Quantum Mechanics I

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1 Linear Algebra

1.1 Vector Spaces

Definition 1 (Vector Space)**.** A vector space consists of vectors $|\alpha\rangle, |\beta\rangle, \ldots$ and scalars $a, b, \ldots$, scalar multiplication and addition, vector adition, scalar with vector multiplication and the following axioms:

1. Identity element of addition exists.
$$|\alpha\rangle + |0\rangle = |\alpha\rangle$$
2. Inverse element of addition exists.
$$|\alpha\rangle + |-\alpha\rangle = |0\rangle$$
3. Identity of scalar vector multiplication exists.
$$1\,|\alpha\rangle = |\alpha\rangle$$
4. Associativity of addition.
$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$
5. Commutivity of addition.
$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$
6. Distributivity of scalar/vector multiplication.
$$a(|\alpha\rangle + |\beta\rangle) = a\,|\alpha\rangle + a\,|\beta\rangle$$
7. Distributivity of vector addition.
$$(a+b)\,|\alpha\rangle = a\,|\alpha\rangle + b\,|\alpha\rangle$$
8. Multiplication compatibility.
$$(ab)\,|\alpha\rangle = a(b\,|\alpha\rangle)$$

Definition 2 (Linear Combination)**.** $|\alpha\rangle$ is a linear combination of $|\beta\rangle$ and $|\gamma\rangle$ if there are scalaers a, b such that
$$|\alpha\rangle = a\,|\beta\rangle + b\,|\gamma\rangle\,.$$

Definition 3 (Linearly Independent)**.** $|\alpha\rangle$ is linearly independent of $|\beta\rangle$ and $|\gamma\rangle$ if there are no scalaers a, b such that
$$|\alpha\rangle = a\,|\beta\rangle + b\,|\gamma\rangle\,.$$

Definition 4 (Span and Dimensions)**.** A set of vectors spans a space if any member of the space can be expressed as a linear combination of them. The dimension of a space is a number of vectors needed to span the space.

Definition 5 (Basis)**.** A set of linearly independent vectors that spans a vector space is called a basis.

Example 1. Possible ways of writing a vector.

- $\mathbf{a}$
- $\vec{a}$
- $|\alpha\rangle$
- $\langle\alpha|$
- $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
- $(1, 2, 3)$
- a_i

Definition 6 (Inner Product)**.** An inner product is an operation in a vector space that takes two vectors, returns a scalar and follows the following axioms:

1. Conjugate symmetric
$$\langle\alpha, \beta\rangle = \langle\beta, \alpha\rangle^*$$
2. Linearity.
$$\begin{aligned} \langle a\,\alpha, \beta\rangle &= a\,\langle\alpha, \beta\rangle \\ \langle\alpha + \beta, \gamma\rangle &= \langle\alpha, \gamma\rangle + \langle\beta, \gamma\rangle \end{aligned}$$
3. Positive definity.
$$\begin{aligned} \langle\alpha, \alpha\rangle &\geq 0 \\ \langle 0, 0\rangle &= 0 \end{aligned}$$

A vector space with an inner product is called an inner product space.

Definition 7 (Norm)**.** A norm of a vector is

$$\sqrt{\langle\alpha, \alpha\rangle} = ||\alpha||$$

A normalized vector is a vector with norm 1.

Definition 8 (Orthogonal)**.** A set of normalized vectors is said to be orthogonal if

$$\langle\alpha, \beta\rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1 (Schwarz Inequality)**.**

$$(\langle\alpha, \beta\rangle)^2 \leq \langle\alpha|\alpha\rangle\,\langle\beta|\beta\rangle$$

Definition 9 (Linear Transformation)**.** A transformation/operator T is called linear if

$$T(a|\alpha\rangle + b|\beta\rangle) = a(T|\alpha\rangle) + b(T|\beta\rangle)$$

Example 2 (Linear transformation in a specific basis)**.** Matrix multiplication.

$$\begin{aligned} |\beta\rangle &= T|\alpha\rangle \\ b_i &= \sum_{j=1}^{n} T_{ij} a_j \\ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} &= \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \end{aligned}$$

Corollary 1. With two general linear transformations T and V:

$$T(V|\alpha\rangle) \neq V(T|\alpha\rangle)$$

or simply

$$TV \neq VT$$

1.2 Matrix operations

Definition 10 (Transpose of a Matrix)**.** The transpose of a matrix T is

$$\begin{array}{ccc} T & \to & \tilde{T} \\ \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} & \to & \begin{pmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} \end{array}$$

A matrix is called symmetrix if $T = \tilde{T}$ and antisymmetric if $T = -\tilde{T}$.

Definition 11 (Complex Conjugate of a Matrix)**.** The complex conjugate of a matrix T is

$$\begin{array}{ccc} T & \to & T^* \\ \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} & \to & \begin{pmatrix} T_{11}^* & T_{12}^* & T_{13}^* \\ T_{21}^* & T_{22}^* & T_{23}^* \\ T_{31}^* & T_{32}^* & T_{33}^* \end{pmatrix} \end{array}$$

A matrix is called real if $T = T^*$ and imaginary if $T = -T^*$.

Definition 12 (Hermitian Conjugate of a Matrix)**.** The hermitian conjugate (or self-adjoint) of a matrix $T^\dagger$ is

$$\begin{array}{ccc} T & \to & T^\dagger \equiv \tilde{T}^* \\ \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} & \to & \begin{pmatrix} T_{11}^* & T_{21}^* & T_{31}^* \\ T_{12}^* & T_{22}^* & T_{32}^* \\ T_{13}^* & T_{23}^* & T_{33}^* \end{pmatrix} \end{array}$$

A matrix is called hermition if $T = T^\dagger$ and skew hermitian if $T = -T^\dagger$.

Definition 13 (Commutator)**.** The commutator of two operators T, V is

$$[T, V] = TV - VT$$

Corollary 2.

$$\tilde{TV} = \tilde{V}\tilde{T}$$

Corollary 3.

$$(TV)^\dagger = V^\dagger T^\dagger$$

Definition 14 (Unit Matrix)**.** The operator I is called the unix matrix

$$I \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 15 (Inverse)**.** The inverse T^{-1} of an matrix T has the property

$$T^{-1}T = TT^{-1} = I$$

The operator of a matrix is invertiable, if and only if $\det(T) \neq 0$. If a matrix is not invertible, it is said to be singular. Note that $\det(T) = (\det(T^{-1})^{-1}$.

Corollary 4.

$$(TV)^{-1} = V^{-1}T^{-1}$$

Definition 16 (Unitary Matrix)**.** A matrix T is unitary if

$$T^\dagger = T^{-1}$$

Corollary 5. Unitary transformations preserve inner products

$$\langle\alpha'|\beta'\rangle = \alpha'^\dagger\beta' = (U\alpha)^\dagger(U\beta) = \alpha^\dagger U^\dagger U\beta = \alpha^\dagger\beta = \langle\alpha|\beta\rangle$$

1.3 Changing Bases

To go from an old basis e_o to a new one e_n we can use a transformation matrix S:

$$a_n = S\, a_o.$$

An orperator transforms as follows

$$T_n = ST_oS^{-1}.$$

Definition 17 (Similarity)**.** Two operators T and U are said to be similar if a non-singular matrix S exists such that

$$T = SUS^{-1}.$$

Corollary 6. The determinant is unchanged in a coordinate transformation:

$$\det T_n = \det T_o.$$

Lecture 1 Ends