Lecture 3

1.5 Vertical structure

We have swept the vertical structure of the accretion disk under the carpet until now. It's actually really simple. We make the assumtion that the scale height of the disk

$$H \ll r,$$

i.e. we are looking at a thin disk. Then the force balance in the z direction is given by

$$\frac{dP}{dz} = -\rho g_z$$

Using the equation of state from earlier, $P \propto \rho$ (for constant temperature), and g_z being linear in z, we have

$$\frac{d\rho}{dz} \propto -\rho z.$$

The solution is an exponential of the form

$$\rho(z) = \rho_0 e^{-z^2/(2H^2)}$$

Using the definition of the sound speed $c_s^2 = \frac{k_b T_c}{\mu m_p}$ and the fact that $g_z = \Omega^2 z$, we get

$$H = \frac{c_s}{\Omega}$$

and

$$\rho_0 = \frac{1}{\sqrt{2\pi}} \frac{\Sigma}{H}.$$

1.6 Disk dispersal

So far, we looked at steady state solutions of accretion disks. At some point in time, the mass supply from the stellar nursery will be cut off. The disk will continue to accrete viscously. This happens on a viscous timescale

$$\tau_\nu = \frac{r^2}{12\nu}$$

After one viscous timescale, the central object will have accreted about half the disk material. For proto-planetary disks, this timescale is about a million years.

Once the disk becomes optically thin (in the radial direction), the disk dispersal becomes a runaway effect. The star is then able to blow away the remnants of the disk. This works particularly well for young stars which have a higher rate of EUV radiation than the current Sun. The mechanism that removes the disk particles is Poynting-Robertson (PR) drag and photo-evaporation. This evaporation is very fast, around 10^5 yrs. What we are left with are the solid materials for which the PR drag is not strong enough: a debris disk and planets.

