Unit-sphere games

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Abstract

This paper introduces a class of games, called unit-sphere games, in which strategies are real vectors with unit 2-norms (or, on a unit-sphere). As a result, they should no longer be interpreted as probability distributions over actions, but rather be thought of as allocations of one unit of resource to actions and the payoff effect on each action is proportional to the *square root* of the amount of resource allocated to that action. The new definition generates a number of interesting consequences. We first characterize the sufficient and necessary condition under which a two-player unit-sphere game has a Nash equilibrium. The characterization reduces solving a unit-sphere game to finding all eigenvalues and eigenvectors of the product matrix of individual payoff matrices. For any unit-sphere game with non-negative payoff matrices, there always exists a unique Nash equilibrium; furthermore, the unique equilibrium is efficiently reachable via Cournot adjustment. In addition, we show that any equilibrium in positive unit-sphere games corresponds to approximate equilibria in the corresponding normal-form games. Analogous but weaker results are obtained in n-player unit-sphere games.

1 Introduction

Consider the following two games.

Example 1. Protecting Manhattan. *Two police stations try to protect Manhattan, which can be visualized as a rectangle, from a terrorist attack. Station A is responsible for protecting all the* streets, *i.e., the horizontal paths across the rectangle; while station B is responsible for protecting all the* avenues, *i.e., the vertical paths. Each police station has one unit of police force, distributes its force among its paths, and derives a positive utility u_{ij}^A (resp. u_{ij}^B) from successfully protecting each subway station S_{ij}, namely, the intersections of street i and avenue j and 0 utility when failing to protect it. The probability of successfully protecting a subway station is $\sqrt{a_i b_j}$, where a_i and b_j are the amount of police force station A and B allocates to street i and avenue j respectively.*

Example 2. Sponsored search for complementary queries. *A user submits a query, say "Yellow Stone national park", to a travel website, hoping to book a hotel and a flight. The query triggers interests from two complementary advertisement agencies: one has a collection of hotel ads and the other airline ads. The website allocates space for two listings of advertisements, one for hotel and the other for airline.*

Each agency derives positive utility u_{ij}^A (resp. u_{ij}^B) if the user successfully clicks a pair of ads (hotel i, airline j). Note that agency A's utility may depend on j since the payment rule may involve both i and j. To achieve a click through rate (CTR, the probability that an ad is clicked) of a_i for hotel ad i, agency A needs to pay $C_A a_i^2$, where C_A is some characteristic constant (e.g., the quality score) with respect to A; similarly, agency B must pay $C_B b_j^2$ to get a CTR of b_j. The rationality of the cost model is further justified in the footnote.[1] The two agencies seek to maximize their utilities with fixed amounts of budget, X_A and X_B respectively.

Formalizing the problem above, we have the following game theoretical model: agency A picks a CTR vector a, where a_i is the CTR that agency A wants to achieve for hotel ad i, subject to the total cost equal to the budget of the agency, i.e., $C_A \|a\|_2^2 = X_A$, where C_A and X_A are constants. Similarly, agency B chooses a vector b of expected CTR of airline ads subject to its own budget constraint. Agency A (resp. B) seeks to maximize its utility, $a^T U^A b$ (resp. $a^T U^B b$), where $U^A = \{u_{ij}^A\}$ and $U^B = \{u_{ij}^B\}$.

The games above may somewhat resemble the *Blotto game* [13], where players wish to jointly ensure a set of outcomes and the probability that an outcome is ensured is a non-linear function in the amount of resource each agent spends on the corresponding action. As we shall see, both examples are instances of *positive unit-sphere games*, a class of games that possess unique, learnable, pure Nash equilibria.

2 Unit-sphere games

Most of the paper deals with *2-player unit-sphere games*. In Section 6, this definition is extended to accommodate any number of players.

Definition 1. *A two-player* unit-sphere game (USG) *is defined by two matrices $A \times B$, where*

- *A is an $m \times n$ payoff matrix for player 1,*

- *B is an $n \times m$ payoff matrix for player 2.*

A unit-sphere strategy x for player 1 is a column vector of real numbers such that $x \in \mathbb{R}^m$, $\|x\|_2 = 1$, while a strategy y for player 2 is a column vector of real numbers such that $y \in \mathbb{R}^n$, $\|y\|_2 = 1$. Given a *strategy profile* (x, y), the utility $u_1(x, y)$ of player 1 is $x^T A y$ while the utility $u_2(x, y)$ of player 2 is $y^T B x$. Other game-theoretical notions, such as best response and Nash equilibrium, follow standard definitions.

Mathematically, the above definition is a 2-player normal-form game, except for the definition of strategy, where the restriction of unit L_1-norm is now replaced by unit L_2-norm. In other words, each unit-sphere strategy is a point on a unit sphere, rather than a probability distribution. It implies that both x and y can be negative on some dimensions, as long as they are on a unit sphere.

It is important to note that a USG can just be thought of as a standard normal-form game, where each pure strategy corresponds to a unit-sphere strategy and there are infinite many such strategies.

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[1]According to certain existing empirical evaluations (see [1]), the CTR of an ad link is concave in payment, where the degree of concavity depends on the keyword length. In our example, the CTR is proportional to the square root of the money spent, which is one special and commonly used concave function. In fact, it is standard to assume that the cost of a certain amount of "effort" e (e.g., CTR in our example) is proportional to e^2. See [7, 6, 10, 8].

