Towards Categorical Diffusion

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Talk Outline

- Many views on diffusion
  - Hodge Laplacian
  - Graph/graph connection Laplacian
  - Combinatorial Hodge Laplacian
- Network sheaves
  - Global sections
  - Sheaf Laplacian
- Quantale-enriched categories
  - Quantales
  - $\mathcal{Q}$-categories
  - Weighted meets/joints
- Categorical network diffusion
  - $\mathcal{Q}\mathscr{C}at$-categories
  - $\underline{\mathcal{Q}\mathscr{C}at}$-valued (co)presheaves
  - Weighted global sections
  - Tarski Laplacian
  - Hodge-Tarski Theorem
  - Tarski Fixed Point Theorem
- Applications

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The Many Facets of Diffusion

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The Many Facets of Diffusion

Diffusion in physics

- Diffusion is central concept in thermodynamics. Heat equation, $\partial_t x = \alpha \nabla^2 x$ with Laplacian ∇^2 models change of temperature or concentration in Euclidean space over time
- Diffusion generalized to *manifolds*. Suppose $\mathbb{M}$ is a m-dimensional Riemannian manifold. The deRahm complex is the complex

$$\Omega^0(\mathbb{M}) \xrightarrow{d} \Omega^1(\mathbb{M}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^m(\mathbb{M}) \xrightarrow{d} 0$$

  where $\Omega^k(\mathbb{M})$ is the Hilbert space of differential forms and d is the exterior derivate.

  - $\Delta = d\partial + \partial d$ where $\partial = d^*$ is the linear adjoint
  - $\omega = \alpha + \beta + \gamma$ where $\alpha \in \operatorname{im} d$, $\beta \in \operatorname{im} \partial$, $\gamma \in \ker \Delta$

  Hodge Theorem. $H^k_{\mathrm{dR}}(\mathbb{M}; \mathbb{R}) \cong \ker \Delta_k$

  $\Delta_0 = d^*d$ is the *Laplace-Beltrami* operator and generalizes the classical Laplacian.

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Diffusion in graph theory

- Suppose $\mathbb{X} = (\mathbb{X}_0, \mathbb{X}_1)$ is an undirected graph with $|\mathbb{X}_0| = n$ and with label function $x : \mathbb{X}_0 \to \mathbb{R}$.
- Two nodes $v, w \in \mathbb{X}_0$ are adjacent, written $v \sim w$, if $(v, w) \in \mathbb{X}_1$. Let $\mathbf{deg}(v)$ be the number adjacent nodes
- The adjacency matrix of a graph is defined

$$A_{v,w} = \begin{cases} 1, & v \sim w \\ 0, & \text{otherwise} \end{cases}$$

- Let $(B_k)_{k\geq 0}$ be a *random walk* on $\mathbb{X}$; B_0 chosen uniformly at random. The transition matrix of this Markov chain is

$$P_{v,w} = \mathbb{P}(B_k = w \,|\, B_{k-1} = v) = \begin{cases} \frac{1}{\deg(v)}, & w \sim v \\ 0, & \text{otherwise} \end{cases}$$

- The matrix $L = I - D^{-1}A$ is the *normalized graph Laplacian for random walks*; leads to heat equations
  - Continuous time, $\partial_t x = -Lx$
  - Discrete time, $U_k = \left(\mathbb{E}\left[x(B_k) \,|\, B_0 = v\right]\right)_{v\in\mathbb{X}_0}$

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The Many Facets of Diffusion

Diffusion in discrete geometry

- Vector diffusion map generalizing random walks on graph with vector features (Singer & Wu 2012)
- Graph connection Laplacian $\mathscr{L}_{con} = I - \mathscr{D}^{-1}\mathscr{A}$ where
$$\mathscr{A}[v, w] = \sum_{w \sim v} w_{v,w} O_{v,w} x_w$$
for parallel transport maps $O_{v,w} \in O(d)$.
- Heat equation is $\partial_t \mathbf{x} = -\mathscr{L}\mathbf{x}$ where $\mathbf{x}(0) = (\mathbb{R}^d)^n$
- Useful in learning representation of vector-field data (Battiloro, R., et al. 2024)

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The Many Facets of Diffusion

