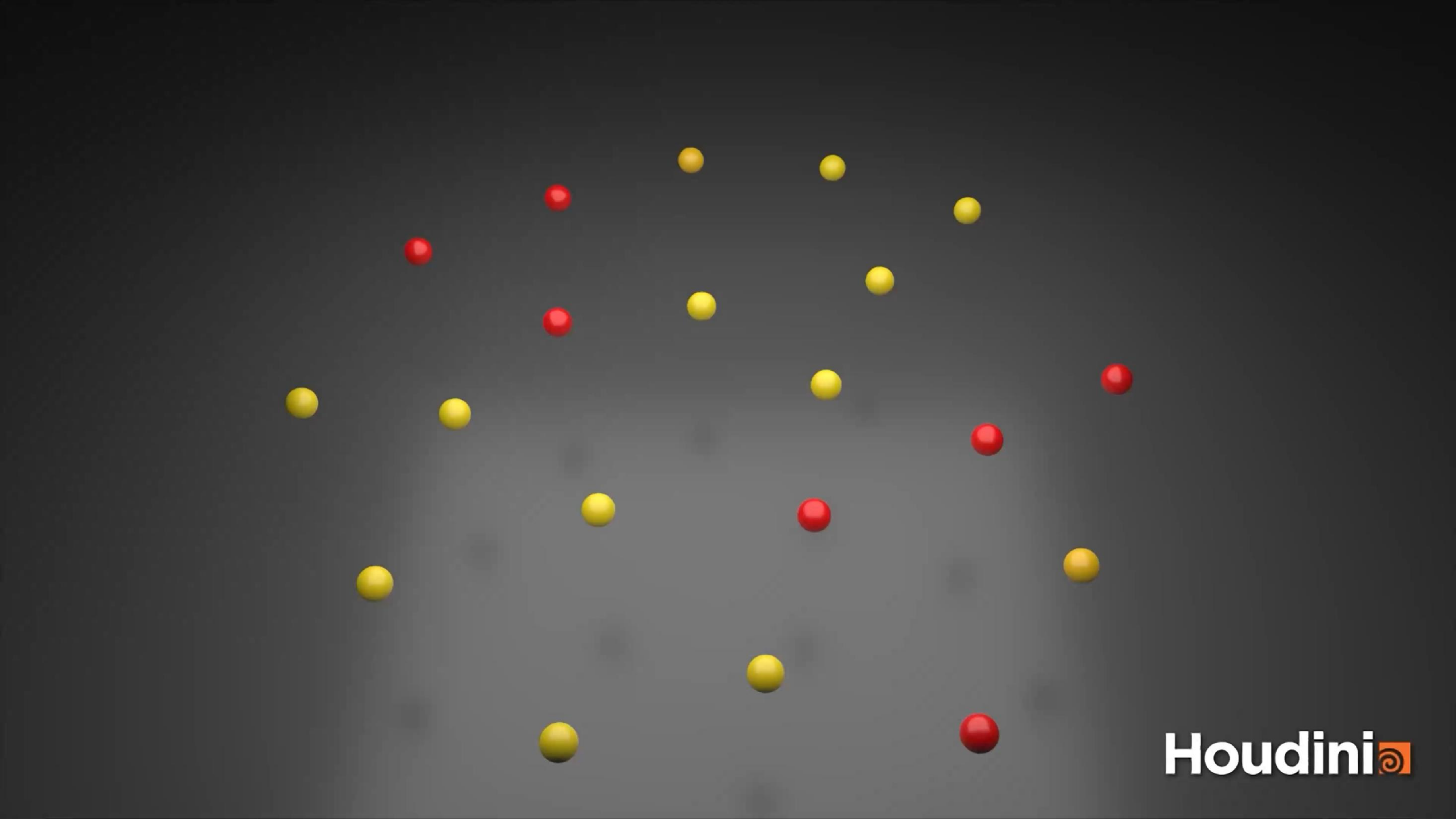
THE TARSKI LAPLACIAN & BEYOND

UF TDA CONFERENCE 2023

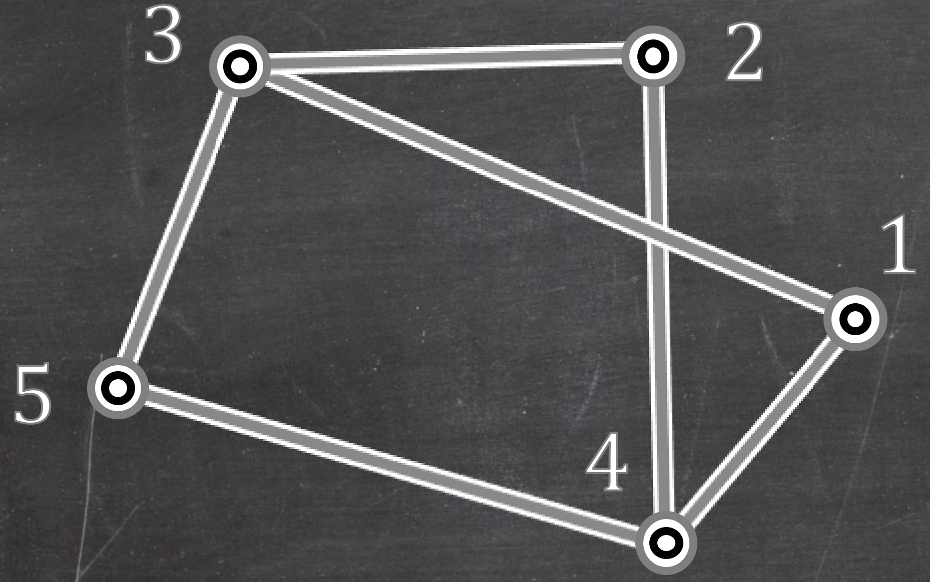
HANS RIESS
DEPARTMENT OF ELECTRICAL/COMPUTER ENGINEERING
Duke
PRATT SCHOOL *of* ENGINEERING

MOTIVATION


Houdini

THE GRAPH LAPLACIAN

$$L = D - A$$



$$L = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

THE GRAPH LAPLACIAN

SO MANY APPLICATIONS...

