证明：设$\vec{u}(t)=\left(u_1(t),u_2(t),\cdots,u_n(t)\right)$，$\vec{v}(t)=\left(v_1(t),v_2(t),\cdots,v_n(t)\right)$，则

(1) 设常向量$\vec{C}=\left(c_1,c_2,\cdots,c_n\right)$，则$\dfrac{d}{dt}\vec{C}=\left(c_1{}',c_2{}',\cdots,c_n{}'\right)=\left(0,0,\cdots,0\right)=\vec{0}.$

(2) 设c是任一常数，则$c\vec{u}(t)=\left(cu_1(t),cu_2(t),\cdots,cu_n(t)\right)$，

$$\frac{d}{dt}\left[c\vec{u}(t)\right]=\left(cu_1{}'(t),cu_2{}'(t),\cdots,cu_n{}'(t)\right)=c\left(u_1{}'(t),u_2{}'(t),\cdots,u_n{}'(t)\right)=c\left[\frac{d}{dt}\vec{u}(t)\right].$$

(3) 因为$\vec{u}(t)\pm\vec{v}(t)=\left(u_1(t)\pm v_1(t),u_2(t)\pm v_2(t),\cdots,u_n(t)\pm v_n(t)\right)$，所以

$$\begin{aligned}\frac{d}{dt}\left[\vec{u}(t)\pm\vec{v}(t)\right]&=\frac{d}{dt}\left(u_1(t)\pm v_1(t),u_2(t)\pm v_2(t),\cdots,u_n(t)\pm v_n(t)\right)\\&=\left(\left(u_1(t)\pm v_1(t)\right)',\left(u_2(t)\pm v_2(t)\right)',\cdots,\left(u_n(t)\pm v_n(t)\right)'\right)\\&=\left(u_1{}'(t)\pm v_1{}'(t),u_2{}'(t)\pm v_2{}'(t),\cdots,u_n{}'(t)\pm v_n{}'(t)\right)\\&=\left(u_1{}'(t),u_2{}'(t),\cdots,u_n{}'(t)\right)\pm\left(v_1{}'(t),v_2{}'(t),\cdots,v_n{}'(t)\right)\\&=\frac{d}{dt}\vec{u}(t)\pm\frac{d}{dt}\vec{v}(t).\end{aligned}$$

(4) 设$\varphi(t)$是可导的数量值函数，则$\varphi(t)\vec{u}(t)=\left(\varphi(t)u_1(t),\varphi(t)u_2(t),\cdots,\varphi(t)u_n(t)\right)$，

$$\begin{aligned}&\frac{d}{dt}\left[\varphi(t)\vec{u}(t)\right]\\&=\left(\left(\varphi(t)u_1(t)\right)',\left(\varphi(t)u_2(t)\right)',\cdots,\left(\varphi(t)u_n(t)\right)'\right)\\&=\left(\varphi(t)'u_1(t)+\varphi(t)u_1{}'(t),\varphi(t)'u_2(t)+\varphi(t)u_2{}'(t),\cdots,\varphi(t)'u_n(t)+\varphi(t)u_n{}'(t)\right)\\&=\left(\varphi(t)'u_1(t),\varphi(t)'u_2(t),\cdots,\varphi(t)'u_n(t)\right)+\left(\varphi(t)u_1{}'(t),\varphi(t)u_2{}'(t),\cdots,\varphi(t)u_n{}'(t)\right)\\&=\varphi(t)'\left(u_1(t),u_2(t),\cdots,u_n(t)\right)+\varphi(t)\left(u_1{}'(t),u_2{}'(t),\cdots,u_n{}'(t)\right)\\&=\frac{d}{dt}\left[\varphi(t)\right]\vec{u}(t)+\varphi(t)\frac{d}{dt}\left[\vec{u}(t)\right].\end{aligned}$$