Diffusion in computational topology

- Let $\mathbb{X}$ be a simplicial complex or a regular cell complex
- The simplicial chain complex

$$C_0(\mathbb{X}) \xleftarrow{\partial} C_1(\mathbb{X}) \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_k(\mathbb{X}) \xleftarrow{\partial} \cdots$$

  where $\partial([i_0 i_1 \cdots i_k]) = \sum_{j=0}^{k} (-1)^j [i_0 i_1 \cdots \hat{i}_j \cdots i_k]$. Let $d = \partial^*$ be the adjoint of the boundary map.
- Eckmann (1994) suggested a Hodge theory with $\Delta = \partial d + d\partial$ where Hodge decomposition and Hodge theorem $\ker \Delta \cong H_k(\mathbb{X}; \mathbb{R})$ still hold.
- ODE $\dot{\mathbf{x}} = -\Delta \mathbf{x}$ converges to a harmonic homology class for any $\mathbf{x}(0) \in C_k(\mathbb{X})$
- This theory is extended to cellular sheaves which generalizes **both the combinatorial Hodge Laplacian** and the **connection Laplacian** (Hansen & Ghrist, 2019)

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Network Sheaves

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Network Sheaf Theory

- Let $\mathbb{X}$ be a graph (general theory of cellular for regular cell complexes). Let $\mathcal{J}^{op} = (\mathbb{X}, \trianglelefteq)$ be a partial order given by the transitive closure of incidence relation

$$v \trianglelefteq e \trianglerighteq w \text{ if } e = (v, w) \text{ is an edge with boundary } \partial(e) = \{v, w\}$$

- Suppose $\mathscr{C}$ is a data category.
    - A *network sheaf* on $\mathbb{X}$ valued in $\mathscr{C}$ is presheaf: $\underline{F} : \mathcal{J}^{\mathrm{op}} \to \mathscr{C}$
    - A *network cosheaf* is a copresheaf: $\overline{F} : \mathcal{J} \to \mathscr{C}$
    - The object $F_v := \underline{F}v = \overline{F}v$ is called the *stalk* at v
    - The maps

$$\underline{F}_{e \trianglerighteq v} : F_v \to F_e$$
$$\overline{F}_{v \trianglelefteq e} : F_e \to F_v$$

  are called *restriction* & *corestriction* maps

- The *global sections* of $\underline{F}$ is defined as $\lim \underline{F}$ which can be identified as the cone

$$\Gamma(\mathbb{X}; \underline{F}) = \left\{ (x_v, x_{v,e})_{v \in V, e \in E} : \underline{F}_{e \trianglerighteq v}(x_v) = x_{e,v}, x_{e,v} = x_{e,w}, \quad \forall e = (v, w) \right\}.$$

Remark. $\underline{F}$ is actually a sheaf if we put the Alexandrov topology on $\mathcal{J}$ and if $\mathscr{C}$ is complete. Category of sheaves on $\mathsf{Alex}(\mathcal{J})$ equivalent to $[\mathcal{J}^{\mathrm{op}}, \mathscr{C}]$ (Curry 2014).

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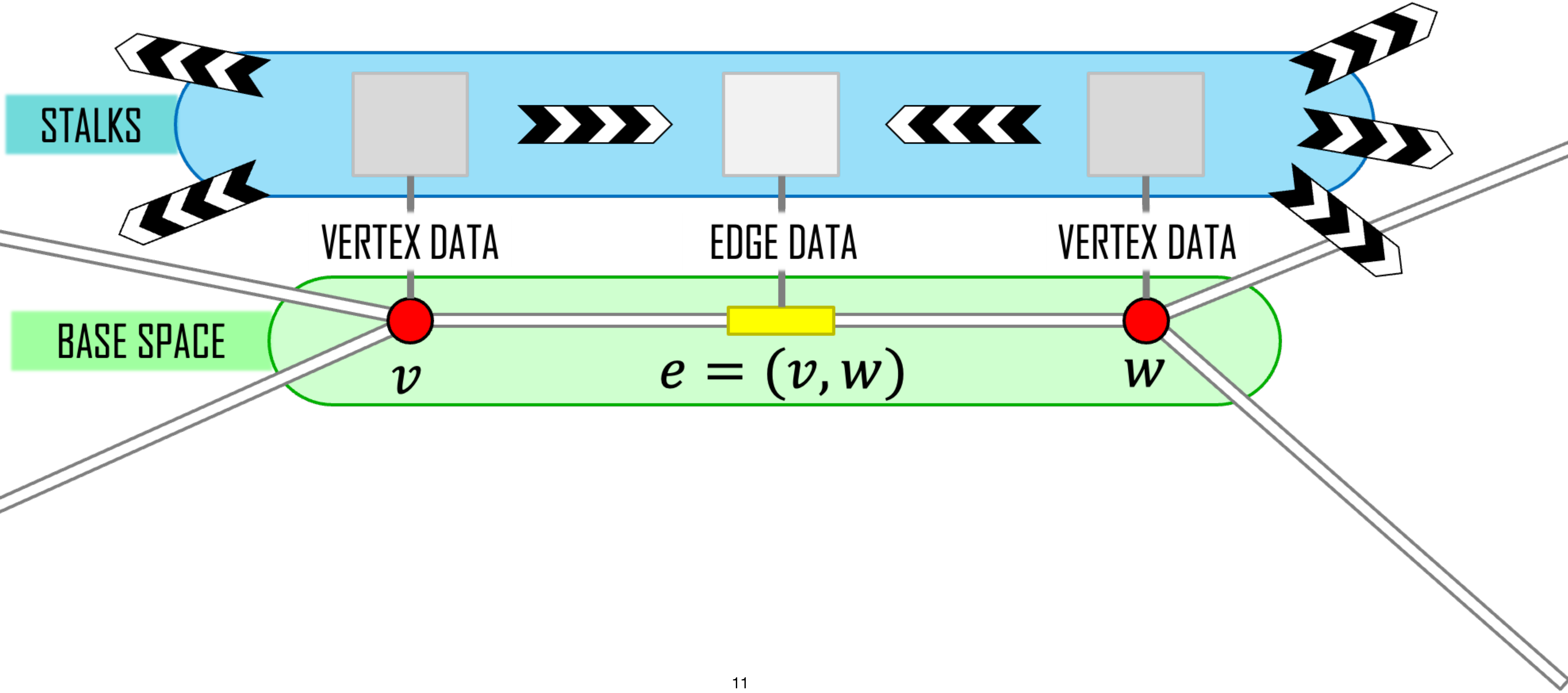
Network Sheaf Theory



