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# ON THE NUMBER OF $\tau$-TILTING MODULES OVER THE AUSLANDER ALGEBRAS OF RADICAL SQUARE ZERO NAKAYAMA ALGEBRAS

BY

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**Abstract.** Let $\Lambda_n$ be a radical square zero Nakayama algebra with $n$ simple modules and $\Gamma_n$ the Auslander algebra of $\Lambda_n$. We calculate the number $|\tau\text{-tilt}\,\Gamma_n|$ of $\tau$-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ of support $\tau$-tilting modules over $\Gamma_n$. In particular, we prove the recurrence relations

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|,$$

from which the exact values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ are derived.

**1. Introduction.** The starting point of tilting theory was the introduction of tilting modules over a hereditary algebra by Happel and Ringel [10]. Ever since, the study of tilting modules has been an important branch in the representation theory of finite-dimensional algebras.

In 2014, Adachi, Iyama and Reiten [1] introduced $\tau$-tilting theory replacing the rigidity condition $\mathrm{Ext}^1_\Lambda(M,M) = 0$ for a tilting module by the weaker condition $\mathrm{Hom}_\Lambda(M,\tau M) = 0$ for a $\tau$-tilting module, where $\Lambda$ is a finite-dimensional algebra and $\tau$ is the Auslander–Reiten translation. The support $\tau$-tilting modules are in bijection with some important objects in representation theory including functorially finite torsion classes introduced in [5], 2-term silting complexes introduced in [13], cluster-tilting objects in the cluster category and left finite semibricks introduced in [3]. Therefore, it is important to calculate the number of support $\tau$-tilting modules over a given algebra.

For hereditary algebras, the (support) $\tau$-tilting modules are exactly the (support) tilting modules. For algebras of Dynkin type, the numbers of these

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modules were first calculated via cluster algebras [7], and later via representation theory [14]. In particular, over a hereditary algebra of type $\mathbb{A}_n$, the number of tilting modules is $C_n$ and the number of support tilting modules is $C_{n+1}$, where $C_i$ is the $i$th Catalan number $\frac{1}{i+1}\binom{2i}{i}$.

Recall from [4, V.3.2] that a finite-dimensional algebra is *Nakayama* if its quiver is one of the following:

$$A_n: \quad 1 \longrightarrow 2 \longrightarrow 3 \to \cdots \to n, \qquad \widetilde{A}_n: \quad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & \cdots \arrow[r] & n \arrow[llll, bend right] \end{tikzcd}.$$

Adachi [2] gave a recurrence relation for the number of $\tau$-tilting modules over Nakayama algebras of type $A_n$. Asai [3] also gave a recurrence relation for the number of support $\tau$-tilting modules over Nakayama algebras $KA_n/\mathrm{rad}^r$ and $K\widetilde{A}_n/\mathrm{rad}^r$. More recently, Gao and Schiffler [9] extended the recurrence relation of Adachi to $\tau$-tilting modules over $K\widetilde{A}_n/\mathrm{rad}^r$.

It was showed in [6] that the number of tilting modules over the Auslander algebra of $K[x]/(x^n)$ is $n!$. Kajita [12] calculated the number of tilting modules over the Auslander algebra of a hereditary algebra of Dynkin type. Iyama and Zhang [11] classified the support $\tau$-tilting modules over the Auslander algebra of $K[x]/(x^n)$, and they also showed that there is a bijection between the set of support $\tau$-tilting modules over the Auslander algebra of $K[x]/(x^n)$ and the symmetric group of degree $n$. More recently, Zhang [16] calculated the number of tilting modules over the Auslander algebra $\mathit{\Gamma}_n$ of a radical square zero Nakayama algebra $\mathit{\Lambda}_n$. In particular, Zhang proved that the number of tilting modules over $\mathit{\Gamma}_n$ is $2^{n-1}$ if $\mathit{\Lambda}_n$ is of type $A_n$; and it is $2^n$ if $\mathit{\Lambda}_n$ is of type $\widetilde{A}_n$.

In this paper, we calculate the number $|\tau\text{-tilt}\,\mathit{\Gamma}_n|$ of $\tau$-tilting modules and the number $|\mathrm{s}\tau\text{-tilt}\,\mathit{\Gamma}_n|$ of support $\tau$-tilting modules over the Auslander algebra $\mathit{\Gamma}_n$ of a radical square zero Nakayama algebra $\mathit{\Lambda}_n$. Our result is as follows.

