5 Generalized Statistical Intepretation

In the following we will define what we understand as the *Generalized Statistical Intepretation* of Quantum Mechanics. This is a core concept of Quantum Mechanics. It is thus very important that you fully understand the meaning of it.

If we measure an observable $Q(x,p)$ of a particle in the state $\Psi(x,t)$ we are certain to get one of the eigenvalue of the operator $Q(x, -i\hbar d/dx)$. If the spectrum is discrete, the probability of getting the eigenvalue q_n (with associated eigenfunction f_n) is

$$|c_n|^2 \qquad \text{where} \qquad c_n = \langle f_n | \Psi \rangle \,.$$

If the spectrum is continuous the probability to get a value in the infinitesimal interval dz is

$$|c(z)|^2 \, dz \qquad \text{where} \qquad c(z) = \langle f_z | \Psi \rangle \,.$$

When a measurement has been performed, the wave function collapses to the corresponding eigenstate.

Let us now see what this concept means in practive by looking at two examples.

Example 9 (Position operator)**.** First, let us look at what those c coefficients look like for the position operator. Recall that the eigenfunctions g_y of the position operator are delta functions. Thus:

$$c(y) = \langle g_y | \Psi \rangle = \int \delta(x-y)\Psi(x,t)\; dx = \Psi(y,t),$$

where we worked in position space. What you can see is that the coefficients c (which is a function of y in this case) are really just the position wave function!

Example 10 (Momentum space wave function)**.** Let's do the same for the momentum operator. In this case the eigenfunctions are $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

$$c(p) = \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar}\Psi(x,t)dx$$

This function is so important that it gets its own name, the momentum space wave function. We will come back to the mometum space wave function in just a bit. For now, notice the symmetry:

$$\begin{aligned}
\Phi(p,t) &= \langle f_p | \Psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar}\Psi(x,t)dx \\
\Psi(x,t) &= \langle g_x | \Phi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int e^{ipx/\hbar}\Phi(p,t)dp
\end{aligned}$$

Some of you might remember seeing a transformation like this. It is a Fourier Transform.

`[Draw examples of simple functions and their Fourier Transform on the board]`

6 Dirac Notation

In the first two lectures, we talked about many mathematical definitions. Of particular interest were vectors which are quantities that live in a vector space. Recall, that vectors are an abstract object; although we often think of them in terms of a basis, they do not need to be expressed in a basis.

`[Draw an arrow on the board. Say this is a vector. Only then draw a basis on the board and use it to define the vector and its components in the given basis.]`

Wavefunctions are also vectors, they living in a special vector space that we already talked about in the second lecture. We called it Hilbert Space. Using the generalized statistical interpretation, we can now make use of this concept of vectors and make sense of what that means *physically*. Just as normal vectors, wavefunctions also do not need to be expressed in a basis to have a physical meaning.

Let us label our wavefunction as $\zeta(t)$. This is the mathematical object that describes our qunatum mechanical system, for example a particle.

Next, let us express it in the basis of position eigenfunctions in the following way:

$$\Psi(x,t) = \langle x|\zeta(t)\rangle$$

We could equivalently expand it in eigenfunctions of the momentum operator

$$\Phi(p,t) = \langle p|\zeta(t)\rangle$$

Or, if we have a discrete spectrum of the Energy operator, in eigenfunctions of the Hamiltonian, $|n\rangle$:

$$c_n(t) = \langle n|\zeta(t)\rangle$$

All those representations describe the *same* vector and therefore the *same* physical state.

Similar to vectors, operators are also basis independent. They are defined by their action on an arbitrary vector $|\alpha\rangle$:

$$|\beta\rangle = Q\,|\alpha\rangle\,.$$

If we define a basis, then we can express both the vectors and the operator in that basis:

$$\begin{aligned}
|\alpha\rangle &= \sum_n a_n\,|e_n\rangle \\
|\beta\rangle &= \sum_n b_n\,|e_n\rangle \\
b_m &= \sum_n Q_{mn} a_n
\end{aligned}$$

We can write Q now in the form of a matrix and the numbers Q_{mn} become the matrix elements.

At the beginning of next week's lecture, we'll review the ideas covered today. We will also go through a detailed example that will help you get used to working with this new notation.

Example 11. Imagine a system with only two linear independent states

$$\begin{aligned} |1\rangle &\;\widehat{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |2\rangle &\;\widehat{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

A general state is a linear combination of the two states

$$|\zeta\rangle \;\; = \alpha\,|1\rangle + \beta\,|2\rangle$$

If we require that the state is normalized (We do! Remember this is a condition to be a physical state!), we have an additional condition

$$|\alpha|^2 + |\beta|^2 = 1$$

Finally, assume the system has the Hamiltonian in this form

$$H \;\; = \;\; \begin{pmatrix} h & g \\ g & h \end{pmatrix}$$

where both g and h are real constants. Don't worry about where this particular form of the Hamiltonian is coming from just yet. This is just an example that illustrates the notation.

Now, let us ask the question of how the system evolves if it is in state $|1\rangle$ at $t = 0$. The time dependent Shroedinger equation says

$$i\hbar \frac{d}{dt}\,|\zeta\rangle \;\; = \;\; H\,|\zeta\rangle$$

The time independent Schroedinger equation is

$$H\,|\zeta\rangle \;\; = \;\; E\,|\zeta\rangle$$

We solve the latter first. So, we are looking for eigenvectors of the Hamiltonian.

$$\det \begin{pmatrix} h - E & g \\ g & h - E \end{pmatrix} = (h - E)^2 - g^2 = 0$$

This is solved for $E_\pm = h \pm g$. The corresponding normalized eigenvectors are

$$|\zeta_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Expressing the initial state as a linear combination of the eigenvectors gives

$$|\zeta(0)\rangle = \frac{1}{\sqrt{2}}\left(|\zeta_+\rangle + |\zeta_-\rangle\right)$$

We can then make use of the standard time-dependence $e^{-iE_n t/\hbar}$ to get the time dependent solution

$$\begin{aligned} |\zeta(t)\rangle \;\; &= \;\; \frac{1}{\sqrt{2}}\left(e^{-i(h+g)t/\hbar}\,|\zeta_+\rangle + e^{-i(h-g)t/\hbar}\,|\zeta_-\rangle\right) \\ &= \;\; e^{-i\hbar t/\hbar} \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix} \end{aligned}$$