Applied Category Theory for Dynamics and Control - Introduction

7/7/2025

James Fairbanks

Spectrum of Scientific Computing Technology

Programs that have explicit representations of the model

Arbitrary Code

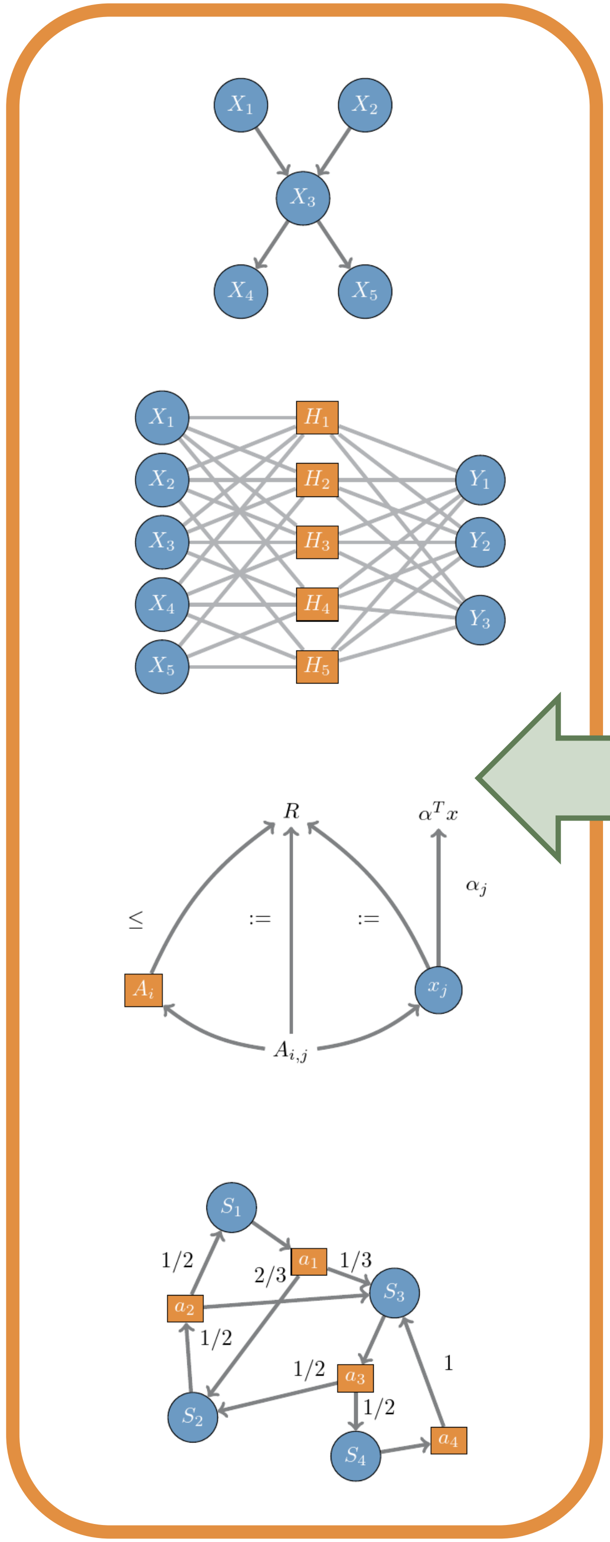
Domain Specific Languages

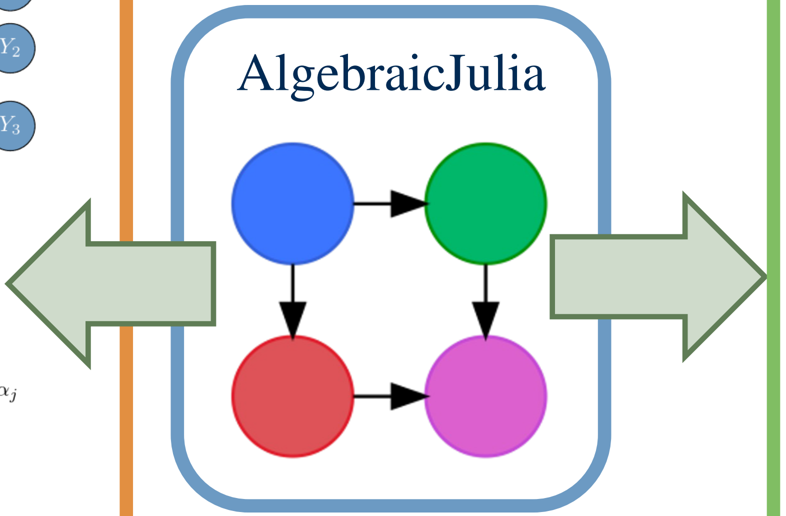
Modeling Frameworks

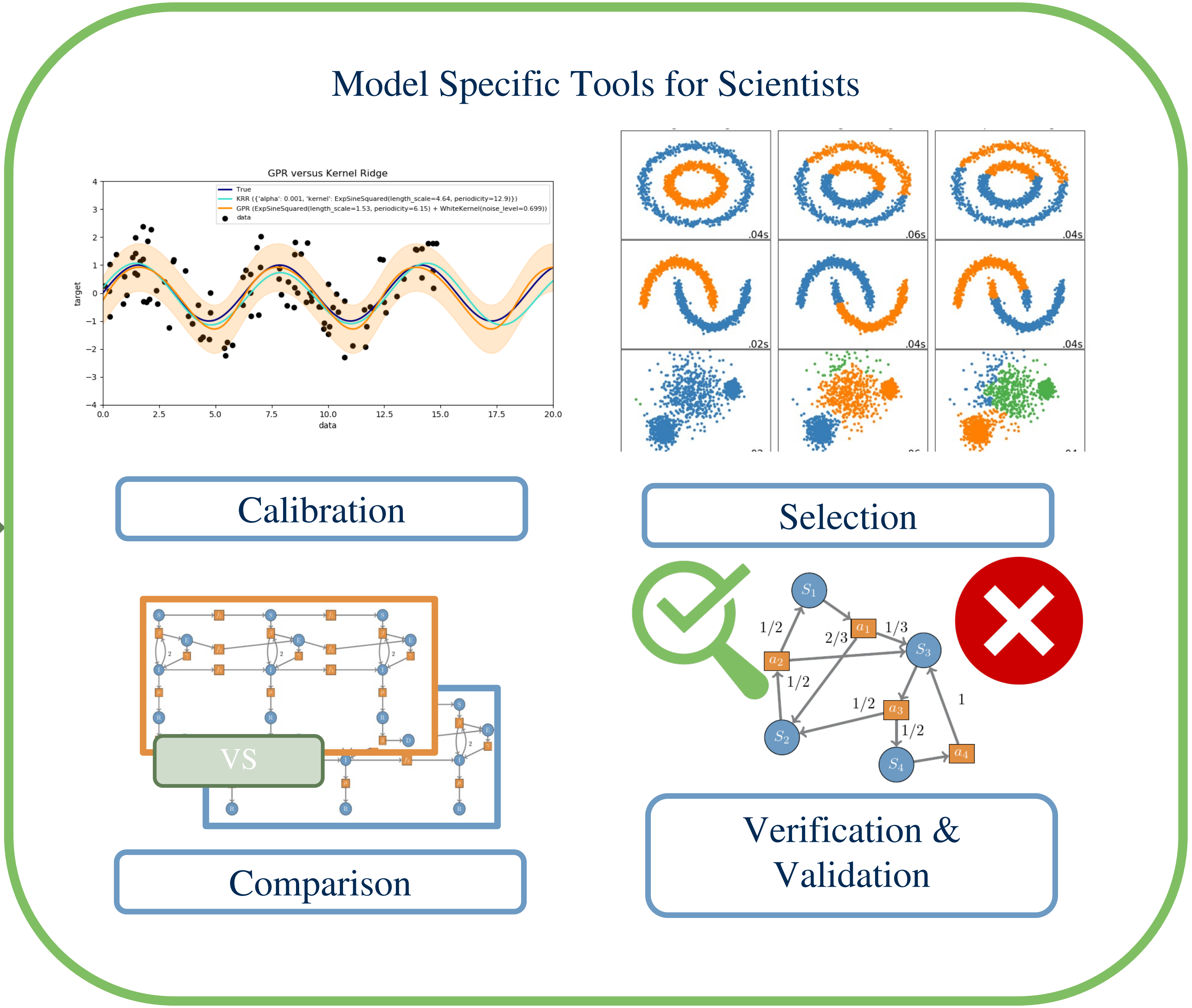
Computer Algebra Systems

SymPy

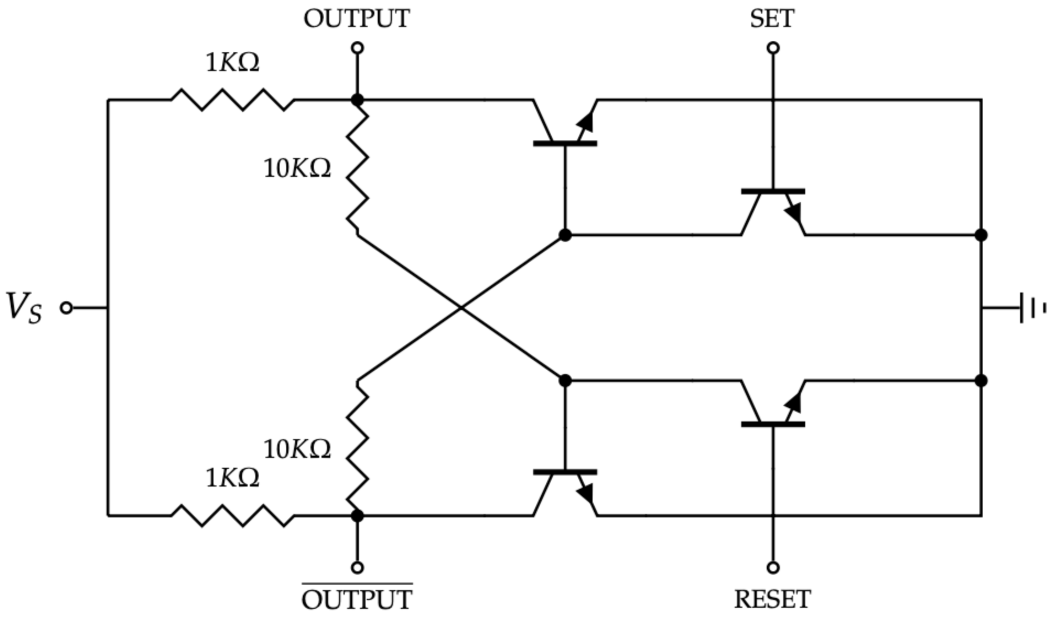
Vision: Model Aware Scientific Computing with Categories

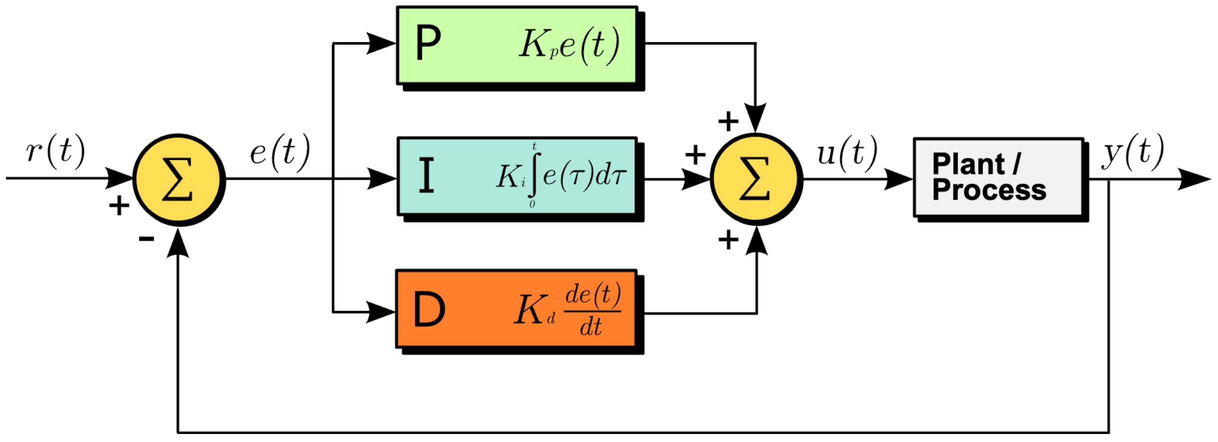


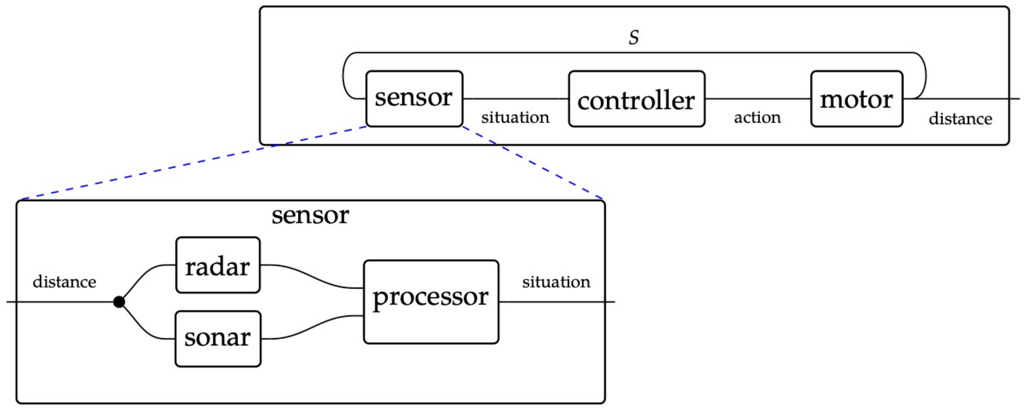


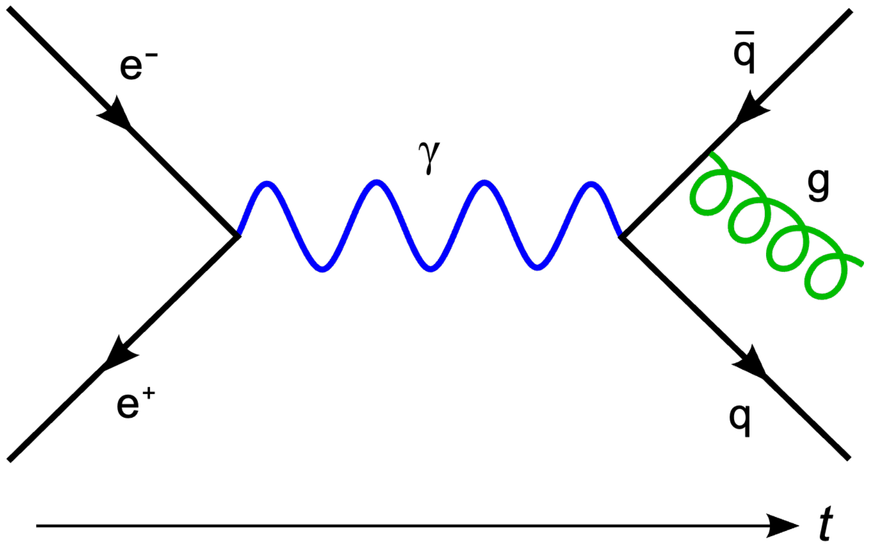


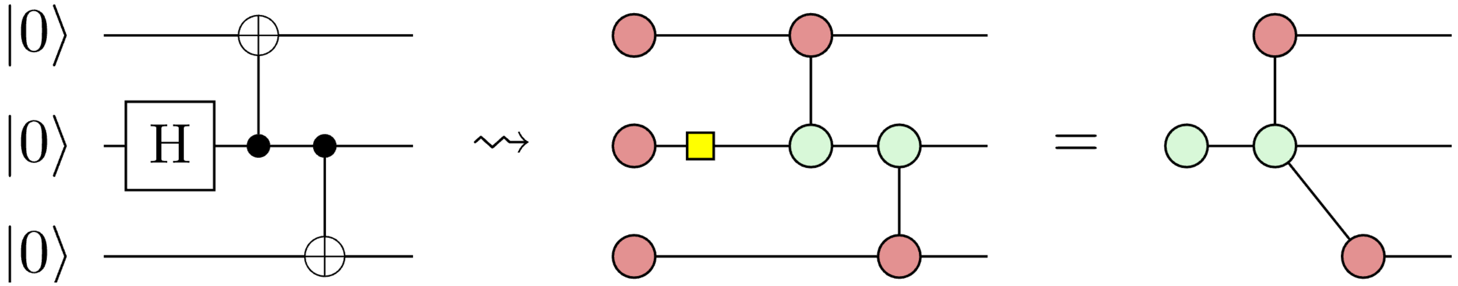
Formal Scientific Diagrams





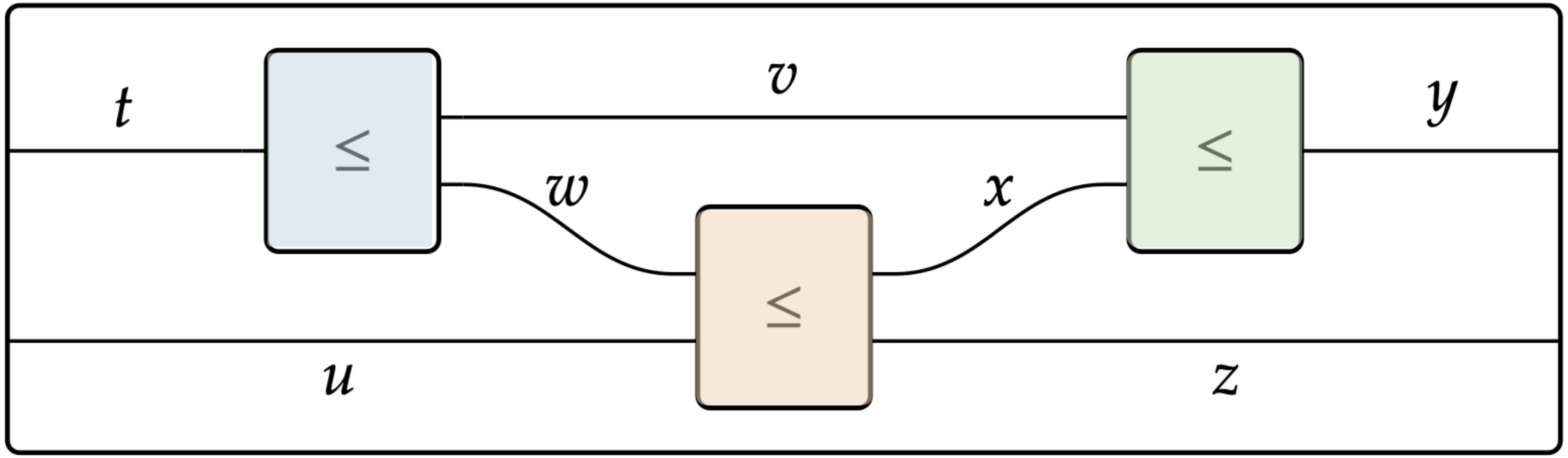






Formal Proofs with Inequalities

- Hypotheses: $t \leq v + w$ $\quad$ $w + u \leq x + z$ $\quad$ $v + x \leq y$
- Proof:

