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A note on the Hasse quiver of τ-tilting modules

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Let Λ be an algebra with an indecomposable projective-injective module. Adachi gave a method to construct the Hasse quiver of support τ-tilting Λ-modules. In this paper, we will show that it can be restricted to τ-tilting modules.

Keywords: τ-tilting modules; Hasse quiver; support τ-tilting quiver.

Mathematics Subject Classification 2020: 16G20, 16G60

1. Introduction

In 2014, Adachi *et al.* [2] introduced the concept of support τ-tilting modules as a generalization of tilting modules. They showed that, in contrast to tilting modules, it is always possible to exchange a given indecomposable summand of a support τ-tilting module for a unique other indecomposable and obtain a new support τ-tilting module. This process, called mutation, is essential in cluster theory. In the same paper, the authors also showed that the support τ-tilting modules are in bijection with several other important classes in representation theory including functorially finite torsion classes introduced in [3], 2-term silting complexes introduced in [4], and cluster-tilting objects in the cluster category.

Let Λ be a finite dimensional K-algebra over an algebraically closed field K. A Λ-module M is called *τ-tilting* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ and $|M| = |\Lambda|$. A module is called *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ. We will denote by $\tau\text{-tilt}\,\Lambda$ (respectively, $\mathrm{s}\tau\text{-tilt}\,\Lambda$) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) Λ-modules. For any two support τ-tilting Λ-modules M, N, we write $M \geq N$ if $\mathrm{Fac}(M) \supseteq \mathrm{Fac}(N)$. Then $\geq$ gives a partial order on support τ-tilting Λ-modules. The associated Hasse quiver (support

τ-tilting quiver) $\mathrm{H}(\Lambda)$ is as follows:

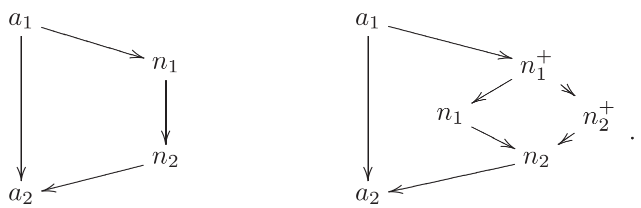
- The set of vertices is all basic support τ-tilting Λ-modules.
- Draw an arrow from M to N if $M > N$ and there is no support τ-tilting Λ-module L such that $M > L > N$.

Let $\mathcal{N}$ be a subposet of $\mathrm{H}(\Lambda)$ and $\mathcal{N}' := \mathrm{H}(\Lambda)\backslash\mathcal{N}$. Adachi define a new quiver $\mathrm{H}(\Lambda)^{\mathcal{N}}$ from $\mathrm{H}(\Lambda)$ as follows (see [1, Definition 3.2]):

- vertices: vertices in $\mathrm{H}(\Lambda)$ and $\mathcal{N}^+$ where $\mathcal{N}^+$ is a copy of $\mathcal{N}$.
- arrows: $\{a_1 \to a_2 \,|\, a_1 \to a_2 \in \mathcal{N}'\} \coprod \{n_2 \to a_2 \,|\, n_2 \to a_2, n_2 \in \mathcal{N}, a_2 \in \mathcal{N}'\}$

$$\coprod \{n_1 \to n_2, n_1^+ \to n_2^+ \,|\, n_1 \to n_2 \in \mathcal{N}\} \coprod \{a_1 \to n_1^+ \,|\, a_1 \to n_1, n_1 \in \mathcal{N}, a_1 \in \mathcal{N}'\}$$

$$\coprod \{n_1^+ \to n_1 \,|\, n_1 \in \mathcal{N}\}.$$



$\mathrm{H}(\Lambda)$ $\qquad\qquad$ $\mathrm{H}(\Lambda)^{\mathcal{N}}$

Suppose that Λ is a basic finite dimensional K-algebra with an indecomposable projective-injective module Q, Adachi shown that $\mathrm{Soc}(Q)$ is a two-sided ideal of Λ(see [1, Proposition 3.1(1)]). Let $\overline{\Lambda} := \Lambda/\mathrm{Soc}(Q)$ and

$$\mathcal{N}_1 := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda} \,|\, Q/\mathrm{Soc}(Q) \in \mathrm{add}\,N \text{ and } \mathrm{Hom}_\Lambda(N, Q) = 0\}.$$

It is shown that there is an isomorphism of posets $\mathrm{H}(\Lambda) \to \mathrm{H}\left(\overline{\Lambda}\right)^{\mathcal{N}_1}$ [1, Theorem 3.3].

