Lecture 5

3 Planet Migration

There are several reasons why we think planet migration is important. Planets form preferentially beyond several AU (beyond the snowline where temperatures in the disk are cooler). However, we see planets in extrasolar planetary systems much closer to their star. They cannot have possibly formed there. A lot of planets are also on eccentric orbits, a property that is hard to explain without migration as planets probably form on circular orbits. In our own Solar System, the population of resonant Kuiper Belt objects indicates significant movement of the gas giants. There is also a timescale problem with Neptune and Uranus.

3.1 Impulse approximation

We start with a quick overview of orbital parameters.

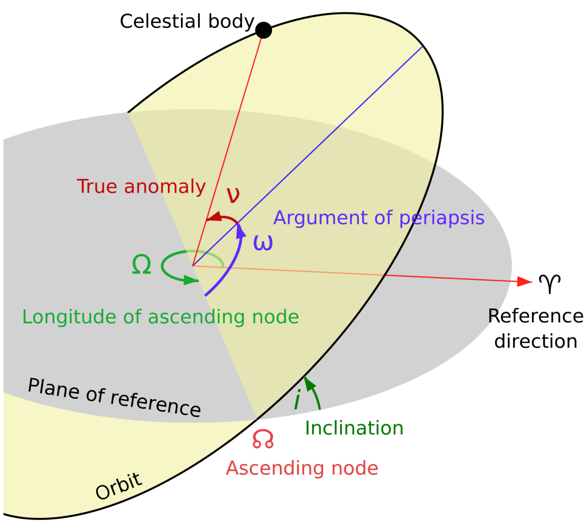

Figure 7: Orbital parameters (wikipedia).

Migration means a change in semi-major axis of a planet. Most commonly migration is discussed in a gas disk. There, a planet changes its semi-major axis by exchanging angular momentum and energy with the disk. We'll mostly talk about the effect of the interaction on the semi-major axis, but any of the orbital parameters can change. Most importantly the eccentricity and inclincation.

Migration in a gas disk is important for planets that are of Mars size or larger. An obvious assumption for migration to happen is that planets and the gas disk must be present at the same time. This is not obvious for terrestrial planets but required for the formation of gas giants.

The exact calculation of the migration speed is tricky, not only mathematically, but also physically. Many small subtle effects could have pottentially huge impacts.

Let us start with an (maybe) obvious statement. An unperturbed axisymmetric disk will not lead to migration. It's the perturbation that drives migration. We will treat the perturbation as a small deflection of a test particle. Clearly, this is an approximation; it's called the impulse approximation.

We consider a mass m moving past another mass M with relative velovity v. Let us label the impact parameter with b. The perpendicular force felt by the mass m is

$$F_{perp} = \frac{GmM}{d^2}\cos\theta$$

where d is the distance between the particles and θ the angle relative to the line during closest approach. This equation can be written as

$$F_{perp} = \frac{GmM}{b^2}\left(1+\left(\frac{vt}{b}\right)^2\right)^{-3/2}$$

where we set $t = 0$ at closest approach.

What we ultimately want is the change in velocity felt by the small mass. Thus, we integrate the force

$$\delta v_{perp} = \int_{-\infty}^{\infty} \frac{F_{perp}}{m} dt = \frac{2GM}{bv}$$

This is the perpendicular velocity change. If we think of a test particle encountering a planet and both are on circular orbits, then this corresponds to a radial kick. A radial kick, however, does not change the angular momentum. We can calculate the parallel velocity change by using the fact that the encounter is conservative. Thus, by energy conservation:

$$v^2 = |\delta v_{perp}|^2 + (v - \delta v_{par})^2$$

and therefore

$$\delta v_{par} \approx \frac{1}{2v}\left(\frac{2GM}{bv}\right)^2$$

Assuming the planet has a semi-major axis of a, then the relative angular momentum change is

$$\delta j = \frac{2G^2M^2a}{b^2v^3}$$

Let's talk about the sign. Material outside the planet is slower than the planet. Thus the planet overtakes it. The force from the gas particle will slow the planet down, it will loose angular momentum and move further in. Gas interior to the planet on the other hand will make the planet move outside. The fine balance between the two effects will determine the overall direction.

The mass of the disk in an annulus from b to $b + \delta b$ near the planet is

$$\delta m = 2\pi a \Sigma \delta b$$

The annulus has a slightly different orbital frequency than the planet, by Kepler's laws. It takes time δt for the entire mass in the ring to encounter the planet once:

$$\delta t = \frac{2\pi}{|\Omega - \Omega_p|}$$

We can calculate this assuming a Keplerian disk

$$|\Omega - \Omega_p| = \frac{3\Omega_p}{2a} b$$

To get the total torque, we need to integrate over b. Let's write down the torque for the material outside of the planet:

$$\frac{dJ}{dt} = -\int_0^{\infty} \frac{8G^2M^2a\Sigma}{9\Omega_p^2}\frac{db}{b^4}$$

This integral is divergent. We need to do something special near the planet (for small b). Typically, people just assume a small lower limit for the impact parameter, then we can do the integrat to get

$$\frac{dJ}{dt} = -\frac{8}{27}\frac{G^2 M^2 a\Sigma}{\Omega_p^2 b_{min}^3}$$

A couple of important things. First, the scale of b_{min} is typically chosen to be of the order of the disk thickness H. There is, however, a considerable uncertainty associated with the small scale physics. This is an active field of research. Second, note that the torque is dependend on M to the power of two. Thus, larger planets will migrate faster. Third, this is the torque from the outside part. There is a torque from the inside two. However, the two torques do in general not balance each other to a high degree of accuracy and we can estimate the migration timescale from the outside torque alone

$$\tau = \frac{J}{|dJ/dt|} \sim 1\mathrm{Myr}$$

for typical disk parameters and an Earth mass planet. For a Jupiter mass planet, we have

$$\tau \sim 0.2\mathrm{Myr}$$

These simple order of magnitude estimate reveal that migration should be important for both Earth and Jupiter mass planets.

Once again, what we calculated from first principles is an approximation. We missed the fact that we are working with a hydrodynamic system, not particles. Thus, pressure did not appear anywhere. One can do a more thorough calculation and get a better estimate (but it's not too far off).

One thing we also didn't consider are co-rotation torques. These occur for gas particles on horseshoe orbits. The magnitude of these torques can be very large. However, the depend on an enourmous number of small details which makes it very difficult to come up with a predictive theory.

There are different regimes for migration.

- **Type I**. In this regime the perturbations on the disk are small. The planet remains fully embedded.

- **Type II**. In this regime the torques from the planet will dominate over the viscous torques. In this case, the material outside the planet will be pushed further outside. The material inside will be pushed further inside. The result is a gap opening in the disk.

To estimate the critical mass for the transition between Type I and II we need to compare the viscous and migration torques. Typically, the transition happens near a Saturn mass:

$$\frac{M}{M_*} \approx \left(\frac{27\pi}{8}\right)^{1/2} \left(\frac{h}{r}\right)^{5/2} \alpha^{1/2}$$

The second assignment asks you to verify this equation numerically.