A note on wide τ-tilting modules and epibricks

Hanpeng Gao[1]

Abstract

As a generalization of support τ-tilting modules, H. Enomoto introduced the notion of wide τ-tilting modules and established a bijection between wide τ-tilting modules and doubly functorially finite ICE-closed subcategories, which extended Adachi-Iyama-Reiten's bijection on torsion classes. In this paper, we consider the relationship between wide τ-tilting modules and some sets of bricks (named epibricks). In particular, we show that there is a bijection between wide τ-tilting modules and epibricks for Nakayama algebras. As a consequence, we get a recurrence relation for the number of wide τ-tilting modules over Nakayama algebras.

Keywords Wide τ-tilting modules · ICE-closed subcategories · Epibricks · Nakayama algebras

Mathematics Subject Classification 16G10 · 16G20

1 Introduction

In 2014, Adachi, Iyama and Reiten[1] introduced *τ-tilting theory* of finite dimensional algebras and showed that it is closely related to torsion theory, silting theory and cluster-tilting theory. In particular, the support τ-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes, left finite wide subcategories, 2-term silting complexes and so on.

Recently, Enomoto[7] studied ICE-closed subcategories which are closed under images, cokernels and extensions. Clearly, torsion classes and wide subcategories are ICE-closed. Moreover, it is shown that a subcategory is ICE-closed if and only if it is a sincere torsion class for some wide subcategory(see,[7, Corollary 2.5, Proposition 4.2]). Note that every functorially finite wide subcategory is equivalent to a module category, Enomoto and Sakai introduced the notion of *wide τ-tilting* modules which are τ-tilting modules over functorially finite wide subcategories. What's more, they also established a bijection between wide τ-tilting modules and doubly functorially finite ICE-closed subcategories, which extended

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Adachi-Iyama-Reiten's bijection between support τ-tilting modules and functorially finite torsion classes.

$$\{wide\ \tau\text{-}tilting\ modules\}$$

$$\|$$

$$\{\tau\text{-}tilting\ modules\ over\ functorially\ finite\ wide\ subcategories\}$$

$$AIR's\ bijection \updownarrow$$

$$\{sincere\ functorially\ finite\ torsion\ classes\ over\ functorially\ finite\ wide\ subcategories\}$$

$$\|$$

$$\{doubly\ functorially\ finite\ ICE\text{-}closed\ subcategories\}.$$

The τ-tilting modules are exactly the tilting modules and the support τ-tilting modules are exactly the support tilting modules for a hereditary algebra. For Dynkin type algebras, these numbers of tilting modules and support tilting modules were first calculated in [8] via cluster algebras, and later in [11] via representation theory. Enomoto[5] has shown that the wide τ-tilting modules are exactly the rigid modules for a hereditary algebra and calculated the number of rigid modules over Dynkin type algebras. In particular, over a hereditary algebra of type $\mathbb{A}_n$, the number of rigid modules is the n-th large Schröder number $LS_n = \sum_{i=0}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}$, where $\binom{n}{i}$ denotes the binomial coefficient.

As generalizations of simple modules and semisimple modules, bricks and semibricks are considered and they have long been studied in [9, 12]. Asai proved that there exists a bijection between the set of support τ-tilting modules and the set of left finite semibricks for any algebra(see [3, Theorem 2.3]). In this article, we will establish a bijection between the set of wide τ-tilting modules and the set of doubly finite epibricks(see, Definition 2.4) for a given algebra. In particular, it will be shown that there is a bijection between the set of wide τ-tilting modules and the set of epibricks for Nakayama algebras. As a consequence, we get a recurrence relation for the number of wide τ-tilting modules over Nakayama algebras. Our main results is as follows.

