τ-Tilting modules over one-point extensions by a simple module at a source point *

Hanpeng Gao

Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China

Abstract

Let B be an one-point extension of a finite dimensional k-algebra A by a simple A-module at a source point i. In this paper, we classify the τ-tilting modules over B. Moreover, it is shown that there are equations

$$|\,\tau\text{-tilt}\,B| = |\,\tau\text{-tilt}\,A| + |\,\tau\text{-tilt}\,A/\langle e_i\rangle| \quad \text{and} \quad |\,\mathrm{s}\tau\text{-tilt}\,B| = 2|\,\mathrm{s}\tau\text{-tilt}\,A| + |\,\mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle|.$$

As a consequence, we can calculate the numbers of τ-tilting modules and support τ-tilting modules over linearly Dynkin type algebras whose square radical are zero.

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1 Introduction

As a generalization of tilting module, the concept of support τ-tilting modules is introduced by Adachi, Iyama and Reiten[2]. They are very important in representation theory of algebras because they are in bijection with some important objects including functorially finite torsion classes, 2-term silting complexes, cluster-tilting objects. It is very interesting to calculate the number of support τ-tilting modules over a given algebra.

For Dynkin type algebras Δ_n, the numbers of tilting modules and support tilting modules were first calculated in [6] via cluster algebras, and later in [8] via representation theory.

Recall that a finite-dimensional K-algebra is said to be a *Nakayama algebra* if every indecomposable projective module and every indecomposable injective module has a unique composition series. May authors calculate the numbers of τ-tilting modules and support τ-tilting modules over Nakayama algebras. In particular, for square radical zero Nakayama algebra Λ_n^2 with n simple modules, there are the following recurrence relations (see, [3, 4, 5]),

$$|\,\tau\text{-tilt}\,\Lambda_n^2| = |\,\tau\text{-tilt}\,\Lambda_{n-1}^2| + |\,\tau\text{-tilt}\,\Lambda_{n-2}^2| \quad \text{and} \quad |\,\mathrm{s}\tau\text{-tilt}\,\Lambda_n^2| = 2|\,\mathrm{s}\tau\text{-tilt}\,\Lambda_{n-}^2| + |\,\mathrm{s}\tau\text{-tilt}\,\Lambda_{n-2}^2|.$$

E-mail: hpgao07@163.com

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In this paper, we consider τ-tilting modules and support τ-tilting modules over the one-point extension B of A by a simple A-module at a source point i. We will show that there is a bijection

$$\tau\text{-tilt}\, B \mapsto \tau\text{-tilt}\, A \coprod \tau\text{-tilt}\, A/\langle e_i\rangle.$$

We also get the following equations,

$$|\,\tau\text{-tilt}\, B| = |\,\tau\text{-tilt}\, A| + |\,\tau\text{-tilt}\, A/\langle e_i\rangle| \quad \text{and} \quad |\,\mathrm{s}\tau\text{-tilt}\, B| = 2|\,\mathrm{s}\tau\text{-tilt}\, A| + |\,\mathrm{s}\tau\text{-tilt}\, A/\langle e_i\rangle|.$$

As an application , we can calculate the numbers of τ-tilting modules and support τ-tilting modules over linearly Dynkin type algebras whose square radical are zero.

Throughout this paper, all algebras will be basic, connected, finite dimensional K-algebras over an algebraically closed field K. For an algebra A, we denote by $\mathrm{mod}\, A$ the category of finitely generated left A-modules and by τ the Auslander-Reiten translation of A. Let P_i be the indecomposable projective module and S_i the simple module of A corresponding to the point i for $i = 1, 2, \cdots, n$. For $M \in \mathrm{mod}\, A$, we also denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M and by $\mathrm{add}\, M$ the full subcategory of $\mathrm{mod}\, A$ consisting of direct summands of finite direct sums of copies of M. For a set X, we denote by $|X|$ the cardinality of X. For two sets X, Y, $X \coprod Y$ means the disjoint union.

