Lattice Theory in Social Choice & Multi-Agent Systems:

a Hodge-podge you never asked for

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BIRS Workshop: Applications of Hodge Theory on Networks

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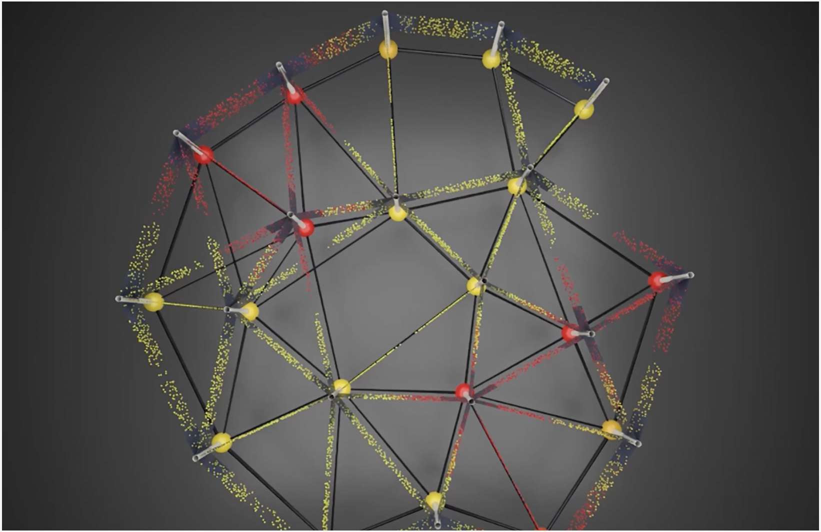
What is a Laplacian?



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Lattices

Definition

A **lattice** is a partially ordered set such that for every $x, y \in P$, there are elements

$$\begin{aligned} x \vee y &= & \min\{z : z \succcurlyeq x,\ z \succcurlyeq y\} \quad \text{(join)} \\ x \wedge y &= & \max\{z : z \preccurlyeq x,\ z \preccurlyeq y\} \quad \text{(meet)} \\ 0 &= & \vee\varnothing \quad \text{(bottom)} \\ 1 &= & \wedge\varnothing \quad \text{(top)} \end{aligned}$$

A lattice is complete if

$$\bigvee_i x_i \qquad \bigwedge_i x_i \tag{1}$$

are defined for arbitrary indexing sets.

Maps Between Lattices

Definition

Suppose $\mathbf{K}$, $\mathbf{L}$ are lattices. A map $f : \mathbf{K} \to \mathbf{L}$ is **monotone** if

- $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$

and **join-preserving** if

- $f\left(\bigvee_i x_i\right) = \bigvee_i f(x_i)$
- $f(0) = 0$

Theorem (Adjoint Functor Theorem)

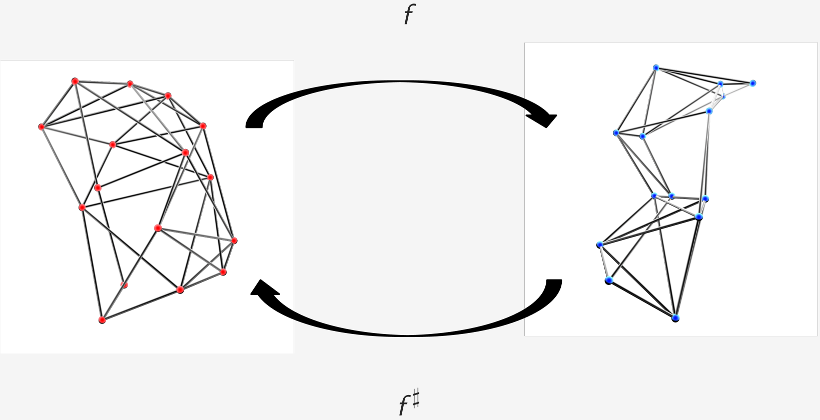
Suppose $f : \mathbf{K} \to \mathbf{L}$ is join-preserving. Then, there exist a map

$$f^\sharp : \mathbf{L} \to \mathbf{K}$$

such that

- $f^\sharp(f(x)) \succcurlyeq x$ for all $x \in \mathbf{K}$
- $f\left(f^\sharp(y)\right) \preccurlyeq y$ for all $y \in \mathbf{L}$

Galois Connections



The adjoint $f^\sharp$ is analogous to the Moore-Penrose inverse in some instances, the linear adjoint in others.

