
Multidimensional Persistence Module Classification via Lattice-Theoretic Convolutions

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Abstract

1 Multiparameter persistent homology has been largely neglected as an input to
2 machine learning algorithms. We consider the use of lattice-based convolutional
3 neural network layers as a tool for the analysis of features arising from multipa-
4 rameter persistence modules. We find that these show promise as an alternative to
5 convolutions for the classification of multidimensional persistence modules.

6 1 Introduction

7 Persistent homology has the ability to discern both the global topology [8] and local geometry [5]
8 of finite metric spaces (e.g. embedded weighted graphs, point clouds in \mathbb{R}^d) making it a befitting
9 feature for the purposes of training a neural network. Single-dimensional homological persistence has
10 drawn recent attention in deep learning [10, 17, 3]. This is, in part, due to a wide range of efficient
11 software libraries [15, 9, 2] for computing persistent homology, as well as a growing cookbook of
12 recipes for featurizing barcodes from single dimensional persistence, including persistence images [1],
13 persistence landscapes [4], and more exotic methods [11].

14 Multidimensional persistence generalizes single-dimensional persistent homology in order to tackle
15 filtrations parameterized in multiple dimensions. Unfortunately, there is no complete compact
16 barcode-like characterization of multidimensional persistence modules [7]. We must make do with
17 incomplete invariants. There are various algebraic invariants; in this paper we will use the *Hilbert*
18 *function* and the *multi-graded Betti numbers*, both of which are \mathbb{N} -valued functions on the parameter
19 space. The Hilbert function is nothing more than the pointwise (topological) Betti numbers¹, and the
20 multi-graded Betti numbers have an interpretation in terms of births and deaths [12].

21 Multidimensional persistence suffers two hindrances to its usefulness in machine learning. First,
22 software for computing multidimensional persistence is scarce. To the authors' knowledge, RIVET is
23 the only available software for computing multidimensional persistence [13]; RIVET specializes to 2-
24 dimensional persistence and is focused on interactive visualization rather than machine-interpretable
25 output. Second, there has been no activity, as far as the authors are aware, studying the extraction of
26 suitable features for multidimensional persistent homology.

27 We hope to ignite an interest in filling both of these gaps. In this paper, we propose a naive
28 featurization of multidimensional persistence modules based on the aforementioned invariants,
29 and design an architecture for classifying these persistence modules. This architecture employs a
30 lattice-theoretic notion of convolution, thereby respecting the order relation of the parameters of
31 the persistence module. We implement our model and compare the performance of our proposed
32 lattice-convolutional architecture with a (simplified) standard convolutional architecture.

¹*Caveat lector*: the multi-graded Betti numbers are not the same as the topological Betti numbers $\beta_i(\mathbb{X}) = \dim(H_i(\mathbb{X}))$.

2 Background

Due to space constraints, we can offer only a brief overview of multiparameter persistent homology. For a primer on persistent homology, see [8, 6]; for multiparameter persistent homology, see [13]. An introduction to lattices may be found in [1].

2.1 Rips complexes and persistent homology

Let (\mathcal{M}, d) be a finite metric space. The *Vietoris-Rips complex* of \mathcal{M} at scale r is the abstract simplicial complex $\text{Rips}_r(\mathcal{M})$ whose simplices are subsets of \mathcal{M} of diameter at most r . There is a natural inclusion $\text{Rips}_r(\mathcal{M}) \rightarrow \text{Rips}_{r'}(\mathcal{M})$ for $r \leq r'$.

Applying the simplicial homology functor (with coefficients in a field k) H_i to $\text{Rips}_r(\mathcal{M})$ produces a sequence of vector spaces $PH_i(r)$. The inclusions $\text{Rips}_r(\mathcal{M}) \rightarrow \text{Rips}_{r'}(\mathcal{M})$ induce maps $PH_i(r) \rightarrow PH_i(r')$, producing the data of *persistence module*. This structure can be compactly described as a functor from \mathbb{R} , viewed as a category via its standard order structure, to the category \mathbf{Vect}_k of vector spaces over k . The simplicity of the category \mathbb{R} gives these persistence modules simple structure: they decompose as direct sums of interval modules $I_{[a,b]}$, which have $I_{[a,b]}(r) = k$ for $a \leq r < b$ and zero otherwise. The maps are the identity where possible and the zero map otherwise.

This representation theoretic fact gives a representation² of \mathbb{R} -indexed persistence modules via *barcodes* or *persistence diagrams*. Each bar $I_{[a,b]}$ in the barcode represents a homology class which is *born* at a and *dies* at b .