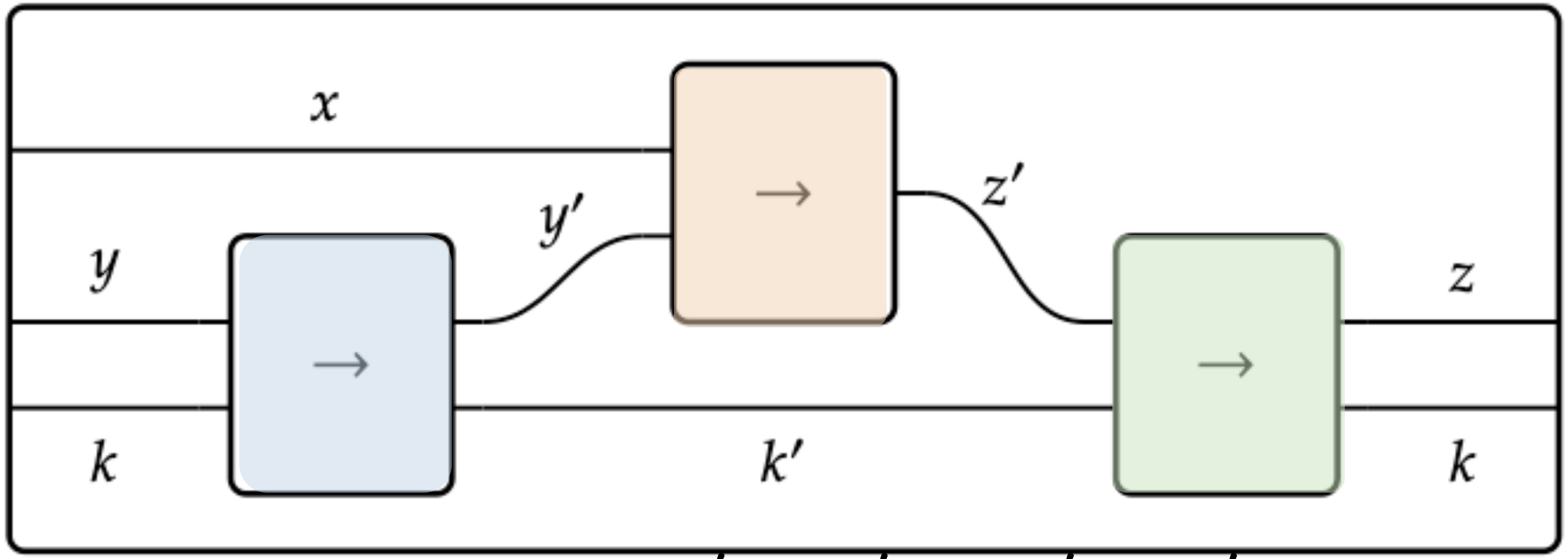


- Conclusion: $\boxed{t + u \leq y + z}$

Automated Reasoning in Chemical Systems

- Primitive Reactions: $y + k \to y' + k'$ $x + y' \to z'$ $z' + k' \to z + k$

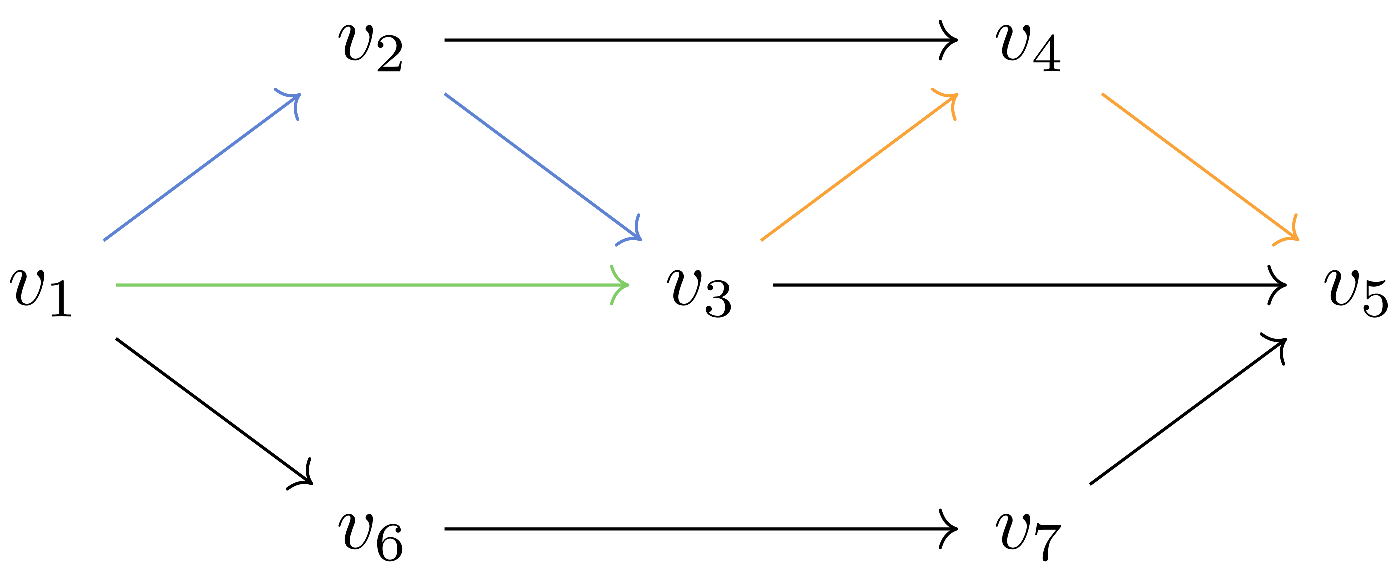
- Chemical Process:



$$x + y + k \to x + y' + k' \to z' + k' \to z + k$$

- Composite Reaction: $\boxed{x + y + k \to z + k}$

Paths in a Graph are Free Categories



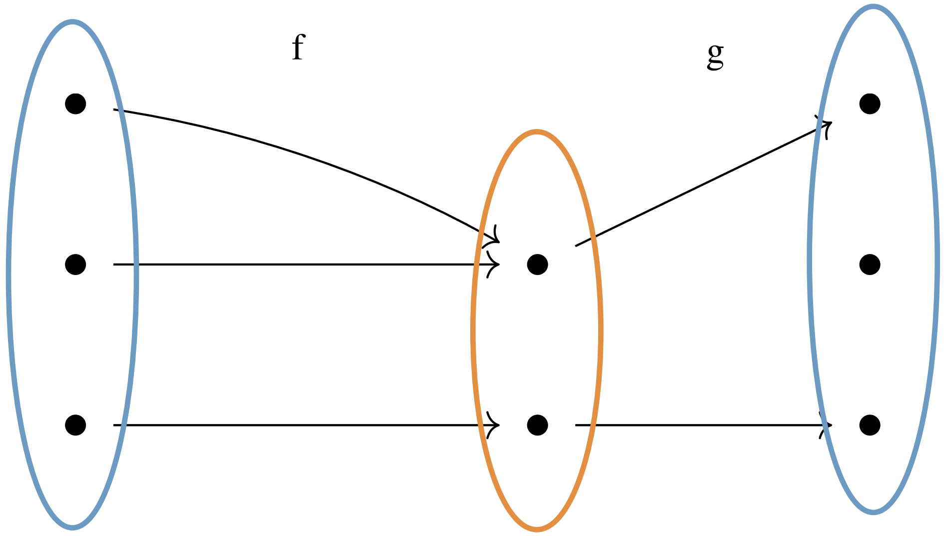
$$v_1 v_2 v_3 \cdot v_3 v_4 v_5 = v_1 v_2 v_3 v_4 v_5$$

$$v_1 v_3 \cdot v_3 v_4 v_5 = v_1 v_3 v_4 v_5$$

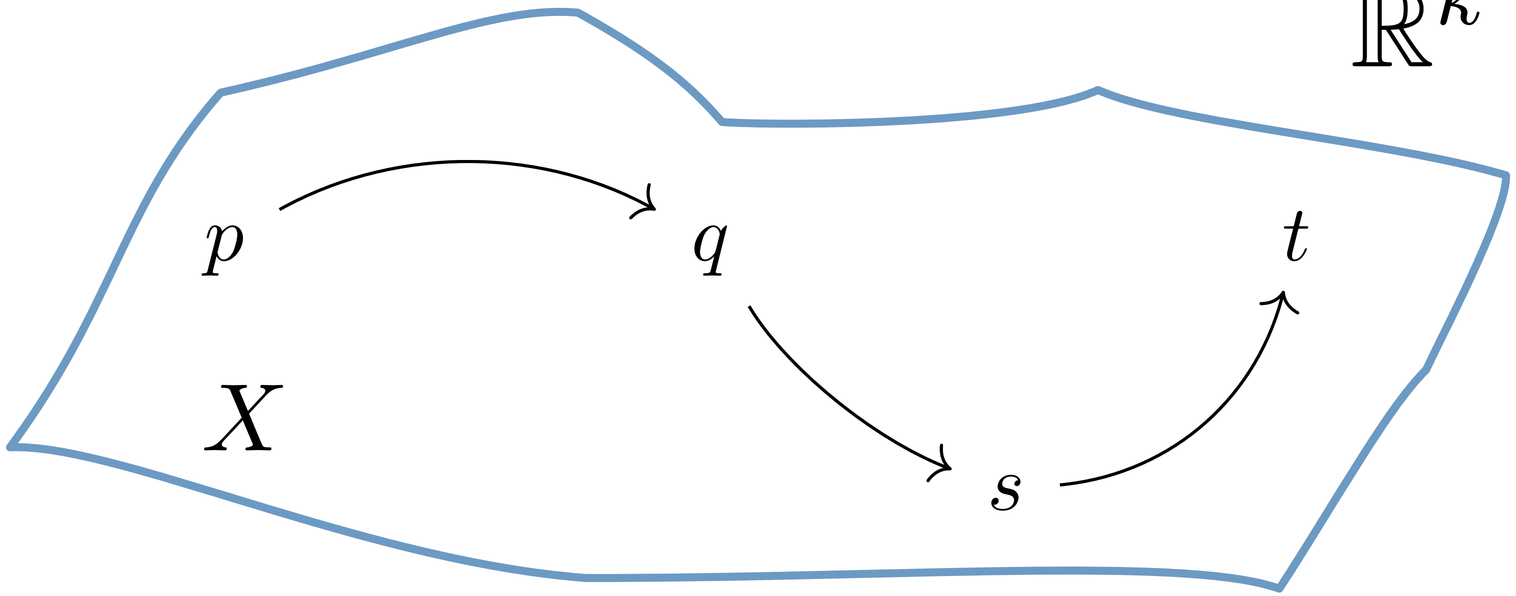
- Every graph presents a free category
- Paths can only compose if they share an endpoint.
- Graph theory studies these categories without the language of CT

Examples

- Sets and Functions
- Finite Sets and Functions
- Vector Spaces and Linear Maps (Matrices)
- Topological Spaces and Continuous Functions
- Convex Spaces and Convex Functions
- Points in a Space and Paths in that Space
- Dynamical Systems and Changes of Coordinates



$$
\begin{tikzcd}
\mathbb{R}^n \arrow[r, "A"] \arrow[dr, "BA"'] & \mathbb{R}^m \arrow[d, "B"] \arrow[dl, phantom, "="] \\
 & \mathbb{R}^k
\end{tikzcd}
$$



Functors

- Every mathematical structure has "structure preserving maps"
- For Categories, these maps are *Functors*
- Map Objects of C to Objects of D and Arrows in C to Arrows in D, such that composition and identities are preserved

$$F_0 : Ob_C \to Ob_D$$

$$F_1 : Hom_C \to Hom_D$$

$$F_1(id(A)) = id(F_0(A))$$

$$F_1(f \circ_C g) = F_1(f) \circ_D F_1(g)$$

Example: The free vector space on a set

$$V : Set \to Vect$$

$$n \mapsto \mathbb{R}^n$$

$$f : n \to m \mapsto A_f : \mathbb{R}^n \to \mathbb{R}^m$$

$$A_f\,(j,i) = \begin{cases} 1 \text{ if } f(i) = j \\ 0 \text{ else} \end{cases}$$

Functorial Semantics in Programming Languages

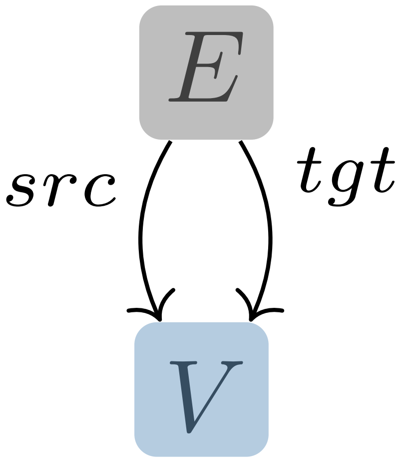
$$\text{Syntax} \xrightarrow{\text{Compile}} \text{Semantics}$$

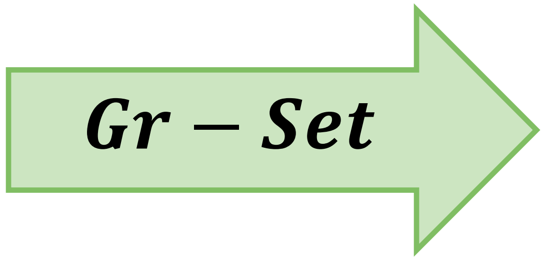
- Syntax is combinatorial trees, graphs, hypergraphs
- Semantics is quantitative, functions, relations, shapes, distributions
- Functor from syntax to semantics recursively generates programs
- Structure of the categories/functor determines allowable language operations

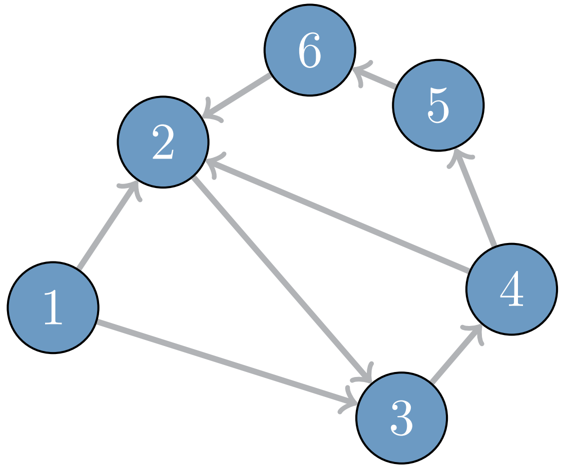
Attributed $\mathcal{C}$-Sets: Categorical Data Structures

