Advanced Macro

Assignment 2

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Competitive equilibrium

Arrow-Debreu competitive equilibrium consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ and the allocations for household $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that,

(i) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the firm allocation $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ solves the firm problem,

$$
\begin{aligned}
\max_{\{y_t, k_t, l_t\}_{t=0}^{\infty}} &\sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t) \\
\text{s.t. } & y_t = z k_t^{\alpha} l_t^{1-\alpha}, \forall t \geq 0; \\
& y_t, k_t, l_t \geq 0, \forall t \geq 0.
\end{aligned}
\tag{1}
$$

(ii) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the household allocation $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves the household problem,

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}} &\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right) \\
\text{s.t. } &\sum_{t=0}^{\infty} p_t(c_t + K_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t); \\
& 0 \leq l_t \leq 1, 0 \leq k_t \leq k_0, c_t \geq 0, k_{t+1} \geq 0, \forall t \geq 0; \\
& k_0 \text{ given.}
\end{aligned}
\tag{2}
$$

(iii) The market clear conditions,

$$
\begin{aligned}
z k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t &= c_t + k_{t+1}; \\
l_t^d &= l_t^s; \\
k_t^d &= k_t^s.
\end{aligned}
$$

Steady state

For firm problem,

$$
\begin{aligned}
r_t &= z\alpha k_t^{\alpha-1} l_t^{1-\alpha} \\
w_t &= z(1-\alpha)k_t^{\alpha} l_t^{-\alpha} \\
r_t k_t + w_t l_t &= z k_t^{\alpha} l_t^{1-\alpha}
\end{aligned}
\tag{3}
$$

Then for household problem,

$$\mathcal{L}(\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}; \lambda_t) = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right) + \lambda_t \left(\sum_{t=0}^{\infty} p_t(z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta)k_t) \right)$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t c_t^{-\sigma} - \lambda_t p_t = 0,$$

$$\frac{\partial \mathcal{L}}{\partial l_t} = -\beta^t \chi l_t^{\eta} + \lambda_t p_t z (1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0, \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial k_t} = \lambda_t p_t (z\alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - \lambda_{t-1} p_{t-1} = 0,$$

$$c_t = z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta)k_t$$

For Steady state, $p_0 = 1$, then,

$$c^{\sigma} l^{\eta} = z(1-\alpha)k^{\alpha}l^{-\alpha}/\chi$$
$$z\alpha k^{\alpha-1} l^{1-\alpha} = 1/\beta - 1 + \delta \tag{5}$$
$$c = z k^{\alpha} l^{1-\alpha} - \delta k$$

$$M = \frac{k}{l} = \left(\frac{z\alpha\beta}{1 - \beta + \beta\delta} \right)^{\frac{1}{1-\alpha}},$$

$$N = \frac{c}{l} = z\left(\frac{k}{l}\right)^{\alpha} - \delta \frac{k}{l} = zM^{\alpha} - \delta M,$$

$$(lN)^{\sigma} l^{\eta} = z(1-\alpha)\left(\frac{k}{l}\right)^{\alpha}/\chi \Rightarrow$$

$$l^{\sigma+\eta} = \frac{z(1-\alpha)M^{\alpha}}{\chi N^{\sigma}} \Rightarrow$$

$$l_{ss} = \left(\frac{z(1-\alpha)M^{\alpha}}{\chi N^{\sigma}} \right)^{\frac{1}{\sigma+\eta}},$$

Then,

$$k_{ss} = Ml$$
$$c_{ss} = Nl$$
$$y_{ss} = z k^{\alpha} l^{1-\alpha} = zM^{\alpha}l$$
$$r_{ss} = \alpha z k^{\alpha-1} l^{1-\alpha} = \alpha z M^{\alpha-1}$$
$$w_{ss} = (1-\alpha)z k^{\alpha} l^{\alpha} = (1-\alpha)zM^{\alpha}.$$

Social planner problem

The problem of the social planner is that, given the initial capital k_0,

$$w(k_0) = \max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta} \right)$$

$$s.t. \ z k_t^{\alpha} l_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta)k_t, \ \forall t \geq 0 \tag{6}$$

$$c_t \geq 0, \ k_t \geq 0, \ 0 \leq l_t \leq 1, \ \forall t \geq 0$$

$$k_0 \text{ is given.}$$

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Bellman equation,

$$V(k) = \max_{\substack{0 \le l \le 1 \\ 0 \le k' \le zk^\alpha l^{1-\alpha} + (1-\delta)k}} \left\{ \frac{(zk^\alpha l^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta} + \beta \mathbb{E} V(k') \right\} \tag{7}$$

