Journal of Algebra and Its Applications
Vol. 20, No. 9 (2021) 2150169 (7 pages)

DOI: 10.1142/S0219498821501693

Extending silted algebras to cluster-tilted algebras

Hanpeng Gao
Department of Mathematics, Nanjing University
No. 22 Hankou Road, Nanjing, Jiangsu 210093, P. R. China
hpgao07@163.com

Received 13 October 2019
Accepted 15 May 2020
Published 26 June 2020

Communicated by J. Chuang

It is well known that the relation extensions of tilted algebras are cluster-tilted algebras. In this paper, we extend the result to silted algebras and prove that some extension of silted algebras are cluster-tilted algebras.

Keywords: Silted algebras; cluster-tilted algebras; tilted algebras; relation extension.

Mathematics Subject Classification 2020: 16G20, 16G60

1. Introduction

Cluster-tilted algebras were introduced by Buan *et al.* in [3] and also in [7] for type $\mathbb{A}$. Let A be a triangular algebra whose global dimension is at most two over an algebraically closed field k. The trivial extension of A by the A-A-bimodule $\mathrm{Ext}^2_A(DA, A)$ is called the *relation extension* [2] of A, where $D{=}\mathrm{Hom}_k(-, k)$ is the standard duality. It is proved that the relation extension of every tilted algebra is cluster-tilted, and every cluster-tilted algebra is of this form in [2].

The concept of silting complexes originated from [13] and 2-term silting complexes are of particular interest and important for the representation of algebra. In [6], the endomorphism algebras of 2-term silting complexes were introduced by Buan and Zhou. They also defined the concept of the silted algebra [5], which is the endomorphism algebras of 2-term silting complex over the derived category of hereditary algebras and proved that an algebra is silted if and only if it is shod [8] (projective dimension or injective dimension of every indecomposable module is at most one). In particular, tilted algebras are silted, indeed, the minimal projective presentation of a tilting module T over the hereditary algebra H gives rise to a 2-term silting complex $\mathbf{P}$ in $\mathrm{K}^b(\mathrm{proj}\, H)$, and there is an isomorphism of algebras $\mathrm{End}_H(T) \cong \mathrm{End}_{\mathcal{D}^b(H)}(\mathbf{P})$.

As a generalization of tilting modules, support τ-tilting modules were introduced by Adachi *et al.* [1]. They also show that there is a bijection between support τ-tilting modules and 2-term silting complexes (see [1, Theorem 3.2]). This result provided that every silted algebra can be described as the triangular matrix algebra $\begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$, where B is a tilted algebra, H_1 is a hereditary algebra and M is a H_1-B-bimodule (see Proposition 3.1). It is a natural question whether silted algebras can be extended to cluster-tilted algebras.

In this paper, we give a positive answer and construct cluster-tilted algebras from silted algebras. We call a silted algebra A with respect to (T, P) for some hereditary algebra H if there exists a 2-term silting complex $\mathbf{P}$ in $\mathcal{D}^b(H)$ corresponding to the support τ-tilting pair (T, P) in $\operatorname{mod} H$ such that $A \cong \operatorname{End}_{\mathcal{D}^b(H)}(\mathbf{P})$. Our main results as follows.

Theorem 1.1. *Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to (T, P) for some hereditary algebra H. Then the matrix algebra*

$$\begin{pmatrix} B \ltimes \operatorname{Ext}^1_H(T, \tau_H^{-1}T) & \operatorname{Hom}_H(P, \tau_H^{-1}T) \\ M & H_1 \end{pmatrix}$$

is a cluster-tilted algebra.

As a consequence, we have the following result.

Theorem 1.2. *Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to (T, P) for some hereditary algebra H. If $\operatorname{Hom}_H(P, \tau_H^{-1}T) = 0$, then the triangular matrix algebra $\begin{pmatrix} B \ltimes \operatorname{Ext}^2_B(DB, B) & 0 \\ M & H_1 \end{pmatrix}$ is a cluster-tilted algebra.*

Note that a tilted algebra is exactly a silted algebra with respect to $(T, 0)$ for some hereditary algebra H, we can easily get the relation extension of every tilted algebra as cluster-tilted.

Throughout this paper, all algebras are finite-dimensional algebras over an algebraically closed field k. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and $\operatorname{proj} A$ the category of finitely generated projective right A-modules. $\mathrm{K}^b(\operatorname{proj} A)$ will stand for the bounded homotopy category of finitely generated projective right A-modules and $\mathcal{D}^b(A)$ is the bounded derived category of finitely generated right A-modules. For a A-module M, $|M|$ is the number of pairwise non-isomorphic direct summands of M. All modules considered basic.