Theorem 1.1 *Let $a_{n,r}$ (respectively $b_{n,r}$) be the number of wide τ-tilting modules of Nakayama algebras $\Lambda_n^r := KA_n/\mathrm{rad}^r$ (respectively $\tilde{\Lambda}_n^r := K\widetilde{A}_n/\mathrm{rad}^r$). Then*

$$a_{n,r} = 2a_{n-1,r} + \sum_{i=2}^{r} LS_{i-1} \cdot a_{n-i,r}.$$

$$b_{n,r} = 2b_{n-1,r} + \sum_{i=2}^{r} LS_{i-1} \cdot b_{n-i,r}.$$

Throughout this paper, all algebras will be basic, connected, finite dimensional K-algebras over an algebraically closed field K. Let Λ be an algebra, mod Λ will be the category of finitely generated right Λ-modules and τ the Auslander-Reiten translation of Λ. We also denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M, $l(M)$ the Loewy length of M, and add$M$ the subcategory consisting of direct summands of finite direct sums of M for $M \in$ mod Λ. Given an algebra $\Lambda = KQ/I$, let P_i be the indecomposable projective module, I_i the indecomposable injective module, S_i the simple module, e_i the primitive idempotent element of an algebra corresponding to the point i. For any $i, j \in \{1, 2, \cdots, n\}$, we will denote by $[i, j] = \{i, i+1, \cdots, j\}$ if $i \leqslant j$. Otherwise,

$[i, j] = \emptyset$. We also write $e_{[i, j]} = e_i + e_{i+1} + \cdots + e_j$. Finally, for a finite set X, we denote by $\#X$ the cardinality of X. $X_1 \coprod X_2$ stand for the disjoint uniou of sets X_1 and X_2.

2 Wide τ-tilting modules and epibricks

Let Λ be an algebra. In this section, we recall some results about wide τ-tilting modules that are needed later.

Let $M \in \mod \Lambda$. M is τ-tilting if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$ and $|M| = |\Lambda|$. M is support τ-tilting if it is a τ-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent e of Λ. Enomoto showed that every functorially finite wide subcategory $\mathcal{W}$ is equivalent to a module category (i.e, there is an algebra Γ such that $\mathcal{W}$ is equivalent to $\mod \Gamma$), and then he introduced the definition of wide τ-tilting modules as follows.

Definition 2.1 ([7, Definition 4.11])

(1) Given a functorially finite wide subcategory $\mathcal{W}$ of $\mod \Lambda$ and $M \in \mathcal{W}$, fix an equivalent $F : \mathcal{W} \simeq \mod \Gamma$. We say M is $\tau_\mathcal{W}$-tilting if $F(M)$ is a τ-tilting Γ-module.
(2) A Λ-module M is called *wide τ-tilting* if there is a functorially finite wide subcategory $\mathcal{W}$ of $\mod \Lambda$ such that M is $\tau_\mathcal{W}$-tilting. The set of all wide τ-tilting Λ-modules will be denoted by $\mathrm{w}\tau$-tilt Λ.

Let $\mathcal{C}$ be a subcategory of $\mod \Lambda$. $\mathcal{C}$ is called ICE-closed if it is closed under images, cokernels and extensions. Both torsion classes and wide subcategories are ICE-closed. Moreover, it is shown that a subcategory $\mathcal{C}$ of $\mod \Lambda$ is ICE-closed if and only if there is a wide subcategory $\mathcal{W}$ such that $\mathcal{C}$ is a torsion class of $\mathcal{W}$. Enomoto called $\mathcal{C}$ *doubly functorially finite* if there is a functorially finite wide subcategory $\mathcal{W}$ such that $\mathcal{C}$ is a functorially finite torsion class of $\mathcal{W}$. The set of all doubly functorially finite ICE-closed subcategories of $\mod \Lambda$ will be denoted by df-ice Λ. The following result extends Adachi-Iyama-Reiten's bijection between support τ-tilting modules and functorially finite torsion classes.