SPECTRAL CLUSTERING — DONATH-HUFFMAN + FIEDLER

CONNECTIVITY CONTROL — ZAVLANOS-PAPPAS

FLOCKING — TANNER-JADBABAIE-PAPPAS + OLFATI

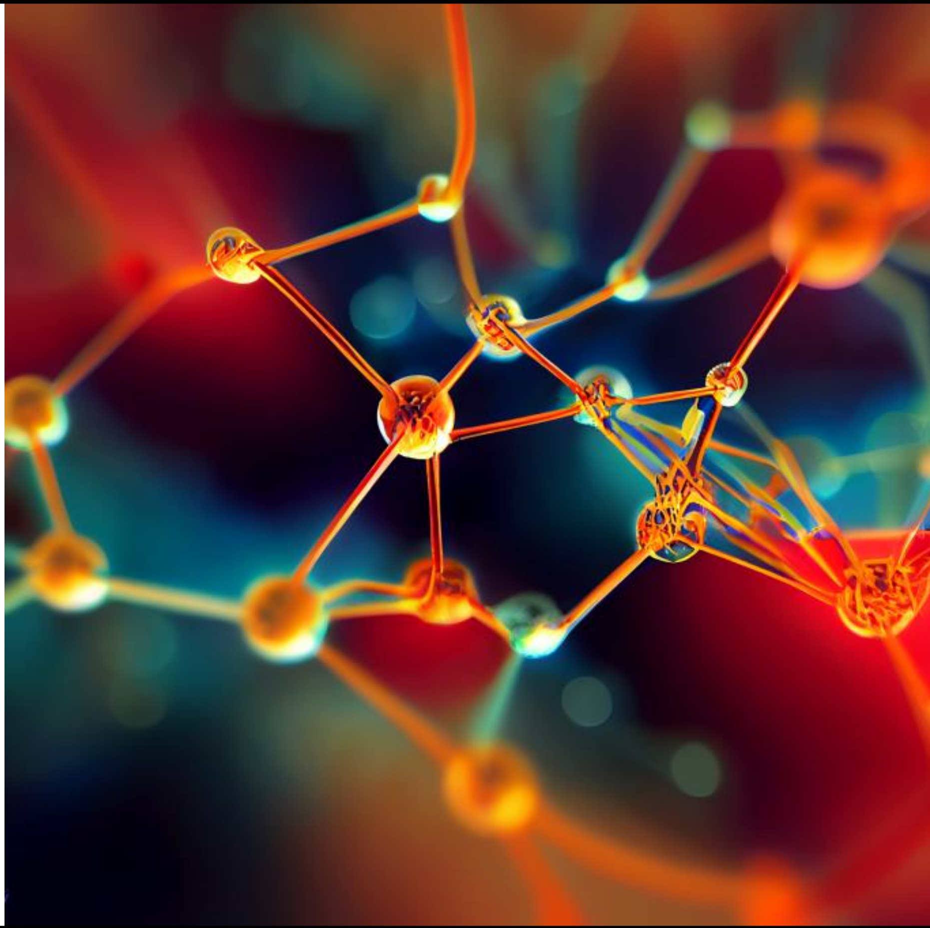
CONSENSUS/COOPERATION — OLFATI

OPINION DYNAMICS — HEGSELMANN-KRAUSS

GRAPH SIGNAL PROCESSING — SHUMAN + MANY OTHERS

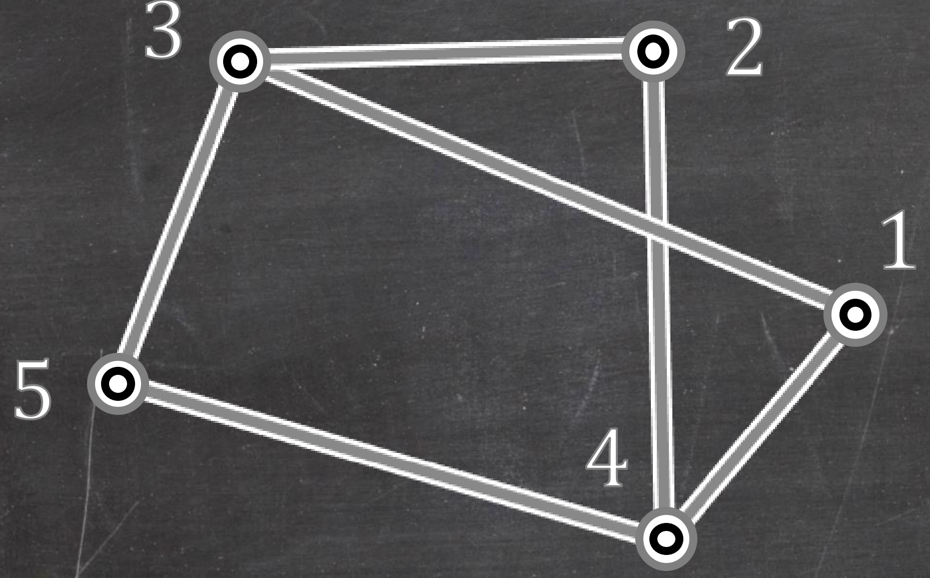
GRAPH NEURAL NETWORKS — GAMA-MARQUES-LEUS-RIBERO

SCALAR DIFFUSION



THE GRAPH CONNECTION LAPLACIAN

$$L = D_1 - A_1$$



$$L = \begin{bmatrix} 2I & & -O_{13} & -O_{14} & \\ & 2I & -O_{23} & -O_{24} & \\ -O_{13}^{\top} & -O_{23}^{\top} & 3I & & -O_{35} \\ -O_{14}^{\top} & -O_{24}^{\top} & & 3I & -O_{45} \\ & & -O_{35}^{\top} & -O_{45}^{\top} & 2I \end{bmatrix}$$

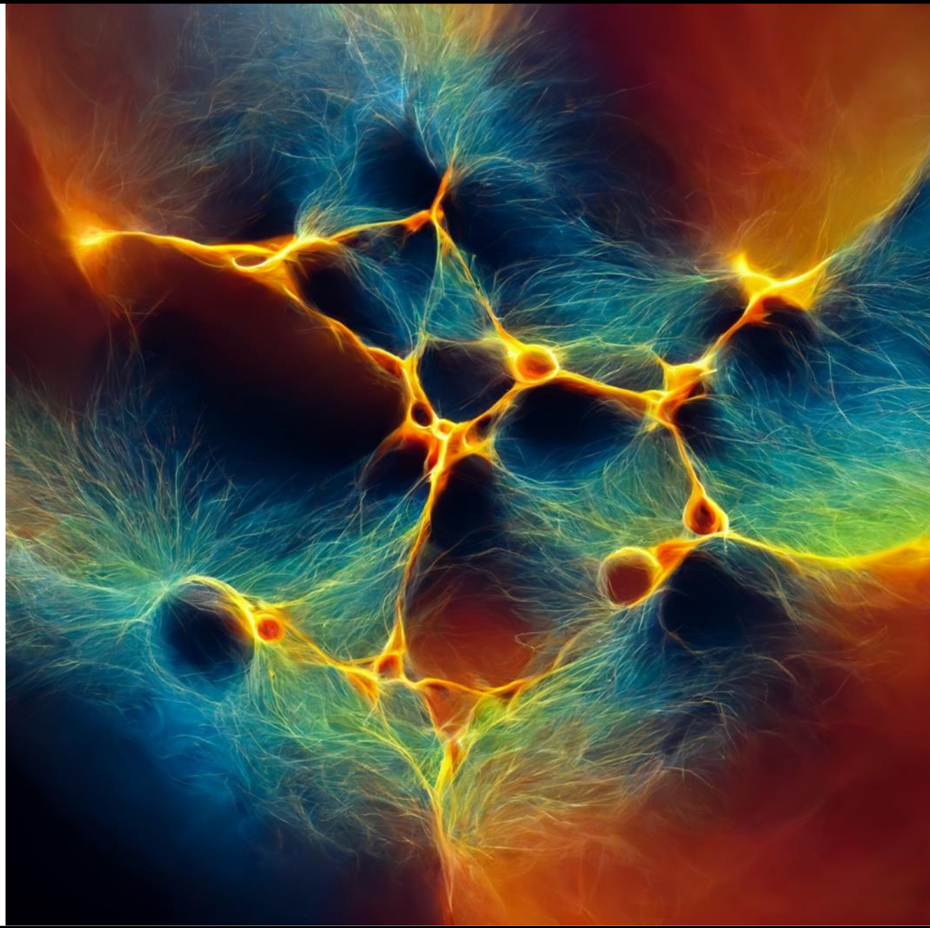
$$O_{ij}^{T} O_{ij} = O_{ij} O_{ij}^{\top} = I$$

THE GRAPH CONNECTION LAPLACIAN

SOME LESSER-KNOWN APPLICATIONS...

DIMENSIONALITY REDUCTION	SINGER-WU
GRAPH NEURAL NETWORKS	BRONSTEIN et AL
SHAPE MATCHING	OVSJANIKOV + OTHERS
SHAPE RETRIEVAL	BRONSTEIN + MANY OTHERS
SYNCHRONIZATION	SINGER + MANY OTHERS

VECTOR DIFFUSION



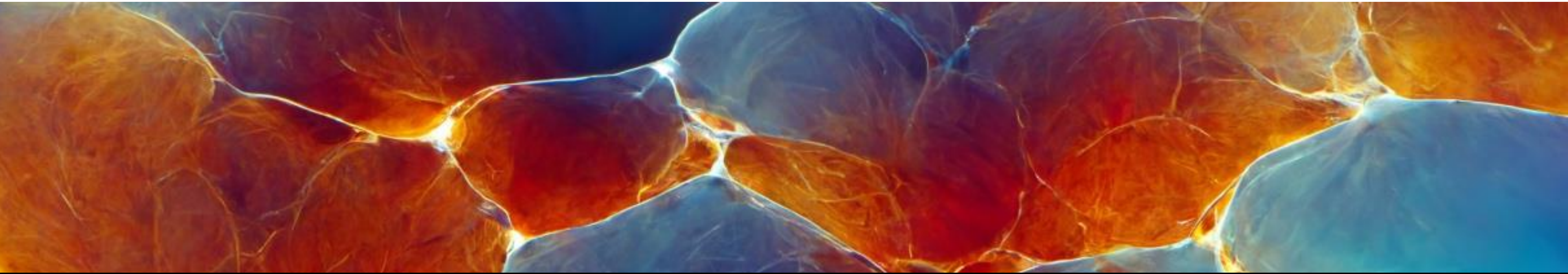
GEOMETRIC ORIGINS

Let $\mathbb{M}$ be a Riemannian manifold in $\mathbb{R}^n$...

LAPLACE-BELTRAMI OPERATOR
[Belkin-Niyogi 2008]

HODGE LAPLACIAN
(cf. de Rham's Theorem)

CONNECTION LAPLACIAN
[Singer-Wu 2012, Singer-Wu 2017]



PARALLEL TRANSPORT

Let $\mathbb{M}$ be a Riemannian manifold in $\mathbb{R}^n$...

