12.4 Addition of Angular Momenta

Let's imagine two spin $\frac{1}{2}$-partiles such as the electron and the proton. If we are in the ground state, there is no orbital angular momentum. How do the spins add-up? Clasically, you just add the vectors.

Quantum mechanically, we only have the following four possible ways to allign the individual spins.

$$\uparrow\uparrow, \downarrow\uparrow, \uparrow\downarrow, \downarrow\downarrow$$

What is the total angular momentum of such an atom? We define

$$S \equiv S^{(1)} + S^{(2)}$$

The first operator only acts on the first particle, the second only acts on the second particle. Let's look at the z component of this operator

$$\begin{aligned} S_z \chi_1 \chi_2 &= (S_z^{(1)} + S_z^{(2)}) \chi_1 \chi_2 \\ &= \hbar(m_1 + m_2) \chi_1 \chi_2 \end{aligned}$$

So the states have the quantum numbers

$$\begin{array}{ll} \uparrow\uparrow & m = 1 \\ \downarrow\uparrow & m = 0 \\ \uparrow\downarrow & m = 0 \\ \downarrow\downarrow & m = -1 \end{array}$$

That doesn't look nice. We are used to seeing only one state with $m = 0$. To understand what is going on, let's apply the lowering operator to the highest state.

$$S_- \uparrow\uparrow \;=\; \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

This motivates us to construct the following triplet combination of states

$$\begin{array}{rcl} |1,1\rangle & = \uparrow\uparrow & s = 1, m = 1 \\ |1,0\rangle & = \frac{1}{\sqrt{2}}(\downarrow\uparrow + \uparrow\downarrow) & s = 1, m = 0 \\ |1,-1\rangle & = \downarrow\downarrow & s = 1, m = -1 \end{array}$$

There is one state left:

$$|0,0\rangle \;=\; \frac{1}{\sqrt{2}}(\downarrow\uparrow - \uparrow\downarrow) \qquad s = 0, m = 0$$

which we call the singlet state. Operating either the raising or lowering operator on this state will give zero.

I labled this state $s = 0$, now, let's prove this. The total spin is given by the operator

$$S^2 = (S^{(1)} + S^{(2)}) \cdot (S^{(1)} + S^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)} \cdot S^{(2)}$$

Note that we are mixing vectors with scalars. Let's look at the last part first

$$\begin{aligned} S^{(1)} S^{(2)} |\uparrow\downarrow\rangle &= S_x^{(1)} S_x^{(2)} |\uparrow\downarrow\rangle + S_y^{(1)} S_z^{(2)} |\uparrow\downarrow\rangle + S_y^{(1)} S_z^{(2)} |\uparrow\downarrow\rangle \\ &= \left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) |\downarrow\uparrow\rangle + \left(-i\frac{\hbar}{2}\right)\left(i\frac{\hbar}{2}\right) |\downarrow\uparrow\rangle + \left(\frac{\hbar}{2}\right)\left(-\frac{\hbar}{2}\right) |\uparrow\downarrow\rangle \\ &= \frac{\hbar^2}{4}(2\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \end{aligned}$$

In the same way, we get

$$S^{(1)}S^{(2)}\left|\downarrow\uparrow\right\rangle = \frac{\hbar^2}{4}(2\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)$$

Then, we have

$$\begin{aligned} S^{(1)}S^{(2)}\left|1,0\right\rangle &= \frac{\hbar^2}{4}\left|1,0\right\rangle \\ S^{(1)}S^{(2)}\left|0,0\right\rangle &= -3\frac{\hbar^2}{4}\left|0,0\right\rangle \end{aligned}$$

And therefore

$$\begin{aligned} S^2\left|1,0\right\rangle &= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right) = 2\hbar\left|1,0\right\rangle \\ S^2\left|0,0\right\rangle &= \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - 2\frac{3\hbar^2}{4}\right) = 2\hbar\left|0,0\right\rangle \end{aligned}$$

We have just shown that the singlet and triplet states are indeed eigenstates of S^2. We have not explicitely shown that for the states $\left|1,1\right\rangle$ and $\left|1,-1\right\rangle$ but it works in the same way.

In essence we decomposed the direct product of two irreducible representations of SU(2) into a direct sum of two irreducible representations that we already know of (spin-0 and spin 1).

In more general terms, if we have two particles with spins s_1 and s_2, then we can get the follwowing spins

$$s = (s_1+s_2), (s_1+s_2-1), \ldots, |s_1-s_2|$$

The combined stated $\left|s.m\right\rangle$ is a linear combination of all composite states

$$\left|s,m\right\rangle = \sum_{m_1+m_2=m} C^{s_1,s_2,s}_{m_1,m_2,m}\left|s_1,m_1\right\rangle\left|s_2,m_2\right\rangle$$

These coefficients are called Clebsch-Gordan coefficients. They tell us, for examplem that

$$\left|3,0\right\rangle = \frac{1}{\sqrt{5}}\left|2,1\right\rangle\left|1,-1\right\rangle + \sqrt{\frac{3}{5}}\left|2,0\right\rangle\left|1,0\right\rangle + \frac{1}{\sqrt{5}}\left|2,-1\right\rangle\left|1,1\right\rangle$$

We can also use the coefficients to go the other way round

$$\left|s_1,m_1\right\rangle\left|s_2,m_2\right\rangle = \sum_s C^{s_1,s_2,s}_{m_1,m_2,m}\left|s,m\right\rangle$$

For example

$$\left|\frac{3}{2},\frac{1}{2}\right\rangle\left|1,0\right\rangle = \sqrt{\frac{3}{5}}\left|\frac{5}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{1}{15}}\left|\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle$$

Lecture 10 Ends