THEOREM 1.1 (Theorems 3.1, 3.5, 4.2 and 4.3). *Let $\mathit{\Gamma}_n$ be the Auslander algebra of a radical square zero Nakayama algebra $\mathit{\Lambda}_n$.*

(1) *If $\mathit{\Lambda}_n$ is of type $A_n$, then*

$$|\tau\text{-tilt}\,\mathit{\Gamma}_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\mathit{\Gamma}_n| = \frac{(3+2\sqrt{3})^n - (3-2\sqrt{3})^n}{2\sqrt{3}}.$$

(2) *If $\mathit{\Lambda}_n$ is of type $\widetilde{A}_n$, then*

$$|\tau\text{-tilt}\,\mathit{\Gamma}_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n},$$

$$|\mathrm{s}\tau\text{-tilt}\,\mathit{\Gamma}_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

The paper is organized as follows. In Section 2, we fix some notations and recall several results about $\tau$-tilting modules and Auslander algebras of radical square zero Nakayama algebras. In Section 3, we show that if $\Lambda_n$ is of type $A_n$, then there are recurrence relations

$$|\tau\text{-tilt}\,\Gamma_n| = 3|\tau\text{-tilt}\,\Gamma_{n-1}| + |\tau\text{-tilt}\,\Gamma_{n-2}|,$$
$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|.$$

In Section 4, we prove the same recurrence relations for $\Lambda_n$ of type $\widetilde{A}_n$. From these recurrence relations the exact values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ are derived. Finally, we list the values of $|\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ for $1 \le n \le 8$ in a table in Section 5.

**2. Preliminaries.** Throughout this paper, all algebras are basic, connected, finite-dimensional $K$-algebras over an algebraically closed field $K$. For an algebra $\Lambda$, we denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated right $\Lambda$-modules and by $\tau$ the Auslander–Reiten translation of $\Lambda$. We use $P_i$, $I_i$ and $S_i$ to denote the indecomposable projective, injective and simple modules of an algebra corresponding to the vertex $i$ respectively. For any $i, j \in \{1, \dots, n\}$, we write $[i, j] = \{i, i+1, \dots, j\}$ if $i \le j$; otherwise, $[i, j] = \emptyset$. Let $e_i$ be the primitive idempotent element of an algebra corresponding to the vertex $i$. We write $e_{[i,j]} := e_i + e_{i+1} + \dots + e_j$.

For a module $M \in \mathrm{mod}\,\Lambda$, we write $|M|$ for the number of pairwise non-isomorphic indecomposable summands of $M$, and use $l(M)$ and $\mathrm{pd}_\Lambda M$ to denote the Loewy length and projective dimension of $M$ respectively. For a finite set $X$, we let $|X|$ denote the cardinality of $X$. For two sets $X_1$ and $X_2$, $X_1 \amalg X_2$ stands for their disjoint union.

DEFINITION 2.1 ([1, Definition 0.1]). Let $\Lambda$ be an algebra and $M \in \mathrm{mod}\,\Lambda$. Then $M$ is called

- *$\tau$-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$;
- *$\tau$-tilting* if it is $\tau$-rigid and $|M| = |\Lambda|$;
- *support $\tau$-tilting* if it is a $\tau$-tilting $\Lambda/\Lambda e\Lambda$-module for some idempotent $e$ of $\Lambda$;
- *proper support $\tau$-tilting* if it is support $\tau$-tilting but not $\tau$-tilting.

Recall that $M \in \mathrm{mod}\,\Lambda$ is called *sincere* if every simple $\Lambda$-module appears as a composition factor in $M$. It is well-known that the $\tau$-tilting modules are exactly the sincere support $\tau$-tilting modules [1, Proposition 2.2(a)].

We denote by $\tau\text{-tilt}\,\Lambda$ (respectively, $\mathrm{s}\tau\text{-tilt}\,\Lambda$, $\mathrm{ps}\tau\text{-tilt}\,\Lambda$) the set of isomorphism classes of basic $\tau$-tilting (respectively, support $\tau$-tilting, proper support $\tau$-tilting) $\Lambda$-modules.

Set

$$\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Lambda := \{M \in \mathrm{ps}\tau\text{-tilt}\,\Lambda \mid M \text{ has no projective direct summands}\}.$$

THEOREM 2.2 ([2, Theorem 2.6]). *Let $\Lambda$ be a Nakayama algebra. Then there is a bijection between $\tau$-tilt $\Lambda$ and $\mathrm{ps}\tau$-tilt$_{\mathrm{np}}\,\Lambda$.*

The following result is useful.