DEFINITION:

Parallel transport along a path $\gamma: [0,1] \to \mathbb{M}$ is a map

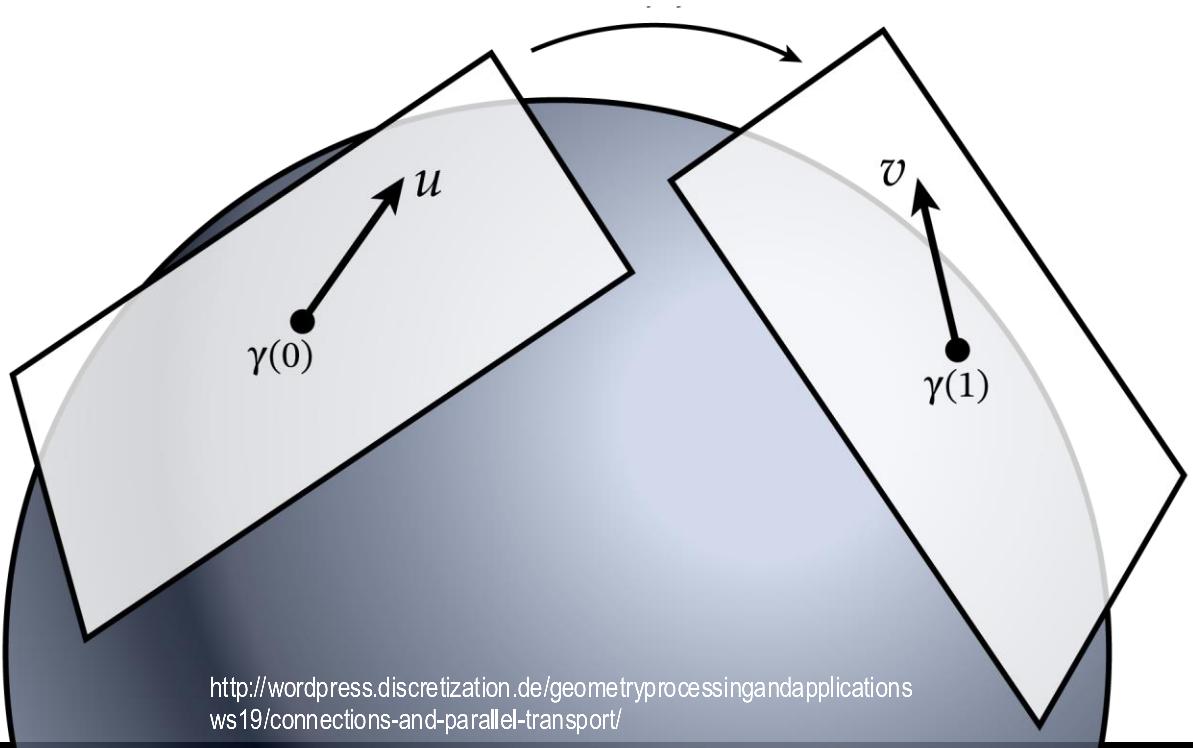
$$\mathcal{P}_\gamma : T_{\gamma(0)}\mathbb{M} \to T_{\gamma(1)}\mathbb{M}$$

satisfying the property

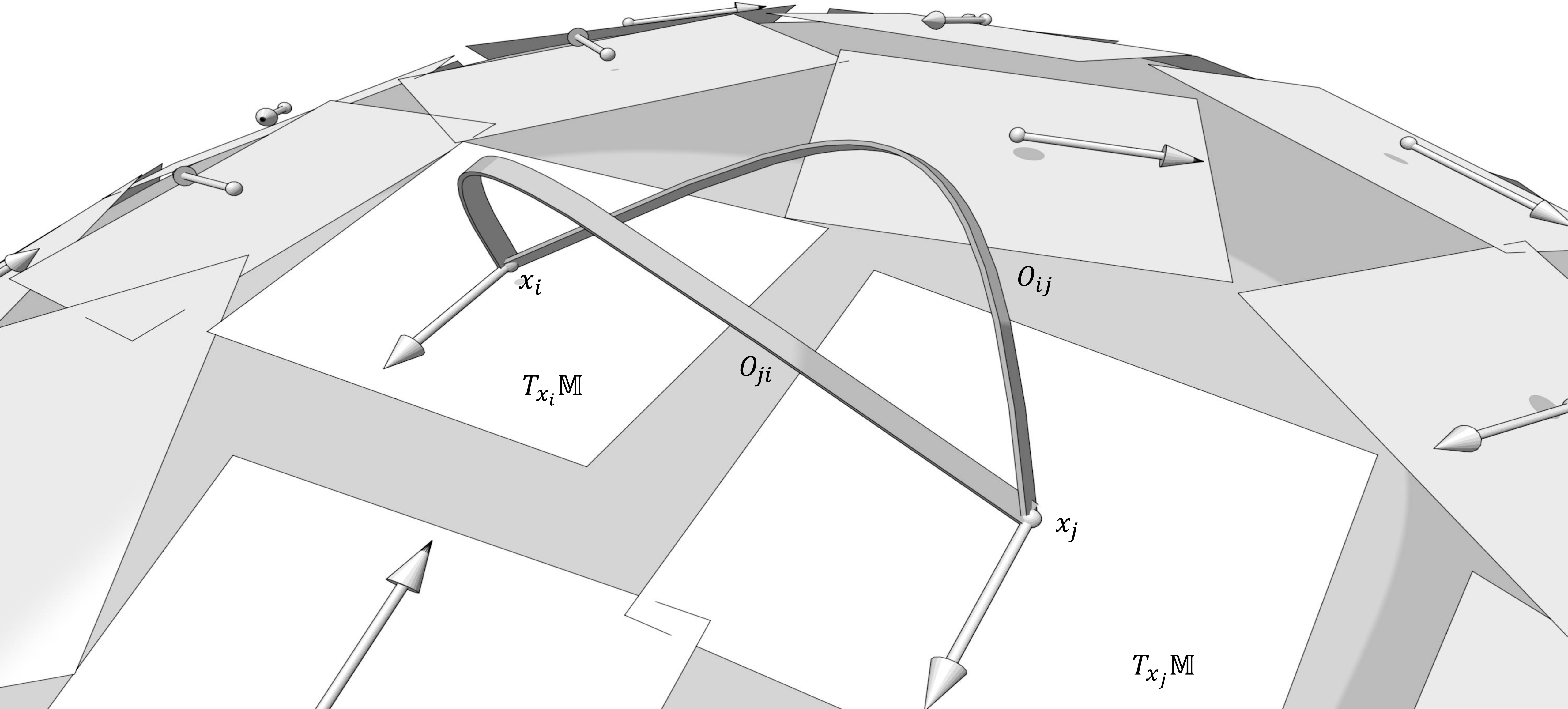
$$\langle u, v \rangle_{T_{\gamma(0)}\mathbb{M}} = \langle \mathcal{P}_\gamma u, \mathcal{P}_\gamma v \rangle_{T_{\gamma(1)}\mathbb{M}}$$

and such that for all composable paths γ_1 and γ_2,

$$\mathcal{P}_{\gamma_1 \cdot \gamma_2} = \mathcal{P}_{\gamma_2} \circ \mathcal{P}_{\gamma_1}$$



http://wordpress.discretization.de/geometryprocessingandapplications ws19/connections-and-parallel-transport/



x_i
O_{ij}
O_{ji}
$T_{x_i}\mathbb{M}$
x_j
$T_{x_j}\mathbb{M}$

BACKGROUND

LATTICES

LATTICES are PARTIALLY ORDERED SETS with a PAIR of OPERATIONS

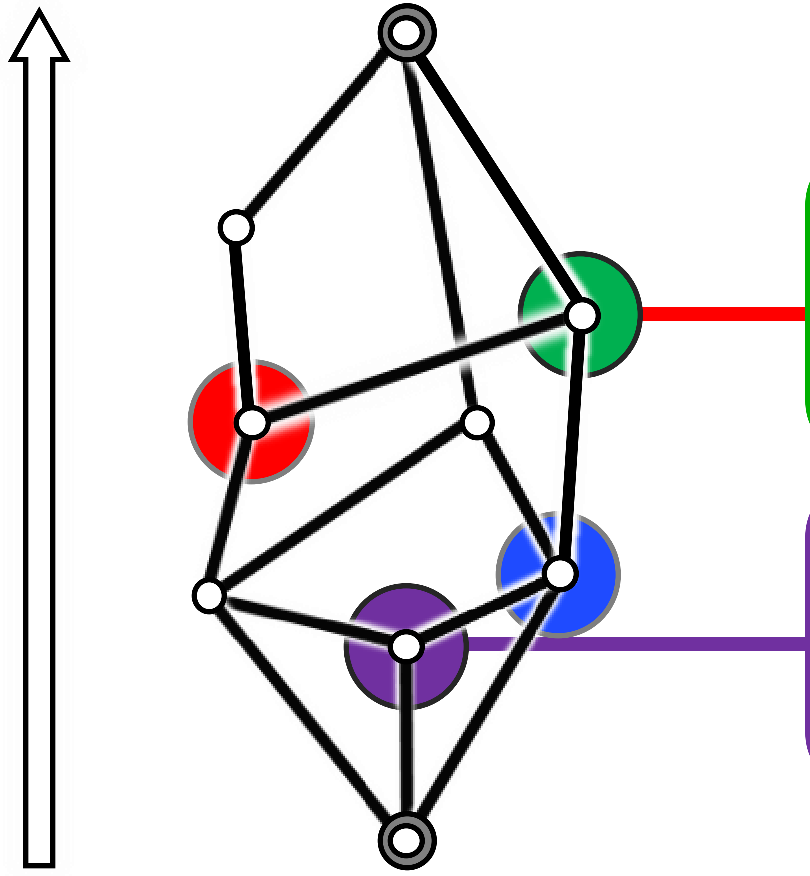
$\vee : \wedge$

COMMUTATIVE

ASSOCIATATIVE

IDEMPOTENT

ABSORBATIVE



$\vee$: JOIN

LEAST UPPER BOUND

UNION ; OR ; MAX ...