2. Preliminaries

2.1. *Tilted algebras*

Let A be an algebra. An A-module T is called *tilting* if (1) the projective dimension of T is at most one, (2) $\operatorname{Ext}^1_A(T, T) = 0$ and (3) $|T| = |A|$. The endomorphism algebra of a tilting module over a hereditary algebra is called a *tilted* algebra [10]. The following result is very useful.

Theorem 2.1 ([9]). *Let H be a hereditary algebra, T a tilting H-module and $B = \mathrm{End}_H(T)$ the corresponding tilted algebra. Then we have the following*:

(1) *The derived functor* $\mathrm{RHom}_H(T, -) : \mathcal{D}^b(H) \to \mathcal{D}^b(B)$ *is an equivalence which maps T to B.*
(2) $\mathrm{RHom}_H(T, -)$ *commutes with the Auslander–Reiten translations and the shifts in the respective categories.*

2.2. *Silted algebras*

Definition 2.1 ([1, Definition 0.1]). Let $T \in \mathrm{mod}\, A$.

(1) T is called *τ-rigid* if $\mathrm{Hom}_A(T, \tau T) = 0$.
(2) T is called *τ-tilting* if it is τ-rigid and $|T| = |A|$.
(3) T is called *support τ-tilting* if it is a τ-tilting A/AeA-module for some idempotent e of A.

Sometimes, it is convenient to view support τ-tilting modules and τ-rigid modules as certain pairs of modules in $\mathrm{mod}\, A$.

Definition 2.2 ([1, Definition 0.3]). Let (T, P) be a pair in $\mathrm{mod}\, A$ with $P \in \mathrm{proj}\, A$.

(1) (T, P) is called a *τ-rigid pair* if M is τ-rigid and $\mathrm{Hom}_A(T, M) = 0$.
(2) (T, P) is called a *support τ-tilting pair* if T is τ-rigid and $|T| + |P| = |A|$.

It is shown in [1, Proposition 2.3] that (T, P) is a support τ-tilting pair in $\mathrm{mod}\, A$ if and only if T is a τ-tilting A/AeA-module with $eA \cong P$.

A complex $\mathbf{P} \in \mathrm{K}^b(\mathrm{proj}\, A)$ is called *silting* [13] if $\mathrm{Hom}_{\mathrm{K}^b(\mathrm{proj}\, A)}(\mathbf{P}, \mathbf{P}[i]) = 0$ for $i > 0$ and if $\mathbf{P}$ generates $\mathrm{K}^b(\mathrm{proj}\, A)$ as a triangulated category. Moreover, $\mathbf{P}$ is called 2-term if it only has nonzero terms in degree 0 and -1.

The next result shows that the relationship between support τ-tilting modules and 2-term silting complexes. For convenience, we denote by $s\tau$-tilt A all support τ-tilting modules over the algebra A and 2-silt A all 2-term silting complexes over $\mathrm{K}^b(\mathrm{proj}\, A)$.

Theorem 2.2 ([1, Theorem 3.2]). *There exists a bijection between* $s\tau - \mathrm{tilt}\, A$ *and* $2 - \mathrm{silt}\, A$ *given by* $(T, P) \in s\tau - \mathrm{tilt}\, A \to P_1 \oplus P \to P_0 \in 2 - \mathrm{silt}\, A$ *and* $\mathbf{P} \in 2 - \mathrm{silt}\, A \to \mathrm{H}^0(\mathbf{P}) \in s\tau - \mathrm{tilt}\, A$, *where $P_1 \to P_0$ is a minimal projective presentation of T.*

We call an algebra A as *silted* [5] if there is a hereditary algebra H and $\mathbf{P} \in$ 2-silt H such that $A \cong \mathrm{End}_{\mathcal{D}^b(H)}(\mathbf{P})$.