From this perspective, the characterization theorems (Theorems 3.1, 3.3, 3.4) are sufficient and necessary conditions for a new class of normal-form games to have (unique) pure Nash equilibria.

In this paper, we do not consider randomized unit-sphere strategies, for the following reasons. First of all, a randomization over unit-sphere strategies is no longer a unit-sphere strategy, thus not well-defined under our new definition. Secondly, it is not hard to see that such a randomized strategy has a L_2-norm less than 1 and is always utility-dominated by some unit-sphere strategy. Last but not least, we are interested in comparing unit-sphere strategy (which is somewhat mixed) to standard mixed strategy, in terms of existence and computation efficiency of Nash equilibrium. Adding another level of mixture makes the comparison less interesting.

One can also view players in a USG as *risk averse agents* whose payoffs, when facing a lottery outcome, are not linear expectations of their utilities on deterministic outcomes in the lottery, but concave expectations (in our case, a square-root function). In general, games with concave utility agents possess a mixed Nash equilibrium but it is in general computationally hard to find such an equilibrium [5, Theorem 1]. Our model and results differ from Fiat and Papadimitriou in that we allow for negative strategies, i.e., x and y can have negative entries, thus the whole strategy set is not necessarily convex, precluding a Nash style proof. Furthermore, when restricting to positive payoff matrices, we show that a *unique* Nash equilibrium exists and easy to compute. Readers are referred to [5] and the references therein for an introduction on non-linear expectations.

Finally, in our definition, adding a positive constant to each payoff function no longer yields an equivalent USG. Intuitively, when adding a large constant to a player's payoff function, the player has more incentive to distribute her resource more evenly among actions. So, it loses generality to restrictions on positive payoff matrices. On the other hand, USGs are scale-invariant in the sense that multiplying a constant to a player's payoff function yields an equivalent USG.

Our results. For 2-player USGs, we show:

- The necessary and sufficient condition for a USG to have a Nash equilibrium (Theorem 3.1). A USG with payoff matrices A and B has an NE if and only if the product of payoff matrices, AB, has a nonnegative eigenvalue.

- Closed forms of a pair of NEs for any positive USG (Theorem 3.2, Theorem 3.3).

- Uniqueness of NE in positive USGs (Theorem 3.4). As a result, the two NEs we give explicitly in Theorem 3.2 are identical (Corollary 1).

- Learnability of the unique NE in repeated positive USGs via Cournot adjustments (Theorem 4.1). The error decreases exponentially fast when both players play their best responses in each round.

- A multiplicative $O\left(\sqrt{\max(m,n)}\right)$-approximation for mixed strategy NE in normal form games via USGs (Theorem 5.1), where m and n are the numbers of pure strategies for the players.

We further generalize our results to m-player positive USGs. We show the existence of NE in multiplayer positive USGs (Theorem 6.1), and discuss subclasses of multiplayer positive USGs, symmetric positive USGs and Markov positive USGs. We present an algorithm to find a symmetric NE for any symmetric positive USG with even number of players, and show that a unique NE

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exists in any Markov positive USG, which can be efficiently reached via Cournot adjustments (Theorem 6.2, Theorem 6.3).

3 Nash Equilibria in USGs

In this section, we characterize the sufficient and necessary conditions for Nash equilibrium (NE) to exist in USGs. In particular, equilibrium exists in all the USGs with positive payoff matrices. It is unique and efficiently computable, via a well-known learning process known as *Cournot adjustment*.

3.1 Structure of NE in USGs

Let us now consider a USG $A \times B$. It is easy to see that the utilities of the two players under strategy profile (x, y) are

$$u_1 = x^T A y = \|Ay\|_2 \cos \alpha,$$

$$u_2 = y^T B x = \|Bx\|_2 \cos \beta,$$

respectively, where α denotes the angle between x and Ay and β denotes the angle between y and Bx.[2] Since both x and y are on the unit-sphere, a strategy profile (x, y) forms an NE if and only if

$$x = \arg\max_{x'} x'^T A y \iff \alpha = 0 \text{ or } \|Ay\| = 0 \iff \lambda x = Ay,$$

and

$$y = \arg\max_{y'} y'^T B x \iff \beta = 0 \text{ or } \|Bx\| = 0 \iff \mu y = Bx,$$

where $\lambda = \|Ay\|_2$, $\mu = \|Bx\|_2$.

By this observation, we derive a necessary condition of existence of NE for two-player USGs.

Lemma 1. *For a USG $A \times B$. If AB and BA do not share a nonnegative eigenvalue, it does not have any NE.*

Proof. We show that an NE exists only if AB and BA share a nonnegative eigenvalue. Consider payoff matrices A and B. For a NE profile (x, y),

$$B\lambda x = BAy \Rightarrow \lambda \mu y = BAy,$$

$$A\lambda y = ABx \Rightarrow \lambda \mu x = ABx.$$

In other words, x is an eigenvector of AB with eigenvalue $\lambda \mu$, and y is an eigenvector of BA with eigenvalue $\lambda \mu$. $\qquad \square$

It is known that AB and BA have the same set of eigenvalues. The following theorem characterizes the sufficient and necessary condition for an NE to exist in any two-player USG.

Theorem 3.1. *For a USG $A \times B$, it has an NE if and only if AB has a nonnegative eigenvalue.*

Proof. The only-if direction follows from Lemma 1. We now prove the if direction. Assume AB has a nonnegative eigenvalue λ with eigenvector x such that $\|x\|_2 = 1$.