$\wedge$: MEET

GREATEST LOWER BOUND

INTERSECTION ; AND ; MIN ...

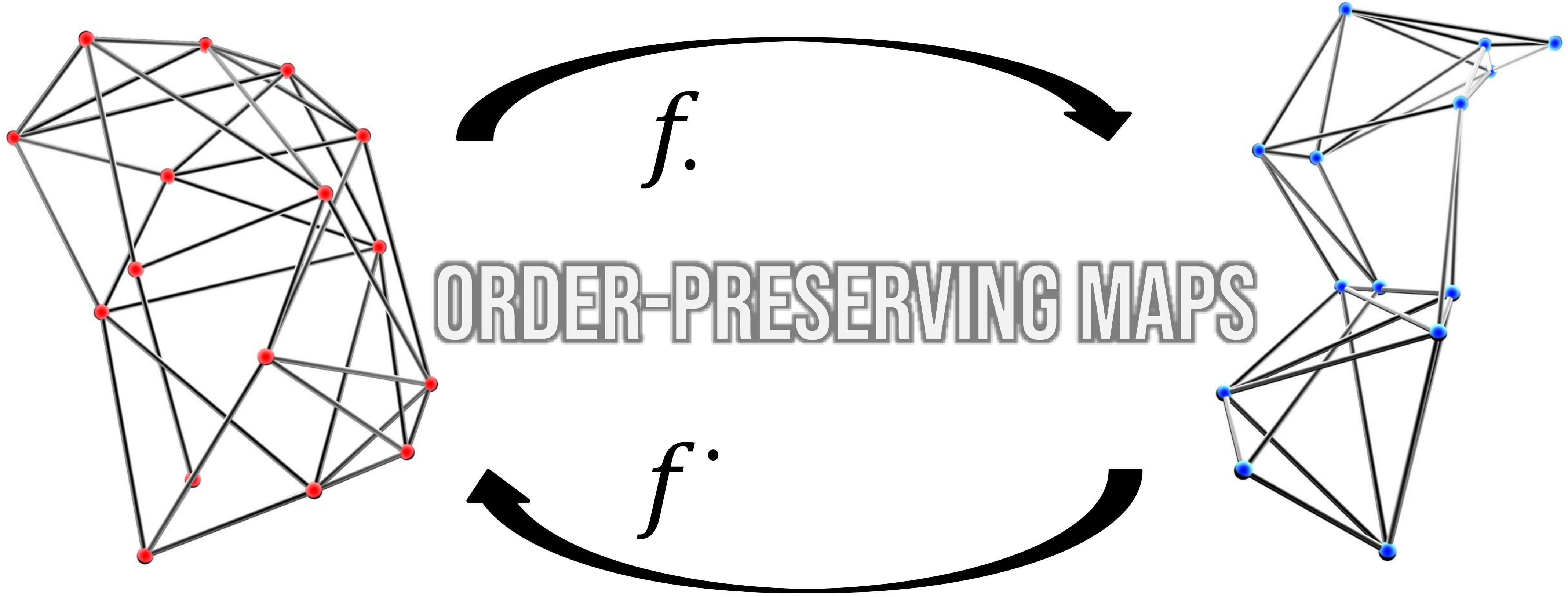
EXAMPLES OF LATTICES

SO MANY…

- POWERSETS
- SUBSPACES
- FORMAL CONCEPTS
- MAX-PLUS VECTOR SPACES
- CONDITIONAL ENTROPY
- STABLE MATCHINGS
- PARTITIONS/SUBPARTITIONS
- INVARIANT SUBSPACES
- PREFERENCE RELATIONS
- BOOLEAN/HEYTING ALGEBRAS
- (CO)CLOSURE SYSTEMS
- M-V LOGICS

LATTICE CONNECTIONS

LATTICE (GALOIS) CONNECTIONS COME IN PAIRS



LATTICE CONNECTIONS

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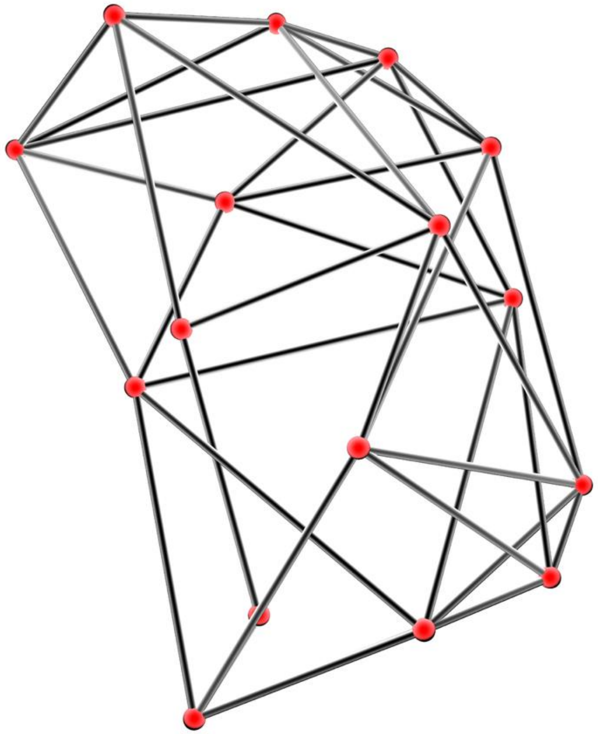
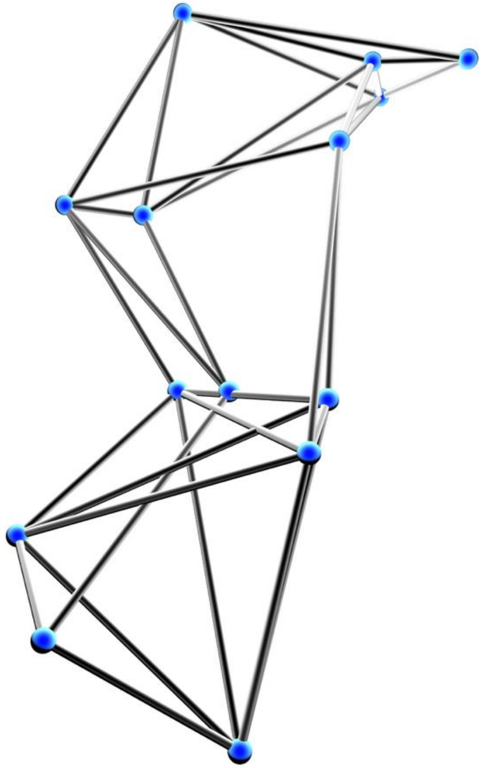


