7 Uncertainty Principle

Theorem 6 (Generalized Uncertainty Principle)**.** For any two observables A and B we have:

$$\begin{aligned}\sigma_A^2 &= \langle (A - \langle A \rangle)\Psi | (A - \langle A \rangle)\Psi \rangle = \langle f | f \rangle \\ \sigma_B^2 &= \langle (B - \langle B \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle = \langle g | g \rangle\end{aligned}$$

where

$$\begin{aligned}f &= (A - \langle A \rangle)\psi \\ g &= (B - \langle B \rangle)\psi\end{aligned}$$

Using the Schwarz inequality, we get

$$\sigma_A^2 \sigma_B^2 = \langle f | f \rangle \langle g | g \rangle \geq | \langle f | g \rangle |^2$$

Note that for any complex number z

$$|z|^2 = Re(z)^2 + Im(z)^2 \geq Im(z)^2 = \left(\frac{1}{2i}(z - z^*) \right)^2$$

Thus

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}(\langle f | g \rangle - \langle g | f \rangle) \right)^2$$

Now, a long calculation:

$$\begin{aligned}\langle f | g \rangle &= \langle (A - \langle A \rangle)\Psi | (B - \langle B \rangle)\Psi \rangle \\ &= \langle \Psi | (A - \langle A \rangle)(B - \langle B \rangle)\Psi \rangle \\ &= \langle \Psi | (AB - B \langle A \rangle - A \langle B \rangle + \langle A \rangle \langle B \rangle)\Psi \rangle \\ &= \langle AB \rangle - \langle A \rangle \langle B \rangle\end{aligned}$$

Similarly

$$\langle g | f \rangle = \langle BA \rangle - \langle A \rangle \langle B \rangle$$

Thus

$$\langle f | g \rangle - \langle g | f \rangle = \langle AB \rangle - \langle BA \rangle \equiv \langle [A, B] \rangle$$

The uncertainty principle is then:

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2$$

Example 12. Let's check the original Heisenberg uncertainty principle. We have

$$\begin{gathered}A = x \\ B = \frac{\hbar}{i} \frac{d}{dx}\end{gathered}$$

Because

$$[x, p] = i\hbar$$

we have

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2}\right)^2$$

Theorem 7 (Minimum Uncertainty Wave Packet)**.** What kind of solutions make the inequality in the uncertainty principle an equality? There are two appearances of inequalities in the proof of the theorem. Let's make those equalities. The first one is the Schwarz Inequality. It is an equality if the two functions are related by a constant $g(x) = c \cdot f(x)$. The second inequality appeared when we throw out the real part of $\langle f|g\rangle$ (which is now $\langle f|cf\rangle$). Since $\langle f|f\rangle$ is real, c has to be imarginary. Or, rephrasing this, we have the necessary and sufficient condition:

$$f(x) = i \cdot a \cdot g(x)$$

where a is real.

Example 13. For the momentum/position uncertainty principle the previous theorem gives:

$$\left(\frac{\hbar}{i}\frac{d}{dx} - \langle p\rangle\right)\Psi = ia(x - \langle x\rangle)\Psi$$

This is solved by

$$\Psi(x) = Ae^{-a(x-\langle x\rangle)^2/2\hbar}e^{1\langle p\rangle x/\hbar}$$

Theorem 8 (Time derivative of an expectation value)**.** Let's just calculate this

$$\begin{aligned}
\frac{d}{dt}\langle Q\rangle &= \frac{d}{dt}\langle\Psi|Q|\Psi\rangle \\
&= \langle\dot{\Psi}|Q\Psi\rangle + \langle\Psi|\dot{Q}\Psi\rangle + \langle\Psi|Q\dot{\Psi}\rangle
\end{aligned}$$

The Schroedinger Equation says

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

Thus

$$\begin{aligned}
\frac{d}{dt}\langle Q\rangle &= -\frac{1}{i\hbar}\langle H\Psi|Q\Psi\rangle + \langle\Psi|\dot{Q}\Psi\rangle + \frac{1}{i\hbar}\langle\Psi|QH\Psi\rangle \\
&= -\frac{1}{i\hbar}\langle\Psi|HQ\Psi\rangle + \langle\dot{Q}\rangle + \frac{1}{i\hbar}\langle\Psi|QH\Psi\rangle \\
&= \frac{1}{i\hbar}\langle[H, Q]\rangle + \langle\dot{Q}\rangle
\end{aligned}$$

Example 14. Lets look at the uncertainty principle again with the two operators H and Q, where Q does not depend on time:

$$\sigma_Q\sigma_H \geq \frac{\hbar}{2}\left|\frac{d}{dt}\langle Q\rangle\right|$$

If one defines

$$\begin{aligned}
\Delta E &= \sigma_H \\
\Delta t &= \frac{\sigma_Q}{|d\langle Q\rangle/dt|}
\end{aligned}$$

then

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Δt is the time it takes for the expactation value of Q to change by σ_Q.

Lecture 5 Ends