Let us recap the first lecture.

1. We first looked at infalling material in an adiabatic flow. If we assume that the infalling material cannot loose any energy and only converts potential energy into internal energy (heat), then no disk will form. We saw this by estimating the soundspeed and comparing it to the escape-speed. They are comparable.

2. Next, we assumed that the material will radiate away energy. We assume a black body radiation. If we want a steady state sollution, the energy radiated away at the stellar surface must equal the energy from infalling material. This solves the *energy problem*, i.e. now that we do not conserve energy anymore, material can accrete onto the star. This is worth emphasizing as it is important and maybe counter-intuitive: We simply ignore energy conservation! This is related to the concept of an isothermal disk. The temperature (internal energy) of the gas only depends on its radial distance from the central object. If a gas parcel moves from one position to the next, the temperature changes instantly due to some radiative/thermodynamical process.

3. We looked at the limiting case where the opacity of the infalling material begins to play a role. The limiting case arrises when the radiation pressure onto the infalling material equals the gravitational force. In that case no accretion occurs anymore (this is the Eddington Limit).

4. Finally, we started to look at angular momentum. Radiation alone cannot remove angular momentum. So we're once again stuck. To accrete, we need another mechanism to get rid of angular momentum. Viscosity is the natural candidate. To start off, we wrote down equations for mass and angular momentum conservation (without viscosity):

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma\, r\, v_r\right) = 0,$$

$$r\frac{\partial}{\partial t}\left(\Sigma\, r^2\, \Omega\right) + \frac{\partial}{\partial r}\left(\Sigma v_r\, r^3\, \Omega\right) = 0$$

We now have two equations, for two unkowns (Σ and v_r). Before we solve these equations and also add viscosity, let's motivate where they come from.

Consider a thin ring with the inner boundary at radius r and a radial width of Δr. The amount of material Δm that flows in the ring through the radial boundaries at r and $r + \Delta r$ per time unit Δt is

$$\Delta m = \left(v_r(r)\ \Sigma(r)\ 2\pi r - v_r(r+\Delta r)\ \Sigma(r+\Delta r)\ 2\pi(r+\Delta r)\right)\Delta t$$

Noting that $m = \Sigma\ 2\pi((r+\Delta r)^2 - r^2)$ and taking the limit $\Delta r \to 0$, we get

$$m = \Sigma 2\pi r \Delta r$$

and

$$\Delta\Sigma 2\pi r \Delta r = \left(v_r(r)\ \Sigma(r)\ 2\pi r - v_r(r+\Delta r)\ \Sigma(r+\Delta r)\ 2\pi(r+\Delta r)\right)\Delta t$$

after rearranging

$$\frac{\Delta\Sigma}{\Delta t} = \frac{1}{r}\frac{v_r(r)\ \Sigma(r)\ r - v_r(r+\Delta r)\ \Sigma(r+\Delta r)\ (r+\Delta r)}{\Delta r}$$

taking the limit $\Delta r \to 0$ again, as well as $\Delta t \to 0$

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left(\Sigma\, r\, v_r\right).$$

Lecture 2

We can combine the two differential equations into one equation for Σ that is independent of v_r. First, we use the product rule to rewrite the angular momentum equation

$$r^3\Omega\frac{\partial\Sigma}{\partial t}+r\Sigma\frac{\partial}{\partial t}\left(r^2\,\Omega\right)+r^2\Omega\frac{\partial}{\partial r}\left(\Sigma v_r\,r\right)+\Sigma v_r\,r\,\frac{\partial}{\partial r}\left(r^2\,\Omega\right)=0.$$

Then, we plug in the mass conservation

$$r^3\Omega\frac{\partial\Sigma}{\partial t}+r\Sigma\frac{\partial}{\partial t}\left(r^2\,\Omega\right)-r^3\Omega\frac{\partial\Sigma}{\partial t}+\Sigma v_r\,r\,\frac{\partial}{\partial r}\left(r^2\,\Omega\right)=0$$

After simplifying, we get

$$r\Sigma\frac{\partial}{\partial t}\left(r^2\,\Omega\right)+\Sigma v_r\,r\,\frac{\partial}{\partial r}\left(r^2\,\Omega\right)=0$$

The first term is zero since the gravitational potential does not change. Thus we are left with

$$\Sigma v_r\,r\,\frac{\partial}{\partial r}\left(r^2\,\Omega\right)=0$$

Assuming a Keplerian shear, we can calculate the derivate to get

$$\begin{aligned}\Sigma v_r\,r\,\frac{\partial}{\partial r}\left(\sqrt{r\,GM}\right) &= 0\\ \Sigma v_r\,\frac{1}{2}\sqrt{rGM} &= 0.\end{aligned}$$

A solution for this equation requires either $\Sigma=0$ (i.e. no disk) or $v_r=0$ (i.e. no radial motion). Do does this mean there are no accretion disks? No, we just forgot to include an important effect into our accretion disk: viscosity.

