11.2 Eigenfunctions

Let's first rewrite the angular momentum operators in spherical coordinates. Remember the definitions

$$\begin{aligned}\vec{L} &= \frac{\hbar}{i}\left(\vec{r}\times\vec{\nabla}\right)\\ \vec{\nabla} &= \hat{r}\frac{\partial}{\partial r}+\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\end{aligned}$$

$$\vec{L}=\frac{\hbar}{i}\left(r(\hat{r}\times\hat{r})\frac{\partial}{\partial r}+(\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta}+(\hat{r}\times\hat{\phi})\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

Because we have a right handed, orthonormal coordinate system, this can be simplified to

$$\vec{L}=\frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta}-\hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

We can write the unit vectors in terms of the cartesian basis vectors

$$\begin{aligned}\hat{\theta} &= (\cos\theta\cos\phi)\ \hat{e}_x+(\cos\theta\sin\phi)\ \hat{e}_y-\sin\theta\ \hat{e}_z\\ \hat{\phi} &= -\sin\phi\ \hat{e}_x+\cos\phi\ \hat{e}_y\end{aligned}$$

Thus

$$\begin{aligned}L_x &= \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta}-\cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)\\ L_y &= \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta}-\sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)\\ L_z &= \frac{\hbar}{i}\frac{\partial}{\partial\phi}\end{aligned}$$

Similarly, we can calculate L^2

$$L_2=-\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

We have already solved these equations and found the eigenfunctions and eigenvalues!

$$f_l^m=Y_l^m$$

Thus, when we derived the stationary state for the H atom, we actually found simultaneous eigenstates for three operators H,L_z,L^2:

$$\begin{aligned}H\psi &= E\psi\\ L^2\psi &= \hbar^2 l(l+1)\psi\\ L_z\psi &= \hbar m\psi\end{aligned}$$

Finally, note that we can rewrite the spherical Schroedinger Equation using the angular momentum operator

$$\frac{1}{2mr^2}\left(-\hbar^2\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+L^2+V\right)\psi=E\psi$$

12 Spin

If you consider the solar system, then there are two main components to the total angular momentum, the orbital momentum of the planets and the planets' spins. Jupiter is leading that contribution. Although, the Sun has 1000 times more mass than the planets, most of the angular momentum is in the orbital angular momentum of Jupiter. The Sun contains 0.3% of the angular momentum as spin. Classically, there is no real difference between the two. Spin is just the angular momentum of a rigid body. In Quantum mechanics, this distinction is much more fundamental.

Here, we postulate that each elementary particle can have a spin. This is an intrinsic, unchangeable parameter. You can perturb a particle as much as you want, you won't be able to change it's spin (but you might change it's angular momentum, see H-atom).

We use the angular momentum as a blue-print to come up with reasonable commutator relations.

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

Just as before, there are eigenstates that satisfy the equations

$$\begin{aligned} S^2 |s, m\rangle &= \hbar^2 s(s+1) |s, m\rangle \\ S_z |s, m\rangle &= \hbar m |s, m\rangle \end{aligned}$$

As before, we can construct raising anf lowering operators. These are (including the normalization)

$$S_\pm |s.m\rangle \equiv (S_x \pm i\, S_y) |s, m\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)}\, |s, m \pm 1\rangle$$

There is not restriction to integer values of s. We thus have

$$\begin{gathered} s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ m = -s. - s + 1, \ldots, s - 1, s \end{gathered}$$

12.1 Spin 1/2

Electrons have spin 1/2. It's a two dimensional system as there are two states, spin up and spin down. We express an arbitrary state as a spinor

$$\chi = a\chi_+ + b\chi_-$$

with the usual normalization and

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

All operators are now two dimensional complex matricies. We already know what they do

$$\begin{aligned} S^2 \chi_+ &= \frac{3}{4}\hbar\chi_+ \\ S^2 \chi_- &= \frac{3}{4}\hbar\chi_- \end{aligned}$$

If we assume an arbitrary form

$$S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix}$$

we can work out the components. Let students work on this for 10 minutes. The answer is