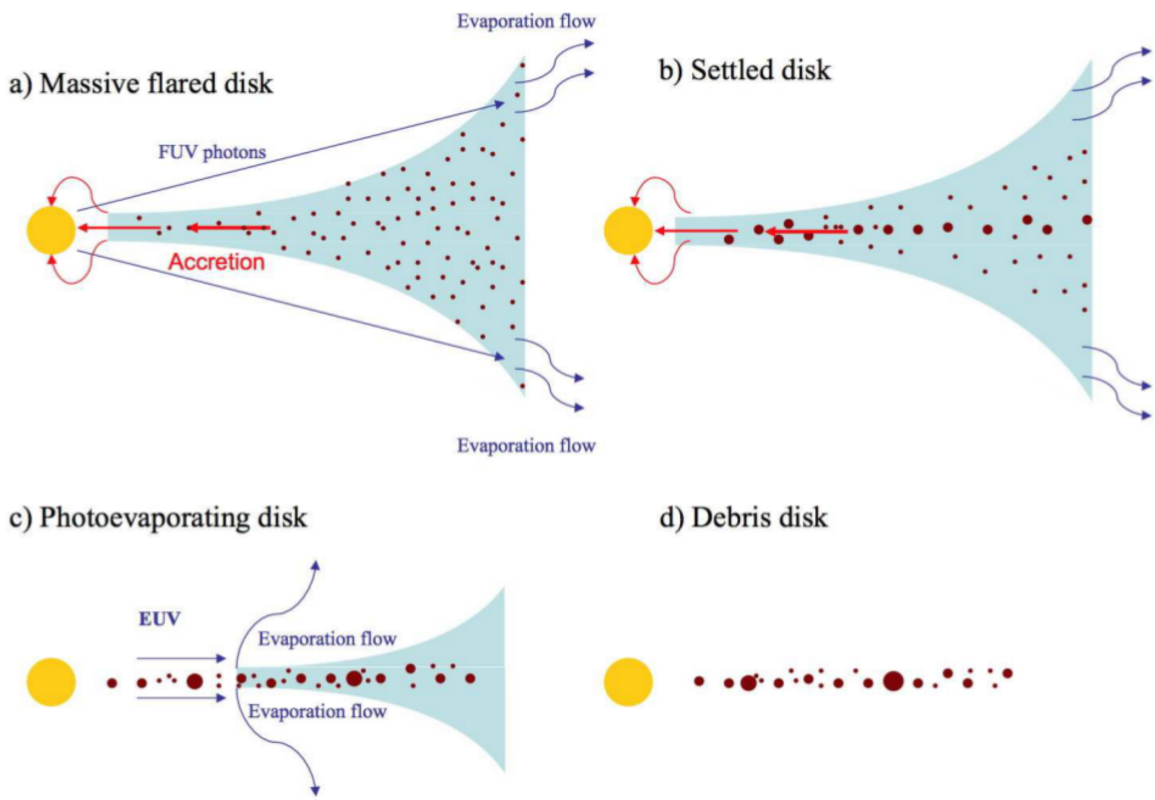

Figure 2: Disk evolution. Image from astrobites.

1.7 Simple Derivation of Poynting-Robertson drag

Consider a dust particle orbiting a star on a circular orbit. In the following we will assume that the dust particle absorbs every photon from star and reemits it in a random direction. Let the dust particle's velocity be $\vec{\mathbf{v}}$ and its position be $r\hat{\mathbf{r}}$, where r is the distance from the star and $\hat{\mathbf{r}}$ the unit vector pointing from the star to the particle. The radiation forces felt by the dust particle can be described as a simple drag force between the dust particle and photons from the star.

Following Burns et al. (1979), we say that the radiation pressure force that would be experienced by a stationary dust particle at this position is a factor β smaller than the gravitational force from the star, and define

$$F_r \equiv \frac{\beta G M_*}{r^2}\,.$$

Here, $\beta \sim 3L_*/(8\pi c\rho G M_* d)$, where L_* is the star's luminosity, c is the speed of light, ρ is the density of the dust grain, G is the gravitational constant, M_* is the star's mass, and d is the diameter of the dust grain. Note that, since F_r scales as r^2, radiation pressure acts in a way that is effectively equivalent to reducing the mass of the star by a factor of β.

We now move on to non-stationary particles. Let us first consider a particle moving radially,

$$\vec{\mathbf{v}}_{\text{radial}} = \dot{r}\hat{\mathbf{r}}.$$

Taking this relative movement between star and dust particle into account, we can calculate the radial component of the radiation force,

$$\vec{\mathbf{F}}_{\text{radial}} = F_r \left(1 - 2\frac{\dot{r}}{c}\right)\hat{\mathbf{r}}\,.$$

There are two contributions to the term involving $\dot{r}/c$, hence the coefficient of 2. Firstly, the energy per photon is Doppler boosted by a factor of $1-\dot{r}/c$. Secondly, the dust particle moves at radial speed $\dot{r}$ relative to the star and therefore encounters photons a factor of $1-\dot{r}/c$ more often. Bringing the two contributions together, the force is increased by $(1-\dot{r}/c)^2 \approx 1 - 2\dot{r}/c$ for $|\dot{r}| \ll c$.

The radial force is not the only force felt by an orbiting dust particle. There is also an azimuthal component, which depends on the azimuthal velocity,

$$\vec{\mathbf{v}}_{\text{azimuthal}} = \vec{\mathbf{v}} - \vec{\mathbf{v}}_{\text{radial}}\,.$$

For $v \equiv |\mathbf{v}| \ll c$, the azimuthal component is just the radial force scaled by the ratio of the azimuthal velocity to the speed of light,

$$\vec{\mathbf{F}}_{\text{azimuthal}} \quad = \quad F_r \frac{-\vec{\mathbf{v}}_{\text{azimuthal}}}{c} + O\left[(v/c)^2\right]\,,$$

which may be rewritten as

$$\begin{aligned}\vec{\mathbf{F}}_{\text{azimuthal}} &= F_r \frac{-(\vec{\mathbf{v}} - \vec{\mathbf{v}}_{\text{radial}})}{c} \\ &= F_r \frac{\dot{r}\hat{\mathbf{r}} - \vec{\mathbf{v}}}{c}\,.\end{aligned}$$

The total force is then the sum of the radial and azimuthal components

$$\begin{aligned}\vec{\mathbf{F}}_{\text{photon}} &= \vec{\mathbf{F}}_{\text{radial}} + \vec{\mathbf{F}}_{\text{azimuthal}} = F_r \left\{\left(1 - 2\frac{\dot{r}}{c}\right)\hat{\mathbf{r}} + \frac{\dot{r}\hat{\mathbf{r}} - \vec{\mathbf{v}}}{c}\right\} \\ &= F_r \left\{\left(1 - \frac{\dot{r}}{c}\right)\hat{\mathbf{r}} - \frac{\vec{\mathbf{v}}}{c}\right\}\end{aligned}$$

Even though this force is due to just one physical effect (photons hitting a dust particle), the different components are often referred to as different processes. The term "radiation pressure" is used for the radial component and the term "Poynting-Robertson drag" is used for the azimuthal component.

