Network Preference Dynamics using Lattice Theory

Hans Riess, *Duke University*
Michael C. Munger *Duke University*
Zachary I. Bell *AFRL*
Gregory Henselman-Petrusek *PNNL*
Robert Ghrist *University of Pennsylvania*
Michael M. Zavlanos *Duke University*

Speaker: **Hans Riess**
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Introduction

Key contributions:

- Preference lattice
  - Algebraic model for preference: meet ($\wedge$) and join ($\vee$) operations
  - Median aggregation: majority rule for lattices
- Message passing preference dynamics
  - Decentralized networked consensus-like algorithm on preference lattices
  - Prove existence/computability of equilibria in a few cases
  - Software implementation of our algorithm

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Problem Formulation

Assumptions.

- Agents have preference ordering over a number of alternatives (i.e. options)
- Agents' preference relations are incomplete (missing comparisons)
  - limited information available
  - all pairs of alternatives
- Agents are organized in a network
  - modeled by an undirected graph
  - can share all preference relations with neighboring agents

Problem 1. Design a decentralized mechanisms for nodes to share, receive and aggregate preferences in order to augment their independent decision-making capabilities.

Problem 2. Analyze and compute equilibria of the above dynamical systems.

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Related Work

Preference lattice on complete preference (Curello & Sinander 2024)

- Based on single-crossing dominance
- Applications in economics (monotone comparative statics, risk-aversion,

Laplacian consensus with lattice theory priors (Ghrist & Riess, 2022)

- Tarski Laplacian (Ghrist & Riess, 2022)
- Synchronization (Riess, Munger, Zavlanos; 2023)

Consensus on lattices

- Aggregation functions on lattices (Karacal & Mesiar, 2017)

Distributed consensus

- Min/max consensus (Tahbaz-Salehi & Jadbabaie, 2006)
- Biased min consensus (Zhang & Li, 2017)
- Minmax consensus (Shi, Xia, Johansson, 2015)

Opinion dynamics (DeGroot 1974; Hegselman & Krause, 2022; etc.)

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Lattice Theory

Definition. A ***poset*** is a set with a binary relation $(\mathscr{P}, \preceq)$ satisfying the following properties:

- $a \preceq b$ and $b \preceq c$ implies $a \preceq c$ (transitivity)
- $a \preceq a$ (reflexivity)
- $a \preceq b$ and $b \preceq a$ implies $a = b$ (anti-symmetry)

Definition. A ***lattice*** is a poset $(\mathscr{L}, \preceq)$ such that

- $a \vee b = \min\{c : c \succeq a,\ c \succeq b\}$ (called *join*)
- $a \wedge b = \max\{c : c \preceq a,\ c \preceq b\}$ (called *meet*)

