4 Eigenfunctions of Observables

Bare with me. We're almost done with the long list of mathematical definitions.

Definition 35 (Discrete spectrum)**.** A spectrum is discrete if the eigenvalues are unique. The eigenfunctions lie in Hilbert space and are physically realizable.

Definition 36 (Continuous spectrum)**.** A spectrum is continuous if the eigenvalues are continuous in a range. In that case the eigenfunction are not normalizable (not in Hilbert space) and therefore nor physical states (think of free particle). However, linear combinations can be physical states (think of wave package).

The discrete case is easier. So let's start with that.

4.1 Discrete spectra

Theorem 4 (Eigenvalues are real.)**.** Suppose $Qf = qf$ and $\langle f|Qf\rangle = \langle Qf|f\rangle$. The latter assumes Q is hermitian and that f is in Hilbert space and therefore the inner product exists. Then

$$q\,\langle f|f\rangle = q^*\,\langle f|f\rangle$$

as one can pull the q (and q^*) in front of the inner product. We also made use of the fact that $f \neq 0$ (which would not be a valid eigenfunction).

Theorem 5 (Eigenfunctions to distinct eigenvalues are orthogonal.)**.** Suppose $Qf = qf$ and $Qg = q'g$, $q \neq q'$. Then

$$\begin{aligned}\langle f|Qg\rangle &= \langle Qf|g\rangle \\ q'\,\langle f|g\rangle &= q^*\,\langle f|g\rangle\end{aligned}$$

Because q is real (see above) it follows that $\langle f|g\rangle = 0$.

Note that this theorem does not tell us anything about degenerate states. In that case, we have to use the Gram-Schmidt orthogonalization procedure to come up with orthogonal states.

Let's do an example of the Gram-Schmidt procedure. It works the same was as you know from linear algebra, we just apply it to functions. Thus, all the inner products become integrals.

Example 6 (Orthogonalization)**.** Both $f(x) = e^x$ and $g(x) = e^{-x}$ are eigenfunctions of d^2/dx^2 with the same eigenvalue. Let us consider the intervale $[-1, 1]$. It is easy to calculate eigenvalue (which is 1):

$$\frac{d^2}{dx^2}e^x = 1 \cdot e^x.$$

The same is true for e^{-x}. However, the functions are not orthogonal.

$$\int_{-1}^{1} f(x)g(x)\,dx = \int_{-1}^{1} e\,dx \neq 0.$$

Let's use the Gram-Schmidt procedure:

1. Normalize $f(x)$:

$$\int_{-1}^{1} f(x)f(x)\,dx = \frac{e^2}{2} - \frac{1}{2e^2}.$$

$$\hat{f}(x) \equiv \frac{f(x)}{\sqrt{\frac{e^2}{2} - \frac{1}{2e^2}}}$$

2. Find projection of $g(x)$ onto $f(x)$ and subtract from $f(x)$:

$$
\begin{aligned}
g'(x) &= g(x) - \int_{-1}^{1} \hat{f}(x) g(x)\ dx\ \hat{f}(x) \\
&= e^{-x} - \left(\frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \right) \int_{-1}^{1} 1\ dx\ e^x \\
&= e^{-x} - \left(\frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \right) 2\ e^x
\end{aligned}
$$

3. Normalize $g'(x)$. Let's skip that because it's super long and boring.

4. Let's check that the two functions are orthogonal (we can do that without normalizing g').

$$
\begin{aligned}
\int_{-1}^{1} \hat{f}(x) g'(x)\ dx &= \frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \int_{-1}^{1} e^x \cdot \left(e^{-x} - \left(\frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \right) 2\ e^x \right)\ dx \\
&= \frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \left[2 - \left(\frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} \right) \left(e^2 - e^{-2} \right) \right] \\
&= \frac{1}{\frac{e^2}{2} - \frac{1}{2e^2}} [2 - 2] = 0
\end{aligned}
$$

In a finite dimensional space there is an additional property of hermitian operators: Their eigenvectors span the space. This does not hold in infinite dimensional spaces, so we make it an axiom (following Dirac):

Axiom 1. The eigenvectors of an observable are complete (they span the space).

4.2 Continuous spectra

Let us now discuss the continuous case. The proofs of the last two theorems fail because the inner products might not exist (remember, the functions are not normalizable). This is a bit subtle, so lets go through this with examples rather than rigorous proofs.

Example 7 (Eigenvectors and eigenfunctions of the momentum operator)**.** We are looking for solutions to the differential equation

$$
\frac{\hbar}{i} \frac{d}{dx} f_p(x) = p\ f_p(x)
$$

The solutions are of the general form

$$
f_p(x) = A\ e^{ipx/\hbar}
$$

Note that this is not square integrable, thus not normalizable and the inner products will not exist. However, we can nevertheless come up with some sort of orthogonality:

$$
\int_{-\infty}^{\infty} f_p^*(x) f_{p'}(x)\ dx = |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}\ dx = |A|^2 2\pi\hbar\delta(p - p')
$$

We can thus set $A = 1/\sqrt{2\pi\hbar}$ and write

$$
\langle f_p | f_{p'} \rangle = \delta(p - p').
$$

We call this Dirac orthonormality. Further note that the eigenfunctions are complete. We can construct any function $f(x)$ as a linear combination of $f_p(x)$:

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\ dp \qquad \text{where}$$
$$c(p) = \int_{-\infty}^{\infty} c(p')\delta(p-p')\ dp' = \int_{-\infty}^{\infty} c(p') \langle f_p | f_{p'} \rangle\ dp' = \langle f_p | f \rangle$$

This is nothing but a Fourier Transform.

`[Make a quiz on the board. Draw a few simple functions and as what the Fourier Transforms are.]`

Example 8 (Eigenvectors and eigenfunctions of the position operator)**.** Let's hae a look for solutions of the equation

$$x\ g_y(x) = y\ g_y(x)$$

`[Explain every component of the equation in detail on the board.]`

This looks weird on first sight. We look for a function that is the same if multiplied by x (a continuous variable) up to a constant y. The solution to this might be surprising. There exists a class of functions that actually solves this:

$$g_y(x) = A\delta(x-y)$$

where we pick $A = 1$ for the normalization constant. Again, we have Dirac orthonormality:

$$\int g^*_{y'}(x) g_y(x) dx = \int \delta(x-y')\delta(x-y)\ dx = \delta(y-y').$$

Thus:

$$\langle g_{y'} | g_y \rangle = \delta(y-y')$$

LECTURE 3 ENDS