2.2 Multiparameter persistence

The Rips construction produces a filtration of simplicial complexes from a finite metric space; it is natural to consider the behavior of the homology functor over a pair of coherent filtrations. Consider a finite metric space (\mathcal{M}, d) and a filtration function $\rho : \mathcal{M} \rightarrow \mathbb{R}$. This data specifies a bifiltration of simplicial complexes given by

$$\mathbb{X}_{r,t} = \text{Rips}_r x \in \mathcal{M} \mid \rho(x) \leq t.$$

There is a natural inclusion $\mathbb{X}_{r,t} \hookrightarrow \mathbb{X}_{r',t'}$ whenever $(r,t) \leq (r',t')$ in the lattice $\mathbb{R} \times \mathbb{R}$. Composing with the homology functor produces a 2-parameter persistence module

$$PH_i : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbf{Vect}; \quad (r,t) \mapsto H_i(X_{r,t}).$$

While there does not exist a discrete set of invariants for M , we can extract meaningful features. Two particularly informative types of features are the *Hilbert function*

$$\text{Hilb} : R_+ \times R \rightarrow \mathbb{Z}_+; \quad \mathbf{x} \mapsto \dim(M_{\mathbf{x}}),$$

and the *multi-graded Betti numbers* $\xi_j : L \rightarrow \mathbb{Z}_+$, for $j = 0, 1, 2$. For $M = PH_i$ as above, the Hilbert function counts the number of connected components ($i = 0$), cycles ($i = 1$), or higher dimensional voids ($i > 1$) of the complex $\mathbb{X}_{r,t}$ at each $(r,t) \in R_+ \times \mathbb{R}$. The multi-graded Betti numbers, on the other hand, capture information about locations of births and deaths of persistence classes.

2.3 Lattice-theoretic signal processing

Classical signal processing proceeds by constructing filters for space- or time-indexed signals using convolutional operators. These implicitly rely on the algebraic properties of the underlying space, in particular the existence of well-behaved translation operators. This fact is exploited in the general framework of algebraic signal processing [18] and extended to more general domains by the theory of graph signal processing [14]. We here describe a similar extension to signals on a finite lattice proposed in [16].

A *lattice* L is a partially ordered set in which every pair of elements x, y has a greatest lower bound (the *meet* $x \wedge y$) and a least upper bound (the *join* $x \vee y$). These operations and their properties

²Pun intended.

75 produce an algebraic characterization of lattices; the order theoretic and algebraic properties of a
 76 lattice can be recovered from each other. The key insight of [16] is that the meet and join operations
 77 on a lattice define two “shift operators” that can be exploited to define convolutional filters for signals
 78 on a lattice.

79 That is, for two signals $f, g : L \rightarrow \mathbb{R}$, where L is a lattice, we define

$$(f *_{\wedge} g)(x) = \sum_{a \in L} f(x \wedge a)g(a) \quad \text{and} \quad (f *_{\vee} g)(x) = \sum_{a \in L} f(x \vee a)g(a).$$

80 A particularly nice class of examples of lattices are given by the sets \mathbb{R}^n viewed as partially ordered
 81 sets, with the ordering $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ whenever $x_i \leq y_i$ for all $1 \leq i \leq n$. These
 82 are, of course, the indexing sets for persistent homology, suggesting that lattice convolutions over
 83 \mathbb{R}^n or its finite sublattices may be useful in analyzing data coming from multiparameter persistence
 84 computations.

85 3 Lattice Convolutional Neural Networks

86 Convolutions over \mathbb{R}^2 (with its abelian group structure) have served as an easily parameterized and
 87 efficient set of linear operations adapted to the structure of images. Their extreme utility in computer
 88 vision problems is owed to the translation equivariance properties of images: humans naturally
 89 recognize an image translated via an additive reparameterization as equivalent to the original.

90 The data of a multidimensional persistence module is also indexed by \mathbb{R}^n or a regular finite subset
 91 thereof, but its natural algebraic structure is not that of an abelian group. Rather, with its partial order
 92 structure, the indexing set is a lattice. In processing signals associated with the persistence module,
 93 it may be useful to take this structure into account rather than imposing the abelian group structure
 94 implied by standard convolutions.