2 Main results

Let A be an algebra. we recall the definiition about support τ-tilting modules.

Definition 2.1. ([2, Definition 0.1]) *Let $M \in \mathrm{mod}\, A$.*

(1) M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.

(2) M is τ-tilting if it is τ-rigid and $|M| = |A|$.

(3) M is support τ-tilting if it is a τ-tilting A/AeA-module for some idempotent e of A.

We will denote by $\tau\text{-tilt}\, A$ (respectively, $\mathrm{s}\tau\text{-tilt}\, A$) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) A-modules.

Let $X \in \mathrm{mod}\, A$. The *one-point extension* of A by X is defined as the following matrix algebra

$$B = \begin{pmatrix} A & X \\ 0 & k \end{pmatrix}$$

with the ordinary matrix addition and the multiplication, and we write $B := A[X]$ with a the extension point.

Let A be an algebra with a source point i, in this paper, we always assume $B := A[S_i]$. In this case, P_a is an indecomposable projective-injective B-module and S_a is a simple injective B-module by [1, Proposition 2.5(c)].

Lemma 2.2. *Let $M \in \mathrm{mod}\, B$. If M is τ-rigid, then $M \oplus P_a$ is also. Moreover, if M is τ-tilting, then it have P_a as a direct summand.*

Proof. Since S_i is simple, there are only two indecomposable B-modules P_a, S_a which have S_a as a composition factor and they are injective, we have τM have no S_1 as a composition factor.Thus, $\mathrm{Hom}_A(P_a, \tau M) = 0$ and, we get $M \oplus P_a$ is τ-rigid. If M is τ-tilting, then it is maximal τ-rigid by [2, Theorem 2.12]. Hence, it have P_a as a direct summand. $\square$

Theorem 2.3. *There is a bijection*

$$\tau\text{-tilt}\, B \longleftrightarrow \tau\text{-tilt}\, A \coprod \tau\text{-tilt}\, A/\langle e_i \rangle.$$

Proof. Let $M \in \tau\text{-tilt}\, A \coprod \tau\text{-tilt}\, A/\langle e_i \rangle$. If $M \in \tau\text{-tilt}\, A$, then τM has no S_a as a composition factor since the vertex a is a source in B, and hence $\mathrm{Hom}_B(P_a, \tau M) = 0$. Therefore, $M \oplus P_a$ is a τ-tilting B-module since it is τ-rigid and $|M \oplus P_a| = |M| + 1 = |A| + 1 = |B|$. If $M \in \tau\text{-tilt}\, A/\langle e_i \rangle$, then M has no S_i as a composition factor and τM has no S_a as a composition factor. Note that there is an almost split sequence $0 \to S_i \to P_a \to S_a \to 0$, we have $\tau S_a = S_i$. Thus,

$$\mathrm{Hom}_B(M \oplus P_a \oplus S_a, \tau(M \oplus P_1 \oplus S_a)) = \mathrm{Hom}_B(M \oplus P_a \oplus S_a, \tau M \oplus S_i) = 0.$$

So, $M \oplus P_a \oplus S_a$ is a τ-tilting B-module.

Conversely, Let $M \in \tau\text{-tilt}\, B$. Then we decompose M as $M = P_a \oplus N$ by Lemma 2.2. If N has no S_a as direct summand, the N is a τ-tilting $B/\langle e_a \rangle (\cong A)$-module, that is, $N \in \tau\text{-tilt}\, A$. If N has S_a as direct summand, then we decompose N as $N = S_a \oplus L$ where L has no S_a as a composition factor. We claim that L has no S_i as a composition factor. Otherwise, there is a summand K of L such that the top of K is S_i since i is a source point. In particular, $\mathrm{Hom}_B(L, S_i) \neq 0$. This implies

$$\mathrm{Hom}_B(M, \tau M) = \mathrm{Hom}_B(L \oplus P_a \oplus S_a, \tau L \oplus S_i) \neq 0.$$

This is a contradiction. Hence, L is a τ-tilting $A/\langle e_i \rangle$-module , that is, $L \in \tau\text{-tilt}\, A/\langle e_i \rangle$. $\square$

Corollary 2.4. *All τ-tilting B-modules are exactly those forms $P_a \oplus M_1$ and $P_a \oplus S_a \oplus M_2$ where M_1 and M_2 are τ-tilting modules over A and $A/\langle e_i \rangle$ respectively.*

The above Corollary give a relation about $|\tau\text{-tilt}\, B|$ and $|\tau\text{-tilt}\, A|$.