Network Sheaf Theory

$\mathscr{C} = \mathscr{H}ilb$

- Suppose $\mathscr{C}$ is the category $\mathscr{H}ilb$ of Hilbert spaces and $\underline{F}$ is a network sheaf over $\mathbb{X}$ valued in $\mathscr{H}ilb$ and suppose $\overline{F}$ is the network cosheaf where $\overline{F}_{v \trianglelefteq e}$ is $\underline{F}^*_{e \trianglerighteq v}$ (linear adjoint)
- $C^0(\mathbb{X}; \underline{F}) = \bigoplus_{v \in \mathbb{X}_0} F_v$ and $C^1(\mathbb{X}; \underline{F}) = \bigoplus_{e \in \mathbb{X}_1} F_e$ are the 0 and 1 -cochains with coboundary map

$$(d\mathbf{x})_e = \sum_{v_e} [v : e] \; \underline{F}_{v \trianglelefteq e}(x_v)$$

  where $[v : e] = \pm 1$ according to orientation
- Then, the *sheaf Laplacian* is the map $\mathscr{L} : C^0(\mathbb{X}; \underline{F}) \to C^0(\mathbb{X}; \underline{F})$ defined $\mathscr{L} = d^* d$, or, explicitly

$$(\mathscr{L}\mathbf{x})_v = \sum_{w \trianglelefteq e \trianglerighteq v} \left(\overline{F}_{v \trianglelefteq e} \circ \underline{F}_{e \trianglerighteq v}\right)(x_v) - \left(\overline{F}_{v \trianglelefteq e} \circ \underline{F}_{e \trianglerighteq w}\right)(x_w)$$

  - $\underline{F}_{e \trianglelefteq v} = \overline{F}_{v \trianglelefteq e} = I$ implies $\mathscr{L}$ is the graph Laplacian
  - $\overline{F}_{v \trianglelefteq e} \underline{F}_{e \trianglerighteq v} = w_{v,w} O_{v,w}$ for $O_{v,w} \in O(d)$, $w_{v,w} > 0$ implies $\mathscr{L}$ is the graph connection Laplacian

Theorem (Ghrist & Hansen 2019; Ghrist & Gould TBD). For any initial condition $\mathbf{x}(0) \in C^0(\mathbb{X}; \underline{F})$, $\dot{\mathbf{x}} = -\mathscr{L}\mathbf{x}$ converges to orthogonal projection onto

$$\left\{\mathbf{x} : \underline{F}_{e \trianglerighteq v}(x_v) = \underline{F}_{e \trianglerighteq w}(x_w), \quad \forall e = (v, w)\right\} \cong \Gamma(\mathbb{X}; \underline{F})$$

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Quantale Enriched Category Theory

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Quantale Enriched Category Theory

Quantales

- A complete lattice Q is a partially ordered set $(Q, \preceq)$ such that the supremum $\bigvee_{s \in S} s$ exists for every subset $S \subseteq Q$.
- In a complete lattice, the meet $(\bigwedge)$ can be always be written as a join $(\bigvee)$ on downsets
- A quantale is a complete lattice with the structure of a monoid $(Q, \otimes, 1)$ such that

$$\begin{array}{c} p \otimes \left(\bigvee_{q \in S} q\right) = \bigvee_{q \in S} (p \otimes q) \\ \left(\bigvee_{q \in S} q\right) \otimes p = \bigvee_{q \in S} (q \otimes p) \end{array}, \quad \forall S \subseteq S \, \forall p \in Q$$

