COLLOQUIUM MATHEMATICUM

VOL. 160 2020 NO. 2

SUPPORT τ-TILTING MODULES UNDER SPLIT-BY-NILPOTENT EXTENSIONS

BY

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Abstract. Let Γ be a split extension of a finite-dimensional algebra Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$, and let (T,P) be a pair in $\operatorname{mod}\Lambda$ with P projective. We prove that $(T\otimes_\Lambda \Gamma_\Gamma, P\otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in $\operatorname{mod}\Gamma$ if and only if (T,P) is a support τ-tilting pair in $\operatorname{mod}\Lambda$ and $\operatorname{Hom}_\Lambda(T\otimes_\Lambda E, \tau T_\Lambda) = 0 = \operatorname{Hom}_\Lambda(P, T\otimes_\Lambda E)$. As applications, we obtain a necessary and sufficient condition for $(T\otimes_\Lambda \Gamma_\Gamma, P\otimes_\Lambda \Gamma_\Gamma)$ to be a support τ-tilting pair for a cluster-tilted algebra Γ corresponding to a tilted algebra Λ; and we also show that if $T_1, T_2 \in \operatorname{mod}\Lambda$ are such that $T_1\otimes_\Lambda \Gamma$ and $T_2\otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules, then $T_1\otimes_\Lambda \Gamma$ is a left mutation of $T_2\otimes_\Lambda \Gamma$ if and only if T_1 is a left mutation of T_2.

1. Introduction. In this paper, all algebras are finite-dimensional basic algebras over an algebraically closed field k. For an algebra Λ, $\operatorname{mod}\Lambda$ is the category of finitely generated right Λ-modules and τ is the Auslander–Reiten translation. We write $D := \operatorname{Hom}_k(-,k)$.

Mutation is an operation for a certain class of objects in a fixed category to construct a new object from a given one by replacing a summand, which is possible only when the given object has two complements. It is well known that tilting modules are fundamental in tilting theory. Happel and Unger [10] gave some necessary and sufficient conditions under which mutation of tilting modules is possible; however, mutation of tilting modules is not always possible. As a generalization of tilting modules, Adachi, Iyama and Reiten [1] introduced support τ-tilting modules and showed that any almost complete support τ-tilting module has exactly two complements. So, in this case, mutation is always possible. Moreover, for a 2-Calabi–Yau triangulated category $\mathcal{C}$, it was shown in [1] that there is a close relation between cluster-tilting objects in $\mathcal{C}$ and support τ-tilting Λ-modules, where Λ is a 2-Calabi–Yau tilted algebra associated with $\mathcal{C}$. Then Liu and Xie [11] proved that a maximal rigid object T in $\mathcal{C}$ corresponds to a support τ-tilting $\operatorname{End}_{\mathcal{C}}(T)$-module.

2020 *Mathematics Subject Classification*: 16G20, 16E30.
Key words and phrases: support τ-tilting modules, split-by-nilpotent extensions, cluster-tilted algebras, left mutations, Hasse quivers.
Received 27 November 2018; revised 22 February 2019.
Published online 6 February 2020.

DOI: 10.4064/cm7774-3-2019

Given two algebras Λ and Γ, it is interesting to construct a (support τ-)tilting Γ-module from a (support τ-)tilting Λ-module. Assem, Happel and Trepode [3] studied how to extend and restrict tilting modules for one-point extension algebras by a projective module. Suarez [12] generalized this result to the case of support τ-tilting modules. More precisely, let $\Gamma = \Lambda[P]$ be the one-point extension of an algebra Λ by a projective Λ-module P, and e the identity of Λ. If M_Λ is a basic support τ-tilting Λ-module, then $\mathrm{Hom}_\Gamma(\Gamma e, M_\Lambda) \oplus S$ is a basic support τ-tilting Γ-module, where S is the simple module corresponding to the new point; conversely, if T_Γ is a basic support τ-tilting Γ-module, then $\mathrm{Hom}_\Gamma(e\Gamma, T_\Gamma)$ is a basic support τ-tilting Λ-module [12, Theorem A].

Let Γ be a split extension of an algebra Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$, that is, there exists a split surjective algebra morphism $\Gamma \to \Lambda$ whose kernel E is contained in the radical of Γ [4, 7]. In particular, all relation extensions [2, 14] and one-point extensions are split ones. There are two functors $-\otimes_\Lambda \Gamma : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma$ and $-\otimes_\Gamma \Lambda : \mathrm{mod}\,\Gamma \to \mathrm{mod}\,\Lambda$. Assem and Marmaridis [4] investigated the relationship between (partial) tilting Γ-modules and (partial) tilting Λ-modules by using these two functors. Analogously, we will investigate the relationship between support τ-tilting Γ-modules and support τ-tilting Λ-modules. This paper is organized as follows.

In Section 2, we give some terminology and some preliminary results.

In Section 3, we first prove the following

Theorem 1.1 (Theorem 3.1). *Let Γ be a split extension of Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$. If (T, P) is a pair in $\mathrm{mod}\,\Lambda$ with P projective, then the following statements are equivalent:*

(1) *$(T \otimes_\Lambda \Gamma_\Gamma, P \otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in $\mathrm{mod}\,\Gamma$.*
(2) *(T, P) is a support τ-tilting pair in $\mathrm{mod}\,\Lambda$ and*

$$\mathrm{Hom}_\Lambda(T \otimes_\Lambda E, \tau T_\Lambda) = 0 = \mathrm{Hom}_\Lambda(P, T \otimes_\Lambda E).$$

As a consequence, if Γ is a cluster-tilted algebra corresponding to a tilted algebra Λ and (T, P) is a pair in $\mathrm{mod}\,\Lambda$ with P projective, then $(T \otimes_\Lambda \Gamma_\Gamma, P \otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in $\mathrm{mod}\,\Gamma$ if and only if (T, P) is a support τ-tilting pair in $\mathrm{mod}\,\Lambda$ and $\mathrm{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_\Lambda, \tau T_\Lambda) = 0 = \mathrm{Hom}_\Lambda(P, \tau^{-1}\Omega^{-1}T_\Lambda)$ (Proposition 3.4).

Moreover, we have the following

Theorem 1.2 (Theorem 3.10). *Let Γ be a split extension of Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$. Let $T_1, T_2 \in \mathrm{mod}\,\Lambda$ be such that $T_1 \otimes_\Lambda \Gamma$ and $T_2 \otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules. Then the following statements are equivalent:*

(1) *$T_1 \otimes_\Lambda \Gamma$ is a left mutation of $T_2 \otimes_\Lambda \Gamma$.*
(2) *T_1 is a left mutation of T_2.*

The Hasse (exchange) quiver $Q(s\tau\text{-tilt}\,\Lambda)$ of Λ consists of the set of vertices which are support τ-tilting Λ-modules T and has arrows from T to all its left mutations. So Theorem 1.2 shows that if $T_1, T_2 \in \operatorname{mod}\Lambda$ are such that $T_1 \otimes_\Lambda \Gamma$ and $T_2 \otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules, then there exists an arrow from $T_1 \otimes_\Lambda \Gamma$ to $T_2 \otimes_\Lambda \Gamma$ in $Q(s\tau\text{-tilt}\,\Gamma)$ if and only if there exists an arrow from T_1 to T_2 in $Q(s\tau\text{-tilt}\,\Lambda)$.