95 To this end, we construct a lattice convolution-based neural network layer suitable for use with
 96 features originating from multidimensional persistence modules. To the authors’ knowledge, such
 97 an architecture has not previously been described, although a special case (where the underlying
 98 lattice is a power set) has been implemented in [19]. We specialize the convolutions described in
 99 Section 2.3 to the particular case of regular finite sublattices of \mathbb{R}^2 . These may be represented (up
 100 to isomorphism) as $L = [m] \times [n]$, where $[n]$ is the ordered set $\{0, 1, \dots, n\}$. The meet and join
 101 operations are easily computed elementwise:

$$(r, t) \wedge (r', t') = (\min(r, r'), \min(t, t')); \quad (r, t) \vee (r', t') = (\max(r, r'), \max(t, t')).$$

102 A lattice convolution layer takes as input an N_{in} -dimensional signal $f : [m] \times [n] \rightarrow \mathbb{R}^{N_{\text{in}}}$ and
 103 outputs an N_{out} -dimensional signal $[m] \times [n] \rightarrow \mathbb{R}^{N_{\text{out}}}$. The layer’s parameters are given by a function
 104 $g : [m] \times [n] \rightarrow \mathbb{R}^{N_{\text{out}} \times N_{\text{in}}}$. If we label the entries of $f(x, y)$ by f_i and the entries of $g(x, y)$ by g_j^i
 105 The layer then acts by

$$\text{MeetConv}(f)(x, y)^j = \sum_i (f_i *_{\wedge} g_j^i)(x, y) = \sum_i \sum_{(a, b) \in [m] \times [n]} f_i(x \wedge a, y \wedge b) g_j^i(a, b)$$

106 in the case of convolution with respect to the meet operation, and

$$\text{JoinConv}(f)(x, y)^j = \sum_i (f_i *_{\vee} g_j^i)(x, y) = \sum_i \sum_{(a, b) \in [m] \times [n]} f_i(x \vee a, y \vee b) g_j^i(a, b)$$

107 in the case of convolution with respect to the join operation.

108 Convolutional neural networks are useful in part because the convolution kernels (here the functions
 109 g) can have very small support, reducing the number of parameters that must be learned. In the
 110 standard convolutional setting, these kernels are implicitly supported in a neighborhood of the origin,
 111 but the location of the kernel is not usually explicitly specified. In the lattice setting, we do need to
 112 specify where the kernel resides. In the abelian group case, the kernel is supported near the identity,
 113 and similarly, when we treat our domain as a lattice, the kernel should be supported near the neutral
 114 element of the operation. That is, for a meet convolution, g should have support near the maximum
 115 (m, n) , and for a join convolution, g should have support near the minimum $(0, 0)$.

Figure 1: Comparison training curves for the lattice neural network and standard convolutional neural network.

4 Experiments

We use a small portion of the Princeton ModelNet dataset [20] as a source of finite metric spaces. This dataset consists of hundreds of 3-dimensional CAD models representing objects from 40 classes. We select two of the classes and sample points from the 3D models to produce finite metric spaces embedded in \mathbb{R}^3 . We then compute the corresponding multidimensional persistence modules, from which we produce features used as an input to a convolutional neural net classifier.

The pipeline thus begins with a 3D model, which is sampled to produce a point cloud in \mathbb{R}^3 . This point cloud then produces a bifiltered simplicial complex, whose persistent homology we calculate using RIVET [13], sampled at a discrete grid of 40×40 points, producing lattice-indexed signals given by the Hilbert function and the multi-graded Betti numbers. These are then passed to the classifier, which produces a class prediction, in this case either “sofa” or “monitor.”

As the filter function on these data sets, we use

$$\rho_{\text{codense}}(x; k) = \frac{k \operatorname{diam}(\mathcal{M})}{\sum_{y \in N_k(x)} d(x, y)},$$

where $N_k(x)$ is the set of the k nearest neighbors to x . This is the *codensity* filtration, so named because the points in the densest regions of \mathcal{M} will appear first.

We compare the performance of two convolutional networks on this classification task. One uses the lattice-convolution based layers described in Section 3, and the other uses standard convolutional layers. Each has three convolutional layers followed by three fully connected layers. The lattice-based convolution layers are of the form $\text{MeetConv}(x) + \text{JoinConv}(x)$. All convolution kernels have dimension 4×4 , hidden convolution layers have 16 features, and the final convolution has 8 features. The fully connected layers have 32 features at each inner layer. The networks are trained using the Adam gradient algorithm with learning rate 0.001 for 50 epochs. We reserve 10% of the data for testing. Results are shown in Figure 1.

The lattice-based convolution and classical convolution networks perform similarly on the training set. However, the lattice convolution network’s predictions generalize more accurately to the testing set.

5 Discussion

Our proposed featurization for persistence modules is rather naive, but well adapted to the use of lattice convolutions as a data processing method. The lattice convolutional neural network shows promise as a method for classifying features arising from a multiparameter persistence module. The algebraic perspective on partially ordered sets, exemplified by lattices, may also offer approaches to featurizing more complex invariants of persistence modules. In particular, the incidence algebra may offer a natural way to represent the rank invariant [7] in a way amenable to convolution-like operations.

We hope with these brief experiments to inspire further work on featurizing multidimensional persistence for use in machine learning algorithms.

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