PROPOSITION 2.3 ([2, Proposition 2.32]). *Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then each $\tau$-tilting $\Lambda$-module has $P_1$ as a direct summand.*

As a consequence, we get

LEMMA 2.4. *Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then each support $\tau$-tilting $\Lambda$-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors has $P_1$ as a direct summand.*

*Proof.* Let $M$ be a support $\tau$-tilting $\Lambda$-module which has $S_1, \dots, S_{l(P_1)}$ as composition factors. If $M$ is $\tau$-tilting, then it has $P_1$ as a direct summand by Proposition 2.3. Now, assume that $M$ has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors but not $S_{j+1}$. Let $N$ be the maximal direct summand of $M$ which only has $S_1, \dots, S_{l(P_1)}, \dots, S_j$ as composition factors. Then $N$ is a $\tau$-tilting $\Lambda/\langle e_{[j+1,n]}\rangle$-module. By Proposition 2.3, $N$ has $P_1$ as a direct summand. ■

THEOREM 2.5 ([2, Theorem 2.33 and Corollary 2.34]). *Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then there are mutually inverse bijections*

$$\tau\text{-tilt}\,\Lambda \leftrightarrow \coprod_{i=1}^{l(P_1)} \tau\text{-tilt}(\Lambda/\langle e_i\rangle)$$

*given by $\tau$-tilt $\Lambda \ni M \mapsto M/P_1$ and $N \mapsto N \oplus P_1 \in \tau$-tilt $\Lambda$. In particular,*

$$|\tau\text{-tilt}\,\Lambda| = \sum_{i=1}^{l(P_1)} C_{i-1} \cdot |\tau\text{-tilt}(\Lambda/\langle e_{[1,i]}\rangle)|.$$

REMARK 2.6. Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then every $\tau$-tilting $\Lambda$-module can be decomposed $M$ as $M = P_1 \oplus N_1 \oplus N_2$ where $N_1$ is a maximal direct summand of $M$ without $S_1$ as composition factors. Moreover, $N_1 \oplus N_2$ is a $\tau$-tilting $\Lambda/\langle e_{j+1}\rangle$-module where $j := l(N_2)$ (see [2, proof of Theorem 2.33]).

An algebra $\Lambda$ is of *finite representation type* if there are only finitely many indecomposable $\Lambda$-modules $X_1, \dots, X_m$ up to isomorphism. In this case, the endomorphism algebra $\mathrm{End}_\Lambda(\bigoplus_{i=1}^m X_i)$ is called the *Auslander algebra* of $\Lambda$.

By a straightforward calculation, we get the quiver of the Auslander algebra of radical square zero Nakayama algebras:

PROPOSITION 2.7.

(1) *The Auslander algebra $\mathit{\Gamma}_n$ of $\mathit{\Lambda}_n := KA_n/\mathrm{rad}^2$ is given by the quiver*

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

*with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$.*

(2) *The Auslander algebra $\mathit{\Gamma}'_n$ of $\mathit{\Lambda}_n := K\widetilde{A}_n/\mathrm{rad}^2$ is given by the quiver*

$$\begin{tikzcd} 1 \arrow[r, "a_1"'] & 2 \arrow[r, "a_2"'] & 3 \arrow[r, "a_3"'] & \cdots \arrow[r] & 2n-1 \arrow[r, "a_{2n-1}"'] & 2n \arrow[lllll, bend right, "a_{2n}"'] \end{tikzcd}$$

*with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n$.*

**3. The case for $\mathit{\Gamma}_n$.** In this section, we will give formulas for $|\tau\text{-tilt}\,\mathit{\Gamma}_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\mathit{\Gamma}_n|$.

Let $\mathit{\Delta}_n$ be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$.

The following result gives a formula for $|\tau\text{-tilt}\,\mathit{\Gamma}_n|$.