Graphs are ubiquitous because they a simple & useful structure

Gr

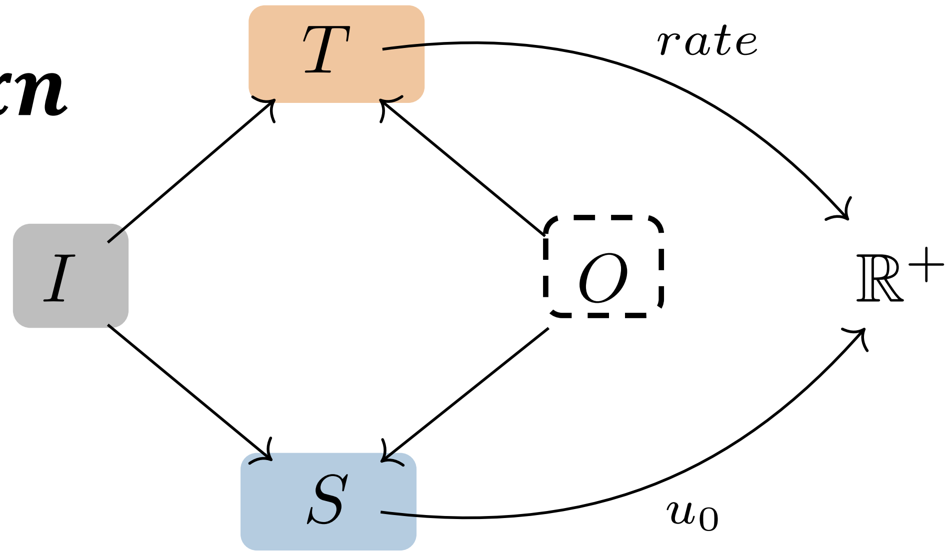


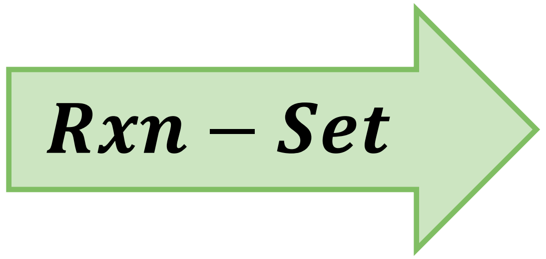


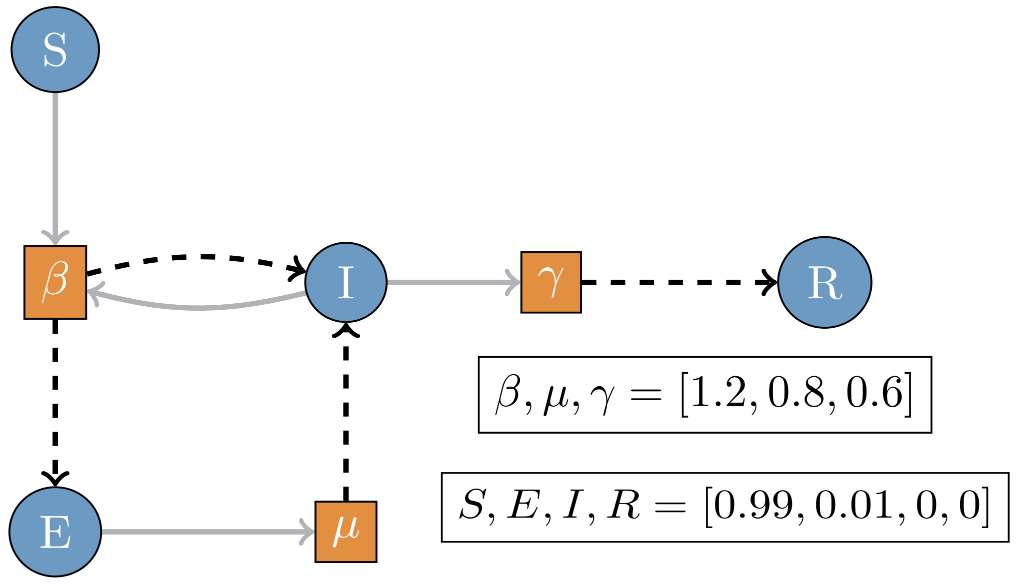


Attributed $\mathcal{C}$-Sets generalize algebraic graph theory

Rxn

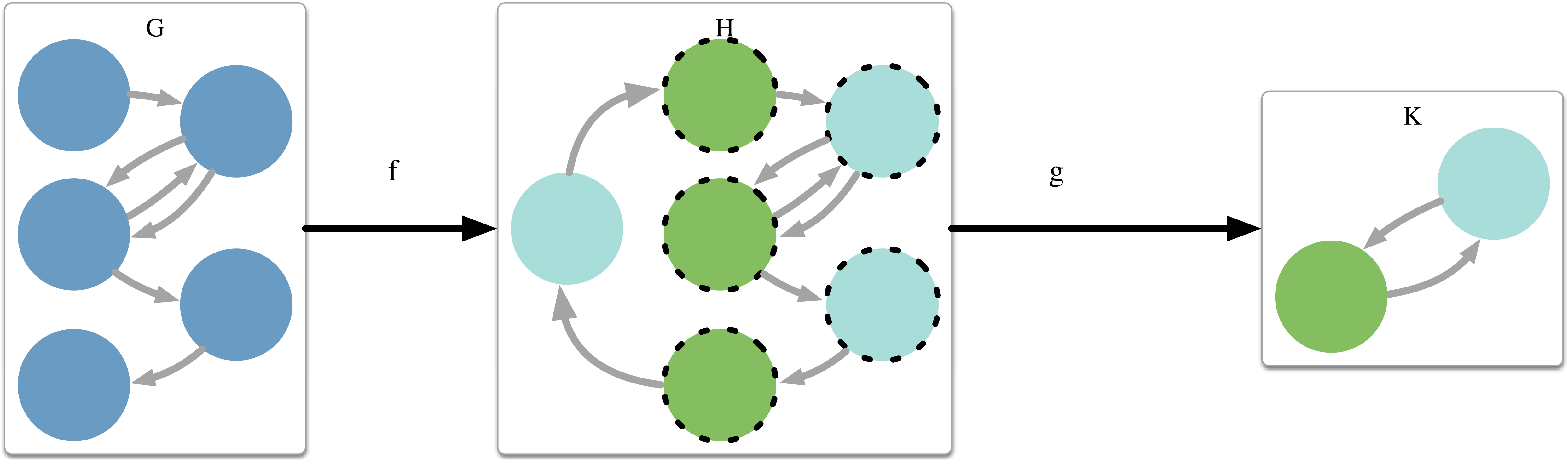




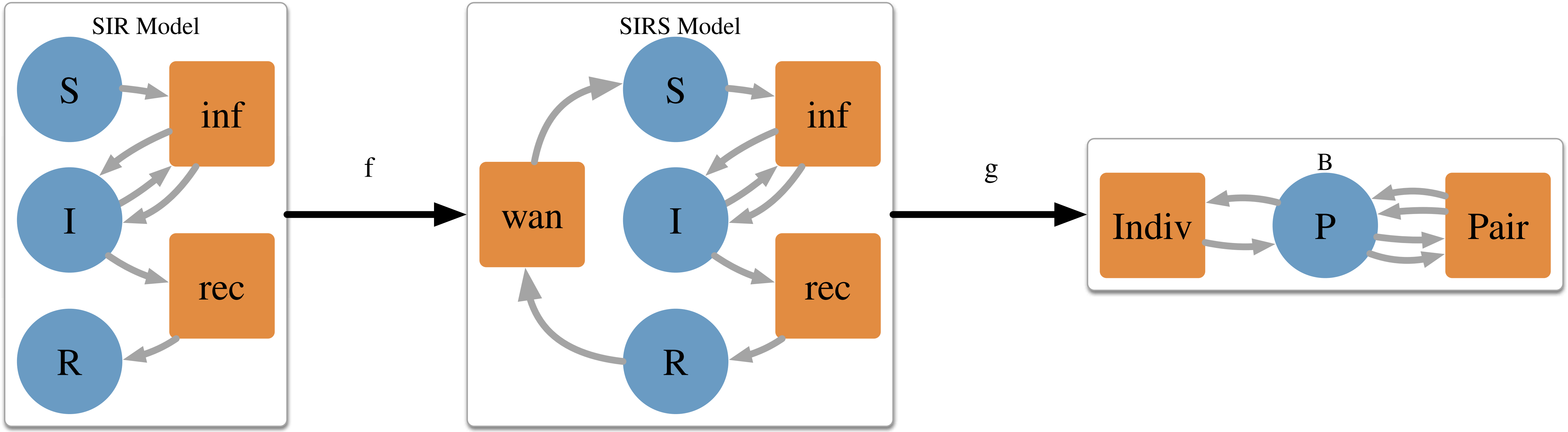


Attributed $\mathcal{C}$-Sets: Categorical Data Structures

Graph Morphisms



Petri Net Morphisms



ACSets generalize Tables and Graphs

Table

$$\begin{tikzcd} & Index \arrow[dl] \arrow[d] \arrow[dr] & \\ C_1 & \cdots & C_n \end{tikzcd}$$

Graph

$$\begin{tikzcd} E \arrow[r, shift left, "src"] \arrow[r, shift right, "tgt"'] & V \end{tikzcd}$$

Weighted, Property Graph

$$\begin{tikzcd} E \arrow[r, shift left, "src"] \arrow[r, shift right, "tgt"'] \arrow[d, "w"'] & V \arrow[d, "prop"] \\ W & P \end{tikzcd}$$

Discrete Dynamical System

$$\begin{tikzcd} State \arrow[loop above, "next"] \end{tikzcd}$$

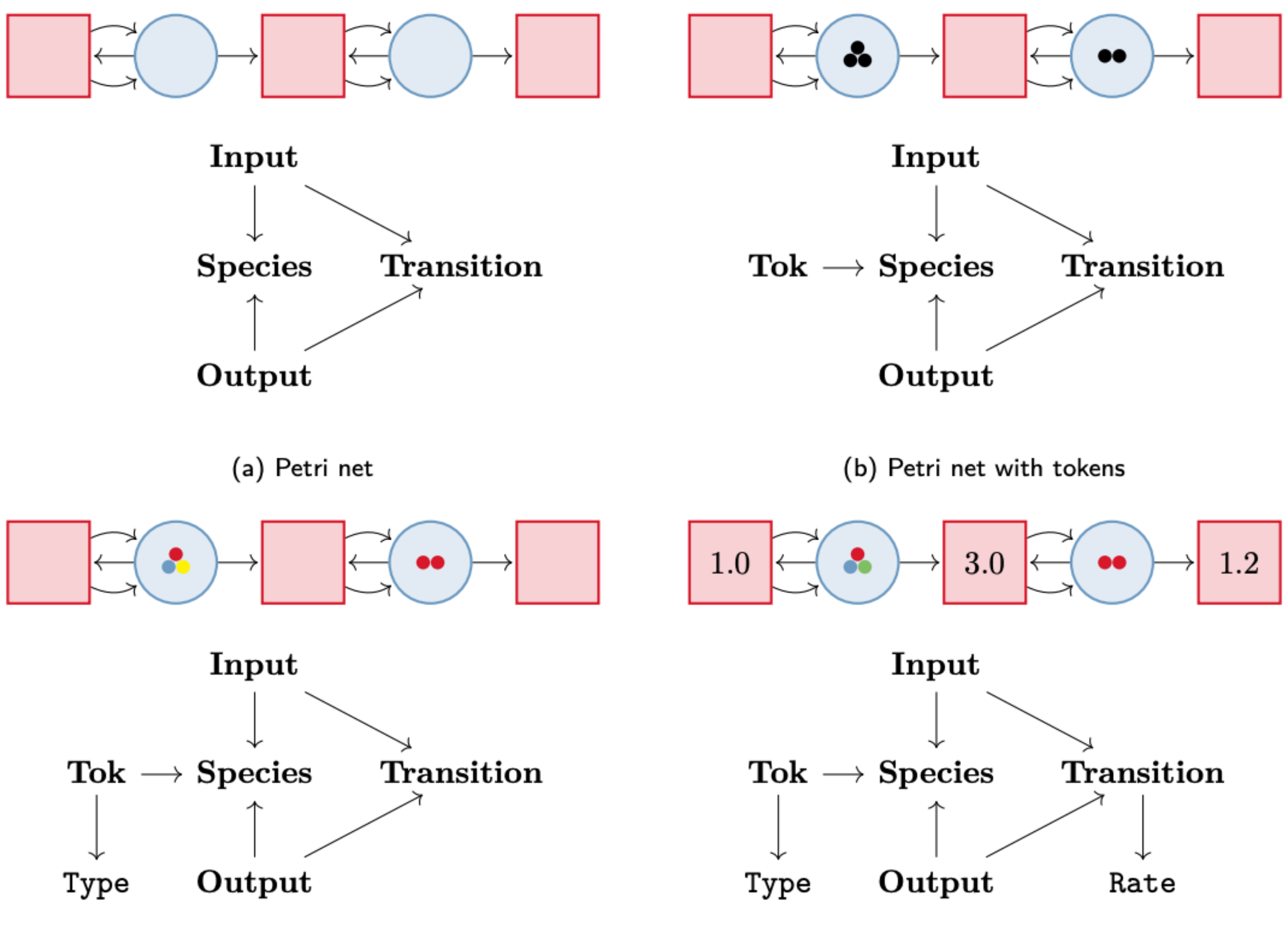
DDS on a Graph

$$\begin{tikzcd} & E \arrow[d, shift left] \arrow[d, shift right] \\ State \arrow[loop above, "next"] & V \arrow[l] \end{tikzcd}$$

Open Circular Port Graphs

$$\begin{tikzcd} Wire \arrow[r, shift left, "src"] \arrow[r, shift right, "tgt"'] & Port \arrow[dl, "box"] \\ Vertex & OPort \arrow[u, "con"'] \end{tikzcd}$$