2.3. *Cluster-tilted algebras*

The *cluster category* $\mathcal{C}_H$ of a hereditary algebra H is the quotient category $\mathcal{D}^b(H)/F$, where $F = \tau_{\mathcal{D}}^{-1}[1]$ and $\tau_{\mathcal{D}}^{-1}$ is the inverse of the Auslander–Reiten translation in $\mathcal{D}^b(H)$. The space of morphisms from $\tilde{X}$ to $\tilde{Y}$ in $\mathcal{C}_H$ is given by $\mathrm{Hom}_{\mathcal{C}_H}(\tilde{X}, \tilde{Y}) = \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^b(H)}(X, F^iY)$. It is shown that $\mathcal{C}_H$ is a triangulated category [11]. An object $\tilde{T} \in \mathcal{C}_H$ is called *tilting* if $\mathrm{Ext}^1_{\mathcal{C}_H}(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals $|H|$. The algebra of endomorphisms $C = \mathrm{End}_{\mathcal{C}_H}(T)$ is called cluster-tilted [4]. It is proved that the relation extension of every tilted algebra is cluster-tilted , and every cluster-tilted algebra is of this form in [2].

3. Main Results

In this section, we prove our main results and give an example to illustrate our results.

Definition 3.1. We call a silted algebra A with respect to (T, P) for some hereditary algebra H if there exists $\mathbf{P} \in 2\text{-silt}\, H$ corresponding to $(T, P) \in s\tau\text{-tilt}\, H$ such that $A \cong \mathrm{End}_{\mathcal{D}^b(H)}(\mathbf{P})$.

Proposition 3.1. *Let A be a silted algebra with respect to (T, P). Then A is a triangular matrix algebra $\left(\begin{smallmatrix} B & 0 \\ M & H_1 \end{smallmatrix}\right)$, where B is a tilted algebra, H_1 is a hereditary algebra and M is a H_1-B-bimodule.*

Proof. Suppose that there is a hereditary algebra H and $\mathbf{P} \in 2\text{-silt}\, H$ corresponding to $(T, P) \in s\tau\text{-tilt}\, H$ such that $A \cong \mathrm{End}_{\mathcal{D}^b(H)}(\mathbf{P})$, then we have

$$\begin{aligned} A &\cong \mathrm{End}_{\mathcal{D}^b(H)}(\mathbf{P}) \\ &\cong \mathrm{End}_{\mathcal{D}^b(H)}(T \oplus P[1]) \text{ (by Theorem 2.2)} \\ &\cong \begin{pmatrix} \mathrm{End}_{\mathcal{D}^b(H)}(T) & \mathrm{Hom}_{\mathcal{D}^b(H)}(P[1], T) \\ \mathrm{Hom}_{\mathcal{D}^b(H)}(T, P[1]) & \mathrm{End}_{\mathcal{D}^b(H)}(P[1]) \end{pmatrix}. \end{aligned}$$

Take $H' = H/HeH$, we have H' as a hereditary algebra, where $eH \cong P$. Therefore, T is a tilting H'-module and $B = \mathrm{End}_{\mathcal{D}^b(H)}(T) \cong \mathrm{End}_H(T) \cong \mathrm{End}_{H'}(T)$ is a tilted algebra. Moreover, $H_1 = \mathrm{End}_{\mathcal{D}^b(H)}(P[1]) \cong \mathrm{End}_H(P) \cong eHe$ is a hereditary algebra. Note that $\mathrm{Hom}_{\mathcal{D}^b(H)}(P[1], T) = 0$ since P is projective and $M = \mathrm{Hom}_{\mathcal{D}^b(H)}(T, P[1]) \cong \mathrm{Ext}^1_H(T, P)$ is a $H_1 - B$-bimodule, we have A as a triangular matrix algebra. □

Lemma 3.1. *Let $\mathcal{C}_H$ be a cluster category of a hereditary algebra H and $T \in$* $\mathrm{mod}\, H$*. Then we have*

$$\mathrm{End}_{\mathcal{C}_H}(\tilde{T}, \tilde{T}) \cong \mathrm{End}_{\mathcal{D}^b(H)}(T) \ltimes \mathrm{Hom}_{\mathcal{D}^b(H)}(T, FT),$$

where $\ltimes$ stand for the trivial extension.

Proof. It follows from [2, Lemma 3.3]. $\square$

Theorem 3.1. *Let* $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ *be a silted algebra with respect to* (T, P) *for some hereditary algebra* H. *Then the matrix algebra*

$$\begin{pmatrix} B \ltimes \mathrm{Ext}^1_H(T, \tau_H^{-1}T) & \mathrm{Hom}_H(P, \tau_H^{-1}T) \\ M & H_1 \end{pmatrix}$$

is a cluster-tilted algebra.