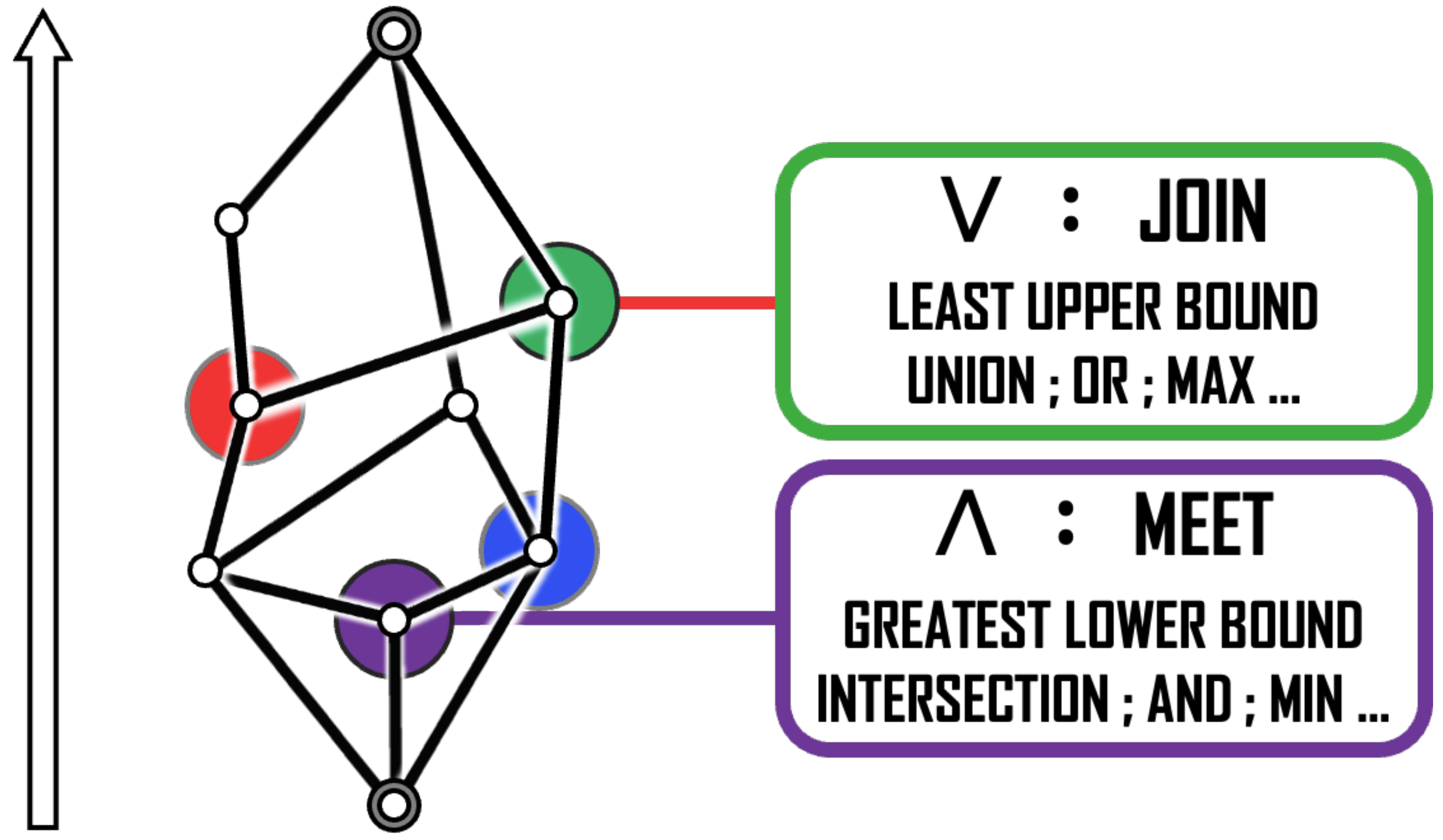
exist for all $a, b, \in \mathscr{L}$.

Write, $a_1 \vee a_2 \vee \cdots \vee a_n = \bigvee a_i$

$a_1 \wedge a_2 \wedge \cdots \wedge a_n = \bigwedge a_i$

Examples.

- $\mathbb{R}^n$ or $\mathbb{Z}^n$, $\wedge$: min , $\vee$: max
- Boolean algebras: $\wedge$: & , $\vee$: |
- Powersets, $\wedge : \cap$, $\vee : \cup$
- Subspaces, subgroups, ideals, etc…
- Convex subsets of a convex set
- Equivalence relations
- **Preference relations**
- Formal Concept Analysis (FCA)



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Preference Relations

Setup:

- $\mathcal{N} = \{1,2,\ldots,N\}$, set of agents
- $\mathcal{A} = \{1,2,\ldots,M\}$, set of alternatives (i.e. options)

Definition. A ***preference relation*** $\gtrsim$ is a binary relation (subset or predicate) satisfying the following:

- $\forall a, a', b \in \mathcal{A}, \quad (a \gtrsim a')\ \&\ (a' \gtrsim b) \to (a \gtrsim b)$ implies $i \gtrsim_u k$ (transitivity);
- $\forall a \in \mathcal{A}, \quad a \gtrsim a$ (reflexivity);

We write $a \gtrsim_i b$ if Agent $i \in \mathcal{N}$ weakly prefers Option $a \in \mathcal{A}$ over Option $b \in \mathcal{A}$.

