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SEMIBRICKS OVER SPLIT-BY-NILPOTENT EXTENSIONS

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Abstract. In this paper, we prove that there is a bijection between the τ-tilting modules and the sincere left finite semibricks. We also construct (sincere) semibricks over split-by-nilpotent extensions. More precisely, let Γ be a split-by-nilpotent extension of a finite-dimensional algebra Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$, and $\mathcal{S} \subseteq \operatorname{mod}\Lambda$. We prove that $\mathcal{S} \otimes_\Lambda \Gamma$ is a (sincere) semibrick in $\operatorname{mod}\Gamma$ if and only if $\mathcal{S}$ is a semibrick in $\operatorname{mod}\Lambda$ and $\operatorname{Hom}_\Lambda(\mathcal{S}, \mathcal{S} \otimes_\Lambda E) = 0$ (and $\mathcal{S} \cup \mathcal{S} \otimes_\Lambda E$ is sincere). As an application, we can construct τ-tilting modules and support τ-tilting modules over τ-tilting finite cluster-tilted algebras.

1. Introduction

Simple modules and semisimple modules are fundamental in the representation theory of a finite dimensional K-algebra Λ. Let S_1, S_2 be two nonisomorphic simple Λ-modules. Schur's Lemma shows that they have the following properties (1) $\operatorname{Hom}_\Lambda(S_i, S_i)$ is a K-division algebra, (2) $\operatorname{Hom}_\Lambda(S_i, S_j) = 0, i \neq j$.

As generalizations of simple modules and semisimple modules, bricks and semibricks are considered and they have long been studied in [9, 12]. A Λ-module M is called a *brick* if $\operatorname{Hom}_\Lambda(M, M)$ is a K-division algebra and a *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Ringel [12] has given a classical result that there is a bijection between the semibricks in $\operatorname{mod}\Lambda$ and the wide subcategories of $\operatorname{mod}\Lambda$ (that is, the subcategories of $\operatorname{mod}\Lambda$ which are closed under taking kernels, cokernels, and extensions). Marks and Šťovíček [11] consider the relationship between wide subcategories of $\operatorname{mod}\Lambda$ and torsion classes of $\operatorname{mod}\Lambda$. In fact, they establish a bijection from functorially finite torsion classes to functorially finite wide subcategories.

In 2014, Adachi, Iyama and Reiten [1] introduced τ-*rigid* modules and it support τ-*tilting modules* which generalize rigid modules and classical tilting

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modules. They also showed that the support τ-tilting modules correspond bijectively to the functorially finite torsion classes. Demont, Iyama and Jasso [8] obtained a bijection from the set of indecomposable τ-rigid modules to the set of isomorphism classes of bricks $\mathcal{B}$ of Λ such that the smallest torsion class $T(\mathcal{B})$ containing $\mathcal{B}$ functorially finite. In [2], the author called a semibrick $\mathcal{S}$ *left finite* if the smallest torsion class $T(\mathcal{S})$ containing $\mathcal{S}$ is functorially finite and he also proved that there exists a bijection between the set $\mathrm{s}\tau\text{-tilt}\,\Lambda$ of support τ-tilting Λ-modules and the set $\mathrm{f_L}\text{-sbrick}\,\Lambda$ of left finite semibricks of Λ given by $M \mapsto \mathrm{ind}(M/\operatorname{rad}_B M)$ where $B = \mathrm{End}_\Lambda(M)$.

Recall that a Λ-module M is called *sincere* if every simple Λ-module appears as a composition factor in M. A τ-tilting Λ-module is exactly sincere support τ-tilting. In this paper, we say that a subset $\mathcal{S}$ of $\mathrm{mod}\,\Lambda$ is sincere if $\mathcal{S}^{\oplus}$ is sincere where $\mathcal{S}^{\oplus}$ stands for the direct sum of all modules in $\mathcal{S}$. Let $\tau\text{-tilt}\,\Lambda$ be the set of all τ-tilting Λ-modules and $\mathrm{sf_L}\text{-sbrick}\,\Lambda$ the set of sincere left finite semibricks in $\mathrm{mod}\,\Lambda$. Our first result is as follows.

Theorem 1.1. *There exists a bijection*

$$\tau\text{-tilt}\,\Lambda \to \mathrm{sf_L}\text{-sbrick}\Lambda$$

defined as $M \mapsto \mathrm{ind}(M/\operatorname{rad}_B M)$.

Let Γ be a split extension of an algebra Λ by a nilpotent bimodule ${}_\Lambda E_\Lambda$, that is, there exists a split surjective algebra morphism $\Gamma \to \Lambda$ whose kernel E is contained in the radical of Γ [4, 5]. Next, we will consider how to construct the (sincere) semibricks in $\mathrm{mod}\,\Gamma$ from the semibricks in $\mathrm{mod}\,\Lambda$. We obtain a sufficient and necessary condition such that $\mathcal{S} \otimes_\Lambda \Gamma$ is a (sincere) semibrick where $\mathcal{S} \subseteq \mathrm{mod}\,\Lambda$.

Theorem 1.2. *Let* Γ *be a split-by-nilpotent extension of an algebra* Λ *by* ${}_\Lambda E_\Lambda$ *and* $\mathcal{S} \subseteq \mathrm{mod}\,\Lambda$. *Then we have*

(1) $\mathcal{S} \otimes_\Lambda \Gamma$ *is a semibrick in* $\mathrm{mod}\,\Gamma$ *if and only if* $\mathcal{S}$ *is a semibrick in* $\mathrm{mod}\,\Lambda$ *and* $\mathrm{Hom}_\Lambda(\mathcal{S}, \mathcal{S} \otimes_\Lambda E) = 0$.

