Advanced Macro

Assignment 2

Hans Martinez

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Competitive equilibrium

Arrow-Debreu competitive equilibrium consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ and the allocations for household $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that,

(i) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the firm allocation $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ solves the firm problem,

$$\begin{aligned} \max_{\{y_t,k_t,l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t) \\ \text{s.t. } & y_t = z k_t^{\alpha} l_t^{1-\alpha}, \forall t \geq 0; \\ & y_t, k_t, l_t \geq 0, \forall t \geq 0. \end{aligned} \tag{1}$$

(ii) Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the household allocation $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves the household problem,

$$\begin{aligned} \max_{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t(c_t + K_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t); \\ & 0 \leq l_t \leq 1, 0 \leq k_t \leq k_0, c_t \geq 0, k_{t+1} \geq 0, \forall t \geq 0; \\ & k_0 \text{ given.} \end{aligned} \tag{2}$$

(iii) The market clear conditions,

$$\begin{aligned} z k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t = c_t + k_{t+1}; \\ l_t^d = l_t^s; \\ k_t^d = k_t^s. \end{aligned}$$

Steady state

For firm problem,

$$\begin{aligned} r_t =& z\alpha k_t^{\alpha-1} l_t^{1-\alpha} \\ w_t =& z(1-\alpha) k_t^{\alpha} l_t^{-\alpha} \\ r_t k_t + w_t l_t =& z k_t^{\alpha} l_t^{1-\alpha} \end{aligned} \tag{3}$$

Then for household problem,

$$\begin{aligned}\mathcal{L}(\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}; \lambda_t) = \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}) + \lambda_t (\sum_{t=0}^{\infty} p_t (z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta) k_t)\\
\frac{\partial \mathcal{L}}{\partial c_t} =& \beta^t c_t^{-\sigma} - \lambda_t p_t = 0,\\
\frac{\partial \mathcal{L}}{\partial l_t} =& - \beta^t \chi l_t^{\eta} + \lambda_t p_t z (1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0,\\
\frac{\partial \mathcal{L}}{\partial k_t} =& \lambda_t p_t (z \alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - \lambda_{t-1} p_{t-1} = 0,\\
c_t =& z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t\end{aligned} \tag{4}$$

For Steady state, $p_0 = 1$, then,

$$\begin{aligned}&c^{\sigma} l^{\eta} = z(1-\alpha) k^{\alpha} l^{-\alpha} / \chi\\
&z \alpha k^{\alpha-1} l^{1-\alpha} = 1/\beta - 1 + \delta\\
&c = z k^{\alpha} l^{1-\alpha} - \delta k\end{aligned} \tag{5}$$

$$M = \frac{k}{l} = \left(\frac{z \alpha \beta}{1 - \beta + \beta \delta} \right)^{\frac{1}{1-\alpha}},$$

$$N = \frac{c}{l} = z (\frac{k}{l})^{\alpha} - \delta \frac{k}{l} = z M^{\alpha} - \delta M,$$

$$\begin{aligned}(lN)^{\sigma} l^{\eta} =& z(1-\alpha)(\frac{k}{l})^{\alpha} / \chi \Rightarrow\\
l^{\sigma+\eta} =& \frac{z(1-\alpha) M^{\alpha}}{\chi N^{\sigma}} \Rightarrow\\
l_{ss} =& \left(\frac{z(1-\alpha) M^{\alpha}}{\chi N^{\sigma}} \right)^{\frac{1}{\sigma+\eta}},\end{aligned}$$

Then,

$$\begin{aligned}k_{ss} =& Ml\\
c_{ss} =& Nl\\
y_{ss} =& z k^{\alpha} l^{1-\alpha} = z M^{\alpha} l\\
r_{ss} =& \alpha z k^{\alpha-1} l^{1-\alpha} = \alpha z M^{\alpha-1}\\
w_{ss} =& (1-\alpha) z k^{\alpha} l^{\alpha} = (1-\alpha) z M^{\alpha}.\end{aligned}$$

Social planner problem

The problem of the social planner is that, given the initial capital k_0,

$$\begin{aligned}w(k_0) = \max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta})\\
s.t. \;\; z k_t^{\alpha} l_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t, \;\; \forall t \geq 0\\
c_t \geq 0, \; k_t \geq 0, \; 0 \leq l_t \leq 1, \;\; \forall t \geq 0\\
k_0 \text{ is given.}\end{aligned} \tag{6}$$