$$f.\, x \preccurlyeq y$$

IF AND ONLY IF

$$f^{\cdot}\, y \succcurlyeq x$$



LATTICE CONNECTIONS

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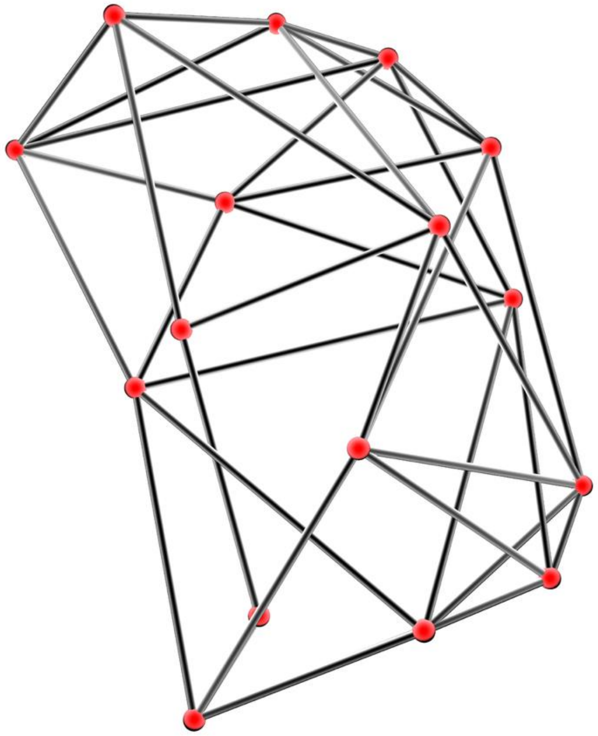
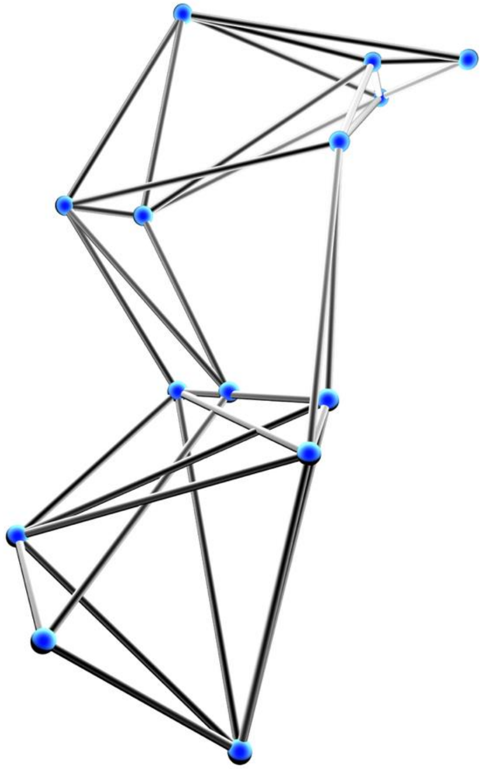


$$f^{\cdot} f_{\cdot}\, x \succcurlyeq x$$

AND

$$f_{\cdot} f^{\cdot}\, y \preccurlyeq y$$



EXAMPLES OF LATTICE CONNECTIONS

SO MANY…

$f_{\cdot}$	$f^{\cdot}$
DIRECT IMAGE (subsets, subspaces)	**INVERSE IMAGE** (subsets, subspaces)
EXISTENTIAL QUANTIFICAITON	**UNIVERSAL QUANTIFICATION**
INTENT (attributes shared by objects)	**EXTENT** (objects instantiating attributes)
MAX-PLUS MATRIX MULTIPLICATION	**MIN-PLUS MATRIX MULTIPLICATION**
INTEGRAL TRANSFORM	**INVERSE INTEGRAL TRANSFORM**

TARSKI FIXED POINT THEOREM

THEOREM [TARSKI 1955] :

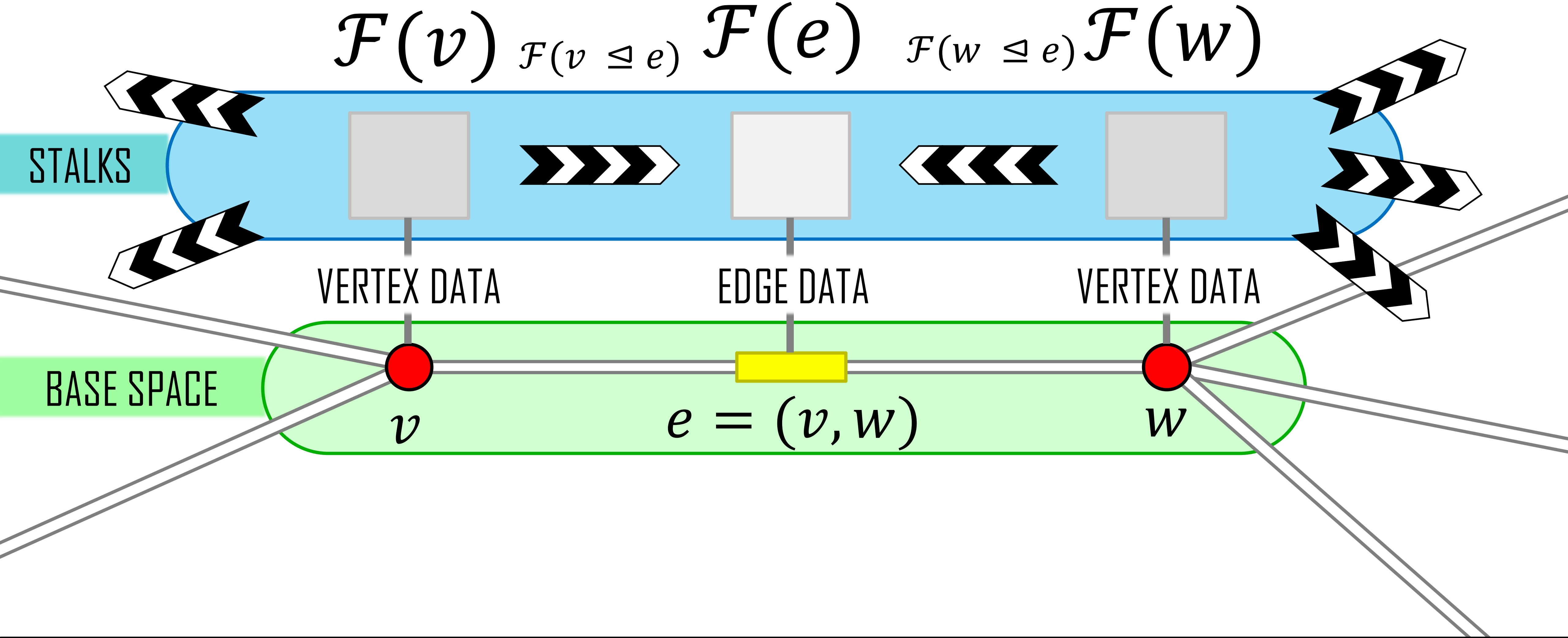
SUPPOSE $\Phi: L \to L$ IS AN ORDER-PRESERVING OPERATOR ON A COMPELTE LATTICE. THEN, THE FIXED POINTS

$$Fix(\Phi) = \{x | \Phi(x) = x\}$$

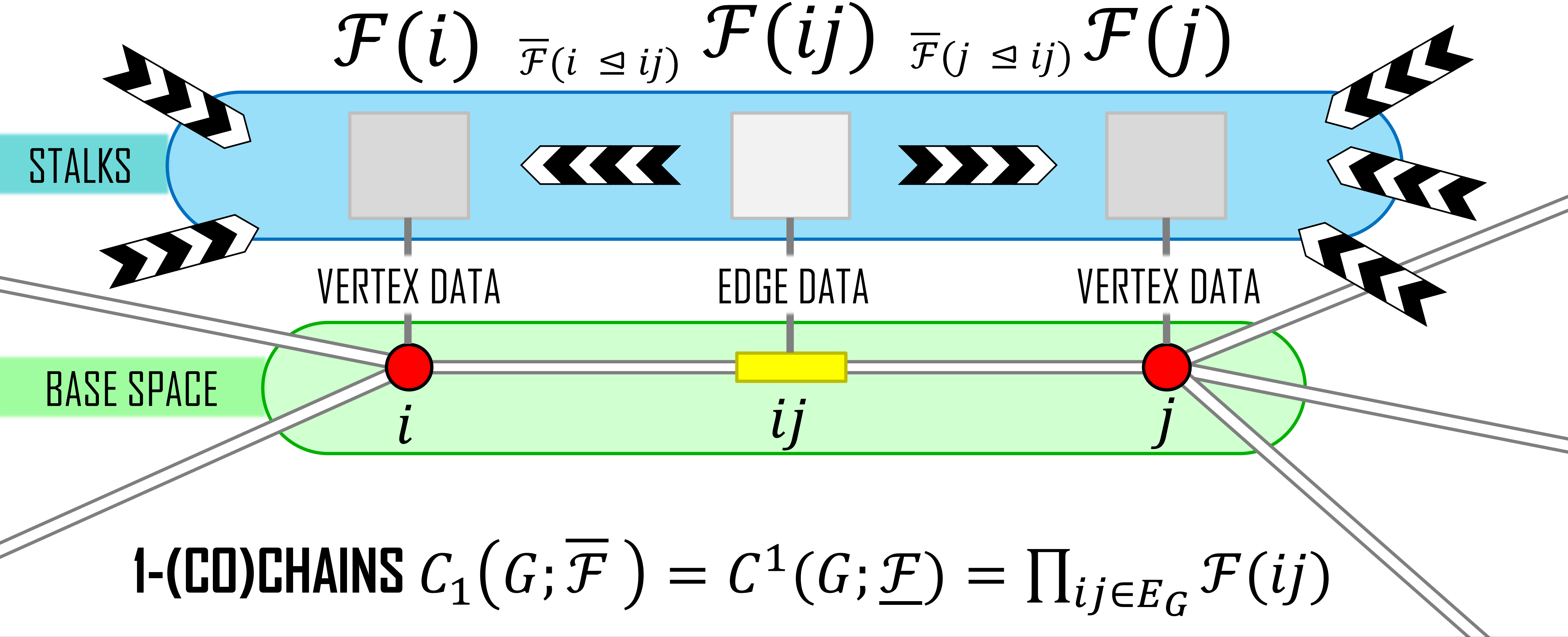
FORMS A COMPLETE LATTICE.