(2) $\mathcal{S} \otimes_\Lambda \Gamma$ *is a sincere semibrick in* $\mathrm{mod}\,\Gamma$ *if and only if* $\mathcal{S}$ *is a semibrick in* $\mathrm{mod}\,\Lambda$, $\mathrm{Hom}_\Lambda(\mathcal{S}, \mathcal{S} \otimes_\Lambda E) = 0$ *and* $\mathcal{S} \cup \mathcal{S} \otimes_\Lambda E$ *is sincere.*

As a consequence, we get that if Γ is a cluster-tilted algebra corresponding to a tilted algebra Λ and $\mathcal{S} \subseteq \mathrm{mod}\,\Lambda$, then $\mathcal{S} \otimes_\Lambda \Gamma$ is a (sincere) semibrick in $\mathrm{mod}\,\Gamma$ if and only if $\mathcal{S}$ is a semibrick in $\mathrm{mod}\,\Lambda$ and $\mathrm{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$ (and $\mathcal{S} \cup \mathcal{S} \otimes_\Lambda E$ is sincere) (see Theorem 3.8).

As an application, we can construct τ-tilting modules and support τ-tilting modules over τ-tilting finite cluster-tilted algebras (they are exactly representation finite cluster-tilted algebras).

The paper is organized as follows. In Section 2, we recall several definitions and results of semibricks and support τ-tilting modules. We study the relationship between semibricks and τ-tilting modules, and then construct the (sincere) semibricks over split-by-nilpotent extensions in Section 3. In Section

4, applying our results to τ-tilting finite cluster-tilted algebras , we construct τ-tilting modules and support τ-tilting modules over them. Finally, we give an example to illustrate our results in Section 5.

Throughout this paper, all algebras will be basic, connected, and finite dimensional K-algebras over an algebraically closed field K. Let Λ be an algebra. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated right Λ-modules and by τ the Auslander-Reiten translation of Λ. For $M \in \operatorname{mod}\Lambda$, we denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M, denote by $\operatorname{ind}(M)$ the set of isoclasses of indecomposable direct summands of M, and denote by $\operatorname{Fac} M$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of modules isomorphic to factor modules of finite direct sums of copies of M. The injective dimension and the first cosyzygy of M are denoted by $\operatorname{id}_\Lambda M$ and $\Omega^{-1}M$ respectively. For two sets $X_1, X_2 \subseteq \operatorname{mod}\Lambda$, we write

$$\operatorname{Hom}_\Lambda(X_1, X_2) = \{\operatorname{Hom}_\Lambda(M, N) \mid M \in X_1, N \in X_2\}.$$

2. Preliminaries

Let Λ be an algebra. In this section, we recall some definitions and facts about semibricks of Λ and support τ-tilting Λ-modules.

Definition ([2, Definition 1.1])**.** Let $S \in \operatorname{mod}\Lambda$.

(1) S is called a *brick* if $\operatorname{Hom}_\Lambda(S, S)$ is a division K-algebra. The set of isoclasses of bricks in $\operatorname{mod}\Lambda$ will be denoted by $\operatorname{brick}\Lambda$.
(2) A subset $\mathcal{S} \subseteq \operatorname{brick}\Lambda$ is called a *semibrick* if $\operatorname{Hom}_\Lambda(S_1, S_2) = 0$ for any $S_1 \neq S_2 \in \mathcal{S}$. The set of semibricks in $\operatorname{mod}\Lambda$ will be denoted by $\operatorname{sbrick}\Lambda$.

By Schur's Lemma, every simple module is a brick, and a set of isoclassees of simple modules is a semibrick.

Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod}\Lambda$. We say that $\mathcal{X}$ is *covariantly finite* if for any $M \in \operatorname{mod}\Lambda$, there exists a morphism $f_M : M \to X_M$ with $X_M \in \mathcal{X}$ such that any morphism $f : M \to X$ with $X \in \mathcal{X}$ factors through f_M. Dually, we can define the concept of *contravariantly finite* subcategories. $\mathcal{X}$ is called *functorially finite* if it is both covariantly finite and contravariantly finite. A full subcategory $\mathcal{T} \subseteq \operatorname{mod}\Lambda$ is said to be a *torsion class* if it is closed under images, direct sums, and extensions. We denote by f-torΛ the set of functorially finite torsion classes of $\operatorname{mod}\Lambda$.

Definition ([2, Definition 1.2(1)])**.** Let $\mathcal{S} \in \operatorname{sbrick}\Lambda$. $\mathcal{S}$ is called *left finite* if $T(\mathcal{S}) \in$ f-torΛ where $T(\mathcal{S})$ stand for the smallest torsion class containing $\mathcal{S}$.

We will write $\mathrm{f_L}$-sbrick Λ for the set of left finite semibricks in $\operatorname{mod}\Lambda$.

Definition ([1, Definition 0.1])**.** Let $M \in \operatorname{mod}\Lambda$.

(1) M is called *τ-rigid* if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$.
(2) M is called *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.

(3) M is called *support τ-tilting* if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ.

Recall that $M \in \operatorname{mod}\Lambda$ is called *sincere* if every simple Λ-module appears as a composition factor in M (equivalently, $\operatorname{Hom}_\Lambda(e_i\Lambda, M) \neq 0, i = 1, 2, \ldots, n$ where $\{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of Λ). It is well-known that the τ-tilting modules are exactly the sincere support τ-tilting modules [1, Proposition 2.2(a)].

We will denote by $\tau\text{-tilt}\,\Lambda$ (respectively, $\mathrm{s}\tau\text{-tilt}\,\Lambda$) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) Λ-modules.

The following result states the relationship between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $\mathrm{f_L}\text{-sbrick}\,\Lambda$.

Theorem 2.1 ([2, Theorem 1.3(2)])**.** *There exists a bijection*

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \to \mathrm{f_L}\text{-sbrick}\,\Lambda$$

defined as $M \mapsto \operatorname{ind}(M/\operatorname{rad}_B M)$ where $B = \operatorname{End}_\Lambda(M)$.

In particular, the support τ-tilting module Λ corresponds to the semibrick consisting of all simple Λ-modules.

3. Main results

In this section, we introduce the concept of the sincere semibricks, establish the relationship between the semibricks and the τ-tilting modules, and then construct (sincere) semibricks over split-by-nilpotent extensions. We start at the following definition.

