# Advanced Macro

Assignment 2

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## Competitive equilibrium

Arrow-Debreu competitive equilibrium consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ and the allocations for household $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ such that,

- Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the firm allocation $\{y_t, k_t^d, l_t^d\}_{t=0}^{\infty}$ solves the firm problem,

$$\begin{aligned} &\max_{\{y_t,k_t,l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t l_t) \\ &\text{s.t. } \; y_t = z k_t^{\alpha} l_t^{1-\alpha}, \forall t \geq 0; \\ &\quad\;\; y_t, k_t, l_t \geq 0, \forall t \geq 0. \end{aligned} \tag{1}$$

- Given a sequence of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the household allocation $\{c_t, k_t^s, l_t^s\}_{t=0}^{\infty}$ solves the household problem,

$$\begin{aligned} &\max_{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}\right) \\ &\text{s.t. } \; \sum_{t=0}^{\infty} p_t(c_t + K_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t l_t); \\ &\quad\;\; 0 \leq l_t \leq 1, 0 \leq k_t \leq k_0, c_t \geq 0, k_{t+1} \geq 0, \forall t \geq 0; \\ &\quad\;\; k_0 \text{ given.} \end{aligned} \tag{2}$$

- The market clear conditions,

$$\begin{aligned} z k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t &= c_t + k_{t+1}; \\ l_t^d &= l_t^s; \\ k_t^d &= k_t^s. \end{aligned}$$

## Steady state

For firm problem,

$$\begin{aligned} r_t =& z\alpha k_t^{\alpha-1} l_t^{1-\alpha} \\ w_t =& z(1-\alpha) k_t^{\alpha} l_t^{-\alpha} \\ r_t k_t + w_t l_t =& z k_t^{\alpha} l_t^{1-\alpha} \end{aligned} \tag{3}$$

Then for household problem,

$$
\begin{aligned}
\mathcal{L}(\{c_t, k_{t+1}, l_t\}_{t=0}^{\infty}; \lambda_t) = \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}) + \lambda_t (\sum_{t=0}^{\infty} p_t (z k_t^{\alpha} l_t^{1-\alpha} - c_t - k_{t+1} + (1-\delta) k_t) \\
\frac{\partial \mathcal{L}}{\partial c_t} =& \beta^t c_t^{-\sigma} - \lambda_t p_t = 0, \\
\frac{\partial \mathcal{L}}{\partial l_t} =& -\beta^t \chi l_t^{\eta} + \lambda_t p_t z (1-\alpha) k_t^{\alpha} l_t^{-\alpha} = 0, \\
\frac{\partial \mathcal{L}}{\partial k_t} =& \lambda_t p_t (z \alpha k_t^{\alpha-1} l_t^{1-\alpha} + 1 - \delta) - \lambda_{t-1} p_{t-1} = 0, \\
c_t =& z k_t^{\alpha} l_t^{1-\alpha} - k_{t+1} + (1-\delta) k_t
\end{aligned}
\tag{4}
$$

For Steady state, $p_0 = 1$, then,

$$
\begin{aligned}
& c^{\sigma} l^{\eta} = z(1-\alpha) k^{\alpha} l^{-\alpha} / \chi \\
& z \alpha k^{\alpha-1} l^{1-\alpha} = 1/\beta - 1 + \delta \\
& c = z k^{\alpha} l^{1-\alpha} - \delta k
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
M = \frac{k}{l} = \left( \frac{z \alpha \beta}{1 - \beta + \beta \delta} \right)^{\frac{1}{1-\alpha}}, \\
N = \frac{c}{l} = z (\frac{k}{l})^{\alpha} - \delta \frac{k}{l} = z M^{\alpha} - \delta M, \\
(lN)^{\sigma} l^{\eta} =& z(1-\alpha)(\frac{k}{l})^{\alpha} / \chi \Rightarrow \\
l^{\sigma+\eta} =& \frac{z(1-\alpha) M^{\alpha}}{\chi N^{\sigma}} \Rightarrow \\
l_{ss} =& \left( \frac{z(1-\alpha) M^{\alpha}}{\chi N^{\sigma}} \right)^{\frac{1}{\sigma+\eta}},
\end{aligned}
$$