Corollary 2.5. *We have*

$$|\tau\text{-tilt}\, B| = |\tau\text{-tilt}\, A| + |\tau\text{-tilt}\, A/\langle e_i \rangle|.$$

Let A be an algebra and $M \in \mathrm{mod}\, A$. M is called a (classical) *tilting* module if

(1) The projective dimension of M is at most one.

(2) $\mathrm{Ext}^1_A(M, M) = 0$.

(3) $|M| = |A|$.

Hence, an A-module M is tilting if and only if it is a τ-tilting and its projective dimension is at most one by the Auslander-Reiten formula. The set of all tilting A-modules will be denoted by $\mathrm{tilt}\, A$.

Corollary 2.6. *Let A be an algebra with a source i. Assume that i is not a sink and $B := A[S_i]$. All tilting B-modules are exactly those forms $P_a \oplus M_1$ where M_1 is a tilting module over A. In particular, $|\operatorname{tilt} B| = |\operatorname{tilt} A|$.*

Proof. By Corollary 2.4, All τ-tilting B-modules are exactly those forms $P_a \oplus M_1$ and $P_a \oplus S_a \oplus M_2$ where M_1 and M_2 are τ-tilting modules over A and $A/\langle e_i \rangle$ respectively. Note that the projective dimension of M_1 as A-module is equal to the projective dimension of M_1 as B-module since a is a source of B. Hence $P_a \oplus M_1$ is a tilting B-module if and only if M_1 is a tilting A-module.

Since there is an exact sequence $0 \to S_i \to P_a \to S_a \to 0$ in $\operatorname{mod} B$, we have the projective dimension of S_a is at most two since S_i is not projective when i is not a sink. Hence $P_a \oplus S_a \oplus M_2$ is not tilting. Thus, $|\operatorname{tilt} B| = |\operatorname{tilt} A|$. $\square$

Example 2.7. Let B be a algebra given by the quiver

$$\begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

with $\operatorname{rad}^2 = 0$. Assume that A is the path algebra given by the quiver $3 \leftarrow 2 \rightarrow 4$, we have $B = A[2]$. There are five τ-tilting A-modules as follows (there are exactly all tilting-A-modules since A is hereditary)

$$3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad 3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 2, \quad 2\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}.$$

We only have one τ-tilting $A/\langle e_2 \rangle$-module $3\ 4$. Hence, we get all τ-tilting B-modules by Corollary 2.4.

$$\begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 2, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 2\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 1\ 3\ 4.$$

By Corollary 2.6,

$$\begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ 2, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 2\ \begin{smallmatrix}&2&\\3&&4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}$$

are all tilting B-modules

Next, we will consider the relationship between $\operatorname{s\tau-tilt} B$ and $\operatorname{s\tau-tilt} A$. We need the following notions.

Let A be an algebra. The *support τ-tilting quiver*(or Hasse quiver) $\mathrm{H}(A)$ of A is defined as follows (more detail can be found [2, Definition 2.29])

- vertices : the isomorphisms classes of basic support τ-tilting A-modules.
- arrows: from a module to its left mutation.

It is well known that $\mathrm{H}(A)$ is a poset. Let $\mathcal{N}$ be a subposet of $\mathrm{H}(A)$ and $\mathcal{N}' := \mathrm{H}(A) \backslash \mathcal{N}$. We define a new quiver $\mathrm{H}(A)^{\mathcal{N}}$ from $\mathrm{H}(A)$ as follows.