In Section 4, we give two examples to illustrate our results.

2. Preliminaries. Let Λ be an algebra. For a module $M \in \operatorname{mod}\Lambda$, $|M|$ is the number of pairwise non-isomorphic direct summands of M, $\operatorname{add} M$ is the full subcategory of $\operatorname{mod}\Lambda$ consisting of modules isomorphic to direct summands of finite direct sums of copies of M, and $\operatorname{Fac} M$ is the full subcategory of $\operatorname{mod}\Lambda$ consisting of modules isomorphic to factor modules of finite direct sums of copies of M. The injective dimension and the first cosyzygy of M are denoted by $\operatorname{id}_\Lambda M$ and $\Omega^{-1}M$ respectively.

2.1. τ-tilting theory

DEFINITION 2.1 ([1, Definition 0.1]). A module $M \in \operatorname{mod}\Lambda$ is called

(1) *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$;
(2) *τ-tilting* (respectively, *almost complete τ-tilting*) if it is τ-rigid and if $|M| = |\Lambda|$ (respectively, $|M| = |\Lambda| - 1$);
(3) *support τ-tilting* if it is a τ-tilting $\Lambda/\langle e\rangle$-module for some idempotent e of Λ.

The next result shows a τ-rigid module may be extended to a τ-tilting module.

THEOREM 2.2 ([1, Theorem 2.10]). *Any basic τ-rigid Λ-module is a direct summand of a τ-tilting Λ-module.*

LEMMA 2.3 ([1, Proposition 2.4]). *Let $X \in \operatorname{mod}\Lambda$ and*

$$P_1 \xrightarrow{f_0} P_0 \to X \to 0$$

be a projective presentation of X in $\operatorname{mod}\Lambda$. For any $Y \in \operatorname{mod}\Lambda$, if $\operatorname{Hom}_\Lambda(f_0, Y)$ is epic, then $\operatorname{Hom}_\Lambda(Y, \tau X) = 0$. Moreover, the converse holds if the projective presentation is minimal.

Sometimes, it is convenient to view support τ-tilting modules and τ-rigid modules as certain pairs of modules in $\operatorname{mod}\Lambda$.

DEFINITION 2.4 ([1, Definition 0.3]). Let (M, P) be a pair in $\operatorname{mod}\Lambda$ with P projective.

(1) The pair (M, P) is called a *τ-rigid pair* if M is τ-rigid and $\operatorname{Hom}_\Lambda(P, M) = 0$.

(2) The pair (M, P) is called a *support τ-tilting pair* (respectively, *almost complete τ-tilting pair*) if it is τ-rigid and $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

Note that (M, P) is a support τ-tilting pair if and only if M is a τ-tilting $\Lambda/\langle e\rangle$-module, where $e\Lambda \cong P$. Hence, M is a τ-tilting Λ-module if and only if $(M, 0)$ is a support τ-tilting pair.

Let (U, Q) be an almost complete τ-tilting pair and let $X \in \operatorname{mod} \Lambda$ be indecomposable. We say that $(X, 0)$ (respectively, $(0, X)$) is a *complement* of (U, Q) if $(U \oplus X, Q)$ (respectively, $(U, Q \oplus X)$) is support τ-tilting. It follows from [1, Theorem 2.18] that any basic almost complete τ-tilting pair in $\operatorname{mod} \Lambda$ has exactly two complements. Two support τ-tilting pairs (T, P) and $(\widetilde{T}, \widetilde{P})$ in $\operatorname{mod} \Lambda$ are called *mutations* of each other if they have the same direct summand (U, Q) which is an almost complete τ-tilting pair. In this case, we write $(\widetilde{T}, \widetilde{P}) = \mu_X(T, P)$ (or simply $\widetilde{T} = \mu_X T$) if the indecomposable module X satisfies either $T = U \oplus X$ or $P = Q \oplus X$.

Definition 2.5 ([1, Definition 2.28]). Let $T = U \oplus X$ and $\widetilde{T}$ be support τ-tilting Λ-modules such that $\widetilde{T} = \mu_X T$ with X indecomposable. Then $\widetilde{T}$ is called a *left mutation* (respectively, *right mutation*) of T, written $\widetilde{T} = \mu_X^- T$ (respectively, $\widetilde{T} = \mu_X^+ T$), if $X \notin \operatorname{Fac} U$ (respectively, $X \in \operatorname{Fac} U$).

Definition 2.6 ([1, Definition 2.29]). The *support τ-tilting quiver* $Q(s\tau\text{-tilt}\, \Lambda)$ of Λ is defined as follows.

(1) The set of vertices consists of the isomorphism classes of basic support τ-tilting Λ-modules.
(2) We draw an arrow from T to each of its left mutations.

2.2. Split-by-nilpotent extensions. Let Λ and Γ be two algebras.

Definition 2.7 ([7, Definition 1.1]). We say that Γ is a *split extension of Λ by the nilpotent bimodule ${}_\Lambda E_\Lambda$*, or simply a *split-by-nilpotent extension* if there exists a split surjective algebra morphism $\Gamma \to \Lambda$ whose kernel E is contained in the radical of Γ.

Let Γ be a split-by-nilpotent extension of Λ by the nilpotent bimodule ${}_\Lambda E_\Lambda$. Clearly, the short exact sequence of Λ-Λ-bimodules

$$0 \to {}_\Lambda E_\Lambda \to {}_\Lambda \Gamma_\Lambda \to \Lambda \to 0$$

splits. Therefore, there exists an isomorphism ${}_\Lambda \Gamma_\Lambda \cong \Lambda \oplus_\Lambda E_\Lambda$. The module categories over Λ and Γ are related by the following functors:

$$- \otimes_\Lambda \Gamma : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma, \quad - \otimes_\Gamma \Lambda : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda,$$
$$\operatorname{Hom}_\Lambda(\Gamma_\Lambda, -) : \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma, \quad \operatorname{Hom}_\Gamma(\Lambda_\Gamma, -) : \operatorname{mod} \Gamma \to \operatorname{mod} \Lambda.$$

Moreover, we have

$$-\otimes_\Lambda \Gamma_\Gamma \otimes_\Gamma \Lambda \cong 1_{\operatorname{mod}\Lambda}, \qquad \operatorname{Hom}_\Gamma(\Lambda_\Gamma, \operatorname{Hom}_\Lambda(\Gamma_\Lambda, -)) \cong 1_{\operatorname{mod}\Lambda}.$$

LEMMA 2.8. *Let Γ be a split-by-nilpotent extension of Λ. Then for any $M \in \operatorname{mod}\Lambda$, we have:*