Definition. A subset $\mathcal{S}$ of $\operatorname{mod}\Lambda$ is called *sincere* if $\mathcal{S}^{\oplus}$ is sincere where $\mathcal{S}^{\oplus}$ stands for the direct sum of all modules in $\mathcal{S}$.

Let $\text{s-sbrick}\,\Lambda$ stand for the set of all sincere semibricks in $\operatorname{mod}\Lambda$ and $\mathrm{sf_L}\text{-sbrick}\,\Lambda$ stand for the set of all sincere left finite semibricks in $\operatorname{mod}\Lambda$.

Theorem 3.1. *There exists a bijection*

$$\tau\text{-tilt}\,\Lambda \to \mathrm{sf_L}\text{-sbrick}\,\Lambda$$

defined as $M \mapsto \operatorname{ind}(M/\operatorname{rad}_B M)$.

Proof. By Theorem 2.1, there exists a bijection

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \to \mathrm{f_L}\text{-sbrick}\,\Lambda$$

defined as $M \mapsto \operatorname{ind}(M/\operatorname{rad}_B M)$. Note that

$$(\operatorname{ind}(M/\operatorname{rad}_B M))^{\oplus} = M/\operatorname{rad}_B M$$

and the τ-tilting Λ-modules are exactly the sincere support τ-tilting Λ-modules, we only need to show M is sincere if and only if $M/\operatorname{rad}_B M$ is also. For any idempotent $e_i \in \Lambda$, we have the following isomorphisms

$$\operatorname{Hom}_\Lambda(e_i\Lambda, M/\operatorname{rad}_B M) \cong (M/\operatorname{rad}_B M)e_i \cong Me_i/\operatorname{rad}_B(Me_i)$$

which implies $M/\operatorname{rad}_B M$ is sincere if and only if Me_i is nonzero (that means M is sincere). $\square$

Corollary 3.2. *Let $\mathcal{S}$ be a sincere left finite semibrick in* $\operatorname{mod}\Lambda$*. Then $T(\mathcal{S})$ is a sincere functorially finite torsion class of* $\operatorname{mod}\Lambda$*.*

Proof. It follows from Theorem 3.1 that there is a τ-tilting Λ-module M such that $\mathcal{S} = \operatorname{ind}(M/\operatorname{rad}_B M)$. Hence, we have $T(\mathcal{S}) = \operatorname{Fac}(M)$ by [2, Lemma 1.5(5)]. Note that $\operatorname{Fac}(M)$ is sincere functorially finite by [1, Corollary 2.8], and hence the result holds. $\square$

An algebra Λ is called *τ-tilting finite* [8, Definition 1.1] if there are only finitely many isomorphism classes of basic τ -tilting Λ-modules (It is equivalent to every torsion class in $\operatorname{mod}\Lambda$ being functorially finite[8, Theorem 3.8]). In this case, we have $\mathrm{sf_L}$-sbrick Λ=s-sbrick Λ.

Corollary 3.3. *Let Λ be a τ-tilting finite algebra. Then there exists a bijection*

$$\tau\text{-tilt}\,\Lambda \to \text{s-sbrick}\,\Lambda.$$

Let Λ and Γ be two algebras. We say that Γ is a *split extension of Λ by the nilpotent bimodule ${}_\Lambda E_\Lambda$*, or simply a *split-by-nilpotent extension* [5, Definition 1.1] if there exists a split surjective algebra morphism $\Gamma \to \Lambda$ whose kernel E is contained in the radical of Γ. There is a short exact sequence of Λ-Λ-bimodules

$$0 \longrightarrow {}_\Lambda E_\Lambda \longrightarrow {}_\Lambda\Gamma_\Lambda \longrightarrow \Lambda \longrightarrow 0$$

which splits. Therefore, we have an isomorphism ${}_\Lambda\Gamma_\Lambda \cong \Lambda \oplus_\Lambda E_\Lambda$. Moreover, if $\{e_1, e_2, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of Λ, then $\{e_1, e_2, \ldots, e_n\}$ is also a complete set of primitive orthogonal idempotents of Γ since ${}_\Lambda E_\Lambda$ is nilpotent.

Next, we will construct (sincere)semibricks in $\operatorname{mod}\Gamma$ from $\operatorname{mod}\Lambda$.

For a simple Λ-module S, $S \otimes_\Lambda \Gamma$ may not be simple. Indeed, let i be a sink of $\Lambda = KQ/I$, then $S_i \cong P_i$ is simple corresponding to the point i. Hence, $S_i \otimes_\Lambda \Gamma \cong P_i \otimes_\Lambda \Gamma$ is the projective Γ-module corresponding to the point i. It may not be simple since i may not be a sink of Γ. For example, let Λ be the algebra given by the quiver $1 \longrightarrow 2$ and Γ the algebra given by the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with the relation $\alpha\beta = 0$. Then 2 is a sink of Λ, however, it is not a sink of Γ. We will show that $S \otimes_\Lambda \Gamma$ is a brick in $\operatorname{mod}\Gamma$ under some condition.

The following lemma is very important in this paper.

Lemma 3.4. *Let Γ be a split-by-nilpotent extension of Λ by ${}_\Lambda E_\Lambda$. For any $M, N \in \operatorname{mod}\Lambda$, we have*

$$\operatorname{Hom}_\Gamma(M \otimes_\Lambda \Gamma, N \otimes_\Lambda \Gamma) \cong \operatorname{Hom}_\Lambda(M, N) \oplus \operatorname{Hom}_\Lambda(M, N \otimes_\Lambda E).$$

Proof. Let $M, N \in \operatorname{mod}\Lambda$. Then we have the following isomorphism

$$\begin{aligned}
\operatorname{Hom}_\Gamma(M \otimes_\Lambda \Gamma, N \otimes_\Lambda \Gamma) &\cong \operatorname{Hom}_\Lambda(M_\Lambda, \operatorname{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, N \otimes_\Lambda \Gamma)) \\
&\cong \operatorname{Hom}_\Lambda(M_\Lambda, N \otimes_\Lambda \Gamma_\Lambda) \\
&\cong \operatorname{Hom}_\Lambda(M_\Lambda, N \otimes_\Lambda (\Lambda \oplus E)_\Lambda)
\end{aligned}$$

$$\cong \mathrm{Hom}_\Lambda(M_\Lambda, N_\Lambda) \oplus \mathrm{Hom}_\Lambda(M_\Lambda, N \otimes_\Lambda E). \qquad \square$$