- vertices : vertices in $\mathrm{H}(A)$ and $\mathcal{N}^+$ where $\mathcal{N}^+$ is a copy of $\mathcal{N}$.
- arrows: $\{a_1 \to a_2 \mid a_1 \to a_2 \in \mathcal{N}'\} \coprod \{n_2 \to a_2 \mid n_2 \to a_2, n_2 \in \mathcal{N}, a_2 \in \mathcal{N}'\}$
$\coprod \{n_1 \to n_2, n_1^+ \to n_2^+ \mid n_1 \to n_2 \in \mathcal{N}\} \coprod \{a_1 \to n_1^+ \mid a_1 \to n_1, n_1 \in \mathcal{N}, a_1 \in \mathcal{N}'\}$
$\coprod \{n_1^+ \to n_1 \mid n_1 \in \mathcal{N}\}$.

$$
\begin{tikzcd}
a_1 \arrow[dd] \arrow[r] & n_1 \arrow[d] \\
 & n_2 \arrow[dl] \\
a_2 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a_1 \arrow[ddd] \arrow[rr] & & n_1^+ \arrow[dl] \arrow[dr] & \\
 & n_1 \arrow[dr] & & n_2^+ \arrow[dl] \\
 & & n_2 \arrow[dll] & \\
a_2 & & &
\end{tikzcd}
.
$$

$\mathrm{H}(A)$ $\qquad\qquad$ $\mathrm{H}(A)^{\mathcal{N}}$

Suppose that A is an algebra with an indecomposable projective-injective module Q. Let $\overline{A} := A/\operatorname{soc}(Q)$ and

$$\mathcal{N} := \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{A} \mid Q/\operatorname{soc}(Q) \in \operatorname{add} N \text{ and } \operatorname{Hom}_A(N, Q) = 0\}.$$

The following Lemma can be found in [4, Theorem 3.3].

Lemma 2.8. *Let A be an algebra with an indecomposable projective-injective module Q. Then there is an isomorphism of posets*

$$\mathrm{H}(A) \longleftrightarrow \mathrm{H}\,(\overline{A})^{\mathcal{N}}.$$

Applying this result to the algebra B, we have the following

Proposition 2.9. *Let $\mathcal{N} := \{S_a \oplus L \mid L \in \mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle\}$. Then there is an isomorphism of posets*

$$\mathrm{H}(B) \longleftrightarrow \mathrm{H}\,(A \times k)^{\mathcal{N}}.$$

Proof. Take $Q = P_a$ which is an indecomposable projective-injective B-module. Since $\operatorname{soc} P_a \cong S_i$, we have $\overline{B} = B/S_i \cong A \times k$ and $P_a/S_i \cong S_a$. We only need to show $\mathcal{N} = \{S_a \oplus L \mid L \in \mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle\}$ in Lemma 2.8. Note that

$$
\begin{aligned}
\mathcal{N} &= \{N \in \mathrm{s}\tau\text{-tilt}\,\overline{B} \mid Q/\operatorname{soc}(Q) \in \operatorname{add} N \text{ and } \operatorname{Hom}_B(N, Q) = 0\} \\
&= \{N \in \mathrm{s}\tau\text{-tilt}(A \times k) \mid S_a \in \operatorname{add} N \text{ and } \operatorname{Hom}_B(N, P_a) = 0\} \\
&= \{S_a \oplus L \mid L \in \operatorname{silt} A \text{ and } \operatorname{Hom}_B(L, P_a) = 0\} \\
&= \{S_a \oplus L \mid L \in \operatorname{silt} A \text{ and } \operatorname{Hom}_A(L, S_i) = 0\}.
\end{aligned}
$$

Since i is a source point of A, this implies L has no S_i as a composition factor and hence it is exactly a support τ-tilting $A/\langle e_i\rangle$-module. Thus, $\mathcal{N} = \{S_a \oplus L \mid L \in \mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle\}$. $\square$