(1) *There exists a bijective correspondence between the isomorphism classes of indecomposable summands of M in $\operatorname{mod}\Lambda$ and the isomorphism classes of indecomposable summands of $M_\Lambda \otimes_\Lambda \Gamma$ in $\operatorname{mod}\Gamma$, given by $N_\Lambda \mapsto N_\Lambda \otimes_\Lambda \Gamma$.*
(2) $|M_\Lambda| = |M_\Lambda \otimes_\Lambda \Gamma|$.
(3) *Any indecomposable projective module in $\operatorname{mod}\Gamma$ is the form $P \otimes_\Lambda \Gamma$, where P is indecomposable projective in $\operatorname{mod}\Lambda$. In particular, $|\Lambda| = |\Gamma|$.*

Proof. Assertion (1) is [4, Lemma 1.2], and the last two assertions follow immediately from (1). ∎

LEMMA 2.9 ([4, Lemma 2.1]). *Let Γ be a split-by-nilpotent extension of Λ. Then for any $M \in \operatorname{mod}\Lambda$,*

$$\tau(M \otimes_\Lambda \Gamma) \cong \operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \tau M_\Lambda).$$

3. Main results. In this section, assume that Γ is a split extension of Λ by the nilpotent bimodule ${}_\Lambda E_\Lambda$.

3.1. τ-tilting and τ-rigid modules. The following result is a τ-version of [4, Theorem A].

THEOREM 3.1. *Let (T, P) be a pair in $\operatorname{mod}\Lambda$ with P projective. Then the following statements are equivalent:*

(1) *$(T \otimes_\Lambda \Gamma_\Gamma, P \otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in $\operatorname{mod}\Gamma$.*
(2) *(T, P) is a support τ-tilting pair in $\operatorname{mod}\Lambda$ and*

$$\operatorname{Hom}_\Lambda(T \otimes_\Lambda E, \tau T_\Lambda) = 0 = \operatorname{Hom}_\Lambda(P, T \otimes_\Lambda E).$$

Proof. By Lemma 2.8(2), we have $|T|+|P| = |T\otimes_\Lambda \Gamma|+|P\otimes_\Lambda \Gamma|$. Hence, $|T| + |P| = |\Lambda|$ if and only if $|T \otimes_\Lambda \Gamma| + |P \otimes_\Lambda \Gamma| = |\Gamma|$ by Lemma 2.8(3).

Let $T, P \in \operatorname{mod}\Lambda$. Then

$$\begin{aligned}
&\operatorname{Hom}_\Gamma(T \otimes_\Lambda \Gamma, \tau(T \otimes_\Lambda \Gamma))\\
&\qquad\cong \operatorname{Hom}_\Gamma(T \otimes_\Lambda \Gamma, \operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \tau T_\Lambda)) && \text{(by Lemma 2.9)}\\
&\qquad\cong \operatorname{Hom}_\Lambda(T \otimes_\Lambda \Gamma \otimes_\Gamma \Gamma_\Lambda, \tau T_\Lambda) && \text{(by adjunction)}\\
&\qquad\cong \operatorname{Hom}_\Lambda(T \otimes_\Lambda \Gamma_\Lambda, \tau T_\Lambda)\\
&\qquad\cong \operatorname{Hom}_\Lambda(T \otimes_\Lambda (\Lambda \oplus E)_\Lambda, \tau T_\Lambda)\\
&\qquad\cong \operatorname{Hom}_\Lambda(T, \tau T_\Lambda) \oplus \operatorname{Hom}_\Lambda(T \otimes_\Lambda E, \tau T_\Lambda),
\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_\Gamma(P\otimes_\Lambda \Gamma, T\otimes_\Lambda \Gamma) &\cong \mathrm{Hom}_\Lambda(P_\Lambda, \mathrm{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, T\otimes_\Lambda \Gamma)) \text{ (by adjunction)}\\ &\cong \mathrm{Hom}_\Lambda(P_\Lambda, T\otimes_\Lambda \Gamma_\Lambda)\\ &\cong \mathrm{Hom}_\Lambda(P_\Lambda, T\otimes_\Lambda (\Lambda\oplus E)_\Lambda)\\ &\cong \mathrm{Hom}_\Lambda(P_\Lambda, T_\Lambda)\oplus \mathrm{Hom}_\Lambda(P_\Lambda, T\otimes_\Lambda E). \ \blacksquare\end{aligned}$$

Note that T is a τ-tilting Λ-module if and only if $(T,0)$ is a support τ-tilting pair in $\operatorname{mod}\Lambda$. The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *For a module $T\in\operatorname{mod}\Lambda$, the following statements are equivalent:*

(1) *$T\otimes_\Lambda \Gamma_\Gamma$ is τ-tilting in $\operatorname{mod}\Gamma$.*
(2) *T is τ-tilting in $\operatorname{mod}\Lambda$ and $\mathrm{Hom}_\Lambda(T\otimes_\Lambda E, \tau T_\Lambda)=0$.*

Let $T\in\operatorname{mod}\Lambda$ be τ-rigid. Assume that E_Λ is generated by T, that is, there exists an epimorphism $T^{(n)}\to E_\Lambda\to 0$ in $\operatorname{mod}\Lambda$ for some $n\geq 1$. Applying the functor $\mathrm{Hom}_\Lambda(-,\tau T_\Lambda)$ to it yields a monomorphism

$$0\to \mathrm{Hom}_\Lambda(E_\Lambda,\tau T_\Lambda)\to \mathrm{Hom}_\Lambda(T^{(n)},\tau T_\Lambda)=0.$$

So $\mathrm{Hom}_\Lambda(E_\Lambda,\tau T_\Lambda)=0$, and hence

$$\mathrm{Hom}_\Lambda(T\otimes_\Lambda E,\tau T_\Lambda)\cong \mathrm{Hom}_\Lambda(T_\Lambda, \mathrm{Hom}_\Lambda({}_\Lambda E_\Lambda,\tau T_\Lambda))=0.$$

Thus by Theorem 3.1 and Corollary 3.2, we have the following result.

Corollary 3.3. *Let (T,P) be a pair in $\operatorname{mod}\Lambda$ with P projective. If E_Λ is generated by T, then the following statements are equivalent:*

(1) *$(T\otimes_\Lambda \Gamma_R, P\otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in $\operatorname{mod}\Gamma$.*
(2) *(T,P) is a support τ-tilting pair in $\operatorname{mod}\Lambda$ and $\mathrm{Hom}_\Lambda(P, T\otimes_\Lambda E)=0$.*

Moreover, $T\otimes_\Lambda \Gamma_\Gamma$ is τ-tilting in $\operatorname{mod}\Gamma$ if and only if T is τ-tilting in $\operatorname{mod}\Lambda$.

