Categories & Sheaves for Optimization:

From Multi-Stage to Distributed

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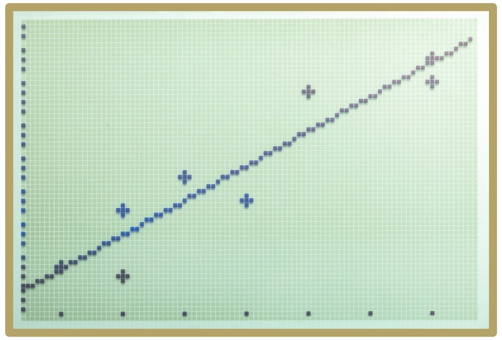
Motivation

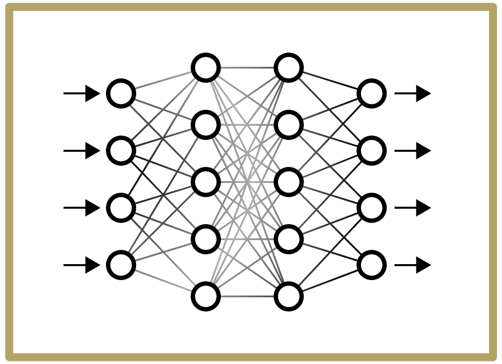
Optimization Techniques in Data Science

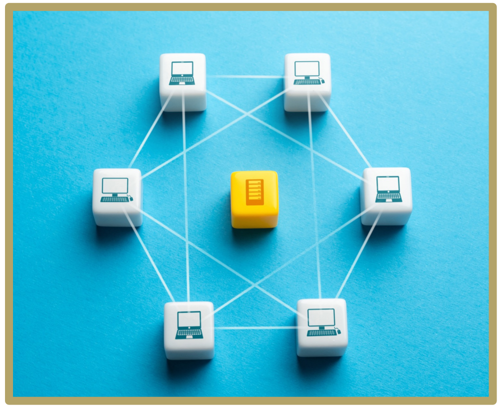
Convex optimization
- Linear regression
- Logistic regression
- SVM (soft margin)

Nonconvex optimization
- k-means
- Deep learning

Distributed optimization
- Large-scale machine learning (distributed data stores)
- Recommendation systems (e.g. Netflix)
- Federated learning (mobile devices)







Optimization is the backbone of data science.

Optimization Techniques in Robotics & Control

Convex optimization

- Linear MPC for trajectory tracking

Nonconvex optimization

- Nonlinear MPC for trajectory tracking with obstacle avoidance
- Simultaneous localization and mapping (SLAM)

Distributed optimization

- Multi-agent MPC (local MPC subject to coupling constraints)

Mathematical Framework

Mathematical themes in optimization

- Compositionality
- Distributed/parallel computing

Compositionality → enriched categories

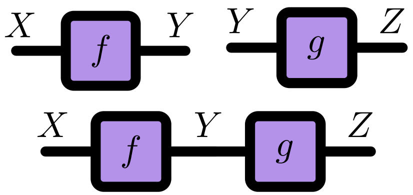
Distributive computing → cellular sheaves

Enriched categories

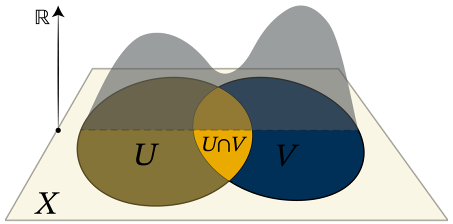
- Compositionality
- Numerical analysis

Cellular sheaves

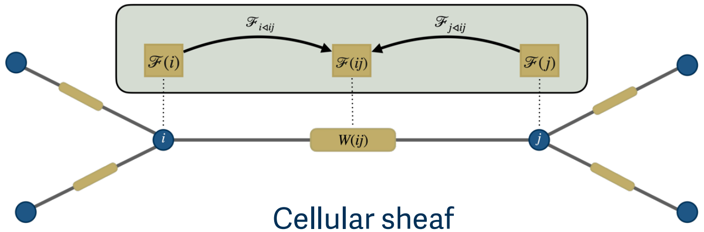
- Sheaf theory describes local-to-global behavior
  - Can function on a space be defined locally by “patches”?
  - Can a problem be solved by “stitching together” solutions to subproblems?
- Sheaf Laplacians
  - Needed computational backbone for sheaves
  - Very new ideas (Hansen 2019, Riess 2022) but old origins



Composition



Classical sheaf



Cellular sheaf

Roadmap

- Part 1: Multi-stage Optimization
  - Enriched category theory
  - Linear model predictive control (MPC)
  - Nonlinear model predictive control (MPC)
- Part 2: Distributed Optimization
  - Cellular sheaves & sheaf Laplacians
  - Multi-Agent Coordination with homological programming
- Part 3: Future Directions

We will explore optimization techniques from the perspectives of enriched category theory and cellular sheaves.

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Part 1:
Multi-Stage Optimization

Joint work by Tyler Hanks, Baike She, Evan Patterson, Matthew Klawon, James Fairbanks & Matthew Hale

Linear Model Predictive Control (MPC)

- Stage objective (convex)
- Stage constraints (convex)
- Dynamics (affine)

- Variables
  - states, x_t
  - control Inputs, u_t
  - time horizon, T
  - current time, t_0

Example: LQR

- $\ell(x_t, u_t) = x_t^{\mathsf{T}} Q x_t + u_t^{\mathsf{T}} R u_t$
- $f(x_t, u_t) = A x_t + B u_t$
- $u_{min} \leq u_t \leq u_{max}$

$$
\begin{array}{lll}
\text{minimize} & \sum_{t=t_0}^{t_0+T-1} \ell(x_t, u_t) & \\
\text{subject to} & x_{t+1} = f(x_t, u_t), & t = t_0, \ldots, t_0 + T - 1, \\
& g(x_t, u_t) \leq 0, & t = t_0, \ldots, t_0 + T - 1,
\end{array}
$$