Proposition 3.5. *Let Γ be a split-by-nilpotent extension of $\Lambda = KQ/I$ by ${}_\Lambda E_\Lambda$ which has basis $\alpha_1, \alpha_2, \dots, \alpha_l$ as a K-vector space. Suppose that there is no path from i to i in E (equivalently, $e_i\alpha_j e_i = 0$, $j = 1, 2, \dots, l$), then $S_i \otimes_\Lambda \Gamma$ is a brick in $\mathrm{mod}\,\Gamma$ where S_i is the simple Λ-module corresponding to the point i.*

Proof. By Lemma 3.4, we have to show $\mathrm{Hom}_\Lambda(S_i, S_i \otimes_\Lambda E)$ is zero. Since S_i is generated by e_i which is an idempotent corresponding to the point i, we have $S_i \otimes_\Lambda E$ has basis $e_i\alpha_1, e_i\alpha_2, \dots, e_i\alpha_l$. For any $f \in \mathrm{Hom}_\Lambda(S_i, S_i \otimes_\Lambda E)$, we have $f(e_i) = w$ for some $w \in S_i \otimes_\Lambda E$. Hence, $f(e_i) = f(e_i^2) = w^2 = 0$ since there is no path from i to i in E. Thus, $\mathrm{Hom}_\Lambda(S_i, S_i \otimes_\Lambda E) = 0$. $\square$

Theorem 3.6. *Let Γ be a split-by-nilpotent extension of an algebra Λ by ${}_\Lambda E_\Lambda$ and $\mathcal{S} \subseteq \mathrm{mod}\,\Lambda$. Then we have*

(1) *$\mathcal{S} \otimes_\Lambda \Gamma$ is a semibrick in $\mathrm{mod}\,\Gamma$ if and only if $\mathcal{S}$ is a semibrick in $\mathrm{mod}\,\Lambda$ and $\mathrm{Hom}_\Lambda(\mathcal{S}, \mathcal{S} \otimes_\Lambda E) = 0$ where $\mathcal{S} \otimes_\Lambda \Gamma = \{S \otimes_\Lambda \Gamma \mid S \in \mathcal{S}\}$.*

(2) *$\mathcal{S} \otimes_\Lambda \Gamma$ is a sincere semibrick in $\mathrm{mod}\,\Gamma$ if and only if $\mathcal{S}$ is a semibrick in $\mathrm{mod}\,\Lambda$, $\mathrm{Hom}_\Lambda(\mathcal{S}, \mathcal{S} \otimes_\Lambda E) = 0$ and $\mathcal{S} \cup \mathcal{S} \otimes_\Lambda E$ is sincere.*

Proof. (1) For any $S_i, S_j \in \mathcal{S}$, it follows from Lemma 3.4 that we have

$$\mathrm{Hom}_\Gamma(S_i \otimes_\Lambda \Gamma, S_j \otimes_\Lambda \Gamma) \cong \mathrm{Hom}_\Lambda(S_i, S_j) \oplus \mathrm{Hom}_\Lambda(S_i, S_j \otimes_\Lambda E).$$

If $i = j$, then we have $\mathrm{Hom}_\Gamma(S_i \otimes_\Lambda \Gamma, S_i \otimes_\Lambda \Gamma)$ is a K-division algebra if and only if $\mathrm{Hom}_\Lambda(S_i, S_i)$ is a K-division algebra and $\mathrm{Hom}_\Lambda(S_i, S_i \otimes_\Lambda E) = 0$.

If $i \neq j$, then $\mathrm{Hom}_\Gamma(S_i \otimes_\Lambda \Gamma, S_j \otimes_\Lambda \Gamma) = 0$ if and only if $\mathrm{Hom}_\Lambda(S_i, S_j) = 0$ and $\mathrm{Hom}_\Lambda(S_i, S_j \otimes_\Lambda E) = 0$. Therefore, the assertion holds.

(2) Note that $(\mathcal{S} \otimes_\Lambda \Gamma)^\oplus \cong \mathcal{S}^\oplus \otimes_\Lambda \Gamma$, we have the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}_\Gamma(e_i\Gamma, (\mathcal{S} \otimes_\Lambda \Gamma)^\oplus) &\cong \mathrm{Hom}_\Gamma(e_i\Lambda \otimes_\Lambda \Gamma, \mathcal{S}^\oplus \otimes_\Lambda \Gamma) \\
&\cong \mathrm{Hom}_\Lambda(e_i\Lambda, \mathrm{Hom}_\Gamma({}_\Lambda\Gamma_\Gamma, \mathcal{S}^\oplus \otimes_\Lambda \Gamma)) \\
&\cong \mathrm{Hom}_\Lambda(e_i\Lambda, \mathcal{S}^\oplus \otimes_\Lambda \Gamma_\Lambda) \\
&\cong \mathrm{Hom}_\Lambda(e_i\Lambda, \mathcal{S}^\oplus \otimes_\Lambda (\Lambda \oplus E)_\Lambda) \\
&\cong \mathrm{Hom}_\Lambda(e_i\Lambda, \mathcal{S}^\oplus \oplus \mathcal{S}^\oplus \otimes_\Lambda E).
\end{aligned}$$

Hence, $\mathcal{S} \otimes_\Lambda \Gamma$ is sincere if and only if $\mathcal{S} \cup \mathcal{S} \otimes_\Lambda E$ is sincere. Finally, the assertion follows (1). $\square$

The following result gives a converse construction of Theorem 3.6.