Proof. Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to (T, P) for some hereditary algebra H. Then $\tilde{T} \oplus \tilde{P}[1]$ is a cluster-tilting object in $\mathcal{C}_H$. For any two H-modules X and Y, we have $\mathrm{Hom}_{\mathcal{D}^b(H)}(X, Y[i]) = 0$ for all $i \geqslant 2$ since H is hereditary. Hence, we have

$$\begin{aligned}
&\mathrm{End}_{\mathcal{C}_H}(\tilde{T} \oplus \tilde{P}[1]) \\
&\quad\cong \begin{pmatrix} \mathrm{End}_{\mathcal{C}_H}(\tilde{T}) & \mathrm{Hom}_{\mathcal{C}_H}(\tilde{P}[1], \tilde{T}) \\ \mathrm{Hom}_{\mathcal{C}_H}(\tilde{T}, \tilde{P}[1]) & \mathrm{End}_{\mathcal{C}_H}(\tilde{P}[1]) \end{pmatrix} \\
&\quad\cong \begin{pmatrix} \mathrm{End}_{\mathcal{D}^b(H)}(T) \ltimes \mathrm{Hom}_{\mathcal{D}^b(H)}(T, FT) & \mathrm{Hom}_{\mathcal{C}_H}(\tilde{P}[1], \tilde{T}) \\ \mathrm{Hom}_{\mathcal{C}_H}(\tilde{T}, \tilde{P}[1]) & \mathrm{End}_{\mathcal{C}_H}(\tilde{P}[1]) \end{pmatrix} \quad \text{(by Lemma 3.1)} \\
&\quad\cong \begin{pmatrix} B \ltimes \mathrm{Ext}^1_H(T, \tau_H^{-1}T) & \mathrm{Hom}_{\mathcal{C}_H}(\tilde{P}[1], \tilde{T}) \\ \mathrm{Hom}_{\mathcal{C}_H}(\tilde{T}, \tilde{P}[1]) & \mathrm{End}_{\mathcal{C}_H}(\tilde{P}[1]) \end{pmatrix} \\
&\quad\cong \begin{pmatrix} B \ltimes \mathrm{Ext}^1_H(T, \tau_H^{-1}) & \mathrm{Hom}_{\mathcal{D}^b(H)}(P[1], FT) \\ \mathrm{Hom}_{\mathcal{D}^b(H)}(T, P[1]) & \mathrm{End}_{\mathcal{D}^b(H)}(P[1]) \end{pmatrix} \\
&\quad\cong \begin{pmatrix} B \ltimes \mathrm{Ext}^1_H(T, \tau_H^{-1}T) & \mathrm{Hom}_H(P, \tau_H^{-1}T) \\ M & H_1 \end{pmatrix},
\end{aligned}$$

which is a cluster-tilted algebra. $\square$

As a consequence, we have the following result.

Corollary 3.1. *Let* A *be a silted algebra with respect to* (T, P) *for some hereditary algebra* H. *If* T *is injective, then* A *is hereditary. In particular, a tilted algebra induced by a injective tilting module is hereditary.*

Proof. Since T is injective, we have $\tau_H^{-1}T = 0$. By Theorem 3.1, A is a cluster-tilted algebra whose global dimension is at most three. Note that every cluster-tilted algebra is 1-Gorenstein [12]. Since the projective dimension of every module over a 1-Gorenstein algebra is at most one or infinite, we get the global dimension of A as at most one, and so A is hereditary. $\square$

Theorem 3.2. *Let $A = \left(\begin{smallmatrix} B & 0 \\ M & H_1 \end{smallmatrix}\right)$ be a silted algebra with respect to (T, P) for some hereditary algebra H. If $\mathrm{Hom}_H(P, \tau_H^{-1}T) = 0$, then the triangular matrix algebra $\left(\begin{smallmatrix} B \ltimes \mathrm{Ext}_B^2(DB, B) & 0 \\ M & H_1 \end{smallmatrix}\right)$ is a cluster-tilted algebra.*

Proof. Take $H' = H/HeH$, we have $\tau_H^{-1}T$ as a H'-module since $\mathrm{Hom}_H(P, \tau_H^{-1}T) = 0$, where $eH \cong P$. Therefore, we have

$$\begin{aligned} \mathrm{Ext}_H^1(T, \tau_H^{-1}T) &\cong \mathrm{Ext}_{H'}^1(T, \tau_{H'}^{-1}T) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(H')}(T, F'T) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(B, F''B) \quad \text{(by Lemma 2.1)} \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(\tau_{\mathcal{D}^b(B)}B[1], B[2]) \\ &\cong \mathrm{Hom}_{\mathcal{D}^b(B)}(DB, B[2]) \\ &\cong \mathrm{Ext}_B^2(DB, B), \end{aligned}$$

where $F' = \tau_{\mathcal{D}^b(H')}^{-1}[1]$ and $F'' = \tau_{\mathcal{D}^b(B)}^{-1}[1]$ is the functor corresponding to F' in the derived category $\mathcal{D}^b(B)$. $\square$

Note that a tilted algebra is exactly silted algebra with respect to $(T, 0)$ for some hereditary algebra H, we can easy get the following result.