MPC used in chemical engineering since the 1980s

Convex Bifunction Composition

Convex program ↦ convex bifunction

$$\begin{array}{lll} \text{minimize} & f_0(\mathbf{x}) & \\ \text{subject to} & f_i(\mathbf{x}) \leq y_i, & i = 1, \ldots, r \\ & f_i(\mathbf{x}) = y_i, & i = r+1, \ldots, m \end{array} \quad \mapsto \quad \begin{array}{l} F(\mathbf{x}, \mathbf{y}) = \\ f_0(\mathbf{x}) + \sum_{i=1}^{r} \mathbb{1}_{\{f_i(\mathbf{x}) \leq y_i\}} + \sum_{i=r+1}^{m} \mathbb{1}_{\{f_i(\mathbf{x}) = y_i\}} \end{array}$$

Composing convex bifunctions

$$(G \circ F)(x, z) = \inf_{y \in \mathcal{B}} F(x, y) + G(y, z)$$

Example

- Let $F \colon \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be the convex bifunction $F(x, y) = x^\top Q x + 1_{\{Ax=y\}}$
- Then, $(F \circ F)(x, z) = \inf_{y \in \mathbb{R}^n} x^\top Q x + 1_{\{Ax=y\}} + y^\top Q y + 1_{\{Ay=z\}}$ which corresponds to the program

$$\begin{array}{ll} \underset{x, y \in \mathbb{R}^n}{\text{minimize}} & x^\top Q x + y^\top Q y \\ \text{subject to} & y = Ax, z = Ay. \end{array}$$

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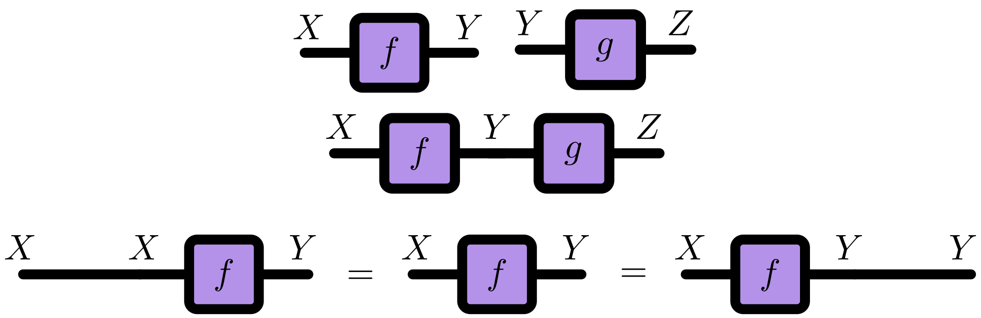
Categories

A category $\mathcal{C}$ is a collection of

- objects denoted $obj(\mathcal{C})$
- set of morphisms $\mathcal{C}(X,Y)$ for every pair of objects $X, Y \in obj(\mathcal{C})$
  - composition: $\circ: \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$
  - identity: $1_x \in \mathcal{C}(X,X)$



Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories. A functor $\mathrm{F}: \mathcal{C} \to \mathcal{D}$ is a map sending

- objects of $\mathcal{C}$ to objects of $\mathcal{D}$
- morphisms $X \xrightarrow{f} Y$ to morphism $F(X) \xrightarrow{F(f)} F(Y)$

such that

- $F(g \circ f) = F(g) \circ F(f)$
- $F(1_X) = 1_{F(X)}$

Symmetric Monoidal Categories

A symmetric monoidal category $(\mathcal{V}, \otimes, I)$ is a category equipped with

- a monoidal product, $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$
- identity object, $I \in obj(\mathcal{V})$

(Notation: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and $\mathbb{B} = \{\top, \bot\}$)

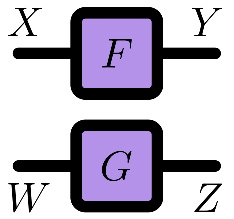
Suppose $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories. A functor $F : \mathcal{V} \to \mathcal{W}$ is monoidal if $F(X \otimes Y) \simeq F(X) \otimes F(Y)$

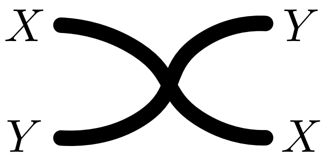
Examples of monoidal categories

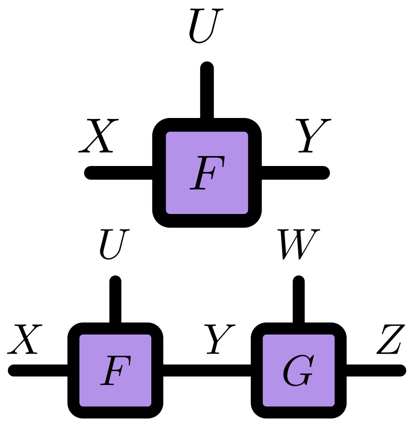
- $\mathcal{V} = (\mathrm{Set}, \times, \{*\})$
- $\mathcal{V} = (\mathbb{R}\mathrm{Vect}, \otimes, \mathbb{R})$
- $\mathcal{V} = (\overline{\mathbb{R}}, +, -\infty)$
- $\mathcal{V} = (\mathbb{B}, \wedge, \top)$

Suppose $(\mathcal{V}, \otimes, I)$ is a symmetric monoidal category. Let $\mathrm{Para}(\mathcal{V})$ be the category with

- objects, $obj(\mathrm{Para}(\mathcal{V})) = obj(\mathcal{V})$
- morphisms, pair (P, f) with object $P \in obj(\mathcal{V})$ & $\mathcal{V}$-morphism $f : P \otimes X \to Y$







Enriched Categories

Given a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, a $\mathcal{V}$-category $\mathcal{C}$ consists of the following data:

- a collection of objects $\mathrm{obj}(\mathcal{C})$
- object $\mathcal{C}(x, y) \in obj(\mathcal{V})$ for every $x, y \in obj(\mathcal{C})$
- morphism
  $$\circ_{x,y,z}: \mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$
  for every $x, y, z \in obj(\mathcal{C})$
- Morphism $I \to \mathcal{C}(x, x)$ for every $\mathrm{c} \in obj(\mathcal{C})$

subject to coherence conditions.