THEOREM 3.1. *We have*

$$|\tau\text{-tilt}\,\mathit{\Gamma}_n| = 3|\tau\text{-tilt}\,\mathit{\Gamma}_{n-1}| + |\tau\text{-tilt}\,\mathit{\Gamma}_{n-2}|$$

*with* $|\tau\text{-tilt}\,\mathit{\Gamma}_1| = 1$ *and* $|\tau\text{-tilt}\,\mathit{\Gamma}_2| = 3$. *Hence*

$$|\tau\text{-tilt}\,\mathit{\Gamma}_n| = \frac{(3+\sqrt{13})^n - (3-\sqrt{13})^n}{\sqrt{13}\cdot 2^n}.$$

*Proof.* Applying Theorem 2.5 to $\mathit{\Gamma}_n$ and $\mathit{\Delta}_n$, we have

$$\begin{aligned}(1)\qquad |\tau\text{-tilt}\,\mathit{\Gamma}_n| &= C_0 \cdot |\tau\text{-tilt}(\mathit{\Gamma}_n/\langle e_1\rangle)| + C_1 \cdot |\tau\text{-tilt}(\mathit{\Gamma}_n/\langle e_1+e_2\rangle)| \\ &= |\tau\text{-tilt}\,\mathit{\Delta}_{n-1}| + |\tau\text{-tilt}\,\mathit{\Gamma}_{n-1}|\end{aligned}$$

and

$$\begin{aligned}(2)\qquad |\tau\text{-tilt}\,\mathit{\Delta}_n| &= C_0 \cdot |\tau\text{-tilt}(\mathit{\Delta}_n/\langle e_0\rangle)| + C_1 \cdot |\tau\text{-tilt}(\mathit{\Delta}_n/\langle e_0+e_1\rangle)| \\ &\quad + C_2 \cdot |\tau\text{-tilt}(\mathit{\Delta}_n/\langle e_0+e_1+e_2\rangle)| \\ &= |\tau\text{-tilt}\,\mathit{\Gamma}_n| + |\tau\text{-tilt}\,\mathit{\Delta}_{n-1}| + 2|\tau\text{-tilt}\,\mathit{\Gamma}_{n-1}|.\end{aligned}$$

The formula (1) implies

$$|\tau\text{-tilt}\,\mathit{\Delta}_{n-1}| = |\tau\text{-tilt}\,\mathit{\Gamma}_n| - |\tau\text{-tilt}\,\mathit{\Gamma}_{n-1}|.$$

Applying it to (2), we have

$$(3)\qquad |\tau\text{-tilt}\,\mathit{\Gamma}_n| = 3|\tau\text{-tilt}\,\mathit{\Gamma}_{n-1}| + |\tau\text{-tilt}\,\mathit{\Gamma}_{n-2}|$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 3x - 1 = 0$. The proof is finished. ■

Let $\Lambda$ be an algebra. Recall that a module $M \in \operatorname{mod} \Lambda$ is called *tilting* if

- $\operatorname{pd}_\Lambda M \leq 1$;
- $\operatorname{Ext}^1_\Lambda(M, M) = 0$;
- $|M| = |\Lambda|$.

Thus a module $M \in \operatorname{mod} \Lambda$ is tilting if and only if it is $\tau$-tilting and $\operatorname{pd}_\Lambda M \leq 1$, by the Auslander–Reiten formula.

The set of all tilting $\Lambda$-modules is denoted by $\operatorname{tilt} \Lambda$. The following result is part of [16, Theorem 2.8]. Here we give another proof.

PROPOSITION 3.2. $|\operatorname{tilt} \Gamma_n| = 2^{n-1}$.

*Proof.* Note that $P_1$ is the unique $\Gamma_n$-module which has $S_1$ as a composition factor and its projective dimension is at most 1. By Remark 2.6 and the above argument, $P_1 \oplus N_1$ is a tilting $\Gamma_n$-module if and only if $N_1$ is a tilting $\Gamma_n/\langle e_1\rangle$-module, since $\operatorname{pd}_{\Gamma_n} N_1 = \operatorname{pd}_{\Gamma_n/\langle e_1\rangle} N_1$. Thus

$$|\operatorname{tilt} \Gamma_n| = |\operatorname{tilt}(\Gamma_n/\langle e_1\rangle)| = |\operatorname{tilt} \Delta_{n-1}|.$$

Note that $P_0$ and $S_0$ are the only two $\Delta_n$-modules which have $S_0$ as a composition factor and their projective dimension is at most 1. Similarly, we get

$$|\operatorname{tilt} \Delta_n| = |\operatorname{tilt}(\Delta_n/\langle e_0\rangle)| + |\operatorname{tilt}(\Delta/\langle e_0 + e_1\rangle)| = |\operatorname{tilt} \Gamma_n| + |\operatorname{tilt} \Delta_{n-1}|.$$