Corollary 2.10. *We have*

$$|\,\mathrm{s}\tau\text{-tilt}\,B| = 2|\,\mathrm{s}\tau\text{-tilt}\,A| + |\,\mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle|.$$

Proof. According to the definition of $\mathrm{H}\,(A \times k)^{\mathcal{N}}$, we have

$$|\,\mathrm{H}\,(A \times k)^{\mathcal{N}}| = |\,\mathrm{H}\,(A \times k)| + |\mathcal{N}| = 2|\,\mathrm{H}(A)| + |\mathcal{N}| = 2|\,\mathrm{s}\tau\text{-tilt}\,A| + |\,\mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle|.$$

Therefore, $|\,\mathrm{s}\tau\text{-tilt}\,B| = |\,\mathrm{H}(B)| = 2|\,\mathrm{s}\tau\text{-tilt}\,A| + |\,\mathrm{s}\tau\text{-tilt}\,A/\langle e_i\rangle|$ by Proposition 2.9. $\square$

Now, it is easy to draw the quiver of $\mathrm{H}(B)$ from the quiver of $\mathrm{H}(A)$ as follows.

$$\mathrm{H}(A) \to \mathrm{H}(A \times k) \to \mathrm{H}\left(A \times k\right)^{\mathcal{N}} \cong \mathrm{H}(B).$$

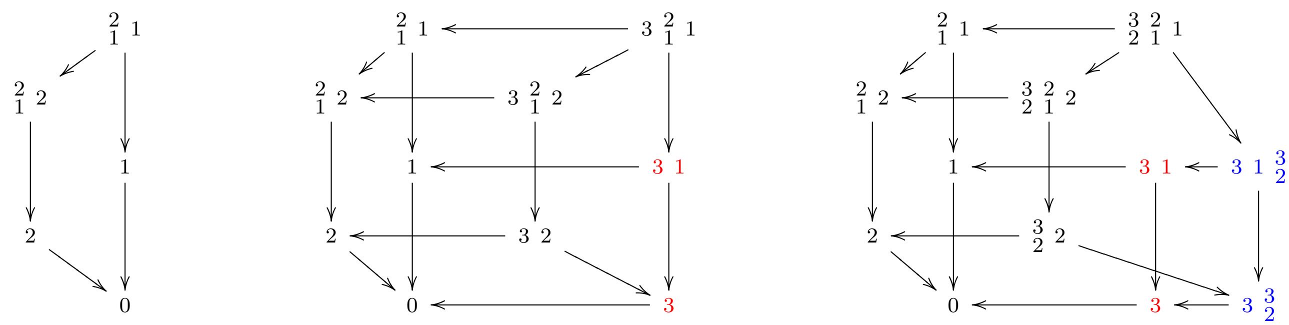
Example 2.11. Let A be a finite dimensional k-algebra given by the quiver

$$2 \longrightarrow 1.$$

Considering the one-point extension of A by the simple module corresponding to the point 2, the algebra $B = A[2]$ is given by the quiver

$$3 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 1$$

with the relation $\alpha\beta = 0$. We can get the Hasse quiver $\mathrm{H}(B)$ of B as follows where $\mathcal{N}$ is remarked by red and $\mathcal{N}^+$ by blue.



$$\mathrm{H}(A) \longrightarrow \mathrm{H}(A \times k) \longrightarrow \mathrm{H}(B)$$

The linearly Dynkin type algebras be the following quivers.

$$A_n: \quad n \to n-1 \to \cdots \to 2 \to 1$$

$$D_n: \quad \begin{tikzcd}[row sep=small] & & & & 1 \\ n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & 3 \arrow[ur] \arrow[dr] & \\ & & & & 2 \end{tikzcd}$$

Take $A_n^2 := kA_n/\operatorname{rad}^2$ and $D_n^2 := kD_n/\operatorname{rad}^2$. Applying our results, we can give recurrence relations about the numbers of τ-tilting modules and support τ-tilting modules over A_n^2 and D_n^2.

