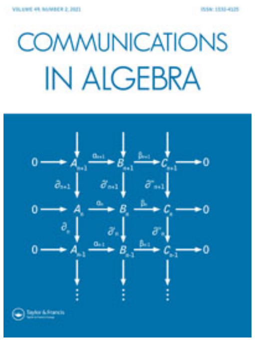

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Support τ-tilting modules over one-point extensions

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ABSTRACT

Let B be the one-point extension algebra of A by an A-module X. We proved that every support τ-tilting A-module can be extended to be a support τ-tilting B-module by two different ways. As a consequence, it is shown that there is an inequality

$$|s\tau\text{-tilt}\, B| \geqslant 2|s\tau\text{-tilt}\, A|.$$

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1. Introduction

Tilting modules are very important in the representation theory of finite dimensional algebras. Mutation is an effective way to construct a new tilting module from a given one. Unfortunately, mutation of tilting modules may not be realized.

In 2014, Adachi et al. [1] introduced the concept of support τ-tilting modules as a generalization of tilting modules, and they showed that mutation of support τ-tilting modules is always possible. The authors also proved that support τ-tilting modules are in bijection with some important classes in representation theory (such as, functorially finite torsion classes, 2-term silting complexes, and cluster-tilting objects in the cluster category).

A new (support τ)-tilting module can be constructed by algebra extensions. In [3], Assem et al. studied how to extend and restrict tilting modules for one-point extension algebras by a projective module. In [8], Suarez generalized this result for the context of support support τ-tilting modules. More precisely, let $B = A[P]$ be the one-point extension of an algebra A by a projective A-module P and e the identity of A. If M is a support τ-tilting A-module, then $\mathrm{Hom}_B(eB, M) \oplus S_a$ is a support τ-tilting B-module, where S_a is the simple module corresponding to the new point a (see [8, Theorem A]). An example shown that $\mathrm{Hom}_B(eB, M) \oplus S_a$ may not be a support τ-tilting B-module if P is not projective (see [8, Example 4.7]).

Bricks and semibricks are considered in [5, 6]. An A-module M is called *brick* if $\mathrm{Hom}_A(M, M)$ is a k division. A *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Let $\mathcal{S}$ be a sembrick and $T(\mathcal{S})$ the smallest torsion class containing $\mathcal{S}$. In [2], the author called a semibrick $\mathcal{S}$ is *left finite* if $T(\mathcal{S})$ is functorially finite and he also proved that there exists a

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bijection $\Phi : \mathrm{s}\tau\text{-tilt}\, A \mapsto \mathrm{f}_L - \mathrm{sbrick}\, A$ between the set of support τ-tilting A-modules and the set of left finite semibricks of A.

In this paper, we construct semibricks over the one-point extension B of an algebra A by an A-module X (may not be projective) and use the bijection to get support τ-tilting B-modules.

Proposition 1.1. (see Proposition 3.2) *Let B be the one-point extension algebra of A by an A-module X and $\mathcal{S}$ be a semibrick in* $\mathrm{mod}\, A$. *Then both $\mathcal{S}$ and $\mathcal{S} \cup S_a$ are semibricks in* $\mathrm{mod}\, B$, *where S_a stands for the simple module corresponding to the extension point a.*

Moreover, it is shown that $\mathcal{S}$ is left finite implies $\mathcal{S} \cup S_a$ is also. We say an A-module M is a support τ-tilting module with respect to the semibrick $\mathcal{S}$ if $\Phi(M) = \mathcal{S}$. As an application, we can construct support τ-tilting modules over one-point extensions from support τ-tilting A-modules.

Proposition 1.2. (see Proposition 3.7) *Let B be the one-point extension algebra of A by an A-module X and $M \in \mathrm{mod}\, A$ be a support τ-tilting module with respect to the semibrick $\mathcal{S}$. Then both $P(T(\mathcal{S}))$ and $P(T(\mathcal{S} \cup S_a))$ are support τ-tilting B-modules.*

As a consequence, we have the following inequality

Corollary 1.3. $|\mathrm{s}\tau\text{-tilt}\, B| \geqslant 2|\mathrm{s}\tau\text{-tilt}\, A|$.

Moreover, we have the following.

Theorem 1.4. (see Theorem 3.9) *Let B be the one-point extension algebra of A by an A-module X and M be a support τ-tilting module in* $\mathrm{mod}\, A$.

