Quantum Mechanics I
Problem Set 1

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Due: Monday, January 11th 2015, noon

Problem 1.1

Vector spaces are often constructed in a way that lets us think of vector space elements as familiar real numbers. Here, we construct a vector space that uses real numbers as its vectors and scalars but has rather unusual definitions of addition and multiplication. Let x, y be vectors of the vector space and r be a scalar. We define the addition of two vectors as

$$x \oplus y \quad = \quad x + y + 7.$$

The $\oplus$ represents a vector addition. $+$ is the standard addition of two real numbers. Thus, $4 \oplus 8 = 19$. The multiplication of a scalar and a vector, $\odot$, is defined as

$$r \odot x = r \cdot x + 7 \cdot (r - 1)$$

The $\cdot$ is the standard multiplication of two real numbers. All other operations in the vector space are what you would expect.

(a) Proof/disproof that all axioms of a vector space are satified.

(b) What is the identity element for scalar multiplication?

(c) What are the identity and inverse elements of vector addition?

Problem 1.2

A three-dimensional vector space is spanned by the orthonormal basis vectors $|1\rangle, |2\rangle, |3\rangle$. The two vectors $|a\rangle$ and $|b\rangle$ are given by

$$\begin{aligned} |a\rangle &= \frac{1}{\sqrt{2}}|1\rangle + \frac{i}{2}|2\rangle + \frac{1}{2}|3\rangle \\ |b\rangle &= \frac{1}{\sqrt{3}}|1\rangle + \frac{i}{\sqrt{3}}|2\rangle \end{aligned}$$

(a) Are the vectors $|a\rangle, |b\rangle$ normalized? If not, normalize them.

(b) Are the vectors $|a\rangle$ and $|b\rangle$ orthogonal to each other?

Problem 1.3

The projection operator $P_{|\beta\rangle}$ is defined as

$$P_{|\beta\rangle} |\alpha\rangle = \frac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle} |\beta\rangle$$

(a) Make a sketch of the effect of $P_{|\beta\rangle}$ in $\mathbb{R}^2$.

(b) Calculate $P^2_{|\beta\rangle} |\alpha\rangle \equiv P_{|\beta\rangle} \left(P_{|\beta\rangle} |\alpha\rangle\right)$.

(c) Using matrix notation and using the example from the previous problem, write down the projection operators $P_{|1\rangle}$ and $P_{|b\rangle}$ in the basis $|1\rangle, |2\rangle, |3\rangle$.

(d) Show (explicitly or in general) that $P_{|1\rangle}$ and $P_{|b\rangle}$ are hermitian.

Problem 1.4

Proof the Schwarz Inequality:

$$|\langle\alpha, \beta\rangle|^2 \leq \langle\alpha|\alpha\rangle \langle\beta|\beta\rangle .$$

Hint: Consider $|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle} |\alpha\rangle$.