CONSTRUCTIVE FIXED-POINT THEOREMS POSSIBLE

NETWORK SHEAVES



NETWORK COSHEAVES



1-(CO)CHAINS $C_1(G;\overline{\mathcal{F}}) = C^1(G;\underline{\mathcal{F}}) = \prod_{ij\in E_G} \mathcal{F}(ij)$

GLOBAL SECTIONS

DEFINITION:

LET $\underline{\mathcal{F}}$ BE A SHEAF OF COMPLETE LATTICES AND JOIN-PRESERVING MAPS OVER A GRAPH. THEN, $H^0(G;\underline{\mathcal{F}}) =$

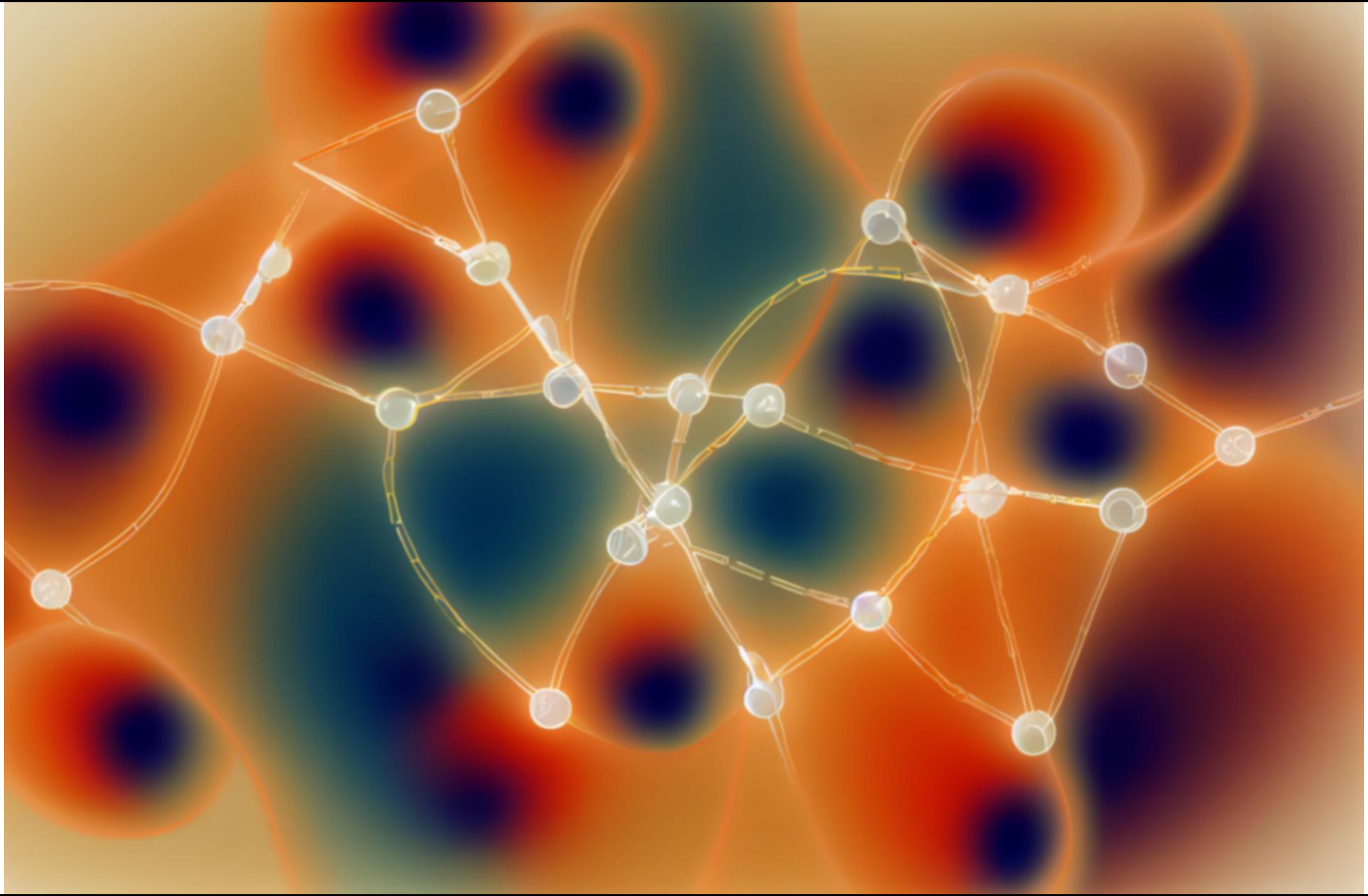
$$\left\{ \mathbf{x} \in C^0(G;\mathcal{F}) \mid \underline{\mathcal{F}}(i \trianglelefteq ij)(x_i) = \underline{\mathcal{F}}(j \trianglelefteq ij)(x_j) \right\}$$

GLOBAL SECTIONS ARE CONSISTENT ASSIGNMENTS OF DATA

HARMONIC STATES

$$H^0(G;\underline{\mathcal{F}}) \longrightarrow C^0(G;\underline{\mathcal{F}}) \underset{\delta_-}{\overset{\delta_+}{\rightrightarrows}} C^1(G;\underline{\mathcal{F}}) \longrightarrow H^1(G;\underline{\mathcal{F}})$$

LATTICE DIFFUSION?



THE TARSKI LAPALCIAN

LATTICE CONSENSUS

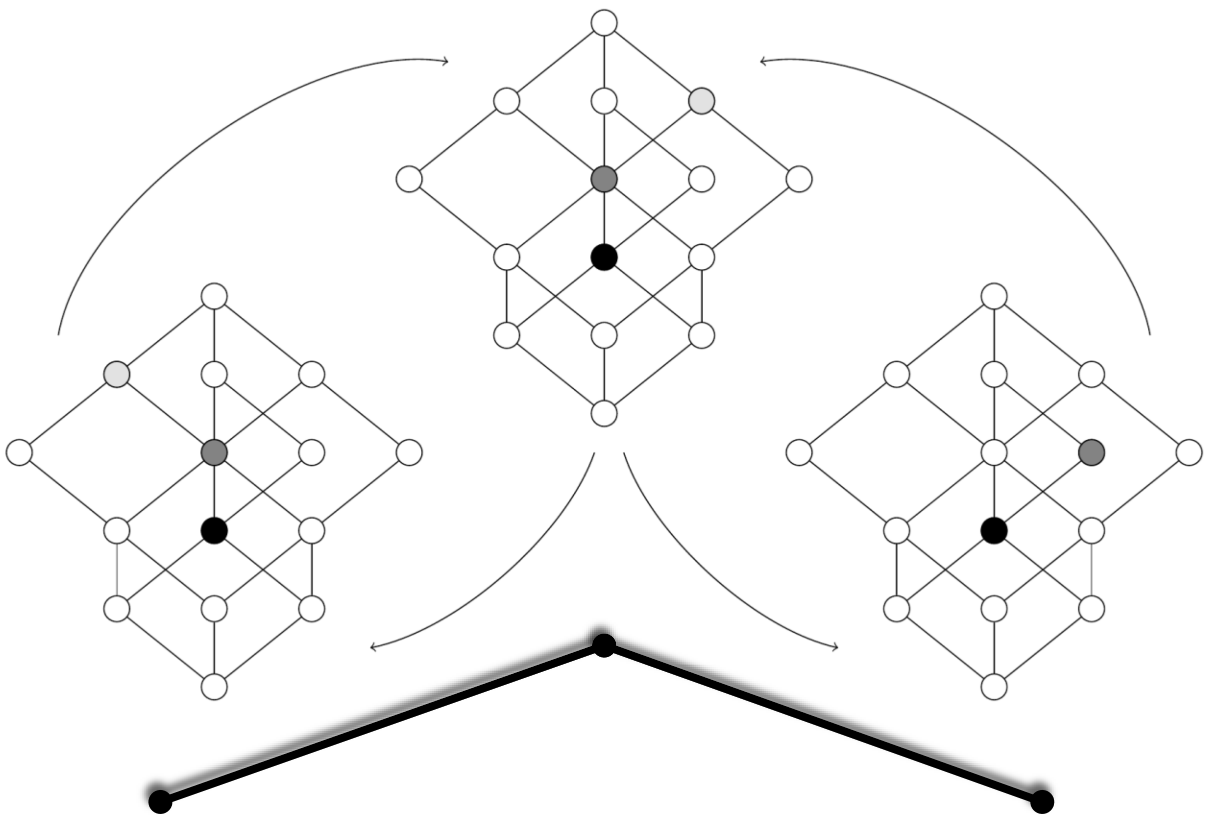
CONSENSUS on a LATTICE over a NETWORK is a GLOBAL SECTION of the CONSTANT SHEAF.