Theorem 2.2 ([7, Theorem 4.13]) *Let Λ be a finite dimensional algebra. Then there is a bijection:*

$$\mathrm{w}\tau\text{-tilt } \Lambda \underset{P(-)}{\overset{cok(-)}{\rightleftarrows}} \text{df-ice } \Lambda$$

where $cok(M)$ denotes the subcategory of $\mod \Lambda$ consisting of cokernels of morphisms in $\mathrm{add}M$, and $P(\mathcal{C})$ denotes the maximal Ext-projective object of $\mathcal{C}$.

A Λ-module S is said to be a brick if $\mathrm{Hom}_\Lambda(S, S)$ is a K-division algebra. Let $\mathcal{S}$ be a set of isomorphism classes of bricks in $\mod \Lambda$. $\mathcal{S}$ is called an *epibrick* if every morphism between elements of $\mathcal{S}$ is zero or a surjection in $\mod \Lambda$. We will denote by $\mathrm{ebrick}\Lambda$ the set of epibricks in $\mod \Lambda$. A subcategories $\mathcal{C}$ of $\mod \Lambda$ is *right Schur* if it is closed under extensions and, for every simple object M in $\mathcal{C}$, every morphism $X \to M$ with $X \in \mathcal{C}$ is zero or a surjection in $\mod \Lambda$. We denote the set of right Schur subcategories of $\mod \Lambda$ by $\mathrm{Schur}_R \Lambda$. In particular, every ICE-closed subcategory is right Schur. The following Lemma follows from the dual of [6, Theorem 2.11].

Lemma 2.3 *Let Λ be a finite dimensional algebra. Then $\mathrm{Filt}(-)$ given a bijection between $\mathrm{ebrick}\Lambda$ and $\mathrm{Schur}_R \Lambda$ with inverse $\mathrm{Sim}(-)$, where $\mathrm{Filt}(\mathcal{S})$ stand for the minimal Extension-closed subcategory which contains $\mathcal{S}$ for $\mathcal{S} \in \mathrm{ebrick}\Lambda$ and $\mathrm{Sim}(\mathcal{C})$ stand for the set of all simple object of the right Schur subcategory $\mathcal{C}$.*

Table 1 wτ-tiltΛ,df-iceΛ and df-eprickΛ

wτ-tilt Λ	df-ice Λ	df-ebrick Λ
$1 \oplus \frac{2}{1} \oplus \frac{3}{2}$	mod Λ	$\{1, 2, 3\}$
$2 \oplus \frac{2}{1} \oplus \frac{3}{2}$	add$\{2, \frac{2}{1}, \frac{3}{2}, 3\}$	$\{2, \frac{2}{1}, 3\}$
$1 \oplus \frac{3}{2} \oplus 3$	add$\{1, \frac{3}{2}, 3\}$	$\{1, \frac{3}{2}, 3\}$
$1 \oplus \frac{2}{1}$	add$\{1, \frac{2}{1}, 2\}$	$\{1, 2\}$
$2 \oplus \frac{3}{2}$	add$\{2, \frac{3}{2}, 3\}$	$\{2, 3\}$
$\frac{3}{2} \oplus 3$	add$\{\frac{3}{2}, 3\}$	$\{\frac{3}{2}, 3\}$
$2 \oplus \frac{2}{1}$	add$\{2, \frac{2}{1}\}$	$\{2, \frac{2}{1}\}$
$\frac{2}{1} \oplus 3$	add$\{\frac{2}{1}, 3\}$	$\{\frac{2}{1}, 3\}$
$\frac{3}{2} \oplus 1$	add$\{\frac{3}{2}, 1\}$	$\{\frac{3}{2}, 1\}$
$1 \oplus 3$	add$\{1, 3\}$	$\{1, 3\}$
$\frac{2}{1}$	add$\{\frac{2}{1}\}$	$\{\frac{2}{1}\}$
$\frac{3}{2}$	add$\{\frac{3}{2}\}$	$\{\frac{3}{2}\}$
2	add$\{2\}$	$\{2\}$
3	add$\{3\}$	$\{3\}$
1	add$\{1\}$	$\{1\}$
0	add$\{0\}$	$\{0\}$

Therefore, if $M \in$ wτ-tilt Λ, cok(M) is doubly functorially finite ICE-closed subcategory by Theorem 2.2 and hence Sim(cok(M)) is an epibrick by Lemma 2.3 .