A ***preference profile*** is a tuple $\Pi = \left(\gtrsim_i\right)_{i \in \mathcal{N}}$

Proposition. Pre($\mathcal{A}$), the set of all preference relations forms a lattice with the following operation:

- $a \left(\gtrsim_i \wedge \gtrsim_j \right) b$ if and only if $\left(a \gtrsim_i b\right)\ \&\ \left(a \gtrsim_j b\right)$
- $a \left(\gtrsim_i \vee \gtrsim_j \right) b$ iff $\exists \{a_n\}_{n=0}^{\ell} \in \mathcal{A}$ for $a_0 = a, a_\ell = b$ such that $\left(a_n \gtrsim_i a_{n+1}\right) \Big| \left(a_n \gtrsim_j a_{n+1}\right)\ \forall n \in \{0,\ldots,\ell-1\}$

Pre($\mathcal{A}$) is called the ***preference lattice***.

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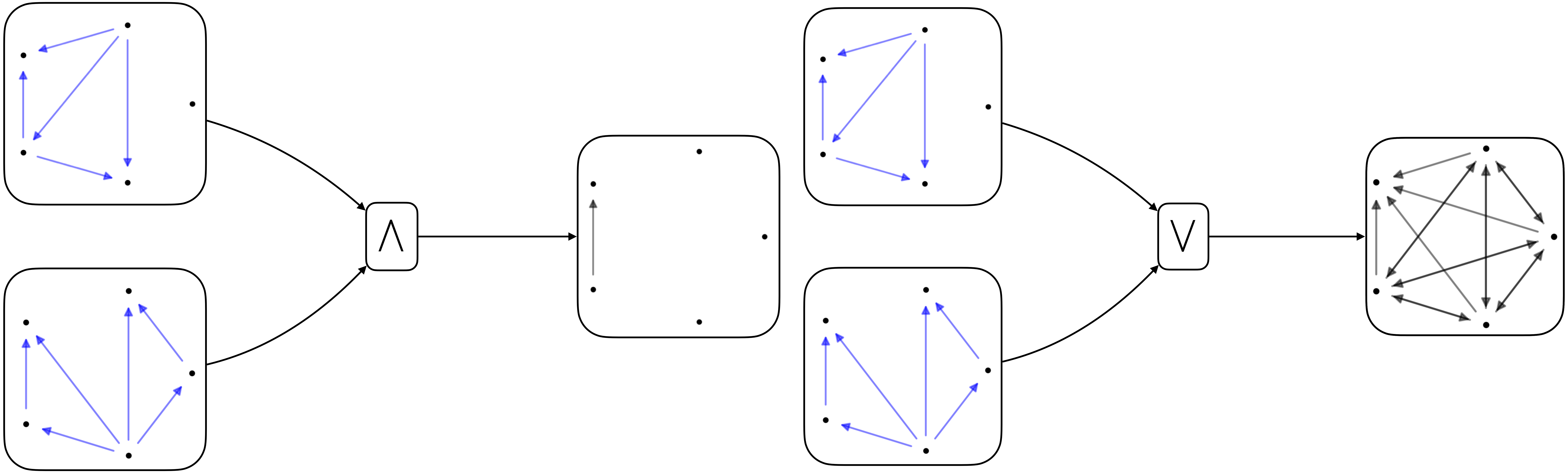
Preference Lattice



$$a\left(\succsim_i \vee \succsim_j \right) b \text{ iff } \left(a_n \succsim_i a_{n+1}\right) \Big| \left(a_n \succsim_j a_{n+1}\right)$$

$$a\left(\succsim_i \wedge \succsim_j \right) b \text{ iff } \left(a \succsim_i b\right) \,\&\, \left(a \succsim_j b\right)$$