(1) *M is a support τ-tilting B-module.*
(2) *Assume that $M \in \mathrm{mod}\, A$ is a support τ-tilting module with respect to the semibrick $\mathcal{S}$, then $P(T(\mathcal{S} \cup S_a))$ has M as direct summand.*
(3) *If $X \in \mathrm{Fac}M$, then $P_a \oplus M$ is a support τ-tilting B-module.*
(4) *If $\mathrm{Hom}_A(X, \mathrm{Fac}M) = 0$, then $S_a \oplus M$ is a support τ-tilting B-module.*

Throughout this paper, all algebras will be basic connected finite dimensional k-algebras over an algebraically closed field k and all modules are basic. Let A be an algebra. The category of finitely generated left A-modules will be denote by $\mathrm{mod}\, A$ and the Auslander-Reiten translation of A will be denote by τ. For $M \in \mathrm{mod}\, A$, we denote by $\mathrm{ind}(M)$ the set of isoclasses of indecomposable direct summands of M, and by $\mathrm{Fac}M$ the full subcategory of $\mathrm{mod}\, A$ consisting of modules isomorphic to factor modules of finite direct sums of copies of M. For a finite set J, $|J|$ stands for the cardinality of J. In particular, we write $|M| = |\mathrm{ind}(M)|$. $\mathbb{N}$ will be the set of all natural numbers.

2. Preliminaries

Let A be an algebra. In this section, we recall some definitions about support τ-tilting modules and semibircks over $\mathrm{mod}\, A$.

Definition 2.1. ([1, Definition 0.1]) Let $M \in \mathrm{mod}\, A$.

(1) M is called τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
(2) M is called τ-tilting if it is τ-rigid and $|M| = |A|$.
(3) M is called support τ-tilting if it is a τ-tilting $A/\langle e\rangle$-module where e is an idempotent of A.

We will denote by $\tau\text{-tilt}\, A$ (respectively, $\mathrm{s}\tau\text{-tilt}\, A$) the set of isomorphism classes of τ-tilting A-modules (respectively, support τ-tilting A-modules).

Definition 2.2. ([1, Definition 0.3]) Let (M, P) be a pair in $\mathrm{mod}\, A$ with P projective.

(1) The pair (M, P) is called a *τ-rigid pair* if M is τ-rigid and $\mathrm{Hom}_A(P, M) = 0$.
(2) The pair (M, P) is called a *support τ-tilting pair* if it is τ-rigid and $|M| + |P| = |A|$.

Note that (M, P) is a support τ-tilting pair if and only if M is a τ-tilting $A/\langle e\rangle$-module, where $eA \cong P$ [1, Proposition 2.3]. Hence, M is a τ-tilting A-module if and only if $(M, 0)$ is a support τ-tilting pair.

The following result is very useful.

Lemma 2.3. ([4, Proposition 5.8]) *For $M \in \mathrm{mod}\,A$, M is τ-rigid if and only if* $\mathrm{Ext}^1_A(M, \mathrm{Fac}M) = 0$.

Definition 2.4. ([2, Definition 2.1]) Let $\mathcal{S} \subseteq \mathrm{mod}\,A$. $\mathcal{S}$ is called a *semibrick* if

$$\mathrm{Hom}_A(S_i, S_j) = \begin{cases} k\text{-}division\ algebra & if\quad i = j \\ 0 & otherwise \end{cases}$$

for any $S_i, S_j \in \mathcal{S}$.

By Schur's Lemma, a set of isoclasses of some simple modules is a semibrick.

Let $\mathcal{Y}$ be a full subcategory of $\mathrm{mod}\,A$ and $M \in \mathrm{mod}\,A$. A homomorphism $f_M : M \to Y_M$ is called left $\mathcal{Y}$-approximation of M with $Y_M \in \mathcal{Y}$ if any morphism $f : M \to Y$ with $Y \in \mathcal{Y}$ factors through f_M. We say that $\mathcal{Y}$ is *covariantly finite* if for any $M \in \mathrm{mod}\,A$, there exists a left $\mathcal{Y}$-approximation of M. Dually, we can define the concepts of right $\mathcal{Y}$-approximation of M and *contravariantly finite* subcategories. $\mathcal{Y}$ is called *functorially finite* if it is both covariantly finite and contravariantly finite.