A $\mathcal{V}$-functor $F: \mathcal{C} \to \mathcal{D}$ is a

- function $obj(F): obj(\mathcal{C}) \to obj(\mathcal{D})$
- $\mathcal{V}$-morphism $F_{x,y}: \mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$ for every $x, y \in obj(\mathcal{C})$

Examples of enriched categories

- a Set-category = category
- a $\overline{\mathbb{R}}$-category ≈ metric space
- a $\mathbb{B}$-category = preorder
- one-object $\mathbb{R}$Vect-category = $\mathbb{R}$-algebra

Suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are $\mathcal{V}$-categories. A $\mathcal{V}$-profunctor is a $\mathcal{V}$-functor
$$F: \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V}$$

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Enriched Category Theory Perspective

Convex bifunctions

$$
\begin{array}{lll}
\text{minimize} & f_0(\mathbf{x}) & \\
\text{subject to} & f_i(\mathbf{x}) \le y_i, & i = 1, \ldots, r \\
 & f_i(\mathbf{x}) = y_i, & i = r+1, \ldots, m
\end{array}
\quad \mapsto \quad
\begin{array}{l}
F(\mathbf{x}, \mathbf{y}) = \\
f_0(\mathbf{x}) + \sum_{i=1}^{r} \mathbb{1}_{\{f_i(\mathbf{x}) \le y_i\}} + \sum_{i=r+1}^{m} \mathbb{1}_{\{f_i(\mathbf{x}) = y_i\}}
\end{array}
$$

Monoidal category of $\mathcal{V}$-categories and $\mathcal{V}$-profunctors, Dist($\mathcal{V}$)

- Given $F\colon \mathcal{A}^{op} \times \mathcal{B} \to \mathcal{V}$ and $G\colon \mathcal{B}^{op} \times \mathcal{C} \to \mathcal{V}$, define the composition of profunctors

$$(G \circ F)(x, z) = \int^{y \in \mathcal{B}} F(x, y) \otimes G(y, z)$$

- Let $\mathcal{A} \mathbin{\overline{\otimes}} \mathcal{B}$ be the $\mathcal{V}$-category with
    - objects $obj(\mathcal{A} \mathbin{\overline{\otimes}} \mathcal{B}) = obj(\mathcal{A}) \times obj(\mathcal{B})$
    - morphisms

$$\mathcal{A} \mathbin{\overline{\otimes}} \mathcal{B}\big((x_1, y_1), (x_2, y_2)\big) = \mathcal{A}(x_1, x_2) \otimes \mathcal{B}(y_1, y_2)$$

Subcategory Conv ⊆ Dist($\overline{\mathbb{R}}$)

- Objects are discrete $\overline{\mathbb{R}}$-categories with object sets = Euclidean spaces
- $\mathcal{E}(\mathcal{A}, \mathcal{B})$ is the set of convex bifunctions between $\mathbb{R}^m = obj(\mathcal{A})$ and $\mathbb{R}^m = obj(\mathcal{B})$
- Parallel composition is given by

$$(F_1 \bar{\otimes} F_2)\big((x_1, y_1), (x_2, y_2)\big) = F_1(x_1, y_1) + F_2(x_2, y_2)$$

- Series composition is given by

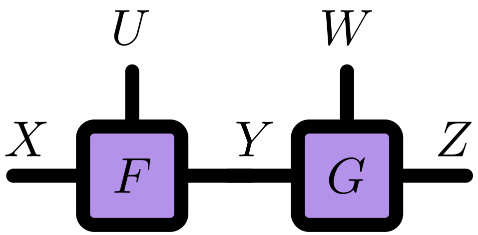
$$(G \circ F)(x, z) = \inf_{y \in \mathcal{B}} F(x, y) + G(y, z)$$

Modeling MPC with Para(Conv)

Consider the category Para(Conv)

$$U \times X \nrightarrow Y$$

$$(u, w, x, z) \mapsto \inf_{y \in Y} \Big(F(u, x, y) + G(w, y, z) \Big)$$

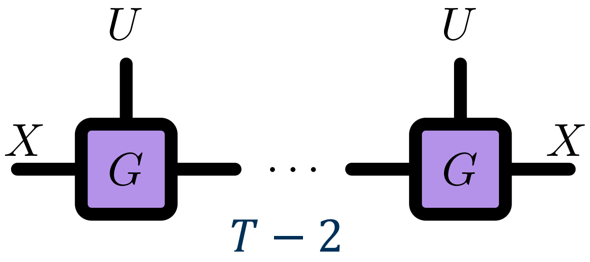


$$x_0 \mapsto \begin{array}{ll} \text{minimize} & \ell(x_0, u_0) \\ \text{subject to} & \boxed{x_1} = f(x_0, u_0), \\ & g(x_0, u_0) \leq 0. \end{array} \qquad \boxed{x_1} \mapsto \begin{array}{ll} \text{minimize} & \ell(x_1, u_1) \\ \text{subject to} & x_2 = f(x_1, u_1), \\ & g(x_1, u_1) \leq 0. \end{array}$$

$\Big\downarrow$ Compose

$$x_0 \mapsto \begin{array}{ll} \text{minimize} & \ell(x_0, u_0) + \ell(x_1, u_1) \\ \text{subject to} & x_1 = f(x_0, u_0), \\ & x_2 = f(x_1, u_1) \\ & g(x_0, u_0) \leq 0 \\ & g(x_1, u_1) \leq 0. \end{array}$$

Modeling MPC with Para(Conv)



$$
\begin{array}{ll}
\text{minimize} & \sum_{t=t_0}^{t_0+T-1} \ell(x_t, u_t) \\
\text{subject to} & x_{t+1} = f(x_t, u_t), \quad t = t_0, \ldots, t_0 + T - 1, \\
& g(x_t, u_t) \leq 0, \qquad\quad t = t_0, \ldots, t_0 + T - 1,
\end{array}
$$

