Definition 18 (Trace)**.** The trace of an operator T is defined as

$$\mathrm{Tr}(T) = \sum_{i=1}^{n} T_{ii}.$$

Corollary 7. The trace is unchanged in a coordinate transformation:

$$\mathrm{Tr}(T_n) = \mathrm{Tr}(T_o).$$

Example 3 (Geometric interpretations of the trace of an operator)**.** Here are some additional interpretations of the trace.

- Assume we have a projection operator $A^2 = A$. Then, the trace is the dimension of the space that is being projected onto.
- If we consider the vector space of all operators (rather then the vector space on which operators act on), then the trace of two operators A, B defines an inner product:

$$\langle A, B \rangle = \mathrm{Tr}(A^\dagger B)$$

- Note that

$$\det(e^{tA}) = 1 + t\mathrm{Tr}(A) + O(t^2)$$

- In Quantum mechanics it will become important when we discuss the expectation value of non-pure states and density matricies.

1.4 Eigenvectors and Eigenvalues

Definition 19 (Eigenvector and Eigenvalue)**.** A vector $|\alpha\rangle \neq |0\rangle$ is called an eigenvector of the linear operator T if a scalar (the eigenvalue) $\lambda \neq 0$ exists such that

$$T\,|\alpha\rangle = \lambda\,|\alpha\rangle\,.$$

Corollary 8. Rewriting the definition of an eigenvector as

$$(T - \lambda\mathbf{1})\,|\alpha\rangle = 0$$

and using the fact that $|\alpha\rangle \neq |0\rangle$, we conclude that

$$\det\left(T - \lambda\mathbf{1}\right) = 0.$$

Definition 20 (Spectrum)**.** The set of eigenvalues of an operator T is called the spectrum of T.

Definition 21 (Degenerate)**.** If one ore more eigenvectors share the same eigenvalue, the spectrum is said to be degenerate.

Example 4. Let's find the eigenvectors and eigenvalue of the following matrix

$$M = \begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix}$$

To find the eigenvalues, we solve the characteristic equation:

$$\det\left|\begin{pmatrix} 2-\lambda & 0 & -2 \\ -2i & i-\lambda & 2i \\ 1 & 0 & -1-\lambda \end{pmatrix}\right| = -\lambda^3 + (1+i)\lambda^2 - i\lambda = 0$$

The solutions are $0, 1, i$. We can now solve for the eigenvectors. For example the eigenvector corresponding to the eigenvalue 0 is found by

$$\begin{pmatrix} 2 & 0 & -2 \\ -2i & i & 2i \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving the equation gives

$$a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The other vectors are

$$b = \begin{pmatrix} 2 \\ 1-i \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Definition 22 (Diagonalizable)**.** A matrix is said to be diagonalizable if a change of basis bring the matrix into diagonal form:

$$T_{ij} = 0 \quad \text{unless} \quad \text{j} = \text{i}.$$

In that case, the similarity matrix is simple the collection of normalized eigenvectors.

Corollary 9. A matrix is diagonalizable if and only if the eigenvectors span the vector space.

Corollary 10. A matrix is certainly diagonalizable if the characteristic equation has n different roots.

Definition 23 (Normal matrix)**.** A matrix is said to be normal if:

$$[N^\dagger, N] = 0$$

Corollary 11. Sufficient conditions for a diagonizability:

1. A normal matrix is diagonalizable.
2. A hermitian matrix is diagonalizable.
3. A unitary matrix is diagonalizable.

Corollary 12. Two matricies can be diagonalized with the same similarity matrix if and only if the two mattricies commute.

1.5 Hermitian Transformation

Definition 24 (Hermitian conjgate of an operator)**.** The hermitian conjugate $T^\dagger$ of a linear operator T is such that

$$\langle T^\dagger \alpha | \beta \rangle = \langle \alpha | T \beta \rangle$$

for all vectors $\langle \alpha |$ and $| \beta \rangle$.

Theorem 2 (Properties of Hermitian Transformations)**.** A Hermitian transformation ($T = T^\dagger$) has the following properties:

1. The eigenvalue are real. Proof: Assume $T\,|\alpha\rangle = \lambda\alpha$. Then

$$\langle\alpha|T\alpha\rangle = \langle\alpha|\lambda\alpha\rangle = \lambda\,\langle\alpha|\alpha\rangle$$

   and because $T = T^\dagger$

$$\langle\alpha|T\alpha\rangle = \langle T\alpha|\alpha\rangle = \langle\lambda\alpha|\alpha\rangle = \lambda^*\,\langle\alpha|\alpha\rangle$$

   It follows that λ is real.