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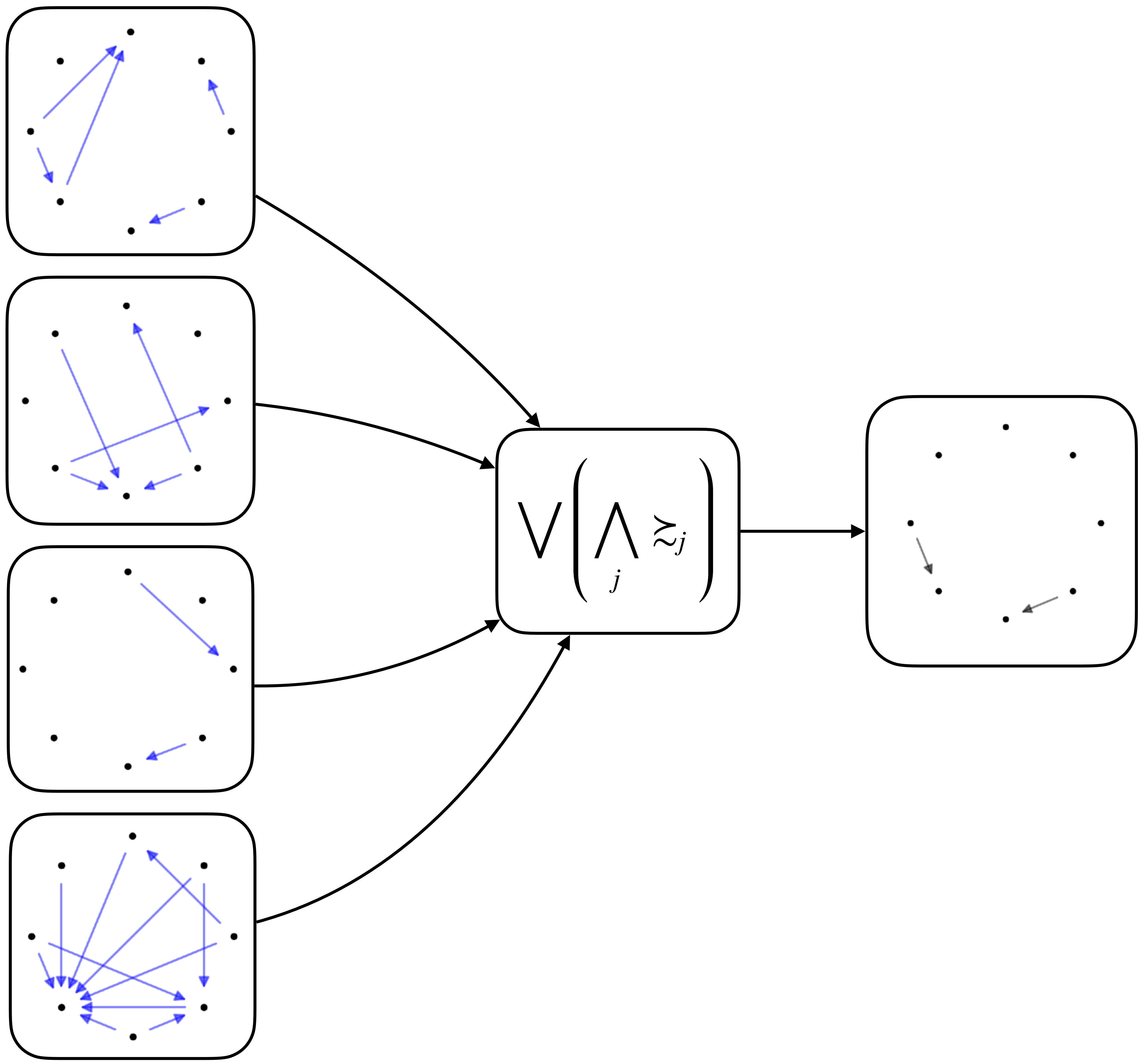
Aggregating Preferences: Centralized



$r = 2$

Definition. (Lattice median, Leclerc 1990, 1993):

- Suppose $\pi = (\succsim_i)_{i \in \mathcal{N}}$ is a preference profile

$$\mathrm{Median}_r(\pi) = \bigvee_{\substack{J \subseteq \{1,2,\ldots,n\} \\ |J| \geq r}} \left(\bigwedge_{j \in J} \succsim_j \right)$$

$r \in \{1,2,\ldots,N\}$, majority threshold

- Join consensus, $\mathrm{Median}_1(\pi) = \vee_{j \in \mathcal{N}} \succsim_j$
- Meet consensus, $\mathrm{Median}_N(\pi) = \wedge_{j \in \mathcal{N}} \succsim_j$

Properties:

- *anonymity*
- *unanimity*
- *r-middle*

Can we extend the lattice median to the decentralized setting?

