Mathematical Details for SFQLIB

Kangbo Li

May 20, 2018

1 Overview.

For background on Single Flux Quantum (SFQ) Pulse Train, please refer to [1]. In short, we are trying to achieve an $\frac{\pi}{2}$ or π rotation using two basic operations.

1. Precession for one clock period (U_{free}).
2. Apply an SFQ pulse ($\mathrm{U_{SFQ}}$).

Assumming a three level qubit, the time evolution operators for the two operations are shown in eqution 1 and 2.

$$U_{free} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j\omega_{01}\tau} & 0 \\ 0 & 0 & e^{-j\omega_{02}\tau} \end{bmatrix} \tag{1}$$

$$U_{SFQ} = \begin{bmatrix} -1\alpha & -1\beta & -\sqrt{2}\alpha \\ 1\beta & -3\alpha & -\sqrt{2}\beta \\ \sqrt{2}\alpha & \sqrt{2}\beta & -2\alpha \end{bmatrix} \tag{2}$$

Where $\alpha = \left(-\frac{1}{3}cos\left(-\sqrt{3}\frac{\delta\theta}{2}\right) + \frac{1}{3}\right)$, $\beta = \left(\frac{1}{\sqrt{3}}sin\left(\sqrt{3}\frac{\delta\theta}{2}\right)\right)$, τ is a clock period, and ω_{01}, ω_{02} are qubit energy splits divided by $\hbar$. In practice, U_{SFQ} is obtained using numerical matrix exponential. Interested readers can find the the derivation in section 2. However, the mathematical details are not important for this software.

Equipped with U_{free} and U_{SFQ}, the problem reduces to contructing an approximation to the ideal rotation operator 3 with the two operators.

$$U_{SFQ}^{ideal} = \begin{bmatrix} cos\left(\frac{\theta}{2}\right) & sin\left(\frac{\theta}{2}\right) & 0 \\ -sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) & 0 \\ 0 & 0 & e^{i\phi} \end{bmatrix} \tag{3}$$

To solve this problem, sfqlib currently offers three features.

- Automate and optimize the construction and multiplication of U_{free} and U_{SFQ}, as well as measure the fidelity of the gate constructed.
- Decompose the spin-$\frac{1}{2}$ rotation matrix into Euler angles α, β, γ. For more details, please refer to section 4.
- Visualize the trajectory of the six cardinal states under consecutive application of U_{free} and U_{SFQ}.

2 Derivation of time evolution operators.

The qubit evolves according to the free precession Hamiltonian H_0 and the SFQ pulses H_{SFQ}

$$H_0 = \hbar \begin{bmatrix} 0 & 0 & 0 \\ 0 & w_{01} & 0 \\ 0 & 0 & w_{02} \end{bmatrix} \quad H_{SFQ} = iC_cV(t)\sqrt{\frac{\hbar\omega_{01}}{2C}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \tag{4}$$

$$\int_0^\tau H_{SFQ}dt = \hbar\frac{\delta\theta}{2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix} \quad \text{where} \quad \delta\theta = \int_0^\tau 2C_cV(t)\sqrt{\frac{\omega_{01}}{2C\hbar}}dt \tag{5}$$

One can exponentiate both matrices in closed form to obtain the time evolution operator $U_0(t) = \int_0^t H_0 dt'$ and U_{SFQ}, providing that the precession can be ignored during SFQ pulses. $U_0(t)$ is trivially obtained as in 6.

$$U_0(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-j\omega_{01}t} & 0 \\ 0 & 0 & e^{-j\omega_{02}t} \end{bmatrix} \tag{6}$$

To obtain U_{SFQ}, using the property $\epsilon^3 = -3\epsilon$, where

$$\epsilon = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

One can show that

$$\epsilon^n = (-3)^{\frac{n-2}{2}}\epsilon^2 \quad \text{if } n \text{ is even} \tag{7}$$

$$\epsilon^n = (-3)^{\frac{n-1}{2}}\epsilon \quad \text{if } n \text{ is odd} \tag{8}$$