Then,

$$
\begin{aligned}
k_{ss} =& Ml \\
c_{ss} =& Nl \\
y_{ss} =& z k^{\alpha} l^{1-\alpha} = z M^{\alpha} l \\
r_{ss} =& \alpha z k^{\alpha-1} l^{1-\alpha} = \alpha z M^{\alpha-1} \\
w_{ss} =& (1-\alpha) z k^{\alpha} l^{\alpha} = (1-\alpha) z M^{\alpha}.
\end{aligned}
$$

## Social planner problem

The problem of the social planner is that, given the initial capital $k_0$,

$$
\begin{aligned}
w(k_0) = \max_{\{c_t, k_t, l_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\frac{c_t^{1-\sigma}}{1-\sigma} - \chi \frac{l_t^{1+\eta}}{1+\eta}) \\
s.t. \;\; z k_t^{\alpha} l_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t, \;\; \forall t \geq 0 \\
c_t \geq 0, \; k_t \geq 0, \; 0 \leq l_t \leq 1, \;\; \forall t \geq 0 \\
k_0 \text{ is given.}
\end{aligned}
\tag{6}
$$

Bellman equation,

$$V(k) = \max_{\substack{0 \leq l \leq 1 \\ 0 \leq k' \leq z k^{\alpha} l^{1-\alpha} + (1-\delta)k}} \left\{ \frac{(z k^{\alpha} l^{1-\alpha} + (1-\delta)k - k')^{1-\sigma}}{1-\sigma} - \chi \frac{l^{1+\eta}}{1+\eta} + \beta \mathbb{E} V(k') \right\} \tag{7}$$

## VFI

Julia code: click here.

```julia
# Julia code
# See A2.jl
```

| Section | ncalls | time | %tot | avg | alloc | %tot | avg |
|---|---|---|---|---|---|---|---|
| Plain VFI n_k=100 | 1 | 287s | 72.0% | 287s | 84.5GiB | 70.9% | 84.5GiB |
| Plain VFI n_k=50 | 1 | 72.6s | 18.2% | 72.6s | 21.1GiB | 17.7% | 21.1GiB |
| Plain VFI n_k=20 | 1 | 13.1s | 3.30% | 13.1s | 3.38GiB | 2.84% | 3.38GiB |
| VFI-HPI n_k=500 | 1 | 11.9s | 3.00% | 11.9s | 5.89GiB | 4.94% | 5.89GiB |
| VFI-MPB n_k=500 | 1 | 8.47s | 2.13% | 8.47s | 2.71GiB | 2.27% | 2.71GiB |
| VFI-HPI n_k=200 | 1 | 2.57s | 0.65% | 2.57s | 839MiB | 0.69% | 839MiB |
| VFI-MPB n_k=200 | 1 | 2.05s | 0.52% | 2.05s | 653MiB | 0.54% | 653MiB |
| VFI-HPI n_k=50 | 1 | 286ms | 0.07% | 286ms | 67.5MiB | 0.06% | 67.5MiB |
| VFI-MPB n_k=50 | 1 | 141ms | 0.04% | 141ms | 34.0MiB | 0.03% | 34.0MiB |
| VFI-HPI n_k=20 | 1 | 133ms | 0.03% | 133ms | 21.0MiB | 0.02% | 21.0MiB |
| VFI-MPB n_k=20 | 1 | 62.8ms | 0.02% | 62.8ms | 10.9MiB | 0.01% | 10.9MiB |