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Aggregating Preferences: Dentralized

Message-passing preference dynamics:

$$\succsim_i' = \varphi\Big(\succsim_i , \mathrm{Median}_r \big[\psi(\succsim_i , \succsim_j)\big]_{j \in \mathcal{N}_i} \Big)$$

- ψ, message function
  - $\psi(\precsim_i , \precsim_j) = \precsim_j$ (truthful)
  - $\psi(\precsim_i , \precsim_j) = \precsim_i$ (mimicking)
- φ, updates function
  - $\varphi(\succsim_i , \succsim) = \succsim_i \wedge \succsim$ (confirming)
  - $\varphi(\succsim_i , \succsim) = \succsim_i \vee \succsim$ (learning)
  - $\varphi(\succsim_i , \succsim) = \succsim_i$ (single-minded)
  - $\varphi(\succsim_i , \succsim) = \succsim$ (open-minded)

Local updates lead to operator on preference profiles:

$$F : \mathrm{Pre}(\mathscr{A})^N \to \mathrm{Pre}(\mathscr{A})^N$$

Algorithm.

Inputs: Initial preference relations

1. Agent j sends "message" $\psi(\precsim_i , \precsim_j)$ (preference relation) to Agent i
2. Agent i collects messages received by neighbors
3. Agent i aggregates messages with median
4. Agent i updates prior preference $\succsim_i$ with $\varphi(\succsim_i , -)$
5. Go to back to 1.

Operator Theory on Lattices

- Suppose $\mathscr{L}$ is a lattice

Definition. A map $f : \mathscr{L} \to \mathscr{L}'$ is ***monotone*** if $x_1 \preceq x_2$ implies $f(x_1) \preceq f(x_2)$.

Definition. A map $f : \mathscr{L} \to \mathscr{L}$ is ***inflationary*** if $f(x) \succeq x$, ***deflationary*** if $f(x) \preceq x$

Lemma (Tarski Fixed Point Theorem). Suppose $f : \mathscr{L} \to \mathscr{L}$ is monotone and $\mathscr{L}$ a complete* lattice. Then, $\mathrm{Fix}(f)$ is a complete lattice.

- We apply this result to our preference dynamics model

Theorem. Suppose the message (ψ) and update (φ) functions are monotone. Then, the equilibrium points (preference profile) of the global message-passing preference dynamical system form a (non-empty) lattice.

Theorem. Suppose the update function (φ) is inflationary or deflationary. Then, for every initial preference profile, the dynamics given locally by $\succsim_i' = \varphi(\succsim_i, \mathrm{Median}_r[\psi(\succsim_i, \succsim_j)]_{j \in \mathcal{N}_i})$ converge to an equilibrium preference profile.