A *torsion class* of $\mathrm{mod}\,A$ is a full subcategory of $\mathrm{mod}\,A$ closed under images, direct sums, and extensions. Recall that a semibrick $\mathcal{S}$ of $\mathrm{mod}\,A$ is *left finite* [2] if $T(\mathcal{S})$ is functorially finite, where $T(\mathcal{S})$ is the smallest torsion class containing $\mathcal{S}$. The set of all left finite semibricks of $\mathrm{mod}\,A$ will be denoted by f_L-$\mathrm{sbrick}\,A$.

The following result states the relationship between $\mathrm{s}\tau$-tilt A and f_L-$\mathrm{sbrick}\,A$.

Theorem 2.5. [2, Theorem 1.3(2)] *there exists a bijection*

$$\Phi : \mathrm{s}\tau\text{-tilt } A \mapsto \mathrm{f}_L - \mathrm{sbrick}\,A$$

given by $M \mapsto \mathrm{ind}(M/\mathrm{rad}_\Gamma M)$ *where* $\Gamma = \mathrm{End}_A(M)$.

Recall that $M \in \mathrm{mod}\,A$ is called *sincere* if every simple A-module appears as a composition factor in M. A τ-tilting A-module is exactly a sincere support τ-tilting. We say a semibrick $\mathcal{S}$ of $\mathrm{mod}\,A$ is *sincere* if $T(\mathcal{S})$ is sincere. Let sf_L-$\mathrm{sbrick}\,A$ stand for all sincere left finite semibricks of $\mathrm{mod}\,A$. We have the following result due to Asai in [2].

Corollary 2.6. *There exists a bijection* $\Phi : \tau\text{-tilt } A \mapsto \mathrm{sf}_L - \mathrm{sbrick}\,A$.

3. Main results

Let $X \in \mathrm{mod}\,A$. The *one-point extension* of A by X is defined as the following matrix algebra

$$B = \begin{pmatrix} A & X \\ 0 & k \end{pmatrix}$$

with the ordinary matrix addition and the multiplication induced by the module structure of X. We write $B := A[X]$ with a the extension point. All B-modules can be viewed as $\begin{pmatrix} M \\ k^n \end{pmatrix}_f$ where $M \in \mathrm{mod}\,A, n \in \mathbb{N}$ and $f \in \mathrm{Hom}_A(X \otimes_k k^n, M)$(see, [7, XV.1]). In particular, $S_a = \begin{pmatrix} 0 \\ k \end{pmatrix}_0$ and $P_a = \begin{pmatrix} X \\ k \end{pmatrix}_1$. Moreover, the morphisms from $\begin{pmatrix} M \\ k^n \end{pmatrix}_f$ to $\begin{pmatrix} M' \\ k^{n'} \end{pmatrix}_{f'}$ are pairs of $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ such that the following diagram

$$\begin{tikzcd} X \otimes_k k^n \arrow[r, "f"] \arrow[d, "X\otimes\beta"'] & M \arrow[d, "\alpha"] \\ X \otimes_k k^{n'} \arrow[r, "f'"] & M' \end{tikzcd}$$

commutes, where $\alpha \in \mathrm{Hom}_\Lambda(M, M')$ and $\beta \in \mathrm{Hom}_\Gamma\left(k^n, k^{n'}\right)$. A sequence

$$0 \longrightarrow \begin{pmatrix} M_1 \\ k^{n_1} \end{pmatrix}_{f_1} \xrightarrow{\binom{\alpha_1}{\beta_1}} \begin{pmatrix} M_2 \\ k^{n_2} \end{pmatrix}_{f_2} \xrightarrow{\binom{\alpha_2}{\beta_2}} \begin{pmatrix} M_3 \\ k^{n_3} \end{pmatrix}_{f_3} \longrightarrow 0$$

in $\bmod B$ is exact if and only if

$$0 \longrightarrow M_1 \xrightarrow{\alpha_1} M_2 \xrightarrow{\alpha_2} M_3 \longrightarrow 0$$

is exact in $\bmod A$ and

$$0 \longrightarrow k^{n_1} \xrightarrow{\beta_1} k^{n_2} \xrightarrow{\beta_2} k^{n_3} \longrightarrow 0$$

is exact in $\bmod k$.