Theorem 2.12. *Let Λ_n^2 be an algebra (A_n^2 or D_n^2). Then we have*

(1) $|\tau\text{-tilt}\,\Lambda_n^2| = |\tau\text{-tilt}\,\Lambda_{n-1}^2| + |\tau\text{-tilt}\,\Lambda_{n-2}^2|$.

(2) $|\mathrm{s}\tau\text{-tilt}\,\Lambda_n^2| = 2|\mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1}^2| + |\mathrm{s}\tau\text{-tilt}\,\Lambda_{n-2}^2|$.

Proof. Since Λ_n^2 is the one-point extension of Λ_{n-1}^2 by simple module S_{n-1} and $\Lambda_{n-1}^2/\langle e_{n-1}\rangle \cong \Lambda_{n-2}^2$. Now, the result follows from Corollary 2.5 and Corollary 2.10. $\square$

Corollary 2.13.

(1) $|\operatorname{tilt} A_n^2| = 2 \quad (n \geqslant 2)$.

(2) $|\tau\text{-tilt}\, A_n^2| = \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{\sqrt{5}\cdot 2^{n+1}}$.

(3) $|\mathrm{s}\tau\text{-tilt}\, A_n^2| = \frac{(1+\sqrt{2})^{n}-(1-\sqrt{2})^{n}}{2\sqrt{2}}$.

(4) $|\operatorname{tilt} D_n^2| = 5$.

(5) $|\tau\text{-tilt}\, D_n^2| = \frac{(2\sqrt{5}-1)(1+\sqrt{5})^{n-1}+(2\sqrt{5}+1)(1-\sqrt{5})^{n-1}}{\sqrt{5}\cdot 2^{n-1}}$.

(6) $|\mathrm{s}\tau\text{-tilt}\, D_n^2| = \frac{(3\sqrt{2}-1)(1+\sqrt{2})^{n-1}+(3\sqrt{2}+1)(1-\sqrt{2})^{n-1}}{\sqrt{2}}$.

Example 2.14. *We give some examples of the numbers of τ-tilting modules and support τ-tilting modules over A_n^2 and D_n^2 in the following tables.*

n	1	2	3	4	5	6	7	8	9	10
$\lvert\tau\text{-tilt}\, A_n^2\rvert$	1	2	3	5	8	13	21	34	55	89
$\lvert\mathrm{s}\tau\text{-tilt}\, A_n^2\rvert$	2	5	12	29	70	169	408	985	2378	5741

n	4	5	6	7	8	9	10
$\lvert\tau\text{-tilt}\, D_n^2\rvert$	6	11	17	28	45	73	118
$\lvert\mathrm{s}\tau\text{-tilt}\, D_n^2\rvert$	32	78	118	454	1026	2506	6038

References

[1] Auslander, M. Reiten, I. and Smalø, S. O., Representation Theory of Artin Algebras, Corrected reprint of the 1995 original, Cambridge Stud. in Adv. Math. 36, Cambridge Univ. Press, Cambridge, 1997.

[2] Adachi, T., Iyama, O. and Reiten, I., τ-tilting theory. Compos. Math., 50: 415–452 (2014)

[3] Asai, S., Semibricks. Int. Math. Res. Not. IMRN., 00(0): 1–62 (2018)

[4] Adachi, T., The classification of τ-tilting modules over Nakayama algebras. J. Algebra, 452: 227–262 (2016)

[5] Gao, H. and Schiffler, R., On the number of τ-tilting modules over Nakayama algebras. SIGMA, 16, 058, 13 pages (2020)

[6] Fomin, S. and Zelevinsky, A., Y-sytstems and generalized associahedra. Ann. Math., 158: 977–1018 (2003)

[7] Keller, B. and Vossieck, D., Aisles in derived categories. Bull. Soc. Math. Belg. Sér. A , 40(2): 239–253 (1988)

[8] Obaid, A., Nauman, S. K. ,Fakieh, W. M. and Ringel, C. M. , The number of support-tilting modules for a Dynkin algebra. J. Integer Seq., 18(10), Article 15.10.6, 24 pp.(2015)