Building Rich Data Structures as ACSets

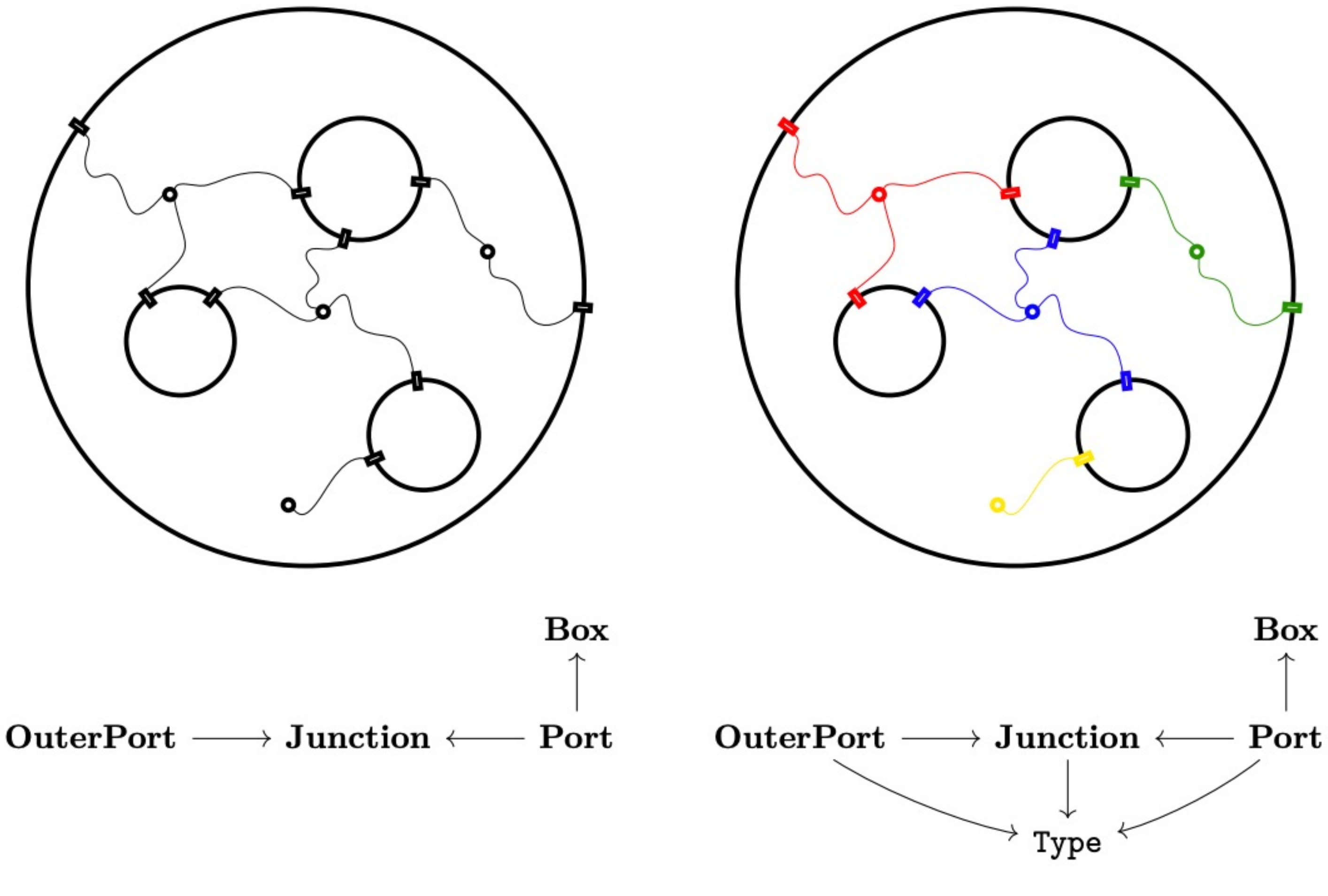


(a) Petri net

(b) Petri net with tokens

(c) Petri net with typed tokens

(d) Petri net with typed tokens and rates



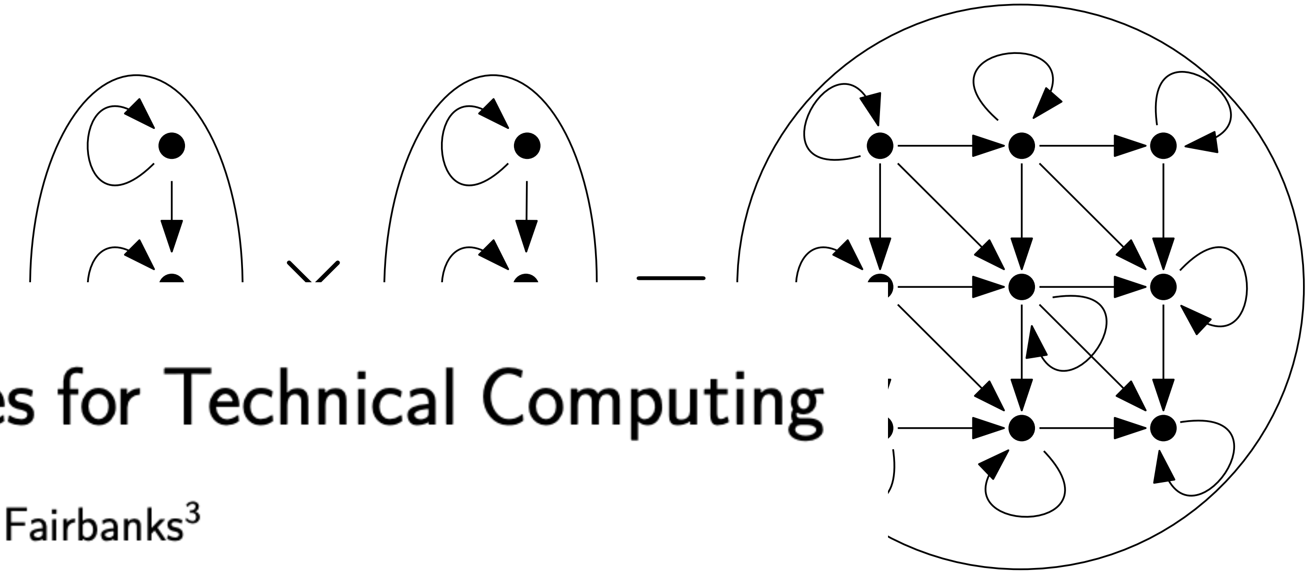
(a) Undirected wiring diagram

(b) Typed undirected wiring diagram

- The data of a mathematical model can usually be stored as an ACSet
- Highly generic code implies low complexity cost for adding features

ACSets have many nice features

- All Finite Limits:
  - An initial (singleton) object
  - Products (products)
  - Pullbacks
- All Finite C
  - A terminal
  - Coproducts (disjoint unions)
  - Pushouts (unions)
- Limits and Colimits have point-wise formulas



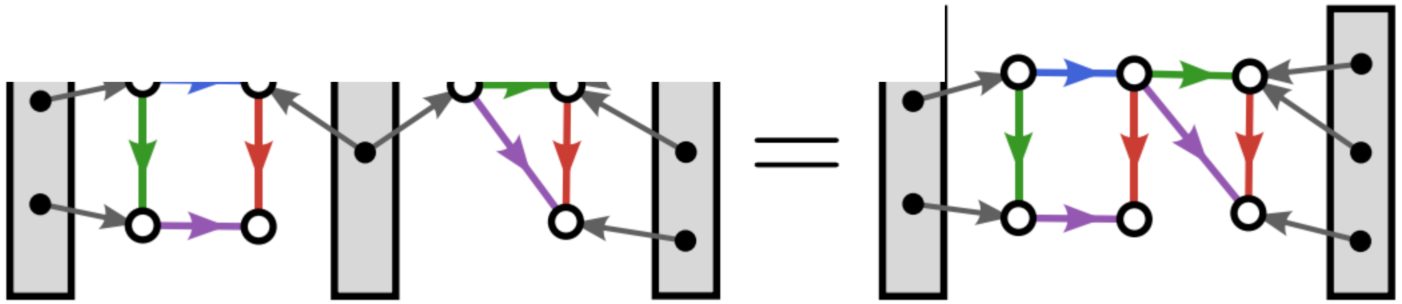
Categorical Data Structures for Technical Computing

Evan Patterson[1], Owen Lynch[2], and James Fairbanks[3]

[1]Topos Institute, California, USA

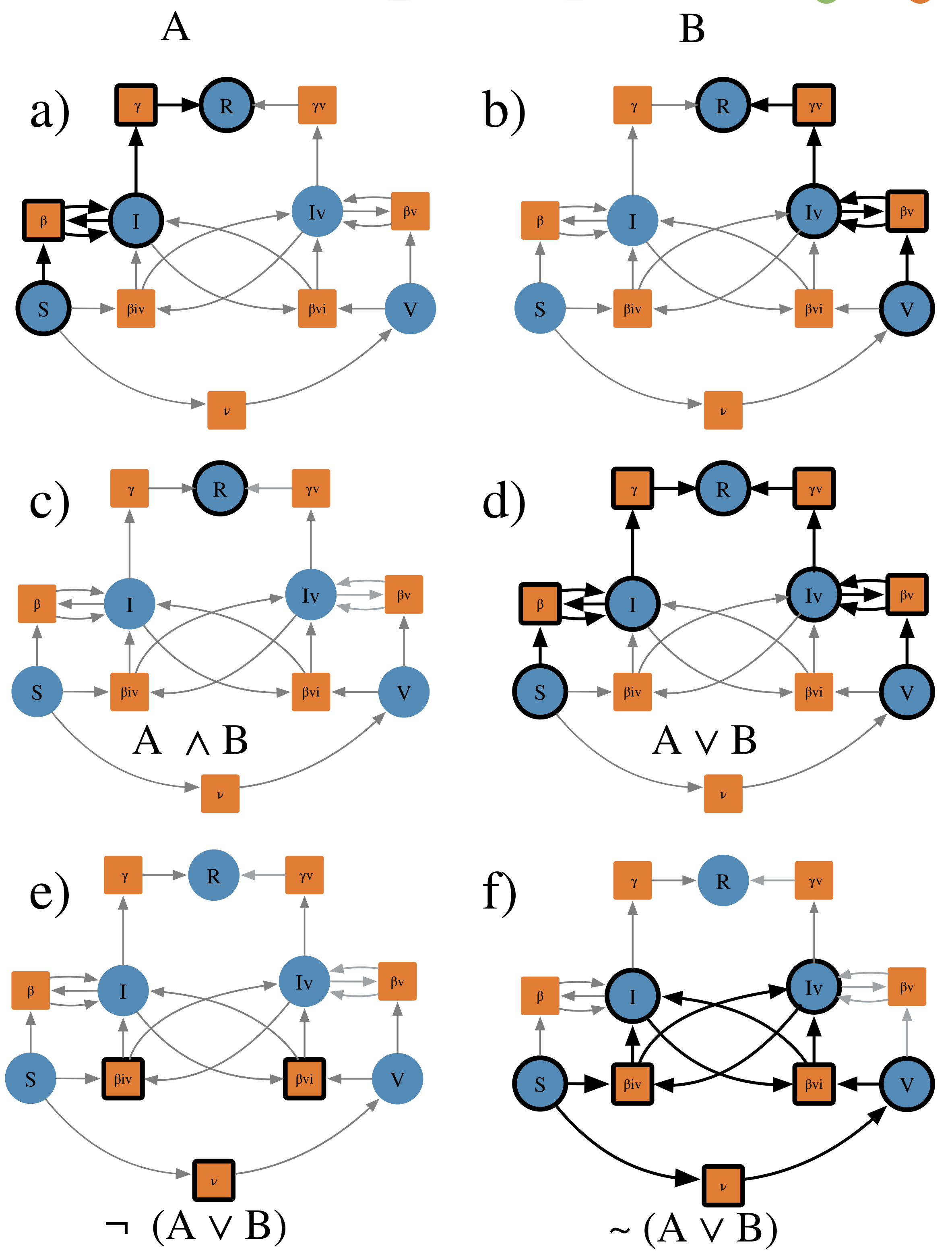
[2]Universiteit Utrecht, Mathematics Department, Utrecht, The Netherlands

[3]University of Florida, Computer & Information Science & Engineering, Florida, USA



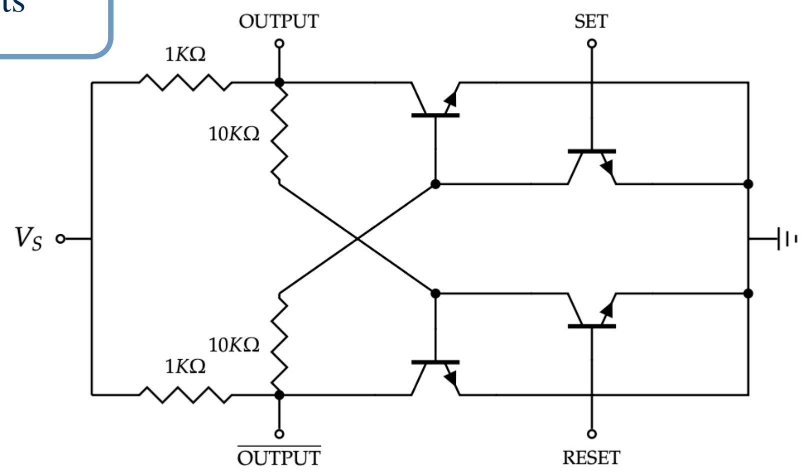
ACSets are Combinatorial, Logical, and Topological

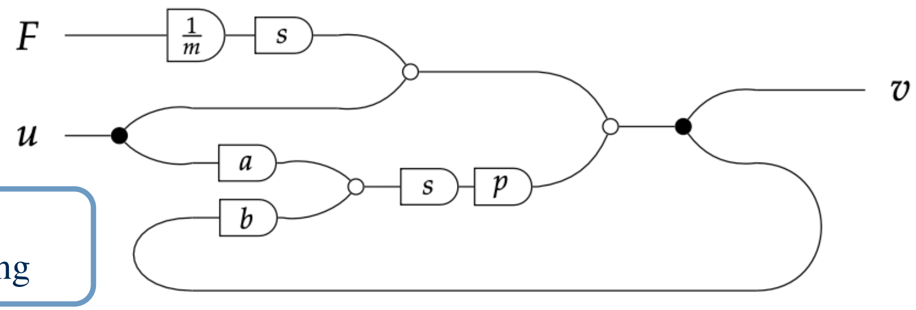
- A C-Set is a bi-Heyting Topos
- For a fixed object, you look at all the subobjects
- They form a lattice with intersection and union
- Logic is intuitionistic because LEM fails.
- Two types of negation
- Combinatorial Boundary Operator



Wiring Diagrams are ACSets

Circuits

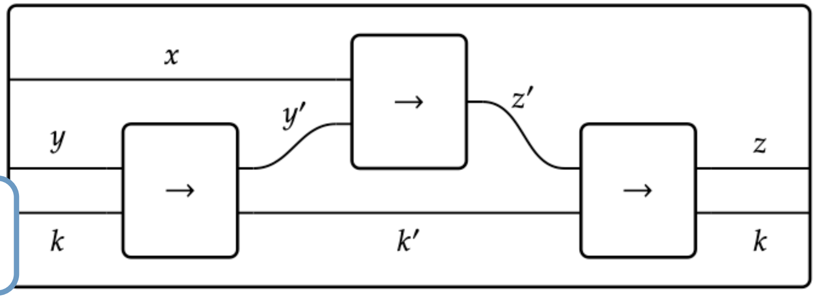




Signal Processing

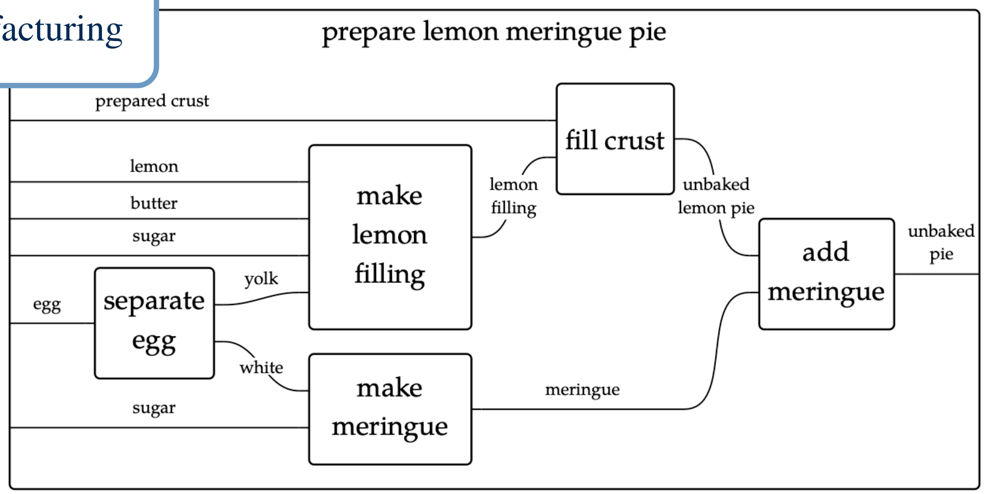
$$v = \int \frac{1}{m} F(t)dt + u(t) + p \int au(t) + bv(t)dt.$$

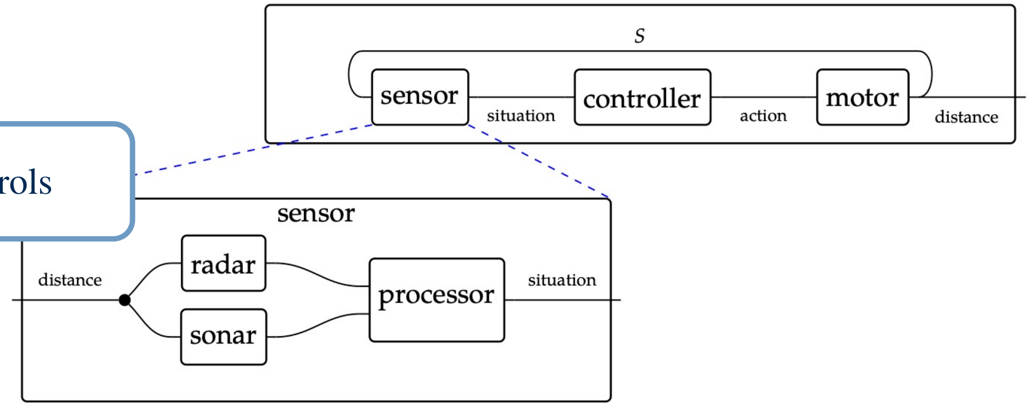
$$y + k \to y' + k' \qquad x + y' \to z' \qquad z' + k' \to z + k$$