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[2] In cases where $Ay = 0$ (resp. $Bx = 0$), one may set α (resp. β) arbitrarily.

- If $Bx \neq 0$, let $y = \frac{Bx}{\|Bx\|_2}$. (x, y) is an NE for the game, because $\frac{\lambda}{\|Bx\|_2} x = Ay$, and $\|Bx\|_2 y = Bx$.

- If $Bx = 0$, $y \neq 0$ can be chosen such that either $Ay = kx$ for some $k > 0$, when $\det A \neq 0$, or $Ay = 0$, when $\det A = 0$. Also we assume $\|y\|_2 = 1$. Again (x, y) is an NE for the game, because $kx = Ay$ for some $k \geq 0$, and the utility of player 2, $y^T Bx$, is always 0.

$\square$

As stated in Theorem 3.1, to solve a USG $A \times B$, i.e., to find all NEs or to ensure that no NE exists, it is equivalent to calculate all eigenvalues of AB and the corresponding eigenvectors. Solving USGs is reduced to the *eigenvalue problem*, one of the most well-studied problems in linear algebra. Refer to [14] for efficient algorithms.

3.2 Positive USGs

We now focus on a general, intuitive class of USGs where there always exists a unique NE.

Definition 2. *A USG $A \times B$ is* positive *if $A, B > 0$, and any strategy satisfies $x, y \geq 0$[3].*

Positive USGs (PUSGs) have many interesting properties that general USGs do not necessarily possess. Before we state these properties, we need the following lemma from linear algebra.

Lemma 2. *(Perron-Frobenius [2]): For any square matrix $A > 0$, we have*

- *A has an eigenvalue $\lambda > 0$. Moreover, for any other eigenvalue μ of A, $|\lambda| > |\mu|$. We call λ the Perron-Frobenius value, or spectral radius of A, denoted as $\lambda = \rho(A)$.*

- *The eigenvalue λ has algebraic and geometric multiplicity one. There is an eigenvector $x > 0$ of A with an eigenvalue of λ. Moreover, the only positive eigenvectors of A have the form kx for some $k > 0$, and all positive eigenvectors have corresponding eigenvalue λ.*

Lemma 3. *For payoff matrices $A > 0$, $B > 0$, AB and BA share at least one positive eigenvalue, which is their spectral radius.*

Proof. Clearly, AB and BA are square matrices. Let $x > 0$ be an eigenvector of AB with eigenvalue $\lambda = \rho(AB) > 0$, whose existence is guaranteed by Lemma 2. Note that

$$BA(Bx) = B(ABx) = \lambda(Bx).$$

Namely, Bx is an eigenvector of BA with eigenvalue λ. It follows that AB and BA share the same positive eigenvalue $\lambda > 0$. Now suppose $\rho(BA) > \lambda$. By the same argument, we can see that $\rho(BA)$ is an eigenvalue of AB, a contradiction. $\square$

With Lemma 3, we are now able to derive a pair of NEs for all PUSGs.

Theorem 3.2. *There exists two NEs (x_1, y_1), (x_2, y_2) for any PUSG, where*

- *$x_1 > 0$ is the unit eigenvector of AB with eigenvalue $\lambda = \rho(AB)$.*

---

[3]We say a matrix $A > 0$ if $A_{ij} > 0$ for all (i, j), and a vector $x \geq 0$ if $x_i \geq 0$ for all i.

- $y_1 = \frac{Bx_1}{\|Bx_1\|_2}$.

  where the utilities of the players obtained from (x_1, y_1) are $\left(\frac{\lambda}{\|Bx_1\|_2}, \|Bx_1\|_2\right)$.

- $y_2 > 0$ is the unit eigenvector of BA with eigenvalue $\lambda = \rho(BA)$.[4]

- $x_2 = \frac{Ay_2}{\|Ay_2\|_2}$.

  where the utilities of the players obtained from (x_2, y_2) are $\left(\|Ay_2\|_2, \frac{\lambda}{\|Ay_2\|_2}\right)$.

Proof. We prove for the case of (x_1, y_1). The case of (x_2, y_2) is symmetric. By Lemma 3, it is always feasible to pick x_1 as stated in the theorem. For player 1,

$$
\begin{aligned}
u_1(x', y_1) = x'^T A y_1 &= \frac{1}{\|Bx_1\|_2} x'^T A B x_1 \\
&= \frac{\lambda}{\|Bx_1\|_2} x'^T x_1 \leq \frac{\lambda}{\|Bx_1\|_2} x_1^T x_1 \\
&= \frac{\lambda}{\|Bx_1\|_2}.
\end{aligned}
$$

For player 2,

$$
\begin{aligned}
u_2(x_1, y') = y'^T B x_1 &= y'^T \|Bx_1\|_2 y_1 \\
&\leq \|Bx_1\|_2 y_1^T y_1 = \|Bx_1\|_2.
\end{aligned}
$$

In other words, neither player has profitable deviation in (x_1, y_1). $\qquad\square$

Theorem 3.2 derives a pair of NEs for any PUSG. One might wonder whether the two NEs are identical? This is indeed the case. We dedicate Subsection 3.3 to this result.

In fact, there is a symmetric NE in a PUSG if the payoff matrices satisfy a certain additional condition. Before we state these conditions, we need the following technical lemma.