Thus $|\operatorname{tilt} \Gamma_n| = 2|\operatorname{tilt} \Gamma_{n-1}|$ with $|\operatorname{tilt} \Gamma_1| = 1$, and so $|\operatorname{tilt} \Gamma_n| = 2^{n-1}$. ■

As generalizations of simple modules and semisimple modules, bricks and semibricks were introduced and studied in [8, 15]. Let $\Lambda$ be an algebra. A $\Lambda$-module $M$ is called a *brick* if $\operatorname{Hom}_\Lambda(M, M)$ is a $K$-division algebra, and a *semibrick* is a set consisting of isoclasses of pairwise Hom-orthogonal bricks. Recall from [3] that a semibrick $\mathcal{S}$ is called *left finite* if the smallest torsion class $T(\mathcal{S})$ containing $\mathcal{S}$ is functorially finite. There exists a bijection between $\mathrm{s}\tau\text{-tilt}\, \Lambda$ and the set of left finite semibricks of $\Lambda$ [3, Theorem 2.3]. Note that every torsion class is functorially finite for a representation-finite algebra. So, for a Nakayama algebra $\Lambda$, there exists a bijection between $\mathrm{s}\tau\text{-tilt}\, \Lambda$ and the set $\operatorname{sbrick} \Lambda$ of semibricks of $\Lambda$, and hence $|\mathrm{s}\tau\text{-tilt}\, \Lambda| = |\operatorname{sbrick} \Lambda|$. Asai gave a method to calculate the number of semibricks over $KA_n/\operatorname{rad}^r$. In fact, we have the following more general result.

PROPOSITION 3.3. *Let $\Lambda$ be a Nakayama algebra of type $A_n$. Then*

(1) $|\mathrm{s}\tau\text{-tilt}\, \Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum\limits_{i=2}^{l(I_n)} C_{i-1}|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)|$,

(2) $|\mathrm{s}\tau\text{-tilt}\, \Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1\rangle)| + \sum\limits_{i=2}^{l(P_1)} C_{i-1}|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_{[1,i]}\rangle)|$.

*Proof.* (1) For a given brick $X$ of $\Lambda$ with $\operatorname{top} X = S_i$ and $\operatorname{soc} X = S_j$, we will denote $S_{i,j} := X$.

We define $W_0$ as the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks without $S_n$ as a composition factor. It is clear that $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Let $W_i$ $(i = 1, \dots, l(I_n))$ be the subset of $\operatorname{sbrick}\Lambda$ consisting of the semibricks which contain the brick $S_{n-i+1,n}$.

First, there is a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_n\rangle)$$

defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_{n,n}\}$. So $|W_0| = |\operatorname{sbrick}(\Lambda/\langle e_n\rangle)|$.

Secondly, for $i = 2, 3, \dots, l(I_n)$, there exists a bijection

$$W_1 \mapsto \operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle) \times \operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)$$

defined by

$$\begin{aligned} \mathcal{S} \mapsto (\{S \in \mathcal{S} \mid \operatorname{Supp} S \cap [n-i+1, n] = \emptyset\}&, \\ \{S \in \mathcal{S} \mid \operatorname{Supp} S \subset [n-i+2, n-1]\}&), \end{aligned}$$

where $\operatorname{Supp} S$ stands for the support of $S$. Note that $\operatorname{sbrick}\Lambda = \bigcup_{i=0}^{l(I_n)} W_i$. Thus we obtain

$$\begin{aligned} |\text{s}\tau\text{-tilt}\,\Lambda| = |\operatorname{sbrick}\Lambda| = \sum_{i=0}^{l(I_n)} |W_i| \\ &= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)| \\ &\quad + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(\Lambda/\langle 1 - e_{[n-i+2,n-1]}\rangle)| \\ &= 2|\operatorname{sbrick}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\operatorname{sbrick}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\operatorname{sbrick}(KA_{i-2})| \\ &= 2|\text{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} |\text{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)| \cdot |\text{s}\tau\text{-tilt}(KA_{i-2})| \\ &= 2|\text{s}\tau\text{-tilt}(\Lambda/\langle e_n\rangle)| + \sum_{i=2}^{l(I_n)} C_{i-1} \cdot |\text{s}\tau\text{-tilt}(\Lambda/\langle e_{[n-i+1,n]}\rangle)|. \end{aligned}$$

(2) Note that there is a bijection between $\text{s}\tau\text{-tilt}\,\Lambda$ and $\text{s}\tau\text{-tilt}\,\Lambda^{\mathrm{op}}$ [1, Theorem 2.14]). Now the assertion follows from (1). ■

We give the following example to illustrate Proposition 3.3.