Lemma 3.1. *For any $M \in \bmod A$, we have*

(1) $\mathrm{Hom}_B(S_a, M) = 0$;
(2) $\mathrm{Hom}_B(M, S_a) = 0$.

Proof. It is clear since ${}_BM \cong \begin{pmatrix} M \\ 0 \end{pmatrix}_0$. Hence, $\mathrm{Hom}_B(S_a, M) \cong \mathrm{Hom}_B\left(\begin{pmatrix} 0 \\ k \end{pmatrix}_0, \begin{pmatrix} M \\ 0 \end{pmatrix}_0\right) = 0$. Similarly, we can get $\mathrm{Hom}_B(M, S_a) = 0$. □

Proposition 3.2. *Let $\mathcal{S}$ be a semibrick in $\bmod A$. Then both $\mathcal{S}$ and $\mathcal{S} \cup S_a$ are semibricks in $\bmod B$.*

Proof. It follows from Lemma 3.1. □

Lemma 3.3. *Let $\mathcal{S}$ be a semibrick in $\bmod A$. Then*

$$T(\mathcal{S} \cup S_a) = \left\{ \begin{pmatrix} M \\ k^n \end{pmatrix}_f \mid \forall n \in \mathbb{N},\ M \in T(\mathcal{S}) \quad and \quad f \in \mathrm{Hom}_A(X \otimes_k k^n, M) \right\}.$$

Proof. Since $\mathcal{S}$ and S_a belong to $T(\mathcal{S} \cup S_a)$, we have $\left\{ \begin{pmatrix} M \\ 0 \end{pmatrix}_0 \mid M \in T(\mathcal{S}) \right\} \subset T(\mathcal{S} \cup S_a)$ and $\begin{pmatrix} 0 \\ k^n \end{pmatrix} \in T(\mathcal{S} \cup S_a)$ for all $n \in \mathbb{N}$. Note that $\forall n \in \mathbb{N}$, $M \in T(\mathcal{S})$ and $f \in \mathrm{Hom}_A(X \otimes_k k^n, M)$, there exists the following exact sequence in $\bmod B$

$$0 \longrightarrow \begin{pmatrix} M \\ 0 \end{pmatrix}_0 \longrightarrow \begin{pmatrix} M \\ k^n \end{pmatrix}_f \longrightarrow \begin{pmatrix} 0 \\ k^n \end{pmatrix}_0 \longrightarrow 0$$

this implies $\begin{pmatrix} M \\ k^n \end{pmatrix}_f \in T(\mathcal{S} \cup S_a)$. It is clear that $\left\{ \begin{pmatrix} M \\ k^n \end{pmatrix}_f \mid \forall n \in \mathbb{N},\ M \in T(\mathcal{S}) \text{ and } f \in \mathrm{Hom}_A(X \otimes_k k^n, M) \right\}$ is closed under image, direct sum and extension. Thus it is a torsion class. Hence $T(\mathcal{S} \cup S_a) = \left\{ \begin{pmatrix} M \\ k^n \end{pmatrix}_f \mid \forall n \in \mathbb{N},\ M \in T(\mathcal{S}) \text{ and } f \in \mathrm{Hom}_A(X \otimes_k k^n, M) \right\}$. □

Proposition 3.4. *Let $\mathcal{S}$ be a semibrick in* $\operatorname{mod} A$. *If $\mathcal{S}$ is left finite, then $\mathcal{S} \cup S_a$ is also.*

Proof. We only show that $T(\mathcal{S} \cup S_a)$ is covariantly finite. It is dually to prove $T(\mathcal{S} \cup S_a)$ is contravariantly finite.

Let $\binom{M}{k^n}_f \in \operatorname{mod} B$. Then M has a left $T(\mathcal{S})$-approximation $h_M : M \to Z_M$ in $\operatorname{mod} A$ since $T(\mathcal{S})$ is covariantly finite. Take $g = \ h_M \circ f$. The following commutative diagram

$$\begin{tikzcd} X \otimes_k k^n \arrow[r, "f"] \arrow[d, equal] & M \arrow[d, "h_M"] \\ X \otimes_k k^n \arrow[r, "g"] & Z_M \end{tikzcd}$$

