Periodica Mathematica Hungarica (2023) 86:503–513
https://doi.org/10.1007/s10998-022-00485-3

The classification of τ-tilting modules over algebras of type D_n

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Accepted: 30 November 2021 / Published online: 22 July 2022

Abstract

Let Λ be an algebra whose quiver is

$$D_n : 1 \to 2 \to \cdots \to n-2 \begin{array}{l} \nearrow n-1 \\ \searrow n. \end{array}$$

In this paper, we classify the τ-tilting modules over Λ when $l(P_1) \leq n-2$. Moreover, the following recurrence formula for the number of τ-tilting Λ-modules holds:

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{-tilt}\,\Lambda/\langle e_{\leq i}\rangle|,$$

where $e_{\leq i} := e_1 + e_2 + \cdots + e_i$ and $C_i = \frac{1}{i+1}\binom{2i}{i}$ is the ith Catalan number.

Keywords τ-tilting modules · Support τ-tilting modules · Nakayama algebras

Mathematics Subject Classification 16G10 · 16G20

1 Introduction

Adachi et al. [1] introduced the concept of support τ-tilting modules over finite-dimensional algebras as a generalization of tilting modules. The support τ-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes introduced in [5], 2-term silting complexes introduced in [11], cluster-tilting objects in the cluster category introduced in [6] and left finite semibricks introduced in [3]. Therefore, it is important to calculate the number of support τ-tilting modules over a given algebra.

This work was partially supported by NSFC (Grant No. 11971225).

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For a hereditary algebra, the τ-tilting modules are exactly the tilting modules and the support τ-tilting modules are exactly the support tilting modules. For Dynkin type algebras Δ_n, these numbers were first calculated in [7] via cluster algebras, and later in [12] via representation theory (see the following table).

A finite-dimensional algebra is Nakayama if and only if its quiver is one of the following:

$$A_n: \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \qquad\qquad \widetilde{A}_n: \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend right] \end{tikzcd}$$

Δ_n	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\lvert\text{tilt}\Delta_n\rvert$	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{3n-4}{2n-2}\binom{2n-2}{n-2}$	418	2431	17342
$\lvert s\text{-tilt}\Delta_n\rvert$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\frac{3n-2}{2n-1}\binom{2n-1}{n-1}$	833	4160	25080

see [4, V.3.2]. Throughout the paper, let

$$\Lambda_n^r = KA_n/\text{rad}^r \quad \text{and} \quad \widetilde{\Lambda}_n^r = K\widetilde{A}_n/\text{rad}^r.$$

Moreover, $t_r(n)$ and $\tilde{t}_r(n)$ denote the number of τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$, $s_r(n)$ and $\tilde{s}_r(n)$ denote the number of support τ-tilting modules over Λ_n^r and $\widetilde{\Lambda}_n^r$, respectively. The following recurrence relations can be found in [2, 3, 9]:

$$t_r(n) = \sum_{i=1}^{r} C_{i-1}\cdot t_r(n-i), \qquad s_r(n) = 2\cdot s_r(n-1) + \sum_{i=2}^{r} C_{i-1}\cdot s_r(n-i),$$

$$\tilde{t}_r(n) = \sum_{i=1}^{r} C_{i-1}\cdot \tilde{t}_r(n-i), \qquad \tilde{s}_r(n) = 2\cdot \tilde{s}_r(n-1) + \sum_{i=2}^{r} C_{i-1}\cdot \tilde{s}_r(n-i),$$

where $C_i = \frac{1}{i+1}\binom{2i}{i}$ is the ith Catalan number. (The four formulas hold for all $n \geq 1$ and $1 \leq r \leq n$. Moreover, the formulas for $t_r(n)$ and $s_r(n)$ additionally hold for all $r \geq n$ as long as we set $t_r(n) := 0$, $s_r(n) := 0$ for $n < 0$.)

Let $D_n (n \geq 4)$ be the quiver

$$\begin{tikzcd}[row sep=tiny] & & & & n-1 \\ 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-2 \arrow[ur] \arrow[dr] & \\ & & & & n \end{tikzcd}$$

and $D_n^r := KD_n/\text{rad}^r$. In this paper, we classify the τ-tilting modules over algebras of type D_n when $l(P_1) \leq n-2$ and prove that there are similar recurrence relations for the number of τ-tilting and support τ-tilting modules over D_n^r as for the Nakayama algebras Λ_n^r, $\tilde{\Lambda}_n^r$. The following is our first result.