EXAMPLE 3.4. Let $\Lambda$ be the algebra given by the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \rightarrow 4$$

with the relation $\alpha\beta = 0$. By Proposition 3.3(1), we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Lambda| &= 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_4\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_3+e_4\rangle)| \\ &\quad + 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_2+e_3+e_4\rangle)| \\ &= 2\times 12 + 5 + 2\times 2 = 33.\end{aligned}$$

On the other hand, by Proposition 3.2(2),

$$|\mathrm{s}\tau\text{-tilt}\,\Lambda| = 2|\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1\rangle)| + |\mathrm{s}\tau\text{-tilt}(\Lambda/\langle e_1+e_2\rangle)| = 2\times 14 + 5 = 33.$$

The following result gives a formula for $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$.

THEOREM 3.5. *We have*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|$$

*with* $|\mathrm{s}\tau\text{-tilt}\,\Gamma_1| = 2$ *and* $|\mathrm{s}\tau\text{-tilt}\,\Gamma_2| = 12$. *Hence*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = \frac{(3+2\sqrt{3})^n - (3-2\sqrt{3})^n}{2\sqrt{3}}.$$

*Proof.* Applying Proposition 3.3(2) to $\Gamma_n$ and $\Delta_n$ respectively, we have

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1\rangle)| + C_1\cdot|\mathrm{s}\tau\text{-tilt}(\Gamma_n/\langle e_1+e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + |\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|\end{aligned} \tag{4}$$

and

$$\begin{aligned}|\mathrm{s}\tau\text{-tilt}\,\Delta_n| &= 2|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0\rangle)| + C_1\cdot|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0+e_1\rangle)| \\ &\quad + C_2\cdot|\mathrm{s}\tau\text{-tilt}(\Delta_n/\langle e_0+e_1+e_2\rangle)| \\ &= 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| + |\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|.\end{aligned}$$

This implies

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-2}|. \tag{5}$$

This is a linear homogeneous recurrence relation of degree 2 and its characteristic equation is $x^2 - 6x - 3 = 0$. The proof is finished. ∎

Let $\overline{\Gamma}_n$ be the algebra given by the quiver

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$, and let $\overline{\Delta}_n$ be the algebra given by the quiver

$$0 \xrightarrow{a_0} 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $1 \le k \le n-1$. By using the same argument as in Theorem 3.5, we can obtain

$$|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_n| = 6|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-2}|.$$

**4. The case for $\Gamma'_n$.** In this section, we will give formulas for $|\tau\text{-tilt}\,\Gamma'_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n|$.

Let $X_n$ be the set of all support $\tau$-tilting $\Gamma_n$-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands, and let $Y_n$ be the set of all support $\tau$-tilting $\Delta_n$-modules which do not have $P_0, P_1, \ldots, P_{2n-3}$ as direct summands. Let $X'_n$ be the set of all support $\tau$-tilting $\overline{\Gamma}_n$-modules which do not have $P_1, \ldots, P_{2n-2}$ as direct summands, and let $Y'_n$ be the set of all support $\tau$-tilting $\overline{\Delta}_n$-modules which do not have $P_0, P_1, \ldots, P_{2n-2}$ as direct summands.

We need the following lemma.

Lemma 4.1.

(1) $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ *and* $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.
(2) $|X'_n| = 3|X'_{n-1}| + |X'_{n-2}|$ *and* $|Y'_n| = 3|Y'_{n-1}| + |Y'_{n-2}|$.

*Proof.* (1) By Lemma 2.4, all support $\tau$-tilting $\Gamma_n$-modules which have $S_1, S_2$ as composition factors must have $P_1$ as a direct summand. Hence $X_n$ consists of two parts: the first part comes from all support $\tau$-tilting $\Gamma_n$-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and do not have $S_1$ as a composition factor (their number is exactly $|Y_{n-1}|$); the second part comes from all support $\tau$-tilting $\Gamma_n$-modules which do not have $P_1, \ldots, P_{2n-3}$ as direct summands and have $S_1$ as a composition factor but not $S_2$ (their number is exactly $|X_{n-1}|$). Hence, $|X_n| = |Y_{n-1}| + |X_{n-1}|$. Similarly, we have $|Y_n| = |X_n| + |Y_{n-1}| + 2|X_{n-1}|$. These two equalities imply $|X_n| = 3|X_{n-1}| + |X_{n-2}|$ and $|Y_n| = 3|Y_{n-1}| + |Y_{n-2}|$.

