Lecture 6

4 Terrestrial planet formation

We focus on the physical concepts underlying statistical models of terrestrial planet formation. This step will take us from bodies of planetesimal size to fully formed Earth-mass planets. Important questions to ask are: What is the rate of collisions and what is the evolution of eccentricity and inclination as a function of time (and thus growth).

4.1 Collisions

4.2 Gravitational focusing

Our goal is to calculate the collision rate of planetesimals and later planets. If the bodies are small enough, we can ignore gravity and the collision rate can be calculated by purely geometric means. As the bodies get more massive, the bodies feel each other's gravity as they approach each other.

Let's consider two bodies of mass m moving with relative velocity v. Far away, the energy is

$$\frac{1}{4}mv^2$$

At the point of cloest approach, the energy is

$$mv_{max}^2 - \frac{Gm^2}{R_c}$$

where R_c is the distance at closest approach. The two energies have to be equal.

Also notice that there is no radial component at closest approach. Thus, angular momentum conservation gives

$$v_{max} = \frac{1}{2}\frac{b}{R_c}v$$

where b is the impact parameter.

If the bodies have a physical size R_s, then there will be a collision if $R_c < R_s$. The largest impact parameter leading to a collision is

$$b^2 = R_s^2 + 4\frac{GmR_s}{v^2}$$

which we can express in terms of the escape speed of the body

$$v_{esc}^2 = 4\frac{Gm}{R_s}$$

to get

$$b^2 = R_s^2\left(1 + \frac{v_{esc}^2}{v^2}\right)$$

Using this result, we can define a cross section

$$\Gamma = \pi R_s^2\left(1 + \frac{v_{esc}^2}{v^2}\right)$$

Physically, this makes sense. If $v_{esc} \gg v$, then gravitational focusing matters.

4.3 Shear versus dispersion

In the above calculation, we assumed that the only relevant force is the mutual gravity. We need to make sure the gravity from the star can be ignored.

To do that, let us define the Hill sphere:

$$r_H = \left(\frac{m}{M_*}\right)^{1/3} a$$

If we are inside the Hill sphere the gravity from the mass m dominated over that from the star. A corresponding velocity scale is

$$v_H = \sqrt{\frac{Gm}{r_H}}$$

i.e. the velocity at the Hill radius. If two bodies collide and their relative speed is large compared to the Hill velocity $v \gg v_H$, then their collision is determined by two body dynamics. If the velocity is small comparted to the Hill velocity $v \ll v_H$ then we need to take into account three body dynamics (the two colliding bodies + the Sun). The two regimes are said to be dispersion and shear dominated.

4.4 Accretion versus disruption

The outcome of a collision is typically grouped into three regimes

- Accretion. The main body remains solid. Some material might get ejected, but collision results in net growth.
- Shattering. The body gets broken up in smaller pieces. However, the pieces will reacummulate later, forming a rubber pile.
- Dispersal. The bodies break up in multiple pieces, with the biggest remaining piece being smaller than the originallly largest piece.

Which outcome is going to happen depends on the specific energy of the collision. This is meassured as

$$Q \equiv \frac{mv^2}{2M}$$

where m is the smaller body's mass and M is the larger body's mass. Typically one defines critical specific energies needed to disrupt and shatter bodies as Q_D^* and Q_S^*. Obviously, $Q_D^* > Q_S^*$.

The precise values of these factors depend on many variables: mass, radius, material composition, spin, shape, impact velocity and parameter. However, for any particular body, we can at least identify two disctinct regimes.

- Strength dominated regime. Here, the material strength dominated. Q_D^* descreases with size.
- Gravity dominated regime. Here, the gravity is keeping the body together. As a result the binding energy $Q_B \propto GM/s \propto \rho s^2$. Note that in general the binding energy is not a good estimate for Q_D^*. But the trend stays true, Q_D^* will increase with increasing size.

The following image illustrates these regimes.

Let's look at asteroids in the Solar system. They have an eccentricity of roughly $e = 0.1$. The collision velocity would thus be around 2 km/s. If we assume a mass ratio of the asteroid of $m/M = 0.1$ then the specific energy in a collision is $Q = 2 \cdot 10^9$ erg/g. Thus, looking at the graph, we can see that bodies up to 100 km can get destroyed in collisions. This is consistent with what we see in the asteroid belt (collisions and asteroid families).

As a conclusion, we note that the collision velocities during the formation of terrestrial planets must have been much lower.