(5) 因为$\vec{u}(t)\cdot\vec{v}(t)=\left(u_1(t)v_1(t),u_2(t)v_2(t),\cdots,u_n(t)v_n(t)\right)$，所以

$$\begin{aligned}
&\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right]\\
&=\left(\left(u_1(t)v_1(t)\right)',\left(u_2(t)v_2(t)\right)',\cdots,\left(u_n(t)v_n(t)\right)'\right)\\
&=\left(u_1{}'(t)v_1(t)+u_1(t)v_1{}'(t),u_2{}'(t)v_2(t)+u_2(t)v_2{}'(t),\cdots,u_n{}'(t)v_n(t)+u_n(t)v_n{}'(t)\right)\\
&=\left(u_1{}'(t)v_1(t),u_2{}'(t)v_2(t),\cdots,u_n{}'(t)v_n(t)\right)+\left(u_1(t)v_1{}'(t),u_2(t)v_2{}'(t),\cdots,u_n(t)v_n{}'(t)\right)\\
&=\frac{d}{dt}\left[\vec{u}(t)\right]\cdot\vec{v}(t)+\vec{u}(t)\cdot\frac{d}{dt}\left[\vec{v}(t)\right].
\end{aligned}$$

(6) 因为只有三维向量才能定义向量积，且

$$\vec{u}(t)\times\vec{v}(t)=\left(u_2(t)v_3(t)-u_3(t)v_2(t),u_3(t)v_1(t)-u_1(t)v_3(t),u_1(t)v_2(t)-u_2(t)v_1(t)\right),$$

所以

$$\begin{aligned}
&\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right]\\
&=\left(\left(u_2(t)v_3(t)-u_3(t)v_2(t)\right)',\left(u_3(t)v_1(t)-u_1(t)v_3(t)\right)',\left(u_1(t)v_2(t)-u_2(t)v_1(t)\right)'\right)\\
&=\begin{pmatrix}
u_2{}'(t)v_3(t)+u_2(t)v_3{}'(t)-u_3{}'(t)v_2(t)-u_3(t)v_2{}'(t),\\
u_3{}'(t)v_1(t)+u_3(t)v_1{}'(t)-u_1{}'(t)v_3(t)-u_1(t)v_3{}'(t),\\
u_1{}'(t)v_2(t)+u_1(t)v_2{}'(t)-u_2{}'(t)v_1(t)-u_2(t)v_1{}'(t)
\end{pmatrix}\\
&=\left(u_2{}'(t)v_3(t)-u_3{}'(t)v_2(t),u_3{}'(t)v_1(t)-u_1{}'(t)v_3(t),u_1{}'(t)v_2(t)-u_2{}'(t)v_1(t)\right)\\
&+\left(u_2(t)v_3{}'(t)-u_3(t)v_2{}'(t),u_3(t)v_1{}'(t)-u_1(t)v_3{}'(t),u_1(t)v_2{}'(t)-u_2(t)v_1{}'(t)\right)\\
&=\left(u_1{}'(t),u_2{}'(t),u_3{}'(t)\right)\times\left(v_1(t),v_2(t),v_3(t)\right)+\left(u_1(t),u_2(t),u_3(t)\right)\times\left(v_1{}'(t),v_2{}'(t),v_3{}'(t)\right)\\
&=\frac{d}{dt}\left[\vec{u}(t)\right]\times\vec{v}(t)+\vec{u}(t)\times\frac{d}{dt}\left[\vec{v}(t)\right].
\end{aligned}$$

(7) 因为$\vec{u}\left[\varphi(t)\right]=\left(u_1\left(\varphi(t)\right),u_2\left(\varphi(t)\right),\cdots,u_n\left(\varphi(t)\right)\right)$，所以

$$\begin{aligned}
\frac{d}{dt}\vec{u}\left[\varphi(t)\right]&=\frac{d}{dt}\left(u_1\left(\varphi(t)\right),u_2\left(\varphi(t)\right),\cdots,u_n\left(\varphi(t)\right)\right)\\
&=\left(u_1{}'\left(\varphi(t)\right)\varphi'(t),u_2{}'\left(\varphi(t)\right)\varphi'(t),\cdots,u_n{}'\left(\varphi(t)\right)\varphi'(t)\right)\\
&=\varphi'(t)\left(u_1{}'\left(\varphi(t)\right),u_2{}'\left(\varphi(t)\right),\cdots,u_n{}'\left(\varphi(t)\right)\right)\\
&=\varphi'(t)\left[\frac{d}{dw}\vec{u}(w)\right]_{w=\varphi(t)}.
\end{aligned}$$