Proposition 3.7. *Let Γ be a split extension of an algebra Λ and $\mathcal{S} \subseteq \mathrm{mod}\,\Gamma$. Then we have*

(1) *$\mathcal{S} \otimes_\Gamma \Lambda$ is a semibrick in $\mathrm{mod}\,\Lambda$ if and only if $\mathrm{Hom}_\Gamma(S_i, S_i \otimes_\Gamma \Lambda_\Gamma)$ is a K-division algebra for any $S_i \in \mathcal{S}$ and $\mathrm{Hom}_\Gamma(S_i, S_j \otimes_\Gamma \Lambda_\Gamma) = 0$ for any $S_i \neq S_j \in \mathcal{S}$ where $\mathcal{S} \otimes_\Gamma \Lambda = \{S \otimes_\Gamma \Lambda \mid S \in \mathcal{S}\}$.*

(2) $\mathcal{S} \otimes_\Gamma \Lambda$ *is a sincere semibrick in* $\operatorname{mod}\Lambda$ *if and only if* $\mathcal{S} \otimes_\Gamma \Lambda_\Gamma$ *is a semibrick in* $\operatorname{mod}\Gamma$ *and* $\mathcal{S} \otimes_\Gamma \Lambda_\Gamma$ *is sincere.*

Proof. (1) For any $S_i, S_j \in \mathcal{S}$, we have

$$\begin{aligned}\operatorname{Hom}_\Lambda(S_i \otimes_\Gamma \Lambda, S_j \otimes_\Gamma \Lambda) &\cong \operatorname{Hom}_\Gamma(S_i, \operatorname{Hom}_\Lambda({}_\Gamma\Lambda_\Lambda, S_j \otimes_\Gamma \Lambda_\Lambda)) \\ &\cong \operatorname{Hom}_\Gamma(S_i, S_j \otimes_\Gamma \Lambda_\Gamma).\end{aligned}$$

If $i = j$, then we have $\operatorname{Hom}_\Lambda(S_i \otimes_\Gamma \Lambda, S_i \otimes_\Gamma \Lambda)$ is a K-division algebra if and only if $\operatorname{Hom}_\Gamma(S_i, S_i \otimes_\Gamma \Lambda_\Gamma)$ is a K-division algebra.

If $i \neq j$, then $\operatorname{Hom}_\Lambda(S_i \otimes_\Gamma \Lambda, S_j \otimes_\Gamma \Lambda) = 0$ if and only if $\operatorname{Hom}_\Gamma(S_i, S_j \otimes_\Gamma \Lambda_\Gamma) = 0$. Therefore, the result is obvious.

(2) The following isomorphism

$$\begin{aligned}\operatorname{Hom}_\Lambda(e_i\Lambda, (\mathcal{S} \otimes_\Gamma \Lambda)^{\oplus}) &\cong \operatorname{Hom}_\Lambda(e_i\Gamma \otimes_\Gamma \Lambda, \mathcal{S}^{\oplus} \otimes_\Gamma \Lambda) \\ &\cong \operatorname{Hom}_\Gamma(e_i\Gamma, \operatorname{Hom}_\Lambda({}_\Gamma\Lambda_\Lambda, \mathcal{S}^{\oplus} \otimes_\Gamma \Lambda)) \\ &\cong \operatorname{Hom}_\Gamma(e_i\Gamma, \mathcal{S}^{\oplus} \otimes_\Gamma \Lambda_\Gamma) \\ &\cong \operatorname{Hom}_\Gamma(e_i\Gamma, (\mathcal{S} \otimes_\Gamma \Lambda_\Gamma)^{\oplus})\end{aligned}$$

implies $\mathcal{S} \otimes_\Gamma \Lambda_\Lambda$ is sincere if and only if $\mathcal{S} \otimes_\Gamma \Lambda_\Gamma$ is sincere. The assertion follows (1). □

Let A be a hereditary algebra and $\mathcal{D}^b(\operatorname{mod} A)$ the bounded derived category of $\operatorname{mod} A$. The orbit category of $\mathcal{D}^b(\operatorname{mod} A)$ under the action of the functor $\tau^{-1}[1]$ is called *cluster category* denoted by $\mathcal{C}_A$, where $[1]$ is the shift functor. A *tilting object* $\widetilde{T}$ in $\mathcal{C}_A$ is an object such that $\operatorname{Ext}^1_{\mathcal{C}_A}(\widetilde{T}, \widetilde{T}) = 0$ and $|\widetilde{T}| = |A|$ ([6]). The endomorphism algebra of $\widetilde{T}$ is called *cluster-tilted* ([7]). It was shown that the relation extension $\Gamma = \Lambda \ltimes \operatorname{Ext}^2_\Lambda(D\Lambda, \Lambda)$ of a tilted algebra Λ is cluster-tilted (see [3, Theorem 3.4]). Moreover, all cluster-tilted algebras are of this form. In this case, we say Γ is a cluster-tilted algebra corresponding to the tilted algebra Λ.

For $\mathcal{S} \in \operatorname{mod}\Lambda$, let

$$\tau^{-1}\Omega^{-1}\mathcal{S} = \{\tau^{-1}\Omega^{-1}S \mid S \in \mathcal{S}\} \text{ and } \operatorname{id}\mathcal{S} = \operatorname{Max}\{\operatorname{id}_\Lambda S \mid S \in \mathcal{S}\}.$$

We have the following:

Theorem 3.8. *Let* Γ *be a cluster-tilted algebra corresponding to the tilted algebra* Λ *and* $\mathcal{S} \subseteq \operatorname{mod}\Lambda$. *Then*

(1) $\mathcal{S} \otimes_\Lambda \Gamma$ *is a semibrick in* $\operatorname{mod}\Gamma$ *if and only if* $\mathcal{S}$ *is a semibrick in* $\operatorname{mod}\Lambda$ *and* $\operatorname{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$.

(2) $\mathcal{S} \otimes_\Lambda \Gamma$ *is a sincere semibrick in* $\operatorname{mod}\Gamma$ *if and only if* $\mathcal{S}$ *is a semibrick in* $\operatorname{mod}\Lambda$, $\operatorname{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$ *and* $\mathcal{S} \cup \tau^{-1}\Omega^{-1}\mathcal{S}$ *is sincere.*

Proof. Since the global dimension of the tilted algebra Λ is at most 2, we have

$$S \otimes_\Lambda \operatorname{Ext}^2_\Lambda(D\Lambda, \Lambda) \cong \tau^{-1}\Omega^{-1}S$$

for any $S \in \mathcal{S}$ by [13, Proposition 4.1]. Now the assertions follow from Theorem 3.6. $\square$

If $\mathrm{id}\mathcal{S} \leq 1$, then $\tau^{-1}\Omega^{-1}\mathcal{S} = 0$. Hence, we have the following result by Theorem 3.8.

