Algebraic Methods in Complex Systems

Dr. Hans Riess
hans@hansriess.com

About me

In 2016, I received a B.S. in pure mathematics from Duke University. In 2022, supervised by Robert Ghrist, I received a Ph.D. in Electrical and Systems Engineering from the University of Pennsylvania. Currently, I am a Postdoctoral Associate in the Autonomous Systems Lab directed by Michael M. Zavlanos at Duke University.

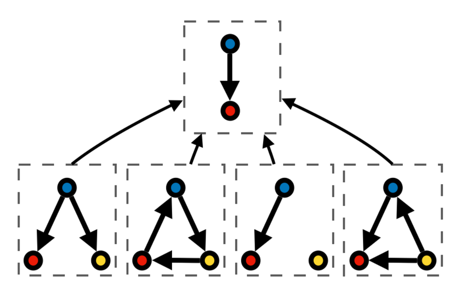
Research summary

I develop theories and algorithms for the analysis and design of multi-agent systems and machine learning models. The central theme of my work is **networks** and **discrete-time dynamical systems** on them. My research style incorporates branches of abstract mathematics that have not yet matured in engineering. My most significant theoretical contribution of my work has been the development of a Laplacian for order lattices—as opposed to vector spaces which Laplace operators ordinarily act on—called the **Tarski Laplacian.**

Applications

Some applications of the Tarski Laplacian include:

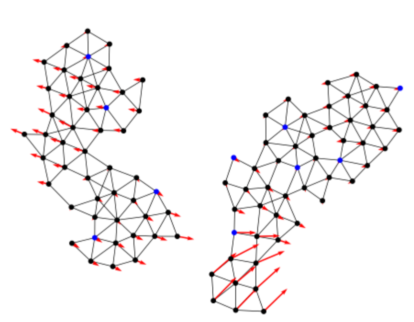
- Network dynamics of preference orderings
- Information diffusion via epistemic logic
- Synchronization of discrete event systems



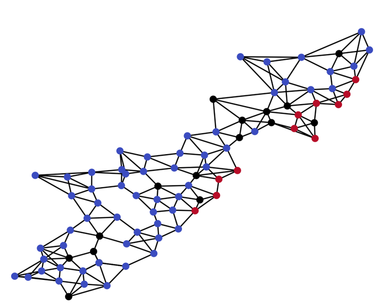
Aggregation of preference relations

Complex Systems

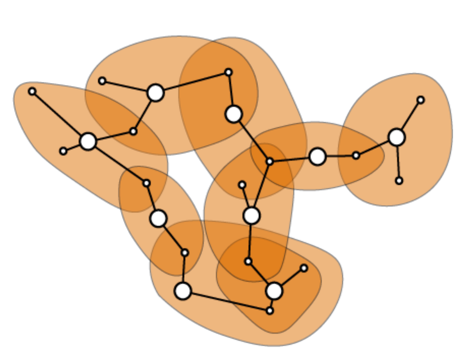
Navigating complex relationships between components of a system, or features in a dataset, requires algorithms as well as data structures that facilitate the propagation of different kinds of information across a system. In a system with agents or data-sources represented as nodes, we model the interactions via the topology of a graph or hypergraph. Once information about the states of agents propagates, this information can be used to synthesize control policies, a particularly-simple example being consensus. Similarly, propagation of features via graph diffusion has shown its advantages in machine learning.



