ECS 132 Fall 2020: Assignment 6 - 10 Points

November 14, 2020

Point designations now appear in the answers section

Instructions

1. You may in no circumstances upload your homework to private tutoring websites such as CourseHero or Chegg. Remember all material related to this course is a property of the University of California and posting them is a violation of the copyright laws.
2. If you refer to a source (either a book or the internet), you must cite it.
3. You are highly urged to work on these problems on your own. See the homework grading policy. Not getting the answer correct has very low penalty. However, trying it and then figuring out where you went wrong will really help you understand the material better and you will be much better prepared for the exams. If you do discuss with others, you must list their names.
4. Write your answers in R Markdown and submit the knitted pdf on Gradescope; for due date and other details see the Homework Policy and Schedule.

Rubric

- For prob 1.3, half the points is for getting first part of the solution right and half for the second part.
- For prob 2.2, half the points for using two rexp()'s to sample correctly and the rest half for approximating $P(W > 5)$ - you get no points for the latter half if you computed expected value or some other nonrelevant statistics.
- For all the other problems/subproblems, you get half for starting with the right approach/formula, and half for computing the answer correctly (i.e. simplifying and executing the right formula).

1 Problem

The probability density function of X, the lifetime of a certain type of electronic device (measure in hours) is given by

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

1. Find $P\{X > 20\}$.
2. What is the cumulative distribution function of X?
3. What is the probability that of 6 such types of devices at least 3 will function for at least 15 hours? What assumption are you making?

1.1 Answers - 4 points

1. Find $P\{X > 20\}$ - **1 pt**

$$\begin{aligned}
P\{X > 20\} &= 1 - P\{X \leq 20\} \\
&= 1 - \int_{10}^{20} f(x)dx \\
&= 1 - [-10x^{-1}|_{10}^{20}] \\
&= 1 - (1 - 0.5) \\
&= 0.5
\end{aligned}$$

2. What is the cumulative distribution function of X? - **1 pt**

$$\begin{aligned}
P\{X \leq a\} &= \int_{10}^{a} f(x)dx \\
&= [-10x^{-1}|_{10}^{a}] \\
&= 1 - \frac{10}{a}
\end{aligned}$$

3. What is the probability that of 6 such types of devices at least 3 will function for at least 15 hours? What assumption are you making? - **2 pts**

Probability that a device will last at least 15 hours if given by $P\{X > 15\}$ which is given by

$$\begin{aligned}
P\{X > 15\} &= 1 - P\{X \leq 15\} \\
&= 1 - F_X(15) \\
&= 1 - (1 - \frac{10}{15}) \\
&= 2/3
\end{aligned}$$

Let $p = 2/3$ denote the probability that the device lasts for at least 15 hours. Hence, the probability that it does not last for 15 hours is $1 - p = 1/3$. Given $n = 6$ devices, let Y be the random variable that denotes the number of devices that last for at least 15 hours. If we assume that the devices are independent, then Y is a Binomial random variable and we need to find $P\{Y \geq 3\}$. This given by

$$\begin{aligned}
P\{Y \geq 3\} &= 1 - P\{Y \leq 2\} \\
&= 1 - \sum_{0}^{2} \binom{6}{i} p^i (1-p)^{6-i} \\
&= 1 - pbinom(2, 6, 0.66667) \\
&= 0.8998957
\end{aligned}$$

2 Problem

Jobs arriving to a computer have been found to require CPU time that can be modelled by an exponential distribution with parameter 1/140 per millisecond. The CPU scheduling discipline is quantum-oriented so that jobs not completing within a quantum of 100 milliseconds will be routed back to the tail of the queue of waiting jobs. Find the probability that an arriving job will be forced to wait for a second quantum. Of the 800 jobs coming in during a day, how many are expected to finish within the first quantum.

2.1 Answer - 2 pts

1 pt - Let X_i denote the CPU time required by an arriving job. $X \sim$ Expo(λ) where $\lambda = 1/140$. The probability that will have to wait for a second quantum is

$$\begin{aligned} P(X > 100) &= 1 - P(X \leq 100) \\ &= 1 - F_X(100) \\ &= 1 - (1 - e^{-\lambda 100}) \\ &= e^{-\frac{100}{140}} \\ &\approx 0.49 \end{aligned}$$

1 pt - The probability that job will finish in 1 quantum is $1 - P(X > 100) = 0.51$. Hence for 800 jobs in a day, the number that will finish in 1 quantum = $0.51 \times 800 = 408$ jobs.

3 Problem

Two core in a CPU serve jobs from a single queue. The time it takes one core to serve a randomly selected job is distributed as Expo($\frac{1}{5}$) so that his average service time is 5 minutes. The other, more powerful core's service times are distributed as Expo($\frac{1}{4}$), averaging 4 minutes. Cores work independently. Both cores are busy and a new job arrives in the queue.

1. What is the probability it will be longer than 5 minutes before the new job started to be served?
2. Write a R code to simulate the problem and verify the result in Part 1.

3.1 Answer - 2 pts

1 pt - Let X_1 and X_2 be the random variable that denotes the execution time in core 1 and core 2, respectively. We are given that $X_1 \sim$ Expo($\frac{1}{5}$) and $X_2 \sim$ Expo($\frac{1}{4}$). Let W denote the time that the new job must wait before it is started to be served. Since the new job will be started to be served as soon as any one of the jobs is finished, we can see that $W = \min(X_1, X_2)$. We want to $P(W > 5)$

$$\begin{aligned} P(W > 5) &= P(\min(X_1, X_2) > 5) \\ &= P(X_1 > 5, X_2 > 5) \\ &= \underbrace{P(X_1 > 5) \times P(X_2 > 5)}_{\text{due to independence}} \\ &= e^{-\frac{1}{5}\times 5} e^{-\frac{1}{4}\times 5} \\ &= e^{-\frac{9}{4}} \end{aligned} =$$

$$0.1054$$

1 pt for code

```
}
n = 10000
rates = 1/5
ratef = 1/4

waiting_time = numeric(n)
for (i in 1:n){
    time_slow = rexp(1,rate=rates)
    time_fast = rexp(1,rate=ratef)
    waiting_time[i] = min(time_slow, time_fast)
}
prob = mean(waiting_time > 5)
print(prob)
```

4 Problem

Suppose the network flow size is a normal random variable with parameter $\mu = 71$was GBytes and $\sigma = 2.5$ GBytes. What percentage of the flows are greater than 72 GBytes? Suppose flows that are greater than 72 GBytes are clssified as large flows. What percentage of the large flows are greater than 77 GBytes?

4.1 Answer - 2 pts

1 pt Let X be a random variable that denotes the network flow size. $X \sim N(71, 2.5)$. First, we want to find $P(X > 72)$ which is $1 - P(X \leq 72)$.

$$\begin{aligned} P(X \leq 72) &= P(\frac{X-71}{2.5} \leq \frac{72-71}{2.5}) \\ &= \Phi(0.4) \\ &= 0.6554 \end{aligned}$$

Thus the required probability is $1 - 0.6554 = 0.3446$.

1pt - For the second part we want to find the probability that the flow size is greater than 77 GBytes given that it is a large flow.

$$\begin{aligned} P(X > 77 | X > 72) &= \frac{P(X > 77 \cap X > 72)}{P(X > 72)} \\ &= \frac{P(X > 77)}{P(X > 72)} \\ &= \frac{1 - P(X \leq 77)}{P(X > 72)} \\ &= \frac{1 - \Phi(2.4)}{0.3446} \\ &= \frac{1 - 0.9918}{0.3446} \\ &= 0.024 \end{aligned}$$