implies that $\binom{f_M}{1}$ is a morphism from $\binom{M}{k^n}_f$ to $\binom{Z_M}{k^n}_g$. Next, we will show that $\binom{f_M}{1}$ is left $T(\mathcal{S} \cup S_a)$-approximation of $\binom{M}{k^n}_f$. For any $\binom{M_1}{k^{n_1}}_{f_1} \in T(\mathcal{S} \cup S_a)$ and morphism $\binom{a}{b} : \binom{M}{k^n}_f \to \binom{M_1}{k^{n_1}}_{f_1}$, there is a morphism $h' : Z_M \to M_1$ such that $a = h' \circ h_M$ since h_M is a left approximation. Note that there exists a commutative diagram

$$\begin{tikzcd} X \otimes_k k^n \arrow[r, "f"] \arrow[d, "X\otimes b"'] & M \arrow[d, "a"] \\ X \otimes_k k^{n_1} \arrow[r, "f_1"] & M_1 \end{tikzcd}$$

that is $a \circ f = f_1 \circ (X \otimes b)$. Therefore,

$$f_1 \circ (X \otimes b) = a \circ f = h' \circ h_M \circ f = h' \circ g,$$

that is, the following diagram

$$\begin{tikzcd} X \otimes_k k^n \arrow[r, "g"] \arrow[d, "X\otimes b"'] & Z_M \arrow[d, "h'"] \\ X \otimes_k k^{n_1} \arrow[r, "f_1"] & M_1 \end{tikzcd}$$

commutates. Hence, $\binom{h'}{b}$ is a morphism from $\binom{Z_M}{k^n}_g$ to $\binom{M_1}{k^{n_1}}_{f_1}$, and the following equation holds

$$\begin{pmatrix} h' \\ b \end{pmatrix} \circ \begin{pmatrix} h_M \\ 1 \end{pmatrix} = \begin{pmatrix} h' \circ h_M \\ b \end{pmatrix} = \begin{pmatrix} g \\ b \end{pmatrix}.$$

By Lemma 3.3, $\binom{Z_M}{k^n}_g \in T(\mathcal{S} \cup S_a)$ since $Z_M \in T(\mathcal{S})$. Thus, we were done. □

The following result can be found immediately.

Corollary 3.5. *Let $\mathcal{S}$ be a semibrick in* $\operatorname{mod} A$. *If $\mathcal{S}$ is sincere left finite, then $\mathcal{S} \cup S_a$ is also.*

Let $\mathcal{F}$ be a full subcategory of $\operatorname{mod} A$. An A-module M is called **Ext-projective** in $\mathcal{F}$ if $\operatorname{Ext}^1_A(M, F) = 0$ for all $F \in \mathcal{F}$. If $\mathcal{F}$ is functorially finite in $\operatorname{mod} A$, then there are only finitely many indecomposable Ext-projective modules in $\mathcal{F}$ up to isomorphism. In this case, we will denote by $P(\mathcal{F})$ the direct sum of all Ext-projective modules in $\mathcal{F}$ up to isomorphism.

Definition 3.6. We say that an A-module M is a support τ-tilting module with respect to the semibrick $\mathcal{S}$ if $\Phi(M)=\mathcal{S}$.

Now, we can construct support τ-tilting B-modules from support τ-tilting A-modules.

Proposition 3.7. *Let $M \in \operatorname{mod} A$ be a support τ-tilting module with respect to the semibrick $\mathcal{S}$. Then both $P(T(\mathcal{S}))$ and $P(T(\mathcal{S} \cup S_a))$ are support τ-tilting B-modules. Moreover, if M is τ-tilting, then $P(T(\mathcal{S} \cup S_a))$ is also.*

Proof. Since M is a support τ-tilting module, we have $\mathcal{S}$ is a left finite semibrick of $\operatorname{mod} A$ by Theorem 2.5. Hence, $\mathcal{S}$ is also a left finite semibrick of $\operatorname{mod} B$. Moreover, we have $\mathcal{S} \cup S_a$ is a left finite semibrick of $\operatorname{mod} B$ by Proposition 3.4. Therefore, $T(\mathcal{S})$ and $T(\mathcal{S} \cup S_a)$ are functorially finite torsion classes. Hence, $P(T(\mathcal{S}))$ and $P(T(\mathcal{S} \cup S_a))$ are support τ-tilting B-module by [1, Theorem 2.7]). □

As a consequence, we have the following inequality.

Corollary 3.8. $|s\tau\text{-tilt}\, B| \geqslant 2|s\tau\text{-tilt}\, A|$.