Bellman equation,

$$V(k) = \max_{\substack{0 \leq l \leq 1 \\ 0 \leq k' \leq zk^{\alpha}l^{1-\alpha}+(1-\delta)k}} \{\frac{(zk^{\alpha}l^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi\frac{l^{1+\eta}}{1+\eta} + \beta\mathbb{E}V(k')\} \tag{7}$$

VFI

Julia code: click here.

```
# Julia code
# See A2.jl
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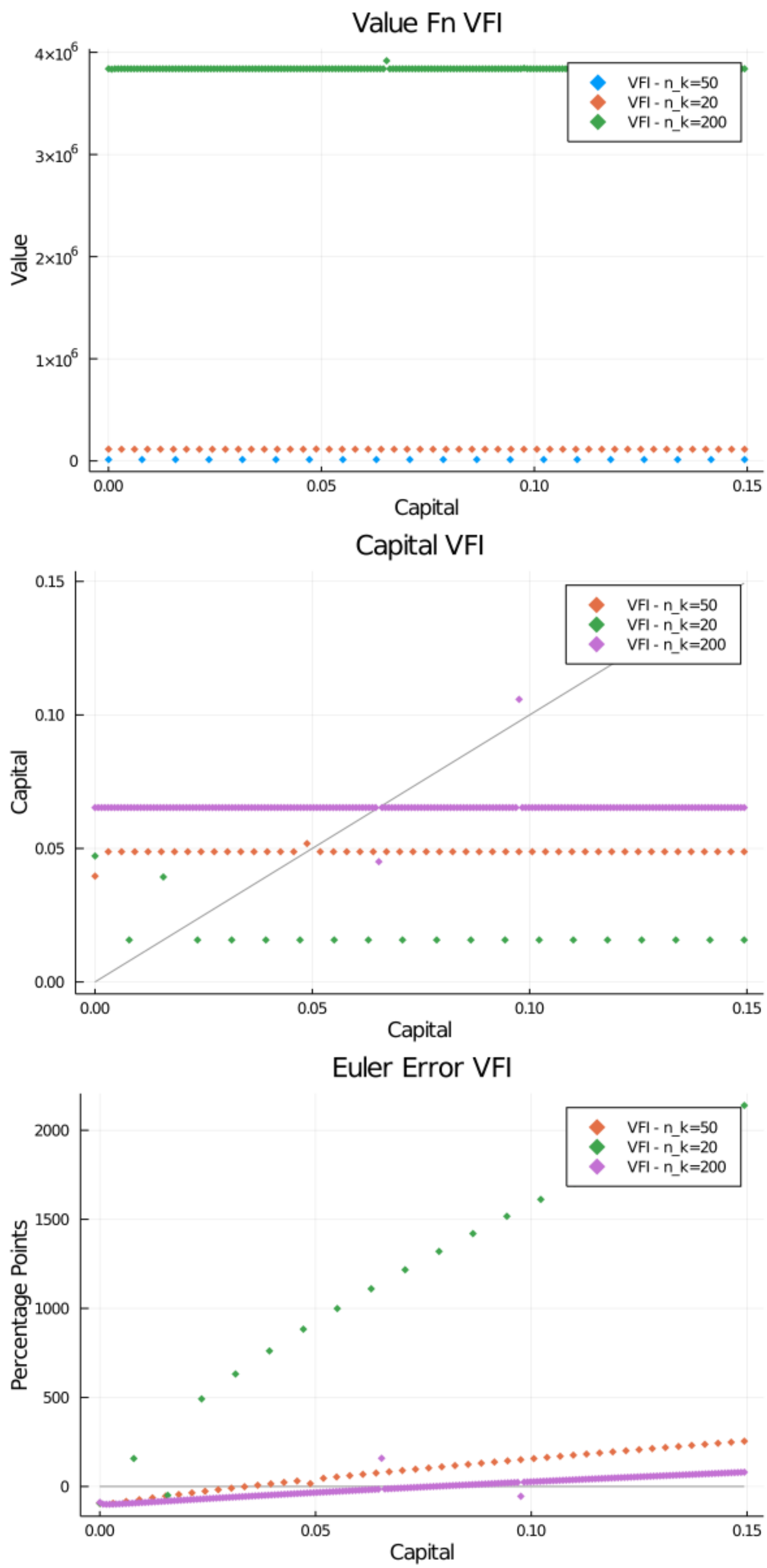

Figure 1: Plain VFI

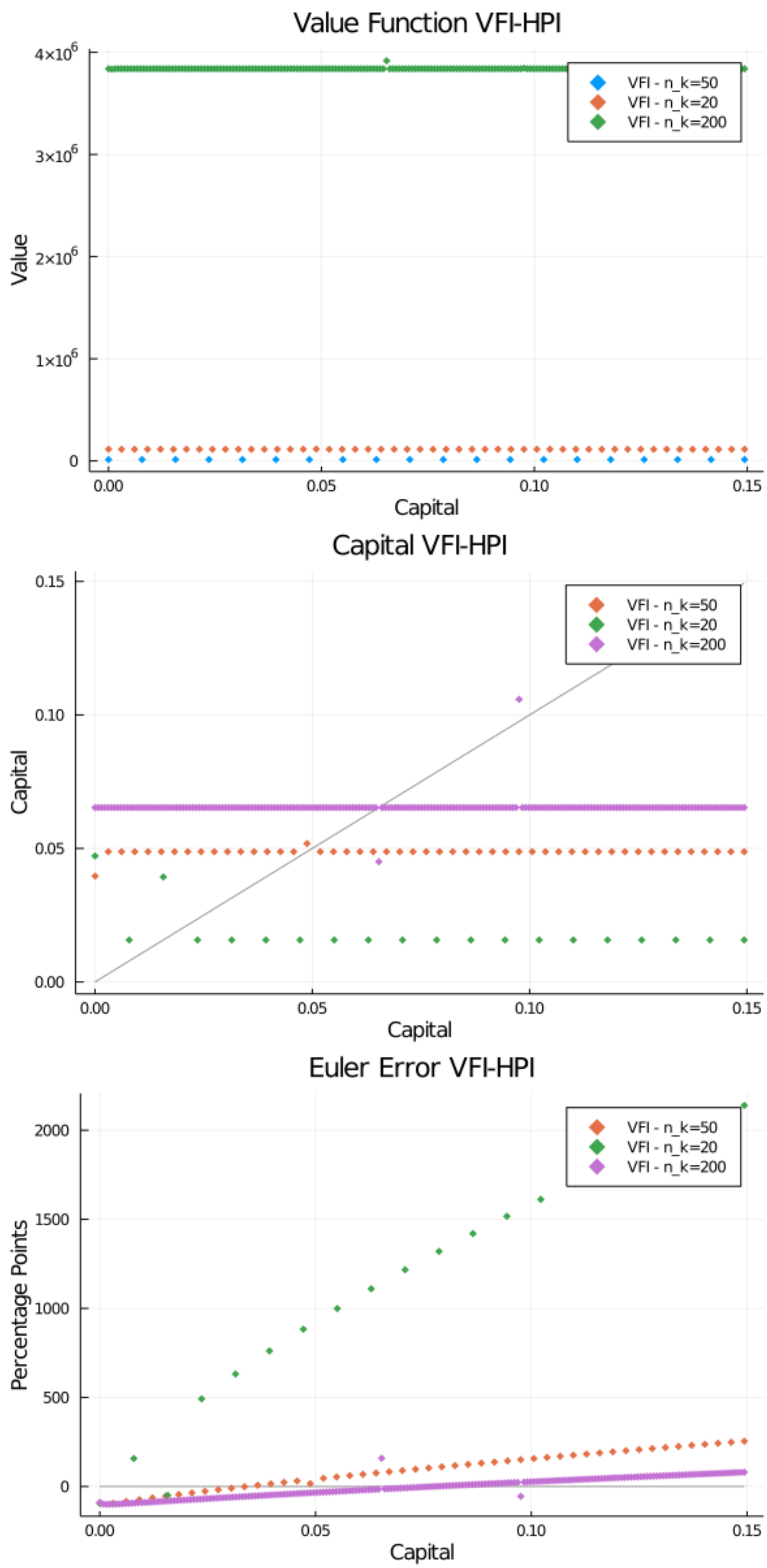

Figure 2: HPI-VFI