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly for S_z Let students work on this for 10 minutes. Answer:

$$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

And, yet again the same procedure to get

$$\begin{aligned} S_+ &= \hbar\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \\ S_- &= \hbar\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\end{aligned}$$

We already know that $S_\pm = S_x \pm iS_y$. Thus

$$\begin{aligned} S_x &= \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ S_y &= \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\end{aligned}$$

What we just derived are the Pauli Spin Matricies:

$$\begin{aligned} \sigma_x &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ \sigma_y &= \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \\ \sigma_z &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\end{aligned}$$

Let's discuss the eigenvectors. The ones for σ_z are trivial. What are the eigenvectors for σ_x? Let students calculate for 5 minutes. Answer is

$$\begin{aligned} \chi^x_+ &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix} \\ \chi^x_- &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}\end{aligned}$$

Let's talk about one example. Say, we prepare a particle in the state

$$\chi = \begin{pmatrix}a\\ b\end{pmatrix}$$

what is the probability of finding the z spin compoment pointing up? Answer: $|a|^2$. What is the probability of finding the x component pointing up? Answer: $|a+b|^2/2$. What does that mean for consequitive measurements?

12.2 Electrons in the magnetic field

A spinning charged particle has a magnetic dipole moment

$$\mu = \gamma S$$

If there is an external magnetic field, there is an energy associated with the orientation of the dipole

$$H = -\mu \cdot B$$

Let's talk about Larmor precession. This occurs in a uniform external magnetific field. For convienience, let's assume that the magnetic field is pointing in the z direction.

$$H = -\gamma B_0 S_z = -\frac{1}{2}\gamma B_0 \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The eigenstates and eigenvalues are

$$\begin{array}{ll} \chi_+ & E_+ = -\frac{1}{2}\gamma B_0 \hbar \\ \chi_- & E_- = +\frac{1}{2}\gamma B_0 \hbar \end{array}$$

Thus, we can use the time dependent Schroedinger equation to solve for the general solution

$$i\hbar \frac{\partial}{\partial t}\chi = H\chi$$

The solution is

$$\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar}$$

or rewritten as

$$\chi(t) = \cos(\alpha/2)\chi_+ e^{-iE_+ t/\hbar} + \sin(\alpha/2)\chi_- e^{-iE_- t/\hbar}$$

We can now calculate

$$\langle S_x \rangle = \chi(t)^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \chi(t) = \frac{\hbar}{2}\sin\alpha\cos(\gamma B_0 t)$$

and

$$\begin{aligned} \langle S_y \rangle &= \chi(t)^\dagger S_y \chi(t) = -\frac{\hbar}{2}\sin\alpha\sin(\gamma B_0 t) \\ \langle S_z \rangle &= \chi(t)^\dagger S_z \chi(t) = \frac{\hbar}{2}\cos\alpha \end{aligned}$$

This means that the expectation value $\langle S \rangle$ processes about the field in the same way as it does clasically. The frequency is the Larmor frequency:

$$\omega = \gamma B_0$$

12.3 Stern-Gerlach experiment

Let's assume an inhomogeneous magnetic field in the form

$$B_z = B_0 + \alpha z$$

In reality there is also a small component of the magnetic field in the x (or y) direction because $\mathrm{div}(B) = 0$. However, this is not important for our discussion.

If a particle enters the magnetic field and leaves after a time T, the Hamiltonian looks like

$$H(t) = -\gamma(B_0 + \alpha z)S_z$$

for $0 < t < T$ and is zero otherwise. If we start out with the same initial state as before and apply the same procedure as before, we get for $t > T$

$$\begin{aligned} \chi(t) &= a\chi_+ e^{-iE_+ T/\hbar} + b\chi_- e^{-iE_- T/\hbar} \\ &= a\chi_+ e^{i\gamma(B_0+\alpha z)T/2} + b\chi_- e^{-i\gamma(B_0+\alpha z)T/2} \end{aligned}$$

This means that the particle has momentum in the z direction. And the momentum depends on the spin.

$$p_z = \pm\frac{\alpha\gamma T\hbar}{2}$$

LECTURE 9 ENDS