In this paper, we will show that this result can be restricted to τ-tilting modules. More precisely, for an algebra Λ, let $Q(X)$ be the full subquiver of $\mathrm{H}(\Lambda)$ consisting of those vertices in X for a sbuset X of $\mathrm{H}(\Lambda)$. Now, considering the set

$$\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda} \,|\, Q/\mathrm{Soc}(Q) \in \mathrm{add}\,N,\ \ \mathrm{Hom}_\Lambda(N, Q) = 0 \text{ and } |N| = |\overline{\Lambda}| - 1\},$$

we have the following result.

Theorem 1.1. *Let Λ be an algebra with an indecomposable projective-injective module Q.*

(1) *If Q is simple, then there is an isomorphism*

$$Q(\tau\text{-tilt}\,\Lambda) \to Q(\tau\text{-tilt}\,\overline{\Lambda}).$$

(2) *If Q is not simple, then there is an isomorphism*

$$Q(\tau\text{-tilt}\,\Lambda) \to Q\left(\tau\text{-tilt}\,\overline{\Lambda} \coprod \mathcal{N}\right).$$

Moreover, if $Q/\mathrm{Soc}(Q)$ has $\mathrm{Soc}(Q)$ as a composition factor, then $\mathcal{N} = \emptyset$. Hence, we have an isomorphism $Q(\tau\text{-tilt}\,\Lambda) \to Q(\tau\text{-tilt}\,\overline{\Lambda})$.

As an application, we can calculate the number of τ-tilting modules over linearly Dynkin type algebras whose square radical are zero.

Throughout this paper, all algebras will be basic, connected, finite dimensional K-algebras over an algebraically closed field K. For an algebra Λ, we denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated right Λ-modules and by τ the Auslander–Reiten translation of Λ. For $M \in \mathrm{mod}\,\Lambda$, we also denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M and by $\mathrm{add}\,M$ the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of finite direct sums of copies of M. For a set X, we denote by $|X|$ the cardinality of X. For two sets X, Y, $X \coprod Y$ means the disjoint union.

2. Main Results

Let Λ be an algebra. We always assume that Λ has an indecomposable projective-injective module Q and $\overline{\Lambda} := \Lambda/\mathrm{Soc}(Q)$. Considering the following functor:

$$\overline{(-)} := - \otimes_\Lambda \overline{\Lambda} : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\overline{\Lambda}.$$

Then we have $\overline{Q} = Q/\mathrm{Soc}(Q)$ and $\overline{M} \cong M$ for all indecomposable Λ-modules M which are not isomorphism to Q by [1, Proposition 3.1(2)]. We will denote by $\alpha(M)$ a basic Λ-module such that $\mathrm{add}(\alpha(M)) = \mathrm{add}\,\overline{M}$.

We need the following lemma.

Lemma 2.1. *Assume Q is not simple and $U \in \mathrm{mod}\,\overline{\Lambda}$ does not have $\overline{Q}$ as a direct summand. Then*