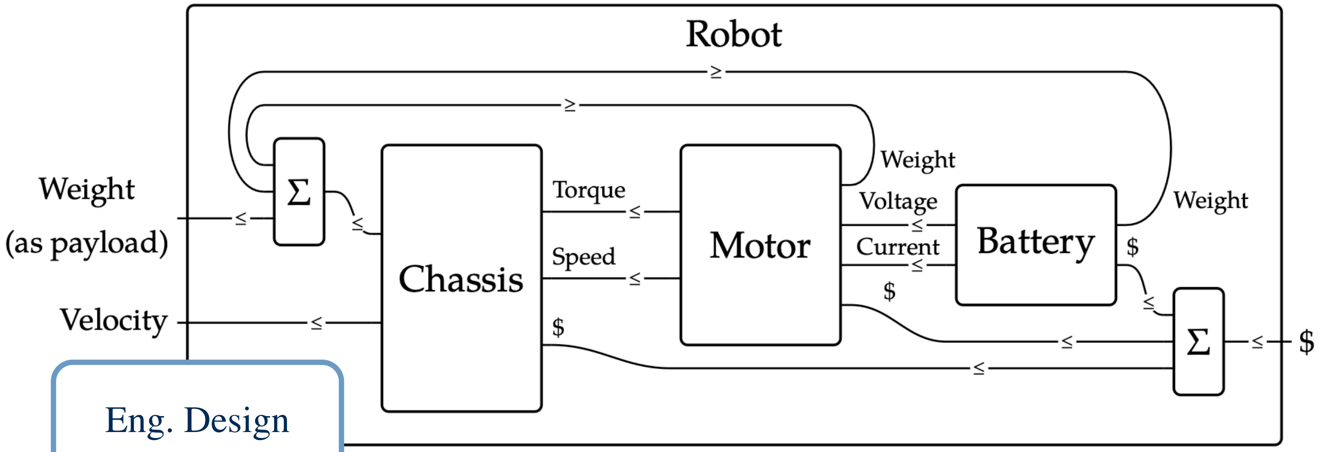
Chemistry

Manufacturing





Controls

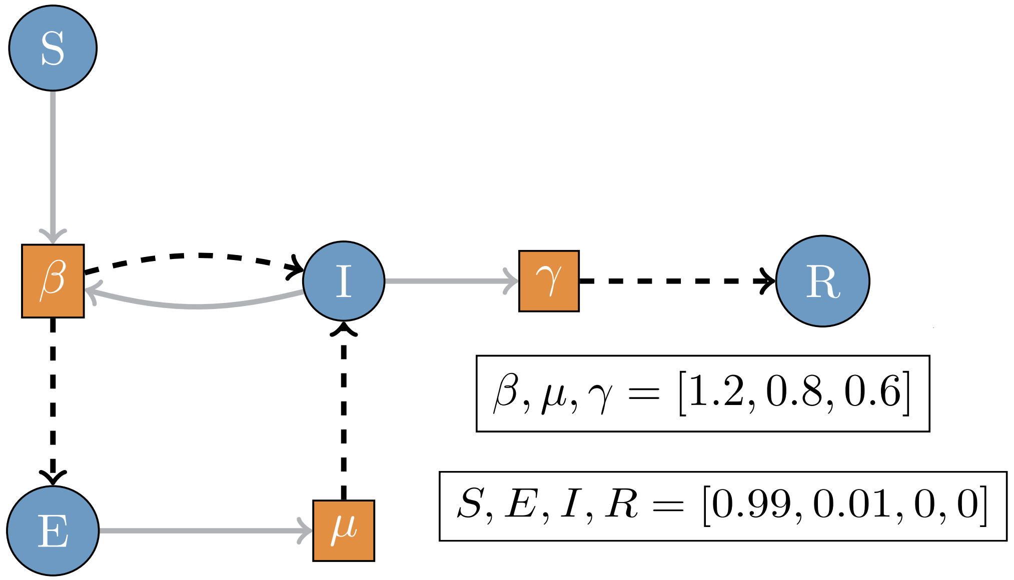


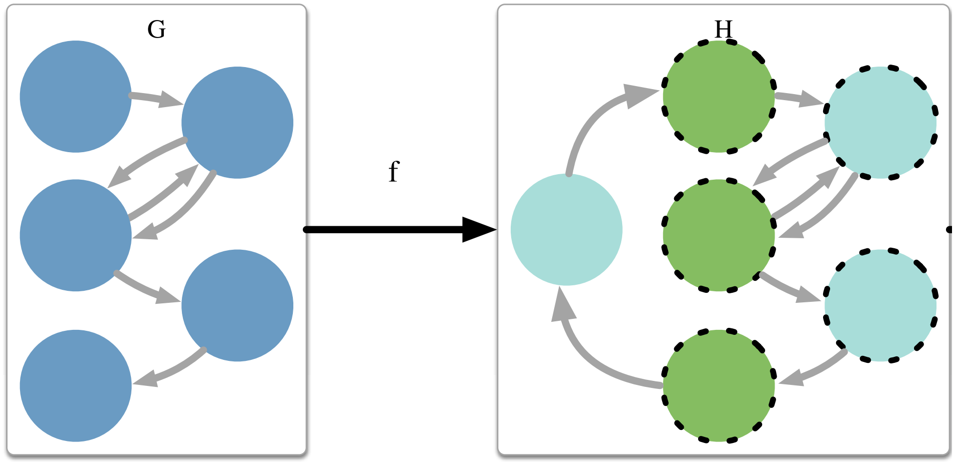
Eng. Design

Figures from Fong and Spivak 2019

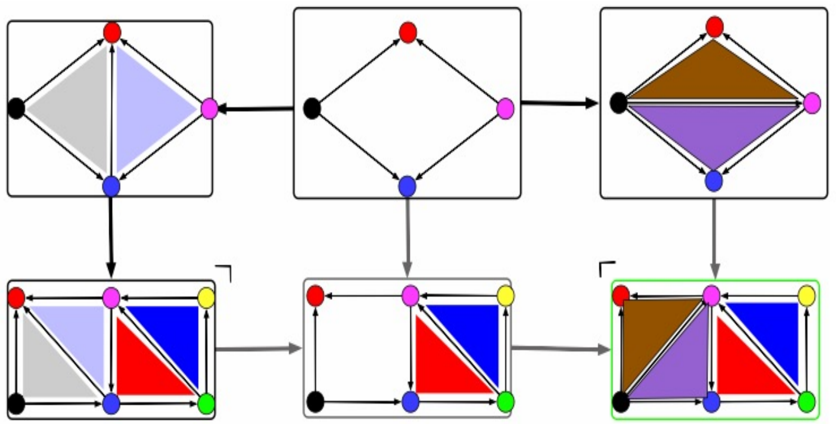
Languages, Data Structures, Algorithms

- Developed infrastructure for software based on Category Theory
- Our generic algorithms are faster than specialized code in SOTA
- Proves that Generality vs Performance tradeoff typical to HPC codes is surmountable
- Data Structures Paper *Compositionality,* premier journal of the Applied Category Theory field.
- Model Structure Modification paper Accepted to *Int. Conf. Graph Transformation* Apr 14th, 2022 (best paper award)





Benchmarks of Catlab vs Graphs.jl (SOTA graph library) showed speedup in 16 of 28 benchmarks. Performance within 2X of SOTA on 10 of 28 benchmarks.



Mesh size	Catlab (s)	ReGraph (s)
2 by 2	1.2×10^{-4}	5.3×10^{-3}
2 by 3	2.7×10^{-4}	8.0
2 by 4	4.7×10^{-4}	1313.3
2 by 5	6.7×10^{-4}	44979.8

Benchmark of Catlab vs ReGraph (a SOTA graph rewriting library). YFA developed software is asymptotically more efficient, achieving over 8 orders of magnitude higher performance on a small problem instance.

From Robotics to Category Theory and Back

ACC Workshop:
Applied Category Theory for Compositional Decision Making
July 7th, 2025

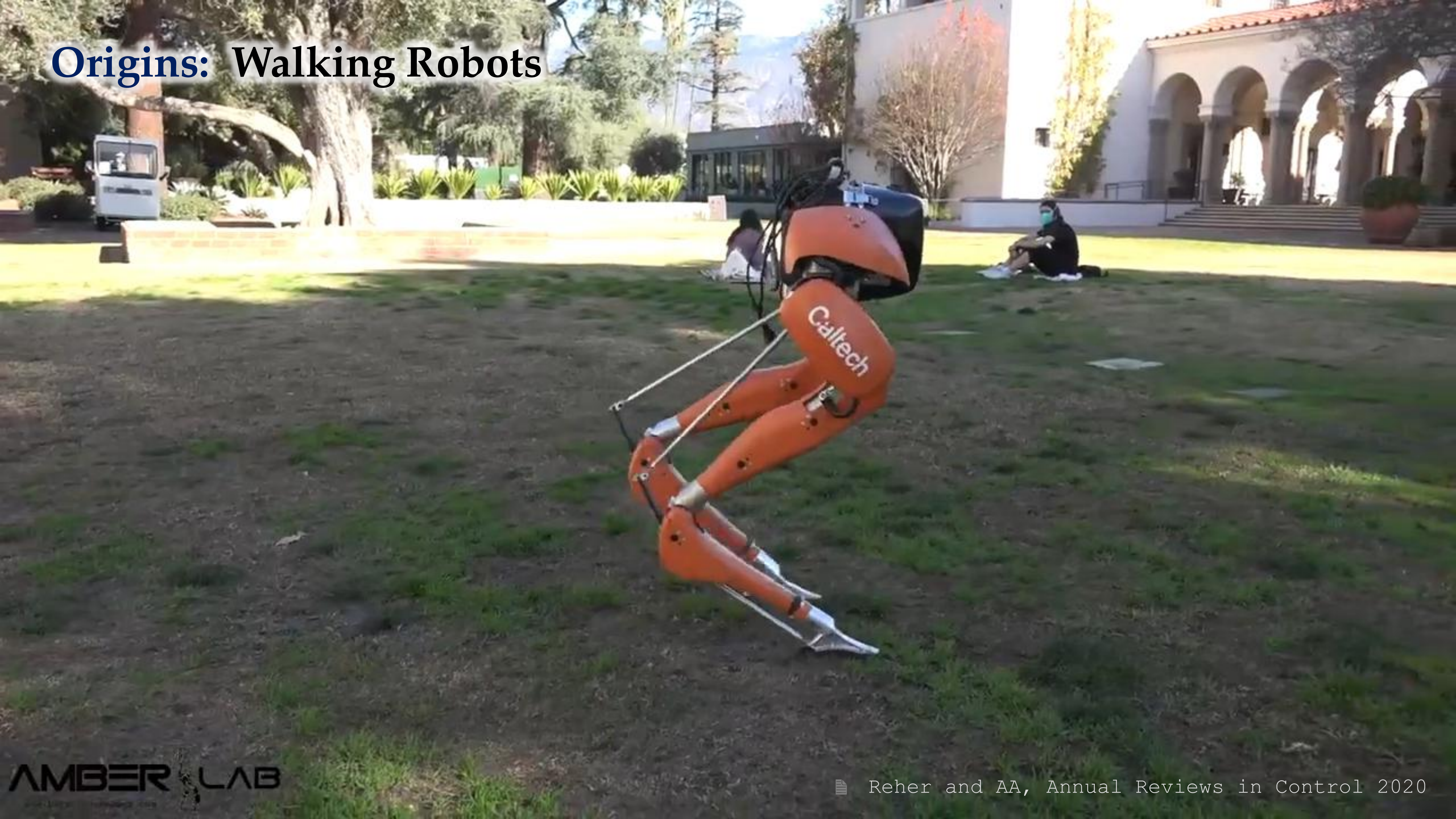
Aaron D. Ames
Bren Professor
Mechanical and Civil Engineering,
Aerospace,
Control and Dynamical Systems,
California Institute of Technology

AMBER LAB

CAST

Caltech

Origins: Walking Robots



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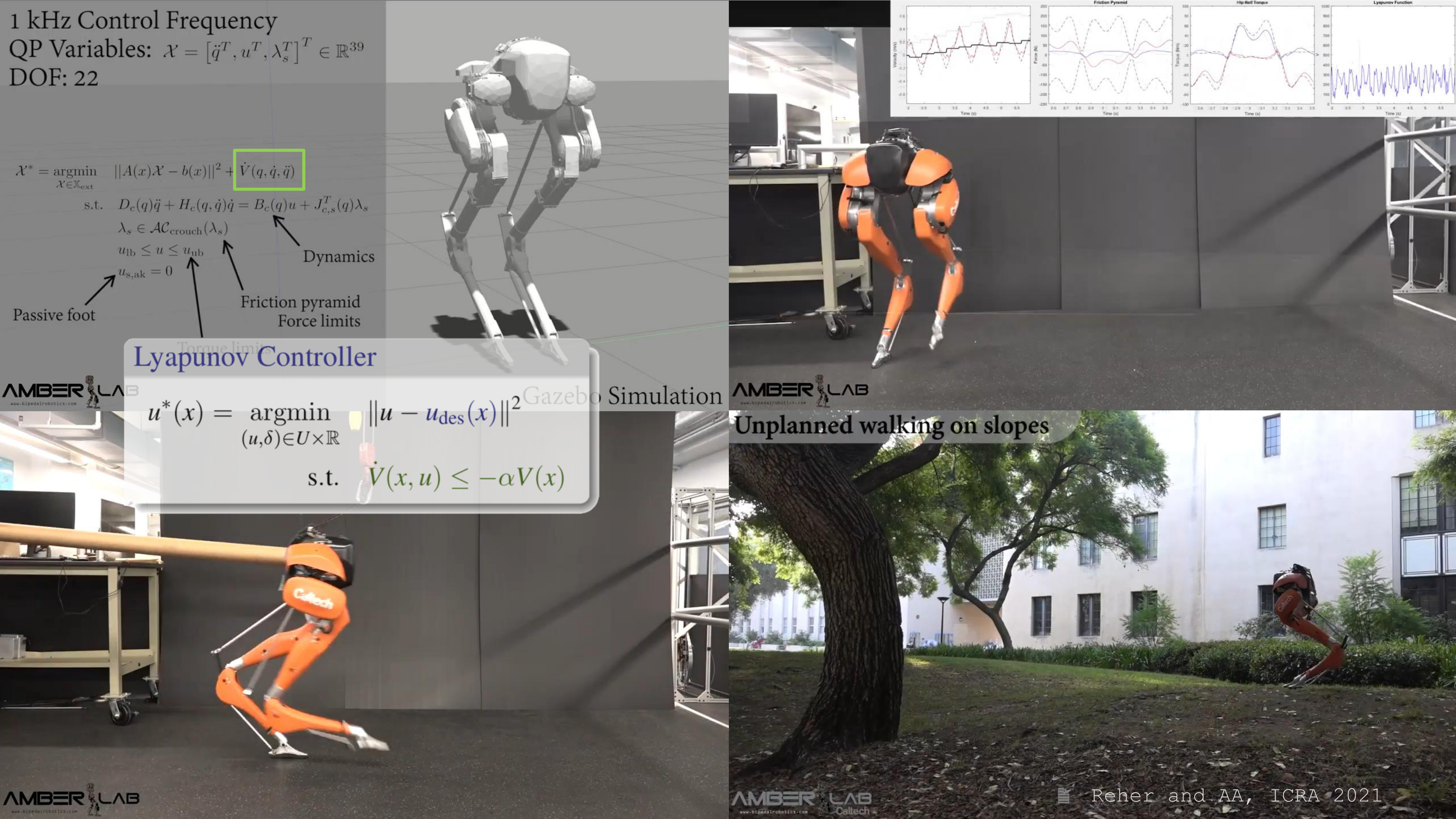
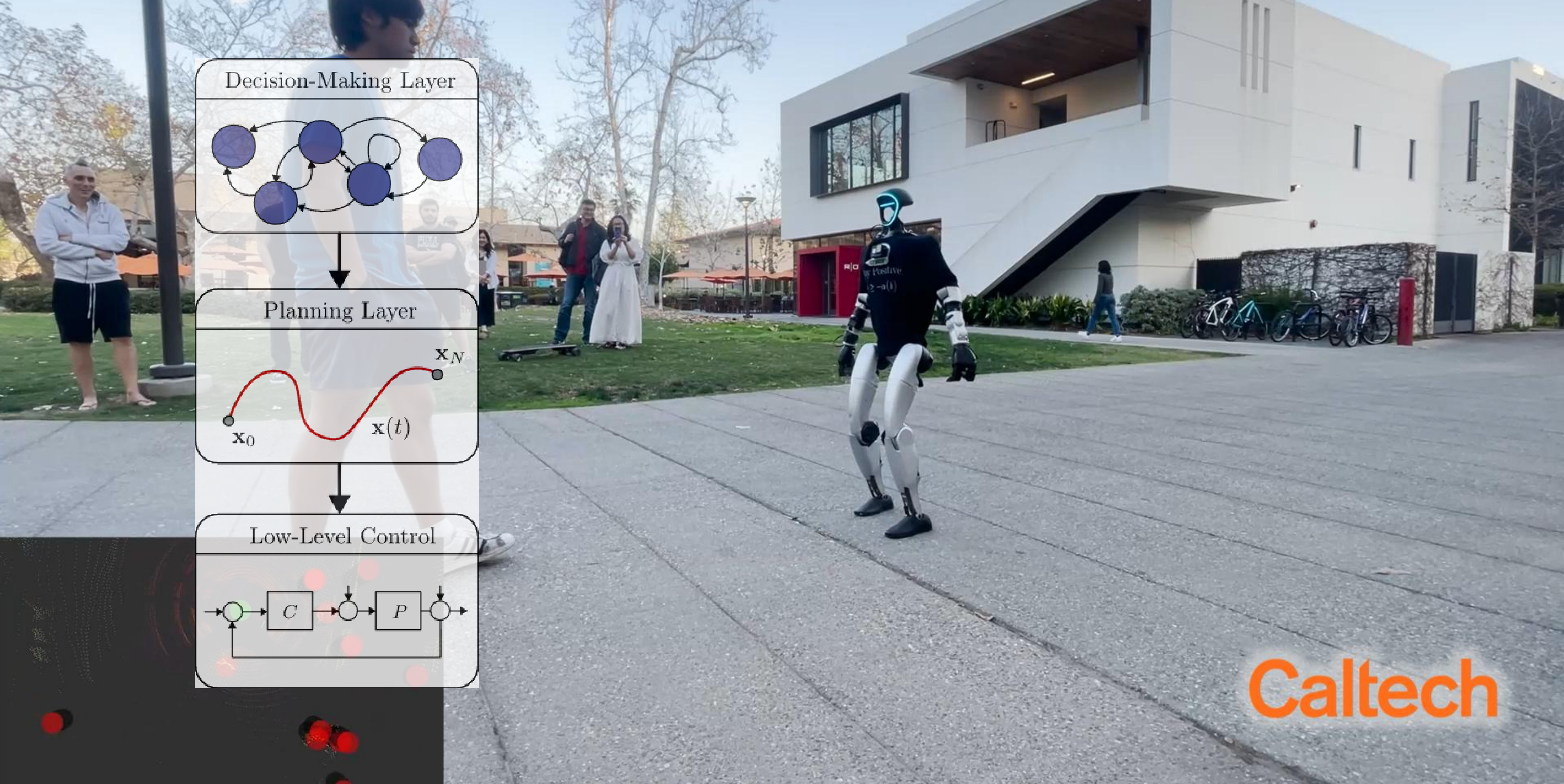
Reher and AA, Annual Reviews in Control 2020