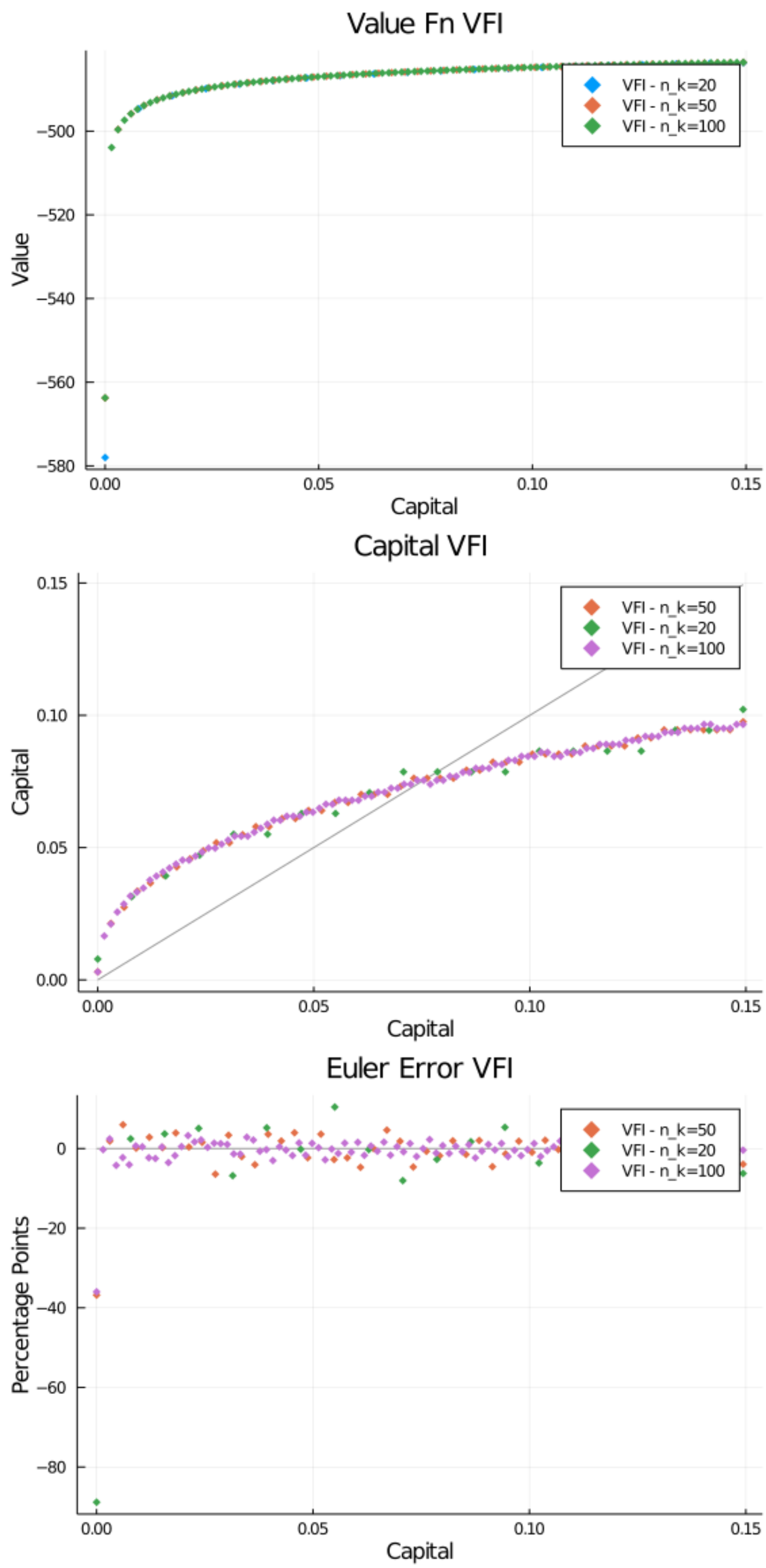


Figure 1: Plain VFI