Let A be a hereditary algebra and $\mathcal{D}^b(\operatorname{mod}A)$ the bounded derived category of $\operatorname{mod}A$. The *cluster category* $\mathcal{C}_A$ is defined by the orbit category of $\mathcal{D}^b(\operatorname{mod}A)$ under the action of the functor $\tau^{-1}[1]$, where $[1]$ is the shift functor; and a *tilting object* $\widetilde{T}$ in $\mathcal{C}_A$ is an object such that $\mathrm{Ext}^1_{\mathcal{C}_A}(\widetilde{T},\widetilde{T})=0$ and $|\widetilde{T}|=|A|$ (see [8]). The endomorphism algebra of $\widetilde{T}$ is called *cluster-tilted* (see [9]). It was shown in [2, Theorem 3.4] that if Λ is a tilted algebra, then the relation extension of Λ by $\mathrm{Ext}^2_\Lambda(D\Lambda,\Lambda)$ is cluster-tilted. Moreover, all cluster-tilted algebras are of this form. In this case, we say Γ is a cluster-tilted algebra corresponding to the tilted algebra Λ.

PROPOSITION 3.4. *Let Γ be a cluster-tilted algebra corresponding to the tilted algebra Λ, and (T, P) a pair in* $\operatorname{mod} \Lambda$ *with P projective. Then the following statements are equivalent:*

(1) *$(T \otimes_\Lambda \Gamma_\Gamma, P \otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in* $\operatorname{mod} \Gamma$.
(2) *(T, P) is a support τ-tilting pair in* $\operatorname{mod} \Lambda$ *and*

$$\operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_\Lambda, \tau T_\Lambda) = 0 = \operatorname{Hom}_\Lambda(P, \tau^{-1}\Omega^{-1}T_\Lambda).$$

Proof. Since the global dimension of the tilted algebra Λ is at most 2, we have

$$T \otimes_\Lambda \operatorname{Ext}^2_\Lambda(D\Lambda, \Lambda) \cong \tau^{-1}\Omega^{-1}T$$

by [13, Proposition 4.1]. Now the assertion follows from Theorem 3.1. ∎

If $\operatorname{id}_\Lambda T \leq 1$, then $\tau^{-1}\Omega^{-1}T = 0$. So by Proposition 3.4, we have the following corollary.

COROLLARY 3.5. *Let Γ be a cluster-tilted algebra corresponding to the tilted algebra Λ, and (T, P) a pair in* $\operatorname{mod} \Lambda$ *with* $\operatorname{id}_\Lambda T \leq 1$ *and P projective. Then the following statements are equivalent:*

(1) *$(T \otimes_\Lambda \Gamma_\Gamma, P \otimes_\Lambda \Gamma_\Gamma)$ is a support τ-tilting pair in* $\operatorname{mod} \Gamma$.
(2) *(T, P) is a support τ-tilting pair in* $\operatorname{mod} \Lambda$.

In particular, $T \otimes_\Lambda \Gamma_\Gamma$ is a τ-tilting Γ-module if and only if T is a τ-tilting Λ-module.

Let $\mathcal{C}$ be a full subcategory of $\operatorname{mod} \Lambda$. We write

$$\mathcal{C}^\perp := \{M \in \operatorname{mod} \Lambda \mid \operatorname{Hom}_\Lambda(C, M) = 0 \text{ for any } C \in \mathcal{C}\},$$
$${}^\perp\mathcal{C} := \{M \in \operatorname{mod} \Lambda \mid \operatorname{Hom}_\Lambda(M, C) = 0 \text{ for any } C \in \mathcal{C}\}.$$

Recall that a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} \Lambda$ is called a *torsion pair* if $\mathcal{T}^\perp = \mathcal{F}$ and ${}^\perp\mathcal{F} = \mathcal{T}$. Every τ-tilting Λ-module T induces a torsion pair $(\mathcal{T}(T), \mathcal{F}(T)) := ({}^\perp(\tau T), T^\perp)$ (see [1]).

PROPOSITION 3.6. *Let $X_\Gamma \in \operatorname{mod} \Gamma$, and let $T \in \operatorname{mod} \Lambda$ be τ-tilting such that $\operatorname{Hom}_\Lambda(T \otimes_\Lambda E, \tau T_\Lambda) = 0$. Then:*

(1) *$X_\Gamma \in \mathcal{T}(T \otimes_\Lambda \Gamma)$ if and only if $X \otimes_\Gamma \Gamma_\Lambda \in \mathcal{T}(T)$.*
(2) *$X_\Gamma \in \mathcal{F}(T \otimes_\Lambda \Gamma)$ if and only if $\operatorname{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, X_\Gamma) \in \mathcal{F}(T)$.*

Proof. Since $\operatorname{Hom}_\Lambda(T \otimes_\Lambda E, \tau T_\Lambda) = 0$, we find that $T \otimes_\Lambda \Gamma$ is a τ-tilting Γ-module by Corollary 3.2 and it will induce a torsion pair. Note that there are isomorphisms

$$\begin{aligned}
\operatorname{Hom}_\Gamma(X_\Gamma, \tau(T \otimes_\Lambda \Gamma)) &\cong \operatorname{Hom}_\Gamma(X_\Gamma, \operatorname{Hom}_\Lambda({}_\Gamma\Gamma_\Lambda, \tau T_\Lambda)) \\
&\cong \operatorname{Hom}_\Lambda(X \otimes_\Gamma \Gamma_\Lambda, \tau T_\Lambda), \\
\operatorname{Hom}_\Gamma(T \otimes_\Lambda \Gamma, X_\Gamma) &\cong \operatorname{Hom}_\Lambda(T_\Lambda, \operatorname{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, X_\Gamma)).
\end{aligned}$$

The result is now obvious. ∎

For a Γ-module U_Γ, $U \otimes_\Gamma \Lambda$ is a Λ-module. If U_Γ is τ-tilting and $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \cong U_\Gamma$, then $U \otimes_\Gamma \Lambda$ is a τ-tilting Λ-module by Theorem 3.1. As a slight generalization of this observation, the following result gives a converse construction of Corollary 3.2.

Proposition 3.7. *Assume that U_Γ is a Γ-module such that $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \in \operatorname{add} U_\Gamma$.*

(1) *If U_Γ is τ-rigid, then $U \otimes_\Gamma \Lambda$ is a τ-rigid Λ-module.*
(2) *If U_Γ is τ-tilting and $U \otimes_\Gamma \Lambda$ is basic, then $U \otimes_\Gamma \Lambda$ is a τ-tilting Λ-module.*

Proof. (1) Let U_Γ be τ-rigid and

$$P_1 \otimes_\Lambda \Gamma \xrightarrow{f_0} P_0 \otimes_\Lambda \Gamma \to U_\Gamma \to 0$$

be a minimal projective presentation of U in $\operatorname{mod} \Gamma$ with P_0, P_1 projective Λ-modules. Applying the functor $- \otimes_\Gamma \Lambda$ to it, we obtain a projective presentation

$$P_1 \xrightarrow{f_0 \otimes 1_\Lambda} P_0 \to U_\Gamma \otimes_\Gamma \Lambda \to 0$$

of $U \otimes_\Gamma \Lambda$ in $\operatorname{mod} \Lambda$. To prove that $U \otimes_\Gamma \Lambda$ is a τ-rigid Λ-module, it suffices to show that $\operatorname{Hom}_\Lambda(f_0 \otimes 1_\Lambda, U \otimes_\Gamma \Lambda)$ is epic by Lemma 2.3.