(2) The proof is similar. ∎

The following result gives a formula for $|\tau\text{-tilt}\,\Gamma'_n|$.

Theorem 4.2. *We have*

$$|\tau\text{-tilt}\,\Gamma'_n| = 3|\tau\text{-tilt}\,\Gamma'_{n-1}| + |\tau\text{-tilt}\,\Gamma'_{n-2}|$$

*with* $|\tau\text{-tilt}\,\Gamma'_1| = 3$ *and* $|\tau\text{-tilt}\,\Gamma'_2| = 11$. *Hence*

$$|\tau\text{-tilt}\,\Gamma'_n| = \frac{(3+\sqrt{13})^n + (3-\sqrt{13})^n}{2^n}.$$

*Proof.* We claim that every proper support $\tau$-tilting $\Gamma'_n$-module $M$ which has $S_1, S_2$ as composition factors must have a projective $\Gamma'_n$-module as a direct summand. Indeed, if $M$ does not have $S_{2n}$ as a composition factor, then it has $P_1$ as a direct summand by Lemma 2.4. Now, assume that $M$ has $S_i, S_{i+1}, \ldots, S_{2n}, S_1, S_2$ as composition factors, but not $S_{i-1}$. Then $M$ has $P_i$ as a direct summand by Lemma 2.4.

Now, $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Gamma'_n$ consists of the following two parts:

(i) $U_1$: the subset of modules which do not have $S_2$ as a composition factor.

(ii) $U_2$: the subset of modules which have $S_2$ as a composition factor, but not $S_1$.

Since $\overline{\Lambda} := \Gamma'_n/\langle e_2\rangle$ is the quiver

$$3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n \xrightarrow{a_{2n}} 1$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \leq k \leq n$, $U_1$ is exactly the set of support $\tau$-tilting $\overline{\Lambda}$-modules which do not have $P_3, P_4, \ldots, P_{2n-1}$ as direct summands, and so $|U_1| = |X_n|$. Note that $\overline{\Gamma} := \Gamma'_n/\langle e_1\rangle$ is the quiver

$$2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \to 2n-2 \xrightarrow{a_{2n-2}} 2n-1 \xrightarrow{a_{2n-1}} 2n$$

with the relations $a_{2k-1}a_{2k} = 0$ for $2 \leq k \leq n-1$. Thus, the number of support $\tau$-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \ldots, P_{2n-2}$ as direct summands is exactly $|Y'_{n-1}|$. Moreover, the number of support $\tau$-tilting $\overline{\Gamma}$-modules which do not have $P_2, P_4, \ldots, P_{2n-2}$ as direct summands and do not have $S_2$ as a composition factor is exactly $|X'_{n-1}|$. Therefore, $|U_2| = |Y'_{n-1}| - |X'_{n-1}|$. By Theorem 2.2, we obtain

$$|\tau\text{-tilt}\,\Gamma'_n| = |\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\,\Gamma'_n| = |U_1| + |U_2| = |X_n| + |Y'_{n-1}| - |X'_{n-1}|.$$

Now, the recurrence relation for $|\tau\text{-tilt}\,\Gamma'_n|$ follows from Lemma 4.1. ∎

The following result gives a formula for $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n|$.

THEOREM 4.3. *We have*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n| = 6|\mathrm{s}\tau\text{-tilt}\,\Gamma'_{n-1}| + 3|\mathrm{s}\tau\text{-tilt}\,\Gamma'_{n-2}|$$

*with* $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_1| = 6$ *and* $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_2| = 42$. *Hence*

$$|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n| = (3+2\sqrt{3})^n + (3-2\sqrt{3})^n.$$

*Proof.* The set $\mathrm{sbrick}\,\Gamma'_n$ of semibricks of $\Gamma'_n$ consists of five parts:

(i) $V_0$: the semibricks without $S_1$ as a composition factor.
(ii) $V_1$: the semibricks which contain $S_1$ but not the brick $I_2$.
(iii) $V_2$: the semibricks which contain $I_1$.
(iv) $V_3$: the semibricks which contain $P_1$.
(v) $V_4$: the semibricks which contain $I_2$.