1 kHz Control Frequency
QP Variables: $\mathcal{X} = [\ddot{q}^T, u^T, \lambda_s^T]^T \in \mathbb{R}^{39}$
DOF: 22
$\mathcal{X}^* = \underset{\mathcal{X} \in \mathbb{X}_{\text{ext}}}{\text{argmin}} \ ||A(x)\mathcal{X} - b(x)||^2 + \boxed{\dot{V}(q, \dot{q}, \ddot{q})}$
s.t. $D_c(q)\ddot{q} + H_c(q, \dot{q})\dot{q} = B_c(q)u + J_{c,s}^T(q)\lambda_s$
$\lambda_s \in \mathcal{AC}_{\text{crouch}}(\lambda_s)$
$u_{\text{lb}} \leq u \leq u_{\text{ub}}$
$u_{\text{s,ak}} = 0$
Dynamics
Passive foot
Friction pyramid
Force limits
Torque limits
Gazebo Simulation
Lyapunov Controller
$u^*(x) = \underset{(u,\delta) \in U \times \mathbb{R}}{\text{argmin}} \ \|u - u_{\text{des}}(x)\|^2$
s.t. $\dot{V}(x, u) \leq -\alpha V(x)$
Friction Pyramid
Hip Roll Torque
Lyapunov Function
Time (s)
Unplanned walking on slopes
AMBER LAB
Caltech
Reher and AA, ICRA 2021

LCA: Layered Control Architecture



LCA: Historical Context

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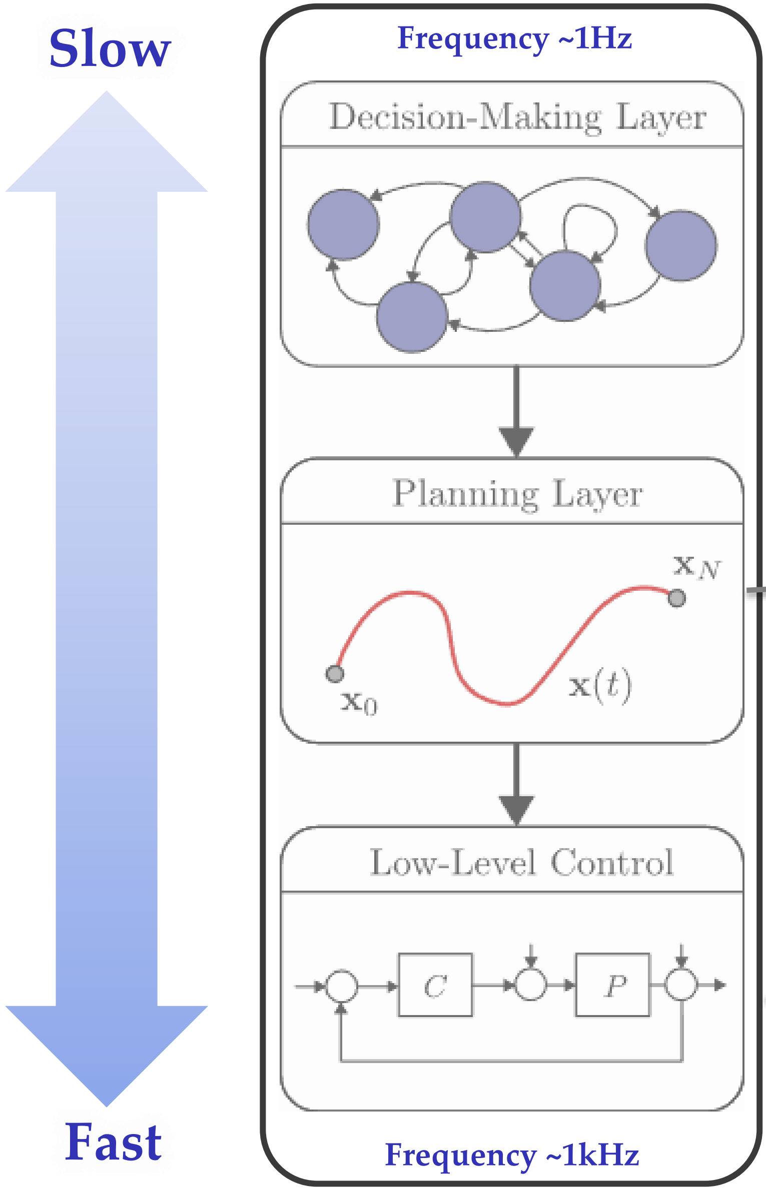
Origins of GNC Stack



Trajectory Planning

Feedback Control

Automatic N... ...cket Vehicle

Copyright, 195... ...yright owner.

Daniel and Florence ... ...gy, Pasadena, Calif.

The primary purpose or the "duty" of the computer is then to properly digest the dynamic and the aerodynamic information of the vehicle and thus to decide the right flight path correction that will insure landing at the chosen destination. The purpose or the duty of the elevator servo is now simply to follow the command of the computer. If the servo is considered to be

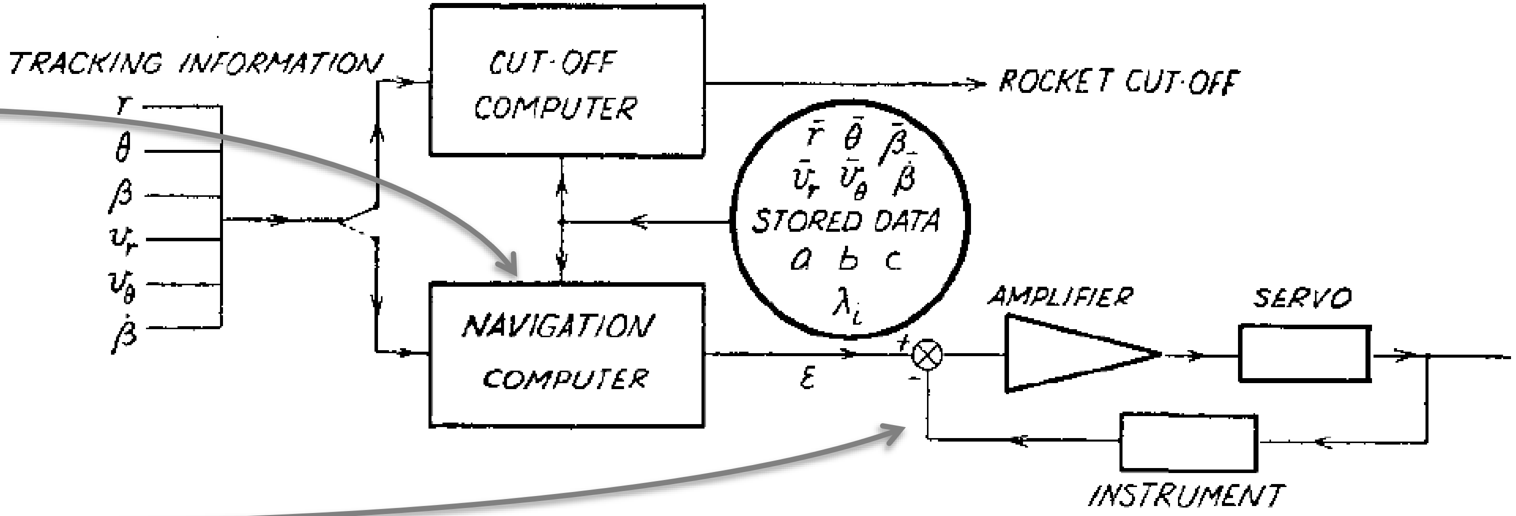
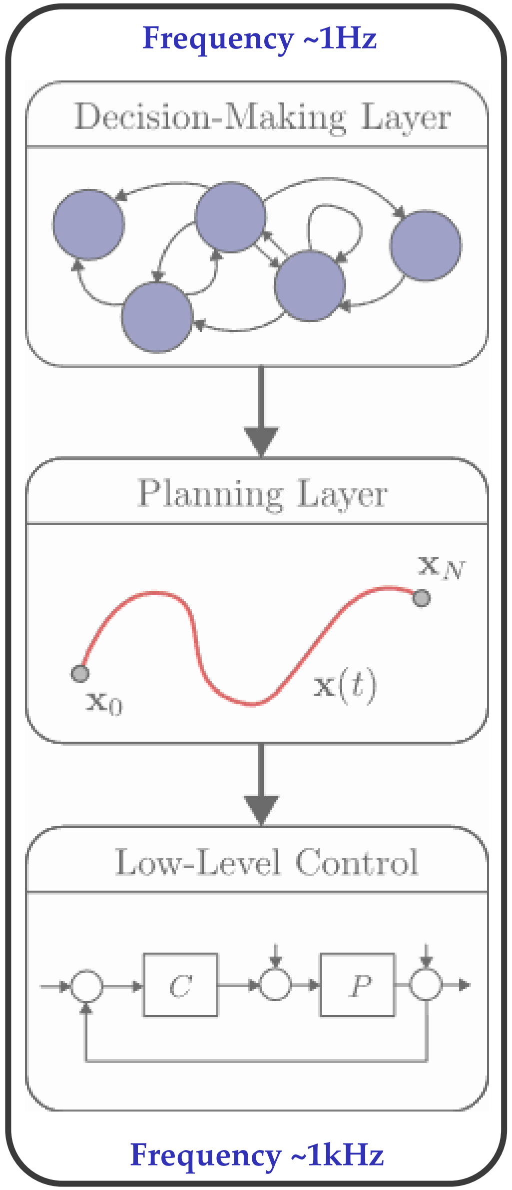


FIG. 2. SYSTEM OF AUTOMATIC NAVIGATION

H.S. Tsien et al., "Automatic navigation of a long range rocket vehicle," 1952

LCA: Historical Context

Slow



Fast

Apollo GNC Stack

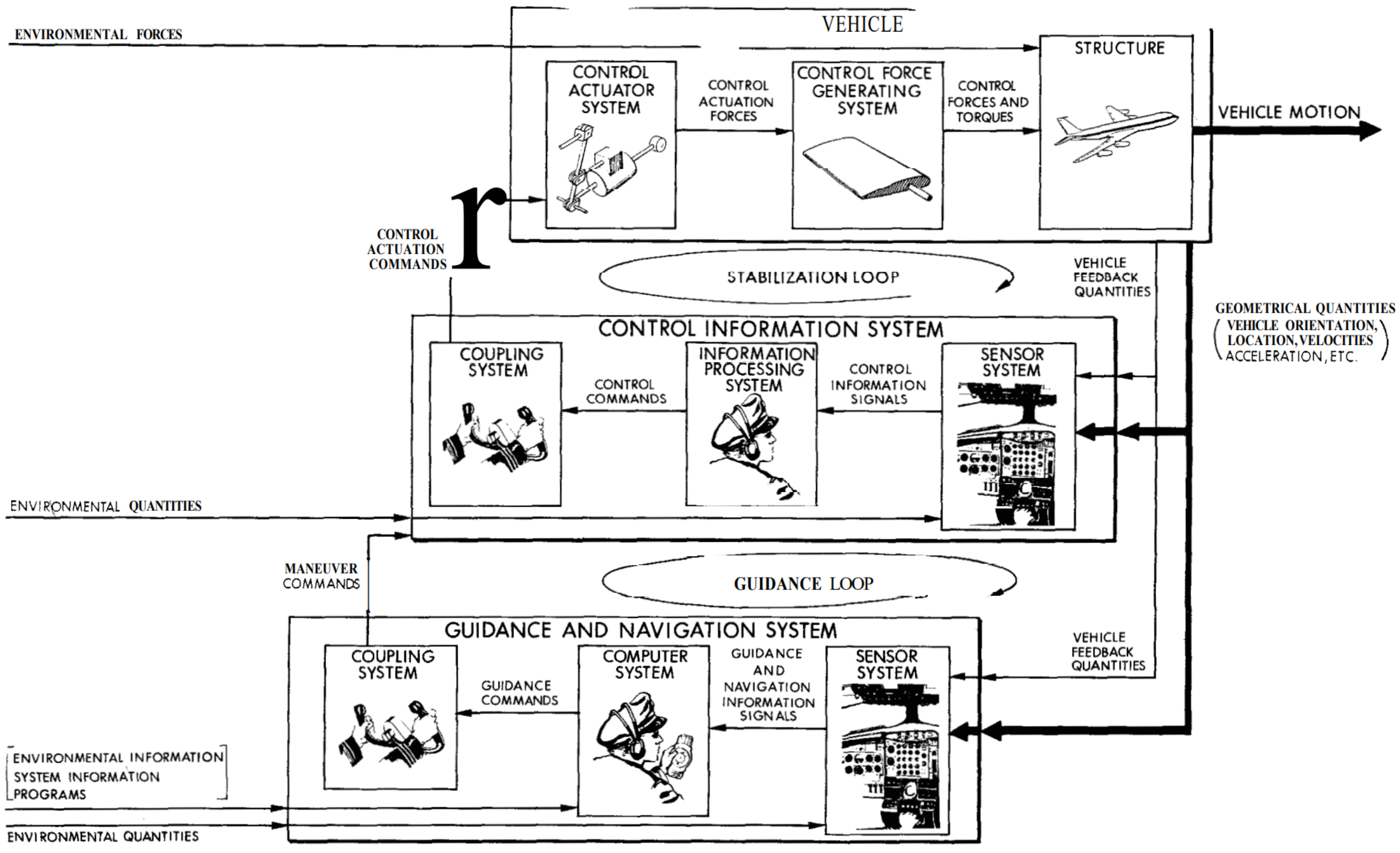
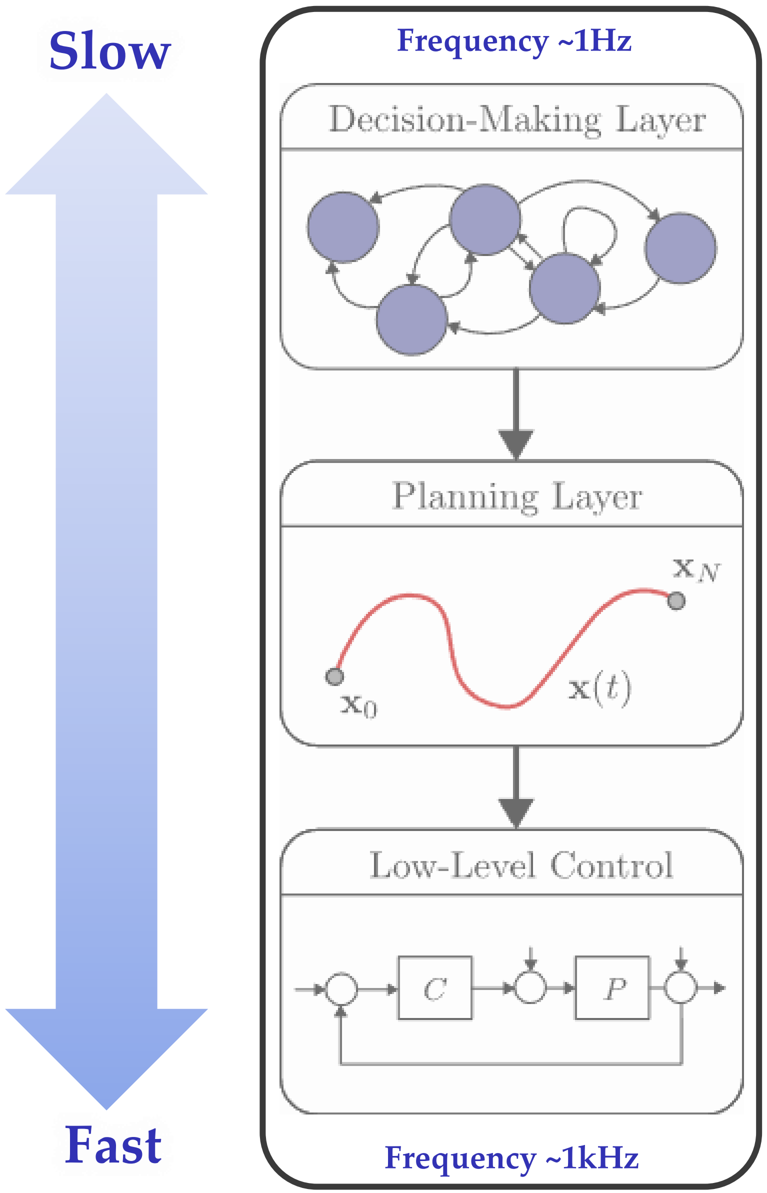
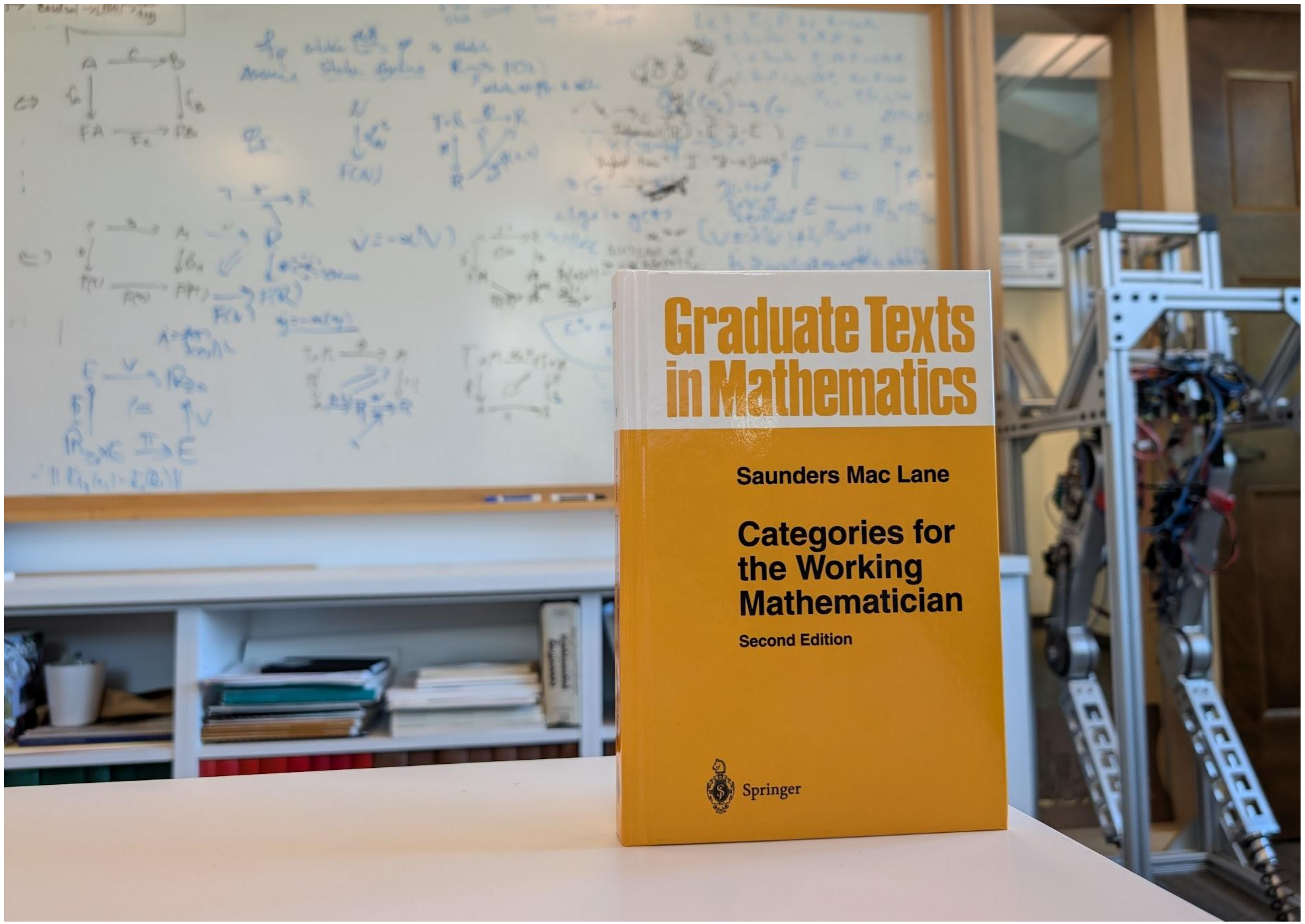