Implementation

Julia library

- AlgebraicControl.jl
- automated creation, modification, and solution of multistage optimal control problems

```
A, B = [0 1; 0.01 0], [0; 1]
Q, R = 2.0*I(2), 3.0
x₀ = [3.0, 1.0]

cost(uₖ, xₖ) = xₖ'*Q*xₖ + uₖ'*R*uₖ
dynamics(uₖ, xₖ) = A*xₖ + B*uₖ
constraints(uₖ,xₖ) = [
    uₖ ≤ 1, uₖ ≥ -1,
    xₖ[1] ≤ 3, xₖ[1] ≥ -3,
    xₖ[2] ≤ 2, xₖ[2] ≥ -2
]

one_step = one_step_bifunction(cost, constraints, dynamics)

N = 10 # prediction horizon
MPC_bifunc = compose(repeat([one_step], N))

us = [Variable(1) for i in 1:N-1]
x_N = Variable(2)

MPC_prob = to_cvx(MPC_bifunc, us, x₀, x_N)

solve!(MPC_prob, SCS.Optimizer)
```

Nonlinear Model Predictive Control (MPC)

- Stage objective (nonconvex)
- Stage constraints (nonconvex)
- Dynamics (nonaffine)

- Variables
  - states, x_t
  - control Inputs, u_t
  - time horizon, T
  - current time, t_0

$$\begin{array}{lll} \text{minimize} & \sum_{t=t_0}^{t_0+T-1} \ell(x_t, u_t) & \\ \text{subject to} & x_{t+1} = f(x_t, u_t), & t = t_0, \ldots, t_0 + T - 1, \\ & g(x_t, u_t) \leq 0, & t = t_0, \ldots, t_0 + T - 1, \end{array}$$

MPC used in chemical engineering since the 1980s

Categorical Model of Nonlinear MPC

Obstacles

- Cannot compose nonconvex bifunctions naively

Assumptions

- Objective functions and constraints are polynomials
  - Semi-algebraic set
  - Stratified Morse theory
- Local minima

$$\operatorname*{local\,min}_{x,x_0} f(x) = \inf_{x \in \mathcal{T}_f(x_0)} f(x)$$

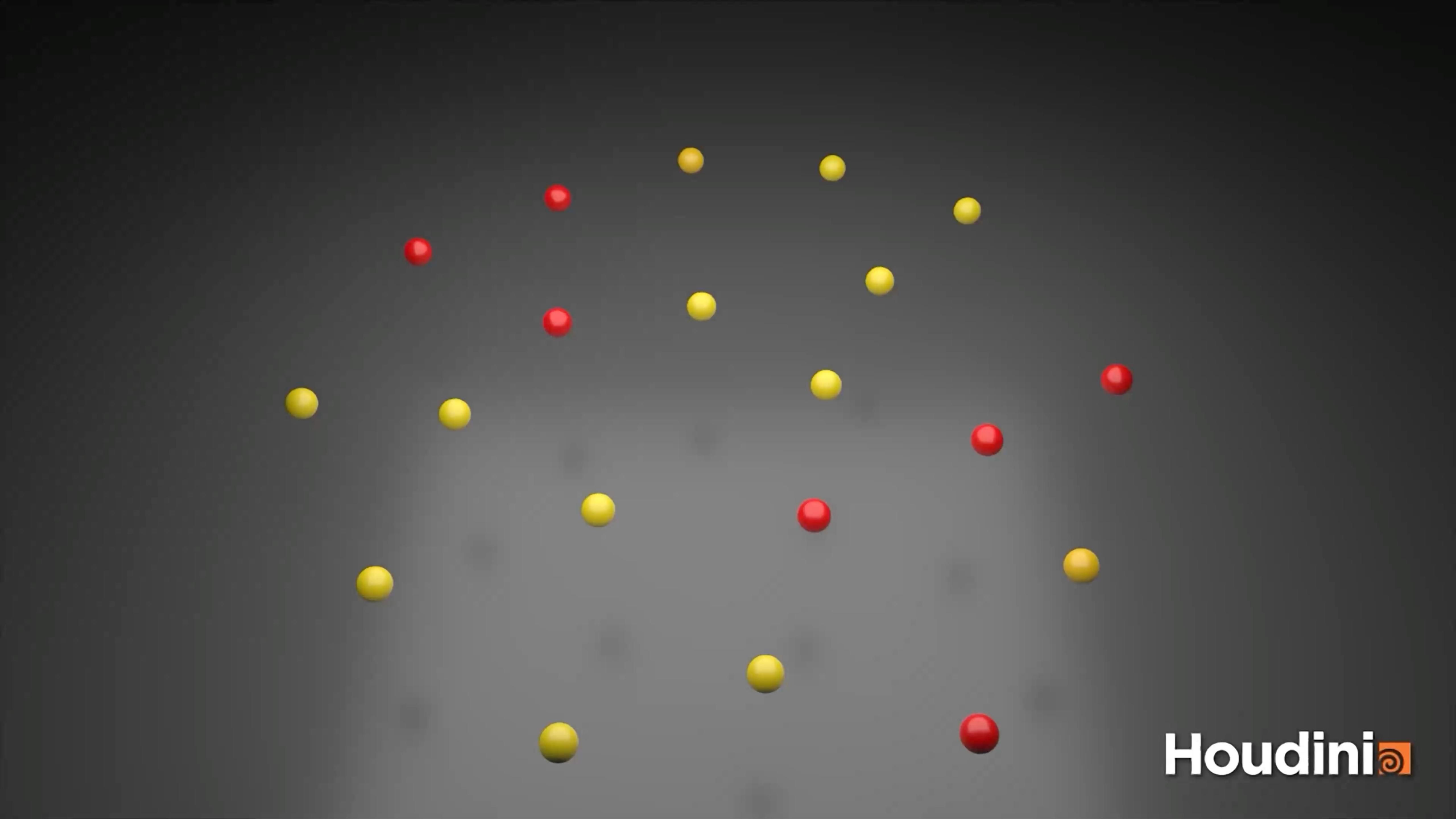
where $\mathcal{T}_f(x_0)$ sends x_0 to the basin of attraction

Define $(G \circ F)(x,z) = \inf_{y \in \mathcal{T}_{F(x,-)+G(-,z)}} F(x,y) + G(y,z)$

- Interpret as profunctor composition with a weighted limit (i.e. weighted coend)

Part 2:
Distributed Optimization

Joint work with Tyler Hanks, Samuel Cohen, Trevor Gross, Matthew Hale, & James Fairbanks.