Corollary 3.9. *Let Γ be a cluster-tilted algebra corresponding to the tilted algebra Λ and $\mathcal{S} \subseteq \operatorname{mod}\Lambda$. If $\mathrm{id}\mathcal{S} \leq 1$, we have*

(1) *$\mathcal{S}\otimes_\Lambda \Gamma$ is a semibrick in $\operatorname{mod}\Gamma$ if and only if $\mathcal{S}$ is a semibrick in $\operatorname{mod}\Lambda$.*
(2) *$\mathcal{S} \otimes_\Lambda \Gamma$ is a sincere semibrick in $\operatorname{mod}\Gamma$ if and only if $\mathcal{S}$ is a sincere semibrick in $\operatorname{mod}\Lambda$.*

4. An application

In this section, we will apply our results to construct τ-tilting modules and support τ-tilting modules over τ-tilting finite cluster-tilted algebras.

Let Γ be a cluster-tilted algebra corresponding to the tilted algebra Λ and $T \in \operatorname{mod}\Lambda$ be a support τ-tilting module. Then there exists a semibrick $\mathcal{S}$ in $\operatorname{mod}\Lambda$ such that $\mathcal{S} = \operatorname{ind}(T/\operatorname{rad}_B T)$ by Theorem 2.1 where $B = \operatorname{End}_\Lambda(T)$. Suppose that $\operatorname{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$, we get a semibrick $\mathcal{S} \otimes_\Lambda \Gamma$ in $\operatorname{mod}\Gamma$ by Theorem 3.8. In addition, if Γ is τ-tilting finite, then $T(\mathcal{S} \otimes_\Lambda \Gamma)$ is a functorially finite torsion class, hence there is a support τ-tilting module $T' \in \operatorname{mod}\Gamma$ such $T(\mathcal{S} \otimes_\Lambda \Gamma) = \operatorname{Fac}(T')$ by using Theorem 2.1 again (in fact, T' is exactly the maximal Ext-projective object in $T(\mathcal{S} \otimes_\Lambda \Gamma)$ by [1, Theorem 2.7]).

Definition. Let Λ be a τ-tilting finite algebra. We say a Λ-module T is a support τ-tilting module with respect to the semibrick $\mathcal{S}$ if $\mathcal{S} = \operatorname{ind}(T/\operatorname{rad}_B T)$. We also say the semibrick $\mathcal{S}$ with respect to T.

Proposition 4.1. *Let Γ be a τ-tilting finite cluster-tilted algebra corresponding to the tilted algebra Λ and $T \in \operatorname{mod}\Lambda$ be a support τ-tilting module with respect to the semibrick $\mathcal{S}$.*

(1) *If $\operatorname{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$, then the maximal Ext-projective object T' in $T(\mathcal{S} \otimes_\Lambda \Gamma)$ is a support τ-tilting Γ-module.*
(2) *If $\operatorname{Hom}_\Lambda(\mathcal{S}, \tau^{-1}\Omega^{-1}\mathcal{S}) = 0$ and $\mathcal{S} \cup \tau^{-1}\Omega^{-1}\mathcal{S}$ is sincere, then the maximal Ext-projective object T' in $T(\mathcal{S} \otimes_\Lambda \Gamma)$ is a τ-tilting Γ-module.*

Proof. (1) Follows from the above discussion.

(2) By Theorem 3.8(2) and Corollary 3.2, $T(\mathcal{S} \otimes_\Lambda \Gamma)$ is a sincere functorially finite torsion class. Hence T' is τ-tilting by [1, Corollary 2.8]. $\square$

Corollary 4.2. *Let Γ be a τ-tilting finite cluster-tilted algebra corresponding to the tilted algebra Λ and $T \in \operatorname{mod}\Lambda$ be a support τ-tilting module with respect to the semibrick $\mathcal{S}$.*

(1) *If $\mathrm{id}\mathcal{S} \leq 1$, then the maximal Ext-projective object T' in $T(\mathcal{S} \otimes_\Lambda \Gamma)$ is a support τ-tilting Γ-module.*

(2) *If* $\mathrm{id}\mathcal{S} \leq 1$ *and* $\mathcal{S}$ *is sincere, then the maximal* Ext-*projective object* T' *in* $T(\mathcal{S} \otimes_\Lambda \Gamma)$ *is a* τ*-tilting* Γ*-module.*

Remark 4.3. (1) The τ-tilting finite cluster-tilted algebras are exactly representation finite cluster-tilted algebras by [14, Theorem 3.1].

(2) In [10], the authors construct τ-tilting modules and support τ-tilting modules over cluster-tilted algebras. More precisely, let Γ be a cluster-tilted algebra corresponding to a tilted algebra Λ and $T \in \operatorname{mod}\Lambda$. They proved that $T \otimes_\Lambda \Gamma_\Gamma$ is support τ-tilting in $\operatorname{mod}\Gamma$ if and only if T is support τ-tilting in $\operatorname{mod}\Lambda$ and

$$\mathrm{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_\Lambda, \tau T_\Lambda) = 0 = \mathrm{Hom}_\Lambda(e\Lambda, \tau^{-1}\Omega^{-1}T_\Lambda),$$

where e is an idempotent of Λ such that T is a τ-tilting $\Lambda/\langle e\rangle$-module [10, Proposition 3.4]. In particular, $T \otimes_\Lambda \Gamma_\Gamma$ is τ-tilting in $\operatorname{mod}\Gamma$ if and only if T is τ-tilting in $\operatorname{mod}\Lambda$ and $\mathrm{Hom}_\Lambda(\tau^{-1}\Omega^{-1}T_\Lambda, \tau T_\Lambda) = 0$. Different from the result, we can construct a τ-tilting Γ-module from a proper support τ-tilting Λ-module (that is, it is a support τ-tilting module but not a τ-tilting Λ-module) by Proposition 4.1(2) (see the example in section 5).