Definition 2.4 Let $\mathcal{S}$ be an epibrick in mod Λ. We say $\mathcal{S}$ is *doubly finite* if Filt($\mathcal{S}$) is a doubly functorially finite ICE-closed subcategory. The set of all doubly finite epibricks in mod Λ will be denoted by df-ebrickΛ.

Now, we get the following result directly by Theorem 2.2 and Lemma 2.3.

Theorem 2.5 *There is a bijection:*

$$\text{w}\tau\text{-tilt } \Lambda \underset{P(\text{Filt}(-))}{\overset{\text{Sim}(\text{cok}(-))}{\rightleftarrows}} \text{df-ebrick } \Lambda$$

Example 2.6 Let Λ be an algebra is given by the quiver $3 \overset{\alpha}{\twoheadrightarrow} 2 \overset{\beta}{\twoheadrightarrow} 1$ with the relation $\alpha\beta = 0$. We list wτ-tilt Λ, df-ice Λ and df-ebrick Λ in table 1.

For a given algebra Λ, wτ-tilt Λ may not be bijection to ebrickΛ even if Λ is a representation-finite algebra.

Example 2.7 Let Λ be an algebra whose quiver is $1 \leftarrow 2 \rightarrow 3$. Then #w$\tau$-tilt $\Lambda = 22$, but #ebrick$\Lambda = 26$. In fact, there are four epibricks $\mathcal{S}_1 = \{2, \frac{2}{1\,3}\}$, $\mathcal{S}_2 = \{2, \frac{2}{3}, \frac{2}{1\,3}\}$, $\mathcal{S}_3 = \{2, \frac{2}{1}, \frac{2}{1\,3}\}$, $\mathcal{S}_4 = \{\frac{2}{3}, \frac{2}{1}, \frac{2}{1\,3}\}$ are not doubly finite since Filt($\mathcal{S}_i$)$(i = 1, 2, 3, 4)$ are right Schur not ICE-closed.

A finite dimensional algebra is Nakayama if and only if its quiver is one of the following

$A_n : \quad 1 \longrightarrow 2 \longrightarrow 3 \rightarrow \cdots \rightarrow n$
$\widetilde{A}_n : \quad 1 \overset{\frown}{\longleftrightarrow} 2 \longrightarrow 3 \rightarrow \cdots \rightarrow n$

see [4, V.3.2]. Next, we will show that $w\tau$-tilt Λ is a bijection to ebrickΛ if Λ is a Nakayama algebra.

Theorem 2.8 *Let Λ be a Nakayama algebra. Then there is a bijection:*

$$w\tau\text{-tilt } \Lambda \underset{P(\mathrm{Filt}(-))}{\overset{\mathrm{Sim}(\mathrm{cok}(-))}{\rightleftarrows}} \text{ebrick}\Lambda$$

Proof [6, Theorem 6.1] stated that all left Schur subcategories are IKE-closed(i.e. closed under extensions, kernels and images) for Nakayama algebras. Now, considering the dual, we have all right Schur subcategories are ICE-closed and hence, all ebricks are doubly finite for Nakayama algebras. Now, the result follows from Theorem 2.5. □

3 On the number of wide τ-tilting modules over Nakayama algebras

Adachi[2] gave a recurrence relation for the number of τ-tilting modules over algebras of type A_n. Asai [3] also gave a recurrence relation for the number of support τ-tilting modules over algebras $\Lambda_n^r := K A_n/\mathrm{rad}^r$ and $\tilde{\Lambda}_n^r := K \widetilde{A}_n/\mathrm{rad}^r$. More recently, Gao and Schiffler[10] have extended the recurrence relations of Adachi to τ-tilting modules over $\widetilde{\Lambda}_n^r$. Next, we will give the number of wide τ-tilting modules by calculating the number of epibricks.

Proposition 3.1 *Let Λ be a Nakayama algebra of type A_n. Then*

$$\#\,w\tau\text{-tilt } \Lambda = 2\#\,w\tau\text{-tilt}(\Lambda/\langle e_n\rangle) + \sum_{i=2}^{l(I_n)} LS_{i-1} \cdot \#\,w\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle).$$

Proof Let $S_{i,j}$ be the brick of Λ with top$S_{i,j} = S_i$ and soc$S_{i,j} = S_j$.

Define W_0 as the subset of ebrickΛ consisting of those epibricks without S_n as a composition factor. It is clear that $\#W_0 = \mathrm{ebrick}(\Lambda/\langle e_n\rangle)$.

Let $W_i(i = 1, 2, \cdots, l(I_n))$ be the subset of ebrickΛ consisting of those epibricks which obtain the brick $S_{n-i+1,n}$.

First, there is a bijection $W_1 \mapsto \mathrm{ebrick}(\Lambda/\langle e_n\rangle)$ defined as $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. Hence, $\#W_1 = \mathrm{ebrick}(\Lambda/\langle e_n\rangle)$.

Second, for $i = 2, 3, ,\cdots, l(I_n)$, there exists a bijection

$$W_i \mapsto \mathrm{ebrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle) \times \mathrm{ebrick}(\Lambda/\langle 1 - e_{[n-i+1,n-1]}\rangle)$$

given by $\mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \mathrm{Supp}S \cap [n-i+1, n] = \emptyset\}, \{S \in \mathcal{S} \mid \mathrm{Supp}S \subset [n-i+1, n-1]\})$ where $\mathrm{Supp}S$ stand for the support of S. Note that ebrick$\Lambda = \bigcup\limits_{i=0}^{l(I_n)} W_i$, we obtain

$$\#\,w\tau\text{-tilt } \Lambda = \#\mathrm{ebrick}\Lambda \quad \text{(by Theorem 2.8)}$$

$$= \sum_{i=0}^{l(I_n)} \#W_i$$

$$= 2\#\mathrm{ebrick}(\Lambda/\langle e_n\rangle) + \sum_{i=2}^{l(I_n)} \#\mathrm{ebrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)$$

$$\cdot \#\mathrm{ebrick}(\Lambda/\langle 1 - e_{[n-i+1,n-1]}\rangle)$$

$$= 2\#\text{ebrick}(\Lambda/\langle e_n\rangle) + \sum_{i=2}^{l(I_n)} \#\text{ebrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)$$

$$\cdot \#\text{ebrick}(KA_{i-1})$$

$$= 2\# \text{w}\tau\text{-tilt}(\Lambda/\langle e_n\rangle) + \sum_{i=2}^{l(I_n)} \# \text{w}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)$$

$$\cdot \# \text{w}\tau\text{-tilt}(KA_{i-1})$$

$$= 2\# \text{w}\tau\text{-tilt}(\Lambda/\langle e_n\rangle) + \sum_{i=2}^{l(I_n)} LS_{i-1} \cdot \# \text{w}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle). \qquad \square$$

In particular, let $a_{n,r} := \# \text{w}\tau\text{-tilt } \Lambda_n^r$, we have the following recurrence relation.

Corollary 3.2

$$a_{n,r} = 2a_{n-1,r} + \sum_{i=2}^{r} LS_{i-1} \cdot a_{n-i,r}.$$

Remark 3.3 In [5], Enomoto shown that the number of wide τ-tilting modules over KA_n is the n-th large Schröder number $LS_n = \sum_{i=1}^{n} \frac{1}{i+1}\binom{n}{i}\binom{n+i}{i}$ by translating his problem to a combinatorial problem on a cluster complex. In fact, we can get the result directly by the proof of Proposition 3.1. Let $a_n := \# \text{w}\tau\text{-tilt}(KA_n) = a_{n,n}$ and $a_0 = 1$. We have the recurrence relation:

$$a_n = 2a_{n-1} + \sum_{i=2}^{n} a_{i-1} \cdot a_{n-i} = a_{n-1} + \sum_{i=0}^{n-1} a_i \cdot a_{n-i-1}.$$

A direct calculation shows that the generating function of a_n is $f(x) = \frac{1-x-\sqrt{x^2-6x+1}}{2x}$, which implies a_n is the n-th large Schröder number LS_n.

Now, let $b_{n,r} := \# \text{w}\tau\text{-tilt } \widetilde{\Lambda}_n^r$. Note that $M \in \text{mod } \widetilde{\Lambda}_n^r$ is a brick if and only if $l(M) \leqslant n$, we get $b_{n,r} = b_{n,n}$ when $r \geqslant n$. Hence, we only need to calculate $b_{n,r}$ for $r \leqslant n$.

Lemma 3.4

$$b_{n,r} = 2a_{n-1,r} + \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r}.$$

Proof Define V_0 as the subset of ebrick $\widetilde{\Lambda}_n^r$ consisting of the epibricks without S_n as a composition factor. It is clear that $\#V_0 = \text{ebrick}(\widetilde{\Lambda}_n^r/\langle e_n\rangle) = a_{n-1,r}$.

Define V_i $(i = 1, 2, \cdots, r)$ as the subset of ebrick $\widetilde{\Lambda}_n^r$ such that there is a brick with S_n as its composition factor and $\max\{l(S)|S \in V_i, n \in \text{Supp} S\} = i$.

Now, let $V_{i,0}$ be the subset of V_i such that $S_{n-i+1,n} \in V_{i,0}$ and $V_{i,k}(k = 1, 2, \cdots, i-1)$ be the subset of V_i such that $S_{n-i+k+1,k} \in V_{i,k}$, where

$$S_{n-i+1,n} = \begin{matrix} n-i+1 \\ n-i+2 \\ \vdots \\ n-1 \\ n \end{matrix} \quad \text{and} \quad S_{n-i+k+1,k} = \begin{matrix} n-i+k+1 \\ \vdots \\ n \\ 1 \\ 2 \\ \vdots \\ k. \end{matrix}$$

Clearly, $V_i = \bigcup\limits_{k=0}^{i-1} V_{i,k}$.

First, there is a bijection $V_{1,0} \mapsto \text{ebrick}(\widetilde{\Lambda_n^r}/\langle e_n \rangle)$ defined as $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. Hence,

$$\#V_1 = \#V_{1,0} = \text{ebrick}(\widetilde{\Lambda_n^r}/\langle e_n \rangle) = a_{n-1,r}.$$

Second, fix i and $k(i = 1, 2, \cdots, r$ and $k = 1, 2, \cdots, i-1)$, there exists a bijection

$$V_{i,k} \mapsto \text{ebrick}(\widetilde{\Lambda_n^r}/\langle e_{[n-i+k+1,n]} + e_{[1,k]}\rangle) \times \text{ebrick}(\widetilde{\Lambda_n^r}/\langle 1 - e_{[n-i+k+1,n]} - e_{[1,k-1]}\rangle)$$

given by

$$\mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \text{Supp} S \cap \{[n-i+k+1, n] \cup [1, k] = \emptyset\}, \{S \in \mathcal{S} \mid$$
$$\text{Supp} S \subset [n-i+k+1, n] \cup [1, k-1]\}).$$

Hence

$$\#V_{i,k} = \#\text{ebrick}(\widetilde{\Lambda_n^r}/\langle e_{[n-i+k+1,n]} + e_{[1,k]}\rangle) \cdot \#\text{ebrick}(\widetilde{\Lambda_n^r}/\langle 1 - e_{[n-i+k+1,n]} - e_{[1,k-1]}\rangle)$$
$$= \#\text{ebrick}\,\Lambda_{n-i}^r \cdot \#\text{ebrick}\,\Lambda_{i-1}^r$$
$$= LS_{i-1} \cdot a_{n-i,r}.$$

Therefore, $\#V_i = \sum\limits_{k=0}^{i-1} \#V_{i,k} = i \cdot LS_{i-1} \cdot a_{n-i,r}$.

Note that $\text{ebrick}\,\widetilde{\Lambda_n^r} = \bigcup\limits_{i=0}^{r} V_i$, we obtain

$$b_{n,r} = \#\text{ebrick}\,\widetilde{\Lambda_n^r} \quad \text{(by Theorem 2.8)}$$
$$= \sum\limits_{i=0}^{r} \#V_i$$
$$= 2a_{n-1,r} + \sum\limits_{i=2}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r}.$$

$\square$

As a consequence, we get the recurrence relation on $b_{n,r}$.

Proposition 3.5 *We have*

$$b_{n,r} = 2b_{n-1,r} + \sum\limits_{i=2}^{r} LS_{i-1} \cdot b_{n-i,r}.$$

Proof It follows from Corollary3.2 and Lemma 3.4,in fact,

$$b_{n,r} - 2b_{n-1,r} - \sum_{i=2}^{r} LS_{i-1} \cdot b_{n-i,r}$$

$$= 2a_{n-1,r} + \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r}$$

$$- 4a_{n-2,r} - 2\sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i-1,r}$$

$$- \sum_{i=2}^{r} LS_{i-1} \cdot \left(2a_{n-i-1,r} + \sum_{j=1}^{r} j \cdot LS_{j-1} \cdot a_{n-i-j,r} \right)$$

$$= 2(a_{n-1,r} - 2a_{n-2,r} - \sum_{i=2}^{r} LS_{i-1} \cdot a_{n-i-1,r})$$

$$+ \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r} - 2\sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i-1,r}$$

$$- \sum_{j=2}^{r} LS_{j-1} \left(\sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i-j,r} \right)$$

$$= \sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i,r} - 2\sum_{i=1}^{r} i \cdot LS_{i-1} \cdot a_{n-i-1,r}$$

$$- \sum_{i=1}^{r} i \cdot LS_{i-1} \left(\sum_{j=2}^{r} \cdot LS_{j-1} \cdot a_{n-i-j,r} \right)$$

$$= \sum_{i=1}^{r} i \cdot LS_{i-1} \left(a_{n-i,r} - 2a_{n-i-1,r} - \sum_{j=2}^{r} LS_{j-1} \cdot a_{n-i-j,r} \right)$$

$$= 0.$$

$\square$

The following proposition and its proof are similar to [3, Theorem 4.1 (3)] and [10, Proposition 3.9]. Leave it to the reader to prove it.

Proposition 3.6 *Let* $\xi_1, \xi_2, \cdots, \xi_r$ *be the roots (not necessarily distinct) of the polynomial* $F_r(X) = X^r - 2X^{r-1} - \sum_{i=2}^{r} LS_{i-1} \cdot X^{r-i}$. *Then we have*

(1) $a_{n,r} = \sum\limits_{\substack{t_1,t_2,\cdots,t_r \in \mathbb{Z}_{\geqslant 0} \\ t_1+t_2+\cdots+t_r=n}} \xi_1^{t_1} \xi_2^{t_2} \cdots \xi_r^{t_r}.$

(2) $b_{n,r} = \sum\limits_{i=1}^{r} \xi_i^n.$

We give examples of the numbers of wide τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$.

Table 2 The number of wide τ-tilting modules of Λ_n^r

$a_{n,r}$					
r	n				
	1	2	3	4	5
1	2	4	8	16	32
2	2	6	16	44	120
3	2	6	22	68	216
4	2	6	22	90	304
5	2	6	22	90	394

Table 3 The number of wide τ-tilting modules of $\widetilde{\Lambda}_n^r$

$b_{n,r}$					
r	n				
	1	2	3	4	5
1	2	4	8	16	32
2	2	8	20	56	152
3	2	8	38	104	332
4	2	8	38	192	552
5	2	8	38	192	1002

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