Lemma 4. *For square matrices $A > 0$, $B > 0$ such that $AB = BA$, A and B share the same one-dimensional eigenspace of spectral radius.*

Proof. Let $\lambda = \rho(A)$, $x > 0$ be an eigenvector of A whose corresponding eigenvalue is λ, then

$$
A(Bx) = B(Ax) = \lambda(Bx),
$$

namely Bx is an eigenvector of A whose eigenvalue is λ. By Lemma 2, the eigenspace of λ is one-dimensional, which implies that $Bx = \mu x$ for some μ. Again by Lemma 2, x belongs to the eigenspace of the spectral radius of B, or equivalently $\mu = \rho(B)$. $\qquad\square$

If $AB = BA$, the corresponding PUSG has a symmetric NE.

Theorem 3.3. *There is a symmetric NE (x, x) for any PUSG with square payoff matrices $A \times B$ such that $AB = BA$. The NE utilities are $(\rho(A), \rho(B))$.*

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[4] Recall that $\rho(AB) = \rho(BA)$

Proof. Let $x > 0$ be the unit eigenvector of A whose corresponding eigenvalue is $\rho(A)$ (and therefore the unit eigenvector of B whose eigenvalue is $\rho(B)$). For player 1,

$$u_1(x', x) = x'^T A x = \rho(A) x'^T x$$
$$\leq \rho(A) x^T x = \rho(A).$$

For player 2,

$$u_2(x', x) = x'^T B x = \rho(B) x'^T x$$
$$\leq \rho(B) x^T x = \rho(B).$$

Neither player has a profitable deviation in (x, x). $\qquad\square$

3.3   Uniqueness of NE in PUSGs

As mentioned, one of the most appealing properties of all PUSGs is that they have unique NE.

Theorem 3.4. *Any PUSG has an unique NE.*

Proof. Let (x, y) be an arbitrary NE of PUSG with payoff matrices A and B, whose existence has been established in Theorem 3.2. By Lemma 1,

$$\exists \lambda > 0, \ \mu > 0, \ \text{s.t.} \ ABx = \lambda x, \ BAy = \mu y$$

We will show that λ is the spectral radius of AB, and x is the corresponding positive unit eigenvector. The case of y is symmetric.

Assume $\lambda \neq \rho(AB)$. By Lemma 2, there must be some $i \in [n]$ such that $x_i = 0$, since there are no other positive eigenvectors beside those of the spectral radius. Note that $\lambda > 0$, $AB > 0$.

$$0 = \lambda x_i = (ABx)_i = \sum_j (AB)_{ij} x_j$$
$$\geq \min(AB)_{ij} \|x\|_1 > 0,$$

a contradiction. Therefore $\lambda = \rho(AB)$. Again by Lemma 2, the eigenspace of λ is one-dimensional. Namely x is the unique positive eigenvector of λ such that $\|x\|_2 = 1$.

The same argument works for y. To conclude, we prove that (x, y) is the unique NE. $\qquad\square$

Note that if we restrict strategies to be strictly positive, Theorem 3.4 then follows directly as a corollary of Lemma 2. When taking nonnegative strategies into consideration, with the additional argument above, we are still able to obtain the uniqueness result.

Corollary 1. *Any PUSG has an unique NE, which has the form stated in Theorem 3.2. Moreover, the two symmetric NEs in Theorem 3.2 are identical.*

Next, we show the unique NE of a PUSG can be efficiently found via a natural learning process.

4   Solving PUSGs via Cournot adjustments

In this section, we show that the unique NE of any PUSG can be resulted when both players follow a well-known learning process called *Cournot adjustments* . This is remarkable property since it states that players can learn to play NE even without any information of each other's payoff matrix.

4.1   Cournot adjustments

Define *Cournot adjustments* as follows,

1. In the first round, each player i plays any positive strategy $s_i^0 > 0$.

2. In round t, each player i observes s_{-i}^t, the strategy of player $-i$.

3. In round $t+1$, each player i plays her best response against s_{-i}^t. Namely

$$s_i^{t+1} = \frac{A_i s_{-i}^t}{\|A_i s_{-i}^t\|_2}.$$

4. Iterate until no player updates her strategy.

Cournot adjustments define a natural protocol for players to learn to play a game over time. It is appealing when players do not know others' payoff matrices and for whatever reason that the players cannot perform equilibrium computation upfront. It is known that, for any standard games, a carefully designed better response dynamics can converge to some mixed-strategy Nash equilibrium (aka. Nash's proof), but may take exponential number of rounds. In the following, we show that this procedure thoroughly exploits the properties of PUSGs and finds efficiently the unique NE for any PUSG in logarithmic number of rounds with respect to the initial error.

4.2   Convergence of Cournot adjustments in PUSGs

To formally state and prove the convergence result, we need the following proposition from numerical analysis.

Lemma 5. *(Convergence of power iteration [12]): For any positive square matrix A whose eigenvalue with the largest modulus is λ and the corresponding eigenspace is E, let x_0 be an arbitrary unit vector such that x is not orthogonal to E. Let*

$$x^t = \frac{Ax^{t-1}}{\|Ax^{t-1}\|_2}.$$

It is guaranteed that x^t converges to x^*, where $Ax^* = \lambda x^*$. Moreover,

$$\forall p \in \mathbb{Z}^+ \cup \{\infty\}, \ \exists r \in (0,1), c \in \mathbb{R}^+, \ \text{s.t.}$$

$$\|x^t - x^*\|_p \le cr^t.$$

We now state a convergence result of Cournot adjustments in PUSGs.