(1) *$U \in \tau\text{-tilt}\,\Lambda$ if and only if $U \in \tau\text{-tilt}\,\overline{\Lambda}$.*
(2) *$Q \oplus \overline{Q} \oplus U \in \tau\text{-tilt}\,\Lambda$ if and only if $\overline{Q} \oplus U \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ and $|\overline{Q} \oplus U| = |\Lambda| - 1$ if and only if $\overline{Q} \oplus U \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda}$, $|\overline{Q} \oplus U| = |\overline{\Lambda}| - 1$ and $\mathrm{Hom}_\Lambda(\overline{Q} \oplus U, Q) = 0$.*
(3) *$Q \oplus U \in \tau\text{-tilt}\,\Lambda$ if and only if $\overline{Q} \oplus U \in \tau\text{-tilt}\,\Lambda$ and $\mathrm{Hom}_\Lambda(\overline{Q} \oplus U, Q) \neq 0$.*

Proof. Note that a support τ-tilting Λ-module M is τ-tilting if and only if $|M| = |\Lambda|$. Hence (1), (2) and (3) follow from [1, Proposition 3.7]. □

Lemma 2.2. *Assume Q is not simple. We have*

$$\{M \in \tau\text{-tilt}\,\Lambda \,|\, Q \notin \mathrm{add}\,M, \overline{Q} \in \mathrm{add}\,M\} = \emptyset.$$

Proof. Let $M \in \{M \in \tau\text{-tilt}\,\Lambda \,|\, Q \notin \mathrm{add}\,M, \overline{Q} \in \mathrm{add}\,M\}$. Write $M = \overline{Q} \oplus X$ where X does not have $Q \oplus \overline{Q}$ as a direct summand, we have $Q \notin \mathrm{Fac}\,X$ since Q is projective. By [1, Proposition 3.7], $Q \oplus \overline{Q} \oplus X \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ and $|Q \oplus \overline{Q} \oplus X| = |\Lambda| + 1$ which implies $Q \in \mathrm{Fac}\, X$. This is a contradiction. $\square$

Now, we decompose $\tau\text{-tilt}\,\Lambda$ as the following three parts.

$$\mathcal{M}_1 := \{M \in \tau\text{-tilt}\,\Lambda \mid Q, \overline{Q} \notin \mathrm{add}\, M\},$$

$$\mathcal{M}_2 := \{M \in \tau\text{-tilt}\,\Lambda \mid Q, \overline{Q} \in \mathrm{add}\, M\},$$

$$\mathcal{M}_3 := \{M \in \tau\text{-tilt}\,\Lambda \mid Q \in \mathrm{add}\, M, \overline{Q} \notin \mathrm{add}\, M\}.$$

If $X \in \tau\text{-tilt}\,\overline{\Lambda}$, then X is sincere by [2, Proposition 2.2(a)]. Hence, X is a sincere Λ-module. Thus $\mathrm{Hom}_\Lambda(N, Q) \neq 0$ since Q is indecomposable injective. Therefore, $\{N \in \tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \mathrm{add}\, N, \mathrm{Hom}_\Lambda(N, Q) = 0\} = \emptyset$. The following proposition can be obtained by Lemma 2.1 immediately.

Proposition 2.1. *Assume that Q is not simple. Then there are bijections*

$$\mathcal{M}_1 \to \mathcal{N}_1, \quad \mathcal{M}_2 \to \mathcal{N}_2, \quad \mathcal{M}_3 \to \mathcal{N}_3$$

given by $M \to \alpha(M)$ where

$$\mathcal{N}_1 := \{N \in \tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \notin \mathrm{add}\, N\},$$

$$\mathcal{N}_2 := \mathcal{N} = \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \mathrm{add}\, N, \mathrm{Hom}_\Lambda(N, Q) = 0 \textit{ and } |N| = |\overline{\Lambda}| - 1\},$$

$$\mathcal{N}_3 = \{N \in \tau\text{-tilt}\,\overline{\Lambda} \mid \overline{Q} \in \mathrm{add}\, N, \mathrm{Hom}_\Lambda(N, Q) \neq 0\}.$$

In particular, there is a bijection

$$\alpha : \tau\text{-tilt}\,\Lambda \to \tau\text{-tilt}\,\overline{\Lambda} \coprod \mathcal{N}.$$

Corollary 2.1. *We have*

$$\tau\text{-tilt}\,\Lambda = \{N \mid N \in \mathcal{N}_1\} \coprod \{Q \oplus N \mid N \in \mathcal{N}_2\} \coprod \{Q \oplus (N/\overline{Q}) \mid N \in \mathcal{N}_3\}.$$

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (1) It is clearly since $\tau\text{-tilt}\,\Lambda = \{Q \oplus M \mid M \in \tau\text{-tilt}\,\overline{\Lambda}\}$ where Q is a simple projective-injective Λ-module.