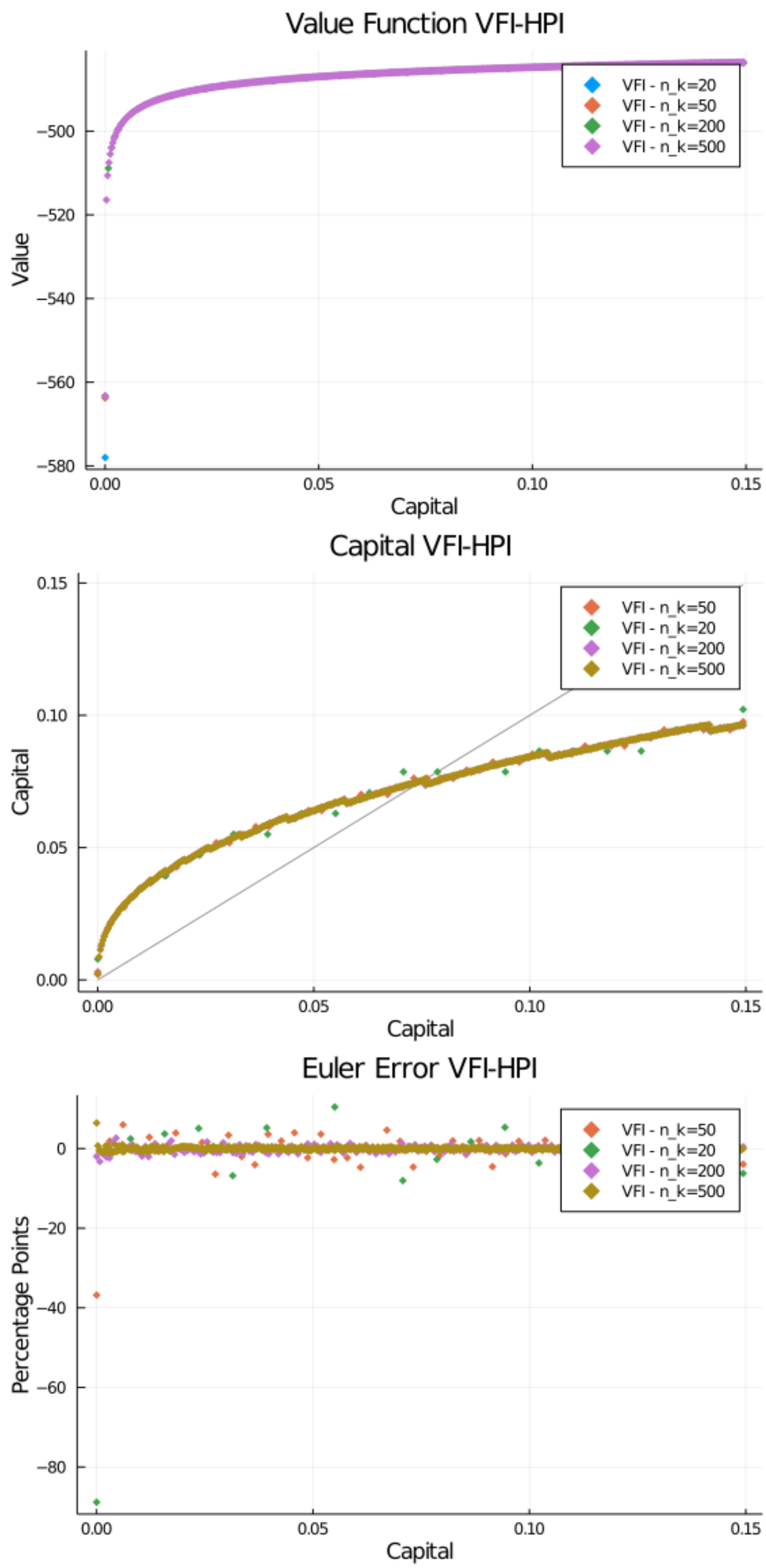


Figure 2: HPI-VFI

