Graduate Course on Planet Formation

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1 Accretion disks

1.1 Introduction

An accretion disk naturally forms when material falls from a large distance, let's call it r_∞, into a gravitational potential. The simplest potential we can study is that of a point mass

$$\Phi = -\frac{GM}{r}.$$

In the above equation G is the gravitational constant, M the mass of the central object and r the distance to it. The infalling material can convert potential energy into kinetic energy. This is the energy source driving the evlolution of accretion disks. Let us assume that the material starts at rest at infinity $r_\infty = \infty$ and moves to a distance r form the central object, also being at rest. The distance r could for example correspond to the surface of a star or planet. In that case the energy difference per unit mass is

$$\Delta E = \Phi(r_\infty) - \Phi(r) = \frac{GM}{r}.$$

If we assume that the material doesn't stay at rest, but enters a circular Keplerian orbit with semi-major axis a, then the specific energy difference is

$$\Delta E = \Phi(r_\infty) - \Phi(a) - \frac{GM}{2a} = \frac{GM}{2a},$$

where we used the fact that a test mass in a circular orbit with semi-major axis a around a central object has the velocity $v_{\mathrm{kep}} = \sqrt{GM/a}$.

1.2 Adiabatic flow

Now, let us assume that the material cannot loose any of its newly gained energy. We call such a flow adiabatic. In that case the energy is converted into internal energy (heat). Let us assume an ideal gas which has the specific energy

$$E = \frac{P}{(\gamma - 1)\rho}$$

and the equation of state

$$P = \frac{\mathcal{R}\rho T}{\mu}$$

where γ is the ratio of specific heats, $\mathcal{R}$ is the gas constant and μ is the mean atomic weight. P and ρ are the pressure and density. If the material has zero[1] internal energy ($T = 0$) at r_∞ and enters into a circular orbit at a semi-major axis a, then it has the temperature

$$T = \frac{1}{2}(\gamma - 1)\underbrace{\frac{GM\mu}{\mathcal{R}a}}_{T_{\rm vir}},$$

where $T_{\rm vir}$ is called the virial temperature. This is an extreme limit, but we can nevertheless use it to come up with some insight into what kind of conditions we can expect in an accretion event.

If we have an adiabatic inflow, then the sound-speed of the gas is close to the escape speed of the system. Specifically, the sound speed is

$$c_s = \sqrt{\gamma \mathcal{R}T/\mu} = \sqrt{\frac{\gamma(\gamma-1)}{2}\frac{GM}{a}}$$

whereas the escape speed is

$$v_{\rm escape} = \sqrt{2\frac{GM}{a}}.$$

What this means is that the gas molecules are very likely to escape from the system. That makes sense as we didn't introduce any method for dissipation and if the molecules fall in from r_∞, they have enough energy to reach r_∞ a second time and therefore escape the system.

Another way to state the same fact is to say that the pressure scale height H is of the order of the semi-major axis a. The pressure scale height is defined as

$$H \equiv \frac{\mathcal{R}T}{\mu g} = \frac{\gamma - 1}{2}a \approx a$$

where g is the gravitational force at the distance a, i.e. $g = GM/a^2$. Note that adiabatic accretion takes place on the orbital timescale

$$\tau_{\rm adiab} = \frac{a}{v_{\rm kep}} = \sqrt{\frac{a^3}{2GM}}$$

This timescale is very fast!

1.3 Radiative loss

We now introduce a dissipative effect into our estimates of accretion flows. Once again, we will make quite a few assumptions that allow us to come up with simple analytical estimates.

First, let's assume that the inflowing gas can radiate away heat. Most of the heat will be radiated away very close to the star (or planet). For that reason, let us assume that all of the energy is radiated away at the that inner boundary, which we will label with the radius R.

We would like to come up with an equilibrium solution that is time independent. In that case heating through accretion has to balance exactly the loss due to radiation. If the surface radiates as a blackbody, then this balance becomes

$$\dot{M}\frac{GM}{R} = \underbrace{4\pi R^2}_{\rm surfacearea}\ \sigma_r T^4$$

where T is the temperature of the blackbody, $\dot{M}$ is the accretion rate and σ_r is the Stefan-Boltzmann constant ($\sigma_r = 5.670373 \cdot 10^8$ W m^2 K^4). We can solve this equation to get the temperature T.

[1] In reality, the temperature of gas in molecular clouds is $10 - 20$K.

When the luminosity of a flow as described above is high and the material is not transparent, it will reabsorb some of the radiation and convert it into heat. There exists a critical luminosity in this scenario. If the luminosity is higher than the critical luminosity, then the inflowing material gets blown out before it can reach the surface at radius R. We can write this as a force balance between gravity $g = GM/r^2$ and the force excerted by the radiative flux F, which we label f_{rad}

$$f_{\mathrm{rad}} = F\frac{\kappa}{c}$$

where c is the speed of light and κ is the opacity of the material. Setting $g = f_{\mathrm{rad}}$ yields the critical flux F_E

$$F_E = \frac{c}{\kappa}\frac{GM}{r^2}$$

which corresponds to the luminosity (assuming spherical symmetry):

$$L_E = \frac{4\pi GMc}{\kappa}.$$

This quantity is known as the Eddington limit.

A comment about the limitations of the Eddington limit. We assumed that the only relevant force other than radiation pressure was gravity. There could be other forces such as those induced by magnetic fields. We assumed that the excess energy is only transported by radiation. There could be other mechanisms such as convection or neutrinos.

1.3.1 Angular Momentum

We have so far ignored an important factor in accretion flows. Besides the energy, there is another conserved quantity: angular momentum. Even if the interstellar medium from which a protoplanetary disk forms is at rest and has effectively zero angular momentum, random small turbulent fluctuations are grown significantly after the material contracts by many order of magnitude from the parsec scale to a few astronomical units.

Because of the finite angular momentum, material cannot fall in radially as we assumed above. Instead, the material forms a disk. Thus, we need to study the evolution of disks next.

By a disk, we mean a flattened object around a central mass. The disk thinkness, or vertical length scale, is the most important quantity describing the geometry of disks. We assume that the disks are circular and that at each radius r, the material is on a Keplerian orbit with orbital frequency

$$\Omega(r) = \sqrt{GM/r^3}$$

where M is the central object's mass. Note that we have ignore the self-gravity of the disk, i.e. we assume the central object is the dominant gravitational force.

We describe the disk using hydrodynamics. For now we assume that the disk is axisymmetric and thin. In cylindrical coordinates, the contintuity equation in the radial direction is then given by

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma\, r\, v_r\right) = 0,$$

where $\Sigma = \int \rho\, dz$ is the surface density and v_r is the radial velocity.

The specific angular momentum of a disk element can be calculated from the standard $\vec{l} = \vec{r} \times \vec{v}$ expression which gives us (assuming axisymmetry again):

$$l = r\, v_\phi = r^2\Omega.$$

Using this, we can write the conservation equation for the angular momentum as

$$r\frac{\partial}{\partial t}\left(\Sigma\, r^2\, \Omega\right) + \frac{\partial}{\partial r}\left(\Sigma v_r\, r^3\, \Omega\right) = 0$$

We now have two equations, for two unkowns (Σ and v_r).

End of Lecture 1