Fig. 1-17 Guidance and Control System - Human Operator's Senses and Power Extended

C. Draper et al., "Guidance and navigation," Massachusetts Inst. Technol., Cambridge, USA, 1965

LCA: Historical Context

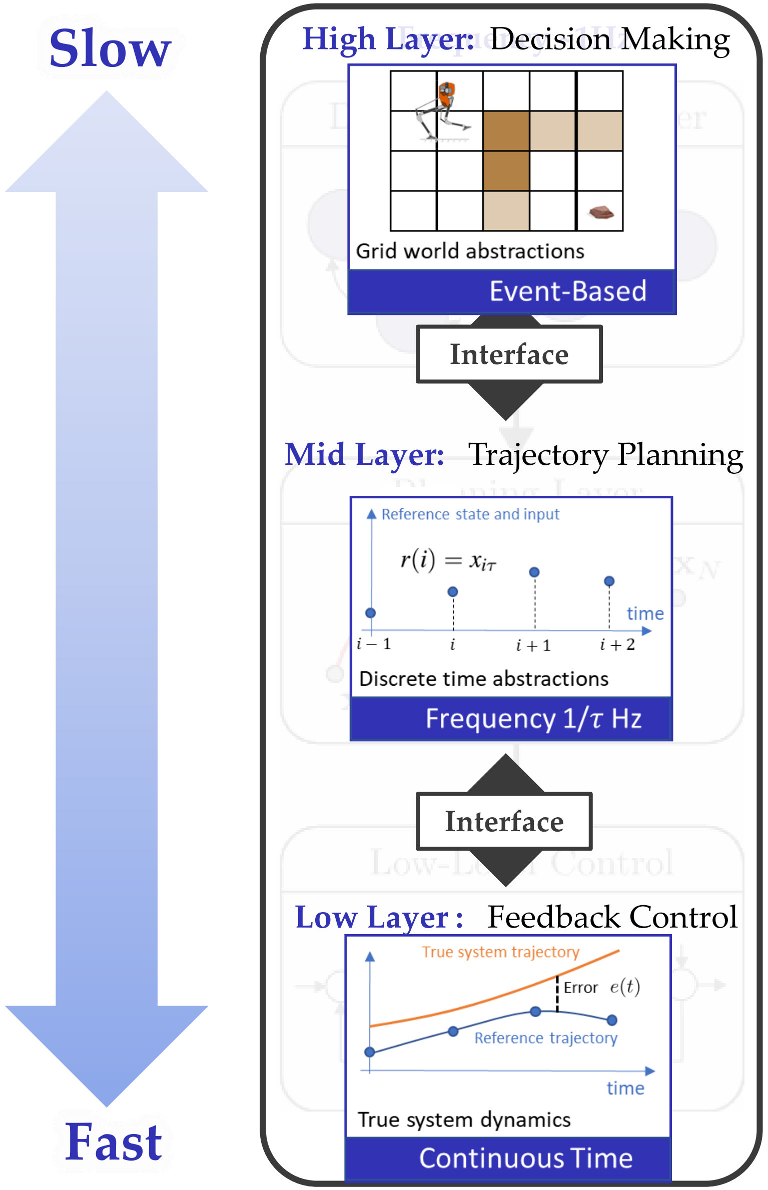




S. Mac Lane, "Categories for the Working Mathematician," Springer, 1971

LCA: Fractured Landscape

Caltech



Formal Methods Worldview:

Specifications: $\xi(s) \models \varphi,$

Dynamics: $s' \in f_{\text{FTS}}(s, a)$

Model Predictive Control Worldview:

Dynamics: $z(i+1) = f_{\text{plan}}(z(i), u_{\text{ff}}(i))$

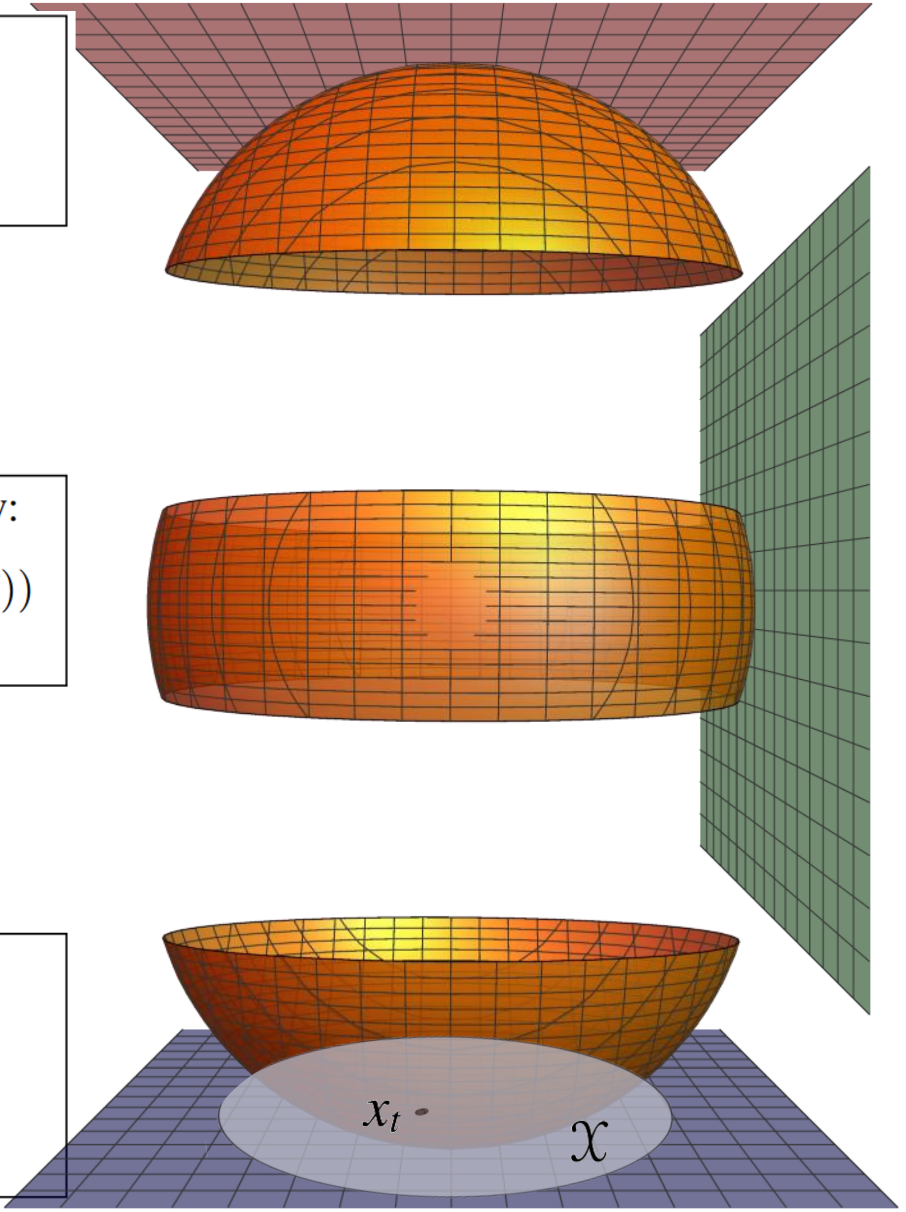
Constraints: $z(k) \in \Pi(\mathcal{X}), \forall k \geq 0$

Nonlinear Control Worldview:

Dynamics: $\dot{x}_t = f(x_t, u_{\text{fb}}(x_t))$

Constraints: $x_t \in \mathcal{X}$

$u_{\text{fb}}(x_t) \in \mathcal{U}, \ \forall t \geq 0$



Rosolia, Singletary, AA, TAC 2022

LCA: Need to Unify

Overall LCA Problem

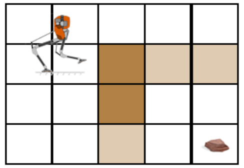
$\xi(x) \models \varphi,$

subject to: $\dot{x}_t = f(x_t, u(x_t)).$

Slow

Fast

High Layer: Decision Making

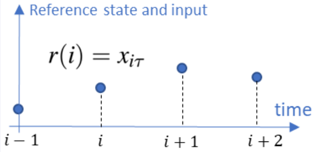


Grid world abstractions

Event-Based

Interface

Mid Layer: Trajectory Planning

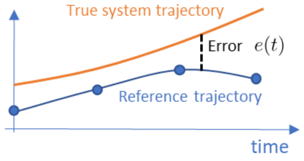


Discrete time abstractions

Frequency $1/\tau$ Hz

Interface

Low Layer: Feedback Control



True system dynamics

Continuous Time

Formal Methods Worldview: / Decision Making

Specifications: $\xi(s) \models \varphi,$

Dynamics: $s' \in f_{\mathrm{FTS}}(s, a)$

Top Interface

$z(k) \in s(\lfloor k/\delta \rfloor)$

Model Predictive Control Worldview: / Trajectory Planning

Dynamics: $z(i+1) = f_{\mathrm{plan}}(z(i), u_{\mathrm{ff}}(i))$

Constraints: $z(k) \in \Pi(\mathcal{X}), \forall k \geq 0$

Bottom Interface

$z(k) = \Pi(x_{k\tau})$

Nonlinear Control Worldview: / Feedback Control

Dynamics: $\dot{x}_t = f(x_t, u_{\mathrm{fb}}(x_t))$

Constraints: $x_t \in \mathcal{X}$

$u_{\mathrm{fb}}(x_t) \in \mathcal{U}, \ \forall t \geq 0$

Satisfy Specifications

$$\operatorname*{satisfy}_{(s,a)} \quad \xi(s,a) := (L(s(k), a(k)))_{k\in\mathbb{N}} \models \varphi$$
$$\text{s.t.} \quad z(k) \in s(\lfloor k/\delta \rfloor)$$

Interface

Optimize Trajectories

$$\operatorname*{minimize}_{z(k), u_{\mathrm{ff}}(k)} \sum_{i=k}^{k+N-1} \mathrm{Cost}(z(i), u_{\mathrm{ff}}(i)) + \mathrm{Cost}_N(z(k+N))$$
$$\text{s.t.} \quad z(i+1) = f_{\mathrm{DT}}(z(i), u_{\mathrm{ff}}(i))$$
$$z(k) = \Pi(x_{k\tau})$$
$$z(i) \in \Pi(\mathcal{X}) \cap s(\lfloor i/\delta \rfloor)$$

Interface

Optimize Real-Time Feedback

$$\operatorname*{minimize}_{u=u_{\mathrm{ff}}+u_{\mathrm{fb}}} \int_{k\tau}^{(k+N)\tau} \left(\|x(s) - z(\lfloor s/\tau \rfloor)\|_Q^2 + \|u_{\mathrm{fb}}\|_R^2\right) ds$$
$$\text{s.t.} \quad \dot{x} = f(x, u_{\mathrm{ff}} + u_{\mathrm{fb}})$$
$$x \in \mathcal{X}$$
$$u_{\mathrm{ff}} + u_{\mathrm{fb}} \in \mathcal{U}$$

Matni, AA, Doyle, CSM 2024

Idea: Unify via Lyapunov

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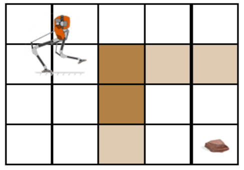
Slow

Fast

High Layer: Decision Making

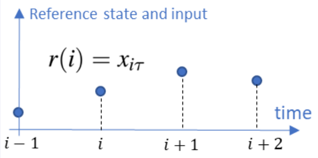


Grid world abstractions

Event-Based

Interface

Mid Layer: Trajectory Planning

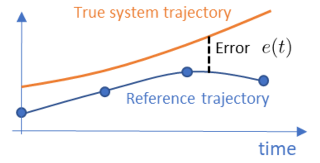


Discrete time abstractions

Frequency $1/\tau$ Hz

Interface

Low Layer: Feedback Control



True system dynamics

Continuous Time

Labeled Transition System:

Dynamics: $s' \in f_{\mathrm{FTS}}(s, a)$

Constraints: $s \in \mathcal{S}$

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

Discrete-Time Dynamical System:

Dynamics: $z(i+1) = f_{\mathrm{plan}}(z(i))$

Constraints: $z(k) \in \mathcal{Z}$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

Continuous-Time Dynamical System:

Dynamics: $\dot{x}(t) = f(x(t))$

Constraints: $x(t) \in \mathcal{X}$

Lyapunov Condition (CT):

$$\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x(t)) \leq 0$$

Idea: Unify via Lyapunov

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Unified Lyapunov Theory:

What is the commonality?