Let $g \in \operatorname{Hom}_\Lambda(P_1, U \otimes_\Gamma \Lambda)$. Then $g \otimes 1_\Gamma \in \operatorname{Hom}_\Gamma(P_1 \otimes_\Lambda \Gamma, U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma)$. By assumption, $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \in \operatorname{add} U_\Gamma$. Without loss of generality, $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma$ is basic, and hence it is a direct summand of U_Γ. Then there exist a canonical embedding $\lambda : U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \to U_\Gamma$ and a canonical epimorphism $\pi : U_\Gamma \to U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma$ such that $\pi\lambda = 1_{U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma}$. Consider the diagram

$$\begin{tikzcd}
P_1 \otimes_\Lambda \Gamma \arrow[r, "f_0"] \arrow[d, "g\otimes 1_\Gamma"'] & P_0 \otimes_\Lambda \Gamma \arrow[r] \arrow[dl, dotted, "\pi i"'] \arrow[ddl, dotted, "i"] & U_\Gamma \arrow[r] & 0 \\
U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \arrow[d, shift left, "\lambda"] & & & \\
U_\Gamma \arrow[u, shift left, "\pi"] & & &
\end{tikzcd}$$

Since $\operatorname{Hom}_\Gamma(f_0, U_\Gamma)$ is epic by Lemma 2.3, it follows that there exists $i \in \operatorname{Hom}_\Gamma(P_0 \otimes_\Lambda \Gamma, U_\Gamma)$ such that $\lambda(g \otimes 1_\Gamma) = i f_0$. Then we have

$$g \otimes 1_\Gamma = 1_{U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma}(g \otimes 1_\Gamma) = \pi\lambda(g \otimes 1_\Gamma) = (\pi i) f_0,$$

and

$$g \cong g \otimes 1_\Gamma \otimes 1_\Lambda \cong ((\pi i) f_0) \otimes 1_\Lambda \cong ((\pi i) \otimes 1_\Lambda)(f_0 \otimes 1_\Lambda).$$

Therefore $\operatorname{Hom}_\Lambda(f_0 \otimes 1_\Lambda, U \otimes_\Gamma \Lambda)$ is epic.

(2) If U_Γ is τ-tilting, then $|U \otimes_\Gamma \Lambda| \geq |U_\Gamma| = |\Gamma| = |\Lambda|$ by Lemma 2.8(3). Thus $U \otimes_\Gamma \Lambda$ is a τ-tilting Λ-module when it is basic by (1) and Theorem 2.2. ∎

However, $U \otimes_\Gamma \Lambda$ may not be basic even if U_Γ is basic. Let $M(U \otimes_\Gamma \Lambda)$ stand for the maximal basic direct summand of $U \otimes_\Gamma \Lambda$, that is, the direct sum of all indecomposable direct summands of $U \otimes_\Gamma \Lambda$ which are pairwise non-isomorphic.

EXAMPLE 3.8. Let Λ be the algebra given by the quiver

$$1 \longrightarrow 2$$

and Γ the algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

with the relation $\alpha\beta = 0$. Then Γ is the split extension of Λ by the nilpotent E generated by β and $U_\Gamma = S_2 \oplus e_2\Gamma$ is a τ-tilting Γ-module, where S_2 is the simple Γ-module corresponding to vertex 2. Applying the functor $-\otimes_\Gamma \Lambda$ to the projective presentation

$$0 \to e_1\Gamma \to (e_2\Gamma)^2 \to U_\Gamma \to 0$$

of U_Γ, we get an exact sequence

$$e_1\Lambda \xrightarrow{0} (e_2\Lambda)^2 \to U \otimes_\Gamma \Lambda \to 0$$

in $\operatorname{mod}\Lambda$. So $U \otimes_\Gamma \Lambda \cong (e_2\Lambda)^2$ and it is not basic. Note that $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \in \operatorname{add} U_\Gamma$ because $U \otimes_\Gamma \Lambda \otimes_\Lambda \Gamma \cong (e_2\Gamma)^2$. Moreover, $M(U \otimes_\Gamma \Lambda) \cong e_2\Lambda$ is a support τ-tilting Λ-module.

We do not know the answer to the following question:

QUESTION 3.9. *Under the conditions of Proposition* 3.7*, if U_Γ is τ-tilting, is $M(U \otimes_\Gamma \Lambda)$ a support τ-tilting Λ-module?*

3.2. Left mutations. Let T be a support τ-tilting Λ-module such that $T\otimes_\Lambda\Gamma$ is a support τ-tilting Γ-module. By Lemma 2.8(1), all indecomposable summands of $T_\Lambda \otimes_\Lambda \Gamma$ are of the form $X \otimes_\Lambda \Gamma$ for some indecomposable summand X of T. We now investigate the relationship between $Q(s\tau\text{-tilt}\,\Lambda)$ and $Q(s\tau\text{-tilt}\,\Gamma)$.

THEOREM 3.10. *Let $T_1, T_2 \in \operatorname{mod}\Lambda$ be such that $T_1\otimes_\Lambda\Gamma$ and $T_2\otimes_\Lambda\Gamma$ are support τ-tilting Γ-modules. Then the following statements are equivalent:*

(1) *$T_1 \otimes_\Lambda \Gamma$ is a left mutation of $T_2 \otimes_\Lambda \Gamma$.*
(2) *T_1 is a left mutation of T_2.*

Proof. (1)⇒(2) Since $T_1 \otimes_\Lambda \Gamma$ and $T_2 \otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules by assumption, T_1 and T_2 are support τ-tilting Λ-modules by Theorem 3.1.

Let $T_1 \otimes_\Lambda \Gamma = \mu^-_{X\otimes_\Lambda\Gamma}(T_2 \otimes_\Lambda \Gamma)$ for some indecomposable Λ-module X. Assume that $(T_1 \otimes_\Lambda \Gamma, P_1 \otimes_\Lambda \Gamma)$ and $(T_2 \otimes_\Lambda \Gamma, P_2 \otimes_\Lambda \Gamma)$ are support τ-tilting pairs having the same almost complete support τ-tilting pair $(U\otimes_\Lambda \Gamma, Q\otimes_\Lambda \Gamma)$, where U and Q are Λ-modules. Then by Lemma 2.8(1), (T_1, P_1) and (T_2, P_2) have the same almost complete support τ-tilting pair (U, Q) and are mutations of each other.

Because $T_2 \otimes_\Lambda \Gamma = (X \otimes_\Lambda \Gamma) \oplus (U \otimes_\Lambda \Gamma)$, we have $T_2 \cong X \oplus U$. It suffices to show that $X \notin \operatorname{Fac} U$. Otherwise, there exists an epimorphism $U^{(n)} \to X \to 0$ in $\operatorname{mod} \Lambda$ for some $n \geq 1$, which yields an epimorphism $U^{(n)} \otimes_\Lambda \Gamma$ $(\cong (U \otimes_\Lambda \Gamma)^{(n)}) \to X \otimes_\Lambda \Gamma \to 0$ in $\operatorname{mod} \Gamma$. This implies that $X \otimes_\Lambda \Gamma \in \operatorname{Fac}(U \otimes_\Lambda \Gamma)$, a contradiction.

