

Let's look at the spectrum of hydrogen. We actually considered stationary states. As the name suggests, they should stay there forever. However, assume we tickle it. Then electrons might jump from one state to another. This is called a transition. The energy difference is

$$E_\gamma = E_i - E_f = -13.6 \text{ eV} \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

The energy of a photon is related to its frequency by

$$E_\gamma = h\nu$$

Whereas the wavelength λ is

$$\lambda = c/\nu$$

We can then write the wavelength of the photons emitted in a transition as

$$\frac{1}{\lambda} = R \left(\frac{1}{n_i^2} - \frac{1}{n_f^2} \right)$$

where R is the Rydberg formula.

$$R \equiv \frac{me^4}{64c\pi^3\hbar^3\epsilon_0^2} = 1.097 \cdot 10^7 \text{ m}^{-1}$$

11 Angular Momentum

Classically, the angular momentum is

$$L = r \times p$$

We make the usual quantum mechanical substitutions

$$\begin{aligned} x &\rightarrow \hat{x} \\ p_x &\rightarrow -i\hbar \frac{\partial}{\partial x} \end{aligned}$$

Note that \hat{L}_x and \hat{L}_y do not commute

$$\begin{aligned} [L_x, L_y] &= i\hbar L_z \\ [L_y, L_z] &= i\hbar L_x \\ [L_z, L_x] &= i\hbar L_y \end{aligned}$$

Thus, we cannot simultaneously measure all three components of the angular momentum. Let's plug this into the uncertainty principle. We get

$$\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} |\langle L_z \rangle|$$

In other words, there are no simultaneous eigenfunctions of L_x and L_y . But we can also define the total angular momentum

$$L^2 \equiv L_x^2 + L_y^2 + L_z^2$$

Note that

$$\begin{aligned}
[L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\
&= L_y [L_y, L_x] + [L_y, L_x] L_y + L_z [L_z, L_x] + [L_z, L_x] L_z \\
&= L_y (-i\hbar L_z) + (-i\hbar L_z) L_y + L_z (i\hbar L_y) + (i\hbar L_y) L_z \\
&= 0
\end{aligned}$$

The same is true for $[L^2, L_y]$ and $[L^2, L_z]$. Thus, we can hope to find simultaneous eigenstates of L^2 and for example L_x .

$$L^2 f = \lambda f \quad L_z f = \mu f$$

11.1 Eigenvalues

We use a ladder operator to find the eigenstates. Just as in the harmonic oscillator case. Let's define

$$L_{\pm} \equiv L_x \pm iL_y$$

The commutators are then

$$[L_z, L_{\pm}] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y) = \pm\hbar L_{\pm}$$

and

$$[L^2, L_{\pm}] = 0$$

Corollary 14. If f is an eigenfunction of L^2 and L_z , then so is $L_{\pm}f$.

Proof:

$$\begin{aligned}
L^2(L_{\pm}f) &= L_{\pm}(L^2f) = L_{\pm}\lambda f = \lambda L_{\pm}f \\
L_z(L_{\pm}f) &= (L_z L_{\pm} - L_{\pm} L_z)f + L_{\pm} L_z f = \pm\hbar L_{\pm}f + L_{\pm}(\mu f) = (\mu \pm \hbar)(L_{\pm}f)
\end{aligned}$$

So, the L_{\pm} operator are raising and lowering L_z while keeping L^2 the same. Thus there has to be a state where we reach the maximum. Otherwise the z -component of the angular momentum would be larger than the total angular momentum. This argument leads to the condition

$$L_+ f_t = 0.$$

Let's call the eigenvalue of that state be $l\hbar$ (justification for that label follows)

$$L_z f_t = \hbar l f_t \quad L^2 f_t = \lambda f_t$$

Let's look at

$$L_{\pm} L_{\mp} = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) = L^2 - L_z^2 \mp i(\hbar L_z)$$

or the other way round

$$L^2 = L_{\pm} L_{\mp} + L_z^2 \mp \hbar L_z$$

Therefore:

$$L^2 f_t = (L_- L_+ + L_z^2 + \hbar L_z) f_t = (0 + \hbar^2 l^2 + \hbar^2 l) f_t = \hbar l(l+1) f_t$$

Thus, $\lambda = \hbar^2 l(l+1)$. This tells us the eigenvalue of L^2 in terms of the maximum eigenvalue of L_z .

For the same reason we have a state that satisfies $L_+f_t = 0$ we have $L_-fb = 0$. Using the same calculation as before we get

$$L^2f_b = (L_+L_- + L_z^2 - \hbar L_z)f_b = (0 + \hbar^2l'^2 - \hbar^2l')f_b = \hbar l'(l' - 1)f_b$$

Thus, $\lambda = \hbar^2l'(l' - 1)$. So either

$$l = -l'$$

or

$$l' = l + 1$$

The first solution is the physical one, the latter would give us a different maximum than minimum. Let's label the eigenvalues of L_z as $m\hbar$ (justification for that label follows). As we just showed, these eigenvalues run from $m = -l$ to l in N integer steps. Because $l = -l + N$, we know that $l = N/2$, so l must be an integer or a half integer.

In summary

$$\begin{aligned} L^2f_l^m &= \hbar^2l(l+1)f_l^m \\ L_zf_l^m &= \hbar mf_l^m \end{aligned}$$

where

$$\begin{aligned} l &= 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \\ m &= -l, l+1, \dots, l-1, l \end{aligned}$$

Discuss illustration in Griffiths, page 165.

LECTURE 8 ENDS