How do we generalize?

Answer:

Saunders Mac Lane

Categories for the Working Mathematician

"Good general theory does not search for the maximum generality, but for the right generality."

–Mac Lane

Lyapunov Condition (TS):

$$\max_{s' \in f_{\text{FTS}}(s,a)} V(s') \leq V(s)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\text{plan}}(z)) - V(z) \leq 0$$

Lyapunov Condition (CT):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\Big|_x f(x(t)) \leq 0$$

Revisiting Classical Lyapunov Theory

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Solutions

- Expressed by commuting diagram:

$$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "Tc"] & \mathcal{X}\times\mathbb{R}^n \\ \mathbb{R} \arrow[u, "1_{\mathbb{R}} \triangleq (\mathrm{id}_{\mathbb{R}},1)"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "(\mathrm{id}_{\mathcal{X}},f) \triangleq f_{\mathcal{X}}"'] \end{tikzcd}$$

where $1(t)\equiv 1$, i.e., 1 is the unit clock: $\dot{t}=1(t)=1$ and

$$Tc(t,t') = (c(t),\dot{c}(t)t')$$

- The *diagram commutes* if $c(t)$ is a solution to $\dot{x}=f(x)$:

$$\begin{aligned} Tc\circ 1_{\mathbb{R}} = f_{\mathcal{X}}\circ c \quad &\Longrightarrow \quad (c(t),\dot{c}(t)1) = (c(t),f(c(t))) \\ &\Longrightarrow \quad \boxed{\dot{c}(t) = f(c(t))} \end{aligned}$$

Lyapunov

- Consider the following diagram that *commutes up to inequality*:

$$\begin{tikzcd} \mathcal{X}\times\mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0}\times\mathbb{R} \\ \mathcal{X} \arrow[u, "\vec{f}"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"'] \end{tikzcd}$$

where $0_{\mathbb{R}_{\geq 0}}$ is zero vector field $0_{\mathbb{R}_{\geq 0}}(r) = (r,0)$, and

$$TV(x,y) \triangleq (V(x), DV(x)y) = \left(V(x), \left.\frac{\partial V}{\partial x}\right|_x y \right).$$

- Saying "diagram commutes up to inequality" means:

$$\begin{aligned} TV\circ f_{\mathcal{X}} \leq 0_{\mathbb{R}_{\geq 0}}\circ V \quad &\Longrightarrow \quad TV(x,f(x)) \leq (V(x),0) \\ &\Longrightarrow \quad \boxed{\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x) \leq 0} \end{aligned}$$

Continuous-Time Dynamical System:

Dynamics: $\dot{x}(t) = f(x(t))$
Constraints: $x(t)\in\mathcal{X}$

Lyapunov Condition (CT):

$$\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x(t)) \leq 0$$

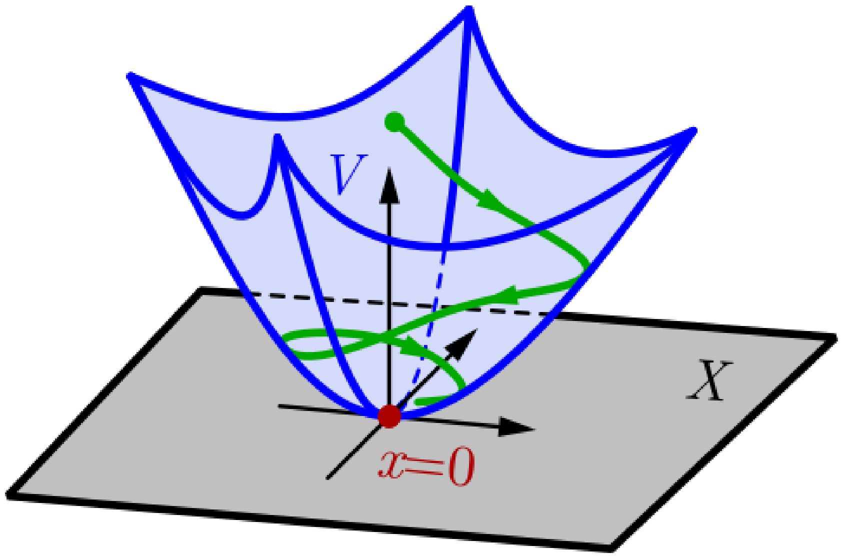
Categorical Lyapunov Theory

Caltech

Classical Conditions

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$
\begin{tikzcd}
\mathbb{R} \times \mathbb{R} \arrow[r, "Tc"] & \mathcal{X} \times \mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0} \times \mathbb{R} \\
\mathbb{R} \arrow[u, "1_{\mathbb{R}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "\vec{f}"'] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"']
\end{tikzcd}
$$



unit clock $1_{\mathbb{R}} = (\mathrm{id}_{\mathbb{R}}, 1)$, zero vector field $0_{\mathbb{R}_{\geq 0}}(r) = (r, 0)$.

Theorem

The system $\dot{x} = f(x)$ is stable iff the diagram (lax) commutes.

Lyapunov Condition (CT):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\bigg|_x f(x(t)) \leq 0$$

Categorical Lyapunov Theory

Coalgebras

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\mathcal{F}(c)"] & \mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
T \arrow[u, "1_T"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & E \arrow[u, "f_E"'] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$

unit clock $1_T : T \to \mathcal{F}(R)$, zero system $0_R : R \to \mathcal{F}(R)$

Theorem

The system $f_E : E \to \mathcal{F}(E)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} : \mathsf{C} \to \mathsf{C}$ an endofunctor, $T =$ time and $R =$ measurement.

On the Stability of Zeno Equilibria*

Aaron D. Ames[1], Paulo Tabuada[2], and Shankar Sastry[1]

HSCC 2006

CATEGORICAL LYAPUNOV THEORY I: STABILITY OF FLOWS

AARON D. AMES, JOE MOELLER, AND PAULO TABUADA

CATEGORICAL LYAPUNOV STABILITY II: STABILITY OF SYSTEMS

AARON D. AMES, SÉBASTIEN MATTENET, AND JOE MOELLER

ArXiv 2025

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Categorical Lyapunov Theory

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Discrete Time

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\mathcal{F}(c)"] & Z \arrow[r, "\mathcal{F}(V)"] & \mathbb{R}_{\geq 0} \\ \mathbb{N} \arrow[u, "1_{\mathbb{N}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & Z \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"'] \end{tikzcd}$$

unit clock $1_{\mathbb{N}}(k) = k + 1$, zero system $0_{\mathbb{R}_{\geq 0}}(r) = r$.

Theorem

The system $z_{k+1} = f_{\text{plan}}(z_k)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} = I$, $\mathsf{C} = \mathsf{Man}$, $T = \mathbb{N}$ and $R = \mathbb{R}_{\geq 0}$.

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\text{plan}}(z)) - V(z) \leq 0$$

Categorical Lyapunov Theory

Transition Systems[1]

Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathcal{P}(A \times A^*) \arrow[r, "\mathcal{P}(A \times c)"] & \mathcal{P}(A \times S) \arrow[r, "\mathcal{P}(A \times V)"] & \mathcal{P}(A \times \mathbb{R}_{\geq 0}) \\
A^* \arrow[u, "1_{A^*}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & S \arrow[u, "f_S"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "0_{\mathbb{R}_{\geq 0}}"']
\end{tikzcd}$$

unit clock $1_{A^*}(a^*) = \{(a, a^*a)\}_{a \in A}$, zero system $0_{\mathbb{R}_{\geq 0}}(r) = A \times \{r\}$.

Theorem

The system $s' \in f_{\mathrm{FTS}}(s,a)$ is stable iff the diagram (lax) commutes.

$\mathcal{F} = \mathcal{P}(A \times -)$, $\mathsf{C} = \mathsf{Set}$, $T = A^*$ (free monoid) and $R = \mathbb{R}_{\geq 0}$.

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Categorical Lyapunov Theory ↔ Ability to Generalize

Caltech

$$\boxed{\textit{system}\, f_E : E \to \mathcal{F}(E)}$$

$$\begin{tikzcd}
\mathcal{F}(T) \arrow[r, "\mathcal{F}(c)"] & \mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
T \arrow[u, "1_T"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & E \arrow[u, "f_E"'] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$

Lyapunov Condition (C):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Lyapunov Condition (TS):

$$\max_{s' \in f_{\mathrm{FTS}}(s,a)} V(s') \leq V(s)$$

$$\textit{system}\ s' \in f_{\mathrm{FTS}}(s,a)$$

Lyapunov Condition (DT):

$$\nabla V(z) = V(f_{\mathrm{plan}}(z)) - V(z) \leq 0$$

$$\textit{system}\ z_{k+1} = f_{\mathrm{plan}}(z_k)$$

Lyapunov Condition (CT):

$$\dot{V}(x) = \left.\frac{\partial V}{\partial x}\right|_x f(x(t)) \leq 0$$

$$\textit{system}\ \dot{x} = f(x)$$

Categorical Lyapunov Theory – What is next?

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Asymptotic Stability

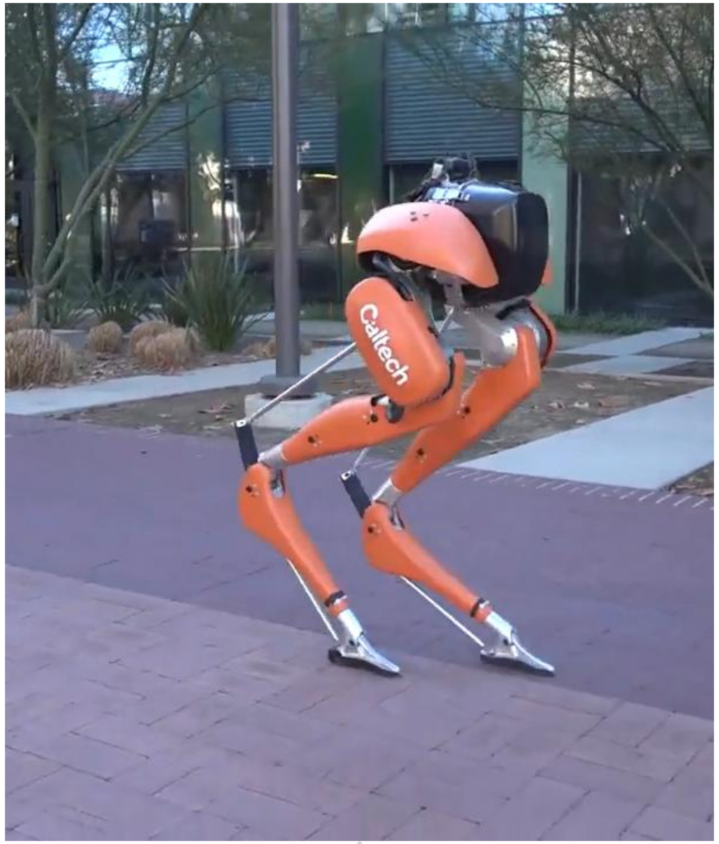
Characterizing **Solutions** and **Lyapunov** via a diagram:

$$\begin{tikzcd}
\mathbb{R}\times\mathbb{R} \arrow[r, "Tc"] & \mathcal{X}\times\mathbb{R}^n \arrow[r, "TV"] & \mathbb{R}_{\geq 0}\times\mathbb{R} \\
\mathbb{R} \arrow[u, "1_{\mathbb{R}}"] \arrow[r, "c"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{X} \arrow[u, "\vec{f}"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & \mathbb{R}_{\geq 0} \arrow[u, "\sigma"']
\end{tikzcd}$$

unit clock $1_{\mathbb{R}} = (\mathrm{id}_{\mathbb{R}}, 1)$, "simplest" stable system $\sigma(r) = (r, -\alpha(r))$.

Theorem

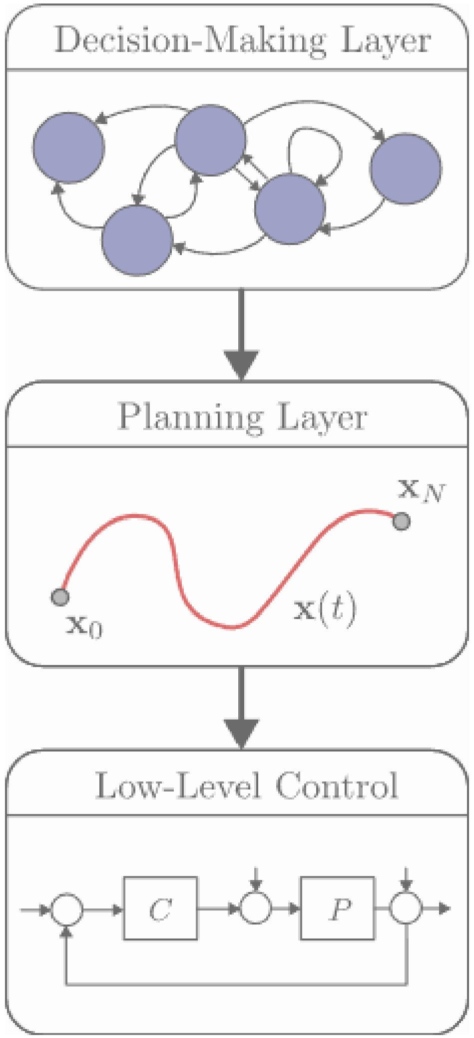
The system $\dot{x} = f(x)$ is asy. stable iff the diagram (lax) commutes.



Lyapunov Condition (Asy.):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\bigg|_x f(x(t)) \leq -\alpha(V(x))$$

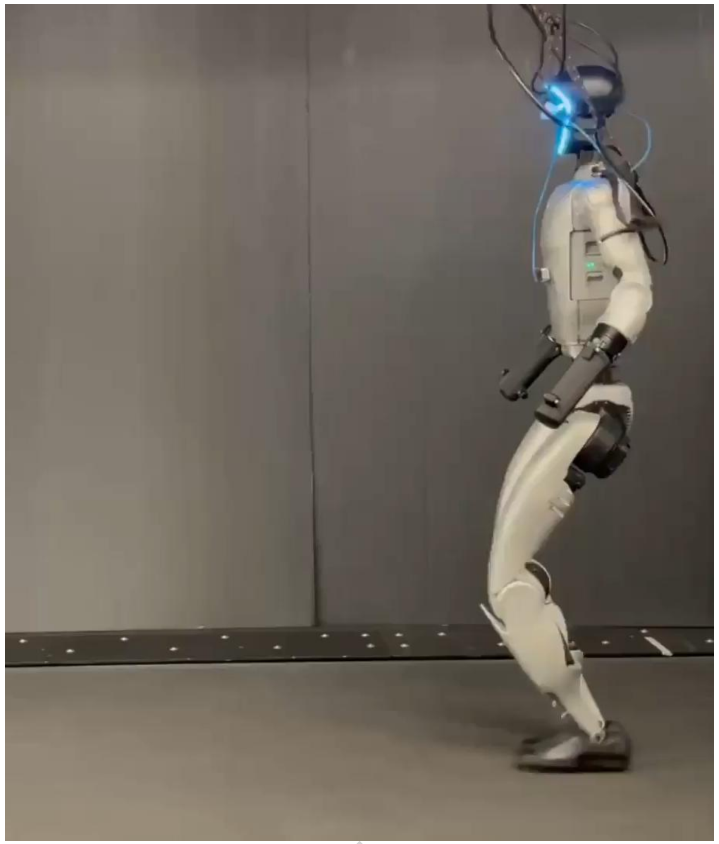
Categorical Theory ↔ Ability to Generalize



Goal: Category theory on robots

$$\begin{tikzcd} \mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] \arrow[dr, phantom, "\leq"] & \mathcal{F}(R) \\ E \arrow[u, "f_E"] \arrow[r, "V"'] & R \arrow[u, "0_R"'] \end{tikzcd}$$

- Robotics: ever increasing *complexity*
- Need for *formal guarantees* to deploy
- Category theory: *formal generalizability*



Lyapunov Condition ($\mathbb{C}$):

$$\mathcal{F}(V) \circ f_E \leq 0_R \circ V$$

Goal: Category Theory on Robots

$$\begin{tikzcd}
\mathcal{F}(E) \arrow[r, "\mathcal{F}(V)"] & \mathcal{F}(R) \\
E \arrow[u, "f_E"] \arrow[r, "V"'] \arrow[ur, phantom, "\leq"] & R \arrow[u, "0_R"']
\end{tikzcd}$$