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Preferences

What is "rationality?"

Suppose X is a set of alternatives. A **preference relation** is a binary relation $\mathcal{P} \subseteq X \times X$ satisfying the axioms

- $x\mathcal{P}y$ or $y\mathcal{P}x$ for all $x, y \in X$ (Connected[1])
- $x\mathcal{P}x$ for all $x \in X$ (Reflexive)
- $x\mathcal{P}y$, $y\mathcal{P}z$ implies $x\mathcal{P}z$ for all $x, y, z \in X$ (Transitive)

Set of rational preference relations, $\mathrm{Ord}(X)$
$i \in \{1, 2, \ldots, n\}$, $x\mathcal{P}_i y$: *voter i prefers x to y*
A **preference profile** is a tuple $\Pi = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$

[1] In the social choice literature, sometimes called **complete**.

Arrow's Impossibility Theorem

Suppose $\Pi = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$ is a preference profile. A function $F : \mathrm{Ord}(X)^n \to \mathrm{Ord}(X)$ is

- Weak Pareto (WP) if for all $i \in \{1, 2, \ldots, n\}$, $x\mathcal{P}_i y$, implies $xF(\Pi)y$
- Independence of Irrelevant Alternatives (IIA) if $\Pi_{|\{x,y\}} = \Pi'_{|\{x,y\}}$ implies $F\left(\Pi_{|\{x,y\}}\right) = F\left(\Pi'_{|\{x,y\}}\right)$
- Dictatorial (D) if there exist $j \in \{1, 2, \ldots, n\}$ such that $F(\Pi) = \mathcal{P}_j$ for all $\Pi \in \mathrm{Ord}(X)$.

Theorem[2]

Any social welfare function $F : \mathrm{Ord}(X)^n \to \mathrm{Ord}(X)$ satisfying

1. Weak Pareto
2. Independence of Irrelevant Alternatives

is necessarily Dictatorial.

[2] Arrow, *Social Choice and Individual Values*.

Preference Lattices

Criticism of Arrovian "rationality?"

- Indifference & indecisiveness indistinguishable[3]
- Human preferences violate transitivity[4]

Lattice Structure

$\mathrm{Rel}(X)$, arbitrary preference relations
$\mathrm{Rel}^\star(X)$, transitive/reflexive preference relations

$$\mathcal{P} \wedge \mathcal{R} = \mathcal{P} \cap \mathcal{R}$$
$$\mathcal{P} \vee \mathcal{R} = (\mathcal{P} \cup \mathcal{R})^\star$$

where $\mathcal{R}^\star = \bigcup_{k=0} \mathcal{R}^{\circ k}$ (transitive closure).

[3] Eliaz and Ok, "Indifference or indecisiveness? Choice-theoretic foundations of incomplete preferences".

[4] Tversky, "Intransitivities of preferences".

Preference Dynamics

Tarski Fixed Point Theorem[5]

Suppose L is a complete lattice and $f : L \to L$ is monotone. Then, $\mathrm{Fix}(f) \subseteq L$ is a complete lattice.

Dynamical Systems Interpretation

Suppose $\Phi : \mathrm{Rel}^\star(X)^n \to \mathrm{Rel}^*(X)^n$ is monotone. Then, the equilibria of the system

$$\begin{aligned} \Pi(t+1) &= \Phi\left(\Pi(t)\right) \\ \Pi(0) &= \Pi_0 \end{aligned}$$

from a complete lattice.