Obviously, $|V_0| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\mathrm{sbrick}\,\overline{\Delta}_{n-1}|$.

There is a bijection $V_1 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)$ defined by $\mathcal{S} \mapsto \mathcal{S} \setminus \{S_1\}$, so

$$|V_1| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1\rangle)| = |\mathrm{sbrick}\,\overline{\Delta}_{n-1}|.$$

Similarly, there are bijections

$$V_2 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1+e_{2n}\rangle) \quad \text{and} \quad V_3 \mapsto \mathrm{sbrick}(\Gamma'_n/\langle e_1+e_2\rangle),$$

so

$$|V_2| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1+e_{2n}\rangle)| = |\mathrm{sbrick}\,\Delta_{n-1}|,$$
$$|V_3| = |\mathrm{sbrick}(\Gamma'_n/\langle e_1+e_2\rangle)| = |\mathrm{sbrick}\,\Delta^{\mathrm{op}}_{n-1}|.$$

Finally, we can define a bijection

$$V_4 \mapsto \operatorname{sbrick}(\Gamma'_n/\langle e_1+e_2+e_{2n}\rangle) \times \operatorname{sbrick}(\Gamma'_n/\langle 1-e_1\rangle)$$

by $V_4 \ni \mathcal{S} \mapsto (\mathcal{S}\setminus\{S_1, I_2\}, S_1\cap\mathcal{S})$. Thus

$$|V_4| = |\operatorname{sbrick}(\Gamma'_n/\langle e_1+e_2+e_{2n}\rangle)|\cdot|\operatorname{sbrick}(\Gamma'_n/\langle 1-e_1\rangle)| = 2|\operatorname{sbrick}\Gamma_{n-1}|.$$

Therefore

$$\begin{aligned}
|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n| = |\operatorname{sbrick}\Gamma'_n| &= \sum_{i=0}^{4}|V_i| \\
&= 2|\operatorname{sbrick}\overline{\Delta}_{n-1}| + |\operatorname{sbrick}\Delta_{n-1}| + |\operatorname{sbrick}\Delta^{\mathrm{op}}_{n-1}| + 2|\operatorname{sbrick}\Gamma_{n-1}| \\
&= 2|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-1}| + |\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + |\mathrm{s}\tau\text{-tilt}\,\Delta^{\mathrm{op}}_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}| \\
&= 2|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}| + 2|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|.
\end{aligned}$$

Note that $|\mathrm{s}\tau\text{-tilt}\,\Delta_{n-1}|$ is a linear combination of $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ and $|\mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1}|$ by (4), so $|\mathrm{s}\tau\text{-tilt}\,\Delta_n|$ has the same recurrence relation as $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$. In particular, $|\mathrm{s}\tau\text{-tilt}\,\overline{\Delta}_n|$, $|\mathrm{s}\tau\text{-tilt}\,\Delta_n|$, $|\mathrm{s}\tau\text{-tilt}\,\Gamma_n|$ have the same recurrence relations, and so $|\mathrm{s}\tau\text{-tilt}\,\Gamma'_n|$ also has the same recurrence relation. ■

**5. Examples.** In this section, we list the numbers of (support) $\tau$-tilting modules over $\Gamma_n$ and $\Gamma'_n$ in the following table. The sequence $|\tau\text{-tilt}\,\Gamma_n|$ is listed in the On-line Encyclopedia of Integer Sequences (OEIS) as the sequence A006190 and $|\tau\text{-tilt}\,\Gamma'_n|$ as A006497.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\tau\text{-tilt}\,\Gamma_n\rvert$ | 1 | 3 | 10 | 33 | 109 | 360 | 1189 | 3927 |
| $\lvert\mathrm{s}\tau\text{-tilt}\,\Gamma_n\rvert$ | 2 | 12 | 78 | 504 | 3258 | 21060 | 136134 | 879984 |
| $\lvert\tau\text{-tilt}\,\Gamma'_n\rvert$ | 3 | 11 | 36 | 119 | 393 | 1298 | 4287 | 114159 |
| $\lvert\mathrm{s}\tau\text{-tilt}\,\Gamma'_n\rvert$ | 6 | 42 | 270 | 17464 | 11286 | 72954 | 471582 | 3048354 |

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