5. An example

In this section, we illustrate our results by the following example. All indecomposable modules are denoted by their Loewy series.

Example 5.1. Let Λ be the tilted algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

with the relation $\alpha\beta = 0$. The cluster-titled algebra Γ corresponding to Λ is given by the following quiver

$$\begin{tikzcd}
& 2 \arrow[dr, "\beta"] & \\
1 \arrow[ur, "\alpha"] & & 3 \arrow[ll, "\gamma"]
\end{tikzcd}$$

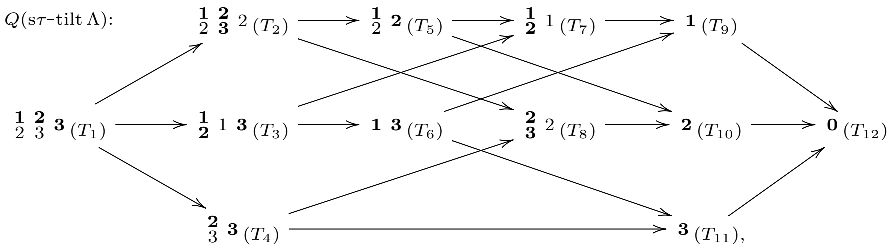
with relations $\alpha\beta = 0$, $\beta\gamma = 0$ and $\gamma\alpha = 0$.

The Auslander-Reiten quiver of Λ is as follows (the dotted horizontal lines from right to left represents Auslander-Reiten translation).

$$\begin{tikzcd}
& \begin{smallmatrix}2\\3\end{smallmatrix} \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & \\
3 \arrow[ur] \arrow[rr, dotted, no head] & & 2 \arrow[ur] \arrow[rr, dotted, no head] & & 1
\end{tikzcd}$$

Note that 3 is the unique indecomposable module in $\operatorname{mod}\Lambda$ with injective dimension two. So for any indecomposable module W not isomorphic to 3, we have $\tau^{-1}\Omega^{-1}W = 0$, and $\tau^{-1}\Omega^{-1}\, 3 = 1$.

The Hasse quiver $Q(\mathrm{s}\tau\text{-tilt}\,\Lambda)$ is as follows and semibricks in $\mathrm{mod}\,\Lambda$ will be marked by black.



Hence, we have the following semibricks in $\mathrm{mod}\,\Lambda$

$$\mathcal{S}_1=\{1,2,3\},\ \mathcal{S}_2=\{1,\begin{smallmatrix}2\\3\end{smallmatrix}\},\ \mathcal{S}_3=\{\begin{smallmatrix}1\\2\end{smallmatrix},3\},\ \mathcal{S}_4=\{2,3\},\ \mathcal{S}_5=\{1,2\},$$
$$\mathcal{S}_6=\{1,3\},\ \mathcal{S}_7=\{\begin{smallmatrix}1\\2\end{smallmatrix}\},\ \mathcal{S}_8=\{\begin{smallmatrix}2\\3\end{smallmatrix}\},\ \mathcal{S}_9=\{1\},\ \mathcal{S}_{10}=\{2\},\ \mathcal{S}_{11}=\{3\},$$
$$\mathcal{S}_{12}=\{0\}.$$

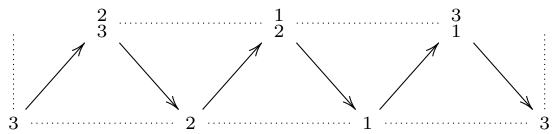
Since $\mathrm{id}\mathcal{S}_i\leq 1$ $(i=2,5,7,8,9,10,12)$, we have the following semibricks in $\mathrm{mod}\,\Gamma$ by Corollary 3.9.

$$\mathcal{S}_2\otimes_\Lambda\Gamma=\{1,\begin{smallmatrix}2\\3\end{smallmatrix}\},\ \mathcal{S}_5\otimes_\Lambda\Gamma=\{1,2\},\ \mathcal{S}_7\otimes_\Lambda\Gamma=\{\begin{smallmatrix}1\\2\end{smallmatrix}\},\ \mathcal{S}_8\otimes_\Lambda\Gamma=\{\begin{smallmatrix}2\\3\end{smallmatrix}\},$$
$$\mathcal{S}_9\otimes_\Lambda\Gamma=\{1\},\ \mathcal{S}_{10}\otimes_\Lambda\Gamma=\{2\},\ \mathcal{S}_{12}\otimes_\Lambda\Gamma=\{0\}.$$

A simple calculation yields

$$\mathrm{Hom}_\Lambda(\mathcal{S}_4,\tau^{-1}\Omega^{-1}\mathcal{S}_4)=0,\mathrm{Hom}_\Lambda(\mathcal{S}_{11},\tau^{-1}\Omega^{-1}\mathcal{S}_{11})=0.$$

Hence, $\mathcal{S}_{11}\otimes_\Lambda\Gamma=\{\begin{smallmatrix}3\\1\end{smallmatrix}\}$ is a semibrick and $\mathcal{S}_4\otimes_\Lambda\Gamma=\{2,\begin{smallmatrix}3\\1\end{smallmatrix}\}$ is a sincere semibrick since $\mathcal{S}_4\cup\tau^{-1}\Omega^{-1}\mathcal{S}_4=\{1,2,3\}$ is sincere by Theorem 3.8. However, $\mathcal{S}_i\otimes_\Lambda\Gamma$ $(i=1,3,6)$ are not semibricks since $\mathrm{Hom}_\Lambda(\mathcal{S}_i,\tau^{-1}\Omega^{-1}\mathcal{S}_i)\neq 0$ $(i=1,3,6)$. The Auslander-Reiten quiver of Γ is as follows(we identify the two copies of 3 along the dotted vertical lines).



