Multi-dimensional Persistent (Finite) Metric Space Classification via Lattice-theoretic Convolutional Filtering

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Abstract

To write...

1 Introduction

Persistent homology has the ability to discern both the global topology [] and local geometry [] of finite metric spaces (e.g. embedded weighted graphs, point clouds in $\mathbb{R}^d$) making it a befitting feature for the purposes of training a neural network. Single-dimensional homological persistence has drawn recent attention in deep learning []. This is, in part, due to a wide range of efficient software libraries [**?**, **?**, **?**] for computing *barcodes* (in the “west-cost" lingo) of filtrations (e.g. Rips, alpha, Čech, sub-levelset). Barcodes provide a compact shape descriptor for metric space data [] that is stable with respect to Gromov-Hausdorff distance [].

In a seminal paper [], Carlsson and Zomorodian show there is no such compact description for multi-parameter persistence.

2 Background

Due to the nature of this paper, limited background will be given. For a primer on persistent homology, see [**?**, **?**]. For an introduction to multi-parameter persistent homology, see [**?**]. For more details on lattice theory, see [**?**].

2.1 Rips complexes and persistence homology

2.2 Multi-parameter persistence

2.3 Lattice-theoretic signal processing

3 Model description

3.1 Features

In our pipeline, we input a finite metric space and a filtration function $(\mathcal{M}, d, \rho)$ where $\rho : \mathcal{M} \to \mathbb{R}$ and $d(\cdot, \cdot)$ is a metric on $\mathcal{M}$. This data specifies a bi-filtration of simplicial complexes with

$$\mathbb{X}_{r,t} = \mathrm{Rips}_r\{x \in \mathcal{M} \mid \rho(x) \leq t\}$$

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This is in fact a filtration as there is an inclusion $\mathbb{X}_{r,t} \hookrightarrow \mathbb{X}_{r',t'}$ whenever $(r,t) \leq (r',t')$. Furthermore, the above filtration begets a 2-parameter persistence module,

$$PH_i : \mathbb{R}_+ \times \mathbb{R} \to \mathsf{Vec};\ (r,t) \mapsto H_i(\mathbb{X}_{r,t})$$

If we assume that (i) PH_i stabilizes for $r \geq R$ and $t \geq T$ for some R, T large enough and (ii) the induced map on homology,

$$H_i(\mathbb{X}_{r,t}) \to H_i(\mathbb{X}_{r',t'})$$

is an isomorphism for all but finitely many $(r,t) \leq (r^{'}, t^{'})$, then we can restrict the domain of our persistence module PH_i—possibly after a re-parameterization of the filtration—to a finite (order) lattice, $\mathsf{L} = [m] \times [n]$ where $[n] = \{0, 1, 2, \ldots, n\}$, obtaining a persistence module $M : \mathsf{L} \to \mathsf{Vect}$. (More generally, our model may accept as inputs signals on any (finite) lattice L with features extracted from any given generalized persistence module supported on L.)

From M, we extract the following features: (i) the *Hilbert function*,

$$\mathrm{Hilb} : \mathsf{L} \to \mathbb{Z}_+;\ \mathbf{x} \mapsto \dim(M_{\mathbf{x}})$$

and (ii) the *multi-graded Betti numbers*[1], $\xi_j : \mathsf{L} \to \mathbb{Z}_+$ for $j = 0, 1, 2$. For $M = PH_i$ as above, the Hilbert function counts the number of connected components ($i = 0$), cycles ($i = 1$), or higher dimensional cycles ($i > 1$) of the complex $\mathbb{X}_{r,t}$ at each (critical) $(r,t) \in \mathsf{L}$. Multi-graded Betti numbers, on the other hand, capture births, deaths

In our experiments, we consider a particular ρ,

$$\rho_{\text{codense}}(x;k) = \frac{k d_{\max}}{\sum_{y \in N_k(x)} d(x,y)};\ N_k(x),\ k\text{-nearest neighbors}$$

where $d_{\max} = \max_{x,x' \in \mathcal{M}} d(x,x')$; we call this filtration *codensity* as the densest points of $\mathcal{M}$ will appear first and the sparsest points will appear last. A folk theorem is that the 2-parameter persistent homology of a Rips/codensity bi-filtration is stable under non-Hausdorff noise—*the (Rips) persistent homology of a point sample and another obtained by adding a small number of additional points at random are close with respect to interleaving distance* (Fig.**??**). (Instability to non-Hausdorff noise is a major limitation of the expressibility of single-parameter persistence of a Rips filtration.)

3.2 Architecture

3.3 Training

4 Experiments

4.1 Software

4.2 Dataset

5 Discussion

[1] *Lector emptor*: multi-graded Betti numbers should not be confused with topological Betti numbers, $\beta_i(\mathbb{X}) = \dim(H_i(\mathbb{X}))$