Similarly, we get (2)⇒(1). ■

As an immediate consequence of Theorem 3.10 and its proof, we get

Corollary 3.11. *Let $T_1, T_2 \in \operatorname{mod} \Lambda$ be such that $T_1 \otimes_\Lambda \Gamma$ and $T_2 \otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules, and let X be the indecomposable Λ-module as in the proof of Theorem* 3.10. *Then the following statements are equivalent:*

(1) $T_1 \otimes_\Lambda \Gamma = \mu^-_{X\otimes_\Lambda \Gamma}(T_2 \otimes_\Lambda \Gamma)$.
(2) $T_1 = \mu^-_X T_2$.

Let $Q = (Q_0, Q_1)$ be a quiver. A subquiver $\widehat{Q} = (\widehat{Q}_0, \widehat{Q}_1)$ of Q is called *full* if $\widehat{Q}_1$ equals the set of all those arrows in Q_1 whose source and target both belong to $\widehat{Q}_0$ [5, Chapter II]. We use $fQ(s\tau\text{-tilt}\,\Gamma)$ to denote the full subquiver of $Q(s\tau\text{-tilt}\,\Gamma)$ whose vertices are $T \otimes_\Lambda \Gamma$ where $T \in Q(s\tau\text{-tilt}\,\Lambda)$, and write $fQ(s\tau\text{-tilt}\,\Lambda)$ for the full subquiver of $Q(s\tau\text{-tilt}\,\Lambda)$ whose vertices are those support τ-tilting Λ-modules T such that $T \otimes_\Lambda \Gamma$ is a support τ-tilting Γ-module. Corollary 3.11 shows that the underlying graphs of $fQ(s\tau\text{-tilt}\,\Lambda)$ and $fQ(s\tau\text{-tilt}\,\Gamma)$ coincide. More precisely, if $T_1, T_2 \in \operatorname{mod} \Lambda$ are such that $T_1 \otimes_\Lambda \Gamma$ and $T_2 \otimes_\Lambda \Gamma$ are support τ-tilting Γ-modules, then there exists an arrow from $T_1 \otimes_\Lambda \Gamma$ to $T_2 \otimes_\Lambda \Gamma$ in $Q(s\tau\text{-tilt}\,\Gamma)$ if and only if there exists an arrow from T_1 to T_2 in $Q(s\tau\text{-tilt}\,\Lambda)$.

3.3. A special case. We now turn to one-point extensions. Let Λ be an algebra and $M \in \operatorname{mod} \Lambda$. The *one-point extension* of Λ by M is defined to be the matrix algebra

$$\Gamma = \begin{pmatrix} \Lambda & 0 \\ M_\Lambda & k \end{pmatrix}$$

with the ordinary matrix addition and multiplication, and we write $\Gamma := \Lambda[M]$ with a the extension point. Let $\Delta := \Lambda \times k$, and let E be the (Δ, Δ)-bimodule generated by the arrows from a to the quiver of Λ. It is easy to see that Γ is a split extension of Δ by the nilpotent bimodule ${}_\Delta E_\Delta$, and $E_\Delta \cong M_\Delta$ while $D({}_\Delta E) \cong S^t$ where S is the simple module corresponding to the point a and $t = \dim_k M$ (see [6]).

In the rest of this subsection, Γ is a one-point extension of Λ by a module M in $\operatorname{mod}\Lambda$, and e_a is the idempotent corresponding to the extension point a and $\Delta := \Lambda \times k$.

REMARK 3.12.

(1) The algebra Γ is a Δ-Δ-bimodule and a Λ-Λ-bimodule.
(2) The algebra Δ is a Λ-Λ-bimodule.
(3) Any Λ-module X can be seen as a Δ-module or a Γ-module. In fact,

$$X_\Gamma \cong X_\Delta \otimes_\Delta \Gamma \cong X_\Lambda \otimes_\Lambda \Gamma.$$

(4) For any Δ-module N, we have

$$N_\Delta \cong Y_\Delta \oplus S^t \quad \text{for some } t \geq 0,$$

where Y is a Λ-module.

We need the following two easy observations.

LEMMA 3.13. *For any $X \in \operatorname{mod}\Lambda$, we have $X \otimes_\Delta E = 0$.*

Proof. Considering the projective presentation

$$e_2\Lambda \to e_1\Lambda \to X \to 0$$

of X in $\operatorname{mod}\Lambda$ with e_1, e_2 idempotents of Λ, we get the projective presentation

$$e_2\Delta \to e_1\Delta \to X \to 0$$

of X in $\operatorname{mod}\Delta$. Applying the functor $-\otimes_\Delta E$ yields the exact sequence

$$e_2E \to e_1E \to X \otimes_\Delta E \to 0.$$

Since E is generated by the arrows from a to the quiver of Λ, we have $e_1E = 0 = e_2E = 0$. Hence $X \otimes_\Delta E = 0$. ∎

LEMMA 3.14. $S \otimes_\Delta E \cong M_\Delta$.

Proof. This follows Lemma 3.13 and the isomorphism

$$M_\Delta \cong E_\Delta \cong \Delta \otimes_\Delta E \cong (S \oplus \Lambda) \otimes_\Delta E \cong (S \otimes_\Delta E) \oplus (\Lambda \otimes_\Delta E). \ ∎$$

Note that basic support τ-tilting modules in $\operatorname{mod}\Lambda$ are exactly those T and $T \oplus S$ where T is a support τ-tilting Λ-module. Hence support τ-tilting pairs in $\operatorname{mod}\Delta$ are exactly $(T, P \oplus S)$ and $(T \oplus S, P)$ where P is a projective Λ-module such that (T, P) is a support τ-tilting pair in $\operatorname{mod}\Lambda$. As a consequence of Theorem 3.1, we also have

PROPOSITION 3.15. *Let Γ be a one-point extension of Λ by a module M in $\operatorname{mod}\Lambda$, and let e_a be the idempotent corresponding to the extension point a. Then for a pair (T, P) in $\operatorname{mod}\Lambda$ with P projective, we have:*

(1) *$(T_\Gamma, P_\Gamma \oplus e_a\Gamma)$ is a support τ-tilting pair in $\operatorname{mod}\Gamma$ if and only if (T_Λ, P_Λ) is a support τ-tilting pair in $\operatorname{mod}\Lambda$.*

(2) $(T_\Gamma \oplus e_a\Gamma, P_\Gamma)$ *is a support* τ*-tilting pair in* $\operatorname{mod}\Gamma$ *if and only if* (T_Λ, P_Λ) *is a support* τ*-tilting pair in* $\operatorname{mod}\Lambda$ *and* $\operatorname{Hom}_\Lambda(M_\Lambda, \tau T_\Lambda) = 0 = \operatorname{Hom}_\Lambda(P_\Lambda, M_\Lambda)$.