Let's go back one step and add a viscous torque term to the angular momentum equation:

$$r\frac{\partial}{\partial t}\left(\Sigma\,r^2\,\Omega\right)+\frac{\partial}{\partial r}\left(\Sigma v_r\,r^3\,\Omega\right)=\frac{1}{2\pi}\frac{\partial T}{\partial r}$$

with the viscous torque being

$$T(r)=2\pi\nu\Sigma r^3\frac{\partial\Omega}{\partial r}$$

where ν is the kinematic viscosity. Going through a similar calculation as above (and assuming a Keplerian shear), we arrive at

$$\frac{\partial\Sigma}{\partial t}=\frac{3}{r}\frac{\partial}{\partial r}\left(\sqrt{r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right)\right).$$

If we know the surface density and the viscosity (which can in principle be a function of radius and time as well), we can calculate the radial velocity

$$v_r(r,t)=-\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}(\nu\Sigma\sqrt{r}).$$

To illustrate what happens, let us study the hypothetical case of an infinitesimally thin ring with mass m and radius r_0. The surface density profile can be written in terms of a delta function:

$$\Sigma(r,t=0)=\frac{m\,\delta(r-r_0)}{2\pi r_0}$$

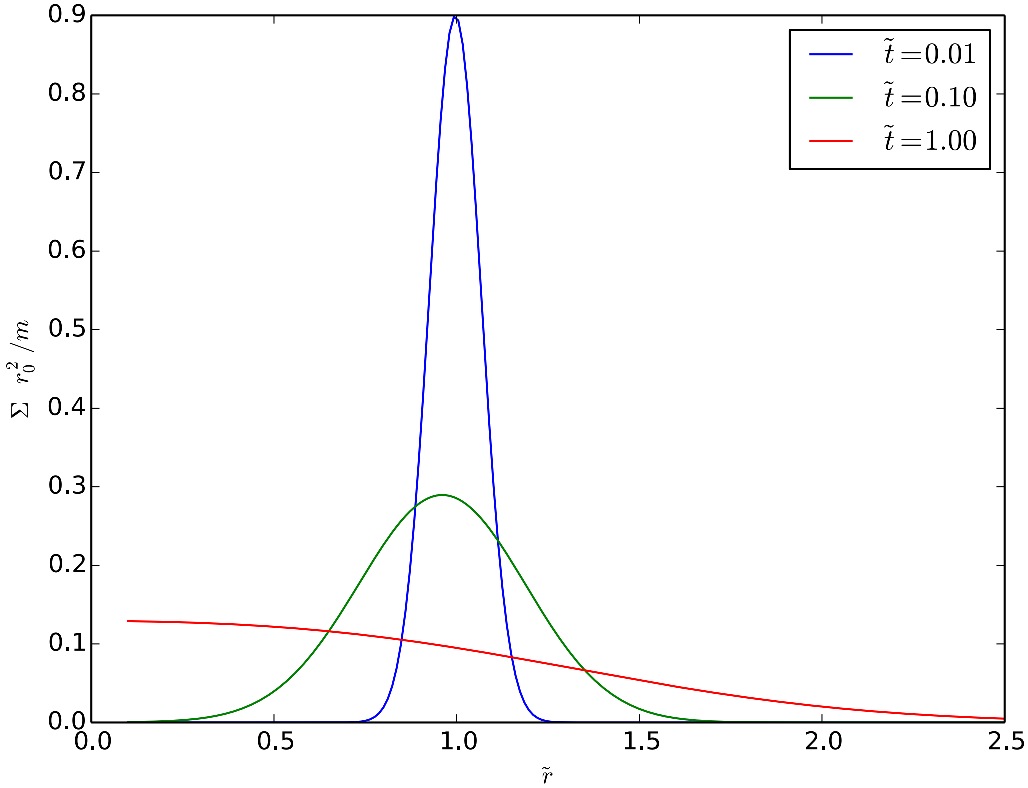

Figure 1: Viscous evolution of an initially infinitesimally thin ring.

Let us introduce dimensionless time $\tilde{t}$ and radius $\tilde{r}$

$$\begin{aligned}\tilde{t} &= t\frac{12\nu}{r_0^2}\\ \tilde{r} &= r/r_0\end{aligned}$$

Now, we can write the solution as

$$\Sigma(\tilde{r},\tilde{t}) = \frac{m}{\pi r_0^2}\tilde{t}^{-1}\tilde{r}^{-1/4}\exp\left(-\frac{1+\tilde{r}^2}{\tilde{t}}\right) I_{1/4}\left(\frac{2\tilde{r}}{\tilde{t}}\right)$$

where $I_{1/4}$ is a modified Bessel function. The evolution is plotted for three times in Figure 1.