multi-agent systems



sensor networks

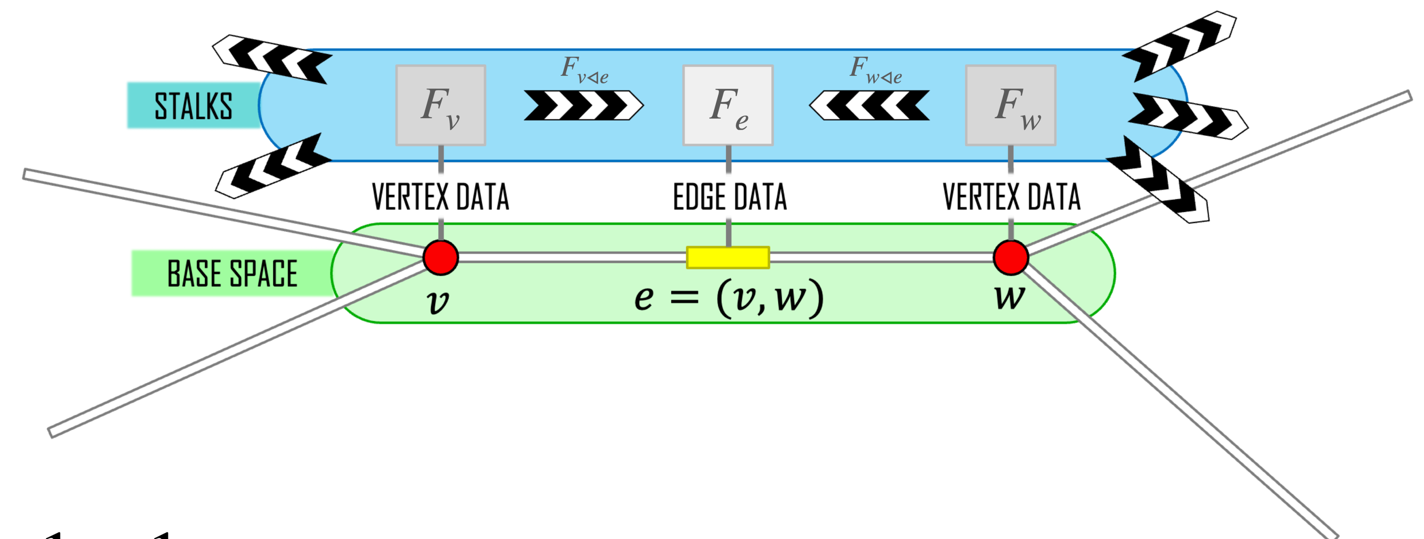


social networks

Network sheaves

Sheaves are a mathematical scaffolding for attaching data to base space such as a (hyper)graph representing a complex system. A sheaf can be valued in any data category such as the category of finite dimensional real vector spaces, or a different category of algebraic structures. Given an undirected graph $G = (V, E)$, a **network sheaf** assigns:

- an object F_v to every node $v \in V$ (e.g. a vector space)
- an object F_e to every edge $e \in E$ (e.g. a vector space)
- a map $F_{v \trianglelefteq e} : F_v \to F_e$ for every edge $e = (v, w)$ (e.g a linear map)

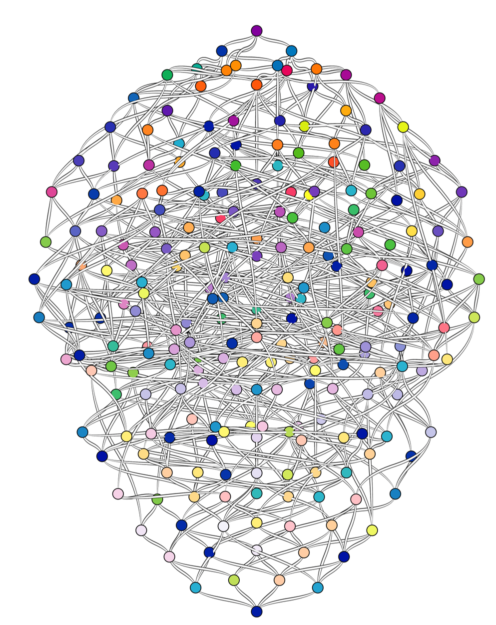


Algebraic structures

While vector spaces have been the cornerstone of much of modern analysis and computation, many systems—especially in multi-agent and cyber-physical settings—demand richer algebraic structures.

Order lattices

Lattices naturally capture order relationships, qualitative states, and decision hierarchies, enabling the modeling of phenomena such as preference dynamics, knowledge diffusion, and synchronization in ways that vector spaces cannot. By expanding beyond vector spaces to incorporate structures like lattices, we unlock new frameworks for understanding and designing systems with complex, discrete, and hierarchical interactions.



Formal concept lattice of an object/attribute table

Enriched categories

Category theory is a formal mathematical theory about abstract arrows usually representing functions and how to compose them. Enriched category theory takes this one step further by attributing objects in another category to each arrow. When categories are enriched in ordered monoids called quantiles, we obtain a composition law:

$$\hom_{\mathsf{C}}(y,z) \otimes \hom_{\mathsf{C}}(x,y) \leq \hom_{\mathsf{C}}(x,z)$$