- $[-,-] : Q \times Q \to Q$ defined by $p \otimes q \preceq r$ iff $q \preceq [p, r]$ (Adjoint Functor Theorem)
- Q is unitally bounded if 1 is the terminal object

Assumption. We assume Q is a *unitally-bounded commutative quantale.*

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Quantale Enriched Category Theory

Quantales

- Facts:
  - If $p \preceq q$, then $r \otimes q \preceq r \otimes q$
  - $\left[p, \bigwedge_{q \in S} q\right] = \bigwedge_{q \in S} [p, q]$
- Examples of quantales:
  - Locales: $p \wedge \left(\bigvee_{q \in S} q\right) = \bigvee_{q \in S} (p \wedge q)$
  - Boolean algebra: $Q = \{0,1\}$
  - Extended positive reals: $[0,\infty]$ with $+$ under the opposite order $\geqslant$
  - Unit interval: $Q = [0,1]$ with a t-norm structure (Hoffman & Reis, 2012)

    $s \otimes t = s \cdot t$

    $s \otimes t = \max(s + t - 1, 0);$

    $s \otimes t = \min(s, t)$

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Quantale Enriched Category Theory

Q-Categories

- Suppose Q is a quantale. A Q-category $\mathscr{C}$ is a category enriched in Q
  - Objects: $(\mathscr{C})_0$ is arbitrary
  - Morphisms: $\hom_{\mathscr{C}}(x, y) \in Q$
  - Composition Law: $\hom_{\mathscr{C}}(y, z) \otimes \hom_{\mathscr{C}}(x, y) \preceq \hom_{\mathscr{C}}(x, z)$
  - Unit Law: $1 \preceq \hom_{\mathscr{C}}(x, x)$ (equality if Q is unitary bounded)
- A Q-funtor between Q-categories $\mathscr{C}$ and $\mathscr{D}$ is a function $F : (\mathscr{C})_0 \to (\mathscr{D})_0$ such that
  $$\hom_{\mathscr{C}}(x, y) \preceq \hom_{\mathscr{D}}(Fx, Fy)$$
  for all $x, y \in (\mathscr{C})_0$
- A Q-adjunction between Q-categories $\mathscr{C}$ and $\mathscr{D}$ are Q-functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ such that $\hom_{\mathscr{D}}(Fx, y) = \hom_{\mathscr{C}}(x, Gx)$
- Examples:
  - $\{0,1\}$-categories are preorders and $\{0,1\}$-functors are monotone maps
  - $[0,\infty]$-categories are Lawvere metric spaces and $[0,\infty]$-functors are non-expansive mappings.

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Quantale Enriched Category Theory

More examples of Q-categories

- $\underline{Q}$ is a Q-category with $(\underline{Q})_0 = Q$ and $\hom_{\underline{Q}}(p,q) = [p,q]$
- Let S be a set. Then, S is a Q-category with $(S)_0 = S$ and $\hom_S(a,b) = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}$
- Let $\mathscr{C}$ be a Q-category. Then, $\mathscr{C}^{op}$ is a Q-category with $(\mathscr{C}^{op})_0 = (\mathscr{C})_0$ and $\hom_{\mathscr{C}^{op}}(x,y) = \hom_{\mathscr{C}}(y,x)$
- Suppose $\left(\mathscr{C}_i\right)_{i\in I}$ is a collection of Q-categories. Then, $\prod_{i\in I}\mathscr{C}_i$ is a Q-category with objects

$$\left(\prod_{i\in I}\mathscr{C}_i\right)_0 = \prod_{i\in I}(\mathscr{C}_i)_0$$

and morphisms

$$\hom_{\prod_{i\in I}\mathscr{C}_i}\left((x_i)_{i\in I}, (y_i)_{i\in I}\right) = \bigwedge_{i\in I}\hom_{\mathscr{C}_i}(x_i, y_i)$$

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Quantale Enriched Category Theory

Q-Categories (continued)

- Suppose $\mathscr{C}$ and $\mathscr{D}$ are Q-categories. Then, $[\mathscr{C}, \mathscr{D}]$ is a Q-category with objects
$$\big([\mathscr{C}, \mathscr{D}]\big)_0 = \big\{F : \mathscr{C} \to \mathscr{D}\big\}$$
and morphism
$$\hom_{[\mathscr{C},\mathscr{D}]}\big(F, G\big) = \bigwedge\nolimits_{x \in (\mathscr{C})_0} \hom_{\mathscr{D}}(Fx, Gx)$$
- Suppose $\mathscr{C}$ is a Q-category. Then, $\hat{\mathscr{C}} := \big[\mathscr{C}^{op}, \underline{Q}\big]$ is the category of *presheaves*.
- Suppose $\mathscr{C}$ is a Q-category. Then, $\check{\mathscr{C}} := \big[\mathscr{C}, \underline{Q}\big]$ is the category of *copresheaves*.