CONSENSUS ALGORITHM:

$$x_i[t+1] = x_i[t] \wedge \left(\bigwedge_{j \in \mathcal{N}_i} x_j[t] \right)$$

$$x_i[0] \in \mathbf{L}$$

"SLOWER" ALGORITHMS: APPROXIMATIONS of the CONSTANT SHEAF.



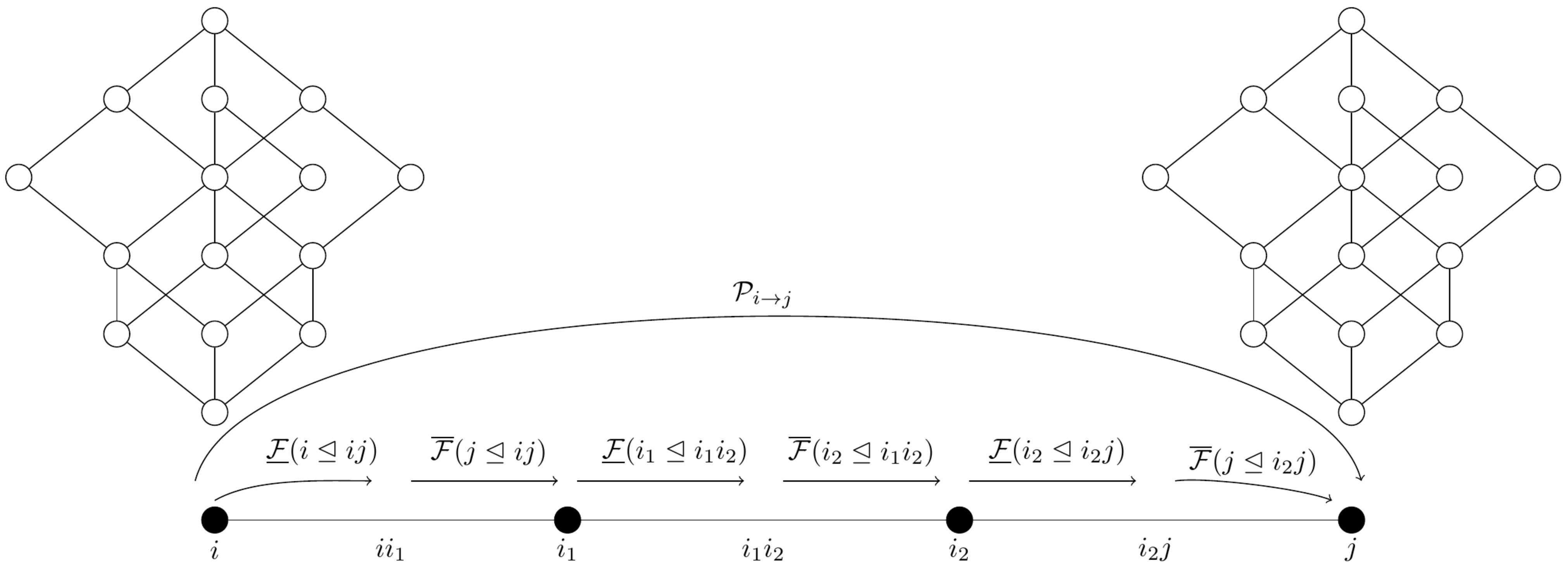
cf. MAX/MIN CONSENSUS

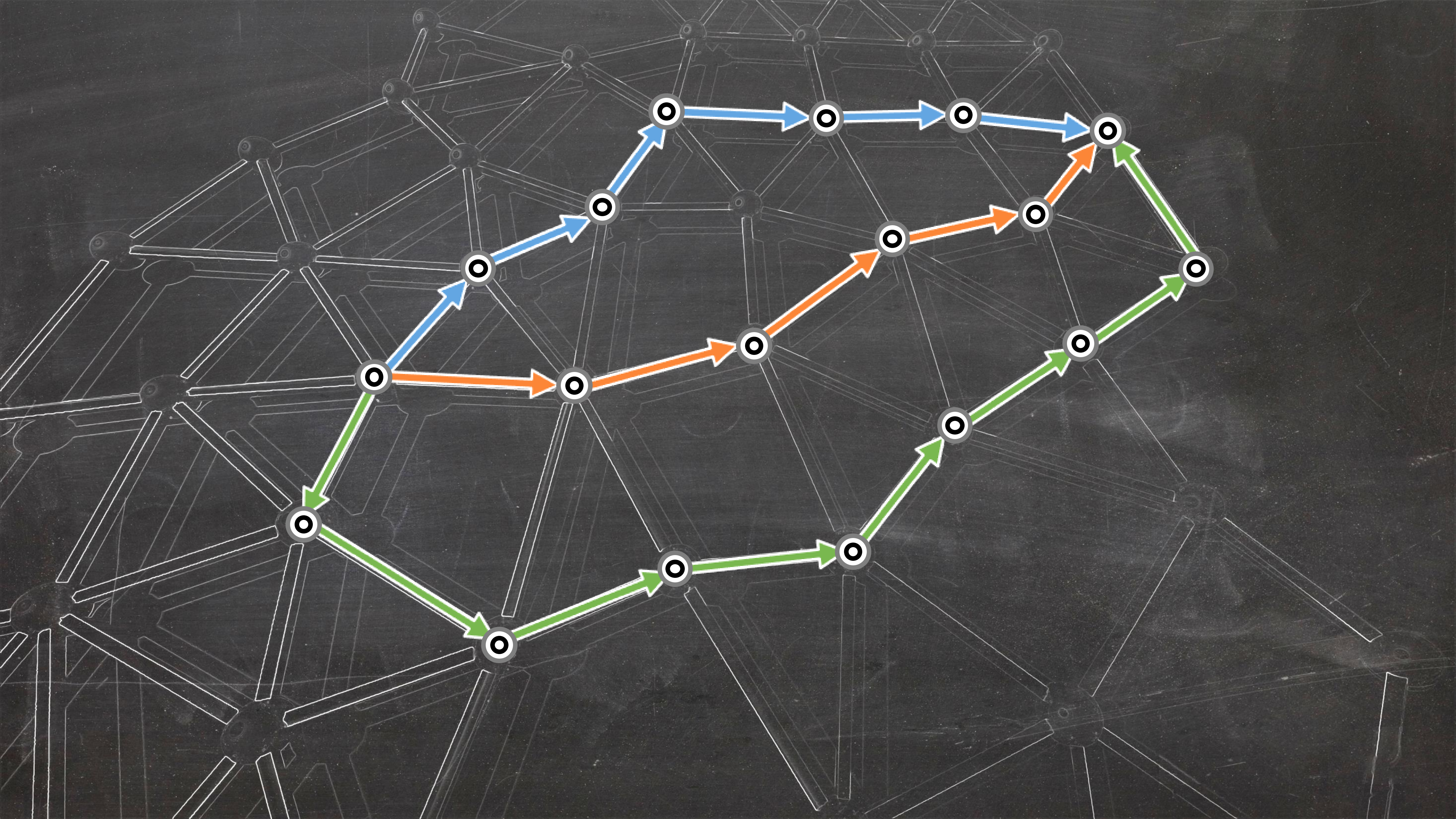
PARALLEL TRANSPORT

DEFINITION: for a NETWORK SHEAF of LATTICES & GALOIS CONENCTIONS $\mathcal{F}$ on G and a PATH from NODE i to NODE j, the PARALLEL TRANSPORT along a PATH $\gamma: i \to j$ is an ORDER-PRESERVING MAP

$$\mathcal{P}_{i\to j}: \mathcal{F}(i) \to \mathcal{F}(j)$$

given by the composition of restriction/corestriction maps along the path.