Corollary 3.2. *The relation extension of every tilted algebra is cluster-tilted.*

Example 3.1. Let H be a hereditary algebra given by the following quiver:

$$\begin{tikzcd} 1 \arrow[dr] & & \\ & 3 \arrow[r] & 4 \\ 2 \arrow[ur] & & \end{tikzcd}$$

The support τ-tilting pair $(T, P) = (P_4 \oplus P_1 \oplus S_1, P_2)$ corresponding to the 2-term silting complex $0 \to P_4 \oplus 0 \to P_1 \oplus P_3 \to P_1 \oplus P_2 \to 0$ induced a silted algebra given as follows:

$$1 \xleftarrow{\gamma} 2 \xleftarrow{\beta} 3 \xleftarrow{\alpha} 4$$

with the relations $\alpha\beta = 0$ and $\beta\gamma = 0$. Note that

$$\dim_k \mathrm{Ext}_H^1(T, \tau_H^{-1}T) = 2, \dim_k \mathrm{Hom}_H(P, \tau_H^{-1}T) = 1.$$

In fact,

$$\dim_k \mathrm{Ext}_H^1(S_1, \tau_H^{-1}P_4) = 1, \dim_k \mathrm{Ext}_H^1(S_1, \tau_H^{-1}P_1) = 1, \dim_k \mathrm{Hom}_H(P_2, \tau_H^{-1}P_1) = 1.$$

By Theorem 3.1, we can construct a cluster-tilted algebra given by the following quiver:

$$\begin{tikzcd}
& 1 \arrow[dl, "\delta"'] & \\
3 \arrow[rr, "\beta"] & & 2 \arrow[ul, "\gamma"'] \arrow[dl, "\epsilon"] \\
& 4 \arrow[ul, "\alpha"] &
\end{tikzcd}$$

with relations $\gamma\delta = \epsilon\alpha, \alpha\beta = 0, \beta\gamma = 0, \beta\epsilon = 0, \delta\beta = 0$.

Acknowledgments

The author would like to thank Professor Ralf Schiffler and Professor Zhaoyong Huang for the helpful discussions. He also thanks the referee for the useful and detailed suggestions. This work was partially supported by the National Natural Science Foundation of China (No. 11971225).

References

[1] T. Adachi, O. Iyama and I. Reiten, τ-tilting theory, *Compos. Math.* **150** (2014) 415–452.

[2] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras as trivial extensions, *Bull. London Math. Soc.* **40** (2008) 151–162.

[3] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, Tilting theory and cluster combinatorics, *Adv. Math.* **204**(2) (2006) 572–618.

[4] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras, *Trans. Amer. Math. Soc.* **359** (2007) 323–332.

[5] A. B. Buan and Y. Zhou, Silted algebras, *Adv. Math.* **303** (2016) 859–887.

[6] A. B. Buan and Y. Zhou, Endomorphism algebras of 2-term silting complexes, *Algebr. Represent. Theory* **21** (2018) 181–194.

[7] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters ($\mathbb{A}_n$ case), *Trans. Amer. Math. Soc.* **358**(3) (2006) 1347–1364.

[8] F. U. Coelho and M. Lanzilotta, Algebras with small homological dimensions, *Manuscripta Mathematica* **100** (1999) 1–11.

[9] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, *London Mathematical Society Lecture Note Series* Vol. 119 (Cambridge University Press, Cambridge, 1988).

[10] D. Happel and C. M. Ringel, Tilted algebras, *Trans. Amer. Math. Soc.* **274**(2) (1982) 399–443.

[11] B. Keller, On triangulated orbit categories, *Documenta Math.* **10** (2005) 551–581.

[12] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* **211**(1) (2007) 123–151.

[13] B. Keller and D. Vossieck, Aisles in derived categories, *Bull. Soc. Math. Belg. Sér. A* **40**(2) (1988) 239–253.