Proof. This follows from Theorem 3.1 and Lemmas 3.13 and 3.14. ∎

Putting $P = 0$ in Proposition 3.15, we get

Corollary 3.16.

(1) T_Γ *is almost complete* τ*-tilting if and only if* T_Λ *is* τ*-tilting.*
(2) $T_\Gamma \oplus e_a\Gamma$ *is* τ*-tilting in* $\operatorname{mod}\Gamma$ *if and only if* T *is* τ*-tilting in* $\operatorname{mod}\Lambda$ *and* $\operatorname{Hom}_\Lambda(M_\Lambda, \tau T_\Lambda) = 0$.

If Γ is a one-point extension of Λ by a non-zero module M_Λ, then there exists an idempotent $e \in \Lambda$ such that $\operatorname{Hom}_\Lambda(e\Lambda, M_\Lambda) \neq 0$. Note that there are τ-tilting $\Lambda/\langle e\rangle$-modules. So, by Proposition 3.15(2), we have

Corollary 3.17. *Let* Γ *be a one-point extension of* Λ *by a non-zero module* M_Λ*. Then there exists a support* τ*-tilting* Λ*-module such that* $T_\Gamma \oplus e_a\Gamma$ *is not support* τ*-tilting.*

4. Examples. In this section, we give two examples to illustrate the results obtained in Section 3. All indecomposable modules are denoted by their Loewy series.

Example 4.1. Let Σ be a finite-dimensional k-algebra given by the quiver

$$1 \to 2 \to 3.$$

Then $T = 1\ \begin{smallmatrix}1\\2\\3\end{smallmatrix}\ 3$ is a tilting Σ-module. The endomorphism algebra Λ of T is a tilted algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

with the relation $\alpha\beta = 0$. The cluster-titlted algebra Γ corresponding to Λ is given by the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"] \end{tikzcd}$$

with the relations $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$, and Γ is a split-by-nilpotent extension of Λ.

Note that 3 is the unique indecomposable module in $\operatorname{mod}\Lambda$ with injective dimension two. So for any indecomposable module W not isomorphic to 3, we have $\tau^{-1}\Omega^{-1}W = 0$. Because

$$0 \to 3 \to \begin{smallmatrix}2\\3\end{smallmatrix} \to \begin{smallmatrix}1\\2\end{smallmatrix} \to 1 \to 0$$

is a minimal injective resolution of 3, we have $\tau^{-1}\Omega^{-1}\,3 = \tau^{-1}\,2 = 1$.

Let (T_i, P_i) be a support τ-tilting pair in $\operatorname{mod}\Lambda$ and $\widetilde{T_i} := T_i \otimes_\Lambda \Gamma$ for each i. We list $\widetilde{T_i}$, $\tau^{-1}\Omega^{-1}T_i$ and $\operatorname{Hom}_\Lambda(P_i, \tau^{-1}\Omega^{-1}T_i)$ in the following table.

T_i	P_i	$\widetilde{T_i} = T_i \otimes \Gamma$	$\tau^{-1}\Omega^{-1}T_i$	$\operatorname{Hom}_\Lambda(P_i, \tau^{-1}\Omega^{-1}T_i)$
$T_1 = \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}\,3$	0	$\widetilde{T_1} = \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}\begin{smallmatrix}3\\1\end{smallmatrix}$	1	0
$T_2 = \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}\,2$	0	$\widetilde{T_2} = \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}\,2$	0	0
$T_3 = \begin{smallmatrix}1\\2\end{smallmatrix}\,1\,3$	0	$\widetilde{T_3} = \begin{smallmatrix}1\\2\end{smallmatrix}\,1\,\begin{smallmatrix}3\\1\end{smallmatrix}$	1	0
$T_4 = \begin{smallmatrix}2\\3\end{smallmatrix}\,3$	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\widetilde{T_4} = \begin{smallmatrix}2\\3\end{smallmatrix}\begin{smallmatrix}3\\1\end{smallmatrix}$	1	$\neq 0$
$T_5 = \begin{smallmatrix}1\\2\end{smallmatrix}\,2$	3	$\widetilde{T_5} = \begin{smallmatrix}1\\2\end{smallmatrix}\,2$	0	0
$T_6 = 1\,3$	$\begin{smallmatrix}2\\3\end{smallmatrix}$	$\widetilde{T_6} = 1\,\begin{smallmatrix}3\\1\end{smallmatrix}$	1	0
$T_7 = \begin{smallmatrix}1\\2\end{smallmatrix}\,1$	3	$\widetilde{T_7} = \begin{smallmatrix}1\\2\end{smallmatrix}\,1$	0	0
$T_8 = \begin{smallmatrix}2\\3\end{smallmatrix}\,2$	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$\widetilde{T_8} = \begin{smallmatrix}2\\3\end{smallmatrix}\,2$	0	0
$T_9 = 1$	$\begin{smallmatrix}2\\3\end{smallmatrix}\,3$	$\widetilde{T_9} = 1$	0	0
$T_{10} = 2$	$\begin{smallmatrix}1\\2\end{smallmatrix}\,3$	$\widetilde{T_{10}} = 2$	0	0
$T_{11} = 3$	$\begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}$	$\widetilde{T_{11}} = \begin{smallmatrix}3\\1\end{smallmatrix}$	1	$\neq 0$
$T_{12} = 0$	$\begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}\,3$	$\widetilde{T_{12}} = 0$	0	0

A simple calculation yields

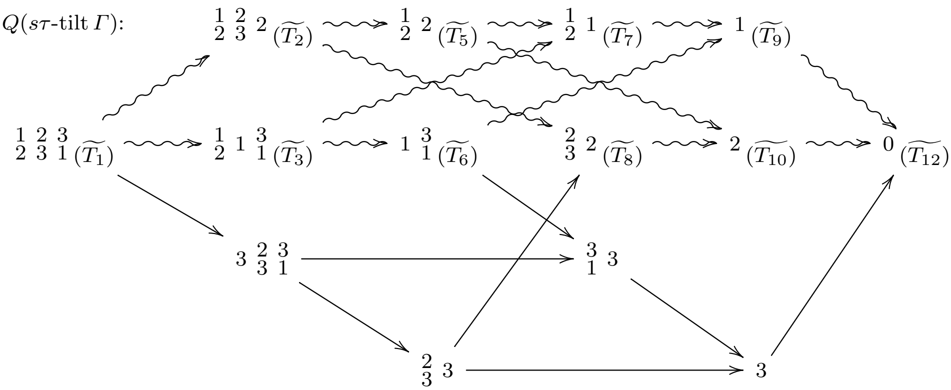
$$\begin{aligned}
\operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_1, \tau T_1) &= 0,\\
\operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_3, \tau T_3) &\cong \operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_3, \tau\, 1)\\
&\cong \operatorname{Hom}_\Lambda(1, 2) = 0,\\
\operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_6, \tau T_6) &\cong \operatorname{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_6, \tau\, 1)\\
&\cong \operatorname{Hom}_\Lambda(1, 2) = 0.
\end{aligned}$$