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Quantale Enriched Category Theory

Weighted meets and joins

Suppose $\mathscr{C}$ is a Q-category, $\mathscr{D}$ is a set, and suppose $D : \mathscr{D} \to \mathscr{C}$ and $W : \mathscr{D} \to \underline{Q}$ are functions.

- The meet of F weighted by W is an object $\bigwedge_{d\in\mathscr{D}}^{W} Dd \in (\mathscr{C})_0$ with the universal property: $\hom_{\mathscr{C}}\left(x, \bigwedge_{d\in\mathscr{D}}^{W} Dd\right) = \bigwedge_{d\in\mathscr{D}} [Wd, \hom_{\mathscr{C}}(x, Dd)]$
- The join of F weighted by W is an object $\bigvee_{d\in\mathscr{D}}^{W} Dd \in (\mathscr{C})_0$ with the universal property: $\hom_{\mathscr{C}}\left(\bigvee_{d\in\mathscr{D}}^{W} Dd, x\right) = \bigwedge_{d\in\mathscr{D}} [Wd, \hom_{\mathscr{C}}(Dd, x)]$

Lemma. Right Q-adjoints preserve weighted meets. Left Q-adjoints preserve weighted joins.

Proof. Suppose $R : \mathscr{C} \to \mathscr{C}'$ and consider the diagram $D : \mathscr{D} \to (\mathscr{C})_0$ with weight $W : \mathscr{D} \to Q$. Then,

$$\begin{aligned} \hom_{\mathscr{C}'}\left(x, R\bigwedge_{d\in\mathscr{D}}^{W} Dd\right) &= \hom_{\mathscr{C}}\left(Lx, \bigwedge_{d\in\mathscr{D}}^{W} Dd\right) \\ &= \bigwedge_{d\in\mathscr{D}} \left[Wd, \hom_{\mathscr{C}}(Lx, Dd)\right] \\ &= \bigwedge_{d\in\mathscr{D}} \left[Wd, \hom_{\mathscr{C}'}(x, RDd)\right] \\ &= \hom_{\mathscr{C}'}\left(x, \bigwedge_{d\in\mathscr{D}}^{W} RDd\right) \end{aligned}$$

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Categorical Network Diffusion

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$Q\mathscr{C}at$-enriched categories

- A $Q\mathscr{C}at$-category consists of
  - a collection $(\mathscr{C})_0$
  - for each pair $X, Y \in (\mathscr{C})_0$ a Q-category $\mathrm{Hom}_{\mathscr{C}}(X, Y)$
  - For every triple $X, Y, Z \in (\mathscr{C})_0$, a Q-functor $\circ_{X,Y,Z} : \mathrm{Hom}_{\mathscr{C}}(Y, Z) \times \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}]}(X, Z)$
- A pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ between $Q\mathscr{C}at$-categories is a function $F : (\mathscr{C})_0 \to (\mathscr{D})_0$ and a Q-functor $F_{X,Y} : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)$ satisfying the compatibility conditions
  - $F_{X,X}(id_X) \cong id_{FX}$
  - $F_{Y,Z}(g) \circ F_{X,Y}(f) \cong F_{X,Z}(g \circ f)$
- Example: $\underline{Q\mathscr{C}at}$ is the category where objects are Q-categories and where hom objects are

$$\mathrm{Hom}_{Q\mathscr{C}at}(\mathscr{C}, \mathscr{D}) = [\mathscr{C}, \mathscr{D}]$$

- Suppose $\mathscr{C}$ and $\mathscr{D}$ are $Q\mathscr{C}at$-categories and $F : \mathscr{C} \to \mathscr{D}$, $W : \mathscr{C} \to \underline{Q\mathscr{C}at}$ are pseudofunctors. Then, the limit of F weighted by W is an object $\lim^W F \in (\mathscr{D})_0$ such that the following is an isomorphism

$$\mathrm{Hom}_{\mathscr{D}}(X, \lim{}^W F) \cong [\mathscr{C}, \underline{Q\mathscr{C}at}]\Big(W, \mathrm{Hom}\big(X, F-\big)\Big)$$