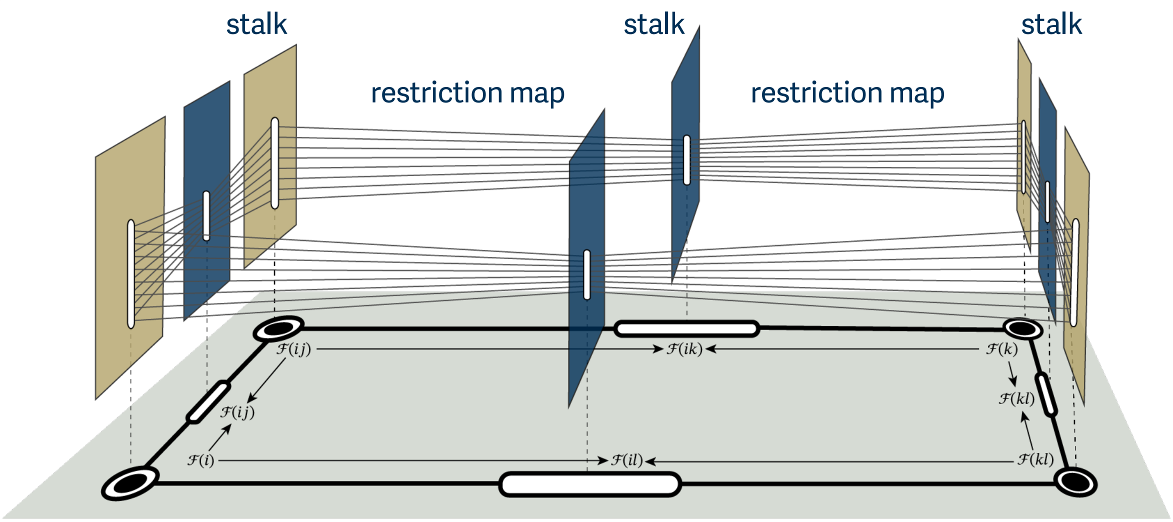

Houdini

Cellular Sheaves

A cellular sheaf is defined over an undirected graph valued in a data category $\mathcal{D}$.

In our case…

- Nodes are agents (e.g., UAVs)
- Edges model bidirectional links (e.g., sonar, RF, etc.)
- Restriction maps encode coordination constraints
- $\mathcal{D}$ is $\mathbb{R}$Vect



Assignments to a sheaf, $x \in \prod_{i\in V} \mathcal{F}(i) \coloneqq C^0(G;\mathcal{F})$

- attributes data (e.g. a state) to each node/edge of the graph
- global sections are consistent assignments to the sheaf

Sheaf Laplacians, $L: C^0(\mathcal{F};G) \to C^0(\mathcal{F};G)$

- act on assignments
- converge to global sections

Example: constant sheaf

- assign $\mathbb{R}$ to every node/edge; assign identity map to every node-edge pair
- global sections = consensus
- sheaf Laplacian = graph Laplacian

Definition. Given an undirected graph $G = (V, E)$ and a data category $\mathcal{D}$, a **cellular sheaf** is a functor that assigns

1. $\mathcal{F}(i)$ for every $i \in V$ (object in $\mathcal{D}$)
2. $\mathcal{F}(ij)$ for every $ij \in E$ (object in $\mathcal{D}$)
3. $\mathcal{F}_{i \trianglelefteq ij} : \mathcal{F}(i) \to \mathcal{F}(ij)$ for every $i \in V$, for every $j \in N_i$ (morphism in $\mathcal{D}$)

Definition. An assignment $\mathbf{x} = (x_i)_{i\in V} \in \prod_{i\in V} \mathcal{F}(i)$ is a **global section** if

$$\mathcal{F}_{i \trianglelefteq ij}(x_i) = \mathcal{F}_{j \trianglelefteq ij}(x_j), \quad \forall i \in V, \quad \forall j \in N_i$$