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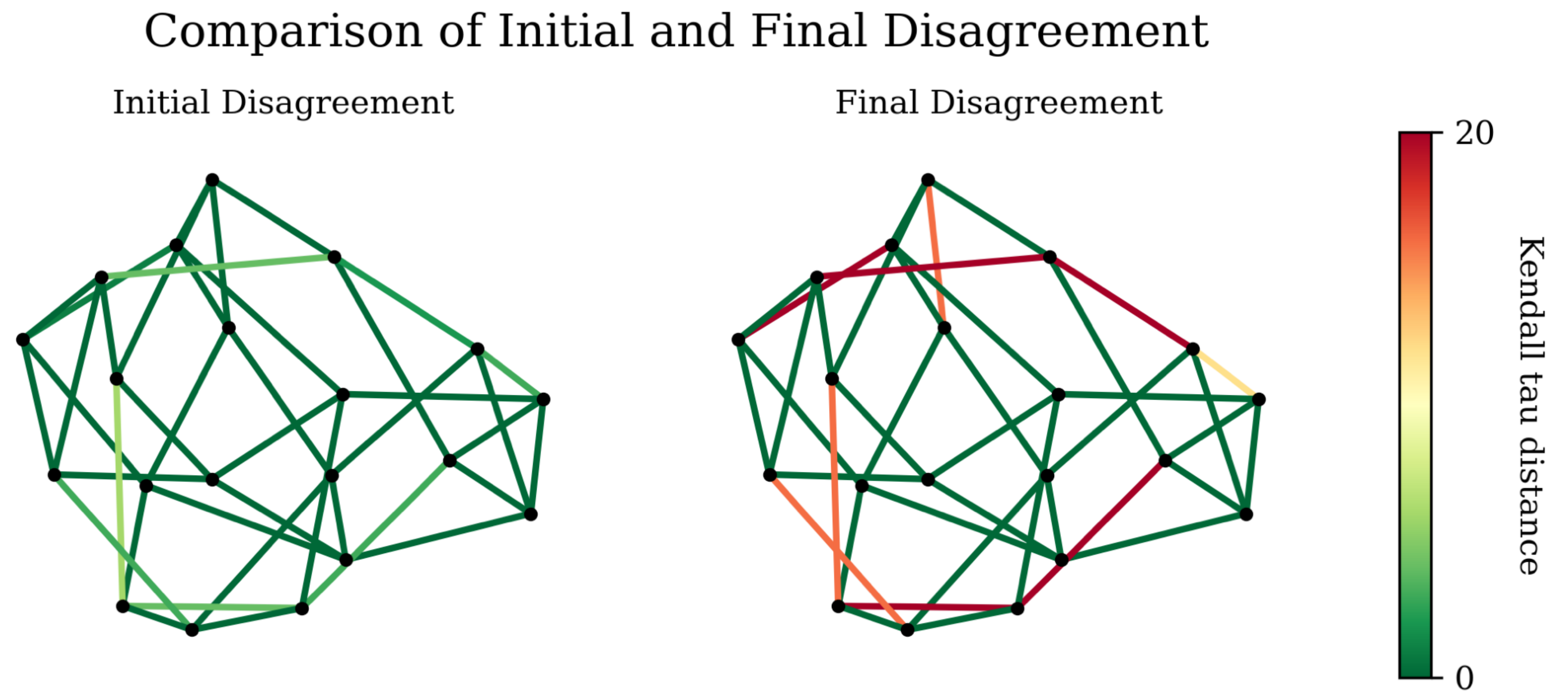
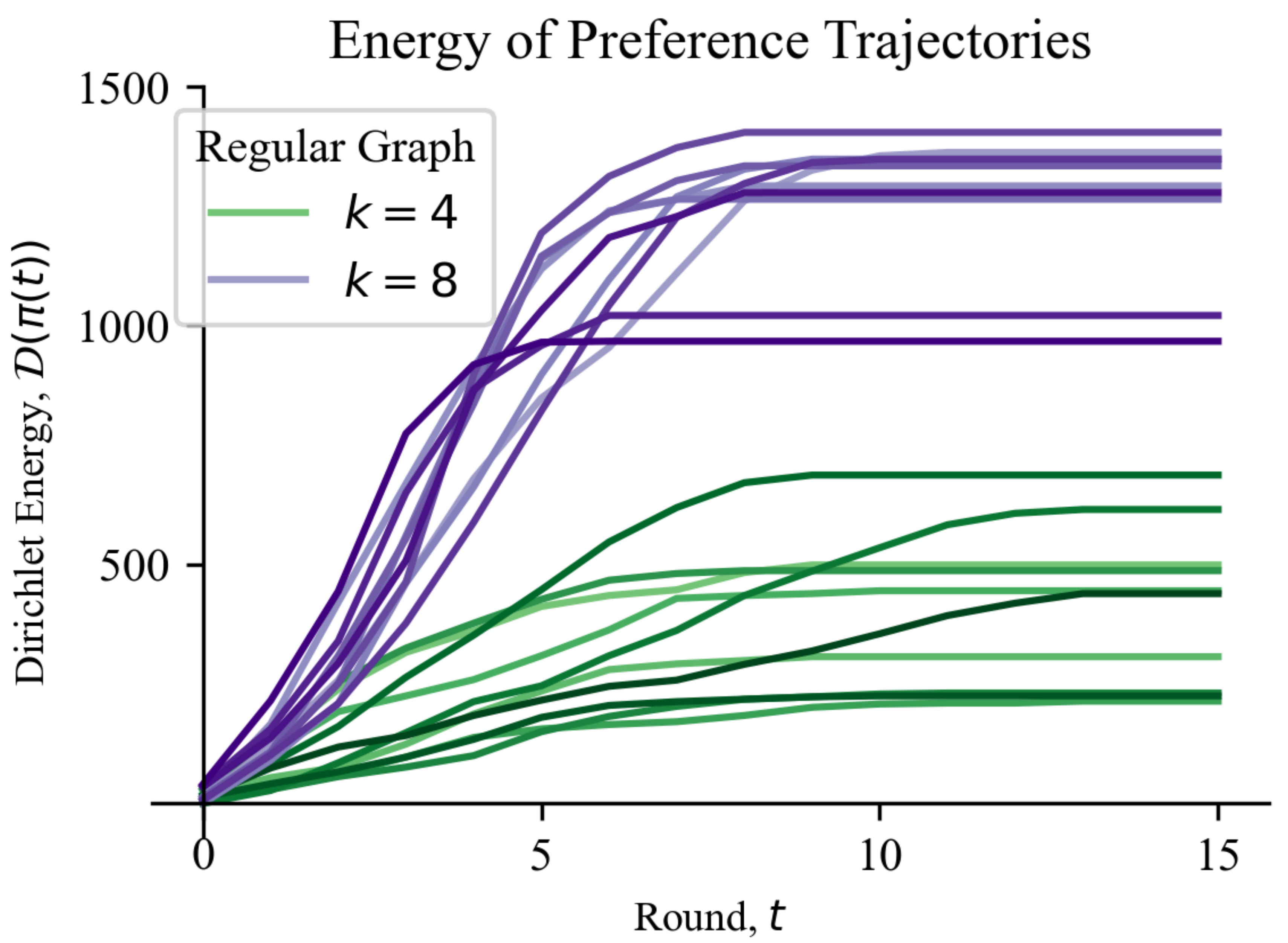
Evaluating Preference Consensus Algorithms