1.8 Alternative Scenarios

Let's finish of this section of accretion disks with a short excursion to planet formation theories that do not require an accretion disk.

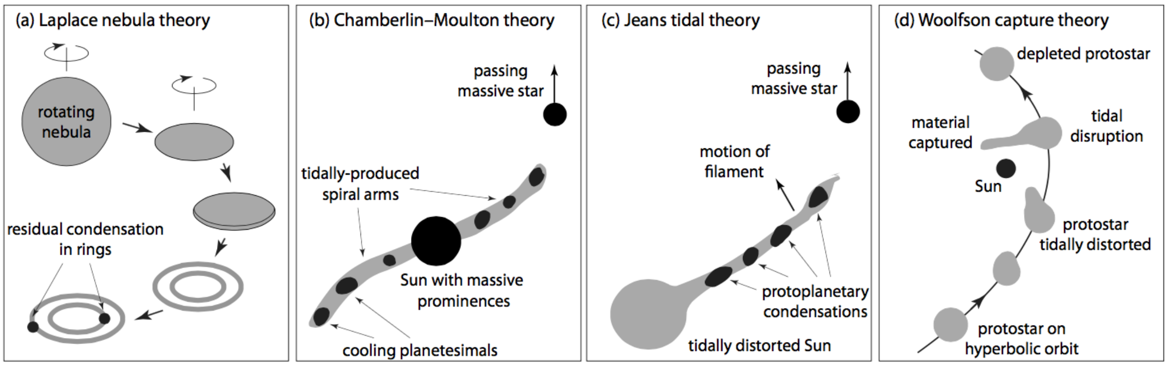

Figure 3: Alternative scenarios. Adapted from Woolfson (1993, Figures 14).

2 Planetesimal formation

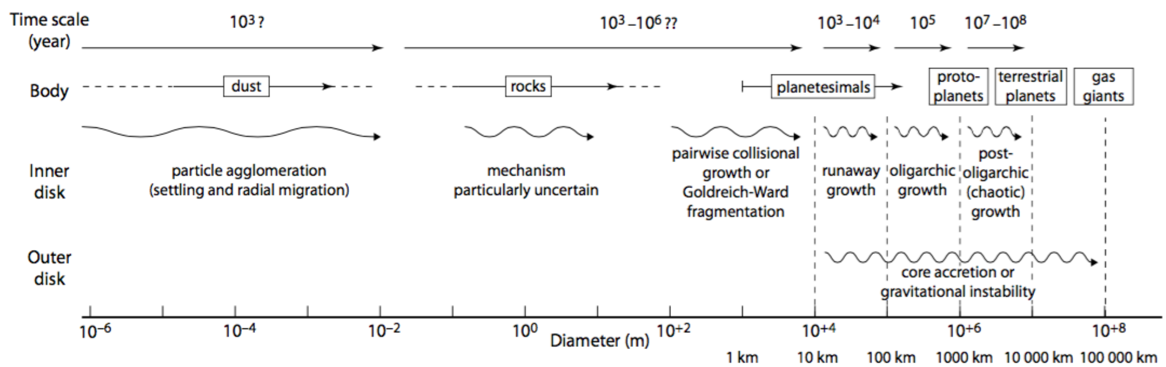

Figure 4: From dust to planets. Adapted from The Exoplanet Handbook (Cambridge University Press).

We've established the framework of planet formation: accretion disks. We can now look at the processes happening within the disk that lead to the formation of planets. Let us start with an overview of scales involved. Figure 4 depicts 14 orders of magnitude that we have to overcome if we want to go from dust particles to terrestrial planets and gas giants.