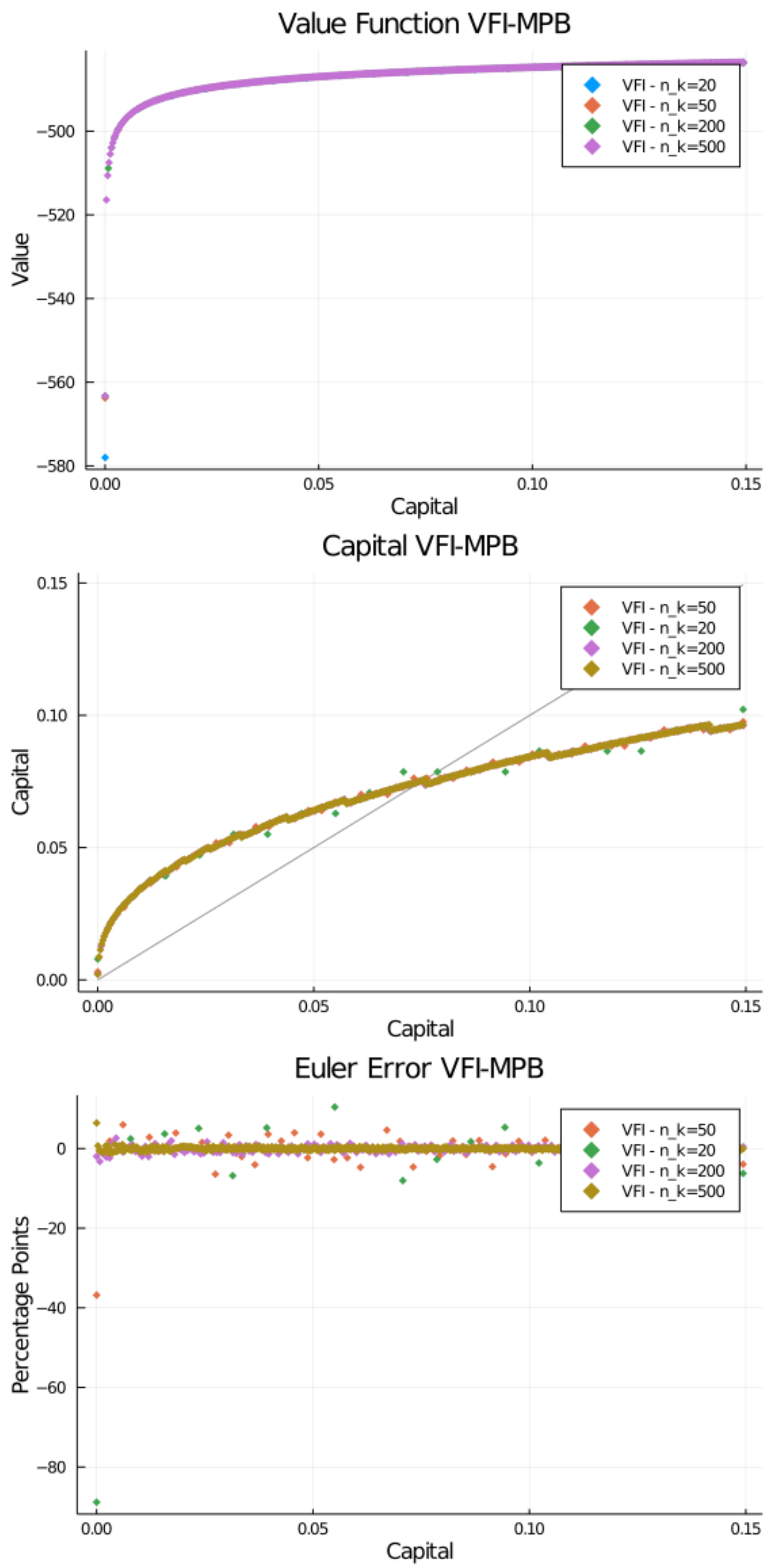


Figure 3: MPB-VFI