[5] Tarski, "A lattice-theoretical fixpoint theorem and its applications".

Choice

Definition

A **choice function** on a set of alternatives X is a map $c : 2^X \to 2^X$ such that $c(A) \subseteq A$ for all $A \subseteq X$.

From Preferences to Choice & Back

Suppose $\mathcal{P}$ is a (weak) preference relation. Then,

$$c_{\mathcal{P}}(A) = \qquad \{x \in X \mid x\mathcal{P}y \ \forall y \in A\}$$

Suppose $c : 2^X \to 2^X$ is a choice function. Then,

$$x\mathcal{R}_c y \quad \text{if and only if} \quad \exists A \subseteq X, x \in c(A) \ \forall y \in A$$

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Multi-Agent Epistemic (MAE) Logic

MAE logic models knowledge or belief in a system with multiple voters.[6]

Syntax

Given a set of atomic proposition Φ, voters $\{1, 2, \ldots, n\}$, the language $\mathcal{L}(\Phi)$ consists of the following syntax defined inductively

$$p \in \Phi \mid \texttt{true} \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \mathsf{K}_i\varphi \mid \neg\mathsf{K}_i\neg\varphi$$

Knowledge v.s. Belief

Knoweldge, $\mathsf{K}_i\varphi \rightarrow \varphi$
Belief, $\neg\mathsf{K}_i\ \texttt{false}$

[6] Fagin et al., *Reasoning about knowledge*.

Semantics of MAE Logic

Kripke Model, $\mathcal{M}$

- S, set of states
- $\mathcal{K}_i \subseteq S \times S \; i = 1, 2, \ldots, n$, binary relations
- Φ, atomic propositions
- $\mu : S \rightarrow 2^{\Phi}$, atomic propositions true at $s \in S$

Semantics

- $(\mathcal{M}, s) \models p$ if $p \in \mu(s)$
- $(\mathcal{M}, s) \models \neg\varphi$ if $(\mathcal{M}, s) \not\models \varphi$
- $(\mathcal{M}, s) \models \varphi_1 \wedge \varphi_2$ if $(\mathcal{M}, s) \models \varphi_1$ and $(\mathcal{M}, s) \models \varphi_2$
- $(\mathcal{M}, s) \models \mathsf{K}_i\varphi$ if $(\mathcal{M}, t) \models \varphi \; \forall t.\; (s, t) \in \mathcal{K}_i, \; i = 1, 2, \ldots, n$

Galois Connection between Knowledge & Possibility

Intent

Suppose $\varphi \in \mathcal{L}(\Phi)$, $\mathcal{M}$ is a Kripke model, then
$\varphi^{\mathcal{M}} = \{s \in S : (\mathcal{M}, s) \models \varphi\}$ is the **intent** of φ.

Galois Connection

For $(S, \mathcal{K}_1, \ldots, \mathcal{K}_n)$ there is residual pair $(\mathcal{K}_i^{\exists}, \mathcal{K}_i^{\forall}) : 2^S \leftrightarrows 2^S$ for each $i = 1, 2, \ldots, n$ given by the formulae
$\mathcal{K}_i^{\exists}(\sigma) = \{t \in S : \ \exists s \in \sigma, \ (s, t) \in \mathcal{K}_i\}$,
$\mathcal{K}_i^{\forall}(\sigma) = \{s \in S : \ | \ \forall t \in S, (s, t) \in \mathcal{K}_i \Rightarrow t \in \sigma\}$.

Lemma

$\mathcal{K}_i^{\exists}\left(\varphi^{\mathcal{M}}\right) = \left(\neg \mathsf{K}_i \neg \varphi\right)^{\mathcal{M}}$ and $\mathcal{K}_i^{\forall}\left(\varphi^{\mathcal{M}}\right) = \left(\mathsf{K}_i \varphi\right)^{\mathcal{M}}$.