Applying Proposition 3.7, we can give those forms of support τ-tilting B-module under certain conditions.

Theorem 3.9. *Let B be the one-point extension of A by X and M be a support τ-tilting module in* $\operatorname{mod} A$.

(1) *M is a support τ-tilting B-module.*
(2) *Assume that $M \in \operatorname{mod} A$ is a support τ-tilting module with respect to the semibrick $\mathcal{S}$, then $P(T(\mathcal{S} \cup S_a))$ has M as direct summand.*
(3) *If $X \in \operatorname{Fac} M$, then $P_a \oplus M$ is a support τ-tilting B-module.*
(4) *If $\operatorname{Hom}_A(X, \operatorname{Fac} M) = 0$, then $S_a \oplus M$ is a support τ-tilting B-module.*

Proof. Assume that $M \in \operatorname{mod} A$ is a support τ-tilting module with respect to the semibrick $\mathcal{S}$, then $T(\mathcal{S}) = \operatorname{Fac} M$ [2, Lemma 2.5(5)].

Note that $\forall n \in \mathbb{N}$, $M' \in T(\mathcal{S})$ and $f \in \operatorname{Hom}_A(X \otimes_k k^n, M')$, there exists the following exact sequence in $\operatorname{mod} B$

$$0 \to \begin{pmatrix} M' \\ 0 \end{pmatrix}_0 \to \begin{pmatrix} M' \\ k^n \end{pmatrix}_f \to \begin{pmatrix} 0 \\ k^n \end{pmatrix}_0 \to 0. \tag{1.1}$$

For any $Y \in \operatorname{mod} B$, applying the functor $\operatorname{Hom}_B(Y, -)$ to (1.1), we have the following exact sequence

$$\operatorname{Ext}^1_B\left(Y, \begin{pmatrix} M' \\ 0 \end{pmatrix}_0\right) \to \operatorname{Ext}^1_B\left(Y, \begin{pmatrix} M' \\ k^n \end{pmatrix}_f\right) \to \operatorname{Ext}^1_B\left(Y, \begin{pmatrix} 0 \\ k^n \end{pmatrix}_0\right) = 0. \tag{1.2}$$

(1) By Proposition 3.7, $P(T(\mathcal{S})) = P(\operatorname{Fac} M) = M$ is a support τ-tilting B-module.
(2) Putting $Y = {}_B M \cong \begin{pmatrix} M \\ 0 \end{pmatrix}_0$ in (1.2), we have $\operatorname{Ext}^1_B\left(M, \begin{pmatrix} M' \\ 0 \end{pmatrix}_0\right) \cong \operatorname{Ext}^1_A(M, M') = 0$ by Lemma 2.3, and hence $\operatorname{Ext}^1_B\left(M, \begin{pmatrix} M' \\ k^n \end{pmatrix}_f\right) = 0$. By Lemma 3.3, M is a Ext-projective object in $T(\mathcal{S} \cup S_a)$. Therefore, $P(T(\mathcal{S} \cup S_a))$ has M as direct summand.
(3) If $X \in \operatorname{Fac} M$, then $P_a \in T(\mathcal{S} \cup S_a)$ by Lemma 3.3. Hence, $P_a \oplus M$ is a direct summand of $P(T(\mathcal{S} \cup S_a))$ by (2). In particular, $P_a \oplus M$ is a τ-rigid B-module. Suppose that (M, P) is a support τ-tilting pair in $\operatorname{mod} A$. Hence, $\operatorname{Hom}_A(P, \operatorname{Fac} M) = 0$ because $\operatorname{Hom}_A(P, M) = 0$.

This implies $\mathrm{Hom}_B(P, P_a) \cong \mathrm{Hom}_A(P, X) = 0$. Therefore, $(P_a \oplus M, P)$ is a support τ-tilting pair in $\mathrm{mod}\, B$ since $|P_a \oplus M| + |P| = 1 + |A| = |B|$.