VFI

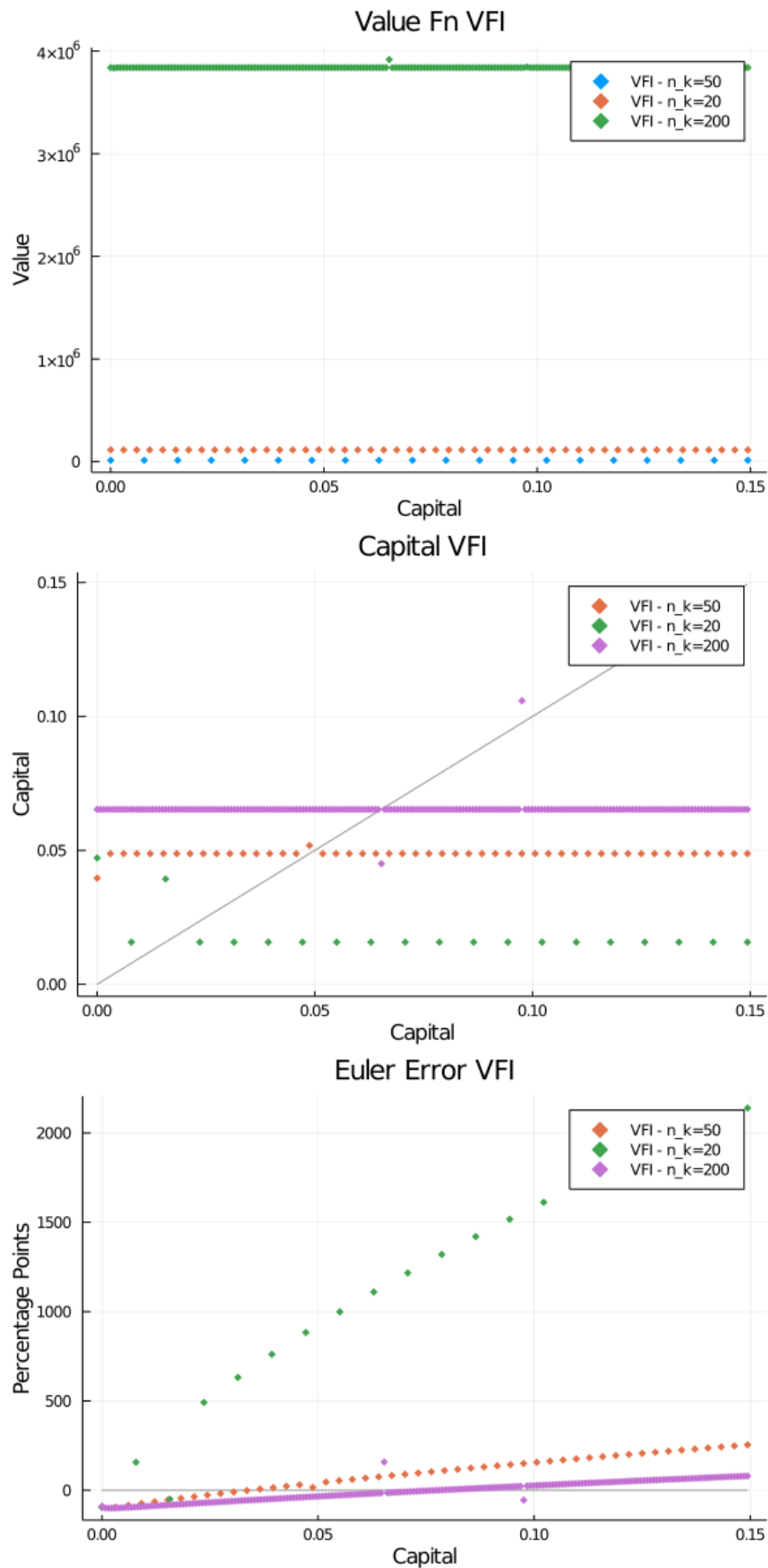Julia code: click here.

```
# Julia code
# See A2.jl
```

Value Fn VFI

Capital VFI

Euler Error VFI

Figure 1: Wage and capital