$\mathcal{K}_i^{\exists}$ sends the semantic content of "φ" to the semantic content of "Agent i believes φ is possible"

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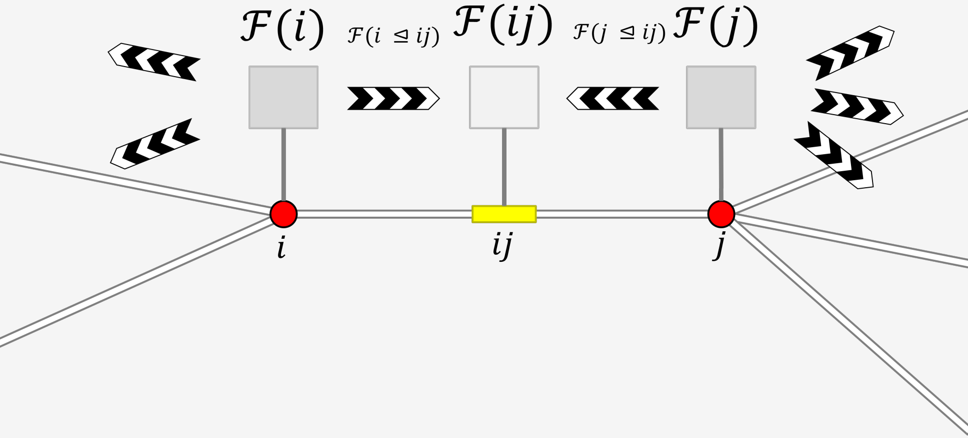


Lattices + Neworks = Sheaves

Network Sheaves

Definition

Suppose $G = (V, E)$ is an undirected graph. A **lattice-valued network sheaf** $\mathcal{F}$ over G is a data structure that assigns:

- A lattice $(\mathcal{F}(i), \wedge_i, \vee_j)$ to every node $i \in V$.
- A lattice $(\mathcal{F}(ij), \wedge_{ij}, \vee_{ij})$ to every edge $ij \in E$.
- Join-preserving maps
  - $\mathcal{F}(i \trianglelefteq ij) : \mathcal{F}(i) \to \mathcal{F}(ij)$
  - $\mathcal{F}(j \trianglelefteq ij) : \mathcal{F}(j) \to \mathcal{F}(ij)$

  for every $ij \in E$.

A 0-**cochain** of $\mathcal{F}$ is an $\mathbf{x} \in \prod_{i \in V} \mathcal{F}(i) = C^0(G; \mathcal{F})$. An 0-cochain is a **global section** if it is in the solution set

$$\Gamma(G; \mathcal{F}) = \left\{ \mathbf{x} \in \prod_{i \in V} \mathcal{F}(i) : \ \mathcal{F}(i \trianglelefteq ij)(x_i) = \mathcal{F}(j \trianglelefteq ij)(x_j) \ \forall ij \in E \right\}$$

Sections are consistent assignments of data.

Tarski Laplacian

Suppose

- $G = (V, E)$ is an undirected graph
- $\mathcal{F}$ is a lattice-valued network sheaf over G

The **Tarski Laplacian** is an operator

$$\begin{aligned} L : \quad & C^0(G;\mathcal{F}) \to C^0(G;\mathcal{F}) \\ (L\mathbf{x})_i = \quad & \bigwedge_{j \in N_i} \mathcal{F}^{\sharp}_{i \trianglelefteq ij} \mathcal{F}_{j \trianglelefteq ij}(x_j) \end{aligned}$$

Time-variant versions of the Tarski Laplacian for asynchronous communications or dynamic networks.[7]

[7] Riess and Ghrist, "Diffusion of Information on Networked Lattices by Gossip".

Hodge-Tarski Theorem

Theorem[8]

Suppose $\mathcal{F}$ is a lattice-valued sheaf over G and L is the Tarski Laplacian of $\mathcal{F}$. Then,

$$\mathrm{Fix}(id \wedge L) = \Gamma(G; \mathcal{F}) \tag{2}$$

Analogy

Suppose Δ_k ($k = 0, 1$ for a graph) is the (combinatorial) Hodge Laplacian in degree k. Compare (2) to the Hodge Theorem

$$\mathrm{Fix}(I - \Delta_k) \cong H^k(G) = H^k(G; \underline{\mathbb{R}}_G)$$

Define $H^0(G; \mathcal{F}) \triangleq \Gamma(G; \mathcal{F})$. What is $H^1(G; \mathcal{F})$?