Theorem 4.1. *If both players follow Cournot adjustments, the strategy sequence (x^t, y^t) linearly converges to the unique NE of the PUSG, where the game matrices $A > 0$, $B > 0$[5].*

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[5] Linear convergence is another way of saying the error diminishes exponentially fast in the number of iterations.

Proof. Let A and B be the payoff matrices. We can explicitly derive the strategy expressions of Cournot adjustments in round t as follows,

$$x^t = \frac{Ay^{t-1}}{\|Ay^{t-1}\|_2}, \ y^t = \frac{Bx^{t-1}}{\|Bx^{t-1}\|_2}.$$

It follows that

$$x^{2k} = \frac{(AB)^k x^0}{\|(AB)^k x^0\|_2}, \ y^{2k} = \frac{(BA)^k y^0}{\|(BA)^k y^0\|_2}, \ \forall k \in \mathbb{N}.$$

Since we choose $x^0 > 0$, $y^0 > 0$, by Lemma 2, it is impossible that x^0 (resp. y^0) is orthogonal to the eigenspace of the spectral radius of AB (resp. BA). By Lemma 5, as k grows, x^{2k} converges to the positive unit eigenvector of AB exponentially fast, and y^{2k} converges to that of BA. Therefore (x^{2k}, y^{2k}) converges to the unique PSNE exponentially fast. As (x^{2k}, y^{2k}) converges, (x^{2k+1}, y^{2k+1}) converges as well, concluding the proof. $\qquad\square$

5 Approximating mixed-strategy equilibrium in standard games via USGs

It is known that computing a mixed-strategy Nash equilibrium (MSNE) in standard two-player games is PPAD-complete [4]. In this section, we show that our understanding of USG can help us to find an approximate MSNE of any standard games.

5.1 Approximation Scheme

Consider any PUSG. By theorems we have derived so far, one can easily compute the unique NE (x, y) of the PUSG. We now normalize x and y to be x' and y', so that $\|x'\|_1 = \|y'\|_1 = 1$. Our main finding in this section is that (x', y') is a multiplicative $O\left(\sqrt{\max(m, n)}\right)$-approximate MSNE[6] for the underlying standard two-player game.

Call this approximation scheme the *simple approximate scheme.*

5.2 Approximation via simple approximate scheme

Once again, before we state and prove our result, we need the following technical lemma.

Lemma 6.
$$\min_{x \in \mathbb{R}^n, \ \|x\|_1 = 1} \left\{ \frac{\|x\|_2^2}{\|x\|_\infty} \right\} = \frac{2}{\sqrt{n} + 1}.$$

---

[6]A *multiplicative k-approximate MSNE* denotes a strategy profile where no player can improve her utility by k times via deviation.

Proof. Let $t = \|x\|_\infty \geq \frac{1}{n}$. We have,

$$\frac{\|x\|_2^2}{\|x\|_\infty} \geq \frac{t^2 + (n-1)\left(\frac{1-t}{n-1}\right)^2}{t}$$

$$= \frac{n}{n-1}t - \frac{2}{n-1} + \frac{1}{t(n-1)}$$

$$\geq \frac{2}{\sqrt{n}+1}.$$

$\square$

We are now ready to state our result of the section.

Theorem 5.1. *For any standard two-player game with payoff matrices A and B, the simple approximation scheme yields a multiplicative $O\left(\sqrt{\max(m,n)}\right)$-approximate MSNE, where m is the number of rows of A, and n is the number of rows of B.*

Proof. Let (x, y) be the NE of the induced PUSG over payoff matrices $A \times B$, and (x', y') be the normalized vectors, as stated in the simple scheme. Since (x, y) is an NE in the PUSG, $\exists \lambda, \mu$, s.t.

$$Ay' = \lambda x', \ Bx' = \mu y'.$$

Consider player one's payoff with or without deviation.

Without deviation, she gets

$$u_1(x', y') = x'^T A y' = \lambda \|x'\|_2^2.$$

By deviation, she gets

$$\max_{\|x_1\|_1=1} u_1(x_1, y') = \max_{\|x_1\|_1=1} x_1^T A y'$$

$$= \lambda \max_{\|x_1\|_1=1} x_1^T x' = \lambda \|x'\|_\infty.$$

By Lemma 6,

$$\min_{(x', y')} \frac{u_1(x', y')}{\max_{\|x_1\|_1=1} u_1(x_1, y')}$$

$$= \min_{(x', y')} \frac{\|x'\|_2^2}{\|x'\|_\infty} \geq \frac{1}{\sqrt{m}} + \frac{m - 2\sqrt{m} + 1}{\sqrt{m}(m-1)}$$

$$= \Omega\left(\frac{1}{\sqrt{m}}\right).$$

Symmetrically, for player two, the approximate factor becomes $\Omega\left(\frac{1}{\sqrt{n}}\right)$. $\square$

6 Multiplayer PUSGs

In this section, we extend our discussion to the general m-player case.

