ECS 132 Fall 2020: Assignment 2 - 10 pts

October 17, 2020

Instructions

1. You may in no circumstances upload your homework to private tutoring websites such as CourseHero or Chegg. Remember all material related to this course is a property of the University of California and posting them is a violation of the copyright laws.

2. If you refer to a source (either a book or the internet), you must cite it.

3. You are highly urged to work on these problems on your own. See the homework grading policy. Not getting the answer correct has very low penalty. However, trying it and then figuring out where you went wrong will really help you understand the material better and you will be much better prepared for the exams. If you do discuss with others, you must list their names.

4. Write your answers in R Markdown and submit the knitted pdf on Gradescope; for due date and other details see the Homework Policy and Schedule.

1 Problem

A person has n keys, of which one will open her door.

1. If she tries the keys at random, discarding those that don't work, what is the probability that she will open the door on her kth try?

2. What if she does not discard previously tried keys?

1.1 Solution

1. If she tries the keys at random, discarding those that don't work, what is the probability that she will open the door on her kth try? On the first try, the probability of getting the right key is $\frac{1}{n}$ and not getting the right key is $\frac{n-1}{n}$. On the second try, the probability of getting the right key is $\frac{1}{n-1}$ since there as $n-1$ remaining keys. On the kth try there are $n-k+1$ remaining keys. Let X denote the random variable which is the number of attempts to get the right key. Hence,

$$P\{X = k\} = (\frac{n-1}{n})(\frac{n-2}{n-1})\ldots(\frac{n-k+1}{n-k+2})(\frac{1}{n-k+1})$$

1.2 Points: 2 pts

- 1/2 pt for properly setting up equations, 1/2 pt for solving part 1.
- 1/2 pt for properly setting up equations, 1/2 pt for solving part 2.

2. What if she does not discard previously tried keys?

   On each try probability that she will get the right key is $1/n$. Let X be the random variable which denotes the number of attempts to get right key. The probability that she will get the right key on the kth try implies that she did not get the right key on the first $k-1$ attempts. Hence,

$$P\{X = k\} = \frac{1}{n}(\frac{n-1}{n})^{k-1} \quad k = 1, 2, \ldots$$

2 Problem

Consider that a certain part of the memory address space is partitioned into 5 sequential sections. Suppose 5 files of different sizes are randomly written into these section (one file in one section). What is the probability that files will be written in order of their sizes in the sections (the largest in the first section, the second largest in the second section, and so on).

2.1 Solutions

Solution 1

There are 5! ways in which files can be arranged. There is one outcome in which they are ordered by their size. Hence, the probability that they will be ordered by their size is 1/5!

Solution 2

Here we use the multiplication rule. Let A denote that the largest file is in position 1, B denote that the second largest files is in position 2, and similarly C, D, and E denote that the third, fourth, and fifth largest files are in the 3rd, 4th, and 5th positions. We want to find the probability $P(A \cap B \cap C \cap D \cap E)$. Using the multiplication rule

$$P(A \cap B \cap C \cap D \cap E) = P(A) \times P(B|A) \times P(C|A \cap B) \times P(D|A \cap B \cap C) + P(E|A \cap B \cap C \cap D)$$

$P(A) = 1/5$ since the largest file could be in one of 5 position. Similarly, $P(B|A) = 1/4$ since the second largest files can be in one of 4 positions given A. Thus

$$\begin{aligned} P(A \cap B \cap C \cap D \cap E) &= (1/5) \times (1/4) \times (1/3) \times (1/2) \times (1/1) \\ &= 1/120 \end{aligned}$$

2.2 Points - 1 pt

- 1/2 pt for recognizing either (a) that there were 5! ways to arrange files, OR (b) properly setting up the probabilities for solution 2.

- 1/2 pt for correct answer

3 Problem

Bridge is a card game in which the deck of 52 cards is randomly and equally divided among the 4 players. So each player gets 13 cards.

1. What is the probability that one player receives all the 13 hearts.
2. What is the probability that each player receives 1 Queen.

As an aside, did you know that one of the greatest bridge players in the world is a (retired) faculty of the Computer Science Department here at UCDavis? Yes! Prof. Charles U. (Chip) Martel who retired a few years ago but still does research (`https://web.cs.ucdavis.edu/~martel/main/` has been the world champion many times [1].

3.0.1 Solution

Let H_i denote the event that hand i has all 13 hearts, $i = 1, 2, 3, 4$. The number of possible outcomes of a hand is $\binom{52}{13}$. So $P(H_i)$ is given by

$$P(H_i) = \frac{1}{\binom{52}{13}} \quad i = 1, 2, 3, 4$$

What we want $P(\cup_{i=1}^{4} H_i$. Note that the events $H_i, i = 1, 2, 3, 4$ are mutually exclusive. If player 1 has all hearts then no other player can have hearts. If E_1 occurs then E_2, E_3, and E_4 cannot occur. and similarly for E_2 and E_3, and E_4. Hence

$$\begin{aligned} P(\cup_{i=1}^{4} H_i) &= \sum_{i=1}^{4} P(H_i) \\ &= 4 * \frac{1}{\binom{52}{13}} \\ &= \sim 6.3 \times 10^{12} \end{aligned}$$

For the second part, we first note that the total number of ways in which 52 cards can be divided among the four players is $\frac{52!}{13!13!13!13!}$. This follows from the Multinomial Theorem. note that these are equally likely outcomes