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Categorical Network Diffusion

$\mathcal{Q}\mathscr{C}at$-valued (co)presheaves

- We consider a pair consisting of a presheaf $\underline{F} : \mathcal{J}^{op} \to \underline{\mathcal{Q}\mathscr{C}at}$ and copresheaf $\overline{F} : \mathcal{J} \to \underline{\mathcal{Q}\mathscr{C}at}$
  - $\underline{F}$ and $\overline{F}$ map nodes/edges to $\mathcal{Q}$-categories
  - We assume that $\underline{F}v = \overline{F}v = F_v$ and $\underline{F}e = \overline{F}e = F_e$ for all nodes $v \in \mathbb{X}_0$ & edges $e \in \mathbb{X}_1$
  - $\underline{F}_{e \trianglelefteq v}$ is a $\mathcal{Q}$-functor between $\mathcal{Q}$-categories F_v and F_e
  - $\overline{F}_{v \trianglelefteq e}$ is a $\mathcal{Q}$-functor between $\mathcal{Q}$-categories F_e and F_v
- We also consider the data of a weighting $W : \mathbb{X}_0 \times \mathbb{X}_0 \to \mathcal{Q}$
- Parallel transport defined for a path $\mathrm{tr}_\gamma : F_{v_1} \to F_{v_\ell}$(Bodnar et al, 2022)

  Consider a path $\gamma = v_1 \trianglelefteq e_1 \trianglerighteq v_2 \trianglelefteq e_2 \trianglerighteq \cdots \trianglelefteq e_{\ell-1} \trianglerighteq v_\ell$ in $\mathbb{X}$. Then, tr_γ is defined

$$\mathrm{tr}_\gamma := \overline{F}_{v_\ell \trianglelefteq e_{\ell-1}} \underline{F}_{e_{\ell-1} \trianglerighteq v_{\ell-1}} \cdots \overline{F}_{v_3 \trianglelefteq e_2} \underline{F}_{e_1 \trianglerighteq v_2} \overline{F}_{v_2 \trianglelefteq e_1} \underline{F}_{e_1 \trianglerighteq v_1}$$

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$$\underline{F} : \mathcal{J}^{op} \to Q\mathcal{C}at$$
$$\overline{F} : \mathcal{J} \longrightarrow Q\mathcal{C}at$$

Categorical Network Diffusion

Weighted Global Sections

- Let $*$ be the 1-object Q-category
- For $e = (v, w)$ let $\Delta(e)$ be the Q-category with objects $(\Delta(e))_0 = \{v, w\}$ and $\hom_{\Delta(e)}(v, w) = W(v, w)$, $\hom_{\Delta(e)}(w, v) = W(w, v)$
- Define $\tilde{W} : \mathcal{J}^{op} \to \underline{Q\mathscr{C}at}$ by sending nodes $v \in (\mathcal{J})_0$ to $*$ and edges $e \in (\mathcal{J})_0$ to$\Delta(e)$ with $\big(\Delta(e)\big)_0 = \partial(e)$ and let $\tilde{W}(e \trianglerighteq v)$ be the functor $* \to \Delta(e)$ which picks out the object v.
- Let $\Gamma^W(\mathbb{X}; \underline{F}) := \lim^W \underline{F}$ which is a Q-category, a subcategory of $\prod_{j \in (\mathcal{J})_0} \underline{F} j$

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Categorical Network Diffusion

Weighted global sections (continued)

- Define W-global sections to be elements $(x_v)_{v \in \mathbb{X}_0} \in \prod_{v \in \mathbb{X}_0} F_v$ such that for every $e = (v, w)$ we have

$$W(v,w) \preceq \hom_{F_e}\left(\underline{F}_{e \trianglerighteq v}(x_v), \underline{F}_{e \trianglerighteq w}(x_w)\right)$$

$$W(w,v) \preceq \hom_{F_e}\left(\underline{F}_{e \trianglerighteq w}(x_w), \underline{F}_{e \trianglerighteq v}(x_v)\right)$$

- Remark: if $W(u,v) = 1$ for all $(u,v) \in \mathbb{X}_0^2$, then

$$\hom_{F_e}\left(\underline{F}_{e \trianglerighteq v}(x_v), \underline{F}_{e \trianglerighteq w}(x_w)\right) = \hom_{F_e}\left(\underline{F}_{e \trianglerighteq w}(x_w), \underline{F}_{e \trianglerighteq v}(x_v)\right) = 1$$

  which implies $\underline{F}_{e \trianglerighteq v}(x_v) = \underline{F}_{e \trianglelefteq w}(x_w)$.