$$1 \leq \hom_{\mathsf{C}}(x,x)$$

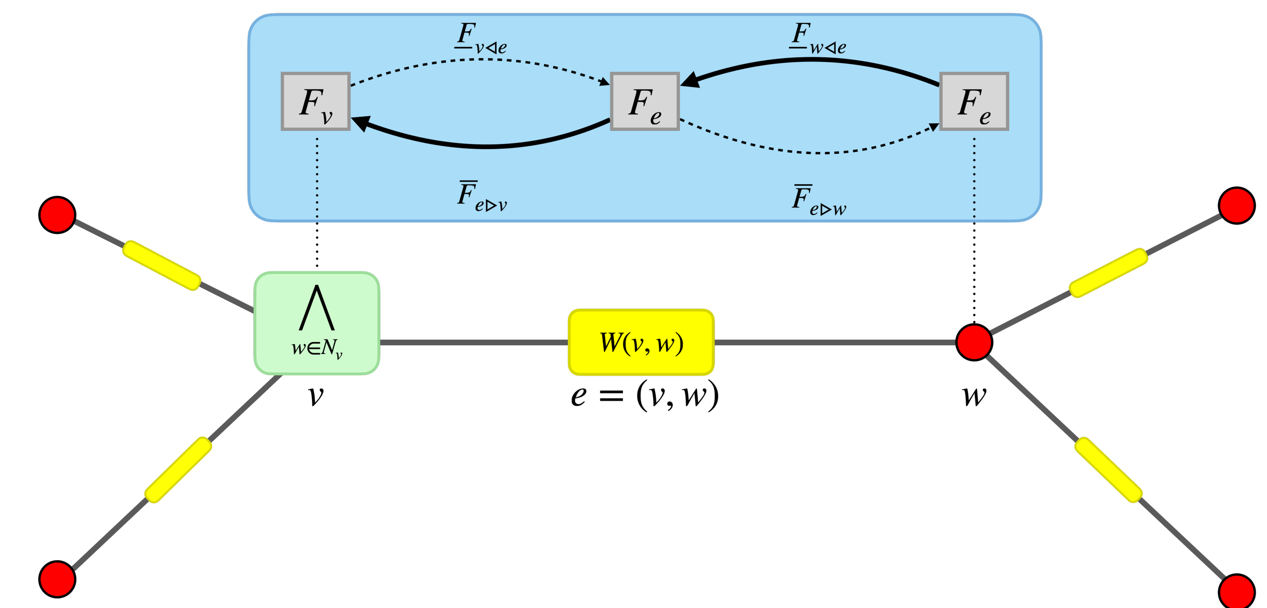
For instance, this law recovers the triangle inequality of metric spaces.

Tarski Laplacian

Consistent assignments $(x_v)_{v\in V} \in \prod_{v \in V} F_v$ of sheaves called, **global sections,** satisfy the local property

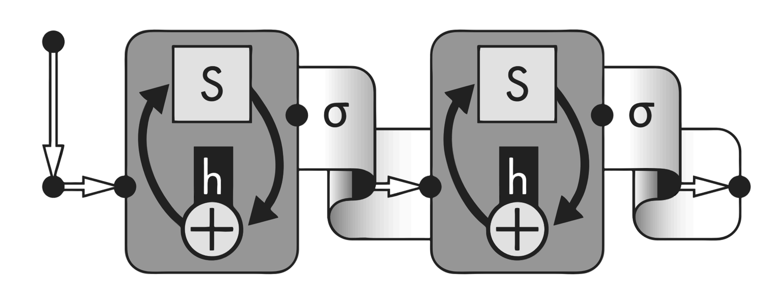
$$F_{v \trianglelefteq e}(x_v) = F_{w \trianglelefteq e}(x_w) \quad \forall e = (v, w)$$

which generalizes consensus. The graph Laplacian is useful for consensus problems such as flocking, but a new Laplacian is needed when the data is not in vector form. The **Tarski Laplacian** is a local operator that leads to a discrete-time dynamical system on a sheaf valued in the category of lattices. The main result is *the fixed points of this dynamical system are identified as the global sections.*

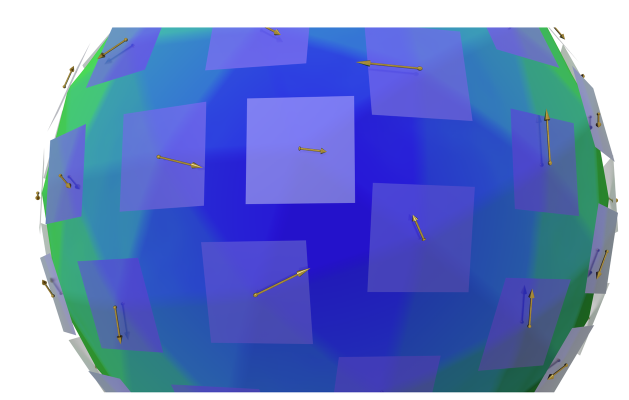


Geometric deep learning

Deep learning has been increasingly important in making predictions about complex systems. Geometric deep learning (GDL) is a new branch of machine learning which exploits geometric structure such as group actions and parallel transport to introduce inductive bias in a model. In the future, we want to build graph neural networks that utilize the Tarski Laplacian for message-passing.



Graph neural network (GNN)



Tangent bundle $T\mathbb{S}^2$

Find out more

Please scan the QR code to be directed to my website (hansriess.com) where you can find links to my papers and contact info.