(2) By Proposition 2.2, we have a bijection

$$\alpha : \tau\text{-tilt}\,\Lambda \to \tau\text{-tilt}\,\overline{\Lambda} \coprod \mathcal{N}.$$

We only need to show that, for any $M, L \in \tau\text{-tilt}\,\Lambda$, $M \geq L$ if and only if $\alpha(M) \geq \alpha(L)$. In fact, if $M \geq L$, then $L \in \mathrm{Fac}\, M$ and we have $\overline{L} \in \mathrm{Fac}\, \overline{M}$ which implies $\alpha(M) \geq \alpha(L)$. Conversely, let $\alpha(M) \geq \alpha(L)$. If both M and L are in $\mathcal{M}_i (i = 1, 2, 3)$, then it is clear that $M \geq L$. Otherwise,

Case 1. If $L \in \mathcal{M}_1$, then $\alpha(L) = L$. Note that $\overline{M} \in \mathrm{Fac}\, M$, we have $M \geq \alpha(M) \geq \alpha(L) = L$.

Case 2. If $L \in \mathcal{M}_2$, then $M \in \mathcal{M}_3$ since $\mathcal{N}_1$ has no $\overline{Q}$ as a direct summand. Thus $\alpha(L) = Q \oplus \overline{L} \in \operatorname{Fac} M$ because $\mathcal{M}_3$ has Q as a direct summand.

Case 3. If $L \in \mathcal{M}_3$, then $M \notin \mathcal{M}_1$ since $\mathcal{N}_1$ has no $\overline{Q}$ as a direct summand. Assume $M \in \mathcal{M}_2$. Then $\alpha(M)$ has no $\operatorname{Soc}(Q)$ as a composition factor and $\alpha(L)$ has $\operatorname{Soc}(Q)$ as a composition factor. This is a contradiction with $\alpha(L) \in \operatorname{Fac} \alpha(M)$.

Thus the assertion follows. □

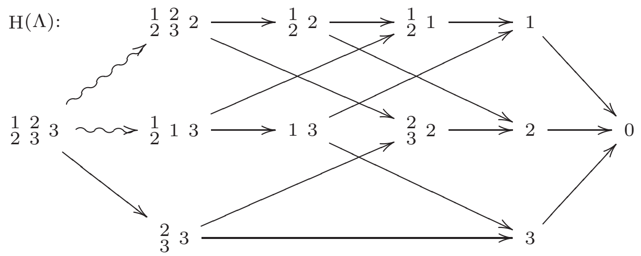
We illustrate Theorem 1.1 with the following example.

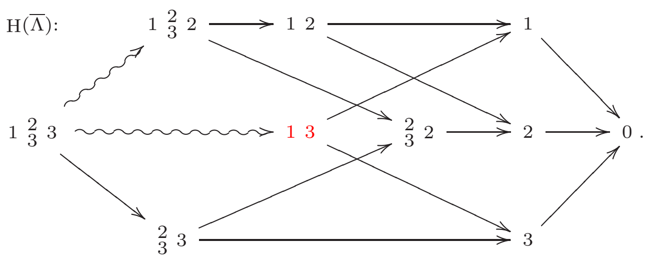
Example 2.1. Let Λ be a finite dimensional K-algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

with the relation $\alpha\beta = 0$. Take $Q = P_1$. Then Q is an indecomposable projective-injective module. The algebra $\overline{\Lambda}$ given by the quiver $1 \ 2 \xrightarrow{\beta} 3$.