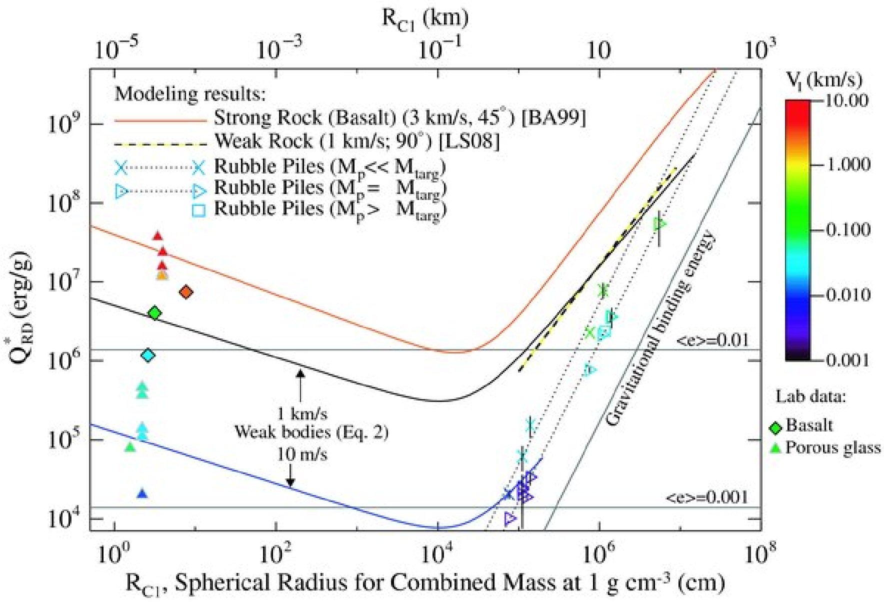

Figure 8: Specific energy for disruption. Image by Stewart amd Leinhardt (2009).

4.5 Statistical growth models

We assume that there are many planetesimals around and that we can thus model them as an ensemble. We define a probability distribution function for the phase space. The pericenters and nodes are distirbuted randomly, thus the only parameters (at a given a) that we need to model are e and i:

$$f(e,i) = 4\frac{\Sigma_p}{m}\frac{ei}{\langle e^2\rangle\langle i^2\rangle}\exp\left(-\frac{e^2}{\langle e^2\rangle}-\frac{i^2}{\langle i^2\rangle}\right)$$

Note that this is a Rayleigh distribution. It would be a Gaussian distribution if we worked in coordinates of v_z and v_r instead of e and i.

We can use the distribution functions to define a random velocity of the planetesimals. There are multiple ways of doing so, i.e.

$$\begin{aligned}
v_{rand} &= \left(e^2+i^2\right)^{1/2} v_k \\
v_{rand} &= \left(\frac{5}{8}e^2+\frac{1}{2}i^2\right)^{1/2} v_k \\
v_{rand} &= \left(\frac{5}{4}e^2+i^2\right)^{1/2} v_k
\end{aligned}$$

Which one to choose depends on the problem. The first one describes the relative velocity with respect to a circular orbit with $i=0$ and the same semi-major axis. The second describes the relative velocity to a locally circular orbit with $i=0$. The third, describes the relative velocity to other planetesimals.

If the mechanism for damping and exciting e and i are related, then we often assume equipartition, i.e.

$$\langle e^2\rangle^{1/2} = 2\langle i^2\rangle^{1/2}$$

Let us now start to look at the growth of protoplanets embedded in a sea of planetesimals. The scale height of the planetesimal disk is

$$h_p \sim \frac{v_{rand}}{\Omega}$$

This allows us to define a volume density of the swarm

$$\rho_{sw} \sim \frac{\Sigma_p}{2h_p}$$

We can write down a differential equation for the evolution of the mass of an embedded protoplanet. For that we use the swarm volume density, the cross section calculated earlier and the enounter velocity at infinity (which we set to v_{rand}).

$$\begin{aligned} \frac{dM}{dt} &= \rho_{sw} v_{rand} \pi R_s^2 \left(1 + \frac{v_{esc}^2}{v_{rand}^2}\right) \\ &= \frac{\sqrt{3}}{2} \Sigma_p \Omega \pi R_s^2 \left(1 + \frac{v_{esc}^2}{v_{rand}^2}\right) \end{aligned}$$

(In the last equal sign, note that we did not calculate the numerical factor accurately, this result is from Lissauer 1993). Rewriting this as an equation for the radius of the planet, we get

$$\frac{dR_s}{dt} = \frac{\sqrt{3}}{8} \frac{\Sigma_p \Omega}{\rho_m} \left(1 + \frac{v_{esc}^2}{v_{rand}^2}\right)$$

Thus,

$$R_s \propto t$$

Ignoring gravitational focusing, this is a very slow process. If we set $\rho_m = 1$ g cm yr^{-1} at $a = 5.2$ AU, it would take 10 Myr to grow a body to a size of 100 km.

So, let's include graviational focussing. And let's assume it's dominant, $v_{esc} \ll v_{rand}$.

$$\frac{dM}{dt} = \rho_{sw} v_{rand} \pi R_s^2 \frac{v_{esc}^2}{v_{rand}^2}$$

by using

$$v_{esc} = \frac{4GM}{R_s}$$

we get

$$\frac{dM}{dt} = \frac{\sqrt{3}\pi G \Sigma_p \Omega M R_s}{v_{rand}^2}$$

Note that we can put all the constants in one parameter (assuming the protoplanet will not excite the swarm of planetesimals and will not accrete a significant part). Then

$$\frac{dM}{dt} = k \cdot M^{4/3}$$

This has important consequences. If there are two proto planets, then the initially more massive planet will grow faster, both in absolute and relative terms. This is called runaway growth.

We ignore a couple of things in this derivation, mostly the feedback from the planet on the planetesimal swarm, but the basic picture remains truw. There is a runaway growth, the initially biggest seed will grow very quickly. And gravitational focusing is important for this effect. Including more physical effects is possible, but one often resorts to numerical simulations nowadays.

Note that the above only works in the dispersion dominated regime. The dynamics are more complicated in the shear dominated regime (three body dynamics).

At some point the shear dominated limit also breaks down. Shear dominated effectively means that the planetesimal disk is cold. If cold enough, then the vertical thickness becomes very small and eventually smaller than the radius of the planet.