DEFINITION: TARSKI LAPLACIAN

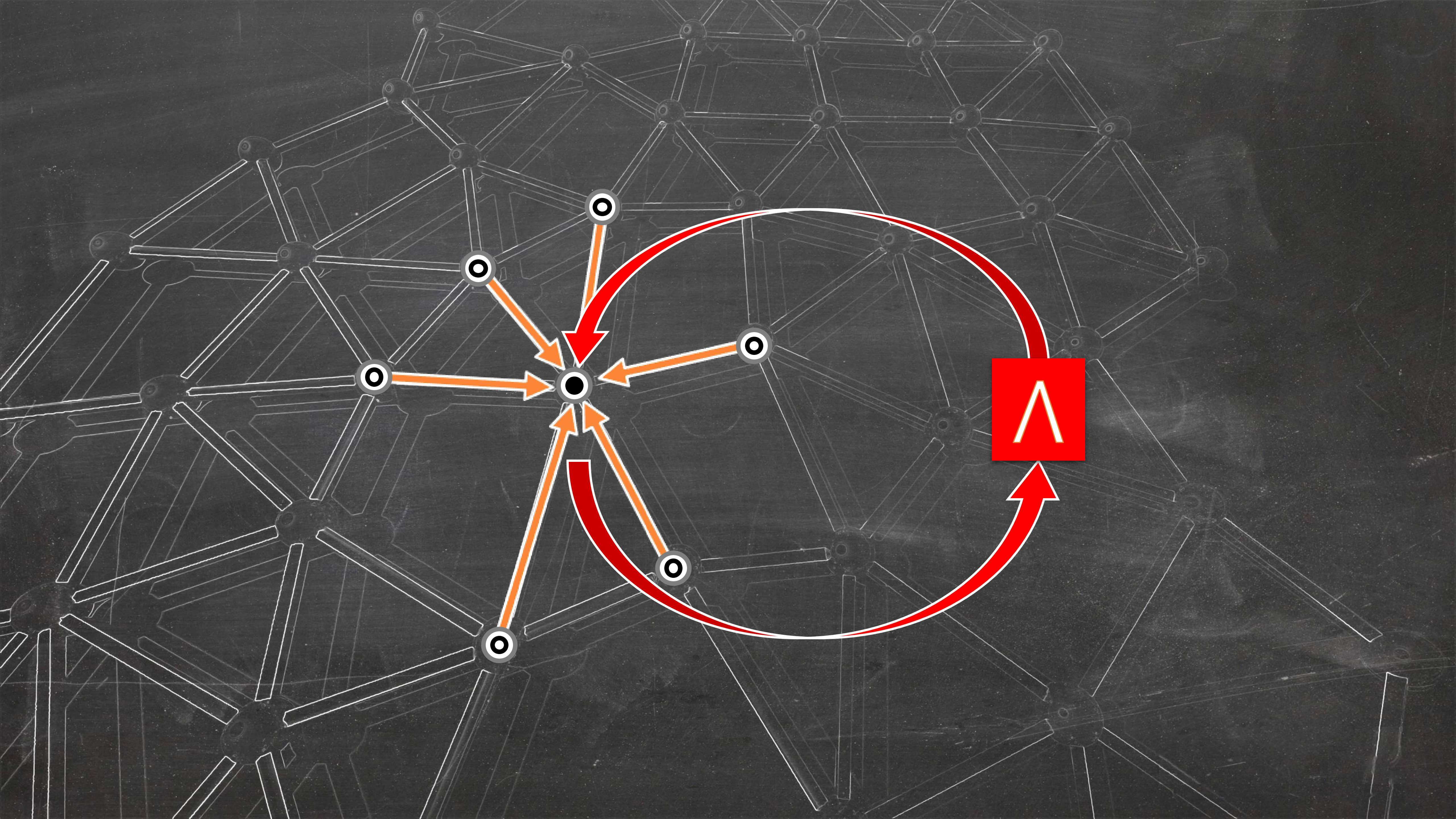
DEFINITION : FOR A SHEAF OF LATTICES ON A GRAPH and $\boldsymbol{x} = (x_i)_{i \in V} \in C^0(G; \underline{\mathcal{F}})$

$$L : C^0(G; \underline{\mathcal{F}}) \to C^0(G; \underline{\mathcal{F}})$$

$$(L\mathbf{x})_i = \bigwedge_{j \in \mathcal{N}_i} \mathcal{P}_{j \to i}(x_j)$$

WHERE

$$\mathcal{P}_{j \to i}(x_j) = \overline{\mathcal{F}}(i \trianglelefteq ij)\underline{\mathcal{F}}(j \trianglelefteq ij)(x_j)$$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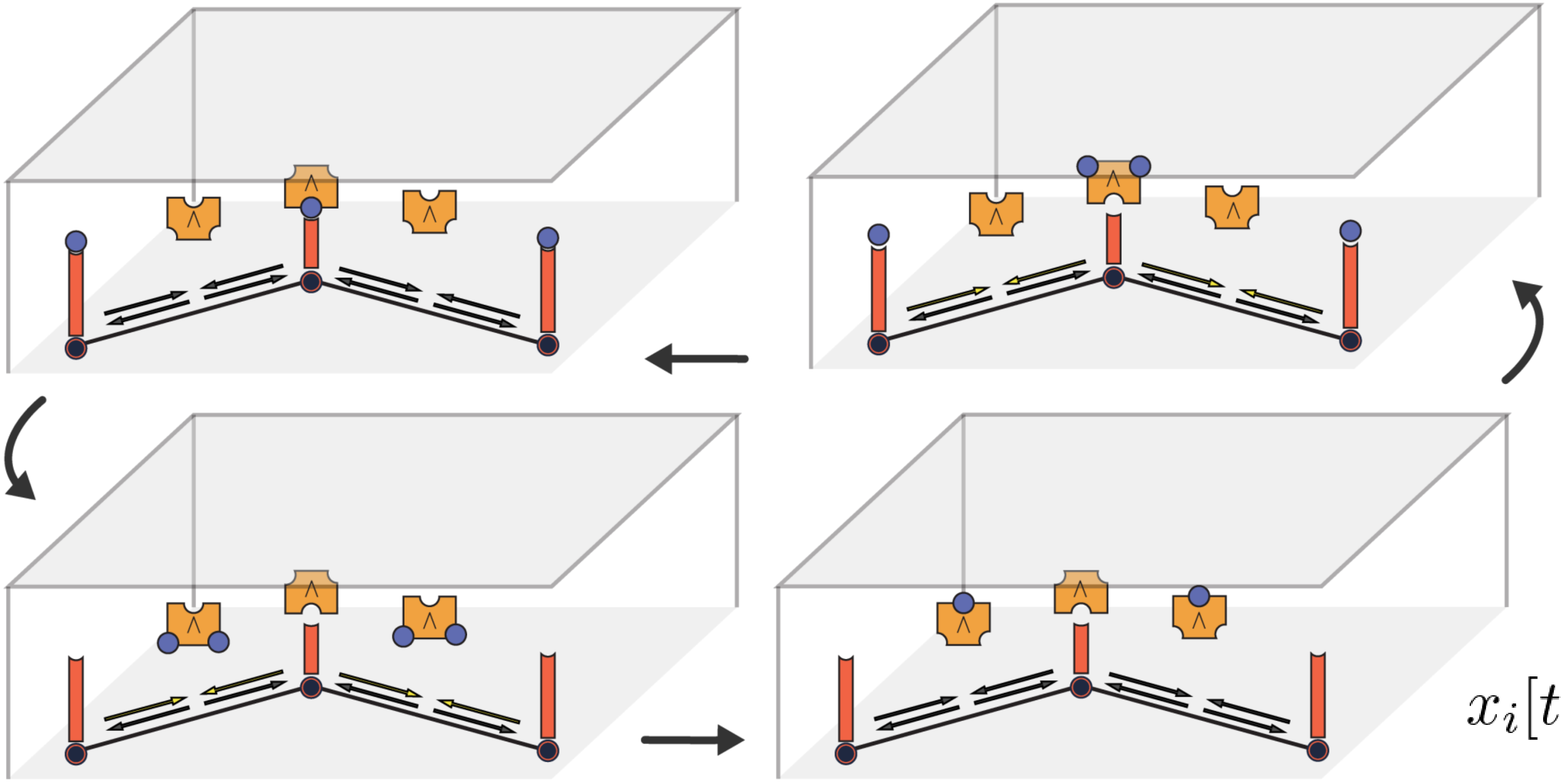
HODGE-TARSKI THEOREM

THEOREM [Ghrist-Riess 2022] : FOR A SHEAF OF COMPLETE LATTICES ON A GRAPH, THE GLOBAL SECTIONS ARE COMPUTED VIA THE TARSKI LAPLACIAN

$$H^0(G;\underline{\mathcal{F}}) = Fix(Id \wedge L)$$
$$= \{\boldsymbol{x} | L(\boldsymbol{x}) \wedge \boldsymbol{x} = \boldsymbol{x}\}$$
$$= \{\boldsymbol{x} | L(\boldsymbol{x}) \succcurlyeq \boldsymbol{x}\}$$

COROLLARY : $H^0(G;\underline{\mathcal{F}})$ IS A (NONEMPTY) COMPLETE LATTICE.

HEAT FLOW



ALGORITHM:

$$x_i[t+1] = x_i[t] \wedge (L\mathbf{x})_i$$

$$x_i[0] \in \mathcal{F}(i)$$

OPEN QUESTIONS