We have torsion classes of $\mathrm{mod}\,\Gamma$ as follows

$$T(\mathcal{S}_2\otimes_\Lambda\Gamma)=\{1,\begin{smallmatrix}2\\3\end{smallmatrix},2,\begin{smallmatrix}1\\2\end{smallmatrix}\},\ T(\mathcal{S}_4\otimes_\Lambda\Gamma)=\{2,\begin{smallmatrix}3\\1\end{smallmatrix},\begin{smallmatrix}2\\3\end{smallmatrix},3\},$$
$$T(\mathcal{S}_5\otimes_\Lambda\Gamma)=\{1,2,\begin{smallmatrix}1\\2\end{smallmatrix}\},\ T(\mathcal{S}_7\otimes_\Lambda\Gamma)=\{\begin{smallmatrix}1\\2\end{smallmatrix},1\},$$
$$T(\mathcal{S}_8\otimes_\Lambda\Gamma)=\{\begin{smallmatrix}2\\3\end{smallmatrix},2\},\ T(\mathcal{S}_9\otimes_\Lambda\Gamma)=\{1\},$$
$$T(\mathcal{S}_{10}\otimes_\Lambda\Gamma)=\{2\},\ T(\mathcal{S}_{11}\otimes_\Lambda\Gamma)=\{\begin{smallmatrix}3\\1\end{smallmatrix},3\},\ T(\mathcal{S}_{12}\otimes_\Lambda\Gamma)=\{0\}.$$

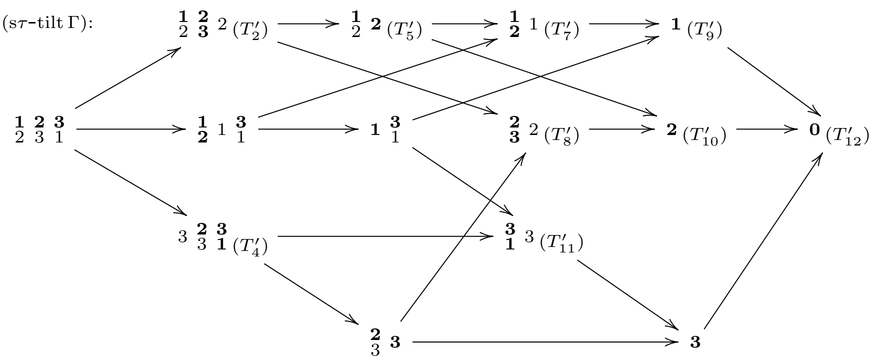
Now, we can get the following support τ-tilting Γ-modules

$$T_2'=\begin{smallmatrix}1\\2\end{smallmatrix}\begin{smallmatrix}2\\3\end{smallmatrix}2,\ T_4'=3\begin{smallmatrix}2\\3\end{smallmatrix}\begin{smallmatrix}3\\1\end{smallmatrix},\ T_5'=\begin{smallmatrix}1\\2\end{smallmatrix}2,\ T_7'=\begin{smallmatrix}1\\2\end{smallmatrix}1,$$
$$T_8'=\begin{smallmatrix}2\\3\end{smallmatrix}2,\ T_9'=1,\ T_{10}'=2,\ T_{11}'=\begin{smallmatrix}3\\1\end{smallmatrix}3,\ T_{12}'=0.$$

The module T_4 is a proper support τ-tilting Λ-module, however, T_4' is a τ-tilting Γ-module.

In fact, all support τ-tilting Γ-modules and semibricks (marked by black) in $\operatorname{mod}\Gamma$ can be found in the following Hasse quiver of Γ.

$Q(\mathrm{s}\tau\text{-tilt}\,\Gamma)$:



References

[1] T. Adachi, O. Iyama, and I. Reiten, *τ-tilting theory*, Compos. Math. **150** (2014), no. 3, 415–452. `https://doi.org/10.1112/S0010437X13007422`

[2] S. Asai, *Semibricks*, to appear in Int. Math. Res. Not. IMRN.

[3] I. Assem, T. Brüstle, and R. Schiffler, *Cluster-tilted algebras as trivial extensions*, Bull. Lond. Math. Soc. **40** (2008), no. 1, 151–162. `https://doi.org/10.1112/blms/bdm107`

[4] I. Assem and N. Marmaridis, *Tilting modules over split-by-nilpotent extensions*, Comm. Algebra **26** (1998), no. 5, 1547–1555. `https://doi.org/10.1080/00927879808826219`

[5] I. Assem and D. Zacharia, *On split-by-nilpotent extensions*, Colloq. Math. **98** (2003), no. 2, 259–275. `https://doi.org/10.4064/cm98-2-10`

[6] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618. `https://doi.org/10.1016/j.aim.2005.06.003`

[7] A. B. Buan, R. Marsh, and I. Reiten, *Cluster-tilted algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 1, 323–332. `https://doi.org/10.1090/S0002-9947-06-03879-7`

[8] L. Demonet, O. Iyama, and G. Jasso, *τ-tilting finite algebras, bricks, and g-vectors*, Int. Math. Res. Not. IMRN **2019**, no. 3, 852–892. `https://doi.org/10.1093/imrn/rnx135`

[9] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[10] H. Gao and Z. Huang, *Support τ-tilting modules under split-by-nilpotent extensions*, Colloq. Math. **160** (2020), no. 2, 247–262. `https://doi.org/10.4064/cm7774-3-2019`

[11] F. Marks and J. Šťovíček, *Torsion classes, wide subcategories and localisations*, Bull. Lond. Math. Soc. **49** (2017), no. 3, 405–416. `https://doi.org/10.1112/blms.12033`

[12] C. M. Ringel, *Representations of K-species and bimodules*, J. Algebra **41** (1976), no. 2, 269–302. `https://doi.org/10.1016/0021-8693(76)90184-8`

[13] R. Schiffler and K. Serhiyenko, *Induced and coinduced modules over cluster-tilted algebras*, J. Algebra **472** (2017), 226–258. `https://doi.org/10.1016/j.jalgebra.2016.10.009`

[14] S. Zito, *τ-tilting finite cluster-tilted algebras*, arXiv:1902.05866.

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