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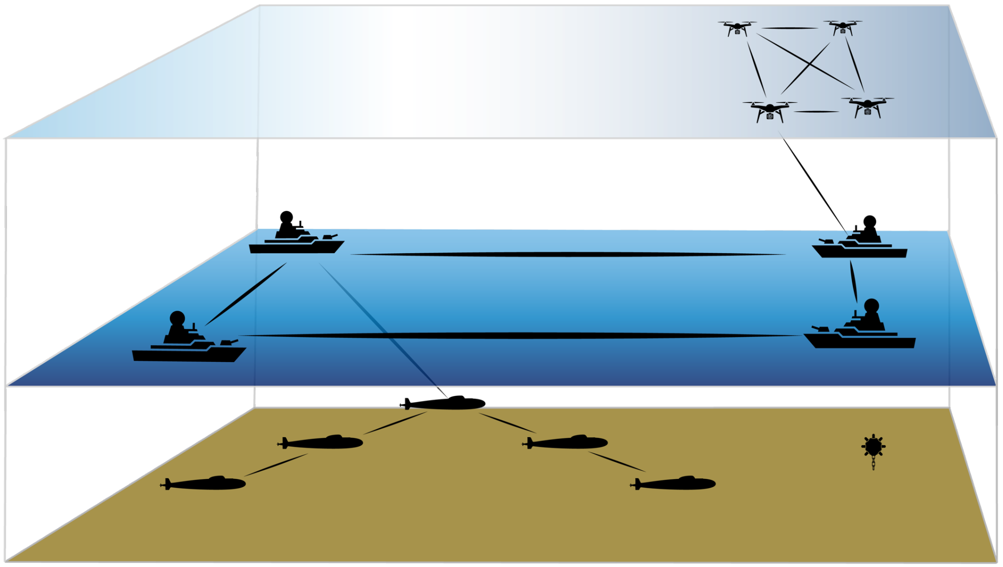
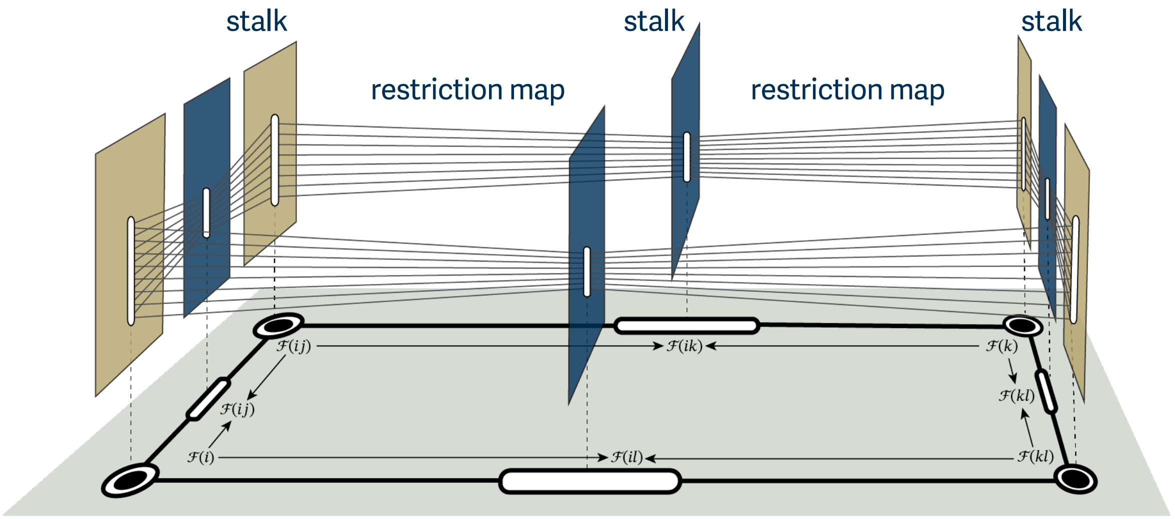
Coordination Sheaves

Sheaf Theory Concepts		
Concept	**Notation**	**Interpretation**
graph	$G = (V, E)$	bidirectional communication network
vertex stalk	$\mathcal{F}(i)$	state space of agent i
local section	x_i	local state
edge stalk	$\mathcal{F}(ij)$	communication space between agent i and j
restriction map	$\mathcal{F}_{i \trianglelefteq ij}$	how agent i sends messages to agent j
sheaf	$\mathcal{F}$	multi-agent communication structure
0-cochains	$C^0(G; \mathcal{F})$	global state space
0-cochain	$\mathbf{x}$	global state
global sections	$\Gamma(G; \mathcal{F})$	feasible global states





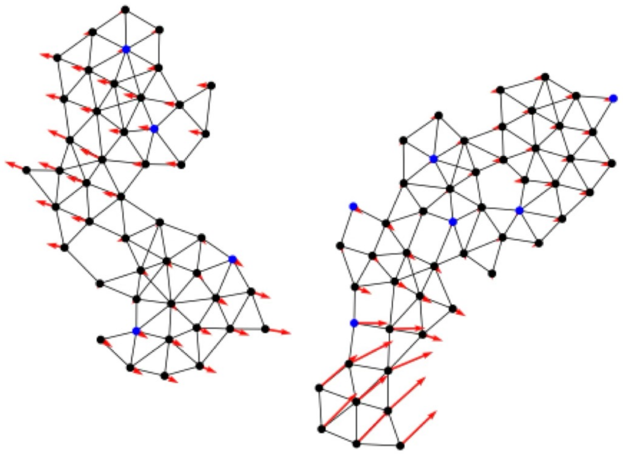
Nonlinear Sheaf Laplacian

Nonlinear sheaf Laplacain

$$(L_{\mathcal{F}}^{\nabla U}\mathbf{x})_i = \sum_{j \in N_i} \mathcal{F}_{i \trianglelefteq ij}^{\top} \left(\nabla U_{ij} \left(\mathcal{F}_{i \trianglelefteq ij}(x_i) - \mathcal{F}_{j \trianglelefteq ij}(x_j) \right) \right).$$

where $U_{ij} : \mathcal{F}(ij) \to \mathbb{R}$ are nonlinear potential functions.

Potential Functions	
$U_e(y)$	**Coordination Goal**
$(1/2)\|y\|_2^2$	Consensus
$y^{\top} A y$	Matrix-weighted consensus
$-(1/2)\|y\|_2^2$	Dissensus
$(1/2)\|y - b\|_2^2$	Reach displacement of b
$(\|y\|_2^2 - r^2)^2$	Reach distance of r



Homological Programming

Definition 4 (Nonlinear Homological Program)**.** A **nonlinear homological program** consists of the following data:

1) Undirected graph $G = (V, E)$;
2) Cellular sheaf $\mathcal{F}$ on G;
3) Objective functions $\{f_i \colon \mathcal{F}(v) \to \overline{\mathbb{R}}\}_{i \in V}$;
4) Potential functions $\{U_e \colon \mathcal{F}(e) \to \mathbb{R}\}_{e \in E}$.

$$\begin{aligned} \underset{\mathbf{x} \in C^0(G;\mathcal{F})}{\text{minimize}} \quad & \textstyle\sum_{i \in V} f_i(x_i) \\ \text{subject to} \quad & L_{\mathcal{F}}^{\nabla U} \mathbf{x} = 0 \end{aligned} \tag{P}$$