Theorem. The objects of $\Gamma^W(\mathbb{X}; \underline{F})$ are W-global sections. Furthermore,

$$\hom_{\Gamma(\mathbb{X};\underline{F})}\left((x_v)_{v \in \mathbb{X}_0}, (y_v)_{v \in \mathbb{X}_0}\right) = \bigwedge_{v \in \mathbb{X}_0} \hom_{F_v}(x_v, y_v)$$

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Categorical Network Diffusion

Tarski Laplacian

Definition. Given the data

$$\begin{array}{l} \underline{F}\colon \mathcal{J}^{op} \to \underline{Q\mathcal{C}at} \\ \overline{F}\colon \mathcal{J} \to \underline{Q\mathcal{C}at} \end{array} \quad W : \mathbb{X}_0 \times \mathbb{X}_0 \to Q,$$

the Tarski Laplacian is the map $\mathscr{L} : \prod_{v \in \mathbb{X}_0} F_v \to \prod_{v \in \mathbb{X}_0} F_v$ given by

$$(\mathscr{L}\mathbf{x})_v = \bigwedge_{v \trianglelefteq e \trianglerighteq w}^{W(v,-)} \overline{F}_{v \trianglelefteq e} \underline{F}_{e \trianglerighteq w}(x_w)$$

where $\mathbf{x} = \left(x_v\right)_{v \in \mathbb{X}_0}$.

Theorem. $\mathscr{L}$ is a functor of Q-categories.

Proof. Need to show $\mathbf{hom}_{\prod_{v \in \mathbb{X}_0}}\left(\mathbf{x}, \mathbf{y}\right) \preceq \mathbf{hom}_{\prod_{v \in \mathbb{X}_0}}\left(\mathscr{L}(\mathbf{x}), \mathscr{L}(\mathbf{y})\right)$.

Use that $\underline{F}$ and $\overline{F}$ are Q-functors and need a lemma about weighted limits.

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$\underline{F}_{e \trianglerighteq v}$
$\underline{F}_{e \trianglerighteq w}$
F_v
F_e
F_w
$\overline{F}_{v \trianglelefteq e}$
$\overline{F}_{w \trianglelefteq e}$
$W(v,-)$
$\bigwedge$
$w \trianglelefteq e \trianglerighteq v$
$W(v, w)$
v
$\trianglelefteq$
e
$\trianglerighteq$
w

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Interlude

Analogy between adjoint linear maps and adjoint functors

- Suppose $\mathscr{C}$ and $\mathscr{D}$ are Q-categories. Recall, $L \dashv R$ is an adjuction when

$$\hom_{\mathscr{D}}(Lx, y) = \hom_{\mathscr{C}}(x, Ry) \text{ for all } x \in (\mathscr{C})_0, y \in (\mathscr{D})_0$$

$$\Updownarrow$$

- Suppose V and W are $\mathbb{R}$-vector spaces. Then, $L : V \to W$ has a linear adjoint when

$$\langle Lx, y\rangle = \langle x, L^*y\rangle \text{ for all } x \in V, y \in W$$

Categorical Network Diffusion

Computing Global Sections

Definition. Suppose $q \in Q$. Let $S_q(\mathscr{L})$ denote the subcategory of $\prod_{v \in \mathbb{X}_0} F_v$ spanned by $\mathbf{x} = (x_v)_{v \in \mathbb{X}_0}$ such that

$$\hom_{\prod_{v \in \mathbb{X}_0} F_v}\left(\mathbf{x}, \mathscr{L}\mathbf{x}\right) \geq q$$

Lemma. Suppose $\underline{F}_{e \trianglerighteq v} \dashv \overline{F}_{v \trianglerighteq e}$ for every incidence $v \trianglelefteq e$ in $\mathscr{J}^{op}$. Then, $\vec{x} \in S_q(\mathscr{L})$ if and only if $\hom_{F_e}\left(\underline{F}_{e \trianglerighteq v}(x_v), \underline{F}_{e \trianglerighteq w}(x_w)\right) \geq q \otimes W(v, w)$ for every $e = (v, w) \in \mathbb{X}_1$

Proof. We have

$$\begin{aligned}
\hom_{\prod_{v \in \mathbb{X}_0} F_v}\left(\vec{x}, \mathscr{L}\vec{x}\right) &= \bigwedge_{v \in \mathbb{X}_0} \hom_{F_v}\left(x_v, \mathscr{L}(\vec{x})_v\right) \\
&= \bigwedge_{v \in \mathbb{X}_0} \hom_{F_v}\left(x_v, \bigwedge^{W(v,-)}_{v \trianglelefteq e \trianglerighteq w} \overline{F}_{v \trianglelefteq e} \underline{F}_{e \trianglerighteq w}(x_w)\right) \\
&= \bigwedge_{v \in \mathbb{X}_0} \bigwedge_{v \trianglelefteq e \trianglerighteq w} \left[W(v, w), \hom_{F_v}\left(x_v, \overline{F}_{v \trianglelefteq e} \underline{F}_{e \trianglerighteq w}(x_w)\right)\right]
\end{aligned}$$