Definition 3. *An m-player PUSG is defined as $(n_1, \ldots, n_m, u_1, \ldots, u_m)$, where n_i is the size of action set of player i, u_i the utility function, where $u_i(x_1, \ldots, x_m)$ is multilinear in $x_1, \ldots, x_m$, and for all $x_1 \geq 0, \ldots, x_m \geq 0$, we have $u_i(x_1, \ldots, x_m) > 0$.*

An equivalent formulation of PUSGs will involve positive tensors. That is, A^k is the payoff tensor for player k, such that $u_k(x_1, \ldots, x_m) = \sum_{i_1, \ldots, i_m} A^k_{i_1, \ldots, i_m} x_{1,i_1} \ldots x_{m,i_m}$, and $A^k_{i_1, \ldots, i_m} > 0$ for all $i_1, \ldots, i_m$. We will further exploit this notation in the following detailed discussion.

6.1 Existence of NE in multiplayer PUSGs

Lemma 7. *(Brouwer's fixed point theorem): For any $n \in \mathbb{Z}^+$, $\Omega \subseteq \mathbb{R}^n$ which is compact and convex, $f : \Omega \to \Omega$ which is continuous, there is some $x^* \in \Omega$ such that $f(x^*) = x^*$.*

Theorem 6.1. *There exists an NE for any m-player PUSG $(A^1, \ldots, A^m)$.*

The proof resembles that of the existence of MSNE in normal form games.

Proof. Let
$$s_i = \sum_{j \leq i} n_j,$$

$$\mathbb{R}^{s_m} \supset \Omega = \left\{ x = (x_1, \ldots, x_m) \in \mathbb{R}^{s_m} \,\middle|\, x_i \in \mathbb{R}^{n_i}, \|x_i\|_1 = 1, x_{i,j} \geq 0, \forall i \in [m], j \in [n_i] \right\}.$$

For all $x = (x_1, \ldots, x_m) \in \Omega$,

$$f(x) = \left(\frac{A^1 x_2 x_3 \ldots x_m}{\|A^1 x_2 x_3 \ldots x_m\|_1}, \frac{A^2 x_1 x_3 \ldots x_m}{\|A^2 x_1 x_3 \ldots x_m\|_1}, \ldots, \frac{A^m x_1 x_2 \ldots x_{m-1}}{\|A^m x_1 x_2 \ldots x_{m-1}\|_1} \right).$$

It is easy to verify that Ω and f satisfy the conditions in Lemma 7. Therefore there is some $x^* = (x_1^*, \ldots, x_m^*)$ satisfying $f(x^*) = x^*$, which implies that there is some λ_i such that $A^i x_1^* \ldots x_m^* = \lambda_i x_i^*$ for all i. So $\left(\frac{x_1^*}{\|x_1^*\|_2}, \ldots, \frac{x_m^*}{\|x_m^*\|_2} \right)$ is an MSNE of the PUSG. $\square$

6.2 Subclasses of multiplayer PUSGs

In this subsection, we investigate several subclasses of multiplayer PUSGs that are easy to solve.

6.2.1 Symmetric PUSGs with even number of players

We first present an algorithm that solves m-player symmetric PUSGs when m is even.

Definition 4. *An m-player symmetric PUSG is a PUSG where $A^i = A^j$ for all $i, j \in [m]$, and $A^k_{i_1, \ldots, i_m} = A^k_{\sigma(i_1), \ldots, \sigma(i_m)}$ for all $k \in [m]$ and $\sigma \in S_n$, where S_n is the symmetric group over $[n]$ and n is the number of actions of each player.*

The method used to find NE in symmetric PUSG is called *SS-HOPM*. SS-HOPM outputs a symmetric NE with a particular payoff which equals the largest *Z-eigenvalue* of the payoff tensor. ($\lambda \in \mathbb{R}$ is a Z-eigenvalue of m-th order n-dimensional symmetric tensor A if there is some $x \in \mathbb{R}^n$, $Ax^{m-1} = \lambda x$ and $\|x\|_2 = 1$.) The linear convergence of SS-HOPM has been originally established in [9] and revised in [3].

The algorithm is as follows,

1. Choose $x^0 > 0$, and the shift constant $\alpha = \lceil m \sum_{i_1,\ldots,i_m} A_{i_1,\ldots,i_m} \rceil$.

2. Let $y^{t+1} = A(x^t)^{m-1} + \alpha x^t$.

3. Compute
$$x^{t+1} = \frac{y^{t+1}}{\|y^{t+1}\|_2}, \quad \lambda^{t+1} = A(x^{t+1})^m.$$

As shown in [3], x^t converges to an symmetric NE x^* while λ^t converges to the payoff of each player under x^*.

6.2.2 Markov PUSGs via Cournot Adjustments

Generalizing our uniqueness result for the two-player case using the techniques in [11], we show that a unique PSNE exists in any *Markov PUSG*, which can be efficiently reached via Cournot adjustments.

Definition 5. *An Markov PUSG $(A^1, \ldots, A^m)$ is a PUSG such that*
$$\sum_{i_k} A^k_{i_1,\ldots,i_m} = c_k$$

for all $k \in [m], i_1 \in [n_1], \ldots, i_{k-1} \in [n_{k-1}], i_{k+1} \in [n_{k+1}], \ldots, i_m \in [n_m]$ and a constant c_k.

In other words, Markov PUSG is a subset of PUSG such that, fixing any other players' strategy profile, the sum of player k's utility over all his/her actions is a constant, for any k. Since every Markov PUSG can be scaled so that for all k, $c_k = 1$, it is without loss of generality to consider games with $c_k = 1$ for all k.