Theorem 1. *Let* $\mathsf{P} = (V, E, \mathcal{F}, \{f_i\}, \{U_e\})$ *be a nonlinear homological program. If* U_e *is differentiable and convex for each* $e \in E$*, and* f_i *is convex for every* $i \in V$*, then* P *is a convex optimization problem.*

Solving Homological Programs with ADMM

Original formulation

$$\begin{aligned} &\text{minimize} \sum_i f_i(x_i) + 1_{\mathcal{C}}(\mathbf{z}) \\ &\text{subject to } \mathbf{x} - \mathbf{z} = 0 \end{aligned}$$

$$\mathcal{C} = \{\mathbf{z} \in C^0(G;\mathcal{F}) \mid L_{\mathcal{F}}^{\nabla U}\mathbf{z} = 0\}$$

Lagrangian relaxation

$$\mathcal{L}_\rho(\mathbf{x},\mathbf{z},\mathbf{y}) = \sum_i f_i(x_i) + \chi_{\mathcal{C}}(\mathbf{z}) + (\rho/2)\|\mathbf{x} - \mathbf{z} + \mathbf{y}\|_2^2.$$

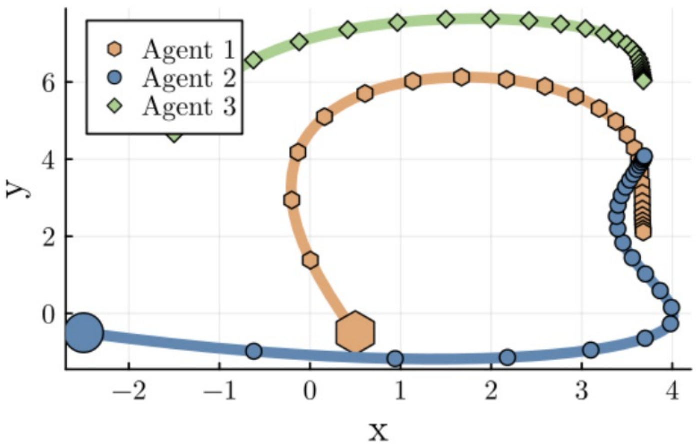
ADMM update rule

$$\begin{aligned} x_i^{k+1} &:= \text{argmin}_{x_i}\ f_i(x_i) + (\rho/2)\|x_i - z_i^k + y_i^k\|_2^2 \\ \mathbf{z}^{k+1} &:= \Pi_{\mathcal{C}}(\mathbf{x}^{k+1} + \mathbf{y}^k) \\ y_i^{k+1} &:= y_i^k + x_i^{k+1} - z_i^{k+1} \end{aligned}$$

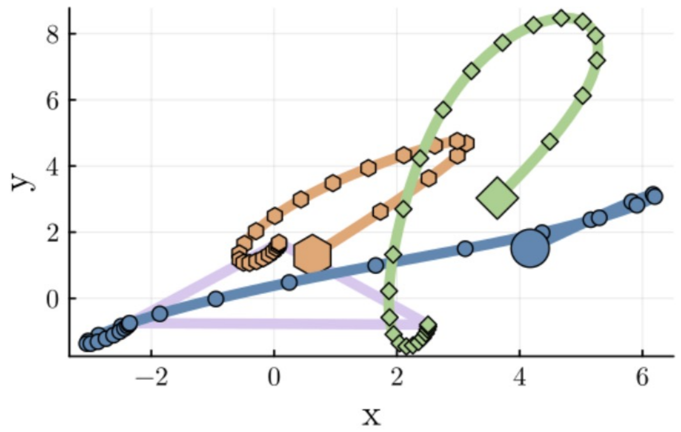
Theorem 3. *Suppose a homological program* P *satisfies assumptions 1 and 2. Assume also that the Lagrangian of Eq. 8 has a saddle point. Then Algorithm 1 applied to* P *has the following properties:*

- *Residual convergence:* $\mathbf{x}^k - \mathbf{z}^k \to 0$ *as* $k \to \infty$
- *Objective convergence:* $\sum_i f_i(x_i) + \chi_{\mathcal{C}}(\mathbf{z}) \to p^*$ *as* $k \to \infty$.
- *Dual variable convergence:* $\mathbf{y}^k \to \mathbf{y}^*$ *as* $k \to \infty$ *where* $\mathbf{y}^*$ *is a dual optimal point.*