2.1 Gas drag

In this subsection we'll consider the effect of gas drag onto a spherical particle. In our case the gas will be the gas in the protoplanetary disk with density ρ. The particle will be a dust particle of size s and density ρ_m. Gas drag occurs when there is a relative velocity between the gas and the particle. Let's call this velocity v. There are two physically different regimes that we need to consider. If $s \leq \lambda$, where λ is the mean free path, then we are in the Epstein regime. If $s \geq \lambda$, then we are in the Stokes regime. Let's talk about the two individually.

2.1.1 Epstein drag

Here, the particles are smaller than the mean free path of the gas. To estimate the force, we need to have the mean velocity of individual gas particles

$$v_{\text{th}} = \sqrt{\frac{8k_BT}{\pi\mu m_H}}$$

where μ is the molecular weight. An order of magntiude estimate of the frequency at which particles collide with the fron of the particle is

$$f_+ \approx \pi s^2(v_{\text{th}} + v)\frac{\rho}{\mu m_H}$$

and for the back side

$$f_- \approx \pi s^2(v_{\text{th}} - v)\frac{\rho}{\mu m_H}$$

Calculating the momentum transerred in every single collusion, the force is then

$$F_D \propto -s^2\rho v_{\text{th}} v$$

The force is linear in the relative speed and quadratic in the size of the particle. A calculation including all the coefficients and assuming a Maxwellian distribution of velocities gives the text-book value of

$$F_D = -\frac{4\pi}{3}\rho s^2 v_{\rm th} v$$

2.1.2 Stokes drag

If the particle is significantly larger than the mean free path of gas particles, then the gas can be treated as a fluid.

$$F_D = -\frac{C_D}{2}\pi s^2 \rho v^2$$

The drag coefficient depends on the shape of the particle. Here, we are assuming a sphere, in which case it really only depends on Reynolds number, and hence the viscosiity. There are different regimes depending on the Reynold number. In the limit of a large Reynolnds number (¿800), C_D becomes a constant of ≈ 0.44.

2.1.3 Dust settling

We can define a timescale describing the interaction of particles with gas, the friction time:

$$t_{\rm fric} = \frac{mv}{F_D}$$

Assume we are in the Eppstein regime we get

$$t_{\rm fric} = \frac{\rho_m}{\rho}\frac{s}{v_{th}}$$

Let's put in realistic numbers $r = 1$ AU, $\rho = 10^{-9}$ g cm^{-3}, $\rho_m = 3$ g cm^{-3}, $v_{th} = 10^5$ cm s^{-1} and $s = 1$ μm:

$$t_{\rm fric} = 3{\rm s}$$

So, small particles are very closely coupled to the gas. Nevertheless, they will settle towards the midplane of the disk in a laminar disk. To see that, let's calculate the terminal vertical speed of a particle in the disk. We get this from the force balance of

$$\begin{aligned} F_G &= m\Omega^2 z \\ F_D &= \frac{4\pi}{3}\rho s^2 v_{\rm th} v \end{aligned}$$

which gives

$$v_{\rm settle} = \frac{\rho_m}{\rho}\frac{s}{v_{\rm th}}\Omega^2 z$$

and thus the settling timescale of

$$t_{\rm settle} = \frac{z}{v_{\rm settle}}$$

This timescale is roughly a hundred thousand years, i.e. shorter than the disk lifetime.

The above equation depends on the gas density. The particles will settle faster for lower densities, thus, taking into account the vertial exponential profile of a disk that we derived earlier, we can write down a more detailed settline time as

$$t_{\rm settle} = \frac{2}{\pi}\frac{\Sigma}{\rho_m s\Omega}\exp\left(-\frac{z^2}{2h^2}\right)$$

This means that particles in the upper layer will settle very quickly towards the midplane.

Note that particles can grow while they slowly sink to the midplane. This speeds up the process and can reduce the timescale by one to two orders of magnitude.

2.1.4 Turbulence

In the presence of turbulence, dust settling changes significantly. Let's look at the limit in which

$$\Omega t_{\rm fric} \ll 1.$$

We can define a diffusion timescale using the diffusion coefficient D

$$t_{\rm dif} = \frac{z^2}{D}$$

If we have an alpha disk model, then

$$D \sim \nu = \frac{\alpha c_s^2}{\Omega}$$

We can now calculate the α for which the diffusion timescale is equal to the settling timescale. The result is

$$\alpha \geq \frac{\pi e^{1/2}}{2} \frac{\rho_m s}{\Sigma}$$

Using again standard values, $\Sigma = 10^{-2}$ g cm^{-2}, $\rho_m = 3$ g cm^{-3}, and $s = 1$ μm, we find the the value for α is very small, 10^{-5}. As we need $\alpha \sim 0.01 - 0.01$ to drive accretion, we conclude that small particles are very well mixed with the gas.