We draw the Hasse quivers $\mathrm{H}(\Lambda)$ and $\mathrm{H}(\overline{\Lambda})$ as follows:





We draw those arrows in $Q(\tau\text{-tilt}\,\Lambda)$ and $Q(\tau\text{-tilt}\,\overline{\Lambda} \coprod \mathcal{N})$ by $\rightsquigarrow$ and $\mathcal{N}$ is marked by red.

Considering the following quivers:

$$A_n : n \longrightarrow n-1 \longrightarrow \cdots \longrightarrow 2 \longrightarrow 1$$

$$\begin{tikzcd} & & & & 1 \\ D_n : n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\ & & & & 2\,. \end{tikzcd}$$

Take $A_n^2 := KA_n/\mathrm{rad}^2$ and $D_n^2 := KD_n/\mathrm{rad}^2$. Applying our results, we can give a recurrence relation about the numbers of τ-tilting modules over A_n^2 and D_n^2.

Theorem 2.1. *Let Λ_n^2 be an algebra (A_n^2 or D_n^2). Then we have*

$$|\tau\text{-tilt}\,\Lambda_n^2| = |\tau\text{-tilt}\,\Lambda_{n-1}^2| + |\tau\text{-tilt}\,\Lambda_{n-2}^2|.$$

Proof. Take $Q = P_n$ which is an indecomposable projective-injective Λ_n^2-module. Since $\mathrm{Soc}(Q) \cong S_{n-1}$, we have $\overline{\Lambda_n^2} = \Lambda_n^2/S_{n-1} \cong \Lambda_{n-1}^2 \times K$ and $Q/S_{n-1} \cong S_n$. Hence

$$\begin{aligned}
\mathcal{N} &= \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{\Lambda_n^2} \mid Q/\mathrm{Soc}(Q) \in \mathrm{add}\,N, \mathrm{Hom}_{\Lambda_n^2}(N,Q) = 0 \text{ and } |N| = n-1\} \\
&= \{N \in \mathrm{s}\tau\text{-tilt}(\Lambda_{n-1}^2 \times K) \mid S_n \in \mathrm{add}\,N, \mathrm{Hom}_{\Lambda_n^2}(N,P_n) = 0 \text{ and } |N| = n-1\} \\
&= \{S_n \oplus L \mid L \in \mathrm{silt}\,\Lambda_{n-1}^2, \mathrm{Hom}_{\Lambda_n^2}(L,P_n) = 0 \text{ and } |L| = n-2\} \\
&= \{S_n \oplus L \mid L \in \mathrm{silt}\,\Lambda_{n-1}^2, \mathrm{Hom}_{\Lambda_{n-1}^2}(L,S_{n-1}) = 0 \text{ and } |L| = n-2\} \\
&= \{S_n \oplus L \mid L \in \mathrm{silt}\,\Lambda_{n-2}^2 \text{ and } |L| = n-2\} \\
&= \{S_n \oplus L \mid L \in \tau\text{-tilt}\,\Lambda_{n-2}^2\}.
\end{aligned}$$

By Theorem 1.1, there is a bijection $Q(\tau\text{-tilt}\,\Lambda_n^2) \to Q(\tau\text{-tilt}(\Lambda_{n-1}^2 \times K) \coprod \mathcal{N})$. Thus

$$|\tau\text{-tilt}\,\Lambda_n^2| = |\tau\text{-tilt}(\Lambda_{n-1}^2 \times K)| + |\mathcal{N}| = |\tau\text{-tilt}\,\Lambda_{n-1}^2| + |\tau\text{-tilt}\,\Lambda_{n-2}^2|. \qquad \square$$

Corollary 2.2.

(1) $|\tau\text{-tilt}\,A_n^2| = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{\sqrt{5}\cdot 2^{n+1}}$.
(2) $|\tau\text{-tilt}\,D_n^2| = \frac{(2\sqrt{5}-1)(1+\sqrt{5})^{n-1}+(2\sqrt{5}+1)(1-\sqrt{5})^{n-1}}{\sqrt{5}\cdot 2^{n-1}}$.

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