Let us now try to find a steady state solution to the evolution equations. In such a case $\dot{\Sigma}=0$ and we find that there exists a constant C with

$$C = \sqrt{r}\frac{\partial}{\partial r}\left(\nu\Sigma\sqrt{r}\right).$$

Plugging this into the equation for the radial velocity we get

$$v_r(r,t) = -\frac{3\,C}{\Sigma r}.$$

The constant C is related to the accretion rate, or inward mass flux by

$$2\pi r\Sigma v_r = -\dot{M} = -6\pi C.$$

and thus

$$v_r = -\frac{\dot{M}}{2\pi\Sigma r}.$$

To get the surface density profile, let us use

$$\begin{aligned}
-\frac{\dot{M}}{2\pi\Sigma r} &= -\frac{3}{\Sigma\sqrt{r}}\frac{\partial}{\partial r}(\nu\Sigma\sqrt{r}) \\
\frac{1}{6\pi}\frac{\dot{M}}{\sqrt{r}} &= \frac{\partial}{\partial r}(\nu\Sigma\sqrt{r}) \\
\int_{r_*}^{r}\frac{1}{6\pi}\frac{\dot{M}}{\sqrt{r}}dr &= \nu\Sigma\sqrt{r} \\
\frac{1}{2\pi}\dot{M}\left(\sqrt{r}-\sqrt{r_*}\right) &= \nu\Sigma\sqrt{r} \\
\Sigma &= \frac{1}{2\pi}\frac{\dot{M}}{\nu}\left(1-\sqrt{r_*/r}\right)
\end{aligned}$$

Finally, let us calculate the energy released due to viscosity. The dissipation per unit time and unit area at a distance r is

$$D = \frac{1}{2}\nu\Sigma\left(r\frac{\partial\Omega}{\partial r}\right)^2$$

The quantity in the brackets is the shear. For a Keplerian disk, this equates to

$$D = \frac{3GM\dot{M}}{4\pi r^3}\left(1-\sqrt{r_*/r}\right).$$

An integration over the entire disk yields the luminosity

$$\begin{aligned}
L &= 2\pi\int_{r_*}^{\infty} r\frac{3GM\dot{M}}{4\pi r^3}\left(1-\sqrt{r_*/r}\right)dr \\
&= \int_{r_*}^{\infty}\frac{3GM\dot{M}}{2r^2}\left(1-\sqrt{r_*/r}\right)dr \\
&= \frac{GM\dot{M}}{2r_*}
\end{aligned}$$

Let's calculate the luminosity for a standard molecular kinematic viscosity. We won't derive the value for the viscosity here but simple assume the literature value of $\nu \approx 0.1\ \mathrm{m^2\,s^{-1}}$. If we change the units to something closer to what we are interested in, this is

$$\nu \approx 10^{-16}\ \mathrm{AU^2\,yr^{-1}}.$$

A protoplanetary disk with the mass of Jupiter and an extend of $r_o \approx 1$ AU has a surface density far fom the star of approximately

$$\Sigma \approx \frac{1}{2\pi}\frac{\dot{M}}{\nu} \approx \frac{M_{\mathrm{jup}}}{2\pi r_o^2}$$

and thus

$$\dot{M} \approx \frac{\nu M_{\mathrm{jup}}}{r_o^2} \approx 8000\mathrm{kg\,yr^{-1}}.$$

At that rate it would take almost 10^{16} yrs to accrete one Jupiter mass. Clearly somehting is wrong.

The answer lies in the nature of the viscosity we used. It is now commonly believed that other sources of viscousity can dominate over the small kinematic viscosity. Primarily accretion is now thought to be driven by turbulence.

1.4 Shakura-Sunayev disk

For a turbulent disk, the viscosity comes from eddies in the flow. The α disk model is the most commonly used way to hide our ignorance towards the complicated turbulent flow. We simply describe everything with one single scalar parameter. To get the order of magnitude right, we have to think about what scales are involved in turbulence. The first limiting scale is the disk thickness H. No eddie can be larger than H and thus we will use H as one of our limiting factors in the α disk model. Another is the sound speed. If the turbulent velocities are larger than the sound speed, we expect dissipation due to hydrodynamic shocks to occur. These shock will then slow down the turbulence. Thus the sound speed acts an an upper boundary. Putting those two scales together, we can construct a quantity which has the same dimensions as the viscosity. The proportionality factor is labelled α and is less than ≈ 1. In summary:

$$\nu = \alpha c_s H.$$

Note that we had to make many assumptions getting to this point. It is very easy to find a reason why you would not want to trust the α disk model. In practice, to get accretion rates that are comparable with observations, we have to use $\alpha \approx 10^{-2}$.