[8] Ghrist and Riess, "Cellular sheaves of lattices and the Tarski Laplacian".

The Heat Equation

Algorithm

Suppose $\mathbf{x}[0] \in \prod_{i \in V} \mathcal{F}(i)$ is an initial condition. The **heat equation** is defined by the following recursion

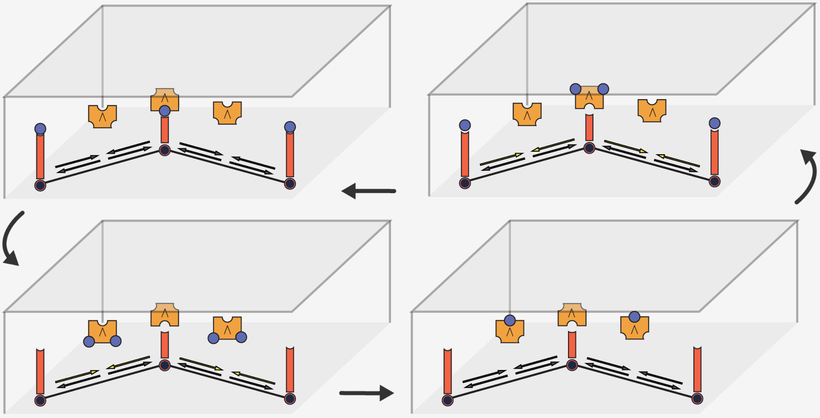
$$\mathbf{x}[t+1] = L(\mathbf{x}) \wedge \mathbf{x}[t] \tag{3}$$

Theorem

The time-independent solutions of the heat equation (3) coincide with the global sections

$$\Gamma(G; \mathcal{F}) = \left\{ \mathbf{x} \in \prod_{i \in V} \mathcal{F}(i) : \mathcal{F}(i \trianglelefteq ij)(x_i) = \mathcal{F}(j \trianglelefteq ij)(x_j) \; \forall ij \in E \right\}$$

Heat Flow



Knowledge Consensus

Sheaf Construction

Suppose $\mathcal{M} = (S, \mathcal{K}_1, \ldots, \mathcal{K}_n, \Phi, \mu)$ is a Kripke model and $G = (V, E)$ is a graph on $\{1, 2, \ldots, n\}$. Construct a sheaf $\mathcal{F}$:

- $\mathcal{I}(i) = 2^S$ for all $i \in V$; $\mathcal{I}(ij) = 2^S$ for all $ij \in E$
- $\mathcal{I}(i \trianglelefteq ij)(\sigma_i) = \mathcal{K}_i^{\exists}(\sigma_i)$
- $\mathcal{I}(i \trianglelefteq ij)^{+}(\sigma_{ij}) = \mathcal{K}_i^{\forall}(\sigma_{ij})$

Semantic Laplacian[9]

$$(L_t \boldsymbol{\sigma})_i = \bigcap_{j \in N_i} \mathcal{K}_i^{\forall} \mathcal{K}_j^{\forall}(\sigma_j), \quad \boldsymbol{\sigma} \in \left(2^S\right)^n$$

[9] Riess and Ghrist, "Diffusion of Information on Networked Lattices by Gossip".

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Open Questions

1. What is a good combinatorial model for preference dynamics?
2. What is preference diffusion? Knowledge diffusion?
3. Is there a Hodge decomposition for $C^0(G; \mathcal{F})$?
4. What is the "Helmholtzian" of a lattice-valued network sheaf?
   Is there a Hodge decomposition for $C^1(G; \mathcal{F})$?

Thank You

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