Now we need to find number of ways in which each had will have a Queen. If we keep aside the 4 Queens, then the remaining 48 cards can be dealt out in $\frac{48!}{12!12!12!12!}$. Now the 4 Queens can be dealt in 4! ways. Thus the total number of ways in which each player receives 1 Queen is $4! \times \frac{48!}{12!12!12!12!}$. This based on the multiplication rule. The the required probability

$$\begin{aligned} P(\text{each player receives 1 Queen}) &= \frac{4! \times \frac{48!}{12!12!12!12!}}{\frac{52!}{13!13!13!13!}} \\ &= 0.105 \end{aligned}$$

[1] `https://alum.mit.edu/slice/meet-chip-martel-75-one-worlds-greatest-bridge-players`

3.1 Points - 2pts

Part 1

- 1/2 pt for properly recognizing that the total possible number of hands is $\binom{52}{13}$ and recognizing that the probability of a hand of all hearts is $\frac{1}{\binom{52}{13}}$
- 1/2 pt for recognizing that the probability is 4x this, and coming to the right numerical solution

Part 2

- 1/2 pt for properly using multinomial theorem to define the possible number of ways to divide a deck of 52 cards amongst four people $\frac{52!}{13!^4}$
- 1/2 pt for defining the number of ways that the queens can be dealt (4!), and the remaining 12 cards/player can be dealt $\frac{48!}{12!^4}$ and finding the numerical solution.

4 Problem

A folder contains n encrypted files of which one is a special file. An attacker has access to the folder and withdraws k of these files one at a time. Each selection is equally likely to be any one of the remaining files at the time. What is the probability that the intruder withdraws the special file?

You must give reasons or derivation of your answer.

4.1 Solution

Let's first figure out the sample space. Since the files are treated in an identical manner, the number of different selections of k files is $\binom{n}{k}$. The number of different selections in which the special file is selected is $1 \times \binom{n-1}{k-1}$. Thus the required probability is given by

$$\begin{aligned} P(\text{Special file is selected}) &= \frac{1 \times \binom{n-1}{k-1}}{\binom{n}{k}} \\ &= \frac{k}{n} \end{aligned}$$

4.2 Points - 1pt

- 1/2 pt for identifying the sample space as $\binom{n}{k}$
- 1/2 pt for identifying the number of selections in which the special file is included
- This problem can be solved in a number of ways, you receive credit if your solution is a reasonable way of doing so that was mentioned in class

5 Problem

Write a simulation of the (hat) matching problem discussed in class. The simulation should calculate the probability of atleast one match for different values of n (the number of people). For each value of n do 1000 experiments to calculate the probability. Plot the probability as a function of n. In the same plot, add a line showing the asymptotic value. Notice how quickly the asymptotic value is reached.

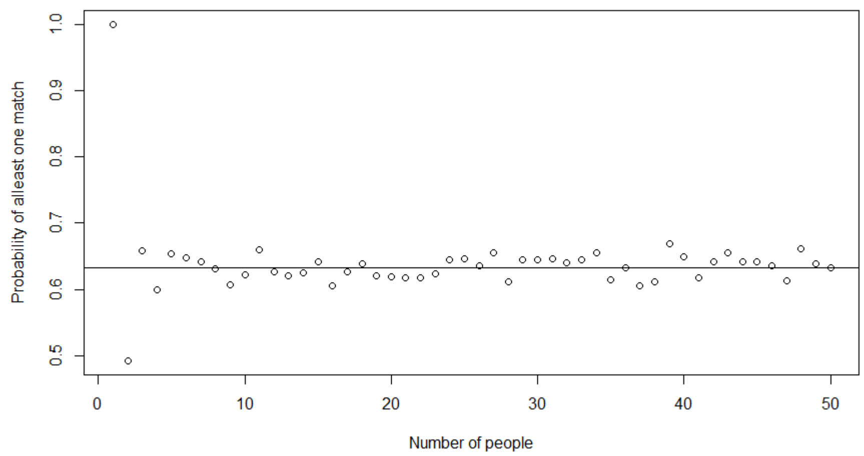
5.1 Solution

```
m = 1000                                      # number of experiments
n = 50                                       # number of people in room
x = numeric(m)                                # vector for numbers of matches 0 no matches
p = numeric(n)

for (k in 1:n)
    {
        for (i in 1:m)
        {
            matches = 0                         # initialize the match count
            b = sample(1:k, k, repl=F)          # n random permutation of numbers 1 throug
            for (j in 1:k)
            {
                if (b[j] == j) {
                 matches = matches +  1
                }
            }
            if (matches > 0)
            {
                x[i] = 1
            }

        }
        p[k] = mean(x==1)                       # probability of atleast 2 matck for k peo
        x = numeric(m)
    }

plot(p, ylab = "Probability of alleast one match", xlab = "Number of people")
abline(h = 0.632)
```



5.2 Points - 3 pts

- 2 pt for making a simulation that effectively models the system
- 1 pt for generating the appropriate figure and finding the asymptote

6 Problem

A couple has two children. What is the probability that both are girls if the older of two is a girl?

6.1 Solution

The sample space $S = \{(B,G),(B,B),(G,B),(G,G)\}$ where the first element of tuple is the elder child. Given that the older of the two child is a girl, the constrained sample space $S_c = \{(G,B),(G,G)\}$. In this sample space the probability both are girls is $\frac{1}{2}$.

6.2 Points - 1 pt

- 1/2 pt for the correct answer (1/2)
- 1/2 pt for any reasonable explanation of this solution – you could discuss that births are independent events, or you could describe the whole sample space.