Lemma 8. *For nonnegative $x_1, \ldots, x_m$ such that $\|x_i\|_1 = 1$, we have $\|A^k x_1 \ldots x_{k-1} x_{k+1} \ldots x_n\|_1 = 1$ for all $k \in [m]$.*

Proof.

$$\|A^k x_1 \ldots x_{k-1} x_{k+1} \ldots x_n\|_1$$
$$= \sum_{i_k} \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} A^k_{i_1,\ldots,i_m} x_{1,i_1} \ldots x_{k-1,i_{k-1}} x_{k+1,i_{k+1}} \ldots x_{m,i_m}$$
$$= \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \sum_{i_k} A^k_{i_1,\ldots,i_m} x_{1,i_1} \ldots x_{k-1,i_{k-1}} x_{k+1,i_{k+1}} \ldots x_{m,i_m}$$
$$= \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} x_{1,i_1} \ldots x_{k-1,i_{k-1}} x_{k+1,i_{k+1}} \ldots x_{m,i_m}$$
$$= 1$$

$\square$

Lemma 9. *Let Ω be as defined above, $f : \Omega \to \Omega$ be such that for $v \in \Omega$,*

$$f(v)_k = A^k v_1 \ldots v_{k-1} v_{k+1} \ldots v_m.$$

For $x = (x_1, \ldots, x_m) \in \Omega$, $y = (y_1, \ldots, y_m) \in \Omega$,

$$\|f(x)_k - f(y)_k\|_1 \le (1 - \delta_k) \left(\sum_{i \in [m],\, i \neq k} \|x_i - y_i\|_1 \right),$$

where

$$\delta_k = \min_{V \subseteq [n_k]} \left[\min_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m} \sum_{i_k \in V} A^k_{i_1, \ldots, i_m} + \min_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m} \sum_{i_k \in V'} A^k_{i_1, \ldots, i_m} \right],$$

and $V' = [n_k] \setminus V$.

Proof.

$$\sum_{i_k \in V_k} (f(x)_{k,i_k} - f(y)_{k,i_k})$$

$$= \sum_{i_k \in V_k} \sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m} A_{i_1, \ldots, i_m} (x_{1,i_1} \ldots, x_{k-1,i_{k-1}}, x_{k+1,i_{k+1}}, \ldots, x_{m,i_m} - $$

$$y_{1,i_1} \ldots, y_{k-1,i_{k-1}}, y_{k+1,i_{k+1}}, \ldots, y_{m,i_m})$$

$$= \sum_{i_k \in V_k} \sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m} A_{i_1, \ldots, i_m} [(x_{1,i_1} - y_{1,i_1}) x_{2,y_2} \ldots, x_{m,i_m} + $$

$$y_{1,i_1} (x_{2,i_2} - y_{2,i_2}) x_{3,i_3} \ldots x_{m,i_m} + \cdots + y_{1,i_1} \ldots y_{m-1,i_{m-1}} (x_{m,i_m} - y_{m,i_m})].$$

Let $V_k \subseteq [n_k]$ be the largest set such that $\forall i_k \in V_k$, $f(x)_{k,i_k} > f(y)_{k,i_k}$, $V_1 \subseteq [n_1]$ the largest set such that $\forall i_1 \in V_1$, $x_{1,i_1} > y_{1,i_1}$. Note that by Lemma 8, $\|x_k\|_1 = \|y_k\|_1 = 1$, and hence

$\sum_{i_k} x_{k,i_k} - y_{k,i_k} = 0$ for all $k \in [m]$. We then have

$$\sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \sum_{i_k \in V_k} A_{i_1,\ldots,i_m}(x_{1,i_1} - y_{1,i_1})y_{2,i_2}\ldots,y_{m,i_m}$$

$$= \sum_{i_1 \in V_1} \sum_{i_2,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \sum_{i_k \in V_k} A_{i_1,\ldots,i_m}(x_{1,i_1} - y_{1,i_1})y_{2,i_2}\ldots,y_{m,i_m}+$$

$$\sum_{i_1 \notin V_1} \sum_{i_2,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \sum_{i_k \in V_k} A_{i_1,\ldots,i_m}(x_{1,i_1} - y_{1,i_1})y_{2,i_2}\ldots,y_{m,i_m}$$

$$\leq \sum_{i_1 \in V_1} \sum_{i_2,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \left(\max_{j_1 \in V_1,j_2,\ldots,j_m} \sum_{j_k \in V_k} A_{j_1,\ldots,j_m} \right)(x_{1,i_1} - y_{1,i_1})y_{2,i_2}\ldots,y_{m,i_m}-$$

$$\sum_{i_1 \notin V_1} \sum_{i_2,\ldots,i_{k-1},i_{k+1},\ldots,i_m} \left(\min_{j_1 \in V_1,j_2,\ldots,j_m} \sum_{j_k \in V_k} A_{j_1,\ldots,j_m} \right)(y_{1,i_1} - x_{1,i_1})y_{2,i_2}\ldots,y_{m,i_m}$$

$$= \left(\max_{j_1 \in V_1,j_2,\ldots,j_m} \sum_{j_k \in V_k} A_{j_1,\ldots j_m} - \min_{j_1 \in V_1,j_2,\ldots,j_m} \sum_{j_k \in V_k} A_{j_1,\ldots,j_m} \right) \times$$

$$\sum_{i_1 \notin V_1} \sum_{i_2,\ldots,i_{k-1},i_{k+1},\ldots,i_m} (y_{1,i_1} - x_{1,i_1})y_{2,i_2}\ldots y_{k-1,i_{k-1}}y_{k+1,i_{k+1}}\cdots y_{m,i_m}$$