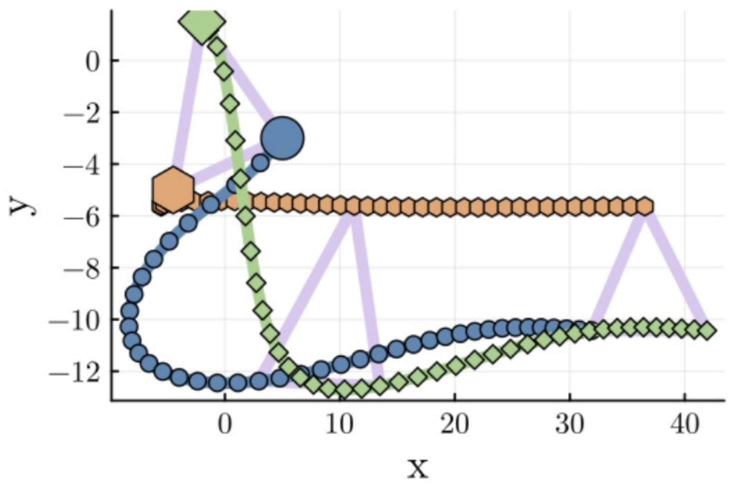
Examples



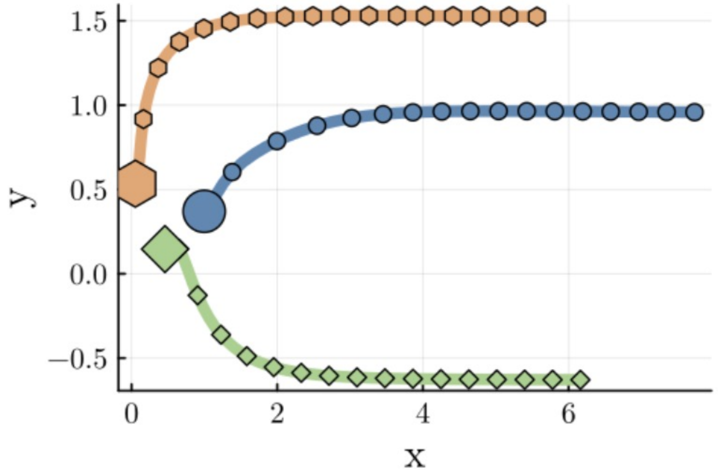
Consensus



Stationary formation



Moving formation



Flocking

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Part 3:
Future Directions

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Sheaves Valued in Category of Convex Bifunctions

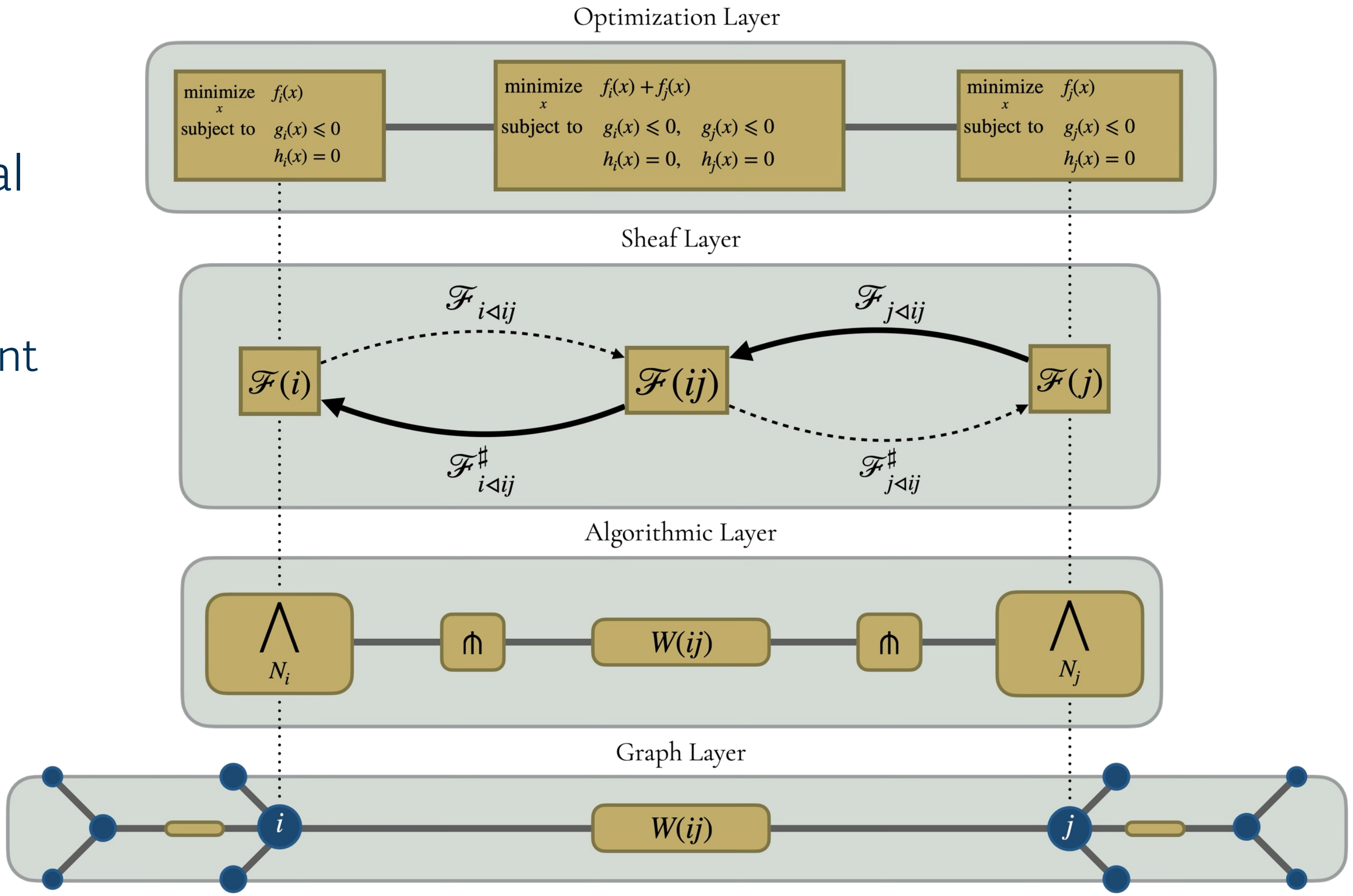
Construct sheaf over graph $G = (V, E)$

- Cellular sheaf valued in category $\mathcal{D} = \mathcal{V}\mathrm{Cat}$
  - $\mathcal{V} = \overline{\mathbb{R}}$ (agreement of convex programs)
- Weighted limit of sheaf = approximate global sections
  - Weight modulates equational satisfaction problem over each edge
  - Weight $W: E \to \mathcal{V}$ indicates level of agreement
- Kan duality

$$\mathcal{A}^{op} \times \mathcal{B} \xrightarrow{F} \mathcal{V} \qquad \longmapsto \qquad [\mathcal{A}, \mathcal{V}] \underset{\varphi^*}{\overset{\varphi_*}{\rightleftarrows}} [\mathcal{B}, \mathcal{V}]$$

Sheaf Laplacian (see [4])

- Adjoints reverse direction of the restriction maps
- Transport information from neighboring stalks
- Aggregation at nodes with weighted limits (via cotensor ⋔)
- Sheaf Laplacian computes approximate global sections



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Thank you!

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