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Categorical Network Diffusion

Computing Global Sections

Proof (continued).

$$\begin{aligned}\hom_{\prod_{v\in\mathbb{X}_0} F_v}\left(\vec{x}, \mathscr{L}\vec{x}\right) &= \bigwedge_{v\in\mathbb{X}_0}\bigwedge_{v\trianglelefteq e\trianglerighteq w}\left[W(v,w), \hom_{F_v}\left(x_v, \overline{F}_{v\trianglelefteq e}\underline{F}_{e\trianglerighteq w}(x_w)\right)\right]\\ &= \bigwedge_{v\in\mathbb{X}_0}\bigwedge_{v\trianglelefteq e\trianglerighteq w}\left[W(v,w), \hom_{F_e}\left(\underline{F}_{e\trianglerighteq w}(x_v), \underline{F}_{e\trianglerighteq w}(x_w)\right)\right]\\ &\geq q\end{aligned}$$

if and only if

$$\left[W(v,w), \hom_{F_e}\left(\underline{F}_{e\trianglerighteq v}(x_v), \underline{F}_{e\trianglerighteq w}(x_w)\right)\right] \geq q \text{ for all } e = (v,w) \in \mathbb{X}_1$$

if and only if

$$\hom_{F_v}\left(\underline{F}_{e\trianglerighteq w}(x_v), \underline{F}_{e\trianglerighteq w}(x_w)\right) \geq q \otimes W(v,w) \text{ for all } e = (v,w) \in \mathbb{X}_1. \square$$

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Categorical Network Diffusion

Hodge-Tarski Theorem

Theorem. Given the data

$$\underline{F}\colon \mathcal{J}^{op} \to \underline{\mathcal{Q}\mathscr{C}at}$$

$$\overline{F}\colon \mathcal{J} \to \underline{\mathcal{Q}\mathscr{C}at} \quad ,$$

$$W\colon \mathbb{X}_0 \times \mathbb{X}_0 \to Q$$

suppose $\underline{F}_{e \trianglerighteq v} \dashv \overline{F}_{v \trianglerighteq e}$ for every incidence $v \trianglelefteq e$ in $\mathcal{J}^{op}$. Then, $\Gamma^W(\mathbb{X}; \underline{F}) \cong S_1(\mathscr{L})$.

- Compare to the Hodge Theorem:

$$H^0_{dR}(\mathbb{M}; \mathbb{R}) \cong \ker \Delta_0$$

- Compare to the Hodge Theorem for network sheaves ($\mathscr{C} = \mathscr{H}ilb$):

$$\Gamma(\mathbb{X}; \underline{F}) \cong \ker \Delta_0$$

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Tarski Fixed Point Theorem

Theorem. Suppose $\mathscr{C}$ is a Q-category and $\mathscr{L} : \mathscr{C} \to \mathscr{C}$ is a Q-functor. Suppose $\mathscr{C}$ has all weighted joins. Then, for every $q \in Q$, the category $S_q(\mathscr{L})$ generated by $x \in (\mathscr{C})_0$ such that $\hom_{\mathscr{C}}(x, \mathscr{L}x) \geq q$ has all weighted meets and joins.

- Work in progress to prove similar result for categories enriched in an arbitrary cosmos

cosmos = bicomplete closed symmetric monoidal

Corollary. Suppose $F_v \in \left(Q\mathscr{C}at\right)_0$ has all weighted meets and joins for all $v \in \mathbb{X}_0$. Then, $\Gamma^W(\mathbb{X}; \underline{F}) \cong S_1(\mathscr{L})$ has all weighted meets and joins.

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Applications

Applications

Logic, engineering, & economics

- Applications with $F_v = [\mathscr{C}^{op}, Q]$ for $\mathscr{C}$ a discrete Q-category
  - $Q = \{0,1\}$, network multi-modal logic (R. & Ghrist, 2022)
  - $Q = [-\infty, \infty]$, synchronization of max-plus linear systems (R., Zavlanos, 2023)
  - $Q = [0,1]$, distributed fuzzy formal concept analysis (Ghrist & Lopez, TBD)
- Other applications
  - Q-valued preference relations
  - network preference dynamics (R., Ghrist, Henselman-Petrusek, Bell, Zavlanos, 2024)

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Thank You

Any questions?

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