Kendall tau-distance: $d(\succsim_i, \succsim_j) = \#\{(a,b) : (a \succsim_i b) \,\&\, (b \succsim_j a)\}$

- $N = 20$ agents; $M = 5$ alternatives
- graph is k-regular; r chosen uniformly at random from $\{1,2,\ldots,k\}$
- φ, truthful; ψ, learning
- Initial preference relations:
  - sample pairs (a, b) with probability p (parameter)
  - discard preference relations violating transitivity

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Open Questions & Future Work

- Dynamics
  - How does the distribution of r effect the structure of the equilibria lattice? Structure of the graph?
  - when do limit cycles emerge?
- Adversarial setting (control inputs)
  - understand strategic manipulation of preferences in network setting
  - what nodes or edges are especially vulnerable?
  - design control inputs on a subset of infected nodes
- Alignment (optimal control)
  - enforce constraints on allowable preference relations
  - barrier functions

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Thanks! Questions

- Hans Riess
  - Duke University, Dept. of Electrical & Computer Engineering
  - Email: hans.riess@duke.edu
  - Website: hansriess.com
  - Twitter/X: @hansmriess
- Michael Munger
  - Duke University, Dept. of Economics
  - Email: munger@duke.edu
- Gregory Henselman-Petrusek
  - Pacific Northwest National Lab
  - Email: gregory.roek@pnnl.gov
- Robert Ghrist
  - University of Pennsylvania, Dept. of Electrical & Systems Engineering
  - Email: ghrist@seas.upenn.edu
- Zachary I. Bell
  - Air Force Research Lab, Elgin Air Force Base
  - Email: zachary.bell.10@us.af.mil
- Michael M. Zavlanos
  - Duke University, Dept. of Mechanical Engineering & Materials Science
  - Email: michael.zavlanos@duke.edu

"Network Preference Dynamics using Lattice Theory"

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Hans Riess[1] Gregory Henselman-Petrusek[2] Michael C. Munger[3] Robert Ghrist[4]
Zachary I. Bell[5] Michael M. Zavlanos[1,6]

***Abstract*—Preferences, fundamental in all forms of strategic behavior and collective decision-making, in their raw form, are an abstract ordering on a set of alternatives. Agents, we assume, revise their preferences as they gain more information about other agents. Exploiting the ordered algebraic structure of preferences, we introduce a message-passing algorithm for heterogeneous agents distributed over a network to update their preferences based on aggregations of the preferences of their neighbors in a graph. We demonstrate the existence of equilibrium points of the resulting global dynamical system of local preference updates and provide a sufficient condition for trajectories to converge to equilibria: stable preferences. Finally, we present numerical simulations demonstrating our preliminary results.**

