ECS 132 Fall 2020: Assignment 7 - 11 pts

November 20, 2020

Instructions

1. You may in no circumstances upload your homework to private tutoring websites such as CourseHero or Chegg. Remember all material related to this course is a property of the University of California and posting them is a violation of the copyright laws.
2. If you refer to a source (either a book or the internet), you must cite it.
3. You are highly urged to work on these problems on your own. See the homework grading policy. Not getting the answer correct has very low penalty. However, trying it and then figuring out where you went wrong will really help you understand the material better and you will be much better prepared for the exams. If you do discuss with others, you must list their names.
4. Write your answers in R Markdown and submit the knitted pdf on Gradescope; for due date and other details see the Homework Policy and Schedule.

Rubric

- Half credit for starting with the right approach/formula, and half for computing the answer correctly (i.e. simplifying and executing the right formula).

1 Problem

Cars cross a certain point in the highway in accordance with a Poisson process with rate $\lambda = 3$ cars per minute.

1. If I run blindly across the highway it takes me $s = 10$ seconds to cross the road. Assume that if I am on the highway when a car passes by, then I will be injured. What is the probability that I will be uninjured?
2. Suppose that I am agile enough to escape from a single car, but if I encounter 2 or more cars while attempting to cross the highways, I will be injured. However, it now takes me $s = 30$ seconds to cross the highway. What is the probability that I will be uninjured while crossing the highway?

1.1 Answer - 2 pts

1pt: The rate of the Poisson process is $\lambda = 3$ cars per minute or $\lambda = 0.05$ cars per second. The Poisson process is denoted by $N(t)$ which denotes the number of arrivals in times t. Based on the definition

$$P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!} \quad k = 0, 1, 2, \dots$$

The probability that I will be uninjured P_u is is the probability that there are no cars arrive in 10 seconds.

$$\begin{aligned} P_u &= P(N(10) = 0) \\ &= \frac{e^{-10\lambda}(10\lambda)^0}{0!} \\ &= e^{-10\times 0.05} \\ &= e^{-0.5} \end{aligned}$$

1pt: In the case I am agile but it takes me 30 seconds to cross

$$\begin{aligned} P_u &= P(N(30) = 0) + P(N(30) = 1) \\ &= \frac{e^{-30\lambda}(30\lambda)^0}{0!} + \frac{e^{-30\lambda}(30\lambda)^1}{1!} \\ &= e^{-1.5} + 1.5 \times e^{-1.5} \end{aligned}$$

2 Problem

Pedestrians approach from the left and right side of the crossing following a Poisson process with average arrival rate λ_l and λ_R arrivals per minute. Each pedestrian then waits until a light is flashed, at which time all waiting pedestrians must cross. We refer to each time the light is flashed as a "dump" and assume that a dump takes zero time (i.e., pedestrians cross instantly). Assume that the left and right arrival processes are independent. Further assume that the dump occurs every T minutes. Find,

1. The expected number of pedestrians crossing left to right on any dump.
2. The probability that zero pedestrians cross (left to right or right to left) on any particular dump.

2.1 Answer - 3 pts

1. **1pt:** Expected number of pedestrian crossing left to right $\lambda_l \times T$.
2. **2pt:** The total arrival rate is $\lambda = \lambda_l + \lambda_r$. Let $N(t)$ denote the number crossing in time t. Probability that there will be 0 crossing is given by

$$P\{N(T) = 0\} = e^{-\lambda T} \tag{1}$$

3 Problem

If X is normally distributed with parameters $\mu = 10$ and $\sigma = 6$, compute

1. $P\{X < 5\}$;
2. $P\{4 < X < 16\}$;
3. $P\{X < 8\}$.

3.1 Answer - 3 pts

1. **1pt:** $P\{X < 5\} = P\{X \leq 5\} \stackrel{a}{=}$ `pnorm(5, mean=10, sd=6);`

$$\begin{aligned}
P\{X<5\} &= P\{\frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\} \\
&= P\{\frac{X-10}{6} < \frac{5-10}{6}\} \\
&= P\{\frac{X-10}{6} < -0.8333333\} \\
&= \Phi(-0.8333333) \\
&= 1-\Phi(0.8333333) \\
&\stackrel{b}{=} 1-0.7967 \\
&= 0.2033
\end{aligned}$$

2. **1pt:** $P\{4 < X < 16\} = P\{4 < X \leq 16\} = P\{X \leq 16\} - P(X \leq 4)$ $\stackrel{a}{=}$ `pnorm(16, mean=10, sd=6) - pnorm(4, mean=10, sd=6);`

$$\begin{aligned}
P\{4<X<16\} &= P\{\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\} \\
&= P\{-1 < \frac{X-10}{6} < 1\} \\
&= \Phi(1)-\Phi(-1) \\
&= \Phi(1)-(1-\Phi(1)) \\
&= 2\times\Phi(1)-1 \\
&\stackrel{b}{=} 2\times 0.8633-1 \\
&= 0.6826
\end{aligned}$$

3. **1pt:** $P\{X < 8\} = P\{X \leq 8\} \stackrel{a}{=}$ `pnorm(8, mean=10, sd=6).`

$$\begin{aligned}
P\{X<8\} &= P\{\frac{X-\mu}{\sigma} < \frac{8-\mu}{\sigma}\} \\
&= P\{\frac{X-10}{6} < \frac{8-10}{6}\} \\
&= P\{\frac{X-10}{6} < -0.33333\} \\
&= \Phi(-0.333333) \\
&= 1-\Phi(0.333333) \\
&\stackrel{b}{=} 1-0.6293 \\
&= 0.3707
\end{aligned}$$

[(a) is using R function, (b) using Z chart]

4 Problem

The annual rainfall (in inches) in a certain region is normally distributed with mean $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches? What assumption are you making?

4.1 Answer - 3 pts

1pt: Let X denote the random variable which denotes the annual rainfall in inches. X is normally distributed with parameters $\mu = 40$ and $\sigma = 4$. We can find the probability that in a certain year the rainfall is more than 50 inches. This is given by

$$\begin{aligned}
P\{X > 50\} &= 1 - P\{X \leq 50\} \stackrel{a}{=} 1 - \texttt{pnorm(50, mean=40, sd=4)} \\
&= 1 - P\{\frac{X-40}{4} \leq \frac{50-40}{4}\} \\
&= 1 - P\{\frac{X-40}{4} \leq 2.5\} \\
&= 1 - \Phi(2.5) \\
&\stackrel{b}{=} 1 - 0.9938 \\
&= 0.0062
\end{aligned}$$

[(a) is using R function, (b) using Z chart]

2pt: Let $p = 0.0062$ denote the probability in year there is more that 50 inches of rainfall (success) and $1 - p = 0.9938$ is the probability of less than 50 inches of rainfall (failure). Let Y denote the number of years before success which implies that Y can take values $\{1, 2, \ldots, \infty\}$. If we assume rainfall in a year is independent of the other years, Y is geometrically distributed and we need to find $P\{Y > 10\}$ which is given by

$$\begin{aligned}
P\{Y > 10\} &= 1 - P\{Y \leq 10\} \\
&= 1 - \sum_{i=1}^{10} (1-p)^{i-1} p
\end{aligned}$$