However, larger particles, ~ 1 mm can settle into the midplane.

2.1.5 Radial drift

The gas has a sub-keplerian velocity in the ϕ direction because it is pressure supported

$$v_{\phi,\rm gas} = v_k \sqrt{1-\eta}$$

where

$$\eta = \frac{n c_s^2}{v_k^2}$$

n is the power-law in radius of the gas pressure: $P \propto r^{-n}$. The dust particles are on a Keplerian orbit because they don't feel the gas pressure. Thus the particles feel a head wind from the gas. This makes them loose angular momentum and drift inwards.

The derivation of the steady state radial velocity is not too hard, but we just quote the result

$$v_r = \frac{\tau_{\rm fric}^{-1} v_{r,\rm gas} - \eta v_k}{\tau_{\rm fric} + \tau_{\rm fric}^{-1}}$$

where

$$\tau_{\rm fric} = t_{\rm fric}\Omega$$

There are two conlcusions from this:

1) Planetesimal formation must be fast
2) Radial redistribution will happen

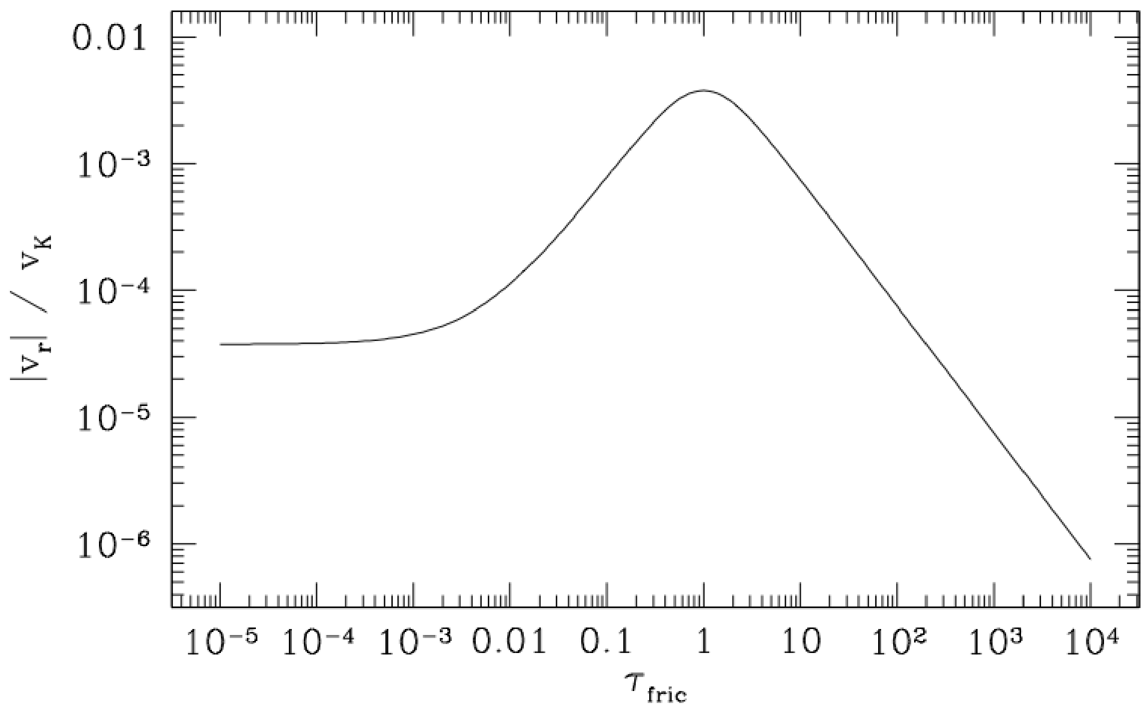

Figure 5: Radial drift of particles in a gas disk. Adapted from Astrophysics of Planet Formation (Cambridge University Press).