Thus all $\widetilde{T_1}$, $\widetilde{T_2}$, $\widetilde{T_3}$, $\widetilde{T_5}$, $\widetilde{T_6}$, $\widetilde{T_7}$, $\widetilde{T_8}$, $\widetilde{T_9}$, $\widetilde{T_{10}}$ and $\widetilde{T_{12}}$ are support τ-tilting, and neither $\widetilde{T_4}$ nor $\widetilde{T_{11}}$ is support τ-tilting by Proposition 3.4. We draw the Hasse quivers $Q(s\tau\text{-tilt}\,\Lambda)$ and $Q(s\tau\text{-tilt}\,\Gamma)$ as follows, where $M_{(T_i)}$ stands for $(T_i = M)$:

$Q(s\tau\text{-tilt}\,\Lambda)$:

$$\begin{tikzcd}
& \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}2\,(T_2) \arrow[r, rightsquigarrow] \arrow[drr, rightsquigarrow] & \begin{smallmatrix}1\\2\end{smallmatrix}2\,(T_5) \arrow[r, rightsquigarrow] \arrow[drr, rightsquigarrow] & \begin{smallmatrix}1\\2\end{smallmatrix}1\,(T_7) \arrow[r, rightsquigarrow] & 1\,(T_9) \arrow[dr, rightsquigarrow] & \\
\begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}3\,(T_1) \arrow[ur, rightsquigarrow] \arrow[r, rightsquigarrow] \arrow[dr] & \begin{smallmatrix}1\\2\end{smallmatrix}1\,3\,(T_3) \arrow[r, rightsquigarrow] \arrow[urr, rightsquigarrow] & 1\,3\,(T_6) \arrow[urr, rightsquigarrow] \arrow[drr] & \begin{smallmatrix}2\\3\end{smallmatrix}2\,(T_8) \arrow[r, rightsquigarrow] & 2\,(T_{10}) \arrow[r, rightsquigarrow] & 0\,(T_{12}) \\
& \begin{smallmatrix}2\\3\end{smallmatrix}3\,(T_4) \arrow[urr] \arrow[rrr] & & & 3\,(T_{11}) \arrow[ur] &
\end{tikzcd}$$

$Q(s\tau\text{-tilt}\,\Gamma)$:



We indicate the arrows in $fQ(s\tau\text{-tilt}\,\Gamma)$ and $fQ(s\tau\text{-tilt}\,\Lambda)$ by $\rightsquigarrow$. Their underlying graphs and corresponding arrows are identical.

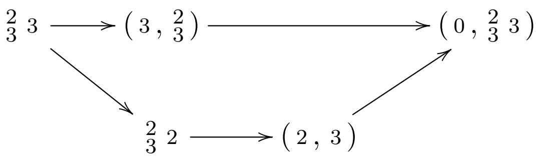
EXAMPLE 4.2. Let Λ be a finite-dimensional k-algebra given by the quiver

$$2 \to 3.$$

Considering the one-point extension of Λ by the simple module corresponding to the point 2, the algebra $\Gamma = \Lambda[2]$ is given by the quiver

$$1 \stackrel{\alpha}{\to} 2 \stackrel{\beta}{\to} 3$$

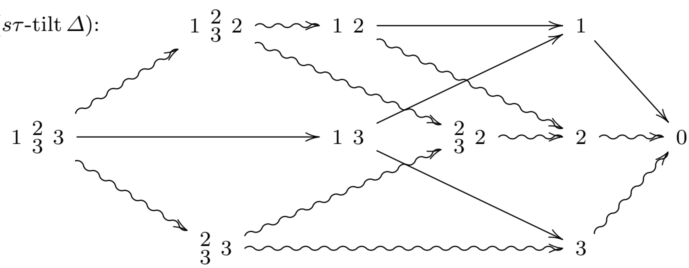
with the relation $\alpha\beta = 0$. Let $\Delta = \Lambda \times k$. The Hasse quiver of Λ is



By Proposition 3.15(1), all $\begin{smallmatrix}2\\3\end{smallmatrix}\,3$, 3, 0, $\begin{smallmatrix}2\\3\end{smallmatrix}\,2$ and 2 are support τ-tilting Γ-modules. From support τ-tilting Λ-modules 3 and 0, it is easy to get two support τ-tilting Δ-pairs $(3\,1, \begin{smallmatrix}2\\3\end{smallmatrix})$ and $(1, \begin{smallmatrix}2\\3\end{smallmatrix}\,3)$. Since $\mathrm{Hom}_\Lambda(\begin{smallmatrix}2\\3\end{smallmatrix}, 2) \neq 0$, it follows from Proposition 3.15(2) that neither $3\,\begin{smallmatrix}1\\2\end{smallmatrix}$ nor $\begin{smallmatrix}1\\2\end{smallmatrix}$ is a support τ-tilting Γ-module. A simple calculation shows that all $\begin{smallmatrix}2\\3\end{smallmatrix}\,3\,\begin{smallmatrix}1\\2\end{smallmatrix}$, $\begin{smallmatrix}2\\3\end{smallmatrix}\,2\,\begin{smallmatrix}1\\2\end{smallmatrix}$ and $2\,\begin{smallmatrix}1\\2\end{smallmatrix}$ are support τ-tilting Γ-modules also by Proposition 3.15(2).

Now we draw $Q(s\tau\text{-tilt}\,\Delta)$ and $Q(s\tau\text{-tilt}\,\Gamma)$:

$Q(s\tau\text{-tilt}\,\Delta)$:



$$Q(s\tau\text{-tilt}\,\Gamma):\quad
\begin{tikzcd}
 & \begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}2 \arrow[r, rightsquigarrow] \arrow[drr, rightsquigarrow] & \begin{smallmatrix}1\\2\end{smallmatrix}2 \arrow[r] \arrow[drr, rightsquigarrow] & \begin{smallmatrix}1\\2\end{smallmatrix}1 \arrow[r] & 1 \arrow[dr] & \\
\begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}3 \arrow[ur, rightsquigarrow] \arrow[r] \arrow[dr, rightsquigarrow] & \begin{smallmatrix}1\\2\end{smallmatrix}1\,3 \arrow[r] \arrow[urr] & 1\,3 \arrow[urr] \arrow[drr] & \begin{smallmatrix}2\\3\end{smallmatrix}2 \arrow[r, rightsquigarrow] & 2 \arrow[r, rightsquigarrow] & 0 \\
 & \begin{smallmatrix}2\\3\end{smallmatrix}3 \arrow[urr, rightsquigarrow] \arrow[rrr, rightsquigarrow] & & & 3 \arrow[ur, rightsquigarrow] &
\end{tikzcd}$$

We also indicate the arrows in $fQ(s\tau\text{-tilt}\,\Delta)$ and $fQ(s\tau\text{-tilt}\,\Gamma)$ by $\rightsquigarrow$. Their underlying graphs and corresponding arrows are identical.

Acknowledgements. The authors thank the referee for useful and detailed suggestions.

This work was partially supported by NSFC (Nos. 11971225, 11571164) and a Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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