(4) Note that there is an exact sequence in $\mathrm{mod}\, B$,

$$0 \to \begin{pmatrix} X \\ 0 \end{pmatrix} \cong X \xrightarrow{f} P_a \to S_a \to 0.$$

For any $Y' \in \mathrm{Fac} M$, applying $\mathrm{Hom}_B(-, Y')$ to it, we have the following exact sequence,

$$\mathrm{Hom}_B(X, Y') \to \mathrm{Ext}^1_B(S_a, Y') \to \mathrm{Ext}^1_B(P_a, Y') = 0.$$

Since $\mathrm{Hom}_A(X, \mathrm{Fac} M) = 0$, we have $\mathrm{Hom}_B(X, Y') = 0$. Hence, $\mathrm{Ext}^1_B(S_a, Y') = 0$. Thus, $\mathrm{Ext}^1_B(S_a, \mathrm{Fac}\, M) = 0$. Putting $Y = S_a$ in (1.2), we have $\mathrm{Ext}^1_B\left(S_a, \begin{pmatrix} M' \\ k^n \end{pmatrix}_f\right) = 0$. By Lemma 3.3, S_a is a Ext-projective object in $T(\mathcal{S} \cup S_a)$. Therefore, $P(T(\mathcal{S} \cup S_a))$ has $S_a \oplus M$ as direct summand. This implies $S_a \oplus M$ is a τ-rigid B-module. Suppose that (M, P) is a support τ-tilting pair in $\mathrm{mod}\, A$. It is easy to get $(S_a \oplus M, P)$ is a support τ-tilting pair in $\mathrm{mod}\, B$ since $\mathrm{Hom}_B(P, S_a) = 0$ and $|S_a \oplus M| + |P| = 1 + |A| = |B|$. Hence, $S_a \oplus M$ is a support τ-tilting B-module. □

Corollary 3.10. *Let B be the one-point extension of A by X and $M \in \mathrm{mod}\, A$ be a τ-tilting module.*

(1) *If $X \in \mathrm{Fac} M$, then $P_a \oplus M$ is a τ-tilting B-module.*
(2) *If $\mathrm{Hom}_A(X, \mathrm{Fac} M) = 0$, then $S_a \oplus M$ is a τ-tilting B-module.*

Example 3.11. Let A be a finite dimensional k-algebra given by the quiver

$$2 \to 3.$$

Considering the one-point extension of A by the simple module corresponding to the point 2, the algebra $B = A[2]$ is given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

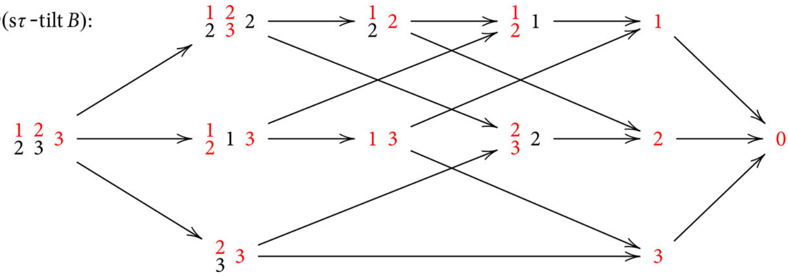
with the relation $\alpha\beta = 0$. The Hasse quiver of A is as follows (semibricks be remarked by red).

$$Q(s\tau\text{-tilt}\, A): \quad \begin{tikzcd} T_1 = \begin{smallmatrix}2\\3\end{smallmatrix}\,3 \arrow[r] \arrow[d] & T_2 = 3 \arrow[r] & T_3 = 0 \\ T_4 = \begin{smallmatrix}2\\3\end{smallmatrix}\,2 \arrow[r] & T_5 = 2. \arrow[ur] & \end{tikzcd}$$

(1) All support τ-tilting A-modules $T_i (i = 1, 2, 3, 4, 5)$ are support τ-tilting B-modules by Theorem 3.9(1).
(2) Since $2 \in \mathrm{Fac} T_i (i = 1, 4, 5)$, we have three support τ-tilting B-modules $P_1 \oplus T_1, P_1 \oplus T_4, P_1 \oplus T_5$ by Theorem 3.9(3). Moreover, $P_1 \oplus T_1, P_1 \oplus T_4$ are τ-tilting B-modules since T_1, T_4 are τ-tilting A-modules by Corollary 3.10.
(3) Since $\mathrm{Hom}_A(2, \mathrm{Fac} T_i) = 0 (i = 2, 3)$, we have two support τ-tilting B-modules $S_1 \oplus T_2, S_1 \oplus T_3$ by Theorem 3.9(4).

In fact, the Hasse quiver $Q(s\tau\text{-tilt}\, B)$ is as follows.

$Q(s\tau\text{-tilt } B)$:



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