Expanding $e^{j\frac{\delta\theta}{2}}$ and grouping the terms with even and odd powers of $i\delta\theta$ give

$$U_{SFQ} = \sum_{even, n\geq 2}^{\infty} \frac{(-3)^{\frac{n-2}{2}}\epsilon^2(i\delta\theta)^n}{n!} + \sum_{odd}^{\infty} \frac{(-3)^{\frac{n-1}{2}}\epsilon(i\delta\theta)^n}{n!} \tag{9}$$

By resumming the sequence, one can show that

$$\epsilon^2\left(-\frac{1}{3}cos\left(-\sqrt{3}\frac{\delta\theta}{2}\right) + \frac{1}{3}\right) + \epsilon\left(\frac{1}{\sqrt{3}}sin\left(\sqrt{3}\frac{\delta\theta}{2}\right)\right) + I \tag{10}$$

By writing $\alpha = \left(-\frac{1}{3}cos\left(-\sqrt{3}\frac{\delta\theta}{2}\right) + \frac{1}{3}\right)$ and $\beta = \left(\frac{1}{\sqrt{3}}sin\left(\sqrt{3}\frac{\delta\theta}{2}\right)\right)$, one can reduce the time evolution operator to

$$U_{SFQ} = \alpha\epsilon^2 + \beta\epsilon + I \tag{11}$$

Expanding U_{SFQ} into matrix form, we find

$$U_{SFQ} = \begin{bmatrix} -1\alpha & -1\beta & -\sqrt{2}\alpha \\ 1\beta & -3\alpha & -\sqrt{2}\beta \\ \sqrt{2}\alpha & \sqrt{2}\beta & -2\alpha \end{bmatrix} \tag{12}$$

If we compare equation 12 with the ideal rotaion,

$$U_{SFQ}^{ideal} = \begin{bmatrix} cos\left(\frac{\theta}{2}\right) & sin\left(\frac{\theta}{2}\right) & 0 \\ -sin\left(\frac{\theta}{2}\right) & cos\left(\frac{\theta}{2}\right) & 0 \\ 0 & 0 & e^{i\phi} \end{bmatrix} \tag{13}$$

there are a few things one can observe from equation 11

- ϵ^2 has a shape of $\begin{bmatrix} X & 0 & X \\ 0 & X & 0 \\ X & 0 & X \end{bmatrix}$ and ϵ has a shape of $\begin{bmatrix} 0 & X & 0 \\ X & 0 & X \\ 0 & X & 0 \end{bmatrix}$ This shows that the coupling dominated by β, with the cross level coupling dominated by α
- Because of the presence of the leakage, the rotation achieved in 01 subspace is not a $\frac{\delta\theta}{2}$ rotation. Rather, the matrix element are approximated by the parameters α and β. For example, $-sin(\frac{\delta\theta}{2})$ is approximated by β.
- If $\delta\theta$ is small, α is also small, then one can ignore the cross level coupling. However, large $\delta\theta$ will lead to substantial cross level leakage.

The analysis is verified numerically.

```
from qutip import Qobj, qeye
from numpy import sqrt, sin, cos
from numpy.linalg import norm
u = Qobj([[0.0, -1.0, 0.0], [1.0, 0.0, -sqrt(2.0)], [0.0, sqrt(2.0), 0.0]])


def analitical(d_theta):
    return pow(u, 2) / 3.0 * (-cos(sqrt(3.0) * d_theta/2.0) + 1.0) +\
        u / (sqrt(3.0)) * sin(sqrt(3.0) * d_theta/2.0) + qeye(3)


def compare(d_theta):
    exp_theta = (u * d_theta/2.0).expm()
    return exp_theta - analitical(d_theta)


return compare(0.1)
```

```
Quantum object: dims = [[3], [3]], shape = (3, 3), type = oper, isherm = True
Qobj data =
[[0. 0. 0.]
 [0. 0. 0.]
 [0. 0. 0.]]
```

3 Gate Fidelity

Fidelity is a very important characterization of a quantum operation. There are currently two methods implemented in sfqlib for measuring the gate fidelity. The first one is suitable for measuring the fidelity of a unitary transformation, whereas the second one is suitable when auxiliary quantum levels are involved [2].