$$\leq \left(1 - \min_{j_1,\ldots,j_m} \sum_{j_k \notin V_k} A_{j_1,\ldots,j_m} - \min_{j_1,\ldots,j_m} \sum_{j_k \in V_k} A_{j_1,\ldots,j_m} \right) \sum_{i_1 \notin V_1}(y_{1,i_1} - x_{1,i_1})$$

$$\leq \frac{1}{2}(1 - \delta_k)\|x_1 - y_1\|_1.$$

We therefore get

$$\|f(x)_k - f(y)_k\|_1 = 2 \sum_{i_k \in V_k}(f(x)_{k,i_k} - f(y)_{k,i_k})$$

$$\leq 2 \sum_{i \neq k} \left[\frac{1}{2}(1 - \delta_k)\|x_i - y_i\|_1 \right]$$

$$= (1 - \delta_k)\left(\sum_{i \neq k} \|x_i - y_i\|_1 \right).$$

$\square$

Theorem 6.2. *There exists an unique NE in any Markov PUSG where $\delta_k > \frac{m-2}{m-1}$ for all k.*

Proof. Assume there are two distinct NE in an m-player game $(A^1,\ldots,A^m)$, x_0 and y_0. Let

$x = \frac{x_0}{\|x_0\|_1}$, $y = \frac{y_0}{\|y_0\|_1}$. By Lemma 9,

$$
\begin{aligned}
&\|x - y\|_1 \\
&= \sum_k \|x_k - y_k\|_1 \\
&= \sum_k \|f(x)_k - f(y)_k\|_1 \\
&\leq \sum_k \sum_{i \neq k} (1 - \delta_k) \left(\|x_i - y_i\|_1 \right) \\
&< \sum_{i \in [m]} (m - 1) \left(1 - \frac{m - 2}{m - 1} \right) \left(\|x_i - y_i\|_1 \right) \\
&= \|x - y\|_1,
\end{aligned}
$$

an contradiction. □

Theorem 6.3. *Cournot adjustments lead to the unique NE in any Markov PUSG where $\delta_k > \frac{m-2}{m-1}$ for all k.*

Proof. For simplicity, we denote strategies by vectors whose L_1-norm are scaled to 1 in the proof. Consider a procedure where player k starts by playing $x_k^0 = (\frac{1}{n_k}, \ldots, \frac{1}{n_k})$. Let x^* be the unique PSNE of the game, guaranteed to exist by Theorem 6.2. Let $\epsilon_0 = \max_{i \in [m]} \|x_i^0 - x_i^*\|_1$, $\delta = \max_{i \in [m]} (1 - \delta_i)$. Clearly, in round t, strategies of player k will be $x_k^t = f(x^{t-1})_k$, where f is the same as stated above.

On the other hand, as shown in Lemma 9,

$$
\|x_k^t - x_k^*\|_1 \leq \delta \left(\sum_{i \neq k} \|x_i^{t-1} - x_i^*\|_1 \right), \forall t \in \mathbb{Z}^+, k \in [m].
$$

By a simple induction, we prove that

$$
\epsilon_t = \max_k \left[\|x_k^t - x_k^*\|_1 \right] \leq (m - 1)^t \delta^t \epsilon_0.
$$

When $t = 0$, it holds obviously that $\epsilon_0 \leq \epsilon_0$. Assume that $\epsilon_{t-1} \leq (m-1)^{t-1} \delta^{t-1} \epsilon_0$, we may show,

$$
\epsilon_t = \max_k \left[\|x_k^t - x_k^*\|_1 \right] \leq \max_k \delta \left(\sum_{i \neq k} \|x_i^{t-1} - x_i^*\|_1 \right) \leq \delta (m - 1) \epsilon_{t-1} \leq (m - 1)^t \delta^t \epsilon_0.
$$

It can be seen easily that ϵ_t goes to 0 exponentially fast considering that $\delta < \frac{1}{m-1}$.

We have shown that the L_1-norm normalized strategies converge to the L_1-norm normalized NE. It follows naturally that the strategies themselves converge to the unique NE. Morevoer, the convergence is linear, i.e., the error decreases exponentially fast. □

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6.3 Multiplicity of NE in multiplayer USGs

In fact, there may be infinitely many NEs in a multiplayer USG. Here is an interesting example [3].

Example 3. *Consider a 4-player USG where game tensors (A^1, A^2, A^3, A^4) are such that $A^1_{1112} = A^1_{2122} = A^2_{1112} = A^2_{1222} = A^3_{1112} = A^3_{2122} = A^4_{1121} = A^4_{2122} = 2$, $A^j_{i_1 i_2 i_3 i_4} = 0$ otherwise. We consider symmetric strategy $x = (x_1, x_2)$. In order for x to be an NE, we need*

$$\begin{cases} 2x_1^2 x_2 = \lambda x_1 \\ 2x_1 x_2^2 = \lambda x_2 \\ x_1^2 + x_2^2 = 1 \end{cases}.$$

By setting $\lambda = 2x_1 x_2$, it appears that any pair of (x_1, x_2) where $x_1^2 + x_2^2 = 1$ forms a symmetric NE. Moreover, any equilibrium payoff $\lambda \in [0, 1]$ can be achieved by some choice of (x_1, x_2).

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