I. INTRODUCTION

When agents or teams of agents in multi-agent systems are responsible for unilateral decision making, a number of feasible options are ranked based on independent information sources as well as the biases of agents. When there is not enough information or time to adequately compare all feasible options, a decision dilemma ensues [1]. In this paper, we design a decentralized mechanism for multi-agent systems to locally share preferences in order to augment each agent's independent decision-making capabilities and, when possible, avoid operational paralysis. As our main contribution, we introduce and offer a qualified solution to the *preference dynamics problem* which asks how preferences of agents evolve as they gather information and interact with other agents. This challenge is distinguished from the *social choice problem*, to design (centralized) preference-aggregation mechanisms that satisfy criteria such as fairness or resistance to strategic manipulation (for instance, see [2]).

In their traditional use in economics and social science, preferences are commonly associated with subjective taste. Here, we take the liberty to extend the notion of preference to arbitrary decision-making contexts, ranging from reinforcement learning (RL) [3] to games [4] to language models [5]. Formally, a *preference relation* is a relative ordering on a set of *alternatives* (i.e. possible choices or options). While preference relations represented by utility functions[1](which is in itself a strong assumption since even complete preferences can be unrepresentable [6]) come with a slew of analytical techniques and computational algorithms, e.g. a Nash equilibrium, raw preference relations have no such exploitable analytical structure. To analyze raw preference relations, in this paper we resort to algebraic techniques, rather than analytic, posing a new set of challenges. However, our primary motivation for avoiding utility theory is that time-variant utility functions cannot capture preference formation since cardinal or ordinal utilities necessarily compare every alternative. Moreover, the distinction between indifference and indecisiveness, i.e., the agent has not made up its mind or faced a choice [7], is captured by incomplete preference relations and not utility functions, although extended preferences are another approach here [8].

We propose an entirely new methodology that relies on imposing an information order on preference relations (Section III) to reason about incomplete preferences which, also, is capable of modeling preference change (Section IV). Mechanisms of preference change include revision, contradiction, as well as addition and subtraction of alternatives [9]. Our message-passing model of preference dynamics, based on the algebraic lattice structure of preference relations, incorporates elements of both revision (the join operation) and contradictions (the meet operation). Avoiding the philosophical question as to whether preference can in fact change (see [10]), our preference dynamics model can be viewed as a process for how preferences are formed in the first place as new information is revealed. The "true" preferences of agents, then, are, perhaps, those that are stable: they are equilibrium points in a preference dynamical system. We characterize the stable preferences and discuss sufficient conditions for when they are computable (Section V).

Closely related to the preference dynamics problem considered here are recent efforts on consensus [11] and synchronization [12] using lattice orders. Other works have used lattice theory to model preferences [13], although the lattice structure they introduce is different from ours. Several authors have studied aggregation or consensus on lattices [14], [15], and there is a corpus of literature addressing the design of preference-aggregation mechanisms, originating with [16], but, to our knowledge, none of these works have approached the problem of preference dynamics from either a decentralized or algebraic point-of-view, both of which we do here. The several efforts to formalize classical consensus

[1]Dept. ECE, Duke University, Durham, NC, USA.
[2]Pacific Northwset National Lab, Richland, WA, USA.
[3]Dept. Political Science, Duke University, Durham, NC, USA.
[4]Dept. ESE, University of Penn., Philadelphia, PA, USA.
[5]Air Force Research Laboratory, Eglin AFB, FL, USA.
[6]Dept. Mech. Eng. & Material Sci, Duke University, Durham, NC, USA.
Corresponding author email: hans.riess@duke.edu.

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[1]A preference relation $\succsim$ is representable by a utility function u if $u(a) \geqslant u(b)$ if and only if $a \succsim b$.

Paper: https://hansriess.com/papes/acc2024.pdf

Slides: https://hansriess.com/slides/acc2024.pdf

Code: https://github.com/hans-riess/prefdynamo/