1. States (Default): Average fidelity of a unitary transformation.

$$F_{avg} = \frac{1}{6}\sum_{\alpha} Tr\{U_G \rho_\alpha U_G^\dagger U_{id} \rho_\alpha U_{id}^\dagger\} \tag{14}$$

   Where $\alpha = \pm x, \pm y, \pm z$.

2. Gates: Subspace averaged fidelity of a unitary transformation.

$$F_{avg} = \frac{1}{6}\left(Tr\{U_G^\dagger \mathcal{P} U_G \mathcal{P}\} + |Tr\{\mathcal{P} U_{ideal}^\dagger U_G\}|^2\right) \tag{15}$$

   Where U_{ideal} is the idealeal gate, and $\mathcal{P}$ is the projection onto qubit subspace.

4 Euler Angles

Given a rotation constructed from U_{free} and U_{SFQ}, there is very little intuition one can obtain from the numerical entries of the matrix. To draw insights from the matrix, one need to decompose it into the form $\mathcal{D}^z(\alpha)\mathcal{D}^y(\beta)\mathcal{D}^z(\gamma)$. The Euler angles α, β, γ are a good description of the action of the matrix.

4.1 Decomposition of spin-$\frac{1}{2}$ rotation

The rotation matrix of spin 1/2 in terms of Euler Angles is shown in Equation. 16

$$\mathcal{D}^z(\alpha)\mathcal{D}^y(\beta)\mathcal{D}^z(\gamma) = \begin{bmatrix} e^{-i(\alpha+\gamma)/2}cos(\beta/2) & -e^{-i(\alpha-\gamma)/2}sin(\beta/2) \\ e^{i(\alpha-\gamma)/2}sin(\beta/2) & e^{i(\alpha+\gamma)/2}cos(\beta/2) \end{bmatrix} \tag{16}$$

Compare this to the matrix

$$\mathcal{D}^z(\alpha)\mathcal{D}^y(\beta)\mathcal{D}^z(\gamma) = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \tag{17}$$

We find

$$|a| = cos(\beta/2) \tag{18}$$
$$|b| = sin(\beta/2) \tag{19}$$
$$a \cdot b = -e^{-i\alpha} sin(\beta/2)cos(\beta/2) \tag{20}$$
$$a/b = -e^{-i\gamma} cot(\beta/2) \tag{21}$$

From which we can solve for

$$\beta = 2arccos(|a|) \tag{22}$$
$$\alpha = ilog\left(-\frac{a \cdot b}{|a| \cdot |b|}\right) \tag{23}$$
$$\gamma = ilog\left(-\frac{a/b}{|a|/|b|}\right) \tag{24}$$
$$\tag{25}$$

4.2 Singularities

1. When $a = 0$ Since this case correponds to a $D^y(\pi)$ rotation, which commutes arbitrary z rotation, α and γ are not fully constrained. One can also see this from equation 26, where $(\alpha + \gamma)$ can be anything. We choose $\alpha = 0$ for as a convention.

$$a = e^{-i(\alpha+\gamma)/2} cos(\beta/2) = 0 \tag{26}$$

   implies that

$$\beta = \pi \tag{27}$$
$$\gamma = -2ilog(-b) \tag{28}$$

2. When $b = 0$ The analysis is similar to the case of $a = 0$. This case corresponds to a free precession. Taking $\alpha = 0$ as the choice.

$$\beta = 0 \tag{29}$$
$$\gamma = 2ilog(a) \tag{30}$$

References

[1] R. McDermott and M. G. Vavilov. "Accurate Qubit Control with Single Flux Quantum Pulses". In: *Physical Review Applied* 2.1 (2014), pp. 1–10. DOI: 10.1103/PhysRevApplied.2.014007.

[2] L. H. Pedersen, N. M. Møller, and K. Mølmer. "Fidelity of quantum operations". In: *Physics Letters, Section A: General, Atomic and Solid State Physics* 367.1-2 (2007), pp. 47–51. DOI: 10.1016/j.physleta.2007.02.069.