2. The eigenvectors belonging to distinct eigenvalues are orthogonal. Proof: Assume $T\,|\alpha\rangle = \lambda\,|\alpha\rangle$ and $T\,|\beta\rangle = \mu\,|\beta\rangle$ with $\lambda \neq \mu$. Then

$$\langle\alpha|T\beta\rangle = \langle\alpha|\mu\beta\rangle = \mu\,\langle\alpha|\beta\rangle$$

   and becayse $T = T^\dagger$

$$\langle\alpha|T\beta\rangle = \langle T\alpha|\beta\rangle = \langle\lambda\alpha|\mu\beta\rangle = \lambda^*\,\langle\alpha|\beta\rangle = \lambda\,\langle\alpha|\beta\rangle$$

   Thus, because $\lambda \neq \mu$ it follows that $\langle\alpha|\beta\rangle = 0$.

3. The eigenvectors span the space. No proof. Not valid in infinite vectorspaces.

2 Hilbert Space

In Quantum Mechanics, we will encounter wavefunctions. We will often think of wavefunctions not as functions but as vectors. But contrary to vectors that you might have encountered in your math courses, these wavefunctions live in an infinite dimensional vectors space. Today, we'll introduce the mathematical foundation of that space. We start of with several definitions. Some of those might sounds trivial, but they will give Quantum Mechanics the stringent mathematical framework that it needs.

Definition 25 (Hilbert Space)**.** The set of all square integrable functions,

$$f(x) \quad \text{such that} \quad \int |f(x)|^2 dx < \infty$$

is a vector space. It is called L_2 or Hilbert space. Wavefunctions live in Hilbert space.

Corollary 13. The set of all functions (not only square integrable ones) is also a vector space, but much too large for our purpose.

Definition 26 (Inner product)**.** We define the inner product on L_2 as

$$\langle f|g\rangle = \int f(x)^* g(x) dx.$$

This definition satisfies all criteria for an inner product. Note that if two functions $f, g \in L_2$, then the inner product is garanteed to exist and is finite (Folllows from Schwarz inequality.)

The star symbol $(*)$ in the definition of the inner product mean complex conjugate and is very important. We work with complex numbers in Quantum Mechanics. In many of the assigments and the exams, you will need to remember to complex conjugate f!

Definition 27 (Normalized function)**.** A function is said to be normalized if the inner product with itself is equal to 1.

Definition 28 (Orthogonal functions)**.** Two functions are said to be orthogonal with respect to each other if their inner product is equal to 0.

Definition 29 (Orthonormal set of functions)**.** A set of functions f_i is said to be orthonormal if

$$\langle f_i|f_j\rangle = \delta_{ij}.$$

Definition 30 (Complete set of functions)**.** A set of functions f_i is said to be complete if any function $f \in L_2$ can be expressed as a linear combination of f_i

$$f(x) = \sum_{i=1}^{\infty} c_i f_i(x)$$

3 Observables

Let us define what we understand by an observable. Operators which the property of being *observable* will play a very important role in Quantum Mechanics. As the name suggests, these correspond to values that we can (at least in principle) observe in an experiment. For now, however, we define what we mean by an observable in a purely mathematical way.

We will later come back to this definition and convince us observable defined this way have just the right properties that we would expect from an observable.

Definition 31 (Observable)**.** Observables are represented by hermitian operators:

$$\langle f|Qf\rangle = \langle Qf|f\rangle \quad \text{for all} \quad f(x)$$

Example 5 (Momentum operator)**.** Let's check that the momentum operator is hermitian.

$$\langle f|p\,g\rangle = \int f^* \frac{\hbar}{i}\frac{dg}{dx}dx = \frac{\hbar}{i} f^* g\Big|_{-\infty}^{\infty} + \int \left(\frac{\hbar}{i}\frac{df}{dx}\right) g\,dx = \langle p\,f|g\rangle$$

Definition 32 (Determinate state)**.** Usually, a measurement on identically prepared particles is not the same each time. This is because of the indeterminancy of quantum mechanics. If every measurement of an operator is certain to return the same value, it is called a determinate state.

Theorem 3 (Determinate states are eigenfunctions.)**.** Determinate states are eigenfunctions of Q. Proof: Assume $\langle q\rangle = q$.

$$\sigma^2 = \langle Q - \langle Q\rangle)^2\rangle = \langle\psi|(Q-q)^2\psi\rangle = \langle(Q-q)\psi|(Q-q)\psi\rangle = 0$$

It follows that $(Q-q)\psi = 0$.

Definition 33 (Spectrum)**.** The collection of eigenvalues of an operator is called its spectrum.

Definition 34 (Degenerate Spectrum)**.** A spectrum is degenerate if two or more eigenfunctions share the same eigenvalue.

Lecture 2 Ends