Theorem 1.1 (see Theorem 2.5). *If Λ is an algebra of type D_n with $l(P_1) \leq n-2$, then there is a bijection*

$$\tau\text{-tilt}\,\Lambda \to \coprod_{i=1}^{l(P_1)} \tau\text{-tilt}\,\Lambda/\langle e_i\rangle$$

given by $\tau\text{-tilt}\,\Lambda \ni M \mapsto M/P_1$ and the inverse map is given by $N \mapsto N \oplus P_1 \in \tau\text{-tilt}\,\Lambda$.

Corollary 1.2 (see Corollary 2.8). *If Λ is an algebra of type D_n with $l(P_1) \leq n-2$, then*

$$|\tau\text{−tilt }\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{−tilt }\Lambda/\langle e_{\leq i}\rangle|,$$

where $e_{\leq i} := e_1 + e_2 + \cdots + e_i$.

Let $d_r(n)$ and $sd_r(n)$ denote the number of τ-tilting modules and support τ-tilting modules over D_n^r, respectively. The following recurrence relations hold:

Theorem 1.3 (see Theorems 2.9 and 2.12). *For $r \leq n-2$,*

(1) $d_r(n) = \sum_{i=1}^{r} C_{i-1} \cdot d_r(n-i)$;

(2) $sd_r(n) = 2 \cdot sd_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot sd_r(n-i)$.

Throughout this paper, Λ will always denote a basic, connected, finite-dimensional K-algebra over an algebraically closed field K. We denote by mod Λ the category of finitely generated right Λ-modules and τ the Auslander–Reiten translation of Λ. Let n be the number of nonisomorphic simple modules over Λ. We also denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M and $l(M)$ the Loewy length of M for $M \in$ mod Λ. For a finite set X, we denote by $|X|$ the cardinality of X. For two sets X_1 and X_2, $X_1 \coprod X_2$ stands for disjoint union. Finally, we will freely use the well-known classification of indecomposable modules over type D algebras given in, e.g., [14, Section 3.3].

2 Main results

Let Λ be an algebra. We first recall the definition of support τ-tilting modules over Λ.

Definition 2.1 ([1, Definition 0.1]). Suppose $M \in$ mod Λ.

(1) M is *τ-rigid* if $\mathrm{Hom}_{\Lambda}(M, \tau M) = 0$.
(2) M is *τ-tilting* if it is τ-rigid and $|M| = |\Lambda|$.
(3) M is *support τ-tilting* if it is a τ-tilting $\Lambda/\langle e\rangle$-module for some idempotent e of Λ.
(4) M is *proper support τ-tilting* if it is a support τ-tilting but not a τ-tilting Λ-module.

Remark 2.2 (1) A $\Lambda/\langle e\rangle$-module is τ-rigid as a $\Lambda/\langle e\rangle$-module if and only if it is τ-rigid as a Λ-module (see [1, Lemma 2.1(b)]).
(2) Recall that $M \in$ mod Λ is called *sincere* if every simple Λ-module appears as a composition factor in M. By [1, Proposition 2.2(a)], the τ-tilting modules are exactly the sincere support τ-tilting modules.

We will denote by τ−tilt Λ (respectively, sτ-tilt Λ, psτ-tilt Λ) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting, proper support τ-tilting) Λ-modules. Obviously, $|\text{s}\tau\text{-tilt }\Lambda| = |\tau\text{−tilt }\Lambda| + |\text{ps}\tau\text{-tilt }\Lambda|$.

Let P_i be the indecomposable projective module, I_i the indecomposable injective module and S_i the simple module of an algebra KQ/I corresponding to the vertex i for $i = 1, 2, \ldots, n$.

Now, we will assume that Λ is an algebra of type D_n with $l(P_1) \leq n-2$.

Lemma 2.3 *If M is a τ-rigid Λ-module, then $M \oplus P_1$ is also a τ-rigid Λ-module. Moreover, each τ-tilting Λ-module has P_1 as a direct summand.*

Proof Since $l(P_1) \leq n-2$ and the vertex 1 is a source, all modules which have S_1 as a composition factor must be injective. Hence, τM has no S_1 as a composition factor. Thus, $\mathrm{Hom}_\Lambda(P_1, \tau M) = 0$. Therefore, $M \oplus P_1$ is a τ-rigid Λ-module.

If N is a τ-tilting Λ-module, then it is τ-rigid and hence $N \oplus P_1$ is also τ-rigid. It follows from [1, Theorems 2.10 and 2.12] that N has P_1 as a direct summand. □

Lemma 2.4 *For $0 < t < n-2$, there is an almost split sequence*

$$0 \longrightarrow \mathrm{rad}\, P_1/\mathrm{rad}^{t+1} P_1 \xrightarrow{\binom{q}{i}} (\mathrm{rad}\, P_1/\mathrm{rad}^{t} P_1) \oplus P_1/\mathrm{rad}^{t+1} P_1 \xrightarrow{(-j,\, p)} P_1/\mathrm{rad}^{t} P_1 \longrightarrow 0$$

in $\mathrm{mod}\,\Lambda$*, where q and p are the canonical epimorphisms and i and j are the inclusion homomorphisms. Hence, $\tau(P_1/\mathrm{rad}^t P_1) = \mathrm{rad}\, P_1/\mathrm{rad}^{t+1} P_1$.*

Proof The given sequence is easily seen to be exact. It is not split and has indecomposable end terms; hence by [4, IV. 1.13], we only need to prove that the homomorphism $g := \binom{q}{i}$ is left almost split in $\mathrm{mod}\,\Lambda$. Let N be an indecomposable Λ-module and $f : \mathrm{rad}\, P_1/\mathrm{rad}^{t+1} P_1 \to N$ be a nonisomorphism. We want to show that f can be factored through g.

Let $\Lambda_1 := \Lambda/\langle e_{n-1} + e_n\rangle$. Then the quiver of Λ_1 is $1 \to 2 \to \cdots \to n-2$ and is a subquiver of Λ. Moreover, Λ_1 is a Nakayama algebra. By [4, V. Theorem 4.1], the above sequence is an almost split sequence in $\mathrm{mod}\,\Lambda_1$. If N has no S_{n-1} and S_n as composition factors, then N is a Λ_1-module and hence f can be factored through g. If N has S_{n-1} or S_n as composition factors, then $f = 0$. So f can be factored through g. □

Now, we can prove our main theorem as follows.

Theorem 2.5 *There is a bijection*

$$\tau\mathrm{-tilt}\,\Lambda \to \coprod_{i=1}^{l(P_1)} \tau\mathrm{-tilt}\,\Lambda/\langle e_i\rangle$$

given by $\tau\mathrm{-tilt}\,\Lambda \ni M \mapsto M/P_1$ and the inverse map is given by $N \mapsto N \oplus P_1 \in \tau\mathrm{-tilt}\,\Lambda$.

Proof Suppose $M \in \tau\mathrm{-tilt}\,\Lambda/\langle e_i\rangle$ for some $1 \leq i \leq l(P_1)$. Then M is a τ-rigid Λ-module by Remark 2.2(1) and hence $M \oplus P_1$ is a τ-tilting Λ-module since it is τ-rigid by Lemma 2.3 and $|M \oplus P_1| = |M| + 1 = |\Lambda|$.

Conversely, suppose $M \in \tau\mathrm{-tilt}\,\Lambda$. Then we decompose M as $M = P_1 \oplus N \oplus L$ by Lemma 2.3 where N is a maximal direct summand of M consisting of $\Lambda/\langle e_1\rangle$-module and $L \ncong P_1$ is a direct summand of M with S_1 as a composition factor. If $L = 0$, then N is a τ-tilting $\Lambda/\langle e_1\rangle$-module clearly. Assume that $L \neq 0$. Since $l(P_1) \leq n-2$, we have $L \in \Lambda/\langle e_{j+1}\rangle$ for $1 \leq j := l(L) < l(P_1)$. Now, we claim that $N \in \mathrm{mod}\,\Lambda/\langle e_{j+1}\rangle$. Indeed, if it does not hold, then there is an indecomposable module Y which is a direct summand of N and has S_{j+1} as a composition factor. Take $X := P_1/\mathrm{rad}^j P_1 \in \mathrm{add}\, L$. By Lemma 2.4, we have $\tau X = \mathrm{rad}\, P_1/\mathrm{rad}^{j+1} P_1$. Thus, $\mathrm{top}\,(\tau X) = S_2$ and $\mathrm{Soc}\,(\tau X) = S_{j+1}$. Note that $\mathrm{mod}\,\Lambda/\langle e_n + e_{n-1}\rangle$ is a full subcategory of $\mathrm{mod}\,\Lambda$ when $l(P_1) \leq n-2$ ($\Lambda_1 := \Lambda/\langle e_n + e_{n-1}\rangle$ is a Nakayama algebra). If $Y \in \mathrm{mod}\,\Lambda_1$, then $\mathrm{Hom}_\Lambda(Y, \tau X) \cong \mathrm{Hom}_{\Lambda_1}(Y, \tau X) \neq 0$ by [2, Lemma 2.4]. If Y has either S_n or S_{n-1} as composition factors, then $Y/\mathrm{Soc}\, Y \in \mathrm{mod}\,\Lambda_1$. Note that there is a exact sequence $0 \to \mathrm{Soc}\, Y \to Y \to Y/\mathrm{Soc}\, Y \to 0$ and $\mathrm{Hom}_\Lambda(\mathrm{Soc}\, Y, \tau X) =$

0, we have $\operatorname{Hom}_{\Lambda}(Y, \tau X) \cong \operatorname{Hom}_{\Lambda}(Y/\operatorname{Soc} Y, \tau X) \cong \operatorname{Hom}_{\Lambda_1}(Y/\operatorname{Soc} Y, \tau X) \neq 0$ since $Y/\operatorname{Soc} Y$ has S_{j+1} as a composition factor. This contradicts that M is τ-rigid. Thus, $N \oplus L$ is a τ-tilting mod $\Lambda/\langle e_{j+1}\rangle$-module. □

Theorem 2.5 gives a method of constructing τ-tilting Λ-modules over algebras of type D_n with $l(P_1) = 2$.

Corollary 2.6 *Let Λ be an algebra of type D_n with $l(P_1) = 2$. Then all τ-tilting Λ-modules are exactly those forms $P_1 \oplus M_1$ and $P_1 \oplus S_1 \oplus M_2$, where M_1 and M_2 are τ-tilting modules over $\Lambda/\langle e_1\rangle$ and $\Lambda/\langle e_1 + e_2\rangle$, respectively.*

Example 2.7 (1) Let Λ be an algebra given by the quiver

$$\begin{tikzcd}[row sep=small] & & 3 \\ 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & 4 \end{tikzcd}$$

with $\operatorname{rad}^2 = 0$. Since $\Lambda/\langle e_1\rangle$ is the algebra given by the quiver $3 \leftarrow 2 \rightarrow 4$, we have five τ-tilting $\Lambda/\langle e_1\rangle$-modules as follows:

$$3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 2, \quad 2\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}.$$

We only have one τ-tilting $\Lambda/\langle e_1 + e_2\rangle$-module: $3\ 4$. Hence, we get all τ-tilting Λ-modules by Corollary 2.6:

$$\begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 2, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 2\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}1\\2\end{smallmatrix}\ 1\ 3\ 4.$$

(2) Let Γ be an algebra given by the quiver

$$\begin{tikzcd}[row sep=small] & & & 3 \\ 5 \arrow[r] & 1 \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\ & & & 4 \end{tikzcd}$$

with $\operatorname{rad}^2 = 0$. Since $\Gamma/\langle e_5\rangle \cong \Lambda$ and $\Gamma/\langle e_5 + e_1\rangle \cong \Lambda/\langle e_1\rangle$, we have all τ-tilting Γ-modules by Corollary 2.6:

$$\begin{gathered}\begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 2, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ 2\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ 1\ 3\ 4, \\ \begin{smallmatrix}5\\1\end{smallmatrix}\ 5\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ 5\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 4, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ 5\ 3\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ 5\ \begin{smallmatrix}2\\4\end{smallmatrix}\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ 2, \quad \begin{smallmatrix}5\\1\end{smallmatrix}\ 5\ 2\ \begin{smallmatrix}2\\3\,4\end{smallmatrix}\ \begin{smallmatrix}2\\3\end{smallmatrix}.\end{gathered}$$

As an application of Theorem 2.5, we immediately get a recurrence relation for $|\tau\text{−tilt}\,\Lambda|$.

Corollary 2.8 *If Λ is an algebra of type D_n with $l(P_1) \leq n - 2$, then*

$$|\tau\text{−tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{−tilt}\,\Lambda/\langle e_{\leq i}\rangle|,$$

where $e_{\leq i} := e_1 + e_2 + \cdots + e_i$.

Proof By Theorem 2.5, we have

$$|\tau\text{−tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} |\tau\text{−tilt}\,\Lambda/\langle e_i\rangle|.$$

Since $l(P_1) \leq n-2$, we get $\Lambda/\langle e_i\rangle \cong \Lambda/\langle e_{\geq i}\rangle \times \Lambda/\langle e_{\leq i}\rangle$, where $e_{\leq i} := e_1 + e_2 + \cdots + e_i$ and $e_{\geq i} := e_i + e_{i+1} + \cdots + e_n$ for $1 \leq i \leq l(P_1)$. Thus, there is a bijection

$$\tau\text{−tilt } \Lambda/\langle e_{\geq i}\rangle \times \tau\text{−tilt } \Lambda/\langle e_{\leq i}\rangle \to \tau\text{−tilt } \Lambda/\langle e_i\rangle$$

given by $(N, L) \mapsto N \oplus L$. Therefore,

$$|\tau\text{−tilt } \Lambda/\langle e_i\rangle| = |\tau\text{−tilt } \Lambda/\langle e_{\geq i}\rangle| \cdot |\tau\text{−tilt } \Lambda/\langle e_{\leq i}\rangle|.$$

Note that $\Lambda/\langle e_{\geq i}\rangle$ is the hereditary algebra of type $\mathbb{A}_{i-1}$, and hence $|\tau\text{−tilt } \Lambda/\langle e_{\geq i}\rangle| = C_{i-1}$. Thus, the assertion follows. □

Recall that $t_r(n)$ stands for the number of τ-tilting modules over $\Lambda_n^r = KA_n/\text{rad}^r$. We use the notations $d_r(n)$ for the number of τ-tilting modules over $D_n^r = KD_n/\text{rad}^r$ $(n \geq 4, r \geq 1)$ and $sd_r(n)$ for the number of support τ-tilting modules over D_n^r. Note that $D_n^1 \cong K^n$, we have $d_1(n) = 1$ and $sd_1(n) = 2^n$ for all $n \geq 4$. For convenience, we also define $D_3^r = K(\bullet \leftarrow \bullet \rightarrow \bullet)$ and $D_2^r = K \times K$ (hence, $d_r(3) = 5, d_r(2) = 1, sd_r(3) = 14, sd_r(2) = 4$,) for all $r \geq 2$.

Theorem 2.9 *We have*

$$d_r(n) = \begin{cases} \sum_{i=1}^{r} C_{i-1} \cdot d_r(n-i) & \text{if } r \leq n-2; \\ \frac{3n-4}{2n-2}\binom{2n-2}{n-2} & \text{if } r \geq n-1. \end{cases}$$

Proof The above discussion shows that the formula holds for $r = 1$.

Now, suppose $r \geq 2$. Take $\Lambda := D_n^r$. If $r \geq n-1$, then Λ is hereditary and so $d_r(n) = \frac{3n-4}{2n-2}\binom{2n-2}{n-2}$ [12, Theorem 1]. If $r \leq n-2$, then $\Lambda/\langle e_{\leq i}\rangle \cong D_{n-i}^r$, hence we get the result by Corollary 2.8. □

The support τ-tilting D_n^r-modules can be calculated by considering the τ-tilting modules over quotient algebras of D_n^r. We will construct a formula for $sd_r(n)$ but for this we need the following lemma.

Lemma 2.10 *If V is the set of all support τ-tilting Λ_n^r-modules which have $S_1, S_2, \ldots, S_{n-1}$ as composition factors, then*

$$|V| = t_r(n-1) + t_r(n).$$

Proof If V_1 is the set of all support τ-tilting Λ_n^r-modules which have $S_1, S_2, \ldots, S_{n-1}$ as composition factors but not S_n, then $V = V_1 \coprod \tau\text{−tilt } \Lambda_n^r$. Hence,

$$|V| = |V_1| + |\tau\text{−tilt } \Lambda_n^r| = t_r(n-1) + t_r(n).$$

□

Proposition 2.11 *For $n \geq 4$,*

$$sd_r(n) = \sum_{i=1}^{n-4} t_r(i-1)sd_r(n-i) + 14t_r(n-4) + 4t_r(n-3) + t_r(n-2) \\ +2t_r(n-1) + d_r(n).$$

Proof Let $\Lambda := D_n^r$ and W_i $(i = 1, 2, \ldots, n)$ denote the set of all support τ-tilting Λ-modules which have $S_1, S_2, \ldots, S_{i-1}$ as composition factors but not S_i.

For a given $1 \leq i \leq n-4$, we have $\Lambda/\langle e_i\rangle \cong \Lambda/\langle e_{\geq i}\rangle \times \Lambda/\langle e_{\leq i}\rangle$ where $e_{\geq i} := e_i + e_{i+1} + \cdots + e_n$ and $e_{\leq i} := e_1 + e_2 + \cdots + e_i$. Hence, there is a bijection

$$\tau\text{−tilt}\, \Lambda/\langle e_{\geq i}\rangle \times s\tau\text{-tilt}\, \Lambda/\langle e_{\leq i}\rangle \to W_i$$

given by $(M, N) \mapsto M \oplus N$. Note that $\Lambda/\langle e_{\geq i}\rangle \cong \Lambda_{i-1}^r$ and $\Lambda/\langle e_{\leq i}\rangle \cong D_{n-i}^r$, therefore

$$\begin{aligned}|W_i| &= |\tau\text{−tilt}\, \Lambda/\langle e_{\geq i}\rangle| \cdot |s\tau\text{-tilt}\, \Lambda/\langle e_{\leq i}\rangle| \\ &= |\tau\text{−tilt}\, \Lambda_{i-1}^r| \cdot |s\tau\text{-tilt}\, D_{n-i}^r| \\ &= t_r(i-1)sd_r(n-i).\end{aligned}$$

Since $\Lambda/\langle e_{n-3}\rangle$ is the product of the algebra Λ_{n-4}^r and the algebra $A_3 :\ n-1 \leftarrow n-2 \rightarrow n$,

$$|W_{n-3}| = |\tau\text{−tilt}\, \Lambda_{n-4}^r| \cdot |s\tau\text{-tilt}\, A_3| = 14t_r(n-4).$$

Note that $\Lambda/\langle e_{n-2}\rangle \cong \Lambda_{n-3}^r \times K \times K$, so

$$|W_{n-2}| = 4|\tau\text{−tilt}\, \Lambda_{n-3}^r| = 4t_r(n-3).$$

$\Lambda/\langle e_{n-1}\rangle$ is the algebra given by the quiver $1 \to 2 \to \cdots \to n-2 \to n$ with $\mathrm{rad}^r = 0$. Hence W_{n-1} is exactly the set of all support τ-tilting $\Lambda/\langle e_{n-1}\rangle$-modules which have $S_1, S_2, \ldots, S_{n-2}$ as composition factors. It follows from Lemma 2.10 that

$$|W_{n-1}| = t_r(n-2) + t_r(n-1).$$

Clearly, $\Lambda/\langle e_n\rangle \cong \Lambda_{n-1}^r$, so

$$|W_n| = t_r(n-1).$$

Note that $ps\tau\text{-tilt}\, \Lambda = \coprod_{i=1}^{n} W_i$, therefore

$$\begin{aligned}|ps\tau\text{-tilt}\, \Lambda| &= \sum_{i=1}^{n} W_i \\ &= \sum_{i=1}^{n-4} t_r(i-1)sd_r(n-i) + 14t_r(n-4) + 4t_r(n-3) \\ &\quad + t_r(n-2) + 2t_r(n-1).\end{aligned}$$

Hence,

$$\begin{aligned}sd_r(n) &= |ps\tau\text{-tilt}\, \Lambda| + |\tau\text{−tilt}\, \Lambda| \\ &= \sum_{i=1}^{n-4} t_r(i-1)sd_r(n-i) + 14t_r(n-4) + 4t_r(n-3) + t_r(n-2) \\ &\quad + 2t_r(n-1) + d_r(n).\end{aligned}$$

□

As a consequence, the following result holds.

Theorem 2.12 *We have*

$$sd_r(n) = \begin{cases} 2 \cdot sd_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot sd_r(n-i) & \text{if } r \leq n-2; \\ \frac{3n-2}{2n-1}\binom{2n-1}{n-1} & \text{if } r \geq n-1. \end{cases}$$

Proof If $r \geq n-1$, then D_n^r is hereditary and so $sd_r(n) = \frac{3n-2}{2n-1}\binom{2n-1}{n-1}$ [12, Theorem 1].
If $r \leq n-2$, then

$$\begin{aligned}
&sd_r(n) - sd_r(n-1) \\
&= \sum_{l=1}^{n-4} t_r(l-1)sd_r(n-l) + 14t_r(n-4) + 4t_r(n-3) + t_r(n-2) + 2t_r(n-1) + d_r(n) \\
&\quad - \left(\sum_{l=1}^{n-5} t_r(l-1)sd_r(n-1-l) + 14t_r(n-5) + 4t_r(n-4)\right. \\
&\quad \left. + t_r(n-3) + 2t_r(n-2) + d_r(n-1)\right) \\
&= sd_r(n-1) + \sum_{l=2}^{n-4} \left(t_r(l-1) - t_r(l-2)\right) sd_r(n-l) + 14\left(t_r(n-4) - t_r(n-5)\right) \\
&\quad + 4\left(t_r(n-3) - t_r(n-4)\right) + \left(t_r(n-2) - t_r(n-3)\right) + 2\left(t_r(n-1)\right. \\
&\quad \left. - t_r(n-2)\right) + \left(d_r(n) - d_r(n-1)\right) \\
&= sd_r(n-1) + \sum_{l=2}^{n-4}\sum_{i=2}^{r} C_{i-1}t_r(l-1-i)sd_r(n-l) + 14\sum_{i=2}^{r} C_{i-1}t_r(n-4-i) \\
&\quad + 4\sum_{i=2}^{r} C_{i-1}t_r(n-3-i) + \sum_{i=2}^{r} C_{i-1}t_r(n-2-i) \\
&\quad + 2\sum_{i=2}^{r} C_{i-1}t_r(n-1-i) + \sum_{i=2}^{r} C_{i-1}d_r(n-i) \\
&= sd_r(n-1) + \sum_{i=2}^{r} C_{i-1}\Big(\sum_{l=1}^{n-i-4} t_r(l-1)sd_r(n-i-l) + 14t_r(n-i-4) \\
&\quad + 4t_r(n-i-3) + t_r(n-i-2) + 2t_r(n-i-1) + d_r(n-i)\Big) \\
&= sd_r(n-1) + \sum_{i=2}^{r} C_{i-1}sd_r(n-i).
\end{aligned}$$

The first and last equations follow from Proposition 2.11. The third equation follows from the recurrence relations of $t_r(n)$ and $d_r(n)$. The fourth equation holds since $t_r(n) = 0$ for $n < 0$.

Hence, $sd_r(n) = 2 \cdot sd_r(n-1) + \sum_{i=2}^{r} C_{i-1} \cdot sd_r(n-i)$. □

Now, we obtain the recurrence relations of $d_2(n)$ and $sd_2(n)$.

Corollary 2.13

$$d_2(n) = d_2(n-1) + d_2(n-2),$$
$$sd_2(n) = 2sd_2(n-1) + sd_2(n-2).$$

Proof It follows from Theorems 2.9 and 2.12. $\square$

As generalizations of simple modules and semisimple modules, bricks and semibricks are considered and they have long been studied in [8, 13]. Let Λ be an algebra. A Λ-module M is called a *brick* if $\mathrm{Hom}_{\Lambda}(M, M)$ is a K-division algebra and a *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. A semibrick $\mathcal{S}$ is called *left finite* [3] if the smallest torsion class $T(\mathcal{S})$ containing $\mathcal{S}$ is functorially finite. It is proved that there exists a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and the set of left finite semibricks of Λ (see [3, Theorem 2.3]). Note that every torsion class is functorially finite for a representation-finite algebra. Hence, there exists a bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and the set $\mathrm{sbrick}\,\Lambda$ of semibricks of Λ for an algebra of type D_n.

Now, we obtain a proof that is different from that of Theorem 2.12.

Proposition 2.14 *If Λ is an algebra of type D_n with $l(P_1) \le n-2$, then*

$$|\mathrm{s}\tau\text{-tilt}\,\Lambda| = 2|\mathrm{s}\tau\text{-tilt}\,(\Lambda/\langle e_1\rangle)| + \sum_{i=2}^{l(P_1)} C_{i-1}\cdot|\mathrm{s}\tau\text{-tilt}\,(\Lambda/\langle e_{\le i}\rangle)|,$$

Proof Suppose X is a brick of Λ such that $\mathrm{top}\,X = S_i$, $\mathrm{Soc}\,X = S_j$ and X has no S_n and S_{n-1} as composition factors. We write $S_{i,j} := X$.

We define W_0 as the subset of $\mathrm{sbrick}\,\Lambda$ consisting of the semibricks without S_1 as a composition factor. It is clear that $|W_0| = |\mathrm{sbrick}\,(\Lambda/\langle e_1\rangle)|$.

Let W_i $(i = 1, 2, \ldots, l(P_1))$ be the subset of $\mathrm{sbrick}\,\Lambda$ consisting of the semibricks which contain the brick $S_{1,i}$.

Firstly, there is a bijection $W_1 \mapsto \mathrm{sbrick}\,(\Lambda/\langle e_1\rangle)$ defined as $\mathcal{S} \mapsto \mathcal{S}\setminus\{S_{1,1}\}$. Hence,

$$|W_1| = |\mathrm{sbrick}\,(\Lambda/\langle e_1\rangle)|.$$

Secondly, there is a bijection $W_2 \mapsto \mathrm{sbrick}\,(\Lambda/\langle e_1+e_2\rangle)$ defined as $\mathcal{S} \mapsto \mathcal{S}\setminus\{S_{1,2}\}$. Hence,

$$|W_2| = |\mathrm{sbrick}\,(\Lambda/\langle e_1+e_2\rangle)|.$$

Thirdly, for $i = 3, \ldots, l(P_1)$, there exists a bijection

$$W_i \mapsto \mathrm{sbrick}\,(\Lambda/\langle e_{[1,i]}\rangle) \times \mathrm{sbrick}\,(\Lambda/\langle 1-e_{[2,i-1]}\rangle)$$

given by $\mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \mathrm{Supp}\,S \cap \{1,2,3,\ldots,i\} = \emptyset\}, \{S \in \mathcal{S} \mid \mathrm{Supp}\,S \subset \{2,3,\ldots,i-1\}\})$, where $\mathrm{Supp}\,S$ stands for the support of S and $e_{[j,k]} = e_j + e_{j+1} + \cdots + e_k$ for all $j \le k$.

Since sbrick $\Lambda = \coprod_{i=0}^{l(P_1)} W_i$, we obtain

$$\begin{aligned}
|s\tau\text{-tilt}\,\Lambda| &= |\text{sbrick}\,\Lambda| \\
&= \sum_{i=0}^{l(P_1)} |W_i| \\
&= 2|\text{sbrick}\,(\Lambda/\langle e_1\rangle)| + |\text{sbrick}\,(\Lambda/\langle e_1+e_2\rangle)| \\
&\quad + \sum_{i=3}^{l(P_1)} |\text{sbrick}\,(\Lambda/\langle e_{[1,i]}\rangle)| \cdot |\text{sbrick}\,(\Lambda/\langle 1-e_{[2,i-1]}\rangle)| \\
&= 2|\text{sbrick}\,(\Lambda/\langle e_1\rangle)| + |\text{sbrick}\,(\Lambda/\langle e_1+e_2\rangle)| \\
&\quad + \sum_{i=3}^{l(P_1)} |\text{sbrick}\,(\Lambda/\langle e_{[1,i]}\rangle)| \cdot |\text{sbrick}\,(KA_{i-2})| \\
&= 2|s\tau\text{-tilt}\,(\Lambda/\langle e_1\rangle)| + |s\tau\text{-tilt}\,(\Lambda/\langle e_1+e_2\rangle)| \\
&\quad + \sum_{i=3}^{l(P_1)} |s\tau\text{-tilt}\,(\Lambda/\langle e_{[1,i]}\rangle)| \cdot |s\tau\text{-tilt}\,(KA_{i-2})| \\
&= 2|s\tau\text{-tilt}\,(\Lambda/\langle e_1\rangle)| + \sum_{i=2}^{l(P_1)} C_{i-1} \cdot |s\tau\text{-tilt}\,(\Lambda/\langle e_{[1,i]}\rangle)| \\
&= 2|s\tau\text{-tilt}\,(\Lambda/\langle e_1\rangle)| + \sum_{i=2}^{l(P_1)} C_{i-1} \cdot |s\tau\text{-tilt}\,(\Lambda/\langle e_{\le i}\rangle)|.
\end{aligned}$$

□

Remark 2.15 (1) A semibrick $\mathcal{S}$ is called *sincere* if the direct sum of all bricks in $\mathcal{S}$ is sincere. It is shown that there is a bijection between the set of τ-tilting Λ-modules and the set of sincere left finite semibricks of Λ (see [10, Corollary 2.6]). Hence, we can obtain Corollary 2.8 similarly to the proof of Proposition 2.14.

(2) Let $D'_n (n \ge 4)$ be the quiver

$$\begin{array}{l} \qquad\qquad\qquad\qquad\quad n-1 \\ 1 \leftarrow 2 \leftarrow \cdots \leftarrow n-2 \;\; \swarrow \\ \qquad\qquad\qquad\qquad\quad\;\; \nwarrow \\ \qquad\qquad\qquad\qquad\qquad n \end{array}$$

and Γ an algebra of type D'_n. Since there is a bijection between $s\tau\text{-tilt}\,\Gamma$ and $s\tau\text{-tilt}\,\Gamma^{op}$ with Γ^{op} the opposite algebra of Γ [1, Theorem 2.14], we have $|s\tau\text{-tilt}\,\Gamma| = 2|s\tau\text{-tilt}\,(\Gamma/\langle e_1\rangle)| + \sum_{i=2}^{l(I_1)} C_{i-1} \cdot |s\tau\text{-tilt}\,(\Gamma/\langle e_{\le i}\rangle)|$ when $l(I_1) \le n-2$ (it can also be obtained by considering the number of semibricks of Γ). It is difficult to classify τ-tilting Γ-modules similarly to Theorem 2.5 since it can be not true that every τ-tilting Γ-module has I_1 as direct summand. For example, let $\Gamma := KD'_n/\text{rad}^2$. Then $P_1 \oplus P_3 \oplus P_4 \oplus I_2$ is τ-tilting but I_1 is not a direct summand. However, we can classify the sincere left finite semibricks of Γ and then get all τ-tilting Γ-modules.

Example 2.16 We give some examples of the numbers of τ-tilting modules and support τ-tilting modules of D^r_n(see Table 1 and Table 2).

Table 1 The number of τ-tilting modules of D_n^r

$d_r(n)$	n				
r	4	5	6	7	8
1	1	1	1	1	1
2	6	11	17	28	45
3	20	27	57	124	235
4	20	77	112	254	620

Table 2 The number of support τ-tilting modules of D_n^r

$sd_r(n)$	n				
r	4	5	6	7	8
1	16	32	64	128	256
2	32	78	118	4548	1026
3	50	120	314	848	2250
4	50	182	458	1258	3588

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