{A} = 0$$

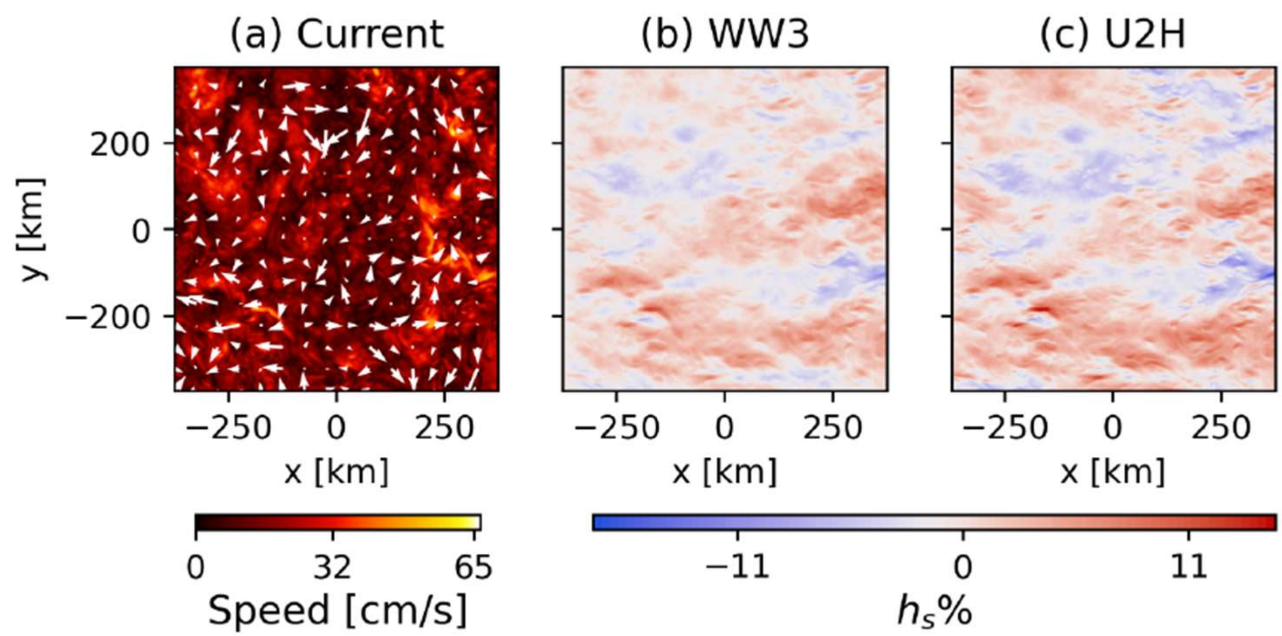
where $\mathcal{A} = \mathcal{A}(\boldsymbol{x}, \boldsymbol{k}, t)$ wave-action density in position-wavevector space

$\omega(\boldsymbol{x}, \boldsymbol{k}) = \sigma(k) + \boldsymbol{k}\cdot\boldsymbol{U}(\boldsymbol{x}),$ deep-water surface gravity waves $\sigma(k) = \sqrt{gk}$

2. weak current

$$\varepsilon = U/c \ll 1$$

# It works well



# Perturbation expansion

- $\partial_t \mathcal{A} + \boldsymbol{\nabla}_{\boldsymbol{k}}\omega \cdot \boldsymbol{\nabla}_{\boldsymbol{x}}\mathcal{A} - \boldsymbol{\nabla}_{\boldsymbol{x}}\omega \cdot \boldsymbol{\nabla}_{\boldsymbol{k}}\mathcal{A} = 0$
- Small parameter $\varepsilon = U/c \ll 1$
- $\omega(\boldsymbol{x}, \boldsymbol{k}) = \sigma(k) + \varepsilon \boldsymbol{k} \cdot \boldsymbol{U}(\boldsymbol{x}).$
  $\mathcal{A}(\boldsymbol{x}, \boldsymbol{k}, t) = \boxed{\bar{A}(\boldsymbol{k})} + \varepsilon a(\boldsymbol{x}, \boldsymbol{k}, t) + O(\varepsilon^2).$

Background, spatially uniform wave action

- $O(\varepsilon)$ terms:

$$(\boxed{\mu} + \boldsymbol{c} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})a = (\boldsymbol{\nabla}_{\boldsymbol{k}}\bar{A} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{U} \cdot \boldsymbol{k}$$

Steady state; dissipation term to enforce causality

# Linear relation

$$(\mu + \boldsymbol{c} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boxed{a} = (\boldsymbol{\nabla}_{\boldsymbol{k}}\bar{A} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{U} \cdot \boldsymbol{k}$$

Change in wave action. Linear in ***U***

# Hs and wave action

- Hs = 4 × sea surface height rms
- Wave action × intrinsic frequency = KE+PE
- For deep water surface gravity waves, KE=PE

$$H_s(\boldsymbol{x},t) = 4\left(g^{-1}\int \sigma(k)\mathcal{A}(\boldsymbol{x},\boldsymbol{k},t)\,\mathrm{d}\boldsymbol{k}\right)^{1/2} .$$

- Expand $H_s(\boldsymbol{x}) = \bar{H}_s + \varepsilon h_\mathrm{s}(\boldsymbol{x}) + O(\varepsilon^2)$

- Therefore $h_\mathrm{s}(\boldsymbol{x}) = \frac{8}{g\bar{H}_s}\int \sigma(k)a(\boldsymbol{x},\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} .$

hs (Changes in Hs) is linear in $a$. Therefore hs is linear in ***U.***

# Linear relation

$$(\mu + \boldsymbol{c} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})a = (\boldsymbol{\nabla}_{\boldsymbol{k}} \bar{A} \cdot \boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{U} \cdot \boldsymbol{k}$$

$$h_{\mathrm{s}}(\boldsymbol{x}) = \frac{8}{g\bar{H}_s} \int \sigma(k) a(\boldsymbol{x}, \boldsymbol{k}) \, \mathrm{d}\boldsymbol{k} \, .$$

Solve in Fourier space

$$\hat{a}(\boldsymbol{q}, \boldsymbol{k}) \stackrel{\text{def}}{=} \int a(\boldsymbol{x}, \boldsymbol{k}) \, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \, \mathrm{d}\boldsymbol{x}$$

Spatial variations of:

wave phase

currents

$$\boldsymbol{k} = k \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \boldsymbol{q} = q \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

# Linear relation

$$\boxed{\begin{aligned}&(\mu + \boldsymbol{c}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}})a = (\boldsymbol{\nabla}_{\boldsymbol{k}}\bar{A}\cdot\boldsymbol{\nabla}_{\boldsymbol{x}})\boldsymbol{U}\cdot\boldsymbol{k}\\ &h_{\mathrm{s}}(\boldsymbol{x}) = \tfrac{8}{g\bar{H}_s}\int\sigma(k)a(\boldsymbol{x},\boldsymbol{k})\,\mathrm{d}\boldsymbol{k}\,.\end{aligned}}$$

Spatial variations of:

wave phase

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Solve in Fourier space

$$\hat{a}(\boldsymbol{q},\boldsymbol{k}) \stackrel{\mathrm{def}}{=} \int a(\boldsymbol{x},\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{x}$$

We get a computable form (after some maneuvers):

$$\boxed{\hat{h}_s(\boldsymbol{q}) = \hat{\boldsymbol{L}}(\boldsymbol{q})\cdot\hat{\boldsymbol{U}}(\boldsymbol{q})\,,}\qquad \hat{\boldsymbol{L}}(\boldsymbol{q}) = \tfrac{16}{g\bar{H}_s}\left(\boldsymbol{e}_{\boldsymbol{q}}^{\perp}\textstyle\sum_{n=-\infty}^{\infty} n(-\mathrm{i})^{|n|}\,2\pi p_n\,\mathrm{e}^{n\mathrm{i}\varphi} - 2\boldsymbol{P}\right)$$

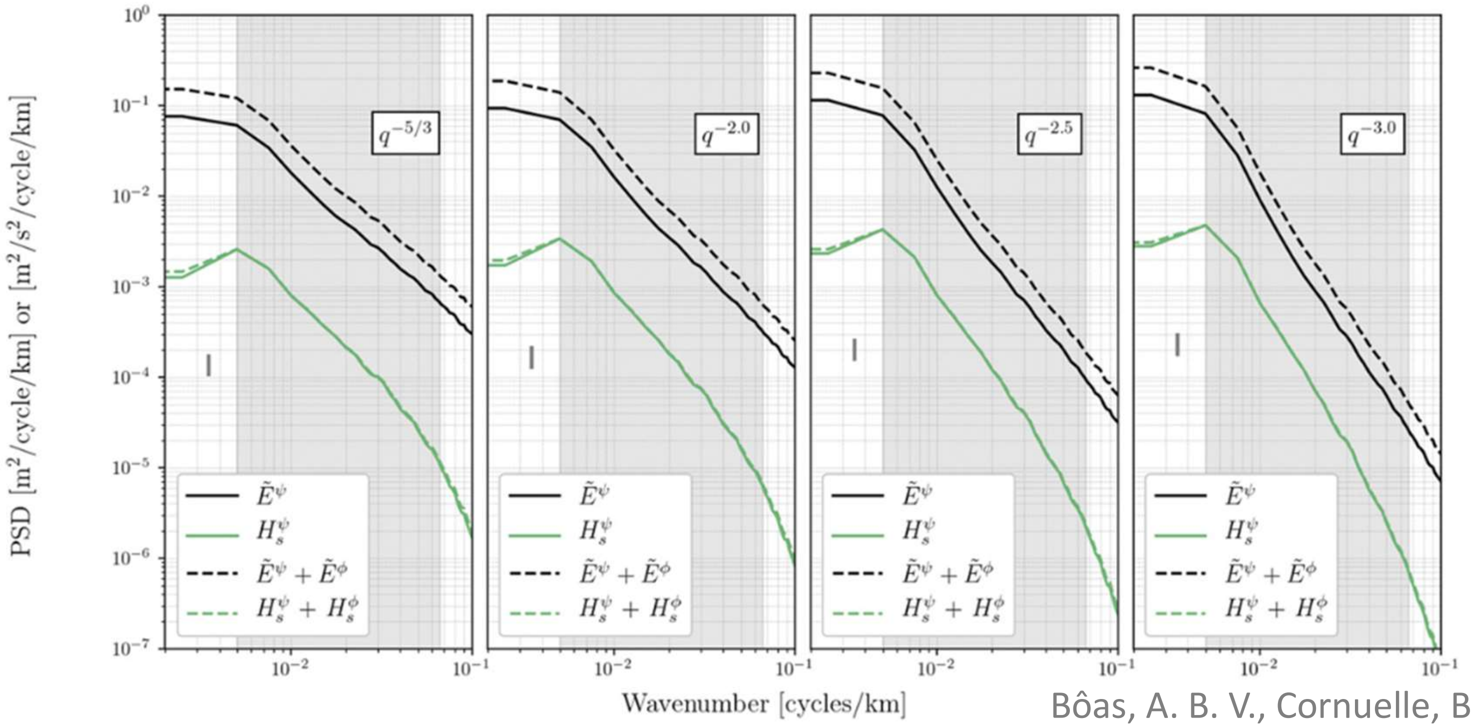
where $\boldsymbol{e}_{\boldsymbol{q}}^{\perp} \stackrel{\mathrm{def}}{=} (-\sin\varphi,\cos\varphi)$, $\boldsymbol{P} \stackrel{\mathrm{def}}{=} \int\bar{A}(\boldsymbol{k})\boldsymbol{k}\,\mathrm{d}\boldsymbol{k}$,

$$\boldsymbol{P} = \oint\mathcal{P}(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\mathrm{d}\theta,\ \mathcal{P}(\theta) = \textstyle\sum_{n=-\infty}^{\infty} p_n\,\mathrm{e}^{n\mathrm{i}\theta}$$

$$\Longrightarrow\quad \hat{\boldsymbol{L}}(\boldsymbol{q}) = \hat{\boldsymbol{L}}(\varphi)$$

$$\hat{h}_s(\boldsymbol{q}) = \hat{\boldsymbol{L}}(\varphi) \cdot \hat{\boldsymbol{U}}(\boldsymbol{q})\,,$$

- Significant wave height should have similar spatial scales as currents

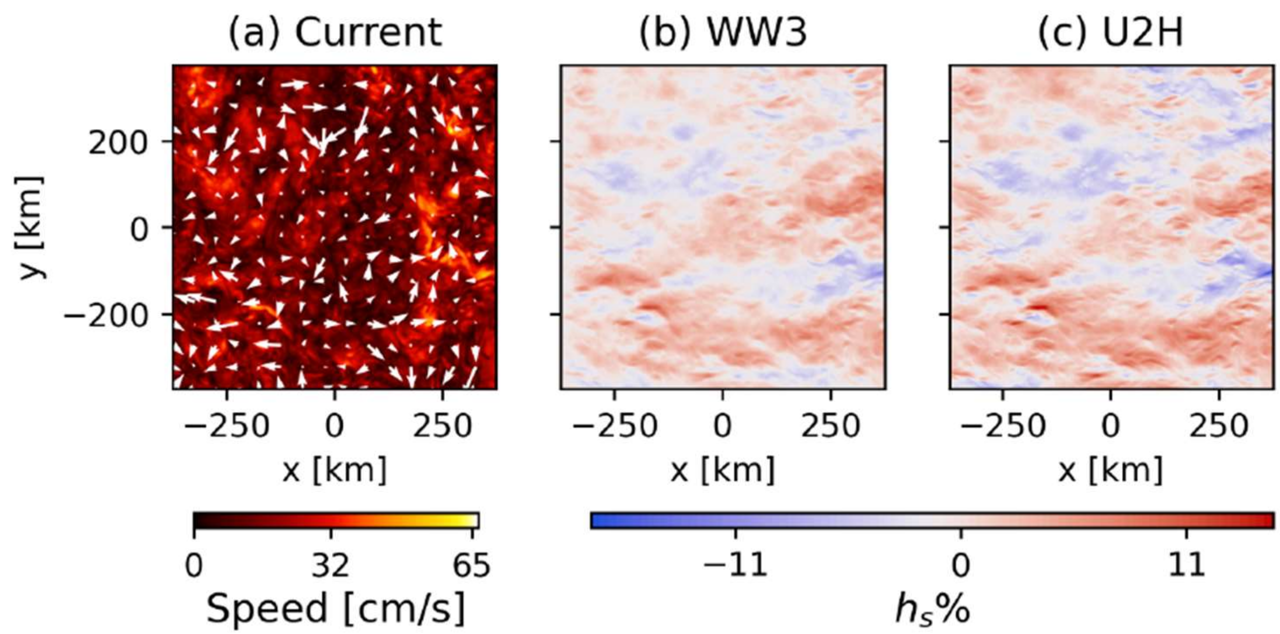



Bôas, A. B. V., Cornuelle, B. D., Mazloff, M. R., Gille, S. T., & Ardhuin, F. (2020).

$$\hat{h}_s(\boldsymbol{q}) = \hat{\boldsymbol{L}}(\varphi) \cdot \hat{\boldsymbol{U}}(\boldsymbol{q})\,,$$

$\hat{\boldsymbol{L}}(\varphi)$ is computable from $\bar{A}(\boldsymbol{k})$

- Can compute significant wave height snapshots from current snapshots



# Swells

Figure from Phillip Capper, Wikipedia

- Long wavelengths (large group velocity) $\varepsilon = U/c \ll 1$
- Highly directional

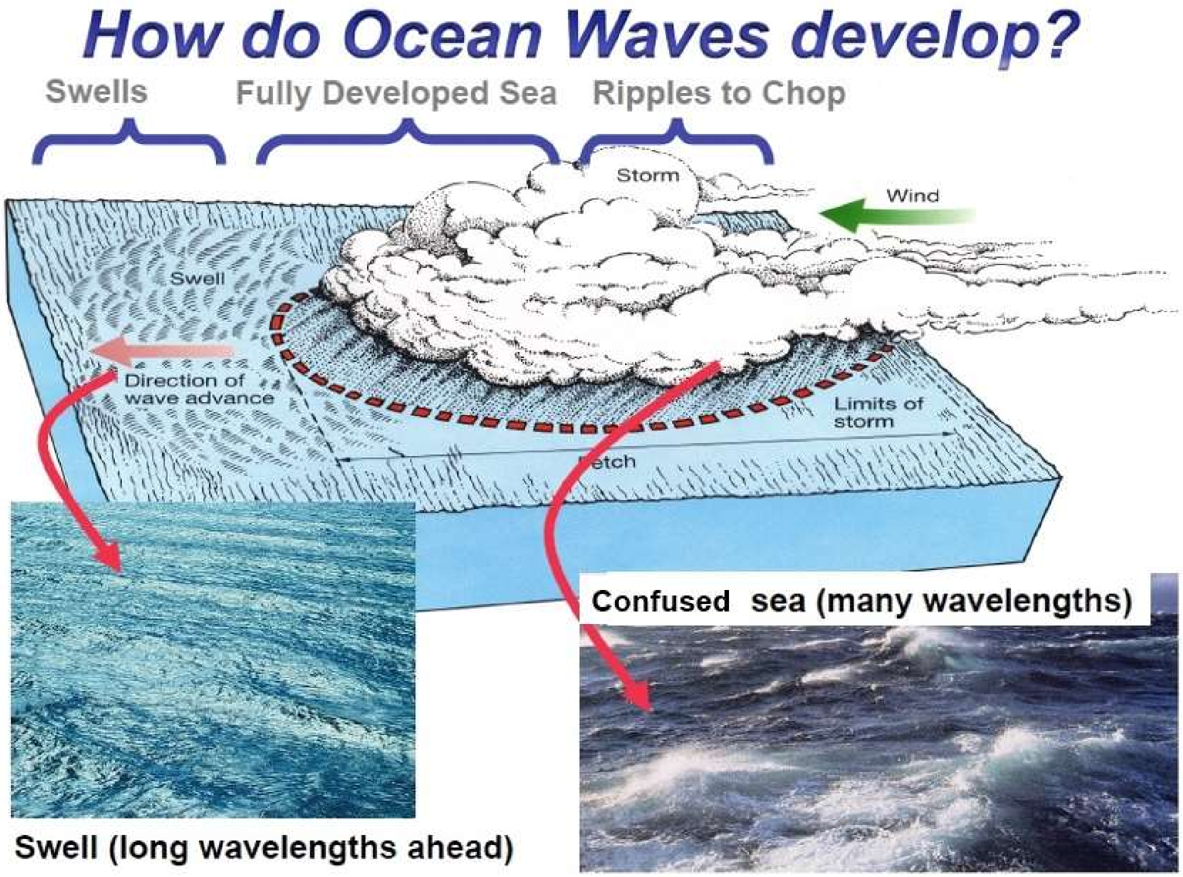



Figure from UBC EOSC 114 course

# Swells

Figure from Phillip Capper, Wikipedia

- Long wavelengths (large group velocity)
  $\varepsilon = U/c \ll 1$
- Highly directional

Another small parameter $\delta \ll 1$

$$\bar{A}(k,\theta) = \delta^{-1}\bar{A}(k,\Theta),$$
$$\Theta = \theta/\delta.$$

# U2H under the swell limit

- Helmholtz decomposition

$$\boldsymbol{U} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}^{\perp}\psi\,,$$

$$\hat{h}_s(\boldsymbol{q}) = \mathrm{i}q\hat{L}_{||}(\varphi)\hat{\phi}(\boldsymbol{q}) + \mathrm{i}q\hat{L}_{\perp}(\varphi)\hat{\psi}(\boldsymbol{q})\,.$$

Under the swell limit $\delta \ll 1$ , we find:

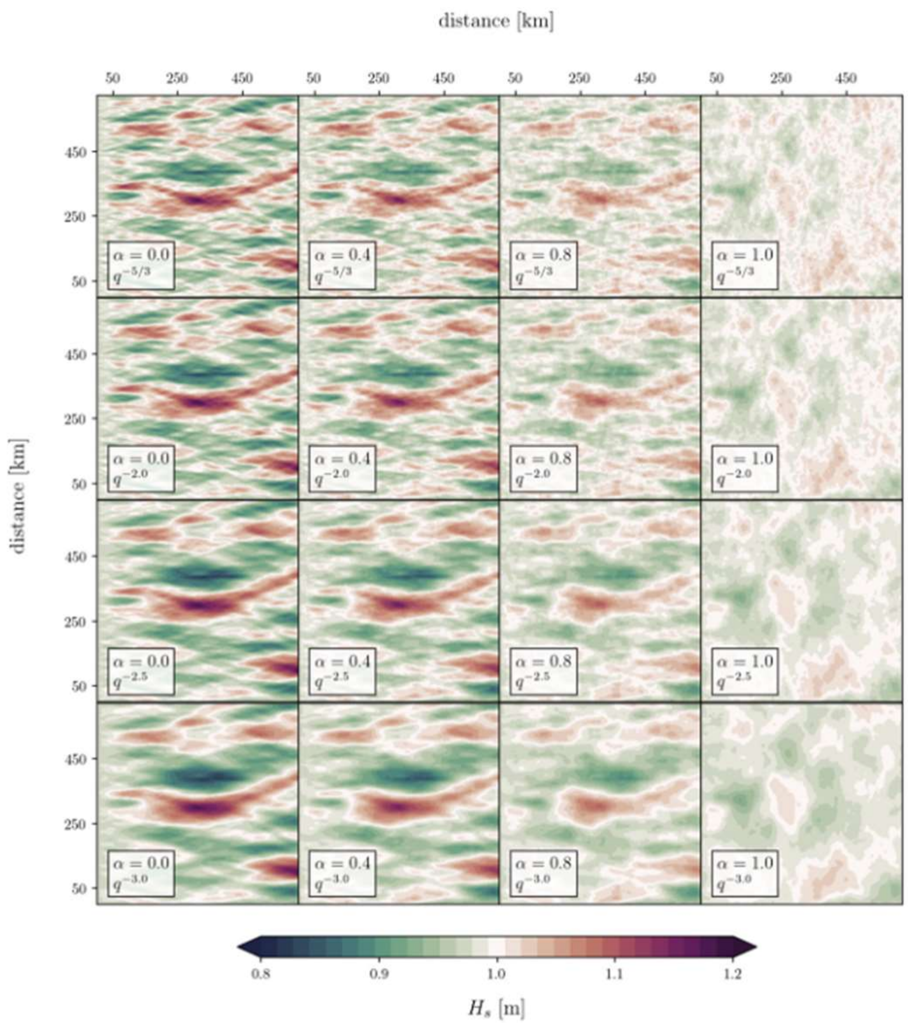
- $\hat{L}_{\perp} = O(\delta^{-2})\hat{L}_{||}$

→ Rotational component of currents leave an asymptotically larger imprint on significant wave height than the divergent currents

- $\hat{\boldsymbol{L}}(\varphi)$ dominant ( $O(\delta^{-2})$ ) in narrow, $O(\delta)$ sectors around $\varphi$= (swell propagation direction $\pm\frac{\pi}{2}$ )

→ streaks: hs elongated in the direction of propagation of the waves

More vorticity → More divergence




All waves are swell-like in this work.

Bôas, A. B. V., Cornuelle, B. D., Mazloff, M. R., Gille, S. T., & Ardhuin, F. (2020).

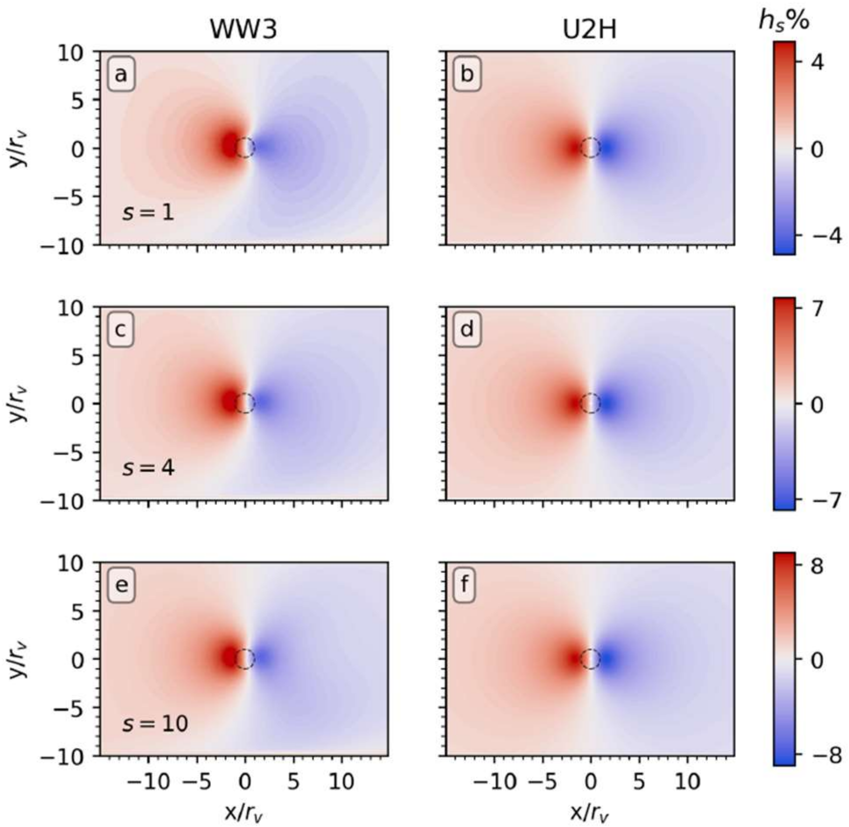
# Impact of divergent currents

$$\boldsymbol{U} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}^{\perp}\psi\,, \quad \hat{h}_s(\boldsymbol{q}) = \mathrm{i}q\hat{L}_{||}(\varphi)\hat{\phi}(\boldsymbol{q}) + \mathrm{i}q\hat{L}_{\perp}(\varphi)\hat{\psi}(\boldsymbol{q})\,.$$

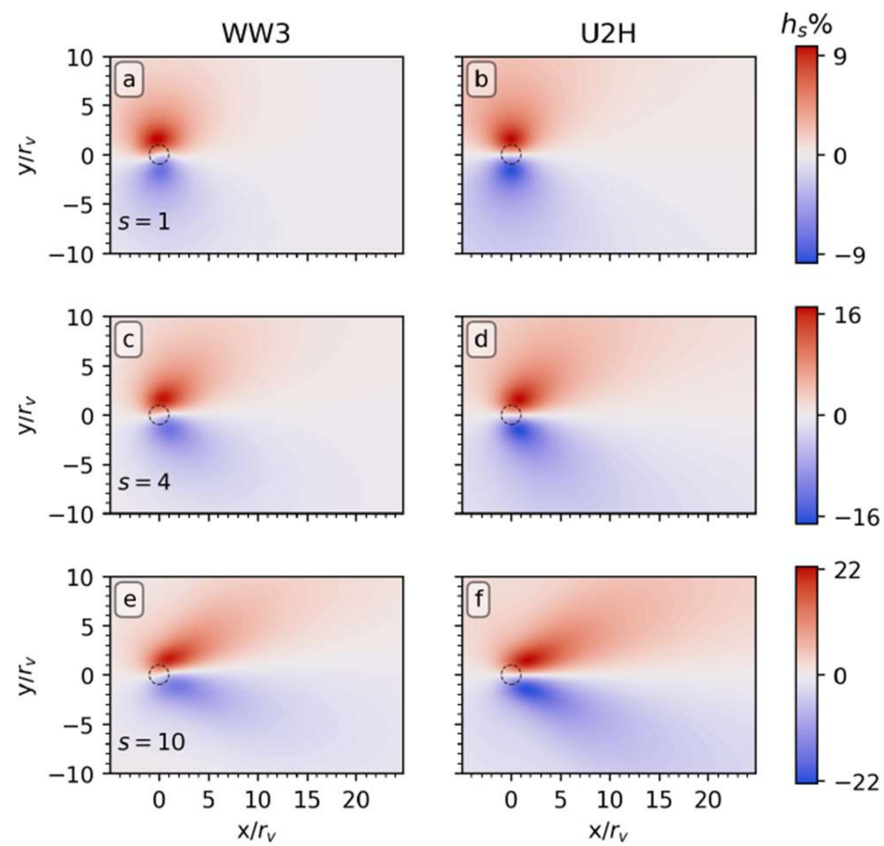
$$\hat{L}_{\perp} = O(\delta^{-2})\hat{L}_{||}$$

But for non-swells where $\delta$ is not small, $\hat{L}_{||}$ can be comparable to $\hat{L}_{\perp}$

Gaussian source currents
(purely divergent)




Gaussian vortex currents
(purely rotational)



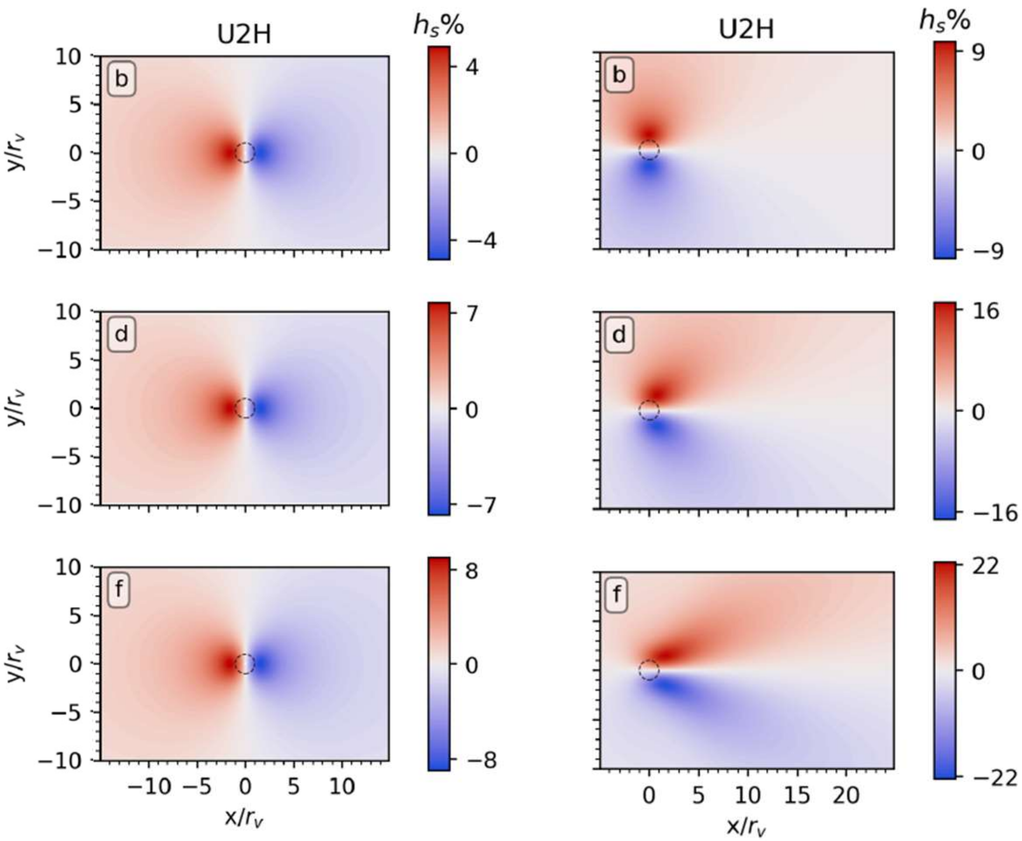

s is a parameter controlling the directional spreading; larger s correspond to smaller $\delta$

Gaussian source currents (purely divergent)

Gaussian vortex currents (purely rotational)

$\delta = O(1)$

$\delta \ll 1$




- Non-swells: divergent flows affect hs to a similar degree as rotational flow
- As $\delta$ decreases (swells), hs from divergent currents stop increasing at some point,

  but the maximum hs from rotational currents keep growing.
- Impacts of divergent flows are local.
- Rotational flows: streaks, nonlocal

# Summary

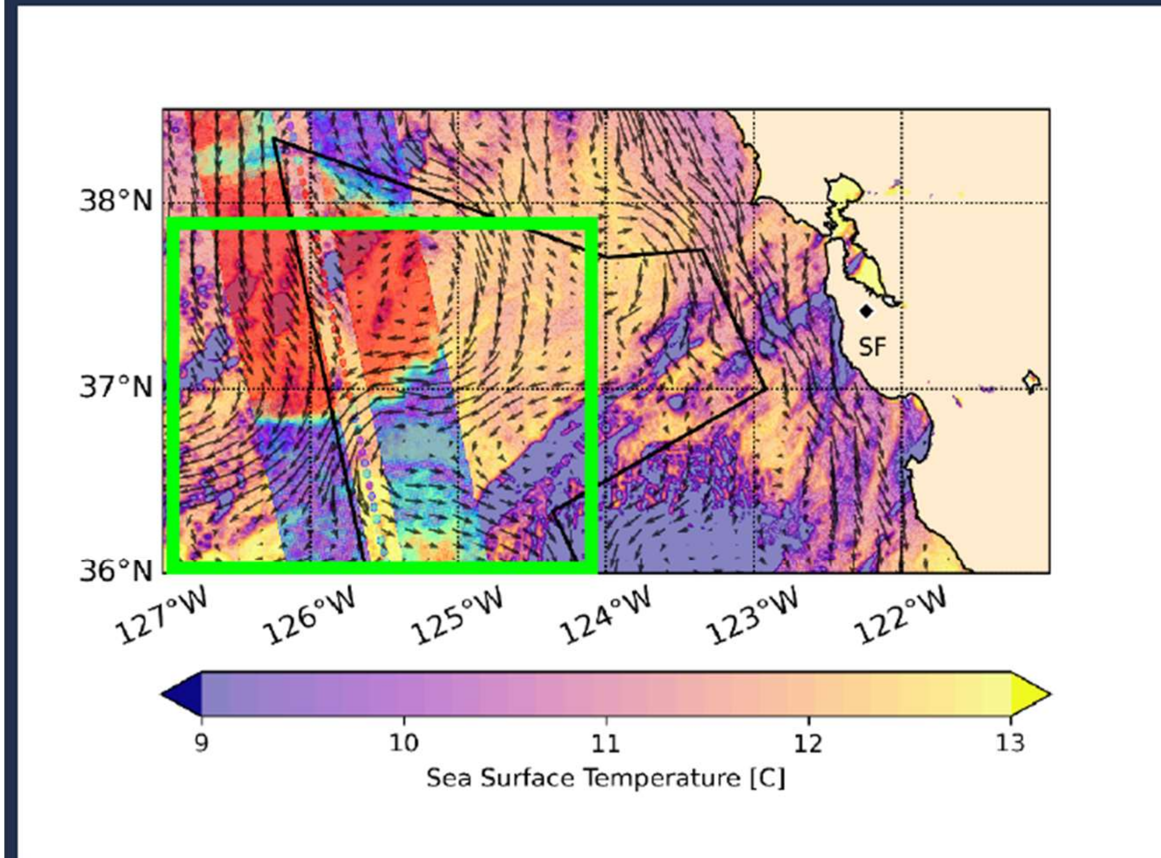
- Linear map from currents to significant wave height (“U2H”)
- Assumptions: scale separation, weak current
- Significant wave height has similar spatial scales as currents
- For non-swells, both divergent and rotational currents can impact hs. Divergent currents’ impacts are local.
- For swells,

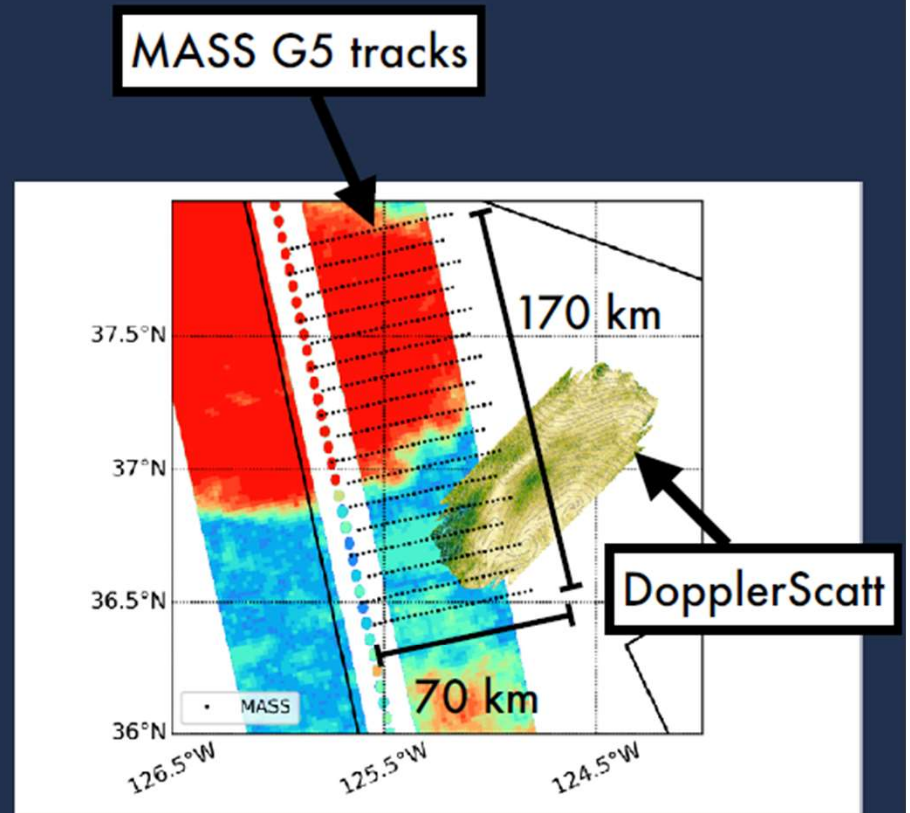
    1). rotational  currents has asymptotically larger impact.

    2). rotational currents’ impacts are nonlocal.

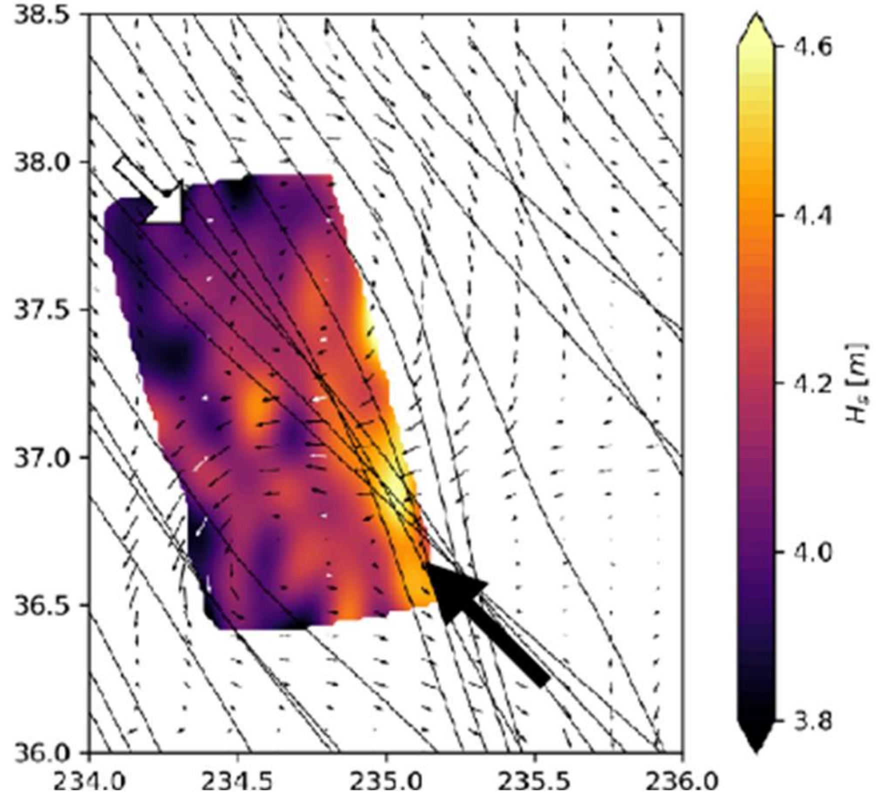
    3). hs are elongated in the waves propagation direction.

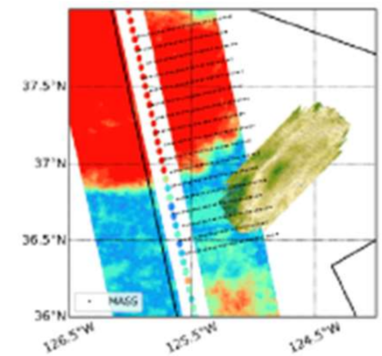
# Overlap between S-MODE and SWOT calval campaign




*Marechal, Villas Bôas, Lenain, and Pizzo (in prep).*




*Marechal